Abstract

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Fine Bruhat intersections for reductive groups have been studied by several authors in connection with Kazhdan-Lusztig theory, canonical bases and Lie Theory. The purpose of this thesis is to study the analogous intersections for reductive monoids. We determine the conditions under which the following intersections

$$\begin{aligned} B\sigma B \cap B^{-}\theta B \\ B\sigma B \cap B^{-}\theta B^{-} \end{aligned}$$

are nonempty. First we study these intersections for the monoid $M_n(K)$. This work gives rise to two new orderings, $\leq_1$ and $\leq_2$, on the monoid of partial permutation matrices. More precisely, $\leq_1$ and $\leq_2$ are orderings that exist within a particular $\mathcal{J}$ -class for a reductive monoid M . The $\mathcal{J}$ -classes of $M_n(K)$ consist of matrices of the same rank. Combinatorial descriptions of the orderings are given and their relation to the Bruhat-Chevalley order is discussed. These results are then generalized to an arbitrary reductive monoid. We show that $\leq_1$ is a partial order on R , the Renner monoid, and that $\leq_2$ is in general not a partial order on elements of R , but rather on equivalence classes of elements in R . We describe the equivalence classes for the matrices and conclude with theorems for the partial permutation matrices of rank $r < n$.

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FINE BRUHAT INTERSECTIONS FOR REDUCTIVE MONOIDS

BY

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Dedication

To my mother

Biography

The author was born in Durham, North Carolina in 1976 and grew up in Danville, VA. She attended Virginia Commonwealth University where she received a Bachelor of Science in Mathematics in 1999. From there she attended graduate school at North Carolina State University and received a Master of Science in Mathematics in 2001.

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Introduction

The study of reductive monoids originated around 1980 and has been developed primarily by Mohan Putcha and Lex Renner. This area of mathematics is a combination of the study of algebraic geometry, semigroups, and algebraic group theory.

Fine Bruhat intersections for reductive groups have been useful in several areas of mathematics such as Kazhdan-Lusztig theory and Lie theory.

The intersections in this paper arise from the following decompositions for a reductive monoid M :

$$M = \bigsqcup_{\sigma, \theta \in R} (B\sigma B \cap B^{-}\theta B)$$

and

$$M = \bigsqcup_{\sigma, \theta \in R} (B\sigma B \cap B^{-}\theta B^{-})$$

where B is a Borel subgroup of the unit group G of M and R is the Renner monoid, the analogue of the Weyl group for a reductive group. In general, one may think of B and B^{-} as abstract analogues of the invertible upper and lower triangular matrices respectively, and of R as an abstract analogue of the permutation matrices. In this paper we determine the conditions under which the following intersections

$$B\sigma B \cap B^{-}\theta B$$

and

$$B\sigma B \cap B^{-}\theta B^{-}$$

are nonempty.

The first chapter is designed to give the reader an overview of the basic background knowledge necessary to understand the remainder of the paper. Examples appear throughout the text to illustrate the theorems and definitions. Where details, proofs, etc. are needed, the appropriate references are given.

In Chapter 2, we consider only the monoid $M_n(K)$, the monoid of all $n \times n$ matrices over a field K . We determine when each of the above intersections are nonempty if σ and θ are rank 1 and rank 2 partial permutation matrices, respectively. We conclude this chapter with theorems for the partial permutation matrices of rank r , for $r < n$.

In Chapter 3, we generalize the results from Chapter 2 to an arbitrary reductive monoid. These results will then be used in the final chapter to prove the theorems stated at the end of Chapter 2.

Chapter 1

Preliminaries

The results obtained in this paper require some background information from semigroup theory, algebraic geometry, algebraic groups and algebraic monoid theory.

In this chapter we sketch most of the prerequisite concepts needed in the subsequent chapters. Theorems are stated without proof, but are accompanied by references and examples to aid in understanding.

1.1 Semigroup Theory

A *semigroup* is a set S together with an associative operation $\mu : S \times S \rightarrow S$. If the set S also contains an element 1 such that $1s = s1 = s$ for all $s \in S$, then S is called a *monoid*. The element 1 is called an *identity element* for S . If the set S does not contain 1 , we may adjoin the identity by $S \cup \{1\}$ and denote it by S^1 .

EXAMPLE 1.1.1. Let S be the set of all integers under multiplication. Then S is a semigroup.

EXAMPLE 1.1.2. $M = M_n(K)$, the set of all $n \times n$ matrices with entries in a field K , with the usual matrix multiplication is a monoid .

The monoid $M_n(K)$ proves to be a very useful concrete example that will be used frequently throughout this paper to provide a better understanding of abstract definitions.

A subsemigroup of S which is a group is called a *subgroup* of S . If $X \subseteq S$ $E(X) = \{e \in X | e^2 = e\}$ is the set of *idempotent elements* of X . Idempotent elements play a key role in the structure theory for monoids. If M is a monoid, then we may give $E(M)$, its set of idempotents, the following partial order

$$f \leq e \Leftrightarrow ef = f = fe \quad (1.1)$$

This ordering will arise later in the chapter in Section 4 when defining the cross-section lattice.

An ideal of S is a nonempty subset I of S such that if $x \in I$ then $S^1 x S^1 \subseteq I$. The idea of an ideal leads naturally to the consideration of some very useful equivalence relations on a semigroup. These equivalence relations, called *Green's Relations*, were first studied by J. A. Green [7]. They allow us to compare and relate elements in a semigroup.

DEFINITION 1.1.1. Let M be a monoid and $a, b \in M$. Then

1. $a\mathcal{R}b$ if $aM = bM$
2. $a\mathcal{L}b$ if $Ma = Mb$
3. $a\mathcal{J}b$ if $MaM = MbM$
4. $a\mathcal{H}b$ if $a\mathcal{R}b$ and $a\mathcal{L}b$
5. $a\mathcal{D}b$ if $a\mathcal{R}c$ and $c\mathcal{L}b$ for some $c \in M$

EXAMPLE 1.1.3. Let $M = M_n(K)$ and $A, B \in M$. Now, the relations $\mathcal{R}, \mathcal{L}, \mathcal{J}$ are familiar ideas from matrix theory. $A\mathcal{L}B$ if and only if A and B are row equivalent. $A\mathcal{R}B$ if and only if A and B are column equivalent and $A\mathcal{J}B$ if and only if A and B have the same rank.

REMARK 1.1.1. If $e, f \in M$ are idempotents, then we see from Green's Relations that [8]:

1. $e\mathcal{R}f$ if and only if $ef = f$ and $fe = e$

2. $e\mathcal{L}f$ if and only if $ef = e$ and $fe = f$
3. $e\mathcal{H}f$ if and only if $e = f$

We will use the following notation to represent an equivalence class defined by Green's Relations:

$$H_e = \{x \in M \mid x\mathcal{H}e\}$$

Similarly, J_e, R_e etc. represent the $\mathcal{J}$ -class of e and the $\mathcal{R}$ -class of e respectively.

An element $a \in S$ is *regular* if $axa = a$ for some $x \in S$. We say that S is *regular* if each of its elements are regular. Let M be a monoid. Sitting inside of every monoid is a set of invertible elements. These invertible elements are called *units* and form what is called the unit group G of M . We say that M is *unit regular* if for each $a \in M$, there exists a $g \in G$ such that $a = aga$.

EXAMPLE 1.1.4. $GL_n(K)$ is the unit group of $M_n(K)$ and $M_n(K)$ is unit regular.

A semigroup S is called an *inverse semigroup* if for each $x \in S$, there exists a unique element $x^* \in S$ such that

$$xx^*x = x \text{ and } x^*xx^* = x^*.$$

This is a very special type of semigroup which will play an important role in this paper.

EXAMPLE 1.1.5. Let N be the set of $n \times n$ matrices with at most one nonzero entry in each row or column. N is an inverse monoid.

REMARK 1.1.2. Denote by R the set of partial permutation matrices, the $n \times n$ matrices with at most one 1 in each row or column. R is a very important monoid which will arise later when we discuss Renner's Decomposition for $M_n(K)$. For more details on semigroup theory see one of several excellent texts: [8], [1], [4].

1.2 Basic Algebraic Geometry

Algebraic geometry is a vast and complicated subject. In this section we will give only a brief overview of the necessary material needed for the remainder of the paper. The reader who wishes to gain a deeper knowledge of algebraic geometry (in the context of algebraic groups) should refer to [9]. Other notable references include [16] and [18].

Throughout the rest of this paper we will assume that K is an algebraically closed field.

We define *affine n -space* over K to be $K^n = K \times K \times \dots \times K$ and denote it by $\mathbb{A}^n$. This is just the set of all n -tuples of elements of K . The topology we give $\mathbb{A}^n$ is called the *Zariski Topology*. This topology is defined by closed sets. A set $X \subseteq \mathbb{A}^n$ is *closed* in $\mathbb{A}^n$ if it is the set of common zeros of a finite collection of polynomials in the polynomial ring $K[x_1, \dots, x_n]$. So, one may think of curves, surfaces and the like. If we think of curves in $\mathbb{A}^2$ as closed sets, it is easy to see that the complement of a curve is open and “big”. It turns out that in the Zariski Topology open sets are dense, hence any two nonempty open sets must intersect. From this we also can see that this topology is compact, however not Hausdorff. This is already evident in the case of $\mathbb{A}^1$.

An *affine variety* is a set that is isomorphic to a closed set in $\mathbb{A}^n$.

EXAMPLE 1.2.1. Let $X = K \setminus \{0\}$. Clearly X is not a closed subset of $K = \mathbb{A}^1$, however, $X \cong \{(x, y) | xy = 1\}$ by sending $x \mapsto (x, \frac{1}{x})$ which is closed in $K \times K = \mathbb{A}^2$. Thus X is an affine variety.

Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be affine varieties. By a *morphism* $\phi : X \rightarrow Y$, we mean a mapping of the form $\phi(x_1, \dots, x_n) = (\psi_1(x), \dots, \psi_m(x))$ where each $\psi_i \in K[X]$. Since affine varieties are defined by polynomial equations a morphism of affine varieties is simply a polynomial map. A morphism f is said to be *dominant* if $\overline{f(X)} = Y$, where closure denotes the Zariski closure.

EXAMPLE 1.2.2. Let $X = Y = M_3(K)$. Define $f : X \rightarrow Y$ by

$$f(A) = \det(A)A$$

Then f is clearly a polynomial map and

$$f(M_3(K)) = GL_3(K) \cup \{0\}$$

thus, $\overline{f(M_3(K))} = M_3(K)$ and hence dominant.

A topological space X is said to be *irreducible* if it cannot be written as the union of two proper, nonempty, closed subsets. Notice that X is irreducible if and only if any two nonempty open sets in X have nonempty intersection, or equivalently, any nonempty open set is dense.

EXAMPLE 1.2.3. $\mathbb{A}^1 = K$ is irreducible since any proper, closed subset of $\mathbb{A}^1$ is finite. More generally, $\mathbb{A}^n$ is irreducible.

EXAMPLE 1.2.4. Let $X = \{(x, y) | xy = 0\} \subseteq \mathbb{A}^2$. X is reducible since $X = \{(x, y) | x = 0\} \cup \{(x, y) | y = 0\}$.

1.3 Algebraic Group Theory

Algebraic group theory is the generalization of the theory of algebraic monoids. This section contains the fundamentals needed to develop the theory of algebraic monoids. In particular, our aim in this section is to provide enough background information to state the Bruhat Decomposition for reductive groups.

Throughout this section all algebraic varieties considered are assumed to be affine.

DEFINITION 1.3.1. A *linear algebraic group* (*algebraic group*) is an affine variety such that the two maps $\mu : G \times G \rightarrow G$, where $\mu(x, y) = xy$, and $i : G \rightarrow G$, where $i(x) = x^{-1}$, are morphisms of varieties (polynomial maps).

REMARK 1.3.1. Note that $G \times G$ is given the Zariski topology rather than the product topology, thus an algebraic group is not a topological group.

EXAMPLE 1.3.1. Let $G = GL_n(K)$. We may consider G as a subset of K^{n^2} in the following way:

$$GL_n(K) = \{(a_{11}, \dots, a_{nn}) | a_{ij} \in K\}$$

As a subset of K^{n^2} , $GL_n(K)$ is not closed, however, $GL_n(K)$ is closed in K^{n^2+1} if we consider $GL_n(K)$ as follows:

$$GL_n(K) = \{(a_{11}, \dots, a_{nn}, b) \mid b = \frac{1}{\det A}\}$$

Therefore $GL_n(K)$ is an algebraic group.

Other classic examples of algebraic groups include:

1. $B_n(K)$ the invertible $n \times n$ upper triangular matrices
2. $T_n(K)$ the invertible $n \times n$ diagonal matrices
3. $U_n(K)$ the unipotent $n \times n$ upper triangular matrices
4. $SP_n(K)$ the symplectic group

Let G be an algebraic group. The irreducible components of G are precisely the connected components (We will see this is not the case for algebraic monoids). Moreover, there is a unique, connected component containing the identity called the *identity component*, denoted by G^0 . G^0 is normal in G and has finite index. For a proof of this see [9]. An algebraic group is connected if $G = G^0$. In algebraic group theory we use the term connected and reserve the term irreducible for group representations.

EXAMPLE 1.3.2. The monoid $M_n(K)$ of all $n \times n$ matrices over K may be identified with $\mathbb{A}^{n^2}$. Hence $GL_n(K)$ may be identified with the open subset defined by the nonvanishing of the determinant function, which is a polynomial equation. Therefore $GL_n(K)$ is connected.

THEOREM 1.3.1. [9] *Let G be a connected algebraic group*

1. *G is solvable if it is isomorphic to a closed subgroup of $B_n(K)$ for some n .*
2. *G is unipotent if it is isomorphic to a closed subgroup of $U_n(K)$ for some n .*
3. *G is a torus if it is isomorphic to a closed subgroup of $T_n(K)$ for some n .*

One special class of algebraic groups is *reductive groups*. These groups will be very important to the discussion of reductive monoids in Section 1.4. In order to define reductive group, we first need to understand the notion of unipotent.

We say that a matrix $u \in GL_n(K)$ is *unipotent* if all its eigenvalues are 1. A subgroup of $GL_n(K)$ is unipotent if each of its elements are unipotent. For example, the group

$$\begin{aligned} U_n(K) &= \{I + \sum_{1 \leq i < j \leq n} c_{ij} E_{ij} \mid c_{ij} \in K\} \\ &= \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \end{aligned}$$

of all unitriangular matrices is a unipotent.

Each algebraic group has a unique maximal solvable subgroup, maximal unipotent subgroup and maximal torus. If $G = GL_n(K)$, then the maximal subgroups of each type are precisely the subgroups stated in Theorem 1.3.3.

DEFINITION 1.3.2. Let G be a connected algebraic group. The maximal, closed, connected, unipotent, normal subgroup is called the *unipotent radical* of G and is denoted by $R_u(G)$.

DEFINITION 1.3.3. An algebraic group G is *reductive* if $R_u(G) = \{1\}$.

The prototype for reductive groups is $GL_n(K)$. Other examples of reductive groups include: $SL_n(K)$, $SP_n(K)$ (the symplectic group), any torus, any semisimple group.

EXAMPLE 1.3.3. Let $G = GL_n(K)$. Now any normal subgroup of G either contains $SL_n(K)$ or is contained in the center $Z(G) = \{cI \mid c \in K^*\}$. Notice, $R_u(G) \not\subseteq SL_n(K)$, for if it did, it would contain many matrices whose eigenvalues are not all 1. Therefore $R_u(G) \subseteq Z(G)$. Clearly the only scalar multiple of the identity with all eigenvalues 1 is the identity, thus $R_u(G) = \{1\}$ and hence $GL_n(K)$ is reductive.

EXAMPLE 1.3.4. Let $G = B_n(K)$ where $n \geq 2$, the invertible upper triangular matrices. Then $R_u(G) = U_n(K)$, thus $B_n(K)$ is not reductive.

One type of subgroup that will be of particular importance to us is the *Borel subgroup*. A *Borel subgroup* of G is a maximal closed connected solvable subgroup. Associated with each Borel subgroup, B , is another Borel subgroup, B^- , called the opposite Borel subgroup. Two Borel subgroups are opposite if $B \cap B^- = T$ where T is a maximal torus. In a group G , Borel subgroups exist and are uniquely determined up to conjugacy [9](Theorem 21.3). Another nice result, [9](Theorems 19.3(b), 19.5), is the following: B can be decomposed as a semidirect product,

$$B = TU \tag{1.2}$$

$$= TR_u(B) \tag{1.3}$$

where T is the maximal torus in B (which is the same as being maximal in G [9](21.2, Corollary A) and U is the normal subgroup of B consisting of the unipotent elements in B .

EXAMPLE 1.3.5. Let $G = GL_n(K)$. Then $B = B_n(K)$, the invertible upper triangular matrices and $B^- = B_n^-(K)$, the invertible lower triangular matrices since $B \cap B^- = T$ where $T = T_n(K)$ the invertible diagonal matrices. And

$$\begin{aligned} B_n(K) &= T_n(K)U_n(K) \\ &= T_n(K)R_u(B) \quad \text{From Example 1.3.4} \end{aligned}$$

In general one can think of Borel subgroups (respectively, opposite Borel subgroups) as abstract analogues of the invertible upper (respectively, lower) triangular matrices and of tori as abstract analogues of the invertible diagonal matrices.

The next important concept is that of the Weyl group of a reductive group G .

DEFINITION 1.3.4. Let G be a group with maximal torus T . The *Weyl group* of G relative to T is given by

$$W(G, T) = N_G(T)/T$$

where $N_G(T)$ is the normalizer of T in G .

To simplify notation a bit, we will denote the Weyl group of G relative to T by $W = N_G(T)/T$.

As usual, let us refer to the matrices for an example.

EXAMPLE 1.3.6. Let $G = GL_2(K)$, $T = T_2(K)$. Then

$$N_G(T) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} \mid ab \neq 0, cd \neq 0 \right\}$$

and

$$T = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid xy \neq 0 \right\}$$

Thus,

$$W = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

which is the group of 2×2 permutation matrices.

In general for $G = GL_n(K)$ the Weyl group $W \simeq S_n$, the permutation group of n letters.

The Weyl group is generated by the set, S , of simple reflections, i.e. $W = \langle S \rangle$; for details on the set of simple reflections the reader should consult [10]. Each element of W may be written in the form

$$w = s_1 s_2 \dots s_k$$

for some $s_i \in S$. The expression of $w \in W$ as a product of generators is called a *word* and is not unique. If k is as small as possible we call it the length of w and denote it by $l(w) = k$. By convention we say that $l(1) = 0$ and denote the longest element of W by w_0 .

Of the many ways to partially order W , the most useful way that is compatible with the length function is called the Bruhat-Chevalley ordering (commonly referred to as the Bruhat order). The Bruhat-Chevalley ordering is defined as follows:

$$\theta \leq \sigma \text{ if } \theta \text{ is a reduced expression of } \sigma$$

Any subset $I \subseteq S$ generates a subgroup $W_I = \langle I \rangle$ of W . Let G be a reductive group with maximal torus T . Let B, B^- be opposite Borel subgroups such that $B \cap B^- = T$. For each $I \subseteq S$ we define P_I and P_I^- as follows

$$\begin{aligned} P_I &= BW_I B \\ P_I^- &= B^- W_I B^- \end{aligned}$$

P_I and P_I^- are subgroups of G containing B and B^- respectively. Subgroups which contain a Borel subgroup are called *parabolic subgroups* (21.2 Corollary B [9]). We define L_I as follows

$$L_I = P_I \cap P_I^-$$

L_I is called a *Levi subgroup* of G . L_I is a reductive group with Weyl group W_I . We denote the opposite Borel subgroups of L_I by B_L and B_L^- where

$$\begin{aligned} B_L &= L_I \cap B \\ B_L^- &= L_I \cap B^- \end{aligned}$$

For the proofs of the following theorems see [9].

THEOREM 1.3.2. $P_I = L_I U_I$ and $P_I^- = L_I U_I^-$.

THEOREM 1.3.3. $B = B_L U_I$ and $B^- = U_I^- B_L$

EXAMPLE 1.3.7. Let $G = GL_n(K)$. Then $W \simeq S_n$ so W is the set of $n \times n$ permutation matrices. If $I \subseteq S$, then in general

$$\begin{aligned} P_I &= \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \\ P_I^- &= \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \end{aligned}$$

where the *'s represent arbitrary matrices of the appropriate size and the diagonal

*'s are, in addition, invertible. Also,

$$U_I = \left\{ \begin{pmatrix} I & & * \\ & \ddots & \\ 0 & & I \end{pmatrix} \right\}$$

$$U_I^- = \left\{ \begin{pmatrix} I & & * \\ & \ddots & \\ 0 & & I \end{pmatrix} \right\}$$

and the intersection of P_I and P_I^- is

$$L_I = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$$

which consists of invertible block diagonals. It is easy to see Theorem 1.3.7, that $P_I = L_I U_I$ and $P_I^- = L_I U_I^-$. Theorem 1.3.8 is clear for the matrices as well.

We now introduce the Bruhat Decomposition for a reductive group. The Bruhat Decomposition is the most important individual fact of the structure theory of reductive groups. It asserts that the double cosets $B \backslash W / B$ correspond bijectively to the elements of the Weyl group W .

THEOREM 1.3.4 (Bruhat Decomposition). [9] *Let G be a reductive group with Borel subgroup $B \supseteq T$. Then G can be written as the following disjoint union:*

$$G = \bigsqcup_{\sigma \in W} B \sigma B$$

where W is the Weyl group of G . For a proof see [9].

There are several translates of the Bruhat Decomposition. One or both of the Borel subgroups B can be replaced with its opposite Borel subgroup B^- . Recall, the longest element of W is denoted by w_0 and that all Borel subgroups are conjugate in G . If we conjugate B by w_0 then we get the following [9]:

$$B^- = w_0 B w_0$$

Note that $w_0^{-1} = w_0$. In doing this we arise at two new decompositions for a reductive group.

$$\begin{aligned}
 G = w_0 G &= \bigsqcup_{\sigma \in W} w_0 B w_0 w_0 \sigma B \\
 &= \bigsqcup_{\sigma \in W} w_0 B w_0 (w_0 \sigma) B \\
 &= \bigsqcup_{\sigma \in W} B^- \sigma B \quad \text{by a change of variable}
 \end{aligned}$$

which is called the *Birkhoff Decomposition*. In replacing both Borel subgroups with B^- we have

$$G = \bigsqcup_{\sigma \in W} B^- \sigma B^-$$

EXAMPLE 1.3.8. Let $G = GL_n(K)$, then the Bruhat Decomposition says that any matrix $A \in GL_n(K)$ can be written as a product of an upper triangular matrix, permutation matrix and another upper triangular matrix where the permutation matrix is unique i.e.

$$A = B_1 P B_2$$

There are two more decompositions we can consider for a reductive group. They are the following:

$$G = \bigsqcup_{\sigma, \theta \in W} (B \sigma B \cap B^- \theta B) \tag{1.4}$$

$$G = \bigsqcup_{\sigma, \theta \in W} (B \sigma B \cap B^- \theta B^-) \tag{1.5}$$

The question that arose in algebraic group theory was when are these intersections nonempty. ie.

$$B \sigma B \cap B^- \theta B \neq \phi \Leftrightarrow ?? \tag{1.6}$$

$$B \sigma B \cap B^- \theta B^- \neq \phi \Leftrightarrow ?? \tag{1.7}$$

The second intersection, 1.7, is always nonempty. Let us take a moment to see why. First we notice that

$$B\sigma B \cap B^{-}\theta B \neq \phi \Leftrightarrow B^{-}B\sigma \cap \theta B^{-}B \neq \phi$$

Now, $B^{-}B$ consists of all matrices whose leading principal minors have nonzero determinants. In other words, $B^{-}B$ is defined by the nonvanishing of the determinant polynomial for the leading principal minors, thus $B^{-}B$ is open in G and therefore dense, hence the intersection is always nonempty.

The first intersection, 1.6, was studied by Deodhar [5]. In particular,

$$B\sigma B \cap B^{-}\theta B \neq \phi \Leftrightarrow \theta \leq \sigma$$

where $\leq$ denotes the Bruhat order.

In the case where $G = GL_n(K)$ and $W = S_n$, there is a combinatorial description of the Bruhat order due to Deodhar (page 119 of [10]). We now describe this ordering since the new orderings that will be defined in Chapter 3 turn out to be closely related to the Bruhat order.

Deodhar's criterion goes as follows. Let $(a_1, a_2, \dots, a_n)$ be a sequence of integers and define $(a_1, \dots, a_n) \leq (b_1, \dots, b_n)$ if $a_i \leq b_i$, for $1 \leq i \leq n$. Denote by $[a_1, a_2, \dots, a_n]$ the sequence rewritten in increasing order. Let $\sigma, \theta \in S_n$, then

$$\theta \leq \sigma \Leftrightarrow [\theta(1), \theta(2), \dots, \theta(k)] \leq [\sigma(1), \sigma(2), \dots, \sigma(k)]$$

for all $1 \leq k \leq n$.

Let us look at an example when $W = S_3$.

EXAMPLE 1.3.9. Let $G = GL_3(K)$, $W = S_3$. Suppose $\sigma, \theta \in W$ where $\sigma = (132)$ and $\theta = (12)$. Is $\theta \leq \sigma$? First let us rewrite σ and θ .

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \theta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

For $k = 1$ we see that

$$\theta(1) = 2 \leq \sigma(1) = 3$$

Now let $k = 2$. Comparing $\theta(1), \theta(2)$ written in increasing order to $\sigma(1), \sigma(2)$ in increasing order we have

$$[1, 2] \leq [1, 3]$$

Certainly if we compare term by term we see that $1 \leq 1, 2 \leq 3$. Finally if $k = 3$ the condition is obviously satisfied, thus $\theta \leq \sigma$ in the Bruhat order. Conversely, we see that $\sigma \not\leq \theta$ since

$$\sigma(1) = 3 \not\leq \theta(1) = 2$$

.

1.4 Algebraic Monoids

In this section we will assemble all of the definitions used to define reductive monoids and lay the ground work for the rest of the paper. The setting for the remaining chapters will be that of the reductive monoid. All of the definitions are taken from [12], [14] and [17].

DEFINITION 1.4.1. A *linear algebraic monoid* (or *algebraic monoid*) M is an affine variety together with an associative morphism $\mu : M \times M \rightarrow M$ and an identity element $1 \in M$ for μ .

Another sometimes easier way to think of algebraic monoids is the following. An algebraic monoid is a submonoid M of $M_n(K)$ which is a Zariski closed set. Recall, by closed set we mean the set of common zeros to a finite collection of polynomials. We may view $M_n(K)$ as an affine space of dimension n^2 with coordinate functions X_{ij} defined by

$$X_{ij} \left(\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \right) = a_{ij} \quad \text{where } a_{ij} \in K$$

EXAMPLE 1.4.1. The monoid $M_n(K)$ is algebraic defined by the empty family of polynomials.

EXAMPLE 1.4.2. $D_n(K)$ the monoid of all diagonal matrices is defined by the vanishing of the X_{ij} when $i \neq j$, thus is algebraic.

EXAMPLE 1.4.3. The monoid of all upper triangular matrices is defined by the vanishing of the X_{ij} when $i > j$, thus is algebraic.

We say an algebraic monoid M is *connected* if it is connected in the Zariski topology. An algebraic monoid is *irreducible* if it cannot be written as a union of two proper Zariski closed sets. Unlike algebraic groups, connected and irreducible are not the same. If M is irreducible then M is connected, but not conversely [12].

EXAMPLE 1.4.4. Let

$$M = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \mid a_1 a_2 = 0 \right\}$$

M is connected, but not irreducible since

$$M = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \mid a_1 = 0 \right\} \cup \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \mid a_2 = 0 \right\}$$

We assume throughout that all algebraic monoids are irreducible. So, how do we construct algebraic monoids? It is very simple. Let M be any submonoid of $M_n(K)$, then the Zariski closure of M is an algebraic monoid. Moreover, if we want the algebraic monoid to have a prescribed unit group G , then we simply take $M = \overline{G}$.

Let M be an algebraic monoid with unit group G . Recall from section 1.1, Green's $\mathcal{J}$ relation, that

$$a\mathcal{J}b \Leftrightarrow MaM = MbM$$

Since $MG \subseteq M$ and $GM \subseteq M$, the group $G \times G$ acts on M by

$$(g, h) \cdot a = gah^{-1}$$

for some $g, h \in G, a \in M$. The orbits for this action are $\mathcal{O} = GaG$. It is clear that $GaG = GbG \Rightarrow MaM = MbM$ and since we are assuming that M is irreducible

$MaM = MbM \Rightarrow GaG = GbG$ [12]. Hence a and b are in the same $\mathcal{J}$ -class if and only if they are in the same $G \times G$ orbit.

A very important result due to Putcha [13] is the existence of the cross-section lattice. Recall from Section 1 that $E(M) = \{e \in M \mid e^2 = e\}$. Let us denote by $U(M)$ the set of all regular $\mathcal{J}$ -classes of M . We say $\Lambda \subseteq E(M)$ is a *cross-section lattice* if

$$|\Lambda \cap J| = 1 \text{ for all } J \in U(M) \text{ and } J_1 \leq J_2 \Rightarrow e_1 \leq e_2$$

where $e_i \in J_i \cap \Lambda$, for $i = 1, 2$. The cross-section lattice is nice since it allows us to choose one idempotent from each $\mathcal{J}$ -class in such a way that the order (1.1 on page 4) on the idempotents is preserved.

EXAMPLE 1.4.5. Let $M = M_n(K)$. The cross-section lattice Λ is given by

$$\Lambda = \left\{ \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \mid 0 \leq r \leq n \right\}$$

A cross-section lattice for a monoid M with unit group G and Borel subgroup $B \supseteq T$, a maximal torus, can be found in the following way:

$$\begin{aligned} \Lambda &= \{e \in E(\overline{T}) \mid Be = eBe\} \\ &= \{e \in E(\overline{T}) \mid eB^- = eB^-e\} \end{aligned}$$

THEOREM 1.4.1. [13]

1. *Cross-section lattice exists.*
2. *Any two cross section lattices are conjugate.*
3. *There is a 1-1 correspondence between the set of all cross-section lattices and with pairs of opposite Borel subgroups.*
4. *Any two cross-section lattices contained in $E(\overline{T})$ are conjugate by an element of W .*

Note that we are able to write M as a union of $\mathcal{J}$ -classes, in particular, the $\mathcal{J}$ -classes of idempotents in Λ [13].

$$M = \bigsqcup_{e \in \Lambda} GeG \quad (1.8)$$

In the last section we defined, for a set $I \subseteq S$ where S is the generating set for W , the subgroups P_I, L_I , etc. We now relate those definitions to the cross-section lattice. Let M be an algebraic monoid with unit group G . Fix $e \in \Lambda$. Let $I = \{s \in S | se = es\}$. Then we have the following:

$$\begin{aligned} W(e) &= \{x \in W | xe = ex\} \\ P(e) = P_I &= \{x \in G | xe = exe\} \\ P^-(e) = P_I^- &= \{x \in G | ex = exe\} \\ L(e) = L_I &= \{x \in G | xe = es\} \end{aligned}$$

Similarly, $U(e) = U_I$, $U^-(e) = U_I^-$. Another very useful fact[12] (which is easily seen for the matrices) is that

$$U_I e = e \quad (1.9)$$

$$e U_I^- = e \quad (1.10)$$

This paper deals only with a special class of algebraic monoids, namely, *reductive monoids*.

DEFINITION 1.4.2. An irreducible algebraic monoid is *reductive* if its unit group is reductive.

EXAMPLE 1.4.6. $M = M_n(K)$ is reductive since its unit group $G = GL_n(K)$ is reductive.

The analogue of the Bruhat Decomposition for reductive monoids is *Renner's Decomposition*. Before we state Renner's Decomposition we first need to define the *Renner monoid*. The Renner monoid is the generalization of the Weyl group, definition 1.3.4 on page 10.

DEFINITION 1.4.3. [14] Let M be a monoid with unit group G and maximal torus T . Then the *Renner monoid* is given by

$$R = \overline{N_G(T)}/T$$

where $\overline{N_G(T)}$ denotes the Zariski closure of $N_G(T)$ in M .

The Renner monoid is a finite inverse monoid with unit group W .

EXAMPLE 1.4.7. Let $M = M_n(K)$. Renner's monoid is the monoid of all partial permutation matrices, i.e. The set of all matrices $A \in M_n(K)$ such that each entry is 0 or 1 and the sum of the entries in each column and row is at most 1.

The Renner monoid can also be written as the following disjoint union,

$$R = \bigsqcup_{e \in \Lambda} WeW.$$

Each element in the Renner monoid can be written in a unique way called *standard form*. From the decomposition above, we know that any element $r \in R$ can be written as $r = w_1ew_2$ where $e \in \Lambda$ and $w_1, w_2 \in W$. To put r in standard form we do the following:

1. Pull out on the right as much as possible of w_1 that can be absorbed by e .
2. Pull out on the left as much as possible of w_2 that commutes with e and move it to the left of e .
3. Cancel with e all that can be cancelled. What is left after all cancelling is standard form.

Before we state the technical definition of standard form we need a few more definitions. Again, fix $e \in \Lambda$.

$$\begin{aligned} W_*(e) &= \bigcap_{f \leq e} W(f) \\ W^*(e) &= \bigcap_{f \geq e} W(f) \\ D(e) &= \{x \in W \mid x \text{ is of minimum length in } xW(e)\} \\ D_*(e) &= \{x \in W \mid x \text{ is of minimum length in } xW_*(e)\} \end{aligned}$$

Note that $W_*(e), W^*(e)$ are subgroups of W and $D(e), D_*(e)$ are sets.

DEFINITION 1.4.4. [11] Let $\sigma \in R$. Then the *standard form* of σ is given by

$$\sigma = xey$$

where $e \in \Lambda$, $x \in D_*(e)$, and $y \in D(e)^{-1}$.

By $D(e)^{-1}$ we mean the set of inverses of elements of $D(e)$.

Let us now look at an example.

EXAMPLE 1.4.8. Let $M = M_3(K)$. Let $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. How do we write

$$\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \theta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ in standard form?}$$

We know

$$\begin{aligned} I = \{(12)\} &= \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \\ W_I = W(e) = \{1, (12)\} &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \\ D(e) = \{1, (23), (123)\} &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\} \\ D_*(e) = W &= \text{all the } 3 \times 3 \text{ permutation matrices} \end{aligned}$$

Therefore

$$\begin{aligned} \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= (132)e(1) \end{aligned}$$

And

$$\begin{aligned}\theta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= (12)e(23)\end{aligned}$$

The set $D_*(e)$ can be written in the following way,

$$D_*(e) = D(e)W^*(e).$$

Let $\sigma \in R$ be an element in standard form. Then

$$\sigma = x_1ey$$

Since $x_1 \in D_*(e)$,

$$x_1 = xw$$

for some $x \in D(e), w \in W^*(e)$. Therefore we can write σ in *modified standard form*,

$$\sigma = xwey$$

Both standard form and modified standard form will be used in the main results in Chapter 3.

In general for a permutation $x \in S_n$, and $e = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ we have the following:

1. $x \in D_*(e) \Leftrightarrow x$ is a matrix such that the last $n-r$ columns are in column echelon form
2. $x \in D(e) \Leftrightarrow x$ is a matrix such that the first r columns are in column echelon form and the last $n-r$ columns are in row echelon form

THEOREM 1.4.2. *The following are equivalent:*

1. $x \in D(e)$
2. $xB_Lx^{-1} \subseteq B$
3. $xB_L^-x^{-1} \subseteq B^-$
4. $x^{-1}Bx \subseteq BU_I^-$
5. $x^{-1}B^-x \subseteq B^-U_I$

For a proof see [3].

We now conclude this section with the analogue of the Bruhat Decomposition for reductive monoids.

THEOREM 1.4.3 (Renner's Decomposition). *[14] Let M be a reductive monoid with unit group G and Borel subgroup $B \supseteq T$. Then M may be written as the following disjoint union*

$$M = \bigsqcup_{\sigma \in R} B\sigma B$$

where R is Renner's monoid.

Just as with the Bruhat Decomposition, one or both of the Borel subgroups, B , can be replaced with its opposite Borel subgroup B^- . Thus,

$$\begin{aligned} M &= \bigsqcup_{\sigma \in R} B^- \sigma B \\ M &= \bigsqcup_{\theta \in R} B^- \theta B^- \end{aligned}$$

This in turn leads to two more decompositions,

$$\begin{aligned} M &= \bigsqcup_{\sigma, \theta \in R} (B\sigma B \cap B^- \theta B) \\ M &= \bigsqcup_{\sigma, \theta \in R} (B\sigma B \cap B^- \theta B^-) \end{aligned}$$

In this paper we determine when the following intersections are nonempty:

$$B\sigma B \cap B^- \theta B \neq \emptyset \Leftrightarrow ?? \tag{1.11}$$

$$B\sigma B \cap B^- \theta B^- \neq \emptyset \Leftrightarrow ?? \tag{1.12}$$

These questions will be answered first for the monoid $M = M_n(K)$ and then for reductive monoids in general. We begin our study on $M_n(K)$ in the following chapter.

Chapter 2

Intersections for $M_n(K)$

A common approach to the study of reductive monoids is to first consider the monoid $M_n(K)$, the $n \times n$ matrices. $M_n(K)$ is a concrete example to work with which allows one to gain insight into the more general problem. Indeed, the matrices foreshadow the general results.

The main purpose of this chapter is to study the following two decompositions for the monoid $M_n(K)$:

$$M = \bigsqcup_{\sigma, \theta \in R} (B\sigma B \cap B^{-}\theta B)$$

and

$$M = \bigsqcup_{\sigma, \theta \in R} (B\sigma B \cap B^{-}\theta B^{-})$$

Our goal is to determine when the following intersections are nonempty:

$$B\sigma B \cap B^{-}\theta B \tag{2.1}$$

and

$$B\sigma B \cap B^{-}\theta B^{-} \tag{2.2}$$

This chapter is divided into two sections; the first sections deals with the intersection (2.1) and the second with the intersection (2.2).

Recall from Chapter 1 that

$$B\sigma B \cap B^{-}\theta B \neq \emptyset \iff B^{-}B\sigma \cap \theta B \neq \emptyset$$

Similarly,

$$B\sigma B \cap B^{-}\theta B^{-} \neq \emptyset \iff B^{-}B\sigma \cap \theta B^{-}B \neq \emptyset$$

Throughout this chapter we will use the following notation:

$M = M_n(K)$ the $n \times n$ matrices

$B = B_n(K)$ the invertible upper triangular matrices

$B^{-} = B_n^{-}(K)$ the invertible lower triangular matrices

$B^{-}B =$ the $n \times n$ matrices with the property that the leading principle minors have nonzero determinant

$R =$ the partial permutation matrices

In each section, we consider the rank one and rank two partial permutation matrices. We state and prove the theorems for both ranks and conclude each section with a theorem for the partial permutation matrices of rank r . In the Chapter 4 we will return to the theorems and prove them.

2.1 Intersection $B\sigma B \cap B^{-}\theta B$

To get started, we define a new ordering on R . Recall that R in this chapter is the monoid of partial permutation matrices.

DEFINITION 2.1.1. ($\leq_1$) Let $\theta, \sigma \in R$. $\theta \leq_1 \sigma$ if $B^{-}\sigma B \cap B^{-}\theta B \neq \emptyset$.

The first thing we notice is that

$$B\sigma B \cap B^{-}\theta B \neq \emptyset \implies \text{rank}(\sigma) = \text{rank}(\theta)$$

Thus we cannot compare σ and θ unless they have the same rank. In section 1.1 we saw that for the monoid $M_n(K)$, $A \mathcal{J} B$ implies $\text{rank}(A) = \text{rank}(B)$. Thus $\leq_1$ is an

order within the particular $\mathcal{J}$ -classes. Since we are working within one $\mathcal{J}$ -class at a time, let us consider first the rank one partial permutation matrices. There are nine rank one partial permutation matrices.

THEOREM 2.1.1. *Let σ, θ be rank one partial permutation matrices, i.e. $\theta = E_{i'_1 j'_1}$ and $\sigma = E_{i_1 j_1}$. $\theta \leq_1 \sigma$ if and only if $i'_1 \leq i_1$ and $j'_1 = j_1$.*

Proof. ($\implies$) We will prove the contrapositive. Suppose that $i'_1 > i_1$ or $j'_1 \neq j_1$. We want to show that

$$B\sigma B \cap B^{-}\theta B = \emptyset$$

which implies $\theta \not\leq_1 \sigma$. Well, suppose

$$B\sigma B \cap B^{-}\theta B \neq \emptyset$$

Recall this is equivalent to

$$B^{-}B\sigma \cap \theta B \neq \emptyset$$

Then there exists $A \in B^{-}B, U \in B$ such that

$$A\sigma = \theta B$$

where A has the property that all leading principal minors have nonzero determinant and U is an invertible upper triangular matrix. Now

$$\begin{aligned} A\sigma &= \theta B \\ \implies \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} j_1 \\ 1 \end{pmatrix}_{i_1=i'_1} &= \begin{pmatrix} j'_1 \\ 1 \end{pmatrix} \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ & u_{22} & \vdots \\ 0 & \ddots & \vdots \\ & & u_{nn} \end{pmatrix} \end{aligned} \quad (2.3)$$

$$\begin{aligned} \implies \begin{pmatrix} a_{1i_1} \\ 0 & a_{2i_1} & 0 \\ \vdots \\ a_{ni_1} \end{pmatrix} &= \begin{pmatrix} 0 \\ u_{j'_1 1} & u_{j'_1 2} & \cdots & u_{j'_1 n} \\ 0 \end{pmatrix} \leftarrow i'_1 \text{ row} \quad (2.4) \\ &\quad \uparrow \\ &\quad j_1 \text{ column} \end{aligned}$$

If $i'_1 > i_1$ then at least the first i_1 entries of column i_1 in A are zero. Hence the determinant of the $i_1 \times i_1$ minor of A is zero. Contradiction, since $A \in B^-B$.

On the other hand, if $j'_1 \neq j_1$ then $u_{j'_1 j'_1} = 0$, another contradiction since U is invertible. Thus,

$$B^-B\sigma \cap \theta B = \emptyset$$

and

$$\theta \leq_1 \sigma.$$

To prove the converse, let us assume that $i'_1 < i_1$ and $j'_1 = j_1$. This time we want to show that

$$B^-B\sigma \cap \theta B \neq \emptyset$$

So, we need to construct matrices $A \in B^-B$ and $U \in B$ such that

$$A\sigma = \theta U.$$

We see from (2.4) that A and U have to satisfy the following conditions respectively:

$$\begin{aligned} a_{ri_1} &= 0 \quad \text{for all } r \neq i'_1 \\ \text{and } u_{j'_1 s} &= 0 \quad \text{for all } s \neq j_1 = j'_1 \end{aligned}$$

Let A be given by

1. $a_{i'_1 i_1} = a_{i_1 i'_1} = 1$
2. $a_{kk} = 1$ for all $k \neq i_1$
3. All other entries 0

i.e.,

$$A = \begin{array}{c} i'_1 \\ i_1 \end{array} \left[\begin{array}{ccc|ccc} & & i'_1 & & i_1 & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \\ \hline & & & & 1 & 0 \\ \hline & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{array} \right]$$

Let U be the identity matrix. Clearly $A \in B^-B, U \in B$ thus

$$B^-B\sigma \cap \theta B \neq \emptyset$$

and

$$\theta \leq_1 \sigma.$$

□

From this theorem we can draw a diagram for the rank one partial permutation matrices showing the order. Recall how we defined $\leq_1$.

$$\theta \leq_1 \sigma \text{ if } B\sigma B \cap B^-\theta B \neq \emptyset$$

Therefore Figure 1 tells us exactly when we have a nonempty intersection.

Figure 2.1: $\leq_1$ order on 3×3 Rank 1 Partial Permutation Matrices

Now we move on to the rank two partial permutation matrices. Before we get started we need some new notation.

Let A be the set of all $n \times n$ rank two partial permutation matrices. Let

$$A_{j_1 j_2} = \{E_{i_1 j_1} + E_{i_2 j_2} \mid 1 \leq j_1 < j_2 \leq n\}$$

Clearly, $A = \bigsqcup_{1 \leq j_1 < j_2 \leq n} A_{j_1 j_2}$. Let $P = \{(i_1, i_2) \mid i_1 \neq i_2\}$.

We now define the following order on P .

DEFINITION 2.1.2. $(i'_1, i'_2) \leq (i_1, i_2)$ if

Case I: $i'_1 < i'_2, i_1 < i_2$

$$i'_1 \leq i_1 \text{ and } i'_2 \leq i_2$$

Case II: $i'_1 < i'_2, i_1 > i_2$

$$i'_1 \leq i_2 \text{ and } i'_2 \leq i_1$$

Case III: $i'_1 > i'_2, i_1 < i_2$

$$i'_2 \leq i_1 \text{ and } i'_1 \leq i_2$$

Case IV: $i'_1 > i'_2, i_1 > i_2$

$$i'_1 \leq i_1 \text{ and } i'_2 \leq i_2$$

PROPOSITION 2.1.1. $\leq$ is a partial order on P .

Proof. Reflexive Suppose $i'_1 < i'_2$ or $i'_2 > i'_1$. Clearly $i'_1 \leq i'_1$ and $i'_2 \leq i'_2$ thus $(i'_1, i'_2) \leq (i'_1, i'_2)$.

Antisymmetric Suppose $(i'_1, i'_2) \leq (i_1, i_2)$ and $(i_1, i_2) \leq (i'_1, i'_2)$. If $i'_1 < i'_2$ and $i_1 < i_2$ then by definition we have $i'_1 \leq i_1, i'_2 \leq i_2$ and $i_1 \leq i'_1, i_2 \leq i'_2$. Hence $i'_1 = i_1$ and $i'_2 = i_2$ so $(i'_1, i'_2) = (i_1, i_2)$. Similar arguments hold for the other cases.

Transitivity Suppose $(i_1, i_2) \leq (k_1, k_2)$ and $(k_1, k_2) \leq (l_1, l_2)$.

Case 1: Suppose $i_1 < i_2, k_1 < k_2$ and $l_1 < l_2$. Then by definition $i_1 \leq k_1, i_2 \leq k_2, k_1 \leq l_1$ and $k_2 \leq l_2$. So we see that $i_1 \leq k_1 \leq l_1$ and $i_2 \leq k_2 \leq l_2$. Therefore $i_1 \leq l_1$ and $i_2 \leq l_2$, hence $(i_1, i_2) \leq (l_1, l_2)$.

Case 2: Suppose $i_1 < i_2, k_1 > k_2$ and $l_1 < l_2$. Then we have $i_1 \leq k_2, i_2 \leq k_1$ and $k_1 \leq l_1$. So $i_1 < i_2 \leq k_1 \leq l_1$ and $i_2 \leq k_1 \leq l_1 < l_2$. Therefore $i_1 < l_1$ and $i_2 < l_2$

hence $(i_1, i_2) \leq (l_1, l_2)$.

The other six cases follow with a similar argument. $\square$

THEOREM 2.1.2. *Fix j_1, j_2 where $j_1 < j_2$. Let $\theta = E_{i'_1 j_1} + E_{i'_2 j_2}$ and $\sigma = E_{i_1 j_1} + E_{i_2 j_2}$. $\theta \leq_1 \sigma$ if and only if $(i'_1, i'_2) \leq (i_1, i_2)$.*

Proof. ($\Leftarrow$) Suppose $(i'_1, i'_2) \leq (i_1, i_2)$. Then we have four cases to consider. In each case, we will need to construct a matrix $A \in B^-B$ and a matrix $U \in B$ such that

$$A\sigma = \theta U$$

This will imply that

$$B^-B\sigma \cap \theta B \neq \emptyset$$

which implies

$$\theta \leq_1 \sigma$$

Let us now look carefully at what restrictions we will have on the matrices A and U in each case. In order for $A\sigma = \theta U$ we must have:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} & j_1 & j_2 \\ 1 & & \\ & & 1 \end{pmatrix}_{i_1 \quad i_2} = \begin{pmatrix} & j_1 & j_2 \\ 1 & & \\ & & 1 \end{pmatrix}_{i'_1 \quad i'_2} \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ & u_{22} & \vdots \\ 0 & \ddots & \vdots \\ & & u_{nn} \end{pmatrix} \quad (2.5)$$

This implies

$$\begin{pmatrix} & a_{1i_1} & a_{1i_2} \\ 0 & a_{2i_1} & 0 & a_{2i_2} & 0 \\ & \vdots & \vdots \\ & a_{ni_1} & a_{ni_2} \end{pmatrix} = \begin{pmatrix} & 0 \\ u_{j_1 1} & \cdots & u_{j_1 n} \\ & 0 \\ u_{j_2 1} & \cdots & u_{j_2 n} \\ & 0 \end{pmatrix} \begin{matrix} \leftarrow i'_1 \\ \leftarrow i'_2 \end{matrix} \quad (2.6)$$

$\begin{matrix} \uparrow & \uparrow \\ j_1 & j_2 \end{matrix}$

$$\left. \begin{aligned} \implies a_{i'_1 i_1} &= u_{j_1 j_1} \\ a_{i'_1 i_2} &= u_{j_1 j_2} \\ a_{i'_2 i_1} &= u_{j_2 j_1} = 0 \quad \text{since } j_1 < j_2 \\ a_{i'_2 i_2} &= u_{j_2 j_2} \\ a_{r i_1} &= 0 \quad \forall r \neq i'_1, i'_2 \\ a_{s i_2} &= 0 \quad \forall s \neq i'_1, i'_2 \\ u_{j_1 t} &= 0 \quad \forall t \neq j_1, j_2 \\ u_{j_2 v} &= 0 \quad \forall v \neq j_1, j_2 \end{aligned} \right\} \quad (2.7)$$

We now begin with case 1.

Case I Suppose $i'_1 < i'_2$ and $i_1 < i_2$. Then by definition of $\leq$ we have $i'_1 \leq i_1$ and $i'_2 \leq i_2$. With these four restrictions on i'_1, i_1, i'_2, i_2 we have the following possible orderings:

1. $i'_1 < i_1 < i'_2 < i_2$
2. $i'_1 < i'_2 < i_1 < i_2$
3. $i'_1 = i_1 < i'_2 < i_2$
4. $i'_1 < i_1 = i'_2 < i_2$
5. $i'_1 < i_1 < i'_2 = i_2$
6. $i'_1 = i_1 < i'_2 = i_2$

Now we construct the matrices A and U separately for each case subject also to the restrictions (2.7). With each of the six orderings we will let U be the identity matrix with $u_{j_1 j_2} = 1$.

1. If $i'_1 < i_1 < i'_2 < i_2$ or $i'_1 < i'_2 < i_1 < i_2$ let A be the matrix such that:

$$(a) \quad a_{i'_1 i_1} = a_{i_1 i'_1} = a_{i'_1 i_2} = a_{i'_2 i_2} = a_{i_2 i'_2} = 1$$

$$(b) \quad a_{kk} = 1 \text{ for all } k \neq i_1, i_2$$

(c) All other entries 0.

i.e.,

$$A = \begin{matrix} & & i'_1 & i_1 & i'_2 & i_2 \\ \begin{matrix} i'_1 \\ i_1 \\ i'_2 \\ i_2 \end{matrix} & \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \end{matrix}$$

$$U = \begin{matrix} & j_1 & & j_2 \\ \begin{matrix} j_1 \\ j_2 \end{matrix} & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \end{matrix}$$

If $i'_1 = i_1 < i'_2 < i_2$ let A be the matrix such that:

(a) $a_{i_1 i_1} = a_{i_1 i_2} = a_{i'_2 i_2} = a_{i_2 i'_2} = 1$

(b) $a_{kk} = 1$ for all $k \neq i_2$

(c) All other entries 0.

If $i'_1 < i_1 < i'_2 < i_2$ let A be the matrix such that:

- (a) $a_{i'_1 i_1} = a_{i'_1 i_2} = a_{i_1 i'_1} = a_{i_1 i_2} = a_{i_2 i'_1} = 1$
- (b) $a_{kk} = 1$ for all $k \neq i_1, i_2$
- (c) All other entries 0.

If $i'_1 < i_1 < i'_2 = i_2$ let A be the matrix such that:

- (a) $a_{i'_1 i_1} = a_{i'_1 i_2} = a_{i_1 i'_1} = a_{i_2 i_2} = 1$
- (b) $a_{kk} = 1$ for all $k \neq i_1$
- (c) All other entries 0.

If $i'_1 = i_1 < i'_2 = i_2$ let A be the matrix such that:

- (a) $a_{i_1 i_1} = a_{i_1 i_2} = a_{i_2 i_2} = 1$
- (b) $a_{kk} = 1$ for all $k \neq i_2$
- (c) All other entries 0.

Now each $A \in B^- B, U \in B$. Thus $B^- B \sigma \cap \theta B \neq \emptyset$ and $\theta \leq_1 \sigma$.

Case II Suppose $i'_1 < i'_2$ and $i_1 > i_2$. Then by definition of $\leq$ we must have $i'_1 \leq i_2$ and $i'_2 \leq i_1$. With these restrictions the possible orderings are:

1. $i'_1 < i'_2 < i_2 < i_1$
2. $i'_1 < i_2 < i'_2 < i_1$
3. $i'_1 = i_2 < i'_2 < i_1$
4. $i'_1 = i_2 < i'_2 = i_1$
5. $i'_1 < i_2 < i'_2 = i_1$
6. $i'_1 < i'_2 = i_2 < i_1$

As in case I, we will again choose U to be the identity with $u_{j_1 j_2} = 1$.
If $i'_1 < i'_2 < i_2 < i_1$ or $i'_1 < i_2 < i'_2 < i_1$ let A be the matrix such that:

1. $a_{i'_1 i_2} = a_{i'_1 i_1} = a_{i'_2 i_2} = a_{i_2 i'_2} = a_{i_1 i'_1} = 1$
2. $a_{kk} = 1$ for all $k \neq i_1, i_2$
3. All other entries 0

i.e.,

$$A = \begin{bmatrix} & i'_1 & i'_2 & & i_1 & & & i_2 & & \\ i'_1 & 1 & & & 1 & & & 1 & & \\ i'_2 & & 1 & & 1 & & & & & \\ & & & \ddots & & & & & & \\ & & & & 1 & & & & & \\ i_2 & & 1 & & 0 & & & & & \\ & & & & & 1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & 1 & & \\ i_1 & 1 & & & 0 & & & & & \\ & & & & & & & 1 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 1 \end{bmatrix}$$

If $i'_1 = i_2 < i'_2 < i_1$, let A be the matrix such that:

1. $a_{i_2 i_2} = a_{i_2 i_1} = a_{i'_2 i_2} = a_{i_1 i'_2} = 1$
2. $a_{kk} = 1$ for all $k \neq i_1$
3. All other entries 0

If $i'_1 = i_2 < i'_2 = i_1$, let A be the matrix such that:

1. $a_{i_2 i_2} = a_{i_2 i_1} = a_{i_1 i_2} = 1$
2. $a_{kk} = 1$ for all $k \neq i_1$
3. All other entries 0

If $i'_1 < i_2 < i'_2 = i_1$, let A be the matrix such that:

1. $a_{i'_1 i_2} = a_{i'_1 i_1} = a_{i_2 i'_2} = a_{i_1 i_2} = 1$
2. $a_{kk} = 1$ for all $k \neq i_1, i_2$
3. All other entries 0

If $i'_1 < i_2 = i'_2 < i_1$, let A be the matrix such that:

1. $a_{i'_1 i_2} = a_{i'_1 i_1} = a_{i_2 i_2} = a_{i_1 i'_1} = 1$
2. $a_{kk} = 1$ for all $k \neq i_1$
3. All other entries 0

Now, each $A \in B^-B$. Thus $B^-B\sigma \cap \theta B \neq \emptyset$ and $\theta \leq_1 \sigma$.

Case III Suppose $i'_1 > i'_2$ and $i_1 < i_2$. By definition of $\leq$, we know that $i'_1 < i_1$. Thus the possible orderings are

1. $i'_2 < i'_1 < i_1 < i_2$
2. $i'_2 < i'_1 = i_1 < i_2$

So we have the following. Let U be as in Case I. If $i'_2 < i'_1 < i_1 < i_2$, let A be the matrix such that:

1. $a_{i'_2 i_2} = a_{i'_1 i_1} = a_{i'_1 i_2} = a_{i_1 i'_1} = a_{i_2 i'_2} = 1$
2. $a_{kk} = 1$ for all $k \neq i_1, i_2$
3. All other entries 0

If $i'_2 < i'_1 = i_1 < i_2$ let A be the matrix such that:

1. $a_{i'_2 i_2} = a_{i'_1 i_1} = a_{i_1 i_2} = a_{i_2 i'_2} = 1$
2. $a_{kk} = 1$ for all $k \neq i_2$

3. All other entries 0

Each choice of $A \in B^-B$. Therefore $B^-B\sigma \cap \theta B \neq \emptyset$ and $\theta \leq_1 \sigma$.

Case IV Suppose $i'_1 > i'_2$ and $i_1 > i_2$. Then by definition of $\leq$, $i'_1 < i_1$ and $i'_2 < i_2$. Again we have six possible orderings.

1. $i'_2 < i'_1 < i_2 < i_1$
2. $i'_2 < i_2 < i'_1 < i_1$
3. $i'_2 < i'_1 = i_2 < i_1$
4. $i'_2 = i_2 < i'_1 = i_1$
5. $i'_2 = i_2 < i'_1 < i_1$
6. $i'_2 < i_2 < i'_1 = i_1$

If $i'_2 < i'_1 < i_2 < i_1$ or $i'_2 < i_2 < i'_1 < i_1$ let A be the matrix such that:

1. $a_{i'_1 i_1} = a_{i_1 i'_1} = a_{i'_1 i_2} = a_{i'_2 i_2} = a_{i_2 i'_2} = 1$
2. $a_{kk} = 1$ for all $k \neq i_1, i_2$
3. All other entries 0.

If $i'_2 < i'_1 = i_2 < i_1$, let A be the matrix such that:

1. $a_{i_2 i_2} = a_{i'_2 i_2} = a_{i_2 i_1} = a_{i_1 i'_2} = 1$
2. $a_{kk} = 1$ for all $k \neq i_1$
3. All other entries 0

If $i'_2 = i_2 < i'_1 = i_1$, let A be the matrix such that:

1. $a_{i_2 i_2} = a_{i'_1 i'_1} = a_{i'_1 i_1} = 1$

2. $a_{kk} = 1$ for all $k \neq i_2$
3. All other entries 0

If $i'_2 = i_2 < i'_1 < i_1$, let A be the matrix such that:

1. $a_{i_2 i_2} = a_{i'_1 i_2} = a_{i'_1 i_1} = a_{i_1 i'_1} = 1$
2. $a_{kk} = 1$ for all $k \neq i_1$
3. All other entries 0

If $i'_2 < i_2 < i'_1 = i_1$, let A be the matrix such that:

1. $a_{i'_2 i_2} = a_{i_2 i'_2} = a_{i_1 i_2} = a_{i_2 i_1} = 1$
2. $a_{kk} = 1$ for all $k \neq i_2$
3. All other entries 0

Clearly each $A \in B^-B, U \in B$ thus $B^-B\sigma \cap \theta B \neq \emptyset$ and $\theta \leq_1 \sigma$.

To show the converse of the theorem, we will prove the contrapositive. Again, there are four cases to consider.

Suppose $(i'_1, i'_2) \not\leq (i_1, i_2)$. We now want to show that $B^-B\sigma \cap \theta B = \emptyset$ which will imply $\theta \not\leq_1 \sigma$.

Case I Suppose $i'_1 < i'_2$ and $i_1 < i_2$. Now $(i'_1, i'_2) \not\leq (i_1, i_2)$ implies that either $i'_1 > i_1$ or $i'_2 > i_2$.

Let us suppose $B^-B\sigma \cap \theta B \neq \emptyset$. Then there exists $A \in B^-B$ and $U \in B$ such

that

$$\begin{aligned}
 & A\sigma = \theta U \\
 \text{From (2.6)} \quad & \Rightarrow \begin{pmatrix} & a_{1i_1} & a_{1i_2} & & \\ 0 & a_{2i_1} & 0 & a_{2i_2} & 0 \\ & \vdots & & \vdots & \\ & a_{ni_1} & a_{ni_2} & & \\ & \uparrow & \uparrow & & \\ & j_1 & j_2 & & \end{pmatrix} = \begin{pmatrix} & 0 & & & \\ u_{j_1 1} & \cdots & u_{j_1 n} & & \\ & 0 & & & \\ u_{j_2 1} & \cdots & u_{j_2 n} & & \\ & 0 & & & \end{pmatrix} \begin{matrix} \leftarrow i'_1 \\ \\ \leftarrow i'_2 \end{matrix} \quad (2.8)
 \end{aligned}$$

If $i'_1 > i_1$, at least the first i_1 entries of the i_1 th column in A are zero. Hence, $\det(i_1 \times i_1)$ minor is zero, contradiction.

If $i'_2 > i_2$, the $i_2 \times i_2$ minor of A has two linearly dependent columns and thus $\det(i_2 \times i_2)$ minor is zero, contradiction. i.e.,

$$\text{the } i_2 \times i_2 \text{ minor} = \begin{pmatrix} & 0 & & 0 & \\ & \vdots & & \vdots & \\ * & a_{i'_1 i_1} & a_{i'_1 i_2} & * & \\ & \vdots & * & \vdots & \\ & 0 & & 0 & \end{pmatrix}$$

where the $*$'s represent arbitrary entries. Thus, $B^-B\sigma \cap \theta B = \emptyset$ and $\theta \not\leq_1 \sigma$.

Case II Suppose $i'_1 < i'_2$ and $i_1 > i_2$. Then $(i'_1, i'_2) \not\leq (i_1, i_2)$ implies that either $i'_1 > i_2$ or $i'_2 > i_1$.

From (2.8) we see that if $i'_1 > i_2$, then at least the first i_2 entries of the i_2 th column of A are zero. Hence $\det(i_2 \times i_2)$ minor is zero, contradiction.

On the other hand, if $i'_2 > i_1$, then the $i_1 \times i_1$ minor of A contains two linearly dependent columns and thus has determinant zero. Contradiction.

Therefore $B^-B\sigma \cap \theta B = \emptyset$ and $\theta \not\leq_1 \sigma$.

Case III Suppose $i'_1 > i'_2$ and $i_1 < i_2$. Then $(i'_1, i'_2) \not\leq (i_1, i_2)$ implies $i'_1 > i_1$. Now we take a closer look at

$$A\sigma = \theta U$$

i.e.,

$$\begin{pmatrix} & a_{1i_1} & & a_{1i_2} & \\ 0 & a_{2i_1} & 0 & a_{2i_2} & 0 \\ & \vdots & & \vdots & \\ & a_{ni_1} & & a_{ni_2} & \\ & \uparrow & & \uparrow & \\ & j_1 & & j_2 & \end{pmatrix} = \begin{pmatrix} & 0 & \\ u_{j_2 1} & \cdots & u_{j_2 n} \\ & 0 & \\ u_{j_1 1} & \cdots & u_{j_1 n} \\ & 0 & \end{pmatrix} \begin{matrix} \\ \leftarrow i'_2 \\ \\ \leftarrow i'_1 \\ \end{matrix} \quad (2.9)$$

At this point we cannot arrive at a contradiction since the ordering of i'_1, i'_2, i_1, i_2 makes a difference.

With the restrictions above, the possible orderings are as follows:

1. $i'_2 \leq i_1 < i'_1 \leq i_2$
2. $i'_2 \leq i_1 < i_2 < i'_1$
3. $i_1 < i'_2 < i'_1 \leq i_2$
4. $i_1 < i'_2 < i_2 < i'_1$
5. $i_1 < i_2 < i'_2 < i'_1$

Consider the $i_1 \times i_1$ minor of A . If the orderings correspond to numbers 4 or 5, then the $i_1 \times i_1$ minor has the following form:

$$i_1 \times i_1 \text{ minor} = \begin{matrix} & i'_2 & & i_1 \\ & & & 0 \\ & i'_2 & & 0 \\ & & * & \vdots \\ & & & \vdots \\ i_1 & & & 0 \end{matrix}$$

Clearly $\det(i_1 \times i_1)$ minor is zero. Contradiction.

Similarly if we consider the other three orderings, then in each case we again have the $i_1 \times i_1$ minor with determinant zero. Contradiction.

Thus $B^-B\sigma \cap \theta B = \emptyset$ and $\theta \not\leq_1 \sigma$.

Case IV Suppose $i'_1 > i'_2$ and $i_1 > i_2$. Then $(i'_1, i'_2) \not\leq (i_1, i_2)$ implies either $i'_1 > i_1$ or $i'_2 > i_2$. From (2.9) in the previous case we see that if $i'_2 > i_2$ then at least the first i_2 entries of the i_2 th column are zero. Hence $\det(i_2 \times i_2)$ minor is zero, contradiction.

Now if we suppose $i'_1 > i_1$ we have to consider the possible orderings as we did in case III. The orderings subject to the restrictions $i'_1 > i'_2, i_1 > i_2, i'_1 > i_1$ are:

1. $i'_2 < i_2 < i_1 < i'_1$
2. $i_2 < i'_2 < i_1 < i'_1$
3. $i_2 < i_1 < i'_2 < i'_1$

Consider the $i_1 \times i_1$ minor of A . With each of these three orderings we arrive at the same contradiction.

i.e.,

$$i_1 \times i_1 \text{ minor} = \begin{matrix} & & i_1 \\ & & 0 \\ & & 0 \\ & * & \vdots \\ & & \vdots \\ i_1 & & 0 \end{matrix}$$

Note: the positions of i'_2, i_2 and i'_1 do not affect the conclusion.

Since $\det(i_1 \times i_1)$ minor is zero, contradiction. Thus $B^-B\sigma \cap \theta B = \emptyset$ and $\theta \not\leq_1 \sigma$.

□

To completely describe when

$$B^-B\sigma \cap \theta B \neq \emptyset$$

we need the following proposition.

PROPOSITION 2.1.2. *Let $\theta = E_{i'_1 j'_1} + E_{i'_2 j'_2}$ and $\sigma = E_{i_1 j_1} + E_{i_2 j_2}$. If $\theta \leq_1 \sigma$ then $j'_1 = j_1$ and $j'_2 = j_2$.*

Proof. We prove the contrapositive. Suppose $j'_1 \neq j_1$ or $j'_2 \neq j_2$. We want to show that $B^- B \sigma \cap \theta B = \emptyset$ and hence $\theta \not\leq_1 \sigma$. So suppose

$$B^- B \sigma \cap \theta B \neq \emptyset$$

Then there exists $A \in B^- B$ and $U \in B$ (where A has the property that the leading principal minors have nonzero determinant and U is an invertible upper triangular matrix) such that

$$A\sigma = \theta U$$

If $i'_1 < i'_2$ and $i_1 < i_2$ then

$$\begin{aligned} & \Rightarrow A \begin{pmatrix} & j_1 & j_2 \\ 1 & & \\ & & 1 \end{pmatrix} \begin{matrix} i_1 \\ i_2 \end{matrix} = \begin{matrix} j'_1 & j'_2 \\ i'_1 & i'_2 \end{matrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} U \\ & \Rightarrow \begin{pmatrix} & a_{1i_1} & a_{1i_2} \\ 0 & a_{2i_1} & 0 & a_{2i_2} & 0 \\ & \vdots & & \vdots & \\ & a_{ni_1} & a_{ni_2} \end{pmatrix} = \begin{pmatrix} & 0 \\ u_{j'_1 1} & \cdots & u_{j'_1 n} \\ & 0 \\ u_{j'_2 1} & \cdots & u_{j'_2 n} \\ & 0 \end{pmatrix} \begin{matrix} \leftarrow i'_1 \\ \leftarrow i'_2 \end{matrix} \\ & \quad \quad \quad \begin{matrix} \uparrow & \uparrow \\ j_1 & j_2 \end{matrix} \\ & \Rightarrow u_{j'_1 r} = 0 \text{ for all } r \neq j_1, j_2 \\ & \text{and } u_{j'_2 s} = 0 \text{ for all } s \neq j_1, j_2 \end{aligned}$$

This implies $u_{j'_1 j'_1} = 0$ and $u_{j'_2 j'_2} = 0$. Contradiction since U is invertible. Therefore we must have $j'_1 = j_1$ and $j'_2 = j_2$ hence $B^- B \sigma \cap \theta B = \emptyset$ and $\theta \not\leq_1 \sigma$.

A similar argument holds for the cases where

1. $i'_1 < i'_2$ and $i_1 > i_2$
2. $i'_1 > i'_2$ and $i_1 < i_2$
3. $i'_1 > i'_2$ and $i_1 > i_2$

□

For the diagram of rank two partial permutation matrices see Figure 2.

Notice, in Figure 2, that the matrices are partitioned according to the columns that contain 1's. i.e. A_{13} consists of all rank two partial permutation matrices with 1's in the first and third columns. Hence we cannot compare matrices that do not lie in the same set, and this is what proposition 2.1.2 asserts.

Let us take a moment and compare the diagram for rank 1, Figure 2.1, with Figure 2.2. Notice that each diagram consists of C_r^n disjoint sets, where n is the size of the matrix and r its rank. Each set has a minimal and maximal element. The minimal elements in each set are in row echelon form.

In general for an $n \times n$ matrix of rank r there are C_r^n possible row echelon forms. As we see from Figure 2.1 (Figure 2.2 respectively) there are $C_1^4 = 4$ ($C_2^3 = 3$ respectively) different row echelon forms. The set of matrices, A_I , which has minimal element of the form

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

is special. All of the other sets are isomorphic to this one. If we pick any other row echelon form, P , and multiply every element in A_I on the right by P , then we get the set containing P .

EXAMPLE 2.1.1. Let us consider the 3×3 rank 2 partial permutation matrices. Then we have the following row echelon forms

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Let us multiply each element in $A_I = A_{12}$ by P_2 and then by P_3 . In doing this we see that

$$A_{12} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow A_{13}$$

Similarly,

$$A_{12} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow A_{23}$$

EXAMPLE 2.1.2. Let $\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Is $\theta \leq_1 \sigma$? Let us use Theorem 2.1.2 to figure this out. First, $\theta = E_{i'_1 j'_1} + E_{i'_2 j'_2} = E_{11} + E_{32}$ and $\sigma = E_{i_1 j_1} + E_{i_2 j_2} = E_{31} + E_{22}$. Our theorem says that we need $(i'_1, i'_2) \leq (i_1, i_2)$ and $j'_1 = j_1, j'_2 = j_2$. Clearly $j'_1 = 1 = j_1$ and $j'_2 = 2 = j_2$. Now we need to check the condition for the indices of the rows.

Note: $(i'_1, i'_2) = (1, 3)$ and $(i_1, i_2) = (3, 1)$

To verify that $(i'_1, i'_2) \leq (i_1, i_2)$ we consult Definition 2.1.2 and see that we are in Case II (where $i'_1 < i'_2$ and $i_1 > i_2$). From this we must have $i'_1 \leq i_2$ and $i'_2 \geq i_1$. Clearly $1 \leq 1$ and $3 \leq 3$, thus $(i'_1, i'_2) \leq (i_1, i_2)$ and $\theta \leq_1 \sigma$.

We have shown that for the rank two partial permutation matrices $\leq_1$ is a partial order. So one might guess that $\leq_1$ is a partial order in general. Indeed, this is the case, as we will see in Chapter 3 when we generalize our results. We conclude this

Figure 2.2: $\leq_1$ order on 3×3 Rank 2 Partial Permutation Matrices

section with a theorem for the rank r partial permutation matrices which will be proved in Chapter 4.

THEOREM 2.1.3. *Let $\theta = E_{i'_1 j'_1} + E_{i'_2 j'_2} + \cdots + E_{i'_r j'_r}$ and $\sigma = E_{i_1 j_1} + E_{i_2 j_2} + \cdots + E_{i_r j_r}$. Then $\theta \leq_1 \sigma$ if and only if $[i'_1, i'_2, \dots, i'_k] \leq [i_1, i_2, \dots, i_k]$ for all $1 \leq k \leq r$ and $j'_1 = j_1, j'_2 = j_2, \dots, j'_r = j_r$.*

Note: $\leq$ is the truncated Bruhat order described on page 13 (Deodhar's Criterion).

2.2 Intersection $B\sigma B \cap B^{-}\theta B^{-}$

In this section we consider the following intersection

$$B\sigma B \cap B^{-}\theta B^{-}$$

Recall from Section 1.3 that this intersection is always nonempty at the group level. We will see that this is not the case for monoids.

DEFINITION 2.2.1. ($\leq_2$) Let $\sigma, \theta \in R$. We define $\theta \leq_2 \sigma$ if $B\sigma B \cap B^{-}\theta B^{-} \neq \emptyset$.

Just as with $\leq_1$ in the previous section, we see that

$$B\sigma B \cap B^{-}\theta B^{-} \neq \emptyset \implies \text{rank}(\sigma) = \text{rank}(\theta)$$

So again we are working within a $\mathcal{J}$ -class. Our approach to this intersection is the same as it was for the first intersection; we start with the rank one partial permutation matrices and then move up to rank two.

THEOREM 2.2.1. *Let $\theta = E_{i'_1 j'_1}$ and $\sigma = E_{i_1 j_1}$. Then $\theta \leq_2 \sigma$ if and only if $i'_1 \leq i_1$ and $j_1 \leq j'_1$.*

Proof. To prove the theorem in this direction we will prove the contrapositive. Suppose $i'_1 > i_1$ or $j_1 > j'_1$. We want to show that $B\sigma B \cap B^{-}\theta B^{-} = \emptyset$ (or $B^{-}B\sigma \cap \theta B^{-}B = \emptyset$). This will imply that $\theta \not\leq_2 \sigma$.

Suppose $B^-B\sigma \cap \theta B^-B \neq \emptyset$. Then there exists $A, C \in B^-B$ such that

$$\begin{aligned} A\sigma = \theta C \\ \Rightarrow \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} j_1 \\ 1 \end{pmatrix}_{i_1} = i'_1 \begin{pmatrix} j'_1 \\ 1 \end{pmatrix} \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \quad (2.10) \end{aligned}$$

$$\begin{aligned} \Rightarrow \begin{pmatrix} a_{1i_1} \\ 0 & a_{2i_1} & 0 \\ \vdots \\ a_{ni_1} \end{pmatrix} = \begin{pmatrix} 0 \\ c_{j'_1 1} & \cdots & c_{j'_1 n} \\ 0 \end{pmatrix} \leftarrow i'_1 \text{ row} \quad (2.11) \\ \uparrow \\ j_1 \text{ column} \end{aligned}$$

If $i'_1 > i_1$, then at least the first i_1 entries of the i_1 th column in A are zero. Therefore $\det(i_1 \times i_1)$ minor is zero, contradiction.

Suppose now that $j_1 > j'_1$. Then we have at least the first j'_1 entries of the j'_1 th row of C equal to zero. Thus $\det(j'_1 \times j'_1)$ minor of C is zero, contradiction since $C \in B^-B$. Hence $B^-B\sigma \cap \theta B^-B = \emptyset$ and $\theta \not\leq_2 \sigma$.

Now let us prove the converse. Suppose $i'_1 \leq i_1$ and $j_1 \leq j'_1$. We need to construct matrices $A, C \in B^-B$ such that $A\sigma = \theta C$. From (2.11) we see that we need the following restrictions on A and C respectively:

$$a_{ri_1} = 0 \text{ for all } r \neq i'_1$$

and

$$c_{j'_1 s} = 0 \text{ for all } s \neq j_1$$

Let A be given by the following:

1. $a_{i'_1 i_1} = a_{i_1 i'_1} = 1$
2. $a_{kk} = 1$ for all $k \neq i_1$
3. All other entries zero.

And C by:

1. $c_{j_1 j'_1} = c_{j'_1 j_1} = 1$
2. $c_{hh} = 1$ for all $h \neq j_1$
3. All other entries 0

i.e.,

$$A = \begin{array}{c} \begin{array}{c} i'_1 \\ i_1 \end{array} \left[\begin{array}{ccc|cc} & & i'_1 & i_1 & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & & 1 \\ \hline & & & & 1 & 0 \\ \hline & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{array} \right] \end{array}$$

$$C = \begin{array}{c} \begin{array}{c} j_1 \\ j'_1 \end{array} \left[\begin{array}{ccc|cc} & & j_1 & j'_1 & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & & 1 \\ \hline & & & & 1 & 0 \\ \hline & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{array} \right] \end{array}$$

Clearly $A, C \in B^-B$. Thus $B^-B\sigma \cap \theta B^-B \neq \emptyset$ and $\theta \leq_2 \sigma$. $\square$

See Figure 3 for the diagram of rank 1 partial permutation matrices. It is not hard to see that $\leq_2$ is a partial order on the set of rank one partial permutation matrices.

Figure 2.3: $\leq_2$ order on 3×3 Rank 1 Partial Permutation Matrices

In the previous section we guessed that $\leq_1$ will be a partial order on R . Let us now see what happens with $\leq_2$.

In the last section we said that

$$A = \bigsqcup_{1 \leq j_1 < j_2 \leq n} A_{j_1 j_2}$$

where $A_{j_1 j_2} = \{E_{i_1 j_1} + E_{i_2 j_2} \mid 1 \leq j_1 < j_2 \leq n\}$.

We now want to describe, A , the set of all rank two partial permutation matrices, in a new way.

DEFINITION 2.2.2. $A^+ = \{E_{i_1 j_1} + E_{i_2 j_2} \mid i_2 < i_1 \text{ and } j_1 < j_2\}$

DEFINITION 2.2.3. $A^- = \{E_{i_1 j_1} + E_{i_2 j_2} \mid i_1 < i_2 \text{ and } j_1 < j_2\}$

DEFINITION 2.2.4. $A = A^+ \sqcup A^-$.

EXAMPLE 2.2.1. Let $\sigma = E_{11} + E_{23}$ and $\theta = E_{31} + E_{22}$. By the definitions above

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in A^-$$

and

$$\theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in A^+$$

An easy way to look at a matrix and tell whether it lies in A^+ or A^- is to connect the 1's in the matrix with a line. If the line has negative slope (positive respectively) then the matrix is an element of A^- (A^+ respectively).

DEFINITION 2.2.5. Let $\sigma = E_{i_1 j_1} + E_{i_2 j_2} \in A^+$. Then $\hat{\sigma} = E_{i_2 j_1} + E_{i_1 j_2} \in A^-$.

REMARK 2.2.1. $\hat{P}$ is the matrix obtained from P by interchanging row i_1 with row i_2 .

THEOREM 2.2.2. Let $\sigma = E_{i_1 j_1} + E_{i_2 j_2}$, $\theta = E_{i'_1 j'_1} + E_{i'_2 j'_2} \in A^-$ and $\hat{\sigma} = E_{i_2 j_1} + E_{i_1 j_2}$, $\hat{\theta} = E_{i'_2 j'_1} + E_{i'_1 j'_2} \in A^+$. The following are equivalent:

1. $(i'_1, i'_2) \leq (i_1, i_2)$ and $(j_1, j_2) \leq (j'_1, j'_2)$
2. $\theta \leq_2 \sigma$
3. $\hat{\theta} \leq_2 \sigma$
4. $\theta \leq_2 \hat{\sigma}$
5. $\hat{\theta} \leq_2 \hat{\sigma}$

where $\leq$ is the standard inequality for ordered pairs. i.e., $i'_1 \leq i_1, i'_2 \leq i_2$.

Proof. We will prove the following first

$$\sim 1 \implies \sim 2$$

$$\sim 1 \implies \sim 3$$

$$\sim 1 \implies \sim 4$$

$$\sim 1 \implies \sim 5$$

Suppose $(i'_1, i'_2) \not\leq (i_1, i_2)$ or $(j_1, j_2) \not\leq (j'_1, j'_2)$. Let us consider first what happens when we assume $(i'_1, i'_2) \not\leq (i_1, i_2)$. We want to show the following:

$$\text{I. } B^-B\sigma \cap \theta B^-B = \emptyset$$

$$\text{II. } B^-B\hat{\sigma} \cap \theta B^-B = \emptyset$$

$$\text{III. } B^-B\sigma \cap \hat{\theta} B^-B = \emptyset$$

$$\text{IV. } B^-B\hat{\sigma} \cap \hat{\theta} B^-B = \emptyset$$

Suppose the intersections in I.-IV. are all nonempty. Then in each case, separately, there exists $A, C \in B^-B$ such that:

$$1. A\sigma = \theta C$$

$$2. A\hat{\sigma} = \theta C$$

$$3. A\sigma = \hat{\theta} C$$

$$4. A\hat{\sigma} = \hat{\theta} C$$

i.e.,

$$\begin{aligned}
1. \quad & \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} j_1 & j_2 \\ 1 & \\ & 1 \end{pmatrix} \begin{matrix} i_1 \\ i_2 \end{matrix} = \begin{matrix} i'_1 \\ i'_2 \end{matrix} \begin{pmatrix} j'_1 & j'_2 \\ 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \\
& \Rightarrow \begin{pmatrix} a_{1i_1} & a_{1i_2} \\ 0 & a_{2i_1} & 0 & a_{2i_2} & 0 \\ \vdots & \vdots \\ a_{ni_1} & a_{ni_2} \end{pmatrix} = \begin{pmatrix} 0 \\ c_{j'_1 1} & \cdots & c_{j'_1 n} \\ 0 \\ c_{j'_2 1} & \cdots & c_{j'_2 n} \\ 0 \end{pmatrix} \begin{matrix} \leftarrow i'_1 \\ \leftarrow i'_2 \end{matrix} \\
& \qquad \qquad \qquad \begin{matrix} \uparrow & \uparrow \\ j_1 & j_2 \end{matrix}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} j_1 & j_2 \\ & 1 \\ 1 & \end{pmatrix} \begin{matrix} i_1 \\ i_2 \end{matrix} = \begin{matrix} i'_1 \\ i'_2 \end{matrix} \begin{pmatrix} j'_1 & j'_2 \\ 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \\
& \Rightarrow \begin{pmatrix} a_{1i_2} & a_{1i_1} \\ 0 & a_{2i_2} & 0 & a_{2i_1} & 0 \\ \vdots & \vdots \\ a_{ni_2} & a_{ni_1} \end{pmatrix} = \begin{pmatrix} 0 \\ c_{j'_1 1} & \cdots & c_{j'_1 n} \\ 0 \\ c_{j'_2 1} & \cdots & c_{j'_2 n} \\ 0 \end{pmatrix} \begin{matrix} \leftarrow i'_1 \\ \leftarrow i'_2 \end{matrix} \\
& \qquad \qquad \qquad \begin{matrix} \uparrow & \uparrow \\ j_1 & j_2 \end{matrix}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} j_1 & j_2 \\ 1 & \\ & 1 \end{pmatrix} \begin{matrix} i_1 \\ i_2 \end{matrix} = \begin{matrix} i'_1 \\ i'_2 \end{matrix} \begin{pmatrix} j'_1 & j'_2 \\ 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \\
& \Rightarrow \begin{pmatrix} a_{1i_1} & a_{1i_2} \\ 0 & a_{2i_1} & 0 & a_{2i_2} & 0 \\ \vdots & \vdots \\ a_{ni_1} & a_{ni_2} \end{pmatrix} = \begin{pmatrix} 0 \\ c_{j'_2 1} & \cdots & c_{j'_2 n} \\ 0 \\ c_{j'_1 1} & \cdots & c_{j'_1 n} \\ 0 \end{pmatrix} \begin{matrix} \leftarrow i'_1 \\ \leftarrow i'_2 \end{matrix} \\
& \qquad \qquad \qquad \begin{matrix} \uparrow & \uparrow \\ j_1 & j_2 \end{matrix}
\end{aligned}$$

$$\begin{aligned}
4. \quad & \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} j_1 & j_2 \\ & 1 \\ 1 & \end{pmatrix} \begin{matrix} i_1 \\ i_2 \end{matrix} = \begin{matrix} i'_1 \\ i'_2 \end{matrix} \begin{pmatrix} j'_1 & j'_2 \\ & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \\
& \Rightarrow \begin{pmatrix} & a_{1i_2} & & a_{1i_1} & & \\ 0 & a_{2i_2} & 0 & a_{2i_1} & 0 & \\ & \vdots & & \vdots & & \\ & a_{ni_2} & & a_{ni_1} & & \\ & \uparrow & & \uparrow & & \\ & j_1 & & j_2 & & \end{pmatrix} = \begin{pmatrix} & 0 & & & & \\ c_{j'_2 1} & \cdots & c_{j'_2 n} & & & \\ & 0 & & & & \\ c_{j'_1 1} & \cdots & c_{j'_1 n} & & & \\ & 0 & & & & \end{pmatrix} \begin{matrix} \leftarrow i'_1 \\ \leftarrow i'_2 \end{matrix}
\end{aligned}$$

Recall, since $\sigma, \theta \in A^-$, we know $i_1 < i_2$ and $i'_1 < i'_2$. Since we are assuming $(i'_1, i'_2) \not\leq (i_1, i_2)$ we must have either $i'_1 > i_1$ or $i'_2 > i_2$. Suppose first that $i'_1 > i_1$. Then in each case (I, II, III and IV) we see that at least the first i_1 entries of column i_1 in A are zero. Hence the determinant of the $i_1 \times i_1$ minor is zero, contradiction since each $A \in B^-B$.

Now suppose that $i'_2 > i_2$. Let us consider the $i_2 \times i_2$ minor of A in each case.

We know $i_1 < i_2$ and $i'_1 < i'_2$ since $\sigma, \theta \in A^-$. We have two possible orderings of i_1, i_2, i'_1 assuming that in addition $i_2 < i'_2$. They are the following:

1. $i_1 < i_2 < i'_1$
2. $i_1 < i'_1 < i_2$

If $i_1 < i_2 < i'_1$ then we have the contradiction above. Thus the only possibility is

$$i_1 < i'_1 < i_2$$

Then the $i_2 \times i_2$ minor has the following form.

$$i_2 \times i_2 \text{ minor} = \begin{matrix} & & & i_2 \\ & & & 0 \\ & & & \vdots \\ & * & a_{i'_1 i_1} & * & a_{i'_1 i_2} \\ & & \vdots & & \vdots \\ i_2 & & 0 & & 0 \end{matrix}$$

Therefore the $i_2 \times i_2$ minor has two linearly dependent columns, hence its determinant is zero, contradiction.

Now suppose $(j_1, j_2) \not\leq (j'_1, j'_2)$. By definition of $\leq$ we see that either $j_1 > j'_1$ or $j_2 > j'_2$.

If $j_1 > j'_1$, then in each case (I, II, III, IV) at least the first j'_1 entries of row j'_1 in C are zero. Hence the $j'_1 \times j'_1$ minor of C has determinant zero, contradiction since $C \in B^-B$.

If $j_2 > j'_2$ then the $j'_2 \times j'_2$ minor of C has two linearly dependent rows, contradiction. This minor has the following form:

$$j'_2 \times j'_2 \text{ minor} = \begin{matrix} & & & & j'_2 \\ & & & * & \\ & 0 & \dots & c_{j'_1 j_1} & \dots & 0 \\ & & & * & \\ j'_2 & 0 & \dots & c_{j'_2 j_1} & \dots & 0 \end{matrix}$$

Hence,

$$\text{I. } B^-B\sigma \cap \theta B^-B = \emptyset$$

$$\text{II. } B^-B\hat{\sigma} \cap \theta B^-B = \emptyset$$

$$\text{III. } B^-B\sigma \cap \hat{\theta} B^-B = \emptyset$$

$$\text{IV. } B^-B\hat{\sigma} \cap \hat{\theta} B^-B = \emptyset$$

and thus

$$\theta \not\leq_2 \sigma$$

$$\theta \not\leq_2 \hat{\sigma}$$

$$\hat{\theta} \not\leq_2 \sigma$$

$$\hat{\theta} \not\leq_2 \hat{\sigma}$$

This concludes the proofs of

$$\sim 1 \implies \sim 2$$

$$\sim 1 \implies \sim 3$$

$$\sim 1 \implies \sim 4$$

$$\sim 1 \implies \sim 5$$

Now we will prove the following:

$$1 \implies 2$$

$$1 \implies 3$$

$$1 \implies 4$$

$$1 \implies 5$$

Suppose $(i'_1, i'_2) \leq (i_1, i_2)$ and $(j_1, j_2) \leq (j'_1, j'_2)$. We want to show that

$$\text{I. } B^-B\sigma \cap \theta B^-B \neq \emptyset \text{ (i.e. } \theta \leq_2 \sigma)$$

$$\text{II. } B^-B\hat{\sigma} \cap \theta B^-B \neq \emptyset$$

$$\text{III. } B^-B\sigma \cap \hat{\theta} B^-B \neq \emptyset$$

$$\text{IV. } B^-B\hat{\sigma} \cap \hat{\theta} B^-B \neq \emptyset$$

Thus in each of the above cases we need to construct matrices $A, C \in B^-B$ such that

$$A\sigma = \theta C$$

$$A\hat{\sigma} = \theta C$$

$$A\sigma = \hat{\theta} C$$

$$A\hat{\sigma} = \hat{\theta} C$$

These are precisely the equalities 1, 2, 3 and 4 respectively from page 53. Let us examine the matrices A and C from 1 on page 53 since the other A matrices differ by interchanging two columns and the C matrices by interchanging two rows.

From 1 on page 53 we must have the following:

$$\begin{cases} a_{i'_1 i_1} &= c_{j'_1 j_1} \\ a_{i'_1 i_2} &= c_{j'_1 j_2} \\ a_{i'_2 i_1} &= c_{j'_2 j_1} \\ a_{i'_2 i_2} &= c_{j'_2 j_2} \end{cases} \quad (2.12)$$

Since we are assuming $(i'_1, i'_2) \leq (i_1, i_2)$ and $(j_1, j_2) \leq (j'_1, j'_2)$, by definition of $\leq$ we have the following restrictions:

$$i'_1 \leq i_1, \quad i'_2 \leq i_2, \quad j_1 \leq j'_1, \quad j_2 \leq j'_2$$

In addition we know that

$$i'_1 < i'_2, \quad i_1 < i_2, \quad j_1 < j_2, \quad j'_1 < j'_2$$

Therefore there are six possible orderings for i'_1, i'_2, i_1, i_2 and j_1, j_2, j'_1, j'_2 . They are as follows:

1. $i'_1 < i_1 < i'_2 < i_2$	1. $j_1 < j'_1 < j_2 < j'_2$
2. $i'_1 < i'_2 < i_1 < i_2$	2. $j_1 < j_2 < j'_1 < j'_2$
3. $i'_1 = i_1 < i'_2 < i_2$	3. $j_1 = j'_1 < j_2 < j'_2$
4. $i'_1 < i_1 < i'_2 = i_2$	4. $j_1 < j'_1 < j'_2 = j_2$
5. $i'_1 < i_1 = i'_2 < i_2$	5. $j_1 < j'_1 = j_2 < j'_2$
6. $i'_1 = i_1 < i'_2 = i_2$	6. $j_1 = j'_1 < j'_2 = j_2$

We choose A according to the orders on i'_1, i_1, i'_2, i_2 and C according to the orders for j_1, j'_1, j_2, j'_2 .

In each $A \in B^-B$ we construct, we will fix the following entries:

$$a_{i'_1 i_1} = a_{i_1 i'_1} = x$$

$$a_{i'_2 i_2} = a_{i_2 i'_2} = w$$

$$a_{i'_1 i_2} = y$$

$$a_{i'_2 i_1} = z$$

Case 1 If $i'_1 < i_1 < i'_2 < i_2$ or $i'_1 < i'_2 < i_1 < i_2$:

1. Choose x, y, z, w such that $x^2 w^2 - xyzw \neq 0$
2. $a_{kk} = 1$ for all $k \neq i_1, i_2$ and all other entries zero

Case 2 If $i'_1 = i_1 < i'_2 < i_2$:

1. Choose x, y, z, w such that $-xw^2 + zwy \neq 0, x \neq 0$
2. $a_{kk} = 1$ for all $k \neq i_1, i_2$ and all other entries zero

Case 3 If $i'_1 < i_1 < i'_2 = i_2$:

1. Choose x, y, z, w such that $-x^2 w + xyz \neq 0$
2. $a_{kk} = 1$ for all $k \neq i_1$ and all other entries zero

Case 4 If $i'_1 < i_1 = i'_2 < i_2$:

1. Choose x, y, z, w such that $yxz - y^2 z \neq 0$
2. $a_{kk} = 1$ for all $k \neq i_2$ and all other entries zero

Case 5 If $i'_1 = i_1 < i'_2 = i_2$:

1. Choose x, y, z, w such that $xw - yz \neq 0, x \neq 0$
2. $a_{kk} = 1$ for all k and all other entries zero

In the same way we constructed the matrices A in each case, we will construct C .

Now in each $C \in B^-B$ that we construct we will choose the following based on the conditions 2.12 on page 57:

$$c_{j'_1 j_1} = c_{j_1 j'_1} = x$$

$$c_{j'_2 j_2} = c_{j_2 j'_2} = w$$

$$c_{j'_2 j_1} = z$$

$$c_{j'_1 j_2} = y$$

Case A If $j_1 < j'_1 < j_2 < j'_2$ or $j_1 < j_2 < j'_1 < j'_2$:

1. Choose x, y, z, w such that $x^2 w^2 - xyzw \neq 0$
2. $c_{hh} = 1$ for all $h \neq j'_1, j'_2$ and all other entries 0

Case B If $j_1 = j'_1 < j_2 < j'_2$:

1. Choose x, y, z, w such that $-xw^2 + yzw \neq 0, x \neq 0$
2. $c_{hh} = 1$ for all $h \neq j'_2$ and all other entries 0

Case C If $j_1 < j'_1 < j'_2 = j_2$:

1. Choose x, y, z, w such that $-xw^2 + xyz \neq 0$
2. $c_{hh} = 1$ for all $h \neq j'_1$ and all other entries 0

Case D If $j_1 < j'_1 = j_2 < j'_2$:

1. Choose x, y, z, w such that $zxw - yz^2 \neq 0, y \neq 0$
2. $c_{hh} = 1$ for all $h \neq j'_2$ and all other entries 0

Case E If $j_1 = j'_1 < j_2 = j'_2$:

1. Choose x, y, z, w such that $xw - zy \neq 0, x \neq 0$
2. $c_{hh} = 1$ for all h and all other entries 0

Recall, from page 57, we want to show the following:

$$\text{I. } B^-B\sigma \cap \theta B^-B \neq \emptyset$$

$$\text{II. } B^-B\hat{\sigma} \cap \theta B^-B \neq \emptyset$$

$$\text{III. } B^-B\sigma \cap \hat{\theta} B^-B \neq \emptyset$$

$$\text{IV. } B^-B\hat{\sigma} \cap \hat{\theta} B^-B \neq \emptyset$$

To show I, choose A from Cases 1–5 and C from Cases A–E based on the orders for i_1, i_2, i'_1, i'_2 and j_1, j_2, j'_1, j'_2 respectively.

To show II, choose A from Cases 1–5, except interchange columns i_1 and i_2 . When we interchange these two columns, the nonzero equations in Cases 1–5 become:

$$\text{Case 1: } -x^2w^2 + xyzw \neq 0, -xy \neq 0$$

$$\text{Case 2: } xw^2 - zwy \neq 0, y \neq 0$$

$$\text{Case 3: } x^2w - xyz \neq 0, -xy \neq 0$$

$$\text{Case 4: } -yxz + y^2z \neq 0, w \neq 0$$

$$\text{Case 5: } -xw + yz \neq 0, y \neq 0$$

To show III, choose A from Cases 1–5. Now we choose C from Cases A–E, except we interchange row j'_1 and row j'_2 . When we interchange these two rows the nonzero equations in each case become the following:

$$\text{Case A: } wxyz - w^2x^2 \neq 0, -zx \neq 0$$

$$\text{Case B: } xw^2 - yzw \neq 0, z \neq 0$$

$$\text{Case C: } wx^2 - yzx \neq 0, -zx \neq 0$$

$$\text{Case D: } -zxw + z^2y \neq 0, w \neq 0$$

$$\text{Case E: } yz - wx \neq 0, z \neq 0$$

Finally, to show IV, choose A as we did in showing II and C as we did in showing III.

Hence we have shown that

$$\theta \leq_2 \sigma$$

$$\theta \leq_2 \hat{\sigma}$$

$$\hat{\theta} \leq_2 \sigma$$

$$\hat{\theta} \leq_2 \hat{\sigma}$$

and this concludes the proof of

$$1 \implies 2$$

$$1 \implies 3$$

$$1 \implies 4$$

$$1 \implies 5$$

□

Let us now take a look at Figure 2.4. Figure 2.4 shows the order $\leq_2$ on the 3×3 rank 2 partial permutation matrices. Notice that at each position in the diagram, two matrices appear rather than one. Let $P, \hat{P}$ be the two matrices in any box. Then $P \leq_2 \hat{P}$ and $\hat{P} \leq_2 P$. Since $P \neq \hat{P}$ we see that $\leq_2$ is not antisymmetric and hence not a partial order on the rank 2 partial permutation matrices. In fact, $\leq_2$ is not a partial order on the permutation matrices of rank r for any $r > 1$. However, if we think of P and $\hat{P}$ as lying in the same equivalence class, then it is not hard to see that $\leq_2$ is a partial order on the equivalence classes of elements of R .

Figure 2.4: $\leq_2$ order on 3×3 Rank 2 Partial Permutation Matrices

EXAMPLE 2.2.2. Let $\theta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in A$. Let's use

Theorem 2.2.2 to verify that $\theta \leq_2 \sigma$. First we write $\theta = E_{12} + E_{23}$ and $\sigma = E_{11} + E_{23}$. Part 1 of our theorem states that we need $(i'_1, i'_2) \leq (i_1, i_2)$ and $(j_1, j_2) \leq (j'_1, j'_2)$. Clearly $(1, 2) \leq (1, 2)$ and $(1, 3) \leq (2, 3)$. Hence $\theta \leq_2 \sigma$.

We conclude this section with a theorem for the rank r partial permutation matrices. We will return to this theorem and prove it in Chapter 4.

THEOREM 2.2.3. Let $\theta = E_{i'_1 j'_1} + E_{i'_2 j'_2} + \cdots + E_{i'_r j'_r}$ and $\sigma = E_{i_1 j_1} + E_{i_2 j_2} + \cdots + E_{i_r j_r}$. Then $\theta \leq_2 \sigma$ if and only if $[i'_1, i'_2, \dots, i'_r] \leq [i_1, i_2, \dots, i_r]$ and $[j_1, j_2, \dots, j_r] \leq [j'_1, j'_2, \dots, j'_r]$.

REMARK 2.2.2. Notice the difference between Theorem 2.2.3 and 2.1.3. In 2.1.3 we have the following criterion $[i'_1, i'_2, \dots, i'_k] \leq [i_1, i_2, \dots, i_k]$ for all $1 \leq k \leq r$. This means the following

$$\begin{aligned} [i'_1] &\leq [i_1] \\ [i'_1, i'_2] &\leq [i_1, i_2] \\ &\vdots \\ [i'_1, i'_2, \dots, i'_r] &\leq [i_1, i_2, \dots, i_r] \end{aligned}$$

where square brackets denotes the sequence rewritten in increasing order, i.e., $[3, 2]$ would mean $(2, 3)$.

In 2.2.2 we need only

$$[i'_1, i'_2, \dots, i'_r] \leq [i_1, i_2, \dots, i_r]$$

This simply means that the set of integers $\{i'_1, \dots, i'_r\}$ written in increasing order is term by term less than the set of integers $\{i_1, \dots, i_r\}$ written in increasing order. Notice this does not imply that $[i'_1] \leq [i_1]$ etc.

Chapter 3

Intersections for Reductive Monoids

In this chapter we generalize the results in Chapter 2 to arbitrary reductive monoids. Since the setting in this chapter will be that of the reductive monoid, we begin by recalling a few important definitions from Section 1.4.

Let M be a reductive monoid with unit group G and cross section lattice Λ . Let $W = \langle S \rangle$, where S is the set of simple reflections, be the Weyl group of G . For $e \in \Lambda$, let

$$\begin{aligned} W(e) &= \langle x \in S \mid xe = ex \rangle \\ W_*(e) &= \langle x \in S \mid xe = ex = e \rangle \\ W^*(e) &= \langle x \in S \mid xe = ex \neq e \rangle \\ D(e) &= \{x \in W \mid x \text{ has minimal length in } xW(e)\} \\ D_*(e) &= \{x \in W \mid x \text{ has minimal length in } xW_*(e)\}. \end{aligned}$$

We study the intersections

$$B\sigma B \cap B^-\theta B \quad \text{and} \quad B\sigma B \cap B^-\theta B^-$$

separately in Sections 3.1 and 3.2 respectively.

3.1 Fine Bruhat Intersection 1

Recall from Chapter 2 that $\leq_1$ is defined as follows:

$$\theta \leq_1 \sigma \Leftrightarrow B\sigma B \cap B^{-}\theta B \neq \emptyset.$$

Let us now take a moment to see how the theorems in Chapter 2 can be generalized to an arbitrary reductive monoid. Let $\theta = E_{i'_1 j'_1} + E_{i'_2 j'_2}$ and $\sigma = E_{i_1 j_1} + E_{i_2 j_2}$ be $n \times n$ rank two partial permutation matrices. From Theorem 2.1.2, we know that

$$\theta \leq_1 \sigma \Leftrightarrow (i'_1, i'_2) \leq (i_1, i_2) \text{ and } j_1 = j_1, j'_2 = j_2$$

where $\leq$ was given by Definition 2.1.2. Now, what we want to do is say something about the following intersection for a reductive monoid M ,

$$B\sigma B \cap B^{-}\theta B$$

when $\sigma, \theta \in R$, the Renner monoid. Recall from Section 1.4, that any element $r \in R$ can be written in a unique way called standard form, i.e.

$$r = xey$$

where $x \in D_*(e)$, $e \in \Lambda$ and $y \in D(e)^{-1}$. Let $\theta = x'e'y'$ and $\sigma = xey$. We saw in Chapter 2 that $\leq_1$ is an ordering within the particular $\mathcal{J}$ -classes of a reductive monoid M . From (1.8), we know that if e, e' are in the same $\mathcal{J}$ -class, i.e.,

$$GeG = Ge'G$$

then we must have $e = e'$. Therefore, we may write

$$\theta = x'ey' \text{ and } \sigma = xey.$$

In Figure 2.2 page 46, we see that the matrices were partitioned according to the columns which contained 1's. Also, we noticed that each disjoint set contained matrices that were row equivalent and that the minimal elements were the different

row echelon forms. So, in order for us to compare σ and θ , we see that they must have the same row echelon form. Renner showed, in [15], that if $\sigma = xey$ is a partial permutation matrix, then ey is the row echelon form of σ . Therefore, we conclude for our theorem that we need $ey = ey'$ which implies $y = y'$ since $y, y' \in D(e)^{-1}$.

Finally, we need to generalize our condition for the indices of the rows of σ and θ . Looking carefully at Theorem 2.1.1 and Definition 2.1.2 that arises in Theorem 2.1.2, we see that the conditions for the indices of the rows appear to match with Deodhar's criterion discussed in Section 1.3. Thus, since $x, x' \in D_*(e) \subset W$, Theorem 2.1.3 suggests that we need $x' \leq x$, where $\leq$ denotes the Bruhat-Chevalley order. This indeed is what happens. In summary, we make the following conjecture:

CONJECTURE 3.1.1. *Let $\theta, \sigma \in R$ be in standard form. i.e., $\theta = x'e'y'$ and $\sigma = xey$. Then $\theta \leq_1 \sigma$ if and only if $x' \leq x$, $e' = e$, and $y' = y$.*

Before we proceed, we state the following lemma that will be used in the proof of the above Conjecture. Let M be a reductive monoid with unit group G and Borel subgroup B . Let $e \in \Lambda$. We denote by

$$C_B(f|f \leq e)$$

the centralizer in B of all idempotents $f \in \Lambda$ such that $f \leq e$. Let us consider the following subgroup of M :

$$H = \{c \in M | ce = e\}.$$

LEMMA 3.1.1. $H \subseteq C_B(f|f \leq e)W_*(e)B$

Proof. Let $c \in H$. Then $ce = e$. Let $e_1 = ec$. Then

$$e_1^2 = ecec = eec = ec = e_1,$$

hence e_1 is an idempotent and $e\mathcal{R}e_1$. Moreover, by Corollary 6.18 [12] $e_1 \in \overline{B}$. Now,

$e_1 = eb$ for some $b \in B$ by Corollary 6.8 [12]. So,

$$\begin{aligned}
 e_1 &= e_1^2 \\
 \Rightarrow eb &= ebeb \\
 \Rightarrow e &= ebe \text{ by multiplying each side by } b^{-1} \\
 \Rightarrow e &= be \text{ since } eBe = Be.
 \end{aligned}$$

Thus, we have

$$ec = e_1 = eb$$

which implies

$$ecb^{-1} = e.$$

Let $d = cb^{-1}$, then

$$ed = e \text{ and}$$

$$\begin{aligned}
 de &= cb^{-1}e \\
 &= ce \text{ since } b^{-1}e = e \\
 &= e.
 \end{aligned}$$

Therefore $ed = e = de$. Let $f \in \Lambda$ where $f \leq e$. Then $fd = f = df$ and hence $d \in C_G(f|f \leq e)$, the centralizer in G of all f such that $f \leq e$. By Cor 7.2 [12] $C_G(f|f \leq e)$ is reductive, therefore by the Bruhat Decomposition,

$$C_G(f|f \leq e) = \bigsqcup_{\sigma \in W_*(e)} C_B(f|f \leq e)\sigma C_B(f|f \leq e)$$

where $C_B(f|f \leq e)$ and $W_*(e)$ are the Borel subgroup and Weyl group of $C_G(f|f \leq e)$ respectively. So, $d \in C_G(f|f \leq e)$ implies

$$d \in C_B(f|f \leq e)W_*(e)C_B(f|f \leq e),$$

but $d = cb^{-1}$ which implies

$$\begin{aligned} db = c &\in C_B(f|f \leq e)W_*(e)C_B(f|f \leq e)B \\ &= C_B(f|f \leq e)W_*(e)B. \end{aligned}$$

Thus, $H \subseteq C_B(f|f \leq e)W_*(e)B$. $\square$

Now we move on to prove our theorem.

THEOREM 3.1.1. *Let $\sigma, \theta \in R$ be in standard form. ie. $\theta = x'e'y'$, $\sigma = xey$. Then $\theta \leq_1 \sigma$ if and only if $x' \leq x$, $e' = e$ and $y' = y$.*

Proof. ($\Rightarrow$) Let G be the unit group of M . Recall from Section 1.4 that

$$M = \bigsqcup_{e \in \Lambda} GeG.$$

i.e. M can be written as the disjoint union of $\mathcal{J}$ -classes of idempotents in the cross-section lattice Λ . Since we are working within one $\mathcal{J}$ -class at a time, $Ge'G = GeG$ implies $e' = e$.

Recall that the Renner monoid R is given by the following disjoint union

$$R = \bigsqcup_{e \in \Lambda} WeW.$$

Moreover,

$$WeW = \bigsqcup_{y \in D(e)^{-1}} Wey.$$

From this we see that $We \simeq Wey$. Thus we prove the forward direction of the theorem by proving the following three statements.

1. Let $\theta \in Wey'$, $\sigma \in Wey$. Then $\theta \leq_1 \sigma$ implies $y' = y$.
2. $x'e \leq_1 xe \Leftrightarrow x'ey \leq_1 xey$.
3. Let $x'e, xe \in We$ where $x, x' \in D_*(e)$. Then $x'e \leq_1 xe \Leftrightarrow x' \leq x$.

Let us first prove #1. Suppose $\theta \leq_1 \sigma$ where $\theta = x'ey'$ and $\sigma = xey$, for some $x, x' \in W$. Then

$$BxeyB \cap B^-x'ey'B \neq \emptyset$$

$\Rightarrow$

$$B^- Bxey \cap x'ey'B \neq \emptyset.$$

So there exists $b_1 \in B^-, b_2, b_3 \in B$ such that

$$b_1 b_2 xey = x'ey'b_3.$$

Thus,

$$b_1 b_2 xey(y'^{-1}ey') = x'ey'b_3(y'^{-1}ey').$$

Since $y' \in D(e)^{-1}$, $y'b_3y'^{-1} = b'_3$ for some $b'_3 \in B$, hence,

$$b_1 b_2 xey(y'^{-1}ey') = x'eb'_3ey'.$$

$\Rightarrow$

$$b_1 b_2 xey(y'^{-1}ey') = x'b_4ey'$$

since $eBe = Be$. Multiply on the right by y'^{-1} and we get

$$b_1 b_2 xeyy'^{-1}e = x'b_4e.$$

Using the fact that $eBe = Be$ once more, we see that,

$$b_1 b_2 xeyy'^{-1}e = x'eb'_4$$

for some $b'_4 \in B$. This implies

$$eyy'^{-1}e = x^{-1}b_2^{-1}b_1^{-1}x'eb'_4. \quad (3.1)$$

Equation (3.1) says that $eyy'^{-1}e \in J$. So, $yy'^{-1} \in W(e)$ and $W(e)y = W(e)y'$. Since $y, y' \in D(e)^{-1}$, $y = y'$.

Now we move on to show #2. Suppose $x'e \leq_1 xe$. Then

$$BxeB \cap B^-x'eB \neq \emptyset.$$

So, there exists $b_1, b_2, b_4 \in B$ and $b_3 \in B^-$ such that

$$b_1xeb_2 = b_3x'eb_4$$

$\implies$

$$b_1 x e b_2 e y = b_3 x' e b_4 e y$$

$\implies$

$$b_1 x b'_2 e y = b_3 x' b'_4 e y$$

for some $b'_2, b'_4 \in B$ since $eBe = Be$. Now, $b'_2, b'_4 \in B = TU_I U_L$, therefore we may write $b'_2 = tu_1 u_2$ and $b'_4 = t' u'_1 u'_2$ for some $t, t' \in T$, $u_1, u'_1 \in U_I$, and $u_2, u'_2 \in U_L$. Thus,

$$b_1 x t u_1 u_2 e y = b_3 x' t' u'_1 u'_2 e y.$$

We know that $U_L = U \cap L$, so $u_2 e = e u_2$ and $u'_2 e = e u'_2$. Thus,

$$b_1 x t u_1 e u_2 y = b_3 x' t' u'_1 e u'_2 y$$

$\implies$

$$b_1 x t e u_2 y = b_3 x' t' e u'_2 y$$

since $U_I e = e$. Also, T commutes with e , so,

$$b_1 x e t u_2 y = b_3 x' e t' u'_2 y.$$

Now, $tu_2, tu'_2 \in B_L$, so let $b_5 = tu_2$ and $b_6 = tu'_2$. Since $y \in D(e)^{-1}$, Theorem 1.4.2 implies that

$$y^{-1} b_5 y = b'_5 \text{ and } y^{-1} b_6 y = b'_6$$

for some $b'_5, b'_6 \in B$. Therefore,

$$b_5 y = y b'_5 \text{ and } b_6 y = y b'_6$$

which implies

$$b_1 x e y b'_5 = b_3 x' e y b'_6$$

and hence $x' e y \leq_1 x e y$.

Conversely, suppose $x' e y \leq_1 x e y$. Then

$$B x e y B \cap B^- x' e y B \neq \emptyset.$$

So there exists $b_1, b_2, b_4 \in B$ and $b_3 \in B^-$ such that

$$b_1 x e y b_2 = b_3 x' e y b_4.$$

Multiplying by y^{-1} on the right yields

$$b_1 x e y b_2 y^{-1} = b_3 x' e y b_4 y^{-1}.$$

Since $y \in D(e)^{-1}$ we know by Theorem 1.4.2 that $y B y^{-1} \subseteq U_I^- B$, so

$$y b_2 y^{-1} = u_1 b \text{ and } y b_4 y^{-1} = u_2 b'$$

for some $u_1, u_2 \in U_I^-$ and $b, b' \in B$. Therefore,

$$b_1 x e u_1 b = b_3 x' e u_2 b'.$$

By 1.10, $e U_I^- = e$, so we have

$$b_1 x e b = b_3 x' e b'$$

which implies

$$B x e \cap B^- x' e B \neq \emptyset$$

and thus $x' e \leq_1 x e$.

Finally we prove #3. Suppose $x' \leq x$. Then by Corollary 1.2 [5],

$$B x B \cap B^- x' B \neq \emptyset.$$

So there exists $b_1, b_2, b_4 \in B$ and $b_3 \in B^-$ such that

$$b_1 x b_2 = b_3 x' b_4$$

$\Rightarrow$

$$b_1 x b_2 e = b_3 x' b_4 e.$$

Since $B = T U_I U_L$, b_2, b_4 can be written as follows, $b_2 = t_1 u_1 u'_1$ and $b_4 = t_2 u_2 u'_2$ for some $u_1, u_2 \in U_I$ and $u'_1, u'_2 \in U_L$. So we have,

$$b_1 x t_1 u_1 u'_1 e = b_3 x' t_2 u_2 u'_2 e.$$

Now, e commutes with both U_L and T and $U_I e = e$, therefore,

$$b_1 x e t_1 u'_1 = b_3 x' e t_2 u'_2.$$

Since $t_1 u'_1, t_2 u'_2 \in B$, let $b'_2 = t_1 u'_1$ and $b'_4 = t_2 u'_2$. Then we have

$$b_1 x e b'_2 = b_3 x' e b'_4$$

which implies $B x e B \cap B^- x' e B \neq \emptyset$ and thus $x' e \leq_1 x e$.

Conversely, let us suppose $x' e \leq_1 x e$. Then,

$$B^- B x e \cap x' e B \neq \emptyset.$$

So there exists $b_1 \in B^-$, $b_2, b_3 \in B$ such that

$$b_1 b_2 x e = x' e b_3$$

which implies

$$b_1 b_2 x e = x' e b_3 e.$$

Since $e B e = B e$,

$$(b_1 b_2 x) e = (x' b_4) e$$

for some $b_4 \in B$. Now let $X = (b_1 b_2 x)$, $Y = x' b_4$. Then $X e = Y e$. Let $Z = X^{-1} Y = (b_1 b_2 x)^{-1} (x' b_4)$. Then, $Z e = e$ and $Z \in H = \{c \in M \mid c e = e\}$. Notice that

$$X Z = Y$$

i.e.

$$b_1 b_2 x Z = x' b_4.$$

By Lemma 3.1.1, $Z \in H \subseteq C_B(f \mid f \leq e) W_*(e) B$ so we may write Z as

$$Z = b_5 w b_6$$

where $b_5 \in C_B(f \mid f \leq e)$, $w \in W_*(e)$ and $b_6 \in B$. Therefore we have

$$b_1 b_2 x b_5 w b_6 = x' b_4.$$

Now $x \in D_*(e)$ and $b_5 \in C_B(f|f \leq e)$ so $xb_5x^{-1} = b'_5$ for some $b'_5 \in B$ which implies $xb_5 = b'_5x$. Hence,

$$b_1b_2b'_5xwb_6 = x'b_4$$

which implies

$$b_1b_2b'_5xw = x'b_4(b_6)^{-1}.$$

Thus $x' \leq xw$, but since $x' \in D_*(e)$ and $w \in W_*(e)$ we have $x' \leq x$ and our result is proved.

These three statements combined give the desired result, i.e. $x' \leq x$ and $y = y'$.

($\Leftarrow$) We now prove the converse of the theorem. Assume $x' \leq x$, $e' = e$ and $y' = y$. Let $\theta = x'ey'$, $\sigma = xey$ where $x, x' \in D_*(e)$, $y, y' \in D(e)^{-1}$. $x' \leq x$ implies

$$BxB \cap B^-x'B \neq \emptyset.$$

So there exists $b_1, b_2, b_4 \in B$ and $b_3 \in B^-$ such that

$$b_1xb_2 = b_3x'b_4.$$

Now multiply both sides on the right by ey , this yields

$$b_1xb_2ey = b_3x'b_4ey. \quad (3.2)$$

By ?? we know that $B = TU = TU_IU_L$, therefore we can rewrite

$$b_2 = tuu_1 \text{ and } b_4 = t'u'u'_1$$

where $t, t' \in T$, $u, u' \in U_I$ and $u_1, u'_1 \in U_L$. Thus, plugging b_2 and b_4 in (3.1) we get

$$b_1xtuu_1ey = b_3x't'u'u'_1ey'.$$

Now $U_L = U \cap L$, thus

$$eu_1 = u_1e \text{ and } eu'_1 = u'_1e.$$

So we have,

$$b_1xtueu_1y = b_3x't'u'eu'_1y.$$

From 1.9 we know that $U_I e = e$. Using this and the fact that T commutes with e we arrive at the following,

$$b_1 x e t u_1 y = b_3 x' e t' u_1' y$$

Now $t u_1, t' u_1' \in B_L$, thus $t u_1 = b_5$ and $t' u_1' = b_6$ for some $b_5, b_6 \in B_L$. So,

$$b_1 x e b_5 y = b_3 x' e b_6 y.$$

Since $y \in D(e)^{-1}$ we have $y^{-1} B_L y \subseteq B$ by Theorem 1.4.2, therefore

$$y^{-1} b_5 y = b'_5 \text{ and } y^{-1} b_6 y = b'_6$$

for some $b'_5, b'_6 \in B$. Hence

$$b_5 y = y b'_5 \text{ and } b_6 y = y b'_6$$

so,

$$b_1 x e y b'_5 = b_3 x' e y b'_6$$

which implies

$$B \sigma B \cap B^- \theta B \neq \emptyset$$

and thus $\theta \leq_1 \sigma$. $\square$

It follows immediately from Theorem 3.1.1 that $\leq_1$ is a partial order on the Renner monoid R .

3.2 Fine Bruhat Intersection 2

Now we consider the second intersection,

$$B \sigma B \cap B^- \theta B^-.$$

In Chapter 2, we defined $\leq_2$ as follows:

$$\theta \leq_2 \sigma \Leftrightarrow B \sigma B \cap B^- \theta B^- \neq \emptyset.$$

We saw that $\leq_2$ was a partial order only on the rank one partial permutation matrices. In Figure 2.4 on page 62, we noticed that $\leq_2$ is not antisymmetric, and hence, not a partial order.

Let us look again at the conditions stated in Theorem 2.2.3. Let $\theta = E_{i'_1 j'_1} + \cdots + E_{i'_r j'_r}$ and $\sigma = E_{i_1 j_1} + \cdots + E_{i_r j_r}$. Then $\theta \leq_2 \sigma$ if and only if $[i'_1, \dots, i'_r] \leq [i_1, \dots, i_r]$ and $[j_1, \dots, j_r] \leq [j'_1, \dots, j'_r]$. Note: the notation $[i'_1, \dots, i'_r] \leq [i_1, \dots, i_r]$ simply means rewrite the integers $i'_1, \dots, i'_r$ and $i_1, \dots, i_r$ in increasing order and check that the first r -tuple is term by term less than the second r -tuple. In Figure 2.4, we notice that elements within an equivalence class differ by a permutation. To understand this in more general terms, we write elements in the Renner monoid in modified standard form. i.e.,

$$r = xwey$$

where $x \in D(e)$, $w \in W^*(e)$, $e \in \Lambda$ and $y \in D(e)^{-1}$. It is precisely the different w 's in $W^*(e)$, that yield the different elements in each equivalence class. The number of elements in each equivalence class will be the order of $W^*(e)$.

It is not hard to see that Deodhar's criterion for permutations implies the conditions stated in the theorem above. Certainly, if $[i'_1, \dots, i'_r] \leq [i_1, \dots, i_r]$ for all $1 \leq r \leq n$, then $[i'_1, \dots, i'_r] \leq [i_1, \dots, i_r]$. Thus, if we let $\theta = x'w'e'y'$ and $\sigma = xwey$, we guess that in order for $\theta \leq_2 \sigma$, we must have $x' \leq x$ and by a similar argument, $y \leq y'$. Therefore our theorem goes as follows.

Let $\theta, \sigma \in R$ be in modified standard form. i.e., $\theta = x'w'e'y'$ and $\sigma = xwey$. Then $\theta \leq_2 \sigma$ if and only if $x' \leq x$, $e' = e$, and $y \leq y'$.

In order to prove this theorem, we need the following lemmas.

LEMMA 3.2.1. $x \leq y$ if and only if $B^-x \subseteq B^-yB^-B$.

Proof. Let w_0 be the longest element in W . Then

$$\begin{aligned}
x \leq y & \\
& \Leftrightarrow w_0 x^{-1} w_0 \leq w_0 y^{-1} w_0 \\
& \Leftrightarrow (w_0 x^{-1} w_0) B \subseteq B^{-} B (w_0 y^{-1} w_0) B \\
& \Leftrightarrow w_0 x^{-1} w_0 B \subseteq w_0 B w_0 B w_0 y^{-1} w_0 B \\
& \Leftrightarrow x^{-1} w_0 B \subseteq B B^{-} y^{-1} w_0 B \\
& \Leftrightarrow x^{-1} w_0 B w_0 \subseteq B B^{-} y^{-1} w_0 B w_0 \\
& \Leftrightarrow x^{-1} B^{-} \subseteq B B^{-} y^{-1} B^{-} \\
& \Leftrightarrow B^{-} x \subseteq B^{-} y B^{-} B.
\end{aligned}$$

□

The following Lemma is the dual of Lemma 3.2.1, which may be proved in a similar fashion.

LEMMA 3.2.2. $x \leq y$ if and only if $xB \subseteq B^{-}ByB$.

We are now ready to prove our theorem.

THEOREM 3.2.1. Let $\theta, \sigma \in R$ be in modified standard form. i.e. $\theta = x'w'e'y'$ and $\sigma = xwey$. $\theta \leq_2 \sigma$ if and only if $x' \leq x$, $e' = e$, and $y \leq y'$.

Proof. Suppose $\theta \leq_2 \sigma$. Since $\leq_2$ is an order within $\mathcal{J}$ -classes, it must be that θ and σ are in the same $\mathcal{J}$ -class. Therefore $GeG = Ge'G$ which implies that $e = e'$. Now, $\theta \leq_2 \sigma$ implies

$$B^{-}Bxwey = x'w'e'y'B^{-}B \neq \emptyset.$$

So there exists $b_1, b_3 \in B^{-}$ and $b_2, b_4 \in B$ such that

$$b_1 b_2 xwey = x'w'e'y'b_3 b_4$$

$\Rightarrow$

$$(x')^{-1} b_1 b_2 xwe = ew'y'b_3 b_4 y^{-1}.$$

Multiplying on the left and on the right separately by e yields

$$e(x')^{-1} b_1 b_2 xwe = ew'y'b_3 b_4 y^{-1}$$

and

$$(x')^{-1}b_1b_2xwe = ew'y'b_3b_4y^{-1}e$$

respectively. Thus we have

$$(x')^{-1}b_1b_2xwe = e(x')^{-1}b_1b_2xwe$$

and

$$ew'y'b_3b_4y^{-1} = ew'y'b_3b_4y^{-1}e$$

which implies that $(x')^{-1}b_1b_2xw \in P(e)$ and $w'y'b_3b_4y^{-1} \in P^-(e)$. First let us consider what it means for $(x')^{-1}b_1b_2xw \in P(e)$. We know from Section 1.3 that

$$P(e) = BW_I B = BW(e)B.$$

Thus,

$$(x')^{-1}b_1b_2xw \in B\tilde{w}B$$

for some $\tilde{w} \in W(e)$. This implies

$$b_1b_2w \in B\tilde{w}Bx^{-1}.$$

By Theorem 7.2.5 and Lemma 7.7.3 of [14]

$$(x')^{-1}b_1b_2w \in B\tilde{w}Bx^{-1} \subseteq \bigcup_{w_1 \leq \tilde{w}} Bw_1x^{-1}B.$$

Therefore we have

$$(x')^{-1}b_1b_2w \in ((x')^{-1}B^-B \cap Bw_1x^{-1}B)$$

for some $w_1 \leq \tilde{w}$. Hence,

$$(x')^{-1}B^-B \cap Bw_1x^{-1}B \neq \emptyset$$

$\implies$

$$B^-B \cap x'Bw_1x^{-1}B \neq \emptyset$$

and thus by Lemma 1.2 [11] we have

$$x' \leq (w_1 x^{-1})^{-1} = x(w_1)^{-1}.$$

Since $x \in D_*(e)$ and $(w_1)^{-1} \in W(e)$ we have $x' \leq x$.

Now we consider $y'w'b_3b_4y^{-1} \in P^-(e)$. Since

$$P^-(e) = B^-W(e)B^-$$

we see that

$$y'w'b_3b_4y^{-1} \in B^- \hat{w} B^-$$

for some $\hat{w} \in W(e)$. This implies

$$w'b_3b_4y^{-1} \in (y')^{-1}B^- \hat{w} B^- \subseteq \bigcup_{w_2 \leq \hat{w}} B^-(y')^{-1}w_2B^-.$$

So we have

$$w'b_3b_4y^{-1} \in (B^-By^{-1} \cap B^-(y')^{-1}w_2B^-).$$

Therefore

$$B^-By^{-1} \cap B^-(y')^{-1}w_2B^- \neq \emptyset$$

$\implies$

$$B^-B \cap B^-(y')^{-1}w_2B^-y \neq \emptyset.$$

Thus by Lemma 1.2 [11]

$$y \leq ((y')^{-1}w_2)^{-1} = (w_2)^{-1}y'$$

which implies $y \leq y'$ since $(w_2)^{-1} \in W(e)$ and $y' \in D(e)^{-1}$.

Conversely, assume $x' \leq x$, $e' = e$ and $y \leq y'$. We know that

$$B_L^- B_L w \cap w' B_L^- B_L \neq \emptyset,$$

thus,

$$x' B_L^- B_L w e y \cap x' w' B_L^- B_L e y \neq \emptyset.$$

So there exists $b_1, b_3 \in B_L^-$, $b_2, b_4 \in B_L$ such that

$$x'b_1b_2wey = x'w'b_3b_4ey.$$

By Corollary 1.2 [5],

$$\begin{aligned} x' &\leq x \\ &\Updownarrow \\ BxB \cap B^-x'B &\neq \emptyset \\ &\Updownarrow \\ x'B &\subseteq B^-BxB. \end{aligned}$$

Therefore, $x'b_1 = b_5b_6xb_7$ for some $b_5 \in B^-$, $b_6, b_7 \in B$. So we have,

$$b_5b_6xb_7wey = x'w'b_3b_4ey.$$

So $b_7 \in B = B_LU_I$ by Theorem 1.3.8, therefore $b_7 = b'_7u$ for some $b'_7 \in B_L$ and $u \in U_I$. Thus,

$$b_5b_6xb'_7uwey = x'w'b_3b_4ey.$$

Now, $x \in D(e)$, which implies by Theorem 1.4.6, $xB_Lx^{-1} \subseteq B$, hence $xb'_7x^{-1} = b_8$ for some $b_8 \in B$ and $xb'_7 = b_8x$. Making this substitution yields,

$$b_5b_6b_8xuwey = x'w'b_3b_4ey.$$

Since $we = ew$ and $U_Ie = e$ we have

$$b_5b_6b_8xwey = x'w'b_3b_4ey.$$

To simplify our notation a little, let $b'_6 = b_6b_8 \in B$. Thus,

$$b_5b'_6xwey = x'w'b_3b_4ey.$$

Now we examine the right side of

$$b_5b'_6xwey = x'w'b_3b_4ey.$$

Since $b_3 \in B_L^-$ and $b_4 \in B_L$, e commutes with both thus,

$$b_5 b'_6 x w e y = x' w' e b_3 b_4 y.$$

By Theorem 1.4.6, $y \in D(e)^{-1}$ implies $y^{-1} B_L y \subseteq B$. Therefore $y^{-1} b_4 y = b'_4$ for $b'_4 \in B$ and

$$b_4 y = y b'_4.$$

So we have,

$$b_5 b'_6 x e w y = x' w' e b_3 y b'_4.$$

Assume $y \leq y'$. From Lemma 3.2.1 we have

$$B^- y \subseteq B^- y' B^- B,$$

so $b_3 y = b_9 y' b_{10} b_{11}$ for some $b_9, b_{10} \in B^-$ and $b_{11} \in B$. Hence,

$$b_5 b'_6 x w e y = x' w' e b_9 y' b_{10} b_{11} b'_4$$

which implies

$$b_5 b'_6 x w e y = x' w' e b_9 y' b_{10} b'_{11}$$

where $b'_{11} = b_{11} b'_4$. Since $y \in D(e)^{-1}$, $(y')^{-1} b_9 y' = b'_9$ for some $b'_9 \in B^-$ and $b_9 y' = y' b'_9$. So,

$$b_5 b'_6 x w e y = x' w' e y' b'_9 b_{10} b'_{11}$$

which implies

$$b_5 b'_6 x w e y = x' w' e y' b'_{10} b'_{11}$$

for some $b'_{10} = b'_9 b_{10} \in B^-$. Thus

$$B^- B x w e y \cap x' w' e y' B^- B \neq \emptyset$$

and $\theta \leq_2 \sigma$.

□

Chapter 4

General Theorems for the Matrices

We now return to the theorems that were stated at the end of each section in Chapter 2. Recall that we proved theorems for rank 1 and rank 2 partial permutation matrices for each of the intersections considered. Let us familiarize ourselves again with the notation used in Chapter 2.

The notations were as follows:

$$\begin{aligned} M &= M_n(K), \text{ the monoid of all } n \times n \text{ matrices} \\ B &= B_n(K), \text{ the invertible upper triangular matrices} \\ B^- &= B_n^-(K), \text{ the invertible lower triangular matrices} \\ B^-B &= \{A \in GL_n(K) \mid \text{all leading principal minors of } A \text{ have nonzero} \\ &\quad \text{determinants}\} \\ R &= \{A \in M_n(K) \mid A \text{ is a partial permutation matrix}\} \end{aligned}$$

4.1 Proof of Theorem 2.1.3

We now focus on the following intersection:

$$B\sigma B \cap B^-\theta B.$$

Both of the theorems for rank one and rank two were proved directly in Chapter 2. As seen already at rank two, the proofs are tedious and long. Since our method of

“brute force” is no longer efficient, we turn to our general theory to prove our theorem. Note that this is a very nice setup. We wanted to gain intuition into the general reductive monoid case, so we started with the monoid $M_n(K)$ and proved the theorems regarding rank one and rank two partial permutation matrices. Observations from this work forecasted both the general reductive monoid theory and the theorems for the partial permutation matrices of rank r . Thus, examining the matrices first, turned out to be invaluable to our understanding of the problem. Also, we are now able to provide a complete example with the matrices.

Throughout the rest of the chapter a partial permutation matrix of rank r will be denoted by

$$E_{i_1 j_1} + E_{i_2 j_2} + \cdots + E_{i_r j_r}$$

where $i_1 \neq i_2 \neq \cdots \neq i_r$ and $j_1 < j_2 < \cdots < j_r$.

The theorem from Section 2.1 was the following:

Let $\theta = E_{i'_1 j'_1} + E_{i'_2 j'_2} + \cdots + E_{i'_r j'_r}$ and $\sigma = E_{i_1 j_1} + E_{i_2 j_2} + \cdots + E_{i_r j_r}$. Then $\theta \leq_1 \sigma$ if and only if $[i'_1, i'_2, \dots, i'_k] \leq [i_1, i_2, \dots, i_k]$ for all $1 \leq k \leq r$ and $j'_1 = j_1, j'_2 = j_2, \dots, j'_r = j_r$.

In order to prove the theorem, we need the following Lemma.

LEMMA 4.1.1. Fix $e \in \Lambda$. Let $xe = E_{i_1 1} + E_{i_2 2} + \cdots + E_{i_r r}$ and $x'e = E_{i'_1 1} + E_{i'_2 2} + \cdots + E_{i'_r r}$. Suppose $x, x' \in D_*(e)$ and that $[i'_1, \dots, i'_k] \leq [i_1, \dots, i_k]$ for $1 \leq k \leq r$ then $x' \leq x$.

Proof. Since $x, x' \in D_*(e)$,

$$x = E_{i_1 1} + E_{i_2 2} + \cdots + E_{i_n n}$$

where for $k \geq r$, $i_{k+1} = \min(\{1, \dots, n\} - \{i_1, \dots, i_k\})$ and

$$x' = E_{i'_1 1} + E_{i'_2 2} + \cdots + E_{i'_n n}$$

where for $k \geq r$, $i'_{k+1} = \min(\{1, \dots, n\} - \{i'_1, \dots, i'_k\})$. Let $k \geq r$ and assume inductively that $[i'_1, \dots, i'_k] \leq [i_1, \dots, i_k]$. We want to show that for $k+1 > r$ that $[i'_1, \dots, i'_{k+1}] \leq [i_1, \dots, i_{k+1}]$. Let us denote the sequence $[i'_1, \dots, i'_k]$ in increasing order by

$$(l'_1, \dots, l'_k)$$

where $l'_1 < l'_2 < \dots < l'_k$. Similarly, $[i_1, \dots, i_k]$ will be denoted by

$$(l_1, \dots, l_k)$$

where $l_1 < l_2 < \dots < l_k$. Since $[i'_1, \dots, i'_k] \leq [i_1, \dots, i_k]$ for $1 \leq k \leq r$ we have

$$l'_1 < l_1, l'_2 < l_2, \dots, l'_k < l_k$$

for $1 \leq k \leq r$.

Case 1: Suppose $l'_1 = 1$ and $l_1 > 1$. Then $i'_{k+1} = p$ for some p and $i_{k+1} = 1$. Therefore we have

$$[i'_1, \dots, i'_k, p] \text{ and } [i_1, \dots, i_k, 1].$$

Now, we insert $p = i'_{k+1}$ into the first gap in the sequence

$$(l'_1, \dots, l'_k).$$

In other words,

$$(1, 2, \dots, p-1, p, l'_p, l'_{p+1}, \dots, l'_k).$$

Now, we compare this, term by term, to the sequence

$$(1, l_1, \dots, l_{p-2}, l_{p-1}, l_p, l_{p+1}, \dots, l_k).$$

It is not hard to see that we still have each term in the first sequence less than the corresponding term in the second sequence. Therefore $[i'_1, \dots, i'_{k+1}] \leq [i_1, \dots, i_{k+1}]$

Case 2: Suppose $l'_1 > 1$ and $l_1 > 1$. Then $i'_{k+1} = 1 = i_{k+1}$. So we consider $[i'_1, \dots, i'_k, 1]$ and $[i_1, \dots, i_k, 1]$. The corresponding sequences written in increasing order are

$$(1, l'_1, l'_2, \dots, l'_k)$$

and

$$(1, l_1, l_2, \dots, l_k)$$

respectively. Clearly our conditions are satisfied and therefore $[i'_1, \dots, i'_{k+1}] \leq [i_1, \dots, i_{k+1}]$

Case 3: Suppose $l'_1 = l_1 = 1$. Then we have the sequences

$$(1, l'_2, \dots, l'_k)$$

and

$$(1, l_2, \dots, l_k)$$

If $l'_2 = 2$ and $l_2 > 2$ then we are in Case 1 again, and thus $[i'_1, \dots, i'_{k+1}] \leq [i_1, \dots, i_{k+1}]$. Similarly if $l'_2 > 2$ and $l_2 > 2$ then we are in Case 2 and we again have $[i'_1, \dots, i'_{k+1}] \leq [i_1, \dots, i_{k+1}]$. Finally, if $l'_2 = 2 = l_2$, then by induction on n we have the desired result.

We now conclude that $[i'_1, \dots, i'_k] \leq [i_1, \dots, i_k]$ for all k and thus by Deodhar's criterion [10] $x' \leq x$. $\square$

Theorem 2.1.3 Let $\theta = E_{i'_1 j'_1} + E_{i'_2 j'_2} + \dots + E_{i'_r j'_r}$ and $\sigma = E_{i_1 j_1} + E_{i_2 j_2} + \dots + E_{i_r j_r}$. Then $\theta \leq_1 \sigma$ if and only if $[i'_1, i'_2, \dots, i'_k] \leq [i_1, i_2, \dots, i_k]$ for all $1 \leq k \leq r$ and $j'_1 = j_1, j'_2 = j_2, \dots, j'_r = j_r$.

Proof. Fix $e \in \Lambda$. Let $\theta = x'ey'$ and $\sigma = xey$ where $x', x \in D_*(e)$ and $y, y' \in D(e)^{-1}$. By Theorem 2.1.1, we know that

$$\theta \leq_1 \sigma \implies x' \leq x \text{ and } y' = y.$$

Since $y = y'$, we know that $ey = ey'$ which implies by [renner] that θ and σ have the same row echelon form. Hence,

$$j'_1 = j_1, j'_2 = j_2, \dots, j'_r = j_r.$$

Now, $x' \leq x$ implies $[i'_1, \dots, i'_k] \leq [i_1, \dots, i_k]$ for all $1 \leq k \leq n$. Since $r < n$ our result is proved.

Conversely, assume that $[i'_1, \dots, i'_k] \leq [i_1, \dots, i_k]$ for $1 \leq k \leq r$ and $j'_1 = j_1, j'_2 = j_2, \dots, j'_r = j_r$. Since $x', x \in D_*(e)$ we know that

$$x'e = E_{i'_1 j'_1} + \dots + E_{i'_r j'_r} \text{ and } xe = E_{i_1 j_1} + \dots + E_{i_r j_r}.$$

Since $[i'_1, \dots, i'_k] \leq [i_1, \dots, i_k]$ for $1 \leq k \leq r$, by Lemma 4.1.1 we know that $x' \leq x$ and hence by Theorem 2.1.1, $x'e \leq_1 xe$. Now, since $j'_1 = j_1, j'_2 = j_2, \dots, j'_r = j_r$ we know that $ey = ey'$ which implies $y = y'$ and by Theorem 2.1.1, $x'ey' \leq_1 xey$, ie. $\theta \leq_1 \sigma$. $\square$

Notice that for all ranks less than n , $\leq_1$ is just the truncated Bruhat order. For rank n , $\leq_1$ is the Bruhat order and is given by Deodhar's criterion in Section 1.3. Thus, it is not hard to see that $\leq_1$ is a partial order on the monoid of all partial permutation matrices and in general a partial order on R , the Renner monoid.

EXAMPLE 4.1.1. Let $M = M_4(K)$. Let

$$\theta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Is $\theta \leq_1 \sigma$? By Theorem 4.1.1 we need to check the following:

1. $[i'_1, \dots, i'_k] \leq [i_1, \dots, i_k]$ for all $1 \leq k \leq 3$
2. $j'_1 = j_1, j'_2 = j_2, j'_3 = j_3$

We rewrite θ and σ as

$$\theta = E_{21} + E_{12} + E_{33}$$

and

$$\sigma = E_{31} + E_{12} + E_{43}.$$

Start with $k = 1$. Clearly $i'_1 = [2] \leq [3] = i_1$. So we move to the case when $k = 2$. Consider i'_1, i'_2 written in increasing order to i_1, i_2 in increasing order. Again,

$$[1, 2] \leq [1, 3]$$

so let $k = 3$. Comparing i'_1, i'_2, i'_3 in increasing order to i_1, i_2, i_3 in increasing order we see that

$$[1, 2, 3] \leq [1, 3, 4],$$

thus condition 1. is satisfied. It is easy to see that condition 2. is satisfied, hence $\theta \leq_1 \sigma$.

4.2 Proof of Theorem 2.2.3

In this section we consider the intersection

$$B\sigma B \cap B^{-}\theta B^{-}.$$

Recall from Chapter 2 that $\leq_2$ is a partial order on the rank one partial permutation matrices, however, it is not for any rank $r \geq 2$. For any rank $r \geq 2$, $\leq_2$ is a partial order on equivalence classes of rank r partial permutation matrices.

Theorem 2.2.3 *Let $\theta = E_{i'_1 j'_1} + \dots + E_{i'_r j'_r}$ and $\sigma = E_{i_1 j_1} + \dots + E_{i_r j_r}$ be $n \times n$ rank r partial permutation matrices. Then $\theta \leq_2 \sigma$ if and only if $[i'_1, \dots, i'_k] \leq [i_1, \dots, i_k]$ and $[j_1, j_2, \dots, j_l] \leq [j'_1, j'_2, \dots, j'_l]$.*

Proof. Suppose that $\theta \leq_2 \sigma$. We may write θ and σ in modified standard form. i.e.

$$\theta = x'w'ey' \text{ and } \sigma = xwey$$

where $x', x \in D(e)$, $w', w \in W_*(e)$ and $y', y \in D(e)^{-1}$. From Theorem 2.2.1,

$$x' \leq x \text{ and } y \leq y'.$$

By Deodhar's Criterion,

$$[i'_1, i'_2, \dots, i'_k] \leq [i_1, i_2, \dots, i_k]$$

for all $1 \leq k \leq n$ and

$$[j_1, j_2, \dots, j_l] \leq [j'_1, j'_2, \dots, j'_l]$$

for all $1 \leq l \leq n$. Clearly this implies

$$[i'_1, i'_2, \dots, i'_r] \leq [i_1, i_2, \dots, i_r]$$

and

$$[j_1, j_2, \dots, j_r] \leq [j'_1, j'_2, \dots, j'_r]$$

since $1 \leq r < n$, hence we have the desired result.

Conversely, suppose $[i'_1, i'_2, \dots, i'_r] \leq [i_1, i_2, \dots, i_r]$ and $[j_1, j_2, \dots, j_r] \leq [j'_1, j'_2, \dots, j'_r]$. We want to show that $\theta \leq_2 \sigma$. As before, we may write $\theta = x'w'ey'$ and $\sigma = xwey$. We will denote the sequences above in increasing order by

$$[i'_1, i'_2, \dots, i'_r] = (l'_1, l'_2, \dots, l'_r)$$

$$[i_1, i_2, \dots, i_r] = (l_1, l_2, \dots, l_r)$$

$$[j_1, j_2, \dots, j_r] = (p_1, p_2, \dots, p_r)$$

$$[j'_1, j'_2, \dots, j'_r] = (p'_1, p'_2, \dots, p'_r).$$

Note that

$$x'e = E_{l'_1 1} + \dots + E_{l'_r r}$$

and

$$xe = E_{l_1 1} + \dots + E_{l_r r}$$

are the column echelon forms of θ and σ respectively. Now,

$$\begin{aligned} [i'_1, i'_2, \dots, i'_r] \leq [i_1, i_2, \dots, i_r] &\Rightarrow (l'_1, l'_2, \dots, l'_r) \leq (l_1, l_2, \dots, l_r) \\ &\Rightarrow [l'_1, l'_2, \dots, l'_k] \leq [l_1, l_2, \dots, l_k] \text{ for all } 1 \leq k \leq r. \end{aligned}$$

Since $x', x \in D(e) \subseteq D_*(e)$, by Lemma 2.1.1, we have that $x' \leq x$.

Similarly,

$$ey' = E_{1p'_1} + E_{1p'_2} + \dots + E_{1p'_r}$$

and

$$ey = E_{1p_1} + E_{1p_2} + \dots + E_{1p_r}$$

are the row echelon forms of θ and σ respectively. So,

$$\begin{aligned} [j_1, j_2, \dots, j_r] \leq [j'_1, j'_2, \dots, j'_r] &\Rightarrow (p_1, p_2, \dots, p_r) \leq (p'_1, p'_2, \dots, p'_r) \\ &\Rightarrow [p_1, p_2, \dots, p_k] \leq [p'_1, p'_2, \dots, p'_k] \text{ for all } 1 \leq k \leq r. \end{aligned}$$

Hence, by the dual of Lemma 2.1.1, $y \leq y'$ and therefore we have $\theta \leq_2 \sigma$. $\square$

Recall, for $M = M_n(K)$, the Renner Monoid, R , consists of all partial permutation matrices. As seen in Figure 4, matrices are in the same equivalence class if the indices of the rows are the same and the indices of the columns are the same. This is precisely why $\leq_2$ is not a partial order on R .

EXAMPLE 4.2.1. Let $M = M_4(K)$. Suppose

$$\theta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Is $\theta \leq_2 \sigma$? What other partial permutation matrices are in the same equivalence class as θ ?

To answer the first question we start by writing

$$\theta = E_{32} + E_{13} + E_{24}$$

and

$$\sigma = E_{42} + E_{23} + E_{14}.$$

By Theorem 2.2.1, we need the following:

1. $[i'_1, i'_2, i'_3] \leq [i_1, i_2, i_3]$
2. $[j_1, j_2, j_3] \leq [j'_1, j'_2, j'_3]$

To check 1. we simply write the row indices for θ and σ in increasing order and compare. Clearly,

$$(1, 2, 3) \leq (1, 2, 4)$$

thus 1. is satisfied. We now write the column indices for first σ then θ and compare. i.e.

$$(2, 3, 4) \leq (2, 3, 4).$$

Therefore we conclude that $\theta \leq_2 \sigma$.

Since θ and σ are 4×4 rank 3 partial permutation matrices, we have 6 matrices in each equivalence class. Therefore the equivalence class containing θ is:

$$\left\{ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

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