ABSTRACT

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Rational homotopy types of simply connected topological spaces have been classified by weak equivalence classes of commutative cochain algebras (Sullivan) and by isomorphism classes of minimal commutative A_∞ -algebras (Kadeishvili).

We classify rational homotopy types of the space X by using the (non-commutative) singular cochain complex, $C^*(X, Q)$, with additional structure given by the homotopies introduced by Baues, $\{E_{1,k}\}$ and $\{F_{p,q}\}$. We show that if we modify the resulting B_∞ -algebra structure on this algebra by requiring that its bar construction be a Hopf algebra up to homotopy, then weak equivalence classes of such algebras classify rational homotopy types.

Singular Cochains and Rational Homotopy Type

by

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INTRODUCTION

Throughout the paper we work over the rationals Q , unless otherwise specified.

The singular cochain algebra $C^*(X, Q)$ of a topological space X almost never is commutative, although it is homotopy commutative. However, we can replace $C^*(X, Q)$ by a genuinely commutative cochain algebra. More precisely, there is a commutative cochain algebra $A_{PL}(X, Q)$ [43] and natural cochain algebra quasi-isomorphisms

$$C^*(X, Q) \xrightarrow{\cong} D(X) \xleftarrow{\cong} A_{PL}(X, Q)$$

where $D(X)$ is a third natural cochain algebra. The contravariant functor $X \mapsto A_{PL}(X, Q)$ serves as the fundamental bridge to transfer problems from topology to algebra.

Quasi-isomorphisms above define a natural isomorphism of graded algebras

$$H^*(X, Q) = H(A_{PL}(X, Q))$$

through which they are always identified; thus for any continuous map, f , we identify $H^*(f, Q) = H(A_{PL}(f, Q))$.

Recall that simply connected spaces X and Y have the same rational homotopy type if and only if there is a chain of rational homotopy equivalences

$$X \rightarrow Z_0 \leftarrow Z_1 \rightarrow Z_2 \leftarrow \cdots \rightarrow Z_k \leftarrow Y$$

(a map $f : W \rightarrow Z$ is a rational homotopy equivalence if $H^*(f, Q)$ is an isomorphism).

There is an analogous notion for commutative cochain algebras: two commutative cochain algebras (A, d) and (B, d) are weakly equivalent if they are connected by the

chain

$$(A, d) \xleftarrow{\simeq} (C(0), d) \xrightarrow{\simeq} (C(1), d) \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} (C(k), d) \xrightarrow{\simeq} (B, d).$$

of quasi-isomorphisms of commutative cochain algebras; such a chain is called a weak equivalence between (A, d) and (B, d) .

It follows that if X and Y have the same rational homotopy type then $A_{PL}(X, Q)$ and $A_{PL}(Y, Q)$ are weakly equivalent; and so $A_{PL}(-, Q)$ defines a map

$$(i) \quad \{\text{rational homotopy types}\} \rightarrow \{\text{weak equivalence classes of commutative cochain algebras}\}$$

Moreover, it was shown in [43] that if simply connected spaces X and commutative cochain algebras (A, d) are such that $H^*(X, Q)$ and $H(A, d)$ have finite type, then this correspondence is a bijection.

Later Kadeishvili described the rational homotopy types in terms of A_∞ -algebras; it was shown that X and Y have the same rational homotopy type if and only if $H(A_{PL}(X, Q))$ and $H(A_{PL}(Y, Q))$ are isomorphic *commutative* A_∞ -algebras [25]; this relation defines a map

$$(ii) \quad \{\text{rational homotopy types}\} \rightarrow \{\text{isomorphism classes of minimal commutative } A_\infty\text{-algebras}\}$$

which turns out to be a bijection if simply connected spaces X and commutative A_∞ -algebras $(A, \{m_i\})$ are such that $H^*(X, Q)$ and $H(A)$ are of finite type, see [25].

In both cases above the bijections were established using *Sullivan's commutative DG-algebra* $A_{PL}(X, Q)$. In our dissertation, we classify the rational homotopy types of simply connected space X by using the (noncommutative) singular cochain algebra, $(C^*(X, Q), d_{C^*(X, Q)}, \smile)$, with additional structure given by the homotopies

$$\{E_{1,k} : C^*(X, Q) \otimes (C^*(X, Q)^{\otimes k}) \rightarrow C^*(X, Q)\}$$

with $E_{1,1} = \smile_1$ and

$$\{F_{p,q} : (C^*(X, Q)^{\otimes p}) \otimes (C^*(X, Q)^{\otimes q}) \rightarrow C^*(X, Q)\}$$

with $F_{1,1} = \smile_2$ of Baues [6], [7].

In Section 3 we introduce the notion of $\tilde{B}_\infty$ -algebra, a modified version of B_∞ -algebra of Getzler and Jones [19], and prove the following

Theorem 74 *The 1-reduced spaces X and Y have the same rational homotopy type if and only if $(C^*(X), d_{C^*(X)}, \smile, \{E_{1,k}^X\}, \{F_{p,q}^X\})$ and $(C^*(Y), d_{C^*(Y)}, \smile, \{E_{1,k}^Y\}, \{F_{p,q}^Y\})$ are weakly equivalent in the $\tilde{B}_\infty$ -algebra category.*

Finally we show that the singular cochain functor C^* induces an equivalence of rational homotopy categories

$$\mathcal{H}oC^* : \mathcal{H}oCW_1 \rightarrow \mathcal{H}o\tilde{B}_\infty\text{-Alg},$$

see Theorems 75 and 76.

1. REVIEW OF NOTATION AND KNOWN RESULTS

In this section we collect the definitions and facts we need; most can be found in [3], [25] and [41]. Since these results are not new, we omit their proofs.

1.1 DG MODULE

Let R be a commutative ring with identity 1_R . Let $\{M_i\}_{i \in \mathbb{Z}}$ be a sequence of R -modules; $M = \sum M_i$ is called a graded R -module (throughout this paper we assume $M_i = 0$ for $i < 0$). An element $x \in M_i$ is said to be homogeneous of degree i , in which case we write $|x| = i$.

A differential graded module (DG -Modules) is a graded module M together with an endomorphism $d \in \text{End}(M)$ of degree $+1$ and square zero. Given DG -Modules M, N and a homomorphism $f : (M, d_M) \rightarrow (N, d_N)$ of degree k , define the differential of f by $Df = d_N f - (-1)^k f d_M$. The category of DG -Modules is denoted by DGM ; its morphisms are chain maps, i.e., maps f of degree 0 with $Df = 0$.

A DGM homomorphism inducing an isomorphism on homology is called a *quasi-isomorphism*.

1.2 DG ALGEBRA

DGA denotes the category of augmented associative DG -algebras with unit. If $(A, d, \mu, \eta, \varepsilon) \in DGA$, then the multiplication $\mu : A \otimes A \rightarrow A$, the unit $\eta : R \rightarrow A$,

the augmentation $\varepsilon : A \rightarrow R$ and the differential $d : A \rightarrow A$ satisfy the usual requirements. We say that A is connected, if $\eta : R \rightarrow A_0$ is an isomorphism. To simplify notation we omit η and ε and write (A, d, μ) for $(A, d, \mu, \eta, \varepsilon)$.

A *morphism* $f : A \rightarrow B$ in DGA is a chain map preserving all structure. A DGA homomorphism inducing an isomorphism on homology is called *quasi-isomorphism*.

Let $IA = \ker(\varepsilon)$ be the augmentation ideal of A . Then the exact sequence $0 \rightarrow IA \xrightarrow{i} A \xrightarrow{\varepsilon} R \rightarrow 0$ defines a functor $I : DGA \rightarrow DGM$; the multiplication μ induces a map $I\mu : IA \otimes IA \rightarrow IA$ given by $i(I\mu) = \mu(i \otimes i)$.

Let $f, g : (A, d_A, \mu_A) \rightarrow (B, d_B, \mu_B) \in DGA$. A degree r R -homomorphism $F : A \rightarrow B$ is an (f, g) -*derivation* if it satisfies

$$F\mu_A = \mu_B(F \otimes g + (-1)^r f \otimes F).$$

A *derivation homotopy* from f to g is an (f, g) -derivation of degree $+1$ further satisfying $f - g = d_B F + F d_A$.

We call the DG-algebra (A, d, μ) 1-connected if it is connected and $A_1 = 0$.

1.3 DG COALGEBRA

DGC denotes the category of coaugmented differential graded coassociative coalgebras (DG Coalgebras) with counit. If $(C, d, \Delta, \eta, \varepsilon) \in DGC$, the comultiplication $\Delta : C \rightarrow C \otimes C$, the counit $\varepsilon : C \rightarrow R$, the coaugmentation $\eta : R \rightarrow C$ and the differential $d : C \rightarrow C$ satisfy the usual requirements. We say that C is connected, if $\varepsilon : C_0 \rightarrow R$ is an isomorphism. To simplify notation we omit η and ε and write (C, d, Δ) for $(C, d, \Delta, \eta, \varepsilon)$.

A *morphism* $f : C \rightarrow D$ in DGC is a chain map preserving all structure. A DGC homomorphism inducing an isomorphism on cohomology is called *quasi-isomorphism*.

Let $JC = \operatorname{coker}(\eta)$ be the coaugmentation ideal of C . Then the exact sequence $0 \rightarrow R \xrightarrow{\eta} C \xrightarrow{p} JC \rightarrow 0$ defines a functor $J : DGC \rightarrow DGM$; the comultiplication Δ induces a map $J\Delta : JC \rightarrow JC \otimes JC$ given by $(J\Delta)p = (p \otimes p)\Delta$.

Let $f, g : (C, d_C, \Delta_A) \rightarrow (D, d_D, \Delta_D) \in DGC$. A degree r R -homomorphism $F : C \rightarrow D$ is an (f, g) -coderivation if it satisfies

$$\Delta_D F = (F \otimes g + (-1)^r f \otimes F) \Delta_C.$$

A *coderivation homotopy* from f to g is an (f, g) -coderivation of degree $+1$ further satisfying $f - g = d_D F + F d_C$.

We call the DG-coalgebra (C, d, Δ) 1-connected if it is connected and $C_1 = 0$.

1.4 DG HOPF ALGEBRA

A *connected differential graded Hopf algebra* (DGHA) is defined to be a quartet (A, d, μ, Δ) such that (A, d, μ) is a connected differential graded algebra, (A, d, Δ) is a connected differential graded coalgebra and $\Delta\mu = (\mu \otimes \mu)(1 \otimes T \otimes 1)(\Delta \otimes \Delta)$. Notice that the last equality is equivalent to saying either that

$$\Delta : A \rightarrow A \otimes A,$$

is a morphism of algebras or that

$$\mu : A \otimes A \rightarrow A$$

is a morphism of coalgebras. The DGHA A is cocommutative if $T\Delta = \Delta$ and commutative if $\mu T = \mu$, where $T : A \otimes A \rightarrow A \otimes A$ is the twisting involution given by $T(a \otimes b) = (-1)^{|a||b|} b \otimes a$.

A *morphism* $f : (A, d, \mu, \Delta) \rightarrow (A', d', \mu', \Delta')$ between DG Hopf algebras is defined to be a graded R -module homomorphism compatible with the both structures; i.e. $d'f = fd$, $f\mu = \mu'(f \otimes f)$ and $\Delta'f = (f \otimes f)\Delta$.

It is easy to check that connected DG Hopf algebras and their morphisms form a category.

1.5 SUSPENSION

The suspension functor $s^r : DM \rightarrow DM$ is given by $(s^r M)_n = M_{n-r}$ and $d_{s^r M} = -d_M$. On an object M , the map $s^r : M \rightarrow s^r M$ has degree r , is the identity in each dimension and $Ds = 0$. If $f : M \rightarrow N$ has degree k , so does $s^r f : s^r M \rightarrow s^r N$ and the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ s^r \downarrow & & s^r \downarrow \\ s^r M & \xrightarrow{s^r f} & s^r N. \end{array}$$

1.6 TENSOR ALGEBRA

Let M be a graded R -module with $M_0 = 0$.

A *tensor algebra* $T(M)$ on a graded module M is the direct sum $T(M) = \sum_{i \geq 0} M^{\otimes i}$, where $M^{\otimes 0} = R$, with multiplication given by

$$\mu((a_1 \cdots a_i) \otimes (a_{i+1} \cdots a_n)) = (a_1 \cdots a_n).$$

Let $i_1 : M \rightarrow T(M)$ be the injection. $T(M)$ has the following universal property: If A is any connected DGA and $\alpha : M \rightarrow IA$ is any homomorphism, there is a unique algebra map $f_\alpha : T(M) \rightarrow A$ which makes the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{i_1} & T(M) \\ \alpha \downarrow & & f_\alpha \downarrow \\ IA & \xrightarrow{i} & A \end{array}$$

An explicit formula for f_α is given by

$$f_\alpha = \mu_A^n(\alpha^{\otimes n} \otimes i^{\otimes n}),$$

where $\mu_A^1 = Id$, $\mu_A^2 = \mu_A$ and $\mu_A^n = \mu_A(\mu_A^{n-1} \otimes Id)$.

The tensor algebra $T(M)$ has the analogous universal property with respect to derivations: for any graded $T(M)$ -module A and for any homomorphism $\beta : M \rightarrow IA$ there exists a derivation $D_\beta : T(M) \rightarrow A$ such that $D_\beta i_1 = \beta$.

Let (C, d, Δ) be a connected DGC. *The Cobar construction* $\Omega(C, d, \Delta)$ due to Adams [2] is defined as a free tensor algebra $T(s^{-1}(JC))$ whose differential d_Ω is defined on generators $s^{-1}JC$ by

$$d_\Omega \cdot i^1 = -i^1(s^{-1}ds) + i^2(s^{-1} \otimes s^{-1})(J\Delta)s,$$

where $i^k : (s^{-1}JC)^{\otimes k} \rightarrow T(s^{-1}JC)$ is the obvious inclusion.

Thus the Cobar construction of a 1-connected DGC is a connected DGA with augmentation $1 : R \rightarrow R$.

1.7 TENSOR COALGEBRA

Let M be a graded R -module with $M_0 = 0$.

A *tensor coalgebra* $T^c(M)$ on a graded module M is the direct sum $T^c(M) = \sum_{i \geq 0} M^{\otimes i}$, where $M^{\otimes 0} = R$, with diagonal given by

$$\Delta(a_1 \cdots a_n) = \sum_{i=0}^n (a_1 \cdots a_i) \otimes (a_{i+1} \cdots a_n).$$

Let $P_1 : T^c(M) \rightarrow M$ be the projection. $T^c(M)$ has the following universal property: If C is a any connected DGC and $\alpha : JC \rightarrow M$ is any homomorphism, there is a unique coalgebra map $f_\alpha : C \rightarrow T^c(M)$ which makes the following diagram

commute:

$$\begin{array}{ccc} T^c(M) & \xrightarrow{P_1} & M \\ f_\alpha \uparrow & & \alpha \uparrow \\ C & \xrightarrow{p} & JC \end{array}$$

An explicit formula for f_α is given by

$$f_\alpha = \sum_{i=1}^{\infty} (p^{\otimes i} \otimes \alpha^{\otimes i}) \Delta_C^i,$$

where $\Delta_C^1 = Id$, $\Delta_C^2 = \Delta_C$ and $\Delta_C^i = (\Delta_C^{i-1} \otimes Id) \Delta_C$.

The tensor coalgebra $T^c(M)$ has the analogous universal property with respect to coderivations: for any graded $T^c(M)$ -comodule C and for any homomorphism $\beta : C \rightarrow M$ there exists a coderivation $D_\beta : C \rightarrow T^c(M)$ such that $P_1 D_\beta = \beta$.

Let (A, d, μ) be a connected DGA. The Bar construction $B(A, d, \mu)$ is the DGC $T^c(s(IA))$ with differential (coderivation)

$$(1) \quad \begin{aligned} d_{BA}[a_1 | \cdots | a_n] &= \sum_{i=0}^{n-1} [\bar{a}_1 | \cdots | \bar{a}_i | da_{i+1} | a_{i+1} | \cdots | a_n] \\ &+ \sum_{i=0}^{n-2} [\bar{a}_1 | \cdots | \bar{a}_i | \mu(\bar{a}_{i+1} \otimes a_{i+1}) | a_{i+k+1} | \cdots | a_n], \end{aligned}$$

where $\bar{a} = (-1)^{|a|+1}a$, and the degree of $[a_1 | \cdots | a_n] \in BA$ is $\sum_{i=1}^n |a_i| + n$.

The Bar construction of a 1-connected DGA is a connected DGC with coaugmentation $1 : R \rightarrow R$.

Let $S_{p,q}$ denote the shuffle permutations in the set of all permutations $\sum_{p+q}$, i.e., $\sigma \in S_{p,q}$ implies $\sigma(1) < \cdots < \sigma(p)$, $\sigma(p+1) < \cdots < \sigma(p+q)$. The shuffle product $\mu_{sh} : BA \otimes BA \rightarrow BA$ is given by

$$\mu_{sh}([a_1 | \cdots | a_p] \otimes [a_{p+1} | \cdots | a_{p+q}]) = \sum_{\sigma \in S_{p,q}} \varepsilon(\sigma) [a_{\sigma^{-1}(1)} | \cdots | a_{\sigma^{-1}(p+q)}],$$

where $\varepsilon(\sigma) = +1$ if the number of transpositions is even, and $\varepsilon(\sigma) = -1$ if the number of transpositions is odd. If DGA $(A, d, \mu, \eta, \varepsilon)$ is commutative, then the

differential $d_{BA} : BA \rightarrow BA$ on the Bar construction is a derivation of μ_{sh} in which case $(BA, \Delta, d_m, \mu_{sh})$ is a DGHA.

1.8 LIE ALGEBRA

In the following two sections we give the main definitions and facts about Lie algebras and Lie coalgebras. References include [1], [3], [8], [25], [35]. For simplicity we consider vector spaces over the field of characteristic 0.

Recall that a Lie algebra consists of a vector space L together with a linear map $[\cdot, \cdot] : L \otimes L \rightarrow L$ (called the bracket) such that

$$(1) \quad [x, x] = 0 \quad x \in L,$$

$$(2) \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad x, y, z \in L.$$

In this form, the definition of Lie algebra is not dualizable: what we must to do is "get rid of the elements". To do so, we write $\varphi : L \otimes L \rightarrow L$ in place of $[\cdot, \cdot] : L \otimes L \rightarrow L$ and

$$\xi : L \otimes L \otimes L \rightarrow L \otimes L \otimes L$$

for the linear map induced by the map

$$L \times L \times L \rightarrow L \times L \times L$$

which cyclically permutes the co-ordinates (i.e. $(x, y, z) \mapsto (y, z, x)$). In effect,

$$\xi = \xi_V : V \otimes V \otimes V \rightarrow V \otimes V \otimes V$$

may be defined for any vector space V as the composite

$$V \otimes V \otimes V \xrightarrow{\cong} V \otimes (V \otimes V) \xrightarrow{T_{V \otimes (V \otimes V)}} (V \otimes V) \otimes V \xrightarrow{\cong} V \otimes V \otimes V.$$

In this notation Jacobi identity becomes

$$\varphi(1_L \otimes \varphi)(1_{L \otimes L \otimes L} + \xi_L + \xi_L^2) = 0$$

and the (1) condition can be replaced by

$$\ker(1 - T) \subset \ker(\varphi).$$

DEFINITION 1. A graded Lie algebra over R is a pair (L, φ) , where L is a graded vector space and $\xi : L \otimes L \rightarrow L$ is a linear map subject to the conditions

$$(1) \quad \ker(1 - T) \subset \ker(\varphi)$$

$$(2) \quad \varphi(1 \otimes \varphi)(1 + \xi + \xi^2) = 0.$$

A graded Lie algebra L is called *t-reduced* if $L_n = 0$ for $n < t$; a 1-reduced Lie algebra is also called *connected*.

DEFINITION 2. A subspace V is called a Lie subalgebra of a Lie algebra L if $\varphi(V \otimes V) \subset V$.

We give here two familiar examples of a Lie algebra which we will use in our further definitions:

Example (1): Let (A, μ) be a connected associative algebra, then $(IA, I\mu - I\mu T)$ is a Lie algebra denoted by $l(A)$. Recall that by IA we denoted an augmentation ideal of algebra A , Subsection 1.2.

Example (2): Let (H, μ, Δ) be a connected Hopf algebra with associative multiplication. Then $P(H)$, the space of primitives of H , carries a structure of a Lie algebra. Recall that $P(H)$ is the space

$$\{x \in H \mid \Delta x = 1 \otimes x + x \otimes 1\}.$$

Since the diagonal Δ of (H, μ, Δ) is an algebra map, we easily find that

$$\Delta[x, y] = 1 \otimes [x, y] + [x, y] \otimes 1$$

where $x, y \in PH$ and $[x, y] = \mu(x \otimes y) - \mu T(x \otimes y)$. As we can see PH is a Lie subalgebra of $l(H)$.

DEFINITION 3. A homomorphism $f : (L_1, \varphi_1) \rightarrow (L_2, \varphi_2)$ is called a Lie algebra homomorphism if $f\varphi_1 = \varphi_2(f \otimes f)$.

The category of Lie algebras and Lie algebra homomorphisms is denoted by LA .

Let V be a vector space with $V_0 = 0$, and (TV, μ, Δ_{sh}) be a connected Hopf algebra over V .

DEFINITION 4. The free Lie algebra over vector space V is defined as the space of primitives $P(TV, \mu, \Delta_{sh})$.

The free Lie algebra over vector space V is denoted by $\mathcal{L}(V)$.

From Subsection 1.6 we know that any graded algebra map $f : T(V) \rightarrow A$, where A is a connected graded algebra, is uniquely defined by its restriction on generators $\alpha = f|_V : V \rightarrow IA$ and $f = f_\alpha$. Let (A, m, Δ) be any connected cocommutative graded Hopf algebra and

$$f_\alpha : (TV, \mu) \rightarrow (A, m)$$

any graded algebra homomorphism. Then $f_\alpha : (T(V), \mu, \Delta_{sh}) \rightarrow (A, m, \Delta)$ is a graded Hopf algebra homomorphism if and only if $Im(\alpha) \subseteq PA$, i.e. $\alpha(v)$ is a primitive for any $v \in V$.

We also know that any derivation $D : TV \rightarrow TV$ is uniquely defined by its restrictions on generators $\beta = D|_V : V \rightarrow I(TV)$ and $D = D_\beta$. Let

$$D_\beta : (TV, \mu) \rightarrow (TV, \mu)$$

be any derivation. Then D_β is also coderivation with respect to Δ_{sh} (i.e. D_β is a Hopf derivation) if and only if $Im(\beta) \subseteq PT(V) = \mathcal{L}(V)$, i.e. $\beta(v)$ is a primitive for any $v \in V$.

The free Lie algebra $\mathcal{L}(V)$ has the analogous universal property as the tensor algebra $T(V)$: if L is any connected graded Lie algebra and $\alpha : V \rightarrow L$ is any homomorphism, then there is a unique Lie algebra map $f_\alpha : \mathcal{L}(V) \rightarrow L$ which makes

the following diagram commute:

$$\begin{array}{ccc} V & \xrightarrow{i} & \mathcal{L}(V) \\ \alpha \searrow & & \downarrow f_\alpha \\ & & L. \end{array}$$

DEFINITION 5. *The universal enveloping algebra UL of a Lie algebra (L, φ) is an object in DGA together with a morphism*

$$i_{UL} : L \rightarrow l(UL)$$

in LA, such that for any $A \in DGA$ and any morphism $f : L \rightarrow l(A)$ in LA, there exists a unique morphism $F : UL \rightarrow A$ of DGA making the diagram

$$\begin{array}{ccc} L & \xrightarrow{i_{UL}} & l(UL) \\ f \searrow & & \downarrow l(F) \\ & & l(A) \end{array}$$

commutative.

Let's look at the construction of UL ; let (TL, μ) be a tensor algebra and I be a smallest ideal of TL containing $Im\{\mu(1-T)(i_{TL} \otimes i_{TL}) - i_{TL}\varphi\}$. Then $UL = \frac{TL}{I}$ and $i_{UL} = Pi_{TL}$, where $i_{TL} : L \rightarrow TL$ and $P : TL \rightarrow UL = \frac{TL}{I}$ are the natural inclusion and the natural projection correspondingly.

In fact, UL inherits from (TL, μ, Δ_{sh}) a coalgebra structure too. We mention once and for all that Δ_{UL} makes UL into cocommutative coassociative Hopf algebra; moreover, there is a natural Hopf algebra isomorphism

$$UL(V) = TV$$

(see [36]).

A *differential graded Lie algebra* is a connected graded Lie algebra (L, φ) together with a derivation δ , i.e. $\delta\varphi = \varphi(\delta \otimes 1 - 1 \otimes \delta)$, of degree -1 whose square is 0.

A differential graded Lie algebra homomorphism

$$f : (L_1, \varphi_1, \delta_1) \rightarrow (L_2, \varphi_2, \delta_2)$$

is a graded Lie algebra homomorphism $f : (L_1, \varphi_1) \rightarrow (L_2, \varphi_2)$ with $f\delta = \delta f$.

The free Lie algebra $\mathcal{L}(V)$ has the analogous universal property with respect to derivations: for any homomorphism $\beta : V \rightarrow \mathcal{L}(V)$ there is a unique derivation $D_\beta : \mathcal{L}(V) \rightarrow \mathcal{L}(V)$ which makes the following diagram commute:

$$\begin{array}{ccc} V & \xrightarrow{i} & \mathcal{L}(V) \\ \beta \searrow & & \downarrow D_\beta \\ & & \mathcal{L}(V). \end{array}$$

The category of DG -Lie algebras and their homomorphisms is denoted by DLA .

In the category of DG -Lie algebras there exists coproducts [8]. The coproduct of L and M is denoted by $L \coprod M$. If $\mathcal{L}V$ is a free Lie algebra over vector space V , then $\mathcal{L}(V) \coprod \mathcal{L}(V) = \mathcal{L}(V \oplus V)$.

Let (L, d, φ) be a DG -Lie algebra. A DG -Lie algebra (L^*, d^*, φ^*) together with homomorphisms

$$(L, d, \varphi) \xrightarrow[j_2]{j_1} (L^*, d^*, \varphi^*) \xrightarrow{p} (L, d, \varphi)$$

is called a *cylinder* on (L, d, φ) if

- (i) $j_1 \coprod j_2 : (L, d, \varphi) \coprod (L, d, \varphi) \rightarrow (L^*, d^*, \varphi^*)$ is injective;
- (ii) $pj_1 = pj_2 = 1$; and
- (iii) p is a quasi-isomorphism.

Two DGA homomorphisms $f, g : (L, d_L, \varphi_L) \rightarrow (M, d_M, \varphi_M)$ are called homotopic, $f \sim g$, if $f \coprod g : (L, d_L, \varphi_L) \coprod (L, d_L, \varphi_L) \rightarrow (M, d_M, \varphi_M)$ extends over some cylinder (L^*, d^*, φ^*) , see [3].

1.9 LIE COALGEBRA

This subsection is completely dual to the previous one; for more details see [1], [25], [35].

DEFINITION 6. A Lie coalgebra over R is a pair (M, ψ) , where M is a vector space and $\psi : M \rightarrow M \otimes M$ is a linear map subject to the conditions

$$(1) \quad \text{Im} \psi \subset \text{Im}(1 - T)$$

$$(2) \quad (1 + \xi + \xi^2)(1 \otimes \psi)\psi = 0$$

A graded Lie coalgebra M is called t -reduced if $M_n = 0$ for $n < t$; a 1-reduced Lie coalgebra is also called *connected*.

DEFINITION 7. A subspace W is called a Lie subcoalgebra of a Lie coalgebra M if $\psi(W) \subset W \otimes W$.

We give here two familiar examples of a Lie coalgebra which we shall use in further definitions:

Example (1): Let (C, Δ) be a connected coassociative coalgebra, then $(JC, J\psi - TJ\psi)$ is a Lie coalgebra denoted by $l^c(C)$. Recall that by JC we denoted a coaugmentation ideal of coalgebra C , Subsection 1.3.

Example (2): Let (H, μ, Δ) be a connected Hopf algebra with coassociative diagonal. The $Q(H)$, the space of indecomposables of H , carries a structure of a Lie coalgebra. Recall that $Q(H)$ can be considered to be a quotient space

$$Q(IH) = \frac{IH}{\text{Im}(I\mu)} = \frac{IH}{IH \cdot IH}.$$

Since the multiplication μ of (H, μ, Δ) is a coalgebra map, we easily check that $(QH, I\Delta - TI\Delta)$ is a Lie coalgebra.

DEFINITION 8. A homomorphism $f : (M_1, \psi_1) \rightarrow (M_2, \psi_2)$ is called a Lie coalgebra homomorphism if $\psi_2 f = (f \otimes f)\psi_1$.

The category of Lie coalgebras and Lie coalgebra homomorphisms is denoted by LC .

Let V be a vector space with $V_0 = 0$, and (T^cV, Δ, μ_{sh}) be a connected Hopf algebra over V .

DEFINITION 9. *The free Lie coalgebra over vector space V is defined as a quotient space $Q(TV, \Delta, \mu_{sh})$.*

The free Lie coalgebra over vector space V is denoted by $\mathcal{L}^c(V)$.

From Subsection 1.7 we know that any graded coalgebra map $f : C \rightarrow T^c(V)$, where C is a connected graded coalgebra, is uniquely defined by the projection

$$\alpha = P_1 f J : JC \rightarrow C \rightarrow T^c(V) \rightarrow V$$

and $f = f_\alpha$. Let (C, ψ, μ) be any connected commutative graded Hopf algebra and

$$f_\alpha : (C, \psi) \rightarrow (T^c(V), \Delta)$$

any graded coalgebra homomorphism. Then $f_\alpha : (C, \psi, \mu) \rightarrow (T^c(V), \Delta, \mu_{sh})$ is a graded Hopf algebra homomorphism if and only if there exists a homomorphism $\alpha' : QC \rightarrow V$ such that $\alpha = \alpha' q$, i.e. the following diagram commutes

$$\begin{array}{ccc} C & \xrightarrow{f_\alpha} & T^c(V) \\ q \downarrow & & \downarrow P_1 \\ QC & \xrightarrow{\alpha'} & V \end{array}$$

where $q : C \rightarrow QC = \frac{C}{C \cdot C}$ is a quotient map. Note that commutativity of the diagram implies that α vanishes on $C \cdot C$.

We also know that any coderivation $D : T^c(V) \rightarrow T^c(V)$ is uniquely defined by the projection

$$\beta = P_1 D J : JT^c(V) \rightarrow T^c(V) \rightarrow T^c(V) \rightarrow V$$

and $D = D_\beta$. Let

$$D_\beta : (T^c(V), \Delta) \rightarrow (T^c(V), \Delta)$$

be any coderivation. Then D_β is also derivation with respect to μ_{sh} (i.e. D_β is a Hopf derivation) if and only if there exists a homomorphism $\beta' : QT^c(V) = \mathcal{L}^c(V) \rightarrow V$ such that $\beta = \beta'q$, i.e. the following diagram commutes

$$\begin{array}{ccc} T^c(V) & \xrightarrow{D_\beta} & T^c(V) \\ q \downarrow & & \downarrow P_1 \\ QT^c(V) & \xrightarrow{\beta'} & V \end{array} .$$

Note that commutativity of the diagram implies that β vanishes on the shuffles.

The free Lie coalgebra $\mathcal{L}^c(V)$ has the analogous universal property as tensor coalgebra $T^c(V)$: if M is any connected graded Lie coalgebra and $\alpha : M \rightarrow V$ is any homomorphism, there is a unique Lie coalgebra map $f_\alpha : M \rightarrow \mathcal{L}^c(V)$ which makes the following diagram commute:

$$\begin{array}{ccc} V & \xleftarrow{P} & \mathcal{L}^c(V) \\ \alpha \nearrow & & \uparrow f_\alpha \\ & & M. \end{array}$$

DEFINITION 10. *The universal coenveloping coalgebra U^cM of a Lie coalgebra (M, ψ) is an object in DGC together with a morphism*

$$\pi_{U^cM} : l^c(U^cM) \rightarrow M$$

in LC, such that for any $C \in DGC$ and any morphism $f : l^c(C) \rightarrow M$ in LC, there exists a unique morphism $F : C \rightarrow U^cM$ of DGC making the diagram

$$\begin{array}{ccc} M & \xleftarrow{\pi_{U^cM}} & l^c(U^cM) \\ f \nearrow & & \uparrow l^c(F) \\ & & l^c(C) \end{array}$$

commutative.

Let us look at the construction of U^cM ; Let (T^cM, Δ) be a tensor coalgebra. The universal coenveloping coalgebra U^cM is the largest subcoalgebra of (T^cM, Δ)

contained in

$$\ker\{(\pi_{T^c M} \otimes \pi_{T^c M})(1 - T)\Delta - \psi\pi_{T^c M}\},$$

where $\pi_{T^c M} : T^c M \rightarrow M$ is the natural projection.

In fact, $U^c M$ inherits from $(T^c M, \Delta, \mu_{sh})$ an algebra structure too. We mention once and for all that $\mu_{U^c M}$ makes $U^c M$ into a commutative associative Hopf algebra; moreover, there is a natural Hopf algebra isomorphism

$$U^c \mathcal{L}^c(M) = T^c M.$$

A *differential graded Lie coalgebra* is a connected graded Lie coalgebra (M, ψ) together with a coderivation δ , i.e. $\psi\delta = (\delta \otimes 1 - 1 \otimes \delta)\psi$, of degree 1 whose square is 0.

A *differential graded Lie coalgebra homomorphism*

$$f : (M_1, \psi_1, \delta_1) \rightarrow (M_2, \psi_2, \delta_2)$$

is a graded Lie coalgebra homomorphism $f : (M_1, \psi_1) \rightarrow (M_2, \psi_2)$ with $f\delta = \delta f$.

The free Lie coalgebra $\mathcal{L}^c(V)$ has the analogous universal property with respect to coderivations: for any homomorphism $\beta : \mathcal{L}^c(V) \rightarrow V$ there is a unique coderivation $D_\beta : \mathcal{L}^c(V) \rightarrow \mathcal{L}^c(V)$ which makes the following diagram commute:

$$\begin{array}{ccc} V & \xleftarrow{P} & \mathcal{L}^c(V) \\ & \beta \nearrow & \uparrow D_\beta \\ & & \mathcal{L}^c(V). \end{array}$$

The category of *DG-Lie coalgebras* and their homomorphisms is denoted by *DLC*.

In the category of *DG-Lie coalgebras* there exist products. The product of M and N is denoted by $M \amalg N$. The universal properties imply that $\mathcal{L}^c(V) \amalg \mathcal{L}^c(V) = \mathcal{L}^c(V \oplus V)$.

Let (N, d, ψ) be a DG -Lie coalgebra. A DG -Lie coalgebra (N^*, d^*, ψ^*) together with homomorphisms

$$(N, d, \psi) \xrightarrow{i} (N^*, d^*, \psi^*) \xrightleftharpoons[p_2]{p_1} (N, d, \psi)$$

is called a *cylinder* on (N, d, ψ) if

- (i) $p_1 \amalg p_2 : (N^*, d^*, \psi^*) \rightarrow (N, d, \psi) \amalg (N, d, \psi)$ is surjective;
- (ii) $p_1 i = p_2 i = 1$; and
- (iii) i is a quasi-isomorphism.

Two DG -Lie coalgebra homomorphisms $f, g : (M, d_M, \psi_M) \rightarrow (N, d_N, \psi_N)$ are called homotopic, $f \sim g$, if $f \amalg g : (M, d_M, \psi_M) \rightarrow (N, d_N, \psi_N) \amalg (N, d_N, \psi_N)$ factors through some cylinder (N^*, d^*, ψ^*) .

1.10 A_∞ -ALGEBRA

We shall use the following notion of A_∞ -algebra (strongly homotopy associative) introduced by Stasheff in [41].

DEFINITION 11. An A_∞ -algebra $(A, \{m_i\})$ is defined as an R -Module A together with a set operations $\{m_i : A^{\otimes i} \rightarrow A \mid \deg(m_i) = 2 - i\}_{i \geq 1}$ satisfying the following relations:

$$\sum_{k=0}^{n-1} \sum_{j=1}^{n-k} (-1)^k m_{n-j+1}(\hat{a}_1, \dots, \hat{a}_k, m_j(a_{k+1}, \dots, a_{k+j}), \dots, a_n) = 0$$

where $\hat{a}_i = (-1)^{|a_i|} a_i$.

There exists an alternative but equivalent set of signs, see [23], [31], etc.

An A_∞ -algebra $(A, \{m_i\})$ is called *minimal* if $m_1 = 0$ [24], [25].

The DGA (A, d, μ) can be considered as an A_∞ -algebra of the following type $(A, \{m_1, m_2, 0, 0, \dots\})$, where $m_1 = d$, and $m_2(a \otimes b) = (-1)^{|a|} \mu(a \otimes b)$.

The Bar construction on an A_∞ -algebra $(A, \{m_i\})$ is the DGC $BA = T^c(sIA)$ with differential

$$(2) \quad d_m[a_1 | \cdots | a_n] =$$

$$\sum_{i=0}^{n-k} \sum_{k=1}^n (-1)^{k+\epsilon_i(k)} [a_1 | \cdots | a_i | m_k(a_{i+1} \cdots a_{i+k}) | a_{i+k+1} | \cdots | a_n],$$

where

$$\epsilon_i(k) = \sum_{t=1}^i (|a_t| + 1) + \sum_{j=1}^{k-1} (|a_{i+j}| + 1)(k - j).$$

Conversely, any differential d on the tensor coalgebra $(T^c(sIA), \Delta)$ on a graded module sIM uniquely defines an A_∞ -algebra structure on M . The operations m_j are given by the composition

$$m_j : M^{\otimes j} \xrightarrow{s^{\otimes j}} (sM)^{\otimes j} \xrightarrow{i} T^c(sIA) \xrightarrow{d} T^c(sIA) \xrightarrow{P_1} sM \xrightarrow{s^{-1}} M$$

The condition $d^2 = 0$ guarantees that such defined $\{m_i\}$ satisfy (2).

DEFINITION 12. Given A_∞ -algebras $(M, \{m_i\})$ and $(N, \{m'_j\})$, an A_∞ -algebra morphism $(M, \{m_i\}) \rightarrow (N, \{m'_j\})$ is a collection of maps

$$\{f_i : M^{\otimes i} \rightarrow N, \deg(f_i) = i - 1\}_{i \geq 1}$$

satisfying the following identities

$$\begin{aligned} & \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} (-1)^k f_{n-j+1}(\hat{a}_1, \cdots, \hat{a}_k, m_j(a_{k+1}, \cdots, a_{k+j}), \cdots, a_n) = \\ & = \sum_{t=1}^n \sum_{k_1 + \cdots + k_t = n} m'_t(f_{k_1}(a_1, \cdots, a_{k_1}), \cdots, f_{k_t}(a_{n-k_t+1}, \cdots, a_n)). \end{aligned}$$

It is clear from the definition that the first component $f_1 : (M, m_1) \rightarrow (N, m'_1)$ is a chain map.

An A_∞ -algebra morphism $\{f_i\} : (M, \{m_i\}) \rightarrow (N, \{m'_i\})$ is called an *isomorphism* (resp. *quasi-isomorphism*) if the first component $f_1 : (M, m_1) \rightarrow (N, m'_1)$ is an *isomorphism* (resp. *quasi-isomorphism*).

DEFINITION 13. A morphism of A_∞ -algebras $\{f_i\} : (M, \{m_i\}) \rightarrow (N, \{\bar{m}_j\})$ induces a DGC map of Bar constructions $B\{f_i\} : BM \rightarrow BN$ given by

$$B\{f_i\} = \sum_{i_1 + \dots + i_k = n} \sum_{k=1}^n [f_{i_1}(s^{-1})^{\otimes i_1} | \dots | f_{i_k}(s^{-1})^{\otimes i_k}].$$

where s^{-1} is a suspension. It is easy to prove that $B\{f_i\}$ is an isomorphism (resp. quasi-isomorphism) if and only if $f_1 : M \rightarrow N$ is an isomorphism (resp. quasi-isomorphism) (see [25], for example).

Conversely, any DG-coalgebra map $f : (BM, d_M, \Delta) \rightarrow (BN, d_N, \Delta)$ uniquely defines A_∞ -algebra morphisms. The operations f_j are given by the composition

$$P_1 f = \{f_j : M^{\otimes j} \xrightarrow{s^{\otimes j}} (sM)^{\otimes j} \xrightarrow{i} BM \xrightarrow{f} BN \xrightarrow{P_1} sN \xrightarrow{s^{-1}} N\}.$$

A_∞ -algebras and A_∞ -algebra morphisms form the A_∞ -algebra category, which we denote by $A_\infty\text{-Alg}$.

There is one more category to consider- category DASH (see [37], [42]); it can be considered as a subcategory of $A_\infty\text{-Alg}$. Objects in DASH are A_∞ -algebras of type $(A, \{m_1, m_2, 0, 0, \dots\})$, i.e. DGA, and morphisms-ordinary morphisms in $A_\infty\text{-Alg}$.

DEFINITION 14. An A_∞ -algebra $(M, \{m_i\})$ is called commutative if

$$m_n s_{i, n-i} = 0$$

for $1 \leq i \leq n-1$, $n \geq 2$, and $s_{i, n-i} \in S_{i, n-i}$; see [24], [25].

Some authors also call such A_∞ -algebra a balanced A_∞ -algebra [33].

The Bar-construction of a commutative A_∞ -algebra $(A, \{m_i\})$ is a DG-Hopf algebra with respect to shuffle products, i.e. $(BA, d_A, \Delta, \mu_{sh}) \in HA$. And conversely, any differential on the DG-Hopf algebra $(BA, d_A, \Delta, \mu_{sh})$ produces commutative A_∞ -algebra on A .

DEFINITION 15. Given two commutative A_∞ -algebras $(M, \{m_i\})$ and $(N, \{\bar{m}_j\})$. An A_∞ -algebra morphism $\{f_i\} : (M, \{m_i\}) \rightarrow (N, \{\bar{m}_j\})$ is called a commutative

A_∞ -algebra morphism if

$$f_n s_{i,n-i} = 0$$

for $1 \leq i \leq n-1$, $n \geq 2$, and $s_{i,n-i} \in S_{i,n-i}$; see [24], [25].

As we have mentioned above, any A_∞ -algebra morphism

$$\{f_i\} : (M, \{m_i\}) \rightarrow (N, \{\bar{m}_j\})$$

induces a DG-coalgebra map of appropriate bar constructions $B\{f\} : (BM, d_M, \Delta) \rightarrow (BN, d_N, \Delta)$. If $\{f_i\} : (M, \{m_i\}) \rightarrow (N, \{\bar{m}_j\})$ is a commutative A_∞ -algebra morphism then

$$B\{f\} : (BM, d_M, \Delta, \mu_{sh}) \rightarrow (BN, d_N, \Delta, \mu_{sh})$$

is a DG-Hopf algebra map. Conversely, any DG-Hopf algebra map

$$f : (BM, d_M, \Delta, \mu_{sh}) \rightarrow (BN, d_N, \Delta, \mu_{sh})$$

produces a commutative A_∞ -algebra morphism

$$f_i : M^{\otimes i} \rightarrow (sM)^{\otimes i} \rightarrow BM \rightarrow BN \rightarrow sN \rightarrow N.$$

Commutative A_∞ -algebras and their morphisms form a commutative A_∞ -algebra category, which is denoted by $C-A_\infty\text{-Alg}$.

1.11 TWISTING COCHAIN

In this section we recall briefly the definition of twisting cochains of Brown [11] and some notions and known facts related with them.

DEFINITION 16. Let $(C, d_C, \Delta, \varepsilon_C, \eta_C)$ be a DGC and $(A, d_A, \mu, \varepsilon_A, \eta_A)$ a DGA. The complex $\text{Hom}(C, A)$ is defined as a DGA with differential

$$Df = d \cdot f - (-1)^{|f|} f \cdot d_C$$

and multiplication

$$f \cdot g = \mu(f \otimes g)\Delta, \quad f, g \in \text{Hom}(C, A).$$

The neutral element in $\text{Hom}(C, A)$ is a composition $e : C \xrightarrow{\varepsilon_C} R \xrightarrow{\eta_A} A$.

It is easy to check that

$$D(f \cdot g) = Df \cdot g + (-1)^{|f|} f \cdot Dg.$$

From here throughout the paper we shall work only with augmentation and coaugmentation ideals of DGA and DGC. Thus we consider the complex $\text{Hom}(JC, IA)$.

DEFINITION 17. The element $t \in \text{Hom}(JC, IA)$ of degree -1 is a twisting cochain in the sense of Brown [11] if it satisfies

$$Dt = -t \cdot t.$$

It is easy to show that for any DGA (A, d_A, μ) and for any DGC (C, d_C, Δ) the natural maps

$$P_1 : BA \rightarrow A \quad \text{and} \quad i_1 : C \rightarrow \Omega C$$

are twisting cochains, see [37].

Let $\alpha : JC \rightarrow IA$ be a map of degree -1. By the universal properties mentioned in Subsections 1.6 and 1.7, any such homomorphism uniquely defines an algebra map $f_\alpha : \Omega C \rightarrow A$ and a coalgebra map $g_\alpha : C \rightarrow BA$. Homomorphisms $f_\alpha : \Omega C \rightarrow A$ and $g_\alpha : C \rightarrow BA$ are chain maps, i.e. maps in DGA and DGC respectively, if and only if $\alpha : JC \rightarrow IA$ is a twisting cochain; see [25], for example. Thus there are bijections

$$(1.11.1) \quad \text{Hom}_{DGC}(C, BA) \leftrightarrow T(JC, IA) \leftrightarrow \text{Hom}_{DGA}(\Omega C, A),$$

where by $T(JC, IA)$ we denote the set of all twisting cochains from JC to IA .

Berikashvili in [9] introduced the following equivalence relation on $T(JC, IA)$.

DEFINITION 18. *Two twisting cochains $\alpha, \alpha' : JC \rightarrow IA$ are equivalent (we write $\alpha \sim \alpha'$) if there exists a homomorphism*

$$\beta : JC \rightarrow IA$$

of degree 0 satisfying the following identity

$$\alpha - \alpha' = \beta d_C - d_A \beta + \beta \cdot \alpha' - \alpha \cdot \beta.$$

The set $T(C, A)$ factored by this equivalence relation is denoted by $D(C, A)$.

Let $f : A \rightarrow B$ be a map in DGA . It is trivial to check that for any $\alpha \in T(JC, IA)$ the composition $If\alpha$, where $If : IA \rightarrow IB$, is a twisting cochain in $T(JC, IB)$. Moreover, if $\alpha \sim \alpha'$, then $If\alpha \sim If\alpha'$. Thus for any DGC C , a DGA map $f : A \rightarrow B$ induces a set map

$$f_* : D(JC, IA) \rightarrow D(JC, IB)$$

defined by $[\alpha] \mapsto [If\alpha]$. Moreover Berikashvili in [10] proved the following

THEOREM 19. *Let $f : A \rightarrow B$ be a map in DGA inducing an isomorphism in homology. Then*

$$f_* : D(JC, IA) \rightarrow D(JC, IB)$$

is a bijection.

Let $f = \{f_i\} : A \rightarrow B$ be a DASH map. Kadeishvili in [26] showed that if $\alpha \in T(JC, IA)$, then

$$\phi = \sum_i Jf_i(\alpha \otimes \cdots \otimes \alpha) \Delta^i$$

belongs to $T(JC, IB)$. Thus for any DGC C , a DASH map $f = \{f_i\} : A \rightarrow B$ induces a set map

$$T(f) : T(JC, IA) \rightarrow T(JC, IB)$$

($\alpha \mapsto \sum_i Jf_i(\alpha \otimes \cdots \otimes \alpha) J\Delta^i$), which by itself induces a set map

$$f_* : DT(JC, IA) \rightarrow D(JC, IB)$$

defined by $[\alpha] \mapsto [\sum_i Jf_i(\alpha \otimes \cdots \otimes \alpha)J\Delta^i]$. Kadeishvili in [26] generalized Theorem 19 in the following way

THEOREM 20. *Let $f = \{f_i\} : A \rightarrow B$ be a map in DASH, such that its first component $f_1 : A \rightarrow B$ induces an isomorphism in homology. Then*

$$f_* : D(JC, IA) \rightarrow DT(JC, IB)$$

is a bijection.

It is known that if $\alpha, \alpha' : JC \rightarrow IA$ are equivalent twisting cochains, then $f_\alpha, f_{\alpha'} : \Omega C \rightarrow A$ (resp. $g_\alpha, g_{\alpha'} : C \rightarrow BA$) are homotopic in DGA (resp. DGC); see [27], for example.

We complete this section with the following

THEOREM 21 ([30]). *Let K be a simplicial set.*

(a) Then in the DGA $A = \text{Hom}(C_(K), \Omega C_*(K) \otimes \Omega C_*(K))$ there exists the element E of degree -1, satisfying the following conditions*

I) $DE = -E \cdot E$ (i.e. E is a twisting cochain)

II) The components $E^{1,0} : C \rightarrow C \otimes R$ and $E^{0,1} : C \rightarrow R \otimes C$ are given by

$$E^{1,0}(x) = -x \otimes 1$$

$$E^{0,1}(x) = -1 \otimes x$$

III) $E^{n,0} = E^{0,n} = 0$ when $n \neq 1$.

(b) Let E and $\bar{E}$ be any twisting cochains in A satisfying II) and III); i.e. $E^{1,0} = \bar{E}^{1,0}$, $E^{0,1} = \bar{E}^{0,1}$, and $E^{n,0} = \bar{E}^{n,0} = 0$, $E^{0,n} = \bar{E}^{0,n} = 0$. Then there exists $P \in A$ of degree 0 such that $P^{0,0} = 0$ and

$$\bar{E} = E + P \cdot E - \bar{E} \cdot P - DP.$$

The notation in the theorem is not very obvious. We clarify it below. Let us denote by $p^{\otimes i} : \Omega C_*(K) \rightarrow C_*(K)^{\otimes i}$, $i = 0, 1, 2, \dots$, the natural projections on the Cobar-construction. Also notice that any homomorphism $F : C_*(K) \rightarrow \Omega C_*(K) \otimes \Omega C_*(K)$

uniquely defines the set of linear maps

$$\{F^{m,n} : C_*(K) \xrightarrow{F} \Omega C_*(K) \otimes \Omega C_*(K) \xrightarrow{p^{\otimes m} \otimes p^{\otimes n}} (C_*(K)^{\otimes m}) \otimes (C_*(K)^{\otimes n})\}$$

called the components of homomorphism F . Part (a) of the theorem says that on $\text{Hom}(C_*(K), \Omega C_*(K) \otimes \Omega C_*(K))$ we have a special *twisting cochain* E (with the components from *II* and *III*); part (b) asserts that this twisting cochain is defined uniquely up to twisting equivalence in sense of Definition 18.

1.12 PERTURBATION LEMMA

We start this section with the description of the category SDR . It is defined as follows: an object in SDR is a set

$$\tau = ((X, d_X) \xrightleftharpoons[\beta]{\alpha} ((Y, d_Y); \nu))$$

where (1) (X, d_X) and (Y, d_Y) are DG -vector spaces with increasing filtration

$$\subset F_p X \subset F_{p+1} X \subset \cdots, \quad \subset F_p Y \subset F_{p+1} Y \subset \cdots,$$

(2) β and α are morphisms of DG -vector spaces preserving filtration and satisfying

$$\beta\alpha = 1_X, \quad \alpha\beta \stackrel{\nu}{\sim} 1_Y, \text{ i.e. } 1_Y - \alpha\beta = d_Y\nu + \nu d_Y,$$

(3)

$$\nu\alpha = 0, \quad \beta\nu = 0, \quad \nu\nu = 0.$$

The morphism in SDR

$$(f, g) : \tau = ((X, d_X) \xrightleftharpoons[\beta]{\alpha} ((Y, d_Y); \nu)) \rightarrow \tau' = ((X', d'_X) \xrightleftharpoons[\beta']{\alpha'} ((Y', d'_Y); \nu'))$$

is defined as a pair of chain maps $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ preserving filtration and satisfying

$$g\alpha = \alpha'f, \quad f\beta = \beta'g, \quad g\nu = \nu'g.$$

Let (C, d) be a chain complex over the field. There is a familiar *SDR*, more usually known as *SDR* data, associated with this chain complex

$$\tau = ((H(C), d = 0) \xrightleftharpoons[g]{f} ((C, d); \nu).$$

Let us discuss this *SDR* more closely. We have 2 splitting short exact sequences

$$0 \rightarrow B_n \xrightarrow{j_n} Z_n \xrightleftharpoons[p_1]{cl} H_n \rightarrow 0$$

and

$$0 \rightarrow Z_n \xrightarrow{i_n} C_n \xrightleftharpoons[p_2]{d_n} B_{n-1} \rightarrow 0$$

where p_1 and p_2 are splitting maps (of course not unique), and cl is a homomorphism assigning to each n -cycle z its homology class $cl(z)$. Then we have two isomorphisms $Z_n \cong B_n \oplus H_n$ and $C_n \cong Z_n \oplus B_{n-1}$. They yield to an isomorphism

$$\gamma_n : C_n \cong B_n \oplus H_n \oplus B_{n-1}$$

given by $\gamma(c_n) = ((c_n - p_2 d_n c_n) - p_1 cl(c_n - p_2 d_n c_n), cl(c_n - p_2 d_n c_n), d_n c_n)$, [40]. Let us define a differential δ on $B_n \oplus H_n \oplus B_{n-1}$ by

$$\delta(b_n, h_n, b_{n-1}) = (b_{n-1}, 0, 0).$$

We see that δ is independent of the choice of the splittings maps, it depends only on d . Clearly, $\delta^2 = 0$ and $\ker(\delta_n)/\text{im}(\delta_{n+1}) = H_n$. It is straightforward to check that $\gamma = \{\gamma_n\}$ is a chain complex isomorphism, i.e. $\gamma_{n-1}\delta_n = d_n\gamma_n$. Once the splitting maps are fixed we identify the chain complexes (C_n, d_n) and $(B_n \oplus H_n \oplus B_{n-1}, \delta_n)$ through γ .

We define the maps $f : H_n \rightarrow C_n$, $g : C_n \rightarrow H_n$, and $\nu : C_n \rightarrow C_{n+1}$ by the following formulas

$$f(h_n) = (0, h_n, 0), \quad g(b_n, h_n, b_{n-1}) = h_n, \quad \nu(b_n, h_n, b_{n-1}) = (0, 0, -b_n).$$

It is immediate from the definitions that $gf = 1$, $fg - 1 = \nu\delta + \delta\nu$, $\nu f = 0$, $g\nu = 0$, $\nu\nu = 0$.

DEFINITION 22. Let (Y, d_Y) be a DG -vector space. A perturbation of the differential d_Y is a homomorphism $t : Y \rightarrow Y$ of degree 1 satisfying $d_Y t + t d_Y + t t = 0$ (this condition guarantees that $(d_Y + t)^2 = 0$, i.e. $d_Y + t$ is a new differential) and $t(F_p Y) \subset F_{p-1} Y$ (decreases filtration).

Let us denote by $SDRP$ the category of objects (τ, t) , where

$$\tau = ((X, d_X) \xrightleftharpoons[\beta]{\alpha} ((Y, d_Y); \nu))$$

is an object in SDR , and t is a perturbation on Y . A morphism in $SDRP$ $(\tau, t) \rightarrow (\tau', t')$ is a SDR morphism $(f, g) : (\tau \rightarrow \tau')$ satisfying $gt = t'g$. We are ready now to state the basic perturbation lemma [12]

LEMMA 23. Let

$$\tau = ((X, d_X) \xrightleftharpoons[\beta]{\alpha} ((Y, d_Y); \nu))$$

be an SDR and $t : Y \rightarrow Y$ any perturbation on Y . Then the formulas

$$t^\infty = \beta t \alpha - \beta t \nu t \alpha + \beta t \nu t \nu t \alpha - \dots$$

$$\alpha^\infty = \alpha - \nu t \alpha + \nu t \nu t \alpha - \dots$$

$$\beta^\infty = \beta - \beta t \nu + \beta t \nu t \nu - \dots$$

$$(3) \quad \nu^\infty = \nu - \nu t \nu + \nu t \nu t \nu - \dots$$

define new (perturbed) SDR

$$\tau^\infty = ((X, d_X + t^\infty) \xrightleftharpoons[\beta^\infty]{\alpha^\infty} ((Y, d_Y + t); \nu^\infty));$$

The construction of τ^∞ is natural; i.e. the correspondence $(\tau, t) \rightarrow \tau^\infty$ defines a functor $F : SDRP \rightarrow SDR$.

Note that we have stated the perturbation lemma above for a DG -vector space.

There is an analogous (co)algebra perturbation lemma for DG -(co)algebras over the field; in that case if τ is (co)algebraic, i.e.,

- (1) (X, d_X) and (Y, d_Y) are DG -(co)algebras;

(2) α and β are DG -(co)algebra maps;

(3) ν is a (co)derivation;

and $t : Y \rightarrow Y$ is (co)derivation, then the resulting τ^∞ is (co)algebraic too [22], [24], [25].

In [23] (see also [39]) was shown that in the homology of 1-connected DG -algebra (A, d, μ) there is an A_∞ -algebra structure. Later using the machinery of the coalgebra perturbation lemma in [25] (see also [21]) it was shown that in homology of the 1-connected commutative A_∞ -algebra $(A, \{m_i\})$ there is a commutative minimal A_∞ -algebra structure (over the rationals). We state the theorem below and give a sketch of the proof.

THEOREM 24. *Given a 1-connected commutative A_∞ -algebra $(A, \{m_i\})$. Then in homology $H(A, m_1 = d)$ there arises a minimal commutative A_∞ -algebra structure and a commutative A_∞ -algebra quasi-isomorphism*

$$\{f_i\} : (H(A), \{m_i^H\}) \rightarrow (A, \{m_i\}).$$

PROOF. Let $\tau = ((H(A), d = 0) \xrightleftharpoons[g]{f} ((A, d_A); \nu)$ be SDR data as discussed above.

Clearly, the given SDR induces a coalgebra SDR

$$B\tau = ((BH(A), d = 0) \xrightleftharpoons[Bg]{Bf} ((BA, Bd); B\nu).$$

(B is a Bar-construction functor).

Let consider the following coderivation (perturbation) on BA

$$t = d_m - Bd$$

where d_m is a differential of BA defined by an A_∞ -algebra structure on A (see definition of differential on Bar construction). Applying the (coalgebra) perturbation lemma to $(B\tau, t)$ we obtain new SDR

$$(B\tau)^\infty = ((BH(A), t^\infty) \xrightleftharpoons[(Bg)^\infty]{(Bf)^\infty} ((BA, Bd + t); (B\nu)^\infty)$$

and according to the formulas in the perturbation lemma $t^\infty = B(g)tB(f)^\infty$.

From Subsection 1.10 we know that any differential t^∞ on $BH(A)$ uniquely defines an A_∞ -algebra structure on $H(A)$. Moreover, since perturbation decreases filtration it was shown in [25] that this is a minimal A_∞ -algebra structure; i.e. the first component $m_1^H = 0$.

So, it remains to show that operations

$$m_j : H(A)^{\otimes j} \xrightarrow{s^{\otimes j}} (sH(A))^{\otimes j} \xrightarrow{i} BH(A) \xrightarrow{t^\infty} BH(A) \xrightarrow{P_1} sH(A) \xrightarrow{s^{-1}} H(A)$$

and

$$f_j : H(A)^{\otimes j} \xrightarrow{s^{\otimes j}} (sH(A))^{\otimes j} \xrightarrow{i} BH(A) \xrightarrow{(Bf)^\infty} BA \xrightarrow{P_1} sA \xrightarrow{s^{-1}} A$$

vanish on the shuffles. Let us assume that operations $f_{j \geq 2}$ vanish on the shuffles; this implies (Subsection 1.10)

$$B(f)^\infty : (BH(A), \mu_{sh}) \rightarrow (BA, \mu_{sh})$$

is a multiplicative map. Then

$$\begin{aligned} t^\infty(a * b) &= B(g)tB(f)^\infty(a * b) = B(g)t[B(f)^\infty(a) * B(f)^\infty(b)] = \\ &= B(g)[tB(f)^\infty(a) * B(f)^\infty(b) + (-1)^{|a|}B(f)^\infty(a) * tB(f)^\infty(b)] = \\ &= B(g)tB(f)^\infty(a) * B(g)B(f)^\infty(b) + (-1)^{|a|}B(g)B(f)^\infty(a) * B(g)tB(f)^\infty(b) = \\ &= t^\infty(a) * b + (-1)^{|a|}a * t^\infty(b). \end{aligned}$$

We have used here that $B(g)B(f)^\infty = 1$, which easily follows from the definition of $B(f)^\infty$ (see formulas in the perturbation lemma) and from the fact that $B(g)B(f) = 1$. Also we have used multiplicativity of

$$B(g) : (BH, \mu_{sh}) \rightarrow (BA, \mu_{sh})$$

which easily follows from the fact that $B(g)$ as a DG -coalgebra map is generated by the A_∞ -algebra map

$$\{g_1 = g, 0, 0, \dots\} : (H, \{m_i^H\}) \rightarrow (A, \{m_i\})$$

with a single non-trivial component $g = g_1 : H \rightarrow A$ and automatically $g_k s_{i,k-i} = 0$ for $k \geq 2$. This implies that $B(g)$ is multiplicative (Subsection 1.10).

Thus assuming $B(f)^\infty$ is multiplicative we obtain t^∞ is a derivation of the shuffle product; i.e. $(BH, t^\infty, \Delta, \mu_{sh}) \in HA$. This implies that t^∞ produces a commutative A_∞ -algebra structure on H (Subsection 1.10). So it remains to show that $B(f)^\infty$ is multiplicative, or equivalently operations

$$f_j : H(A)^{\otimes j} \xrightarrow{s^{\otimes j}} (sH(A))^{\otimes j} \xrightarrow{i} BH \xrightarrow{B(f)^\infty} BA \xrightarrow{P_1} sA \xrightarrow{s^{-1}} A$$

vanish on the shuffles; to check $f_k s_{i,k-i} = 0$ is straightforward, see [25], where it was shown also that this construction is natural and unique. $\square$

REMARK 25. *In the proof of Theorem 24 we constructed DG Hopf algebra map*

$$B(f)^\infty : (BH, t^\infty, \Delta, \mu_{sh}) \rightarrow (BA, d_m, \Delta, \mu_{sh})$$

and its homotopy inverse

$$B(g)^\infty : (BA, d_m, \Delta, \mu_{sh}) \rightarrow (BH, t^\infty, \Delta, \mu_{sh})$$

in DGC such that $B(g)^\infty$ is DG-coalgebra map with $B(g)^\infty B(f)^\infty = 1_{BH}$, and $B(f)^\infty B(g)^\infty \sim 1_{BA}$. It is immediate from Lemma 33 that $B(g)^\infty$ is multiplicative up to homotopy, i.e. there exists a coderivation homotopy from $B(g)^\infty \mu_{sh}$ to $\mu_{sh}(B(g)^\infty \otimes B(g)^\infty)$. In the following sections such maps will be called HAH homomorphisms.

1.13 HOPF ALGEBRAS UP TO HOMOTOPY

In this section we state several fundamental results of Anick from [3] and [4]. Let DGA_0 be a category of connected DG-algebras over R , a commutative ring with unity. Category DGA_0 has push-outs. The push-out of the diagram $B \xleftarrow{f} A \xrightarrow{g} C$ is denoted by $B \amalg_A C$. Most often, we consider push-outs over the trivial graded

$V' \cong V$. To define D , first put $f, f' : TV \rightarrow IA$ by $f(x) = x$ and $f'(x) = x'$. Let S be the unique (f, f') -derivation having $S(x) = sx$ for $x \in V$. Setting $D(x) = fd(x)$ and $D(x') = f'd(x)$ and $D(sx) = x - x' - Sd(x)$, extend D (uniquely) to a $(1, 1)$ derivation of degree -1 on IA . Then $Sd + DS$ is an (f, f') -derivation, $f - f' = Sd + DS$. $D^2 = 0$ and we have constructed the DG-algebra (IA, D) .

PROPOSITION 28. *Let $f, g : (A, d_A, \mu_A) \rightarrow (B, d_B, \mu_B)$ with A free. The following are equivalent*

- (i) $f \sim g$;
- (ii) $f \amalg g : (A, d_A) \amalg (A, d_A) \rightarrow (B, d_B)$ extends over $I(A, d)$
- (iii) there is a derivation homotopy from f to g .

We have a similar proposition for DG-Lie algebras, however, there is no analog for derivation homotopy. The canonical cylinder, due to Tanre [44], is obtained through an entirely new method.

The *Tanre cylinder*, defined for free DG-Lie algebras over Q , is constructed as follows. Given $(\mathcal{L}V, \delta)$, put $I(\mathcal{L}V, \delta) = (I\mathcal{L}V, D)$, where $I\mathcal{L}V = \mathcal{L}(V \oplus sV \oplus DsV)$ and $D|_{\mathcal{L}V} = \delta$. The obvious inclusion $j_1 : \mathcal{L}V \rightarrow I\mathcal{L}V$ is one "end" of the cylinder, and $p : I\mathcal{L}V \rightarrow \mathcal{L}V$ is given on generators by $p(x) = x$, $p(sx) = 0$, and $p(Dsx) = 0$.

The inclusion j_2 , the other "end" of the cylinder is defined as follows; given a pointwise nilpotent derivation θ of degree zero on a graded Lie algebra L , observe that

$$e^\theta = 1 + \sum_{n=1}^{\infty} \frac{\theta^n}{n!}$$

defines a Lie algebra automorphism on L . Define a derivation S of degree +1 on $I\mathcal{L}V$ by defining it on generators as $S(x) = x$, $S(sx) = 0$, and $S(Dsx) = 0$. Then $\alpha = [D, S] = DS + SD$ is a derivation on $I\mathcal{L}V$ of degree zero. Put

$$j_2 = e^\alpha j_1.$$

One sees easily that $j_1 \amalg j_2 : (\mathcal{L}V, \delta) \amalg (\mathcal{L}V, \delta) \rightarrow (I\mathcal{L}V, D)$ injects and that $I\mathcal{L}V \approx \mathcal{L}V \amalg \mathcal{L}V \amalg \mathcal{L}(sV)$ as Lie algebras.

Analogous to Proposition 28 is

LEMMA 29. *Let $f, g : (L, \delta) \rightarrow (L', \delta')$ be two DG-Lie algebra homomorphisms. Then $f \sim g$ if and only if $f \amalg g$ factors (in DLA) through the inclusion $j_1 \amalg j_2$ of L into its Tanre cylinder.*

We state here the Lemaire-Aubry theorem, for proof see [5].

PROPOSITION 30. *Let $f, g : (L, \delta) \rightarrow (L', \delta')$ be two DG-Lie algebra homomorphisms. Then $f \sim g$ if and only if $Uf \sim Ug$ in DGA.*

In any category with a concept of homotopy satisfying Lemma 27, the morphism $f : A \rightarrow B$ is called a *homotopy equivalence* if there exists $g : B \rightarrow A$ for which $gf \sim 1_A$ and $fg \sim 1_B$. Two objects have the same homotopy type if there is a homotopy equivalence between them.

Let $\varepsilon : A \rightarrow R$ be an augmentation for a graded algebra (A, d, μ) and put $\pi_1 = (1 \otimes \varepsilon) : A \otimes A \rightarrow A$, $\pi_2 = (\varepsilon \otimes 1) : A \otimes A \rightarrow A$.

DEFINITION 31. *Let (A, d, μ) be a DG-algebra and $\psi : (A, d, \mu) \rightarrow (A, d, \mu) \otimes (A, d, \mu)$ be a DG-algebra homomorphism. The quartet (A, d, μ, ψ) is called a Hopf algebra up to homotopy (henceforth Hah) over R if there exist four homotopies in DGA*

- (i) *between $\pi_1\psi$ and 1 ;*
- (ii) *between $\pi_2\psi$ and 1 ;*
- (iii) *between $(\psi \otimes 1)\psi$ and $(1 \otimes \psi)\psi$.*
- (iv) *between $T\psi$ and ψ ;*

A homomorphism of Hah's is a DG-algebra homomorphism $f : (A, d, \mu) \rightarrow (A', d', \mu')$ such that $(f \otimes f)\psi \sim \psi'f$. The category of Hah's over R and their homomorphisms is denoted by $HAH(R)$, or simply HAH .

A Hah (A, d, μ, ψ) is called free, r -reduced, etc. if (A, d, μ) is so in DGA. A homomorphism in HAH is called surjection, quasi-isomorphism, etc. if it is so in DGA.

Let us denote by HAH_1 full subcategory of HAH consisting of free 1-reduced objects, i.e. such A 's which are isomorphic to TV for some V with $V_0 = 0$.

The cylinder and homotopy in HAH is defined by essentially copying the definitions for these concepts in DGA . Let $(A, d, \mu, \psi) \in HAH$. Given any DGA cylinder (A^*, d^*, μ^*) on the underlying DG-algebra (A, d, μ) , put $\psi^* = (j_1 \otimes j_2)\psi p$. Then j_1 , j_2 , and p are automatically HAH homomorphisms. The following two lemmas are applications of this definitions.

LEMMA 32. *Let $f : (A, d, \psi, \mu) \rightarrow (A', d', \psi', \mu')$ be an HAH homomorphism and let $g : (A, d, \mu) \rightarrow (A', d', \mu')$ be a DGA Homomorphism. The following are equivalent:*

- (i) $f \sim g$ in DGA ;
- (ii) g is a HAH homomorphism and $f \sim g$ in HAH .

LEMMA 33. *Let $f : (A, d, \psi, \mu) \rightarrow (A', d', \psi', \mu')$ be an HAH homomorphism and suppose $g : (A', d', \mu') \rightarrow (A, d, \mu)$ is a homotopy inverse in DGA . Then $g \in HAH$, and f and g are homotopy inverses in HAH .*

We are ready now to state some theorems from [3]. We state the results over the rationals, although they were proved for any $R \not\supseteq Q$ under certain restrictions on r -mildness of given DG -objects.

PROPOSITION 34. *Let (A, d) be a free DG -algebra, $A = TV$ with $V_0 = 0$. Let ψ be a coproduct on A which is a DG -algebra homomorphism. Suppose ψ is a both cocommutative and coassociative. Then there is a graded Q -module W isomorphic to the graded Q -module V such that $(A, d) = U(\mathcal{L}W, d)$ and $\psi = \Delta_{\mathcal{L}W}$, where $(\mathcal{L}W, d)$ is a free Lie algebra.*

Let introduce some notations which we need in the definitions below.

Let $\mathcal{L}V$ be a free Lie algebra over V . Let $\sigma \in S_k$ and let $\underline{x} = (x_1, \dots, x_k)$ be a list of (homogeneous) elements in $\mathcal{L}V$. Define the graded sign $\sigma : \underline{x}$ of σ relative to $\underline{x}$ to be the sign (i.e. $+1$ or -1) that σ has when viewed as permuting k -blocks of sizes

$|x_1|, \dots, |x_k|$. Put

$$C(\underline{x}) = C(x_1, \dots, x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} [\sigma : \underline{x}] x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(k)}$$

which defines an element in $T(V) = U\mathcal{L}(V)$. Let $U_0(V) = R$ and $U_1(V) = \mathcal{L}(V)$ and

$$U_n(V) = \text{span}\{C(x_1, \dots, x_k) | x_i \in \mathcal{L}(V)\} \in T(V).$$

Define a map $\mu : IT(V) \otimes IT(V) \rightarrow IT(V)$ by

$$(4) \quad \mu(x_1 \otimes C(x_2, \dots, x_k)) = \begin{cases} \frac{1}{k} C(x_1, \dots, x_k) & \text{for } k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Put $E = IT(V)^{\otimes 2} \oplus IT(V)^{\otimes 3}$ and let $\phi = (\phi_1, \phi_2) : T(V)^{\otimes 2} \rightarrow E$, where $\phi_1 = 1 - T$ and $\phi_2 = I(\Delta_{sh}) \otimes 1 - 1 \otimes I(\Delta_{sh})$. Define $\lambda : E \rightarrow T(V)^{\otimes 2}$ as follows

$$(5) \quad \begin{cases} \lambda(a \otimes b) = -\frac{i}{k} T(a \otimes b) & a \in V^{\otimes i}, b \in V^{\otimes k-i}; \\ \lambda(a \otimes b \otimes c) = \frac{i+j}{k} (1 + T)(\mu(a \otimes b) \otimes c) & a \in V^{\otimes i}, b \in V^{\otimes j}, c \in V^{\otimes k-i-j} \end{cases}$$

Let $q : T(V) \rightarrow IT(V)$ denote natural projection. We also allow q to denote any of the homomorphisms $q^{\otimes 2} : T(V)^{\otimes 2} \rightarrow IT(V)^{\otimes 2}$, $q^{\otimes 3} : T(V)^{\otimes 3} \rightarrow IT(V)^{\otimes 3}$, $q^{\otimes 2} \oplus q^{\otimes 3} : T(V)^{\otimes 2} \oplus T(V)^{\otimes 3} \rightarrow IT(V)^{\otimes 2} \oplus IT(V)^{\otimes 3} = E$.

DEFINITION 35. Let (A, d, μ, ψ) be a Hah, $A = T(V)$, $V_0 = 0$. Call (A, d, μ, ψ) n -perfect if

$$(i) \quad (\varepsilon \otimes 1)\psi = 1 = (1 \otimes \varepsilon)\psi$$

$$(ii) \quad \psi = \psi T \text{ and } (\psi \otimes 1)\psi = (1 \otimes \psi)\psi \text{ on } V_{\leq n}$$

(iii) there exist derivation homotopies G_1 from ψ to ψT and G_2 from $(\psi \otimes 1)\psi$ to $(1 \otimes \psi)\psi$ such that $q(G_1, G_2)((\ker J(\psi))_{<n}) \subseteq \text{im}(1_E - \phi\lambda) \subseteq E$.

LEMMA 36. Let (A, d, μ, ψ) be a n -perfect Hah, $A = T(V)$, $V_0 = 0$. Then there exist $\chi \sim \psi$ such that χ coincides with ψ on $T(V_{\leq n})$ and (A, d, μ, χ) is $n+1$ -perfect. Furthermore, the derivation homotopy F from χ to ψ may be chosen so that $F(V_i) = 0$ for $i < n$ and for $i > n+1$.

The initial step of the induction is provided by

PROPOSITION 37. *Let (A, d, μ, ψ) be a Hah. Then there exists $\chi \sim \psi$ such that (A, d, μ, χ) is 1-perfect.*

We combine Proposition 34, Lemma 36 and Proposition 37 to obtain the following (over the rationals)

PROPOSITION 38. *Let (A, d, μ, ψ) be a Hah, $A = T(V)$, $V_0 = 0$. Then there exists on A a strictly coassociative and strictly cocommutative diagonal $\chi \sim \psi$; moreover, the resulting DG-Hopf algebra (A, d, μ, χ) has the form $U\mathcal{L}(W, d)$ for some Q -module W .*

We proceed with a definition, and inductive step, which are analogs of the definition and lemma above.

DEFINITION 39. *Let $(\mathcal{L}V, d)$ and (M, e) be DG-Lie algebras with $V_0 = 0$ and $M_0 = 0$. An HAH homomorphism $f : U(\mathcal{L}V, d) \rightarrow U(M, e)$ is n -perfect if*

$$(i) \ f(V_{\leq n}) \subset M; \text{ and}$$

$$(ii) \text{ there is a derivation homotopy } G \text{ from } \Delta_{UM}f \text{ to } (f \otimes f)\Delta_{\mathcal{L}V} \text{ such that } qG(V_{<n}) \subseteq \text{Im}(1 - I(\Delta_{UM})\mu) \subseteq I(UM)^{\otimes 2}.$$

REMARK 40. *Any such f is automatically 1-perfect from the dimensional viewpoint [3].*

LEMMA 41. *Let $(\mathcal{L}V, d)$ and (M, e) be DG-Lie algebras with $V_0 = 0$, $M_0 = 0$ and let $f : U(\mathcal{L}V, d) \rightarrow U(M, e)$ be n -perfect. There is an $(n + 1)$ -perfect g such that $g \sim f$. If f is an isomorphism, so is g . There is a derivation homotopy F from f to g having $F(V_i) = 0$ for $i < n$ and for $i > n + 1$.*

Note that if the HAH homomorphism $f : U(\mathcal{L}V, d) \rightarrow U(M, e)$ is n -perfect it follows from (i) that $(f \otimes f)\Delta_{U\mathcal{L}(V)} = \Delta_{UM}f$ in dimensions $\leq n$; this implies that if $f : U(\mathcal{L}V) \rightarrow U(M)$ is n -perfect for any $n \in \mathbb{N}$ then it is strictly comultiplicative on all of $U(\mathcal{L}V)$. Hence applying Proposition 41 inductively we obtain the following (over the rationals)

PROPOSITION 42. *Let $f : U(\mathcal{L}V, d) \rightarrow U(\mathcal{L}W, e)$ be a Hah homomorphism, $V_0 = W_0 = 0$. Then there exists a DG-Hopf algebra map $g : U(\mathcal{L}V, d) \rightarrow U(\mathcal{L}W, e)$ such that $g \sim f$. If f is an isomorphism so is g .*

Let us briefly discuss homotopy categories. For any category $\mathcal{C}$ having a notion of homotopy between morphisms satisfying Lemma 27, we may construct the classical homotopy category $\mathcal{Ho}\mathcal{C}$. Objects in $\mathcal{Ho}\mathcal{C}$ are objects of $\mathcal{C}$, but morphisms in $\mathcal{Ho}\mathcal{C}$ are homotopy classes of morphisms in $\mathcal{C}$. Two objects of the same homotopy type in $\mathcal{C}$ become isomorphic in $\mathcal{Ho}\mathcal{C}$. A functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ such that $\mathcal{F}f \sim \mathcal{F}g$ whenever $f \sim g$ induces a functor $\mathcal{Ho}\mathcal{F} : \mathcal{Ho}\mathcal{C} \rightarrow \mathcal{Ho}\mathcal{D}$. In particular, the functor U of Subsection 1.8 (universal enveloping algebra functor) induces $\mathcal{Ho}U$.

A functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence of categories if it induces a bijection on isomorphic classes of objects, as well as bijections $MOR_{\mathcal{C}}(A, B) \rightarrow MOR_{\mathcal{D}}(\mathcal{F}A, \mathcal{F}B)$ for each pair of objects in $\mathcal{C}$. Let us denote by $DLA_1(Q)$ the category of 1 reduced DG-Lie algebras which are free as Lie algebras. We are ready to state the principal result of [3].

THEOREM 43. *$\mathcal{Ho}(U) : \mathcal{Ho}DLA_1(Q) \rightarrow \mathcal{Ho}HAH_1(Q)$ is an equivalence. Furthermore, the functor U itself induces a bijection on isomorphism classes of objects.*

Anick proved this theorem in several intermediate steps, see [3]. We state below only one of the steps (theorem 6.4 in [3]) and refer to it in the last section.

THEOREM 44. *Let $f : (A, d_A, \psi_A, \mu_A) \rightarrow (B, d_B, \psi_B)$ be any homomorphism in $HAH_1(Q)$. There exists a homomorphism $\alpha : (L, d_L) \rightarrow (M, d_M)$ in $DLA_1(Q)$ such that $U(L, d_L)$ may be identified with (A, d_A, μ_A) ; $U(M, d_M)$ may be identified with (B, d_B, μ_B) ; and there exist DGA homotopies $f \sim U\alpha$, $\psi_A \sim \Delta_L$, $\psi_B \sim \Delta_M$. Furthermore, α is an isomorphism if and only if f is an isomorphism.*

We will state below another fundamental theorem from [4], which is a new proof of the fundamental theorem of rational homotopy theory. The fundamental theorem of rational homotopy theory asserts that the homotopy category $\mathcal{Ho}Top_1(Q)$ of simply

connected rational spaces is equivalent to the homotopy category $\mathcal{H}oDLA_1(Q)$ of connected differential graded Lie Q -algebras [38]. Letting $\mathcal{Q}(X)$ be a Quillene's minimal Lie algebra model for simply connected space X , a rephrasing of this could say that a model $\mathcal{Q}$ induces a bijection on weak rational homotopy types, and that two maps $f, g : X \rightarrow Y$ are Q -locally homotopic if and only if $\mathcal{Q}(f) \sim \mathcal{Q}(g)$ in $DLA_1(Q)$. Consequently any statement or problem about $\mathcal{H}oTop_1 Q$ may be translated into an equivalent statement in the purely algebraic category $DLA_1(Q)$. The model $\mathcal{Q}$ is called a "complete rational homotopy invariant" on simply connected spaces.

Let CW_1 denote the category of simply connected CW complexes, with arbitrary pointed maps among them. Anick constructed a model $\mathcal{L}$ from CW_1 to the category DLA_1 of 1-reduced differential graded Lie algebras [3]. Denote by $\mathcal{H}oCW_1 Q$ the homotopy category of simply connected CW complexes by collapsing Q -local homotopy classes of maps and by $\mathcal{H}oDLA_1(Q)$ the homotopy category of 1-reduced DG - Lie algebras as in Theorem 40. Then

THEOREM 45. *The model $\mathcal{L}$ induces an equivalence of rational homotopy categories*

$$\mathcal{H}o\mathcal{L} : \mathcal{H}oCW_1(Q) \longrightarrow \mathcal{H}oDLA_1(Q).$$

1.14 HOMOTOPY G-ALGEBRA

We start this subsection with the definition of Homotopy G -algebra. We slightly change the notation using the sequence of operations $E_{1,q} : A \otimes (\otimes^q A) \rightarrow A$ instead of sequences of multibraces $a\{a_1, \dots, a_q\}$ originally used in [17].

DEFINITION 46. *A Homotopy G -algebra is defined as a DG -algebra (A, d, μ) together with a set of operations*

$$E = \{E_{0,1} = 1, E_{1,q} : A \otimes (\otimes^q A) \rightarrow A, q = 0, 1, 2, \dots\}$$

satisfying

$$\begin{aligned}
(i) \quad & dE_{1,q}(a|b_1, \dots, b_q) - E_{1,q}(da|b_1, \dots, b_q) - \\
& (-1)^{|a|} \sum_{j=1}^q (-1)^{|b_1|+\dots+|b_{j-1}|} E_{1,q}(a|b_1, \dots, db_j, \dots, b_q) = \\
& = (-1)^{|a|+1} \sum_{i=1}^{q-1} (-1)^{|b_1|+\dots+|b_i|} E_{1,q-1}(a|b_1, \dots, b_i \cdot b_{i+1}, \dots, b_q) + \\
& \quad (-1)^{(|a|+1)(|b_1|)} b_1 \cdot E_{1,q-1}(a|b_2, \dots, b_q) + \\
& \quad (-1)^{|a|+|b_1|+\dots+|b_{q-1}|} E_{1,q-1}(a|b_1, \dots, b_{q-1}) \cdot b_q;
\end{aligned}$$

$$\begin{aligned}
(ii) \quad & E_{1,q}(a_1 \cdot a_2|b_1, \dots, b_q) = \\
& = \sum_{k=0}^q (-1)^\varepsilon E_{1,k}(a_1|b_1, \dots, b_k) \cdot E_{1,q-k}(a_2|b_{k+1}, \dots, b_q);
\end{aligned}$$

$$\begin{aligned}
(iii) \quad & E_{1,n}(E_{1,m}(a|b_1, \dots, b_m)|c_1, \dots, c_n) = \\
& = \sum_{0 \leq i_1 \leq \dots \leq i_m \leq n} (-1)^\tau E_{1,\cdot}(a|c_1, \dots, c_{i_1}, E_{1,\cdot}(b_1|c_{i_1+1}, \dots), \dots \\
& \quad \dots, c_{i_m}, E_{1,\cdot}(b_m|c_{i_m+1}, \dots), \dots, c_n)
\end{aligned}$$

where $\varepsilon = (|a_2| + 1) \sum_{i=1}^k |b_i|$, and $\tau = \sum_{p=1}^m |b_p| \sum_{q=1}^{i_p} |c_q|$.

It follows from (i) and (ii) that the map

$$E : (BA \otimes BA, d_{BA \otimes BA}, \Delta_\otimes) \rightarrow (A, d, \mu)$$

given by

$$E|_{(\otimes^p A) \otimes (\otimes^q A)} = \begin{cases} 1, & \text{if } p = 0, q = 1 \\ E_{1,q}, & \text{if } p = 1 \\ 0, & \text{otherwise} \end{cases}$$

is a twisting cochain $dE + Ed_{BA \otimes BA} = E \smile E$, which, according to Subsections 1.7 and 1.11, uniquely defines a *DG*-coalgebra homomorphism

$\mu_E : (BA \otimes BA, d_{BA \otimes BA}, \Delta_\otimes) \rightarrow (BA, d_{BA}, \Delta)$ by

$$\mu_E(b_1, \dots, b_m|c_1, \dots, c_n) =$$

$$\sum_{0 \leq i_1 \leq \dots \leq i_m \leq n} (-1)^\varepsilon c_1, \dots, c_{i_1}, E_{1,\cdot}(b_1|c_{i_1+1}, \dots), \dots \\ \dots, c_{i_m}, E_{1,\cdot}(b_m|c_{i_m+1}, \dots), \dots, c_n);$$

where $\varepsilon = \sum_{p=1}^m |b_p| \sum_{q=1}^{i_p} |c_q|$. The third condition implies that this product is associative.

Thus the bar-construction $(BA, d_{BA}, \Delta, \mu_E)$ over $(A, d, \mu, \{E_{1,q}\})$ is a *DG*-Hopf algebra.

For a 1-reduced simplicial set X , i.e. $X_0 = X_1 = \{*\}$, Baues has constructed a *DG* Hopf algebra map $\phi : \Omega C_*(X, R) \rightarrow \Omega C_*(X, R) \otimes \Omega C_*(X, R)$, see [6]; Gerstenhaber and Voronov Showed that dualization of the restrictions

$$\phi \circ i_1 : C_*(X, R) \rightarrow \Omega C_*(X, R) \rightarrow \Omega C_*(X, R) \otimes \Omega C_*(X, R)$$

yields to a Homotopy G-algebra structure on singular *cochains* $C^*(X, R)$, see corollary 6 in [18]. We describe these dual operations below.

For an n -simplex $\sigma : \Delta(n) \rightarrow X$ with $\Delta(n)$ being the standard n -simplex, let $\sigma(n_0, \dots, n_k)$ denotes its face spanned by the vertices $n_0, \dots, n_k$, where $i \mapsto n_i$ is a monotonic function $\Delta(k) \rightarrow \Delta(n)$. The singular cochain complex $C^*(X, R)$ has a natural Homotopy G-algebra structure; suppose $\varphi \in C^{m_0}(X, R)$, $\varphi_i \in C^{j_i}(X, R)$, $\sigma \in \Delta^{m_0+j_1+\dots+j_k-k}(X)$, then

$$E_{1,k}(\varphi|\varphi_1, \dots, \varphi_k)\sigma = \\ \sum_{n_1, \dots, n_k} (-1)^\varepsilon \varphi \sigma(0, \dots, n_1, n_1 + j_1, \dots, n_2, \dots, n_k + j_k, \dots, n_{k+1} = m) \\ \varphi_1 \sigma(n_1, \dots, n_1 + j_1) \varphi_2 \sigma(n_2, \dots, n_2 + j_2) \dots \varphi_k \sigma(n_k, \dots, n_k + j_k)$$

where $\varepsilon = \sum_{i=1}^{k+1} (i-1)(n_i + 1 - (n_{i-1} + j_{i-1}))$. Note that $E_{1,1}$ coincides with the Steenrod's $\smile_1$ product. For more details see [6], see also [28].

Once we have a Homotopy G-algebra structure $\{E_{1,k}\}$ on $C^*(X, R)$, we know that the Bar construction $(BC^*(X, R), d_{C^*(X,R)}, \Delta)$, together with the multiplication μ_E defined above, becomes a *DG* Hopf algebra.

Later in [7] it was shown that the diagonal $\phi : \Omega C_*(X, R) \rightarrow \Omega C_*(X, R) \otimes \Omega C_*(X, R)$ is cocommutative up to homotopy; i.e. there is a derivation homotopy $\psi : \Omega C_*(X, R) \rightarrow \Omega C_*(X, R) \otimes \Omega C_*(X, R)$ such that $\psi : \phi \sim T\phi$. Baues described the homotopy ψ explicitly on page 475. So $(\Omega C_*(X, R), d_{C_*(X, R)}, \mu, \phi)$ is a Hopf algebra up to homotopy (*Hah*); indeed, we have equalities not just homotopies for first 3 axioms of Definition 31.

We have seen above that the Baues diagonal ϕ on $\Omega C_*(X, R)$ canonically corresponds to a multiplication μ_E on $BC^*(X, R)$. Similarly the direct formula of homotopy ψ and its implications, as it was hypothesized by Baues (see page 476 in [7]), has the dual analogs in the coalgebraic case, i.e. on $BC^*(X, R)$.

Let us describe the "dual" homotopy in the coalgebraic case. Suppose that $f_k \in C^{i_k}(X, R)$, $g_t \in C^{j_t}(X, R)$, $\sigma \in C_m(X, R)$, $m = \sum_{k=1}^p i_k + \sum_{t=1}^q j_t - p - q$, then the set of operations

$$F_{p,q}(f_1, \dots, f_p | g_1, \dots, g_q) \sigma =$$

$$\sum_{\substack{n_1, \dots, n_q \\ k_1, \dots, k_p}} (-1)^\varepsilon f_1 \sigma(0, \dots, n_1, n_1 + j_1, \dots, n_2, \dots, n_{q-1} + j_{q-1}, \dots, n_q, k_1, \dots, n_{q+1})$$

$$g_1 \sigma(n_1, \dots, n_1 + j_1) g_2 \sigma(n_2, \dots, n_2 + j_2) \dots g_{q-1} \sigma(n_{q-1}, \dots, n_{q-1} + j_{q-1})$$

$$g_q \sigma(n_q, \dots, k_1, n_{q+1}, \dots, k_2, k_2 + i_2, \dots, k_3, \dots, k_p + i_p, \dots, k_{p+1} = m)$$

$$f_2 \sigma(k_2, \dots, k_2 + i_2) f_3 \sigma(k_3, \dots, k_3 + i_3) \dots f_p \sigma(k_p, \dots, k_p + i_p),$$

where $\varepsilon = \sum_{t=2}^q (t-1)(n_t + 1 - (n_{t-1} + j_{t-1})) + q(n_{q+1} + 1 - k_1) + (k_2 + 1 - n_{q+1}) + \sum_{t=3}^{p+1} (k_t + 1 - (k_{t-1} + i_{t-1}))$, uniquely defines a coderivation homotopy D_F so that $D_F : \mu_E \sim \mu_E T$.

Thus the Bar construction $(BC^*(X, R), d_{BC^*(X, R)}, \Delta, \mu_E)$ on $C^*(X, R)$ is a coalgebraic *Hah*. By the coalgebraic *Hah*, of course, we mean *DG-coalgebra* (C, d, Δ) together with a *DG-coalgebra* map $\mu : C \otimes C \rightarrow C$ such that the coaugmentation η

of coalgebra C is a two sided homotopy unit for μ , $\mu(\eta \otimes 1) \sim 1$ and $\mu(1 \otimes \eta) \sim 1$, and μ is homotopy associative, $\mu(\mu \otimes 1) \sim \mu(1 \otimes \mu)$, and homotopy commutative, $\mu \sim \mu T$. A homomorphism $f : (C, d, \Delta, \mu) \rightarrow (C', d', \Delta', \mu')$ between two such objects is a DG -coalgebra map $f : (C, d, \Delta) \rightarrow (C', d', \Delta')$ such that $f\mu \sim \mu'(f \otimes f)$; a homomorphism f is called a quasi-isomorphism, isomorphism etc. if it is so in DGC . Throughout the rest of this paper we work with free coalgebras, $C \cong T^c V$, and coderivation homotopies. The coalgebraic Hah 's and their homomorphisms form a category denoted by HAH (from the context it always will be clear whether the given objects belong to the algebraic or coalgebraic HAH , thus the notation HAH is unambiguous).

We have the coalgebraic analogs of the algebraic theorems of Subsection 1.13. In particular, the dual to the Propositions 38 and 42 are the following statements:

PROPOSITION 47. *Let (C, d, Δ, μ) be a coalgebraic Hah , $C = T^c(V)$, $V_0 = 0$. Then there exists on C strictly associative and strictly commutative multiplication $\phi \sim \mu$; moreover, the resulting DG -Hopf algebra (C, d, Δ, ϕ) has the form $U^c \mathcal{L}^c(W, d)$ for some Q -module W .*

PROPOSITION 48. *Let $f : U^c(\mathcal{L}^c V, d) \rightarrow U^c(\mathcal{L}^c W, e)$ be a Hah homomorphism, and $V_0 = W_0 = 0$. Then there exists a DG -Hopf algebra map $g : U^c(\mathcal{L}^c V, d) \rightarrow U^c(\mathcal{L}^c W, e)$ such that $g \sim f$. If f is an isomorphism so is g .*

Definition of the homomorphism in HAH motivates the following

DEFINITION 49. *A Homotopy G -algebra morphism*

$$f : (A, d_A, \mu_A, \{E_{1,q}^A\}) \rightarrow (B, d_B, \mu_B, \{E_{1,q}^B\})$$

is a DASH map

$$f : (A, d_A, \mu_A) \rightarrow (B, d_B, \mu_B)$$

such that the coalgebra map

$$Bf : (BA, d_{BA}, \mu_{E^A}, \Delta) \rightarrow (BB, d_{BB}, \mu_{E^B}, \Delta)$$

of appropriate bar constructions is multiplicative up to homotopy, i.e. there exists a coderivation homotopy from $Bf\mu_{EA}$ to $\mu_{EB}(Bf \otimes Bf)$.

A Homotopy G -algebra morphism

$$f : (A, d_A, \mu_A, \{E_{1,q}^A\}) \rightarrow (B, d_B, \mu_B, \{E_{1,q}^B\})$$

is called quasi-isomorphism, isomorphism, etc. if Bf is so in DGC .

Clearly, the identity map and a composition of Homotopy G -algebra morphisms are Homotopy G -algebra morphisms. Thus Homotopy G -algebras and their morphisms form a category which we call Homotopy G -algebra category.

2. ON THE GUGENHEIM MAP

In this section we give some applications of the Baues operations $\{E_{1,k}\}$ and $\{F_{p,q}\}$.

Let X be a simplicial complex (C^∞ manifold). Then there are two DG -algebras, Sullivan's algebra of rational differential forms (de Rham algebra of differential forms) $A^*(X, Q)$ [16] and the algebra of normalized simplicial cochains (normalized differentiable cochains), $C^*(X, Q)$ corresponding to X .

The transformation $\rho : A^*(X, Q) \rightarrow C^*(X, Q)$ given by

$$\langle \rho\omega, c \rangle = \int_c \omega,$$

where c is any simplicial (singular) chain, was first constructed in the 1930's in the context of smooth manifolds and ordinary differential forms. The de Rham theorem says, whether X is a smooth manifold or simplicial complex, that ρ induces a homology isomorphism

$$H\rho : HA^*(X, Q) \rightarrow HC^*(X, Q)$$

which is a map of algebras [13].

Although ρ itself is not a map of algebras ($A^*(X, Q)$ is commutative, $C^*(X, Q)$ is not) it was shown in [20] that it can be extended to a map of *DASH* ; i.e. there exists a whole family of higher operations

$$\rho_i : A^*(X, Q)^{\otimes i} \rightarrow C^*(X, Q)$$

such that

$$D\rho_i = \sum_{j=1}^{i-1} (-1)^j \{(\rho_j \smile \rho_{i-j}) - \rho_{i-1}(1^{(i-j)} \otimes \wedge \otimes 1^{(i-j-1)})\}.$$

For $i = 1, 2$ we get

$$D\rho_1 = 0, \quad D\rho_2 = \rho_1 \wedge -\rho_1 \smile \rho_1.$$

The latter statement contains, of course, the classical result that $H\rho$ is multiplicative.

From Subsection 1.10 we know that a given *DASH* map $\{\rho_i\}$ canonically defines a coalgebra map

$$B\{\rho_i\} : (BA^*, d_{BA^*}, \Delta) \rightarrow (BC^*, d_{BC^*}, \Delta)$$

of appropriate bar constructions; but both $(BA^*(X, Q), d_{BA^*(X, Q)}, \Delta)$ and $(BC^*(X, Q), d_{BC^*(X, Q)}, \Delta)$ are *DG*-Hopf algebras with respect to shuffle μ_{sh} and Baues μ_E products respectively. We showed in [29] that

THEOREM 50. *Gugenheim's map*

$$B\{\rho_i\} : (BA^*(X, Q), d_{BA^*(X, Q)}, \Delta, \mu_{sh}) \rightarrow (BC^*(X, Q), d_{BC^*(X, Q)}, \Delta, \mu_E)$$

is multiplicative up to homotopy.

Below we recall the construction of the Sullivan functor and state some key lemmas which we use to prove the theorem.

2.1 SULLIVAN FUNCTOR AND SOME KEY LEMMAS

We begin this subsection with short sketch of the Sullivan functor $A(K)$ mainly following [15], [32]. For more details see [16].

A *simplicial object* K with values in a category $\mathcal{C}$ is a sequence $\{K_n\}_{n \geq 0}$ of objects in $\mathcal{C}$, together with $\mathcal{C}$ -morphisms

$$\mathfrak{d}_i : K_{n+1} \rightarrow K_n, \quad 0 \leq i \leq n+1 \quad \text{and} \quad s_j : K_n \rightarrow K_{n+1}, \quad 0 \leq j \leq n,$$

called face and degeneracy operators, satisfying certain identities, see [34] for example.

A *simplicial morphism* $f : K \rightarrow L$ between two such simplicial objects is a sequence of $\mathcal{C}$ -morphisms $\varphi : K \rightarrow L$ commuting with $\mathfrak{d}_i$ and s_j .

A *simplicial set* K is a simplicial object in the category of sets. Thus it consists of a sequence of sets $\{K_n\}_{n \geq 0}$ together with set maps.

A *simplicial cochain algebra* A is a simplicial object in the category of cochain algebras: it consists of a sequence of cochain algebras $\{A_n\}_{n \geq 0}$ with appropriate face and degeneracy operators. Similarly *simplicial cochain complexes*, *simplicial vector spaces* ... are simplicial objects in the category of cochain complexes, vector spaces

Construction of $A(K)$.

Let K be a simplicial set, and let A be a simplicial cochain complex or a simplicial cochain algebra. Then

$$A(K) = \{A^p(K)\}_{p \geq 0}$$

is the "ordinary" cochain complex (or cochain algebra) defined as follows:

- $A^p(K)$ is the set of simplicial set morphisms from K to A^p .

Thus an element $\Phi \in A^p(K)$ is a mapping that assigns to each n -simplex $\sigma \in K_n$ ($n \geq 0$) an

element $\Phi_\sigma \in (A^p)_n$, such that $\Phi_{\mathfrak{d}_i \sigma} = \mathfrak{d}_i \Phi_\sigma$ and $\Phi_{s_j \sigma} = s_j \Phi_\sigma$.

- Addition, the scalar multiplication and the differential are given by

$$(\Phi + \Psi)_\sigma = \Phi_\sigma + \Psi_\sigma, \quad (\lambda \cdot \Phi)_\sigma = \lambda \cdot \Phi_\sigma \quad \text{and} \quad (d\Phi)_\sigma = d\Phi_\sigma.$$

- If A is a simplicial cochain algebra multiplication in $A(K)$ is given by

$$(\Phi \cdot \Psi)_\sigma = \Phi_\sigma \cdot \Psi_\sigma.$$

- If $\varphi : K \rightarrow L$ is a morphism of simplicial sets then $A(\varphi) : A(L) \rightarrow A(K)$ is the morphism of

cochain complexes (or cochain algebras) defined by

$$(A(\varphi)\Phi)_\sigma = \Phi_{\varphi\sigma}.$$

- If $\Theta : A \rightarrow B$ is a morphism of simplicial cochain complexes (or simplicial cochain algebras)

then $\Theta(K) : A(K) \rightarrow B(K)$ is the morphism defined by

$$(\Theta(K)\Phi)_\sigma = \Theta(\Phi_\sigma).$$

- When X is a topological space we write $A(X)$ for $A(S_*(X))$ where $S_*(X) = \{S_n(X)\}_{n \geq 0}$ is simplicial set of singular simplices $\sigma : \Delta^n \rightarrow X$ on the topological space X .

REMARK 51. *Note that the construction $A(K)$ is functorial in A and contrafunctorial in K .*

Let denote by Δ^n the Euclidean n -simplex ($\sum_{i=0}^n t_i = 1, t_i \geq 0$) in R^{n+1} and by $\Delta^n \xrightleftharpoons[\sigma_i]{\mathfrak{d}_i} \Delta^{n+1}$ the ordinary coface and codegeneracy operators.

Let Δ be a category, objects of which are sets Δ^n , and morphisms -compositions of $\mathfrak{d}_i$ and σ_i . The *simplicial object* in a category $\mathcal{B}$ we call any contravariant functor $K : \Delta \rightarrow \mathcal{B}$ (i.e. the family (K_n, d_i, s_i) of objects and morphisms $K_{n-1} \leftarrow K_n \rightarrow K_{n+1}$ in $\mathcal{B}$ satisfying the ordinary relations of commutation); the *simplicial morphism* in $\mathcal{B}$ is a natural transformation of two such functors.

Denote by $A_{DR}(\Delta^n)$ the classical cochain algebra of real C^∞ differential forms on Δ^n . The (differentiable) maps $\mathfrak{d}_i$ and σ_i induce operations

$$A_{DR}(\Delta^{n-1}) \xleftarrow{d_i} A_{DR}(\Delta^n) \xrightarrow{s_i} A_{DR}(\Delta^{n+1})$$

so that the family $(A_{DR}(\Delta^n), d_i, s_i)$ is a commutative simplicial DGA; i.e. the simplicial object in commutative DA over the reals. For the sake of simplicity we suppress Δ^n from notation, writing A_{DR} for $A_{DR}(\Delta^n)$. *The Poincare Lemma allows us to prove that each $A_{DR}(\Delta^n) = (A_{DR})_n$ is acyclic.*

Denote by A_Q a commutative simplicial DGA (over rationals), where $(A_Q)_n$ is a subspace in $(A_{DR})_n$ consisting of those differential forms

$$\sum \omega_{i_1 \dots i_p} dt^{i_1} \wedge \dots \wedge dt^{i_p}$$

whose coefficients $\omega_{i_1 \dots i_p}$ in the barycentric coordinates $(t_0, t_1, \dots, t_n)$ are rational polynomials.

For a classical homotopy operator

$$h : A_{DR}^p(\Delta^n) \rightarrow A_{DR}^{p-1}(\Delta^n),$$

with $1_{(A_{DR})_n} = dh + hd$, $h(A_Q)_n^p \subset (A_Q)_n^{p-1}$. Thus $(A_Q)_n$ is acyclic as well.

DEFINITION 52. *The Sullivan functor on a topological space X is defined as a DGA $A^*(X, Q)$ with*

$$A^p(X, Q) = \text{Hom}(S_*(X), (A_Q)^p),$$

i.e. an element $\Phi \in A^p(X, Q)$ is a function assigning to each singular n -simplex of X a polynomial p -form on Δ^n , $n \geq 0$, compatible with the face and degeneracy maps (see construction of $A(K)$ at the beginning of current subsection). This motivates the following terminology: The commutative cochain algebra, $A^p(X, Q)$, is the cochain algebra of polynomial differential forms on X over rationals.

Analogously over the reals, $A^p(X, R) = \text{Hom}(S_(X), (A_{DR})^p)$.*

Acyclicity of the complex $\text{Hom}(A(X)^{\otimes i}, A(X))$

We continue with the following

LEMMA 53. *The Complex $\text{Hom}(A(X)^{\otimes i}, A(X))$ is acyclic.*

We shall use the machinery of contravariant acyclic models used by Gugenheim in [20]; the machinery of acyclic models was introduced (covariantly) by Eilenberg and MacLane in [14].

For any contravariant functor $K : \mathcal{J} \rightarrow \{\text{vector spaces}\}$ there is a new contravariant functor $\hat{K} : \mathcal{J} \rightarrow \{\text{vector spaces}\}$

$$\hat{K}(X) = \prod_{u: M \rightarrow X} \{K(M), u\},$$

where the product is taken over all maps $u : M \rightarrow X$ of $\mathcal{J}$, and M is a "model", i.e., one of the standard simplices Δ^n . If $f : X \rightarrow Y$ is a map, then $\hat{K}(f) : \hat{K}(Y) \rightarrow \hat{K}(X)$ is defined by

$$\hat{K}(f)\{m_v, v\} = \{m_{f \cdot u}, u\},$$

where $u : M \rightarrow X$, $v : M_v \rightarrow Y$, and $m_v \in K(M_v)$.

Let $K, L : \mathcal{J} \rightarrow \mathcal{V}$ be contravariant functors, and $\Theta : K \rightarrow L$ be a transformation of these functors. Then a new transformation $\hat{\Theta} : \hat{K} \rightarrow \hat{L}$ is defined by

$$\hat{\Theta}(X)\{m, u\} = \{\Theta(M)m, u\}.$$

For any contravariant functor $K : \mathcal{J} \rightarrow \mathcal{V}$ define $\Phi : K \rightarrow \hat{K}$ by

$$\Phi(X)h = \{K(u)h, u\} \quad (h \in K(X)).$$

We easily verify that

$$\hat{\Theta}\Phi = \Phi\Theta.$$

In particular, for the functors A^* and C^* the functors $\hat{A}^*$ and $\hat{C}^*$ are defined by

$$(\hat{A})^n = \widehat{(A^n)}, \quad (\hat{C})^n = \widehat{(C^n)},$$

and we get differentials

$$\hat{d} : \hat{A}^* \rightarrow \hat{A}^*, \quad \hat{d} : \hat{C} \rightarrow \hat{C}^*.$$

To prove Lemma 53 we need the following

PROPOSITION 54. *The Functor A^* is corepresentable, i.e., there is a transformation of functors $\Psi : \hat{A}^* \rightarrow A^*$ such that $\Psi\Phi = \text{the identity}$.*

PROOF. We define $\Psi(X) : \hat{A}^*(X) \rightarrow A^*(X)$ by

$$\langle \Psi(X)\{m_u, u\}, v \rangle = \langle m_v, 1_{M_v} \rangle$$

where $v : M_v \rightarrow X$, and 1_{M_v} denotes the identity map of M_v regarded as a singular chain. Let $h \in A^*(X)$, then

$$\langle \Psi\Phi(h), v \rangle = \langle \Psi\{A^*(u)h, u\}, v \rangle = \langle A^*(v)h, 1_{M_v} \rangle$$

$$= \langle h, v(1_{M_v}) \rangle = \langle h, v \rangle,$$

i.e. $\Psi\Phi = \text{the identity}$. □

We mentioned above that the functor A^* is acyclic on models M . Moreover there is a homotopy operator (contraction)

$$h_M : A^*(M) \rightarrow A^*(M)$$

satisfying $dh_M + h_M d = 1_{A^*(M)}$. It follows from the definition of the tensor product that the functors

$$A^{*\otimes i} : A^{*\otimes i}(M) \rightarrow A^{*\otimes i}(M)$$

are acyclic on models as well. Let us fix for each $i \geq 1$ and a model M , once and for all, a contraction

$$h_M^i : A^{*\otimes i}(M) \rightarrow A^{*\otimes i}(M).$$

In such a case Gugenheim in [20] defined a transformation of functors

$$\hat{h}^i : \hat{A}^{*\otimes i}(X) \rightarrow \otimes^i \hat{A}^{*\otimes i}(X)$$

by $\hat{h}_M^i\{\omega_u, u\} = \{h_M^i\omega_u, u\}$, where $u : M \rightarrow X$. Clearly, $D\hat{h}_M^i = 1_{\otimes^i \hat{A}^*(M)}$.

PROOF OF LEMMA 53. Thus we must to prove that if $Df = 0$ for any $f \in \text{Hom}(A^{\otimes i}(X), A(X))$, $\deg(f) = k$, then there exists $g \in \text{Hom}(A^{\otimes i}(X), A(X))$ of degree $k - 1$ satisfying $Dg = f$.

Let us assume that g is defined in dimension $< n$, i.e. on $(A^{\otimes i}(X))_{< n}$, satisfying $dg_{j-1} - (-1)^{k-1}g_j d = f_{j-1}$ for any $j < n$. We write the defining equation for g_n as

$$g_n d = (-1)^{k-1}(dg_{n-1} - f_{n-1}).$$

For simplicity we denote $\Theta_i = (-1)^{k-1}(dg_{n-1} - f_{n-1})$. Thus Θ_i is defined in dimension $n-1$, and we have the following diagram:

$$\begin{array}{ccccc} & & & & (A^*)_{n-1+k} \\ & & & \nearrow \Theta_i & \\ (A^{*\otimes i})_{n-2} & \xrightarrow{d} & (A^{*\otimes i})_{n-1} & \xrightarrow{d} & (A^{*\otimes i})_n \end{array}$$

Note that $\Theta_i d = 0$; indeed

$$\begin{aligned} (-1)^{k-1}\{dg_{n-1} - f_{n-1}\}d &= (-1)^{k-1}dg_{n-1}d + df_{n-2} = \\ d\{(-1)^{k-1}g_{n-1}d + f_{n-2}\} &= ddg_{n-2} = 0. \end{aligned}$$

We have used here the fact that $Df = 0$, i.e. $(-1)^k f_{n-1}d = df_{n-2}$. Since in the diagram above everything is functorial, we get

$$\begin{array}{ccccc} & & & & (\hat{A}^*)_{n-1+k} \\ & & & \nearrow \hat{\Theta}_i & \nwarrow k_i \\ (\hat{A}^{*\otimes i})_{n-2} & \xrightarrow{\hat{d}} & (\hat{A}^{*\otimes i})_{n-1} & \xrightleftharpoons[\hat{d}]{\hat{h}^i} & (\hat{A}^{*\otimes i})_n \end{array}$$

where $\hat{\Theta}_i \hat{d} = 0$. Now we define $k_i = \hat{\Theta}_i \hat{h}^i$ and obtain

$$k_i \hat{d} = \hat{\Theta}_i \hat{h}^i \hat{d} = \hat{\Theta}_i (1 - \hat{d} \hat{h}^i) = \hat{\Theta}_i.$$

Next we define g_n on $(A^{*\otimes i}(X))_n$ by

$$g_n = \Psi k_i \Phi,$$

and verify

$$g_n d = \Psi k_i \Phi d = \Psi k_i \hat{d} \Phi = \Psi \hat{\Theta}_i \Phi = \Psi \Phi \Theta_i = \Theta_i$$

as required. □

Twisting cochains on $\text{Hom}(BA^*(X) \otimes BA^*(X), A^*(X))$.

For simplicity we will omit X from the notations writing A^* for $A^*(X)$.

Let us represent a homomorphism $E : BA^*(X) \otimes BA^*(X) \rightarrow A^*(X)$ by the collection of its components (restrictions on the tensor powers)

$$\{E^{p,q} = E|_{s^{-1}A^{\otimes p} \otimes s^{-1}A^{\otimes q}}((s^{-1})^{\otimes p} \otimes (s^{-1})^{\otimes q}) : (A^*)^{\otimes p} \otimes (A^*)^{\otimes q} \rightarrow A^*\}$$

Note that if $\deg E = k$, then $\deg E^{p,q} = k - (p + q)$.

In the set of all twisting cochains from $BA^* \otimes BA^*$ to A^* there exists one remarkable element. This is the "generator" $E_{sh} = p_1\mu_{sh}$ of the shuffle product $\mu_{sh} : BA^* \otimes BA^* \rightarrow BA^*$. Indeed, since the multiplication on A^* is commutative, the Bar construction $(BA^*, d_{BA^*}, \mu_{sh}, \Delta)$ of A^* is DGHA (see Subsection 1.10); i.e., $\mu_{sh} \in DC$, then by bijection 1.11.1

$$E_{sh} = p_1\mu_{sh} : BA^* \otimes BA^* \rightarrow A^*$$

is a twisting cochain. It is well known that E_{sh} consists of only two non trivial components: $E_{sh}^{0,1} = 1 : R \otimes A \rightarrow A$ and $E_{sh}^{1,0} = 1 : A \otimes R \rightarrow A$.

Let $F \in \text{Hom}(BA^* \otimes BA^*, A^*)$. Let us denote by F^n the sum of components $F^{p,q}$ of F with $p + q = n$:

$$F^n = \sum_{p+q=n} F^{p,q}.$$

We call the number $p + q$ the filtration of $F^{p,q}$.

REMARK 55. *It is easy to see that the product of two homogenous elements F^{p_1,q_1} and F^{p_2,q_2} in $\text{Hom}(BA^* \otimes BA^*, A^*)$ is homogenous of the type $F^{p_1+p_2,q_1+q_2}$. Thus we see that the multiplication in $\text{Hom}(BA^* \otimes BA^*, A^*)$ adds filtrations of the factors.*

For simplicity we write $\mathcal{A}$ for $\text{Hom}(BA^* \otimes BA^*, A^*)$; if $F \in \mathcal{A}$ has degree m we write $F \in \mathcal{A}^m$.

DEFINITION 56. Let n be a given number. We say that the elements $F_1, F_2 \in \mathcal{A}^m = \text{Hom}^m(BA^* \otimes BA^*, A^*)$ coincide up to n (write $F_1 \stackrel{n}{=} F_2$) if

$$F_1^k = F_2^k$$

for any $k \leq n$.

Let X be a simplicial set. Then the differential D on $\text{Hom}(BA^*(X) \otimes BA^*(X), A^*(X))$ can be expressed as a sum

$$D = D_1 + D_2,$$

where

$$D_1 F = F(d_\wedge \otimes 1 + 1 \otimes d_\wedge) \quad D_2 F = F(d_\otimes \otimes 1 + 1 \otimes d_\otimes) - (-1)^{|F|} d_{A^*(X)} F,$$

$d_\wedge$ and $d_\otimes$ are given by the formula

$$\begin{aligned} d_\wedge[\Phi_1 | \cdots | \Phi_n] &= \sum_{i=0}^{n-2} [\bar{\Phi}_1 | \cdots | \bar{\Phi}_i | (\bar{\Phi}_{i+1} \cdot a_{i+1}) | \Phi_{i+k+1} | \cdots | \Phi_n], \\ d_\otimes[\Phi_1 | \cdots | \Phi_n] &= \sum_{i=0}^{n-1} [\bar{\Phi}_1 | \cdots | \bar{\Phi}_i | d\Phi_{i+1} | \Phi_{i+1} | \cdots | \Phi_n], \end{aligned}$$

where $\Phi_i \in A^*(X)$, $\bar{\Phi}_i = (-1)^{|\Phi_i|+1} \cdot \Phi$ (see Subsection 1.10).

PROPOSITION 57. Let F be an element in $\mathcal{A}$ with $F^{n-1} = 0$. Then equality $DF \stackrel{n}{=} 0$ guarantees that $D_2 F^n = 0$, i.e., F^n is a chain map of (chain) complexes

$$\sum_{p+q=n} (A^*(X) \otimes \cdots^p \otimes A^*(X)) \otimes (A^*(X) \otimes \cdots^q \otimes A^*(X)) \rightarrow A^*(X).$$

PROOF. Clear. □

Now we are ready to prove the following lemma which is analogous to Theorem 21.

LEMMA 58. Let $E, \bar{E} \in \mathcal{A}^1$ satisfy the following conditions

I) $DE = -E \cdot E$ and $D\bar{E} = -\bar{E} \cdot \bar{E}$ (i.e., E are $\bar{E}$ twisting cochains);

II) $E^{1,0} = \bar{E}^{1,0} = 1$, $E^{0,1} = \bar{E}^{0,1} = 1$, and $E^{n,0} = \bar{E}^{n,0} = 0$, $E^{0,n} = \bar{E}^{0,n} = 0$ when

$n \neq 1$.

Then there exists $P \in \mathcal{A}^0$, $P^{0,0} = 0$, satisfying

$$\bar{E} = E + P \cdot E - \bar{E} \cdot P - DP.$$

PROOF. Note that it is enough to construct the sequence of elements in $\mathcal{A}^0$

$$P_{(1)}, P_{(2)}, P_{(3)}, \dots$$

satisfying the following condition

$$I) P_{(1)} = 0$$

$$II) P_{(i)} \stackrel{i}{=} P_{(i+1)}, \quad i = 1, 2, \dots$$

$$III) (P_{(i)})^k = 0 \quad \text{when } k = 0, 1 \text{ or } k > i$$

$$IV) DP_{(i)} \stackrel{i}{=} E - \bar{E} + P_{(i-1)} \cdot E - \bar{E} \cdot P_{(i-1)}.$$

After constructing such a sequence we define P by

$$P \stackrel{i}{=} P_{(i)}, \quad i = 1, 2, \dots$$

Clearly, P satisfies the required identity $\bar{E} = E + P \cdot E - \bar{E} \cdot P - DP$.

Assume that $P_{(1)}, P_{(2)}, \dots, P_{(n-1)}$ have been already constructed. Now we must construct $P_{(n)}$ with the required conditions $I - IV$. Let us consider the following element in $\mathcal{A}^1$

$$\beta(n-1) = E - \bar{E} + P_{(n-1)} \cdot E - \bar{E} \cdot P_{(n-1)} - DP_{(n-1)}.$$

Conditions $I - IV$ and Remark 55 give

$$\begin{aligned} D\beta(n-1) &= \\ &= DE - D\bar{E} + DP_{(n-1)} \cdot E + P_{(n-1)} \cdot DE - D\bar{E} \cdot P_{(n-1)} + \bar{E} \cdot DP_{(n-1)} \stackrel{n}{=} \\ &\stackrel{n}{=} -E \cdot E + \bar{E} \cdot \bar{E} + (E - \bar{E} + P_{(n-2)} \cdot E - \bar{E} \cdot P_{(n-2)}) \cdot E - P_{(n-1)} \cdot E \cdot E + \\ &\quad + \bar{E} \cdot \bar{E} \cdot P_{(n-1)} + \bar{E} \cdot (E - \bar{E} + P_{(n-2)} \cdot E - \bar{E} \cdot P_{(n-2)}) = \\ &= P_{(n-2)} \cdot E \cdot E - P_{(n-1)} \cdot E \cdot E + \bar{E} \cdot \bar{E} \cdot P_{(n-1)} - \bar{E} \cdot \bar{E} \cdot P_{(n-2)} \stackrel{n}{=} 0, \end{aligned}$$

where we used the fact that $P_{(n-2)} \cdot E \cdot E \stackrel{n}{=} P_{(n-1)} \cdot E \cdot E$; indeed, since $P^{(n-2)}, P^{(n-1)}$ and E have no nontrivial components with filtration of 0, multiplication adds filtrations

(Remark 55) and $P^{(n-2)} \stackrel{n-2}{=} P^{(n-1)}$ (condition *II*), then the elements $P^{(n-2)} \cdot E \cdot E$ and $P^{(n-1)} \cdot E \cdot E$ coincide up to n .

By definition, $\beta(n-1)$ has no nontrivial components with filtration $\leq n-1$; then by the Proposition 57

$$D_2\beta(n-1)^n = 0,$$

or equivalently $\beta(n-1)^n$ is a cycle in

$$\sum_{p+q=n} (A^*(X) \otimes \cdot^{\cdot^p} \otimes A^*(X)) \otimes (A^*(X) \otimes \cdot^{\cdot^q} \otimes A^*(X)) \rightarrow A^*(X).$$

Since the complex above is acyclic (see Lemma 53) there exists an element P' such that

$$D_2P' = \beta(n-1)^n.$$

Now we define $P_{(n)}$ by

$$P_{(n)} = P_{(n-1)} + P'.$$

Thus, it remains to show that $P_{(n)}$ together with $P_{(0)}, P_{(1)}, \dots, P_{(n-1)}$ satisfies conditions *I* – *IV*. The first three conditions are immediate. Let us verify condition *IV*. Since $D_2P' = \beta(n-1)^n$ and $\beta(n-1)$ has no nontrivial components with filtration $\leq n-1$, we have

$$DP' \stackrel{n}{=} \beta(n-1),$$

then

$$\begin{aligned} DP_{(n)} &= DP_{(n-1)} + DP' \stackrel{n}{=} DP_{(n-1)} + \beta(n-1) = DP_{(n-1)} - DP_{(n-1)} + \\ &+ E - \bar{E} + P_{(n-1)} \cdot E - \bar{E} \cdot P_{(n-1)} = E - \bar{E} + P_{(n-1)} \cdot E - \bar{E} \cdot P_{(n-1)}. \end{aligned}$$

Thus Lemma 58 is completely proved. $\square$

2.2 MULTIPLICATIVITY UP TO HOMOTOPY

We complete this Section with the proof of Theorem 50.

PROOF OF THEOREM 50. Let us consider the following twisting cochain

$$BA^*(X) \otimes BA^*(X) \xrightarrow{B\{\rho_i\} \otimes B\{\rho_i\}} BC^*(X) \otimes BC^*(X) \xrightarrow{\mu_E} BC^*(X) \xrightarrow{P_1} C^*(X)$$

where μ_E is a Baues multiplication from previous section (the entire composition is a twisting cochain being the composition of coalgebra maps and the twisting cochain P_1 , see Subsection 1.9).

Since the first component $\rho = \rho_1 : A^*(X) \rightarrow C^*(X)$ ($\langle \rho\omega, c \rangle = \int_c \omega$) of Gugenheim's (DASH) map

$$\rho_i : \otimes^i A^*(X) \rightarrow C^*(X) \quad (i \geq 1)$$

induces an isomorphism in homology, we know from Theorem 20 that there exists twisting cochain

$$\bar{E} : BA^*(X) \otimes BA^*(X) \rightarrow A^*(X)$$

satisfying $P_1\mu_E(B\{\rho_i\} \otimes B\{\rho_i\}) \sim \sum_i \rho_i(\bar{E} \otimes \cdots \otimes \bar{E})\Delta^i = P_1B\{\rho_i\}\mu_{\bar{E}}$, where

$$\mu_{\bar{E}} : BA^*(X) \otimes BA^*(X) \rightarrow BA^*(X)$$

is a coalgebra map induced by $\bar{E}$. We know that equivalent twisting cochains on $T(C, A)$ induce homotopic DG coalgebra maps on $Hom(C, BA)$ (see Subsection 1.11); thus $\mu_E(B\{\rho_i\} \otimes B\{\rho_i\}) \sim B\{\rho_i\}\mu_{\bar{E}}$ in DGC , i.e.

$$B\{\rho_i\} : (BA^*(X), d_{BA^*}, \Delta, \mu_{\bar{E}}) \rightarrow (BC^*(X), d_{BC^*}, \Delta, \mu_E)$$

is multiplicative up to homotopy, but the randomly chosen $\mu_{\bar{E}}$ may not be μ_{sh} .

Let us assume that $\bar{E}^{0,1} = 1$, $\bar{E}^{1,0} = 1$ and $\bar{E}^{0,n} = 0$, $\bar{E}^{n,0} = 0$ when $n \neq 1$ (we will show below that we can achieve this). Then we have two twisting cochains E_{sh} (E_{sh} - the generator of shuffle product μ_{sh}) and $\bar{E}$ on $Hom(BA^* \otimes BA^*, A^*)$

satisfying the conditions of Lemma 58; therefore there exists a homomorphism $P \in \text{Hom}(BA^*(X) \otimes BA^*(X), A^*(X))$ such that $E_{sh} \stackrel{P}{\sim} \bar{E}$, hence $\mu_{sh} \sim \mu_{\bar{E}}$, or equivalently

$$1 : (BA^*(X), d_{BA^*}, \Delta, \mu_{sh}) \rightarrow (BA^*(X), d_{BA^*}, \Delta, \mu_{\bar{E}})$$

is multiplicative up to homotopy. Then the composition

$$\begin{aligned} B\{\rho_i\} \cdot 1 : (BA^*(X), d_{BA^*}, \Delta, \mu_{sh}) &\rightarrow (BA^*(X), d_{BA^*}, \Delta, \mu_{\bar{E}}) \\ &\rightarrow (BC^*(X), d_{BC^*}, \Delta, \mu_E) \end{aligned}$$

produces the homotopy multiplicative Gugenheim map

$$B\{\rho_i\} : (BA^*(X), d_{BA^*}, \Delta, \mu_{sh}) \rightarrow (BC^*(X), d_{BC^*}, \Delta, \mu_E).$$

Thus, it remains to show that $\bar{E}^{0,1} = 1$, $\bar{E}^{1,0} = 1$ and $\bar{E}^{0,n} = 0$, $\bar{E}^{n,0} = 0$ when $n \neq 1$. First of all, note that $\mu_{\bar{E}}$ is homotopy commutative, homotopy associative and has a homotopy unit. Let us verify the first assertion, exactly the same argument can be applied to show the rest. Indeed,

$$\begin{aligned} \{\rho_i\}_*(\bar{E}) &= P_1 B\{\rho_i\} \mu_{\bar{E}} \sim P_1 \mu_E (B\{\rho_i\} \otimes B\{\rho_i\}) \sim P_1 \mu_E T(B\{\rho_i\} \otimes B\{\rho_i\}) = \\ &= P_1 \mu_E (B\{\rho_i\} \otimes B\{\rho_i\}) T \sim P_1 B\{\rho_i\} \mu_{\bar{E}} T = P_1 B\{\rho_i\} \mu_{\bar{E}T} = \{\rho_i\}_*(\bar{E}T), \end{aligned}$$

where we used homotopy multiplicativity of the Baues product, $\mu_E \sim \mu_{\bar{E}}T$. Since $\{\rho_i\}_*(\bar{E}) \sim \{\rho_i\}_*(\bar{E}T)$, according to the bijection of Theorem 20 the twisting cochains $\bar{E}$ and $\bar{E}T$ are equivalent, hence they induce homotopic coalgebra maps, $\mu_{\bar{E}} \sim \mu_{\bar{E}}T$; i.e. $\mu_{\bar{E}}$ is homotopy commutative. Now by Proposition 37 $\mu_{\bar{E}}$ can be replaced by the new product

$$\mu_{\tilde{E}} \sim \mu_{\bar{E}} : BA^* \otimes BA^* \rightarrow BA^*$$

which has a strict unit. The latter implies that the twisting cochain

$$\tilde{E} = P_1 \mu_{\tilde{E}} : BA^*(X) \otimes BA^*(X) \rightarrow BA^*(X) \rightarrow A^*(X)$$

has $\tilde{E}^{0,1} = 1$, $\tilde{E}^{1,0} = 1$ and $\tilde{E}^{0,n} = 0$, $\tilde{E}^{n,0} = 0$ when $n \neq 1$. This completes the proof. $\square$

3. $\tilde{B}_\infty$ -ALGEBRA CATEGORY

3.1 B_∞ -ALGEBRA

In this section we recall the notion of a B_∞ -algebra. We show that the singular cochain complex $C^*(X, R)$ endowed with the Baues operations from previous sections is a special case of a B_∞ -algebra; see also Baues' comment on this fact in [7].

The notion of B_∞ -algebra was introduced by Getzler and Jones in [19] as an additional structure on a chain complex H which turns the bar-construction BH into a DG -Hopf algebra. So this structure requires a new differential (which should be a coderivation with respect to standard coproduct on BH) $d_m : BH \rightarrow BH$ and a new associative product $\mu : BH \otimes BH \rightarrow BH$ so that $d_m : BH \rightarrow BH$ is a derivation.

Having in mind the structure of a differential on BH (Subsection 1.10) and of a product $BH \otimes BH \rightarrow BH$ (Subsection 1.14) we can formulate the definition of a B_∞ -algebra as follows

DEFINITION 59. *A B_∞ -algebra is defined as an A_∞ -algebra $(H, \{m_i\})$ together with a set of operations $E = \{E_{p,q} : (\otimes^p H) \otimes (\otimes^q H) \rightarrow H\}$, such that $E_{0,1} = 1$, $E_{1,0} = 1$, $E_{0,n} = E_{n,0} = 0$ $n \neq 1$ and*

$$(i) \sum_{j=0}^{p-i} \sum_{i=1}^p E_{p-i+1,q}(a_1, \dots, a_j, m_i(a_{j+1}, \dots, a_{i+j}), a_{i+j+1}, \dots, a_p | b_1, \dots, b_q) +$$

$$+ \sum_{l=0}^{q-k} \sum_{k=1}^q E_{p,q-k-1}(a_1, \dots, a_p | b_1, \dots, b_l, m_k(b_{l+1}, \dots, b_{k+l}), b_{k+l+1}, \dots, b_q) =$$

$$\begin{aligned}
&= \sum_{\substack{p_1+\dots+p_t=p \\ q_1+\dots+q_t=q}} \sum_{t=1}^{p+q} m_t(E_{p_1,q_1}(a_1, \dots, a_{p_1}|b_1, \dots, b_{q_1}), \dots \\
&\quad \dots, E_{p_t,q_t}(a_{p-p_t+1}, \dots, a_p|b_{q-q_t+1}, \dots, b_q); \\
(ii) \quad &\sum_{\substack{p_1+\dots+p_t=p \\ q_1+\dots+q_t=q}} \sum_{t=1}^{p+q} E_{t,g}(E_{p_1,q_1}(a_1, \dots, a_{p_1}|b_1, \dots, b_{q_1}), \dots \\
&\quad \dots, E_{p_t,q_t}(a_{p-p_t+1}, \dots, a_p|b_{q-q_t+1}, \dots, b_q)|c_1, \dots, c_g) = \\
&\sum_{\substack{r_1+\dots+r_k=q \\ g_1+\dots+g_k=g}} \sum_{k=1}^{q+g} E_{p,k}(a_1, \dots, a_p|E_{r_1,g_1}(b_1, \dots, b_{r_1}|c_1, \dots, c_{g_1}), \dots \\
&\quad \dots, E_{r_k,g_k}(b_{q-r_k+1}, \dots, b_q|c_{g-g_k+1}, \dots, c_g)).
\end{aligned}$$

It follows from (i) that the map $E : (BH \otimes BH, d_{m \otimes m}, \Delta_{\otimes}) \rightarrow (H, \{m_i\})$, given by $E|_{(\otimes^p H) \otimes (\otimes^q H)} = E_{p,q}$, is an A_{∞} -twisting cochain. Thus it generates a DG-coalgebra map $\mu_E : (BH \otimes BH, d_{m \otimes m}, \Delta_{\otimes}) \rightarrow (BH, d_m, \Delta)$, defined by

$$\begin{aligned}
\mu_E(a_1, \dots, a_p|b_1, \dots, b_q) &= \\
&= \sum_{\substack{p_1+\dots+p_t=p \\ q_1+\dots+q_t=q}} \sum_{t=1}^{p+q} E_{p_1,q_1}(a_1, \dots, a_{p_1}|b_1, \dots, b_{q_1}), \dots \\
&\quad \dots, E_{p_t,q_t}(a_{p-p_t+1}, \dots, a_p|b_{q-q_t+1}, \dots, b_q).
\end{aligned}$$

Condition (ii) guarantees that the product defined above is associative; so the Bar-construction (BH, d_m, Δ, μ_E) of a B_{∞} -algebra $(H, \{m_i\}, \{E_{p,q}\})$ is DG-Hopf algebra.

Note that a Homotopy G-algebra $(A, d, \mu, \{E_{1,q}\})$ can be considered as a B_{∞} -algebra $(A, \{m_i\}, \{E_{p,q}\})$ with $m_3 = m_4 = \dots = 0$, and $E_{p,q} = 0$ unless $p = 1$ and $E_{0,1} = 1$. In such a case conditions (i) – (ii) are equivalent to the conditions

(i) – (iii) of the Definition 46. Since we know that $(C^*(X, R), d_{C^*(X, R)}, \smile)$ endowed with Baues operations $\{E_{1,k}\}$ carries the structure of a Homotopy G-Algebra then from the discussion above it is obvious that $(C^*(X, R), d_{C^*(X, R)}, \smile, \{E_{1,k}\})$ is a B_∞ -algebra.

Note that any *commutative* A_∞ -algebra $(A, \{m_i\})$ also can be regarded as a B_∞ -algebra of the following type $(A, \{m_i\}, \{E_{1,0} = E_{0,1} = 1, E_{p,q} = 0 \text{ otherwise}\})$; the first condition in the definition above simply reduces to $m_k s_{i,k-i} = 0$.

DEFINITION 60. *We define a B_∞ -algebra morphism*

$$f : (A, \{m_i^A\}, \{E_{p,q}^A\}) \rightarrow (B, \{m_i^B\}, \{E_{p,q}^B\})$$

as an A_∞ -algebra map

$$f : (A, \{m_i^A\}) \rightarrow (B, \{m_i^B\})$$

such that the coalgebra map

$$Bf : (BA, Bd_{m^A}, \mu_{E^A}, \Delta) \rightarrow (BB, Bd_{m^B}, \mu_{E^B}, \Delta)$$

of appropriate bar constructions is multiplicative up to homotopy, i.e. there exists a coderivation homotopy from $Bf\mu_{E^A}$ to $\mu_{E^B}(Bf \otimes Bf)$.

A B_∞ -algebra morphism

$$f : (A, \{m_i^A\}, \{E_{p,q}^A\}) \rightarrow (B, \{m_i^B\}, \{E_{p,q}^B\})$$

is called quasi-isomorphism, isomorphism, etc. if Bf is so in DGC .

DEFINITION 61. *Two B_∞ -algebra morphisms*

$$f, g : (A, \{m_i^A\}, \{E_{p,q}^A\}) \rightarrow (B, \{m_i^B\}, \{E_{p,q}^B\})$$

are called homotopic, $f \sim g \in B_\infty\text{-Alg}$, if $Bf \sim Bg \in DGC$.

It is immediate that $\sim$ is an equivalence relation.

DEFINITION 62. B_∞ -algebras $(A, \{m_i^A\}, \{E_{p,q}^A\})$ and $(B, \{m_i^B\}, \{E_{p,q}^B\})$ we call weakly equivalent if they are connected by the chain

$$(A, d) \xleftarrow{\simeq} (Z(0), d) \xrightarrow{\simeq} (Z(1), d) \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} (Z(k), d) \xrightarrow{\simeq} (B, d)$$

of quasi-isomorphisms of B_∞ -algebras.

It is easy to check that B_∞ -algebras and their morphisms form a category which we denote by $B_\infty\text{-Alg}$.

We know that the Bar construction $(BC^*(X, Q), d_{BC^*X}, \Delta, \mu_E)$ of a 1-reduced simplicial set X is a DG -Hopf algebra. Moreover we know that the product μ_E is homotopy commutative so that $(BC^*(X, Q), d_{BC^*X}, \Delta, \mu_E) \in HAH$. In the upcoming sections we essentially need homotopy commutative products which encourages us to slightly modify the notion of B_∞ -algebra.

DEFINITION 63. An $(H, \{m_i\}, \{E_{p,q}\})$ is called a $\tilde{B}_\infty$ -algebra if the Bar construction $(BH, d_m, \Delta, \mu_E) \in HAH$.

Note that we do not require in the definition above strict associativity of the multiplication μ_E , as it was the case for B_∞ -algebras. Definition 63 yields to up to homotopy commutative and up to homotopy associative multiplication.

Let X be a 1-reduced simplicial set. In Subsection 1.14 we constructed Baues operations $\{E_{1,k}\}$ and $\{F_{p,q}\}$ on $C^*(X, R)$. We will show below that

$$(C^*(X, R), d_{C^*(X, R)}, \smile, \{E_{1,k}\}, \{F_{p,q}\})$$

is an object in $\tilde{B}_\infty\text{-Alg}$. Indeed we know that operations $\{E_{1,k}\}$ define a strictly associative multiplication μ_E (which also has a strict unit) on the Bar construction $BC^*(X, R)$. Moreover, operations $\{F_{p,q}\}$ uniquely define a coderivation homotopy D_F from μ_E to $\mu_E T$ making the product μ_E homotopy commutative; i.e. the Bar construction $(BC^*(X, R), d_{C^*(X, R)}, \Delta, \mu_E) \in HAH$. Then by Definition 63 we have a $\tilde{B}_\infty$ -algebra structure on $(C^*(X, R), d_{C^*(X, R)}, \smile, \{E_{1,k}\}, \{F_{p,q}\})$.

DEFINITION 64. We define a $\tilde{B}_\infty$ -algebra morphism

$$f : (A, \{m_i^A\}, \{E_{p,q}^A\}) \rightarrow (B, \{m_i^B\}, \{E_{p,q}^B\})$$

as an A_∞ -algebra map

$$f : (A, \{m_i^A\}) \rightarrow (B, \{m_i^B\})$$

such that the coalgebra map

$$Bf : (BA, Bd_{m^A}, \mu_{E^A}, \Delta) \rightarrow (BB, Bd_{m^B}, \mu_{E^B}, \Delta)$$

of appropriate bar constructions is multiplicative up to homotopy, i.e. there exists a coderivation homotopy from $Bf\mu_{E^A}$ to $\mu_{E^B}(Bf \otimes Bf)$.

A $\tilde{B}_\infty$ -algebra morphism

$$f : (A, \{m_i^A\}, \{E_{p,q}^A\}) \rightarrow (B, \{m_i^B\}, \{E_{p,q}^B\})$$

is called a quasi-isomorphism, isomorphism, etc. if Bf is so in HAH .

DEFINITION 65. Two $\tilde{B}_\infty$ -algebra morphisms

$$f, g : (A, \{m_i^A\}, \{E_{p,q}^A\}) \rightarrow (B, \{m_i^B\}, \{E_{p,q}^B\})$$

are called homotopic, $f \sim g \in \tilde{B}_\infty\text{-Alg}$, if $Bf \sim Bg \in HAH$, i.e. there is a coderivation homotopy from Bf to Bg .

It is immediate that $\sim$ is an equivalence relation.

PROPOSITION 66. Let $f, g : (A, \{m_i^A\}, \{E_{p,q}^A\}) \rightarrow (B, \{m_i^B\}, \{E_{p,q}^B\})$ be $\tilde{B}_\infty$ -algebra morphisms. Then $f \sim g \in \tilde{B}_\infty\text{-Alg}$ if and only if $Bf \sim Bg \in HAH$.

PROOF. Directly follows from the Definitions 64 and 65. □

DEFINITION 67. $\tilde{B}_\infty$ -algebras $(A, \{m_i^A\}, \{E_{p,q}^A\})$ and $(B, \{m_i^B\}, \{E_{p,q}^B\})$ are called weakly equivalent if they are connected by the chain

$$(A, d) \xleftarrow{\sim} (Z(0), d) \xrightarrow{\sim} (Z(1), d) \xleftarrow{\sim} \cdots \xleftarrow{\sim} (Z(k), d) \xrightarrow{\sim} (B, d)$$

of quasi-isomorphisms of $\tilde{B}_\infty$ -algebras.

It is easy to check that $\tilde{B}_\infty$ -algebras and their morphisms form a category which we denote by $\tilde{B}_\infty\text{-Alg}$.

Assigning to each 1-reduced simplicial set X an appropriate $\tilde{B}_\infty$ -Algebra

$$X \mapsto (C^*(X), d_{C^*(X)}, \smile, \{E_{1,k}\}, \{F_{p,q}\})$$

defines a correspondence from 1-reduced simplicial sets to 1-connected $\tilde{B}_\infty$ -algebras.

To show that this correspondence is *natural* we need to show that any simplicial map $f : X \rightarrow Y$ induces a $\tilde{B}_\infty$ -Algebra morphism

$$C^*(f) : (C^*(Y), d_{C^*(Y)}, \smile, \{E_{1,k}^Y\}, \{F_{p,q}^Y\}) \rightarrow (C^*(X), d_{C^*(X)}, \smile, \{E_{1,k}^X\}, \{F_{p,q}^X\}).$$

By Definition 64 we need to show that the coalgebra map

$$BC^*(f) : (BC^*(Y), d_{BC^*(Y)}, \Delta, \mu_{E^Y}) \rightarrow (BC^*(X), d_{BC^*(X)}, \Delta, \mu_{E^X})$$

is multiplicative up to homotopy, i.e. $BC^*(f)\mu_{E^Y} \sim \mu_{E^X}(BC^*(f) \otimes BC^*(f))$. But $BC^*(f)$ is strictly multiplicative not just homotopy multiplicative [7]; it is straightforward to check that the singular cochain map $C^*(f) : C^*(Y) \rightarrow C^*(X)$ commutes with operations $\{E_{1,k}\}$ which is a necessary and sufficient condition for multiplicativity of $BC^*(f)$.

We state below some key results of Baues [7], which we will use in the next section.

THEOREM 68. *Let X be a 1-reduced simplicial set. Then there is a product $\mu_{\tilde{E}}$ on the bar construction $BC^*(X, Q)$ such that $(BC^*(X, Q), \mu_{\tilde{E}})$ is a commutative differential Hopf algebra. Moreover there is a derivation homotopy $F : \mu_E \simeq \mu_{\tilde{E}}$. Both $\mu_{\tilde{E}}$ and F are natural for $X \in sSet_1$ so that one gets a commutative diagram of functors*

$$\begin{array}{ccc} sSet_1 & \xrightarrow{BC^*} & DGC_0 \\ (BC^*X)_{\mu_{\tilde{E}}} \searrow & & \nearrow \pi \\ & CHA_0 & \end{array}$$

where $sSet_1$ and CHA_0 are categories of 1-reduced simplicial sets and connected commutative Hopf algebras respectively.

THEOREM 69. *Let X be a 1-reduced simplicial set. Then there is a differential $d = d_{\mathcal{L}^c}$ on the free Lie coalgebra $\mathcal{L}^c(s^{-1}IC^*(X, Q))$ such that $\mathcal{L}^c(X, Q) = (\mathcal{L}^c(s^{-1}IC^*(X, Q)), d_{\mathcal{L}^c})$ defines a functor*

$$\mathcal{L}^c : sSet \longrightarrow DLC_1$$

Here $\mathcal{L}^c$ carries a map $f : X \longrightarrow Y$ to the induced map $\mathcal{L}^c(f) = \mathcal{L}^c(s^{-1}IC^*f)$ which is a differential map. Moreover there is an isomorphism of commutative differential Hopf algebras

$$\beta : (U^c\mathcal{L}^c(X, Q), \mu_{U^c\mathcal{L}^c(X)}) \longrightarrow (BC^*(X, Q), \mu_{\tilde{E}})$$

Here $U^c\mathcal{L}^c(X, Q) = (U^c\mathcal{L}^c(s^{-1}IC^(X, Q)), d)$ is the universal coenveloping coalgebra of the Lie coalgebra $\mathcal{L}^c(s^{-1}IC^*(X, Q))$ and $\mu_{\tilde{E}}$ is the product on the bar construction from the theorem above. The isomorphism β is natural in $X \in sSet_1$ and induces the identity on $s^{-1}IC^*(X, Q)$.*

It is worth mentioning that the differential $d_{\mathcal{L}^c}$ on $\mathcal{L}^c(s^{-1}IC^*(X, Q))$ was defined by $d_{\mathcal{L}^c} = \beta^{-1}d_{BC^*X}\beta$. Consequently we have three naturally isomorphic differential Hopf algebras

$$\begin{aligned} (BC^*(X, Q), d_{BC^*X}, \mu_{\tilde{E}}) &= (U^c\mathcal{L}^c(X, Q), U^cd_{\mathcal{L}^c}, \mu_{U^c\mathcal{L}^c(X)}) \\ &= (BC^*(X, Q), U^cd_{\mathcal{L}^c}, \mu_{sh}) \end{aligned}$$

(see [7], or Subsection 1.9 above).

We know that the differential $U^cd_{\mathcal{L}^c}$ on $(BC^*(X, Q), \mu_{sh}, U^cd_{\mathcal{L}^c})$ canonically defines a *commutative* A_∞ -algebra structure $\{m_i : (C^*(X, Q)^{\otimes i} \rightarrow C^*(X, Q))\}$ on $C^*(X, Q)$ (see Definition 14). Baues computed m_1 and m_2 explicitly.

THEOREM 70. *Let X be a 1-reduced simplicial set. Then the first two components of the differential $U^cd_{\mathcal{L}^c}$ are given by*

$$m_1 = d_{C^*(X, Q)}$$

and

$$m_2 = \frac{1}{2}(\smile + \smile T)$$

where $\smile$ is an Alexander-Whitney multiplication on the singular cochain complex $C^*(X, Q)$.

Clearly m_2 is commutative, i.e. $m_2 s_{1,1} = 0$. We finish this section with the following

REMARK 71. *The functor $\mathcal{L}^c$ in Theorem 69 induces an equivalence of rational homotopy categories*

$$\mathcal{H}o\mathcal{L}^c : \mathcal{H}oSet_1 \rightarrow \mathcal{H}oDLC_1,$$

where $\mathcal{H}o$ denotes a localization with respect to rational cohomology isomorphism, see [7].

3.2 A_∞ -ALGEBRA STRUCTURE IN HOMOLOGY

Let X be a 1-reduced simplicial set and $(A^*(X, Q), d_{A^*(X)}, \wedge)$ be a Sullivan DG -algebra of polynomial differential forms on X , see Section 2. We mentioned above that in homology $H(A^*(X, Q), d_{A^*(X)}, \wedge)$ there is a *commutative* A_∞ -algebra structure denoted by $(HA^*(X, Q), \{m_i^{HA^*(X)}\})$; the commutativity of Sullivan's cochains was strongly applied in the construction.

On the other hand, although the DG -algebra $(C^*(X, Q), d_{C^*(X)}, \smile)$ of the singular cochains almost never is commutative, the Baues' higher homotopies still allow us to construct a *commutative* A_∞ -algebra structure on $H(C^*(X, Q))$. Indeed, from the preceding section we know that there is a commutative A_∞ -algebra structure on $C^*(X, Q)$ with $m_1 = d_{C^*X}$ and $m_2 = \frac{1}{2}(\smile + \smile T)$; then by Theorem 24 we have the following

PROPOSITION 72. *There exist a minimal commutative A_∞ -algebra structure on $HC^*(X, Q)$ and a quasi-isomorphism of commutative A_∞ -algebras*

$$\{f'_i\} : (HC^*(X, Q), \{m_i^{HC^*(X)}\}) \rightarrow (C^*(X, Q), \{m_i\}).$$

Thus for a topological space X we have now two *minimal commutative A_∞ -algebras* on homology $H(X, Q)$; one is obtained through Sullivan's commutative DG-algebra $A^*(X, Q)$ and its appropriate A_∞ -algebra is denoted by $(HA^*(X, Q) = H(X, Q), \{m_i^{HA^*(X)}\})$, and the second is obtained through the singular cochain algebra $C^*(X, Q)$ and its appropriate A_∞ -algebra is denoted by $(H(X, Q), \{m_i^{HC^*(X)}\})$. We show below that they are isomorphic in C - A_∞ -Alg category.

THEOREM 73. *There is a commutative A_∞ -algebra isomorphism*

$$P_1\Sigma : (HA^*(X, Q), \{m_i^{HA^*(X)}\}) \rightarrow (HC^*(X, Q), \{m_i^{HC^*(X)}\}).$$

PROOF. Let us consider the following composition of the bar construction maps

$$\begin{array}{ccc} (BHA^*(X, Q), d_{m^{HA^*(X)}}, \Delta, \mu_{sh}) & & (BHC^*(X, Q), d_{m^{HC^*(X)}}, \Delta, \mu_{sh}) \\ \downarrow Bf & & \uparrow Bg' \\ (BA^*(X, Q), d_{m^{A^*(X)}}, \Delta, \mu_{sh}) & & (BC^*(X, Q), d_{m^{C^*(X)}}, \Delta, \mu_{sh}) \\ \downarrow B\rho & & \uparrow \beta \\ (BC^*(X, Q), d_{m^{C^*(X)}}, \Delta, \mu_E) & \xrightarrow{1} & (BC^*(X, Q), d_{m^{C^*(X)}}, \Delta, \mu_{\tilde{E}}) \end{array}$$

Here Bf is multiplicative and Bg' is multiplicative up to homotopy (see Theorem 24); also β is the Hopf algebra map of Theorem 69 and $B\rho$ is multiplicative up to homotopy by Theorem 50. Therefore the entire composition

$$\Gamma : (BHA^*(X, Q), d_{m^{HA^*(X)}}, \Delta, \mu_{sh}) \rightarrow (BHC^*(X, Q), d_{m^{HC^*(X, Q)}}, \Delta, \mu_{sh})$$

is multiplicative up to homotopy as well. Moreover each of the maps is a quasi-isomorphism, and so is Γ ; but the quasi-isomorphism of the bar constructions over minimal A_∞ -algebras is an isomorphism; i.e. Γ is an isomorphism in HAH . Then by

Proposition 48 there exists DG -Hopf algebra isomorphism

$$\Sigma : (BHA^*(X, Q), d_{m^{HA^*(X)}}, \Delta, \mu_{sh}) \rightarrow (BHC^*(X, Q), d_{m^{HC^*(X)}}, \Delta, \mu_{sh})$$

such that $\Sigma \sim \Gamma$ in HAH . This implies that the underlying commutative A_∞ -algebra morphism

$$P_1 \Sigma : (HA^*(X, Q), \{m_i^{HA^*(X)}\}) \rightarrow (HC^*(X, Q), \{m_i^{HC^*(X)}\})$$

is both commutative and an isomorphism since Σ is (Subsection 1.10). □

4. RATIONAL HOMOTOPY TYPE

By a space below we mean a CW complex; we call a CW complex X 1-reduced if its 1-skeleton X^1 is trivial. For simplicity we suppress Q from the notation, just writing $C^*(X)$, $H\mathcal{A}H$, for $C^*(X, Q)$, $H\mathcal{A}H_1(Q)$ respectively.

THEOREM 74. *The 1-reduced X and Y have the same rational homotopy type if and only if $(C^*(X), d_{C^*(X)}, \smile, \{E_{1,k}^X\}, \{F_{p,q}^X\})$ and $(C^*(Y), d_{C^*(Y)}, \smile, \{E_{1,k}^Y\}, \{F_{p,q}^Y\})$ are weakly equivalent in the $\tilde{B}_\infty$ -algebra category.*

PROOF. Let us assume that X and Y have the same rational homotopy type; this implies that there exists a chain

$$X = T_0 \xrightarrow{f_1} T_1 \xleftarrow{f_2} T_2 \rightarrow \cdots \xleftarrow{f_{n+1}} T_{n+1} = Y$$

of 1-reduced spaces and rational homotopy equivalences, i.e. continuous maps inducing isomorphism on the rational cohomology ring. For simplicity let us assume that there is short such chain:

$$X \xrightarrow{f_1} T \xleftarrow{f_2} Y;$$

the long version is proved exactly by same argument. From Section 3 we know that each of the maps, f_1 and f_2 , induces $\tilde{B}_\infty$ -algebra morphisms C^*f_1 and C^*f_2 . So we have a chain of $\tilde{B}_\infty$ -algebras and $\tilde{B}_\infty$ -algebra morphisms

$$\begin{aligned} (C^*(X), d_{C^*(X)}, \smile, \{E_{1,k}^X\}, \{F_{p,q}^X\}) &\xleftarrow{C^*(f_1)} (C^*(T), d_{C^*(T)}, \smile, \{E_{1,k}^T\}, \{F_{p,q}^T\}) \xrightarrow{C^*(f_2)} \\ &\longrightarrow (C^*(Y), d_{C^*(Y)}, \smile, \{E_{1,k}^Y\}, \{F_{p,q}^Y\}) \end{aligned}$$

So it remains to show that each of the maps, C^*f_1 and C^*f_2 , is a quasi-isomorphism in $\tilde{B}_\infty\text{-Alg}$. By the definition of quasi-isomorphism in $\tilde{B}_\infty\text{-Alg}$ we need to show

that the Bar construction maps BC^*f_1 and BC^*f_2 are quasi-isomorphisms. But they are, since both C^*f_1 and C^*f_2 are quasi-isomorphisms in DGC (see Subsection 1.10, Definition 13). Thus $\tilde{B}_\infty$ -algebras $(C^*(X), d_{C^*(X)}, \smile, \{E_{1,k}^X\}, \{F_{p,q}^X\})$ and $(C^*(Y), d_{C^*(Y)}, \smile, \{E_{1,k}^Y\}, \{F_{p,q}^Y\})$ are weakly equivalent.

Conversely, suppose that the $\tilde{B}_\infty$ -algebras $(C^*(X), d_{C^*(X)}, \smile, \{E_{1,k}^X\}, \{F_{p,q}^X\})$ and $(C^*(Y), d_{C^*(Y)}, \smile, \{E_{1,k}^Y\}, \{F_{p,q}^Y\})$ are weakly equivalent. Again, as above, we assume that there is a single 1-connected $\tilde{B}_\infty$ -algebra $(V, d_V, \cdot, \{E_{p,q}^V\}, \{F_{p,q}^V\})$ connecting them

$$(C^*(X), d_{C^*(X)}, \smile, \{E_{1,k}^X\}) \xleftarrow{f_1} (V, d_V, \cdot, \{E_{p,q}^V\}) \xrightarrow{f_2} (C^*(Y), d_{C^*(Y)}, \smile, \{E_{1,k}^Y\}).$$

From Subsection 3.2 we know that on the homology HC^*X of the space X there exist a minimal commutative A_∞ -algebra structure $(HC^*X, \{m_i\})$ and the following maps in both directions

$$Bf : (BHC^*X, d_m, \Delta, \mu_{sh}) \rightarrow (BC^*X, d_{BC^*X}, \Delta, \mu_{sh})$$

and

$$Bg : (BC^*X, d_{BC^*X}, \Delta, \mu_{sh}) \rightarrow (BHC^*X, d_m, \Delta, \mu_{sh});$$

the first map is strictly multiplicative and the second is multiplicative up to homotopy. Similarly for space Y .

On the other hand, since $(V, d_V, \cdot, \{E_{p,q}^V\}, \{F_{p,q}^V\})$ is a $\tilde{B}_\infty$ -algebra the Bar construction $(BV, d_{BV}, \Delta, \mu_{EV})$ is a Hah . Then by Proposition 47 there is a commutative and associative product $\tilde{\mu}_{EV}$ on BV such that $\tilde{\mu}_{EV} \sim \mu_{EV}$; this implies that the identity map

$$1 : (BV, d_{BV}, \Delta, \tilde{\mu}_{EV}) \rightarrow (BV, d_{BV}, \Delta, \mu_{EV})$$

is multiplicative up to homotopy.

By the same proposition we know that the resulting Hopf algebra $(BV, d_{BV}, \Delta, \tilde{\mu}_{EV})$ (now with commutative and associative multiplication) has the form

$$(BV, d_{BV}, \Delta, \tilde{\mu}_{EV}) = (U^c \mathcal{L}^c W, d, \Delta_{\mathcal{L}^c W}, \mu_{\mathcal{L}^c W}),$$

for some Q -module W that is isomorphic to the Q -module V . But the latter is canonically isomorphic to $(BW, d_{BW}, \Delta, \mu_{sh})$. Then by Theorem 24 on homology $H(W)$ there exists a minimal commutative A_∞ -algebra structure $(H(W), \{m_i\})$ and quasi-isomorphisms in both directions

$$Bf : (BH(W), d_m, \Delta, \mu_{sh}) \rightarrow (BW, d_{BW}, \Delta, \mu_{sh})$$

and

$$Bg : (BW, d_{BW}, \Delta, \mu_{sh}) \rightarrow (BH(W), d_m, \Delta, \mu_{sh}).$$

Therefore the entire composition

$$(BH(W), \mu_{sh}) \xrightarrow{Bf} (BW, \mu_{sh}) = (U^c \mathcal{L}^c W, \mu_{\mathcal{L}^c W}) = (BV, \tilde{\mu}_{EV}) \xrightarrow{1} (BV, \mu_{EV})$$

is multiplicative up to homotopy. Thus we have the following diagram

$$\begin{array}{ccccc} (BHC^*(X), \mu_{sh}) & & (BH(W), \mu_{sh}) & & (BHC^*(Y), \mu_{sh}) \\ \uparrow Bg_X & & \downarrow Bf & & \uparrow Bg_Y \\ (BC^*(X), \mu_{EX}) & \xleftarrow{Bf_1} & (BV, \mu_{EV}) & \xrightarrow{Bf_2} & (BC^*(Y), \mu_{EY}) \end{array}$$

of the quasi-isomorphisms in HAH ; then the compositions

$$F_1 = Bg_X Bf_1 Bf : (BH(W), d_m, \mu_{sh}, \Delta) \rightarrow (BHC^*(X), d_m, \mu_{sh}, \Delta)$$

and

$$F_2 = Bg_Y Bf_2 Bf : (BH(W), d_m, \mu_{sh}, \Delta) \rightarrow (BHC^*(Y), d_m, \mu_{sh}, \Delta)$$

are quasi-isomorphism too. But quasi-isomorphisms of the Bar-constructions over minimal A_∞ -algebras are isomorphisms. By Proposition 48, without loss of generality, we can assume that F_1 and F_2 are DG -Hopf algebra maps. Then the composition

$$F_2 F_1^{-1} : (BHC^*(X), d_m^X, \mu_{sh}, \Delta) \rightarrow (BHC^*(Y), d_m^Y, \mu_{sh}, \Delta)$$

is an isomorphism of DG -Hopf algebras. This implies that the underlying A_∞ -algebra morphism

$$P_1 F_2 F_1^{-1} : (HC^*(X), \{m_i^X\}) \rightarrow (HC^*(Y), \{m_i^Y\})$$

is an isomorphism in $C\text{-}A_\infty\text{-Alg}$. Then by Theorem 73 the commutative A_∞ -algebras $(HA^*(X), \{m_i^X\})$ and $(HA^*(Y), \{m_i^Y\})$ are isomorphic too. The latter implies that X and Y have the same rational homotopy type; see the second bijection in the introduction. $\square$

We know that the correspondence $X \mapsto (C^*(X), d_{C^*(X)}, \smile, \{E_{1,k}^X\}, \{F_{p,q}^X\})$ is natural (see Section 3) and so it defines a transformation of the following sets

$$\begin{aligned} \{\text{rational homotopy types}\} &\rightarrow \{\text{weak equivalent classes of} \\ &\quad \text{1-connected } \tilde{B}_\infty\text{-algebras}\}; \end{aligned}$$

the correctness and injectivity of this correspondence is provided by Theorem 74. Let us assume now that 1-reduced spaces X and 1-connected $\tilde{B}_\infty$ -algebras $(V, \{m_i\}, \{E_{p,q}\})$ are such that $H^*(X, Q)$ and $H^*(V, Q)$ have finite type. Then

THEOREM 75. *The sets*

$$\begin{aligned} \{\text{rational homotopy types}\} &\simeq \{\text{weak equivalence classes of} \\ &\quad \text{1-connected } \tilde{B}_\infty\text{-algebras}\} \end{aligned}$$

are bijective.

PROOF. Let $(V, \{m_i\}, \{E_{p,q}\})$ be a $\tilde{B}_\infty$ -algebra. We want to show that there exists a space X such that $(C^*(X), d_{C^*(X)}, \smile, \{E_{1,k}^X\})$ and $(V, \{m_i\}, \{E_{p,q}\})$ are weakly equivalent $\tilde{B}_\infty$ -algebras.

Since $(BV, d_m^{BV}, \Delta, \mu_E) \in HAH$ we know that there exists a Q -module W such that

$$Bf : (BH(W), d, \Delta, \mu_{sh}) \rightarrow (BV, d_m^{BV}, \Delta, \mu_E)$$

is a quasi-isomorphism in HAH (in more detail we have described this map in the proof of Theorem 74). By construction, the differential d on $BH(W)$ defines a minimal commutative A_∞ -algebra structure $\{m_i\} = \{P_1 d\}$ on $H(W)$, then by the second bijection of the introduction there exists a space X such that $(HA^*(X), \{m_i^X\})$ and

$(H(W), \{m_i\})$ are isomorphic in $C\text{-}A_\infty\text{-Alg}$; i.e there arises a DG Hopf algebra isomorphism

$$G : (BHA^*(X), d_m^{BH^*X}, \Delta, \mu_{sh}) \rightarrow (BH(W), d, \Delta, \mu_{sh})$$

of appropriate Bar constructions.

We show below that for this particular X the $\tilde{B}_\infty$ -algebras $(V, \{m_i\}, \{E_{p,q}\})$ and $(C^*(X), d_{C^*(X)}, \smile, \{E_{1,k}\})$ are weakly equivalent.

Note that we have two quasi-isomorphisms in HAH : first, the composition

$$BfG : (BHA^*(X), d_m^{BH^*X}, \Delta, \mu_{sh}) \rightarrow (BV, d_m^{BV}, \Delta, \mu_E),$$

and the second,

$$\begin{aligned} \beta^{-1}Bf' : (BHC^*(X), d_m^{BH^*X}, \Delta, \mu_{sh}) &\rightarrow (BC^*(X), d_m^{BC^*X}, \Delta, \mu_{sh}) \rightarrow \\ &\rightarrow (BC^*(X), d_{BC^*(X)}, \Delta, \mu_E) \end{aligned}$$

where β^{-1} is the isomorphism of Theorem 69, and Bf' is a coalgebra map induced by the A_∞ -algebra morphism $\{f'_i\}$ of Proposition 72. Then the diagram

$$\begin{array}{ccc} (BHC^*(X), d_m^{BH^*X}, \Delta, \mu_{sh}) & \xrightarrow{\Sigma} & (BHA^*(X), d_m^{BH^*X}, \Delta, \mu_{sh}) \\ \beta^{-1}Bf' \downarrow & & \downarrow BfG \\ (BC^*(X), d_{BC^*(X)}, \Delta, \mu_E) & & (BV, d_m^{BV}, \Delta, \mu_E) \end{array}$$

gives a desired chain of underlying $\tilde{B}_\infty$ -algebras and their quasi-isomorphisms

$$\begin{array}{ccc} (HC^*(X), \{m_i^{HC^*(X)}\}) & \xrightarrow{P_1\Sigma} & (HA^*(X), \{m_i^{HA^*(X)}\}) \\ P_1\beta^{-1}Bf' \downarrow & & \downarrow P_1BfG \\ (C^*(X), d_{C^*(X)}, \smile, \{E_{1,k}\}) & & (V, \{m_i\}, \{E_{p,q}\}) \end{array}$$

where $P_1\Sigma$ is the isomorphism of Theorem 73; i.e. $(C^*(X), d_{C^*(X)}, \smile, \{E_{1,k}^X\})$ and $(V, \{m_i\}, \{E_{p,q}\})$ are weakly equivalent in $\tilde{B}_\infty\text{-Alg}$ as desired. $\square$

Let X and Y be simply connected CW complexes and $f : X \rightarrow Y$ be any point-preserving continuous map. We know that the singular cochain functor induces a $\tilde{B}_\infty$ -algebra morphism

$$C^*f : (C^*(Y), d_{C^*(Y)}, \smile, \{E_{1,k}^Y\}, \{F_{p,q}^Y\}) \rightarrow (C^*(X), d_{C^*(X)}, \smile, \{E_{1,k}^X\}, \{F_{p,q}^X\}).$$

THEOREM 76. *The singular cochain functor induces a bijection of the following sets*

$$\begin{aligned} \{Q\text{-locally homotopy classes} &\simeq \{ \text{homotopy classes of } \tilde{B}_\infty\text{-alg-} \\ \text{of continuous maps } (X, Y) \} &\quad \text{ebra morphisms } (C^*Y, C^*X) \} \end{aligned}$$

PROOF. Let us show first that this correspondence is well defined and is an injection, which is equivalent to say that $f_1, f_2 : X \rightarrow Y$ are Q -locally homotopic maps if and only if $C^*f_1, C^*f_2 : C^*Y \rightarrow C^*X$ are homotopic in $\tilde{B}_\infty\text{-Alg}$.

The continuous maps $f_1, f_2 : X \rightarrow Y$ are Q -locally homotopic if and only if the Lie coalgebra maps $\mathcal{L}^c f_1, \mathcal{L}^c f_2 : \mathcal{L}^c(s^{-1}IC^*(Y)) \rightarrow \mathcal{L}^c(s^{-1}IC^*(X))$ are homotopic in DLC , see Remark 71. We know that $\mathcal{L}^c f_1, \mathcal{L}^c f_2$ are homotopic if and only if

$$U^c \mathcal{L}^c f_1, U^c \mathcal{L}^c f_2 : U^c \mathcal{L}^c(s^{-1}IC^*(Y)) \rightarrow U^c \mathcal{L}^c(s^{-1}IC^*(X))$$

are homotopic in HAH .

On the other hand each of the continuous maps $f_1, f_2 : X \rightarrow Y$ induce HAH morphisms

$$BC^*f_1, BC^*f_2 : (BC^*(Y), d_{BC^*Y}, \Delta, \mu_{E^Y}) \rightarrow (BC^*(X), d_{BC^*X}, \Delta, \mu_{E^X}).$$

We know that $(BC^*(X), d_{BC^*X}, \Delta, \mu_{E^X})$ and $U^c \mathcal{L}^c(s^{-1}IC^*(X))$ are naturally isomorphic in HAH . This isomorphism is obtained by first constructing the strictly commutative and associative product $\mu_{\tilde{E}^X}$ on $(BC^*(X))$ (Theorem 68) and then composing the resulting DG Hopf algebra $(BC^*(X), d_{BC^*X}, \Delta, \mu_{\tilde{E}^X})$ with DG Hopf algebra $U^c \mathcal{L}^c(s^{-1}IC^*(X))$ via the natural Hopf algebra isomorphism β of Theorem 69. Analogously $(BC^*(Y), d_{BC^*Y}, \Delta, \mu_{E^Y})$ is isomorphic to $U^c \mathcal{L}^c(s^{-1}IC^*(Y))$. Clearly, by construction $U^c \mathcal{L}^c f_1, U^c \mathcal{L}^c f_2$ are homotopic in HAH if and only if BC^*f_1, BC^*f_2

are homotopic in HAH . Finally, by Proposition 66 BC^*f_1, BC^*f_2 are homotopic in HAH if and only if the underlying $\tilde{B}_\infty$ -algebra morphisms C^*f_1, C^*f_2 are homotopic in the $\tilde{B}_\infty\text{-Alg}$ category as desired.

Thus it remains to show that this correspondence is a surjection, i.e. for any $\tilde{B}_\infty$ -algebra homomorphism

$$g : (C^*(Y), d_{C^*(Y)}, \smile, \{E_{1,k}^Y\}, \{F_{p,q}^Y\}) \rightarrow (C^*(X), d_{C^*(X)}, \smile, \{E_{1,k}^X\}, \{F_{p,q}^X\})$$

there exists a continuous map $f : X \rightarrow Y$ such that $C^*f \sim g \in \tilde{B}_\infty\text{-Alg}$. Given $\tilde{B}_\infty$ -algebra morphism g induces a HAH homomorphism

$$Bg : (BC^*(Y), d_{BC^*Y}, \Delta, \mu_E) \rightarrow (BC^*(X), d_{BC^*X}, \Delta, \mu_E).$$

We know that $(BC^*(Y), d_{BC^*Y}, \Delta, \mu_E)$ and $(BC^*(X), d_{BC^*X}, \Delta, \mu_E)$ are naturally isomorphic in HAH to $U^c\mathcal{L}^c(s^{-1}IC^*(Y))$ and $U^c\mathcal{L}^c(s^{-1}IC^*(X))$ respectively, through which they are identified. Then by Proposition 48 Bg is homotopic to a DG -Hopf algebra map $U^c\alpha$, $Bg \sim U^c\alpha \in HAH$, for some DG -Lie coalgebra homomorphism

$$\alpha : \mathcal{L}^c(s^{-1}IC^*(Y)) \rightarrow \mathcal{L}^c(s^{-1}IC^*(X))$$

(compare with Theorem 44). By Remark 71 for α there exists a continuous map $f : X \rightarrow Y$ such that $\mathcal{L}^c(s^{-1}IC^*f) \sim \alpha \in DLC$ and hence $U^c\mathcal{L}^c(s^{-1}IC^*f) \sim U^c\alpha \in HAH$. Therefore $U^c\mathcal{L}^c(s^{-1}IC^*f) \sim Bg \in HAH$. But by construction $U^c\mathcal{L}^c(s^{-1}IC^*f) \sim BC^*f \in HAH$, which implies that $BC^*f \sim Bg \in HAH$. It follows now from Definition 65 that $C^*f \sim g \in \tilde{B}_\infty\text{-Alg}$. $\square$

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