ABSTRACT

BAKUNOVA, EUGENIA. Optimal Filtering of Complex Turbulent Systems with Memory Depth through Consistency Constraints. (Under the direction of John Harlim.)

This thesis focuses on finding offline criteria for optimal autoregressive filtering. We show rigorously and numerically that autoregressive parameters satisfying absolute stability and consistency of order two guarantee optimal filtering. This conclusion allows us to develop a new autoregressive model that respects the two consistency constraints. We apply the new model to chaotic timeseries with long memory depth: the truncated Burgers-Hopf (TBH) and the Lorenz 96 models. We find that when absolute stability condition is satisfied, as was the case for the TBH, the filtering skill of the autoregressive model with consistency surpasses the skill of the autoregressive model that does not respect consistency conditions.

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Optimal Filtering of Complex Turbulent Systems with Memory
Depth through Consistency Constraints

by
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A thesis submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Master of Science

Applied Mathematics

Raleigh, North Carolina

2012

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DEDICATION

To my parents.

BIOGRAPHY

Eugenia Bakunova was born in Akademgorodok, Russia on June 13, 1987. She moved with her parents from Russia to Virginia and then to North Carolina. In 2005, Eugenia graduated from East Chapel Hill High School, and in December 2009 she received her Bachelor's degree in Biology and Economics from the University of North Carolina in Chapel Hill. After college, Eugenia started graduate school in the Mathematics Department at North Carolina State University in Raleigh, working towards her Master of Science degree in applied mathematics.

ACKNOWLEDGEMENTS

Foremost, I would like to thank my advisor, Professor John Harlim, for the tremendous effort and time he devoted to supporting me, motivating me to tackle problems, giving advice, and helping me grow not only as a mathematical researcher, but as a multidisciplinary professional. I also want to thank my parents for listening to my countless monologues about math, and for supporting me through hard times.

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Chapter 1

Introduction

Given noisy observations from nature, filtering is the process of finding the best statistical estimate of the true signal. Filtering consists of a two-step predictor-corrector scheme that adjusts a prior estimate to be more consistent with the current observations. The revised estimate is then fed into the model as an initial condition for the future time prediction. This approach of generating initial conditions is also known as data assimilation. In practice, the demand of practical filtering methods for real-time prediction (or state estimation) problem escalates as the model resolution is significantly increased. For example, in the coupled atmosphere-ocean system, the current practical models for prediction of both weather and climate involve general circulation models where the physical equations for these extremely stiff complex flows are discretized in space and time and the effects of unresolved processes are parametrized according to various recipes; the result of these approximations is an extremely unstable chaotic prediction model with several billion degrees of freedom.

In the data assimilation community, many well developed algorithms based on the Bayesian hierarchical modeling [4] and reduced order Kalman filtering strategies [36, 10, 42, 2, 3, 6, 8, 9, 38, 18, 12] have shown some successes in these extremely complex high dimensional nonlinear systems. Having all the above successful ensemble based filtering strategies (despite that most of them are sensitive to variation of parameters such as the ensemble size, localization, variance inflation, etc, so one needs to tune them on an ad-hoc basis), there is still an inherently difficult practical issue for some high dimensional complex problems due to the large computational overhead in generating individual ensemble members through the forward dynamical operator [17]. They are also less successful for more complex phenomena like gravity waves coupled with condensational heating from clouds, which are important for the tropics and severe local weather events.

Alternatively, Majda and Harlim [30] advocated much simpler and computationally faster filtering strategies with linear stochastic models resulting from committing judicious model er-

rors [13, 14]. In particular, the design of their reduced stochastic filters is attributed to the standard approach for modeling turbulent fluctuations [41, 33, 34, 32, 7, 27] with a linear damping and white noise (replacing nonlinear terms). The resulting filtering strategy, the Mean Stochastic Model (MSM-filter), consists of diagonal Langevin equations in Fourier space where the parameters are obtained through an off-line regression fitting to a training data set. The MSM-filter has shown encouraging results in various applications involving more realistic higher dimensional problems with $\mathcal{O}(10^4)$ state variables. In [15], the MSM-filter is superior to the ensemble Kalman filters in estimating midlatitude barotropic flows with baroclinic instability with sparse observations on a numerically stiff regime with shorter radius of deformation, mimicking the ocean turbulence. In [16], the MSM-filter is also used to capture the initiation of moist convectively coupled waves in the tropics as well as the MJO-like traveling waves through partial observation network; there, the MSM is designed for an eight-dimensional system with an appropriate eigenmode basis on each Fourier component (the eight variables include the first two baroclinic winds and potential temperatures, the equivalent boundary layer potential temperature, moisture, and heating rates of three cloud types above the boundary layer).

Despite the encouraging results, the MSM-filter suffers a serious limitation when the underlying dynamical system of interests is weakly chaotic with slow decaying correlation time [13, 21]. Such situation often arises in applications when the underlying true signals can be characterized by coherent interaction of waves of different spatial and temporal scales. For example, the dominant component of the intraseasonal (1-3 months) variability in the tropics is the Madden-Julian Oscillation (MJO); see the review article [43] and [23]. The MJO signal is characterized by deep convection/precipitation that propagates eastward through the portion of Indian and Pacific oceans with an average phase speed of 5m/s; this dominant equatorial wave is observed together with the faster eastward Kelvin waves (with time scale 5-10 days), and westward mixed-Rossby gravity waves (with time scale less than 5 days). Such a multiscale feature with large scale coherent structure is also observed in a toy model for midlatitude weather dynamics, the Lorenz-96 (L-96) model in weak turbulent regime with $F = 6$ [26, 13, 21]. In this idealized context, the energy spectrum is dominated by wavenumber 8 with highly oscillatory autocorrelation function [26, 21]; in this case, the wavenumber 8 has long memory depth. To overcome such a dynamical regime, an autoregressive filter (AR(p)-filter) was introduced in [21]; therein, it was shown that the AR(p)-filter of order $p > 1$ supersedes the MSM-filter in a weakly chaotic dynamical regime. In strongly turbulent regime, both the MSM-filter and AR(p)-filter are comparable. Most importantly, the AR(p)-filter was also shown to be much less sensitive to additional unavoidable intrinsic model errors compared to the standard ensemble Kalman filters.

The goal of this thesis is to find offline mathematical conditions for optimal filtering with model errors through the autoregressive modeling. We argue that an optimal AR(p)-filter can

be achieved by a judicious choice of parameters in the autoregressive models. In particular, we will rigorously show that if the autoregressive parameters are chosen to satisfy absolute stability and a certain subset of the consistency conditions of multistep numerical discretization scheme, then the optimal autoregressive filtering is guaranteed. This result is reminiscent of the Lax-equivalence fundamental theorem in the analysis of finite difference discretization scheme for the numerical solutions of partial differential equations [24].

To find the offline mathematical conditions, we discuss the test model for the complex turbulent signals in Section 2.1, and review the classical discrete multistep scheme and the relevant important properties in Section 2.2. This procedure mimics the classical von Neumann stability analysis [19]. In Sections 2.3 and 2.4, we define the discrete approximate filter with model errors and the multistep Kalman filtering algorithm, respectively. Subsequently, we discuss the main result of this thesis, the consistency conditions for optimal multistep filters, in Chapter 3, in which we start with a one-dimensional numerical example to build an intuition and finish with a mathematically rigorous theorem. In Chapter 4, we develop an algorithm for an improved autoregressive filter; in particular, we will implement the linear regression with consistency constraints to choose the autoregressive model coefficients and show that, if the filter prior model satisfies the absolute stability condition, then the filtered solutions improves significantly compared to the analogous autoregressive filter where the coefficients are obtained from the standard straightforward linear regression. We will test our new approach on two numerical examples: the first Fourier coefficient of the truncated Burgers-Hopf (TBH) model [31, 26, 22] and the wavenumber-8 of the L-96 model in weakly chaotic regime [13, 30, 21], both of which have long memory depth. We end the thesis with a short summary in Chapter 5.

Chapter 2

Multistep filters for complex turbulent signals

2.1 The test model for the true signals

We consider a simple stochastic model for turbulent fluctuations with linear dissipation and white noise forcing to mimic rapid energy transfer due to nonlinear interaction [41, 33, 34, 32, 7, 27]. As a canonical example, the turbulent true signals which will be filtered are determined by solutions of an s -dimensional stochastically forced PDE

$$\begin{aligned}\frac{\partial u(x, t)}{\partial t} &= \mathcal{P}(\partial_x) u(x, t) - \gamma(\partial_x) u(x, t) + f(x, t) + \sigma(x) \dot{W}(t), \\ u(x, 0) &= u_o(x),\end{aligned}\tag{2.1}$$

where $u(x, t) \in \mathbb{R}^s$ represents any vector field of interest. For simplicity, we will only consider scalar u on a one-dimensional spatial coordinate x . One can extend the results below to two- or three-dimensional spatial coordinates. Here the initial data u_o is a Gaussian random field with nonzero covariance. In (2.1), the term $\mathcal{P}(\partial_x)u$ represents the dispersion relation that is typically obtained by linearizing the nonlinear dynamical operator around a constant background field. The external forcing $f(x, t)$ is assumed to be known, damping term $-\gamma(\partial_x)u$ and spatially correlated noise,

$$\sigma(x) \dot{W}(t) \equiv \sum_{k=-\infty}^{\infty} \sigma_k \dot{W}_k(t) e^{ikx},\tag{2.2}$$

are added to represent small scale unresolved turbulent fluctuations and nonlinear interaction. In (2.2), $\sigma_k \in \mathbb{R}$ is the noise strength of Fourier mode k and $W_k(t)$ is the corresponding complex

Wiener process with independent real and imaginary components with variance $t/2$. We non-dimensionalize (2.1) on a 2π -periodic domain such that infinite Fourier series,

$$u(x, t) = \sum_{k=-\infty}^{\infty} \hat{u}_k(t) e^{ikx}, \quad \hat{u}_{-k} = \hat{u}_k^*, \quad \hat{u}_0 \in \mathbb{R}, \quad (2.3)$$

where $\hat{u}_k(t) \in \mathbb{C}$ for $k > 0$ can be utilized in analyzing (2.1) and the related finite approximations.

The operators $\mathcal{P}(\partial_x)$ and $\gamma(\partial_x)$ are defined through unique symbols at a given wavenumber k by

$$\mathcal{P}(\partial_x) e^{ikx} = \tilde{p}(ik) e^{ikx}, \quad (2.4)$$

$$\gamma(\partial_x) e^{ikx} = \gamma(ik) e^{ikx}, \quad (2.5)$$

where $\tilde{p}(ik)$ and $\gamma(ik)$ denote the eigen-solutions of operators $\mathcal{P}(\partial_x)$ and $\gamma(\partial_x)$, respectively. In general, $\mathcal{P}(\partial_x)$ can be any differential operator which is a combination of odd derivatives while $\gamma(\partial_x)$ is a suitable combination of even derivative with positive $\gamma(ik)$ (see Chapter 1 of [35] for various examples of damping operators). Substituting (2.2), (2.3)-(2.5) into initial value problem in (2.1), we reduce our problem to solving the following uncoupled forced Langevin equations on each Fourier mode

$$d\hat{u}_k(t) = [\tilde{p}(ik) - \gamma(ik)] \hat{u}_k(t) dt + \hat{f}_k(t) dt + \sigma_k dW_k, \quad \hat{u}_k(0) = \hat{u}_{k,0}, \quad (2.6)$$

where $\hat{f}_k(t)$ is the Fourier coefficient of the deterministic forcing, $f(x, t)$.

We assume that $\tilde{p}(ik) = i\omega_k$ is wave-like solution with $-\omega_k$ the real valued dispersion relation that measures the oscillation frequency of wavenumber k . In our numerical simulation in Chapter 3, we will choose the Rossby waves linear dispersion in a one-dimensional periodic domain with $\omega_k \sim 1/k$. In this simple context, the statistical equilibrium distribution for (2.6) exists provided that $\hat{f}_k = 0$ and the damping is non-negative,

$$\gamma(ik) > 0 \text{ for all } k \neq 0. \quad (2.7)$$

The damping coefficient, γ_k , is inversely proportional to the correlation time which measures the **memory depth** of the signal. Under these assumptions, the statistical equilibrium for (2.6) is Gaussian with mean zero and variance (or energy spectrum)

$$E_k = \frac{\sigma_k^2}{2\gamma(ik)}, \quad 1 \leq k < +\infty.$$

Mathematically, one needs to require $\sum E_k < \infty$ to define the stochastic solution of (2.1) correctly with a similar requirement on the Gaussian initial data in $u_0(x)$. As discussed in [29], the noise in (2.1) represents the turbulent fluctuations on the mesh scale for both unresolved and resolved features of the nonlinear dynamics ([26, 35] and references there) with a given energy spectrum E_k and decorrelation time γ_k^{-1} at each wave number. In practical problems (as we will see in Section 4 below), quite often the nature of this spectrum and the decorrelation times are roughly known [26, 35, 29]. In this thesis, we only discuss and present results for a single space dimension to avoid cumbersome notation, however, the theory below also applies in several variables.

2.2 The discrete approximation

We consider the standard finite difference approximations operate on a family of equispaced $2N + 1$ mesh points, $x_j = jh, 0 \leq j \leq 2N$, with $(2N + 1)h = 2\pi$. Given real-valued functions f_j on mesh points, we define the complex inner product,

$$(f, g)_h = \frac{h}{2\pi} \sum_{j=0}^{2N} f_j g_j^*, \quad (2.8)$$

where g^* denotes the conjugate of g . The discrete Fourier coefficients of f are defined by $\hat{f}_k = (f, e^{ikx_j})_h$ for $|k| \leq N$, with the well-known properties

$$\begin{aligned} f_j &= \sum_{|k| \leq N} \hat{f}_k e^{ikx_j}, & \hat{f}_{-k} &= \hat{f}_k^*, \\ (f, f)_h &= \sum_{|k| \leq N} |\hat{f}_k|^2. \end{aligned} \quad (2.9)$$

For a standard multistep Adams' method without any random noise, the p th order discrete approximation of (2.1) at time step $m\Delta t$ is expressed in standard fashion [40, 39] in

$$\begin{aligned} \hat{u}_{k,m}^h &= (1 + a_{k,p})\hat{u}_{k,m-1}^h + a_{k,p-1}\hat{u}_{k,m-2}^h + \cdots + a_{k,1}\hat{u}_{k,m-p}^h \\ &\quad + \hat{f}_{k,m} + \eta_{h,k,m}, \end{aligned} \quad (2.10)$$

for wavenumbers $|k| \leq N$. In (2.10), the approximate Fourier coefficients $\hat{u}_k^h$ are defined through the finite Fourier expansion

$$u^h = \sum_{|k| \leq N} \hat{u}_k^h e^{ikx}, \quad (2.11)$$

and $\eta_{h,k,m}$ denote the complex valued Gaussian noises with mean zero covariance

$$\langle \eta_{h,k,m+1} \eta_{h,\ell,m+1}^* \rangle = r_{h,k} \delta_{k+\ell}, \quad |k|, |\ell| \leq N, \quad (2.12)$$

with $r_{h,k}$ the variance at wave number k and δ_j the Kronecker delta function.

In the remainder of this section, we review the consistency conditions for coefficients $a_{k,j}$ of the linear multistep method of the discretization scheme in (2.10) (for consistency conditions of the general multistep method, see Theorem 11.3 in [39]). First, we can rewrite the multistep scheme in (2.10) in vector form:

$$\vec{u}_{k,m}^h = F_{h,k} \vec{u}_{k,m-1}^h + \vec{f}_{k,m} + \vec{\eta}_{h,k,m}, \quad |k| \leq N, \quad (2.13)$$

where $\vec{u}_{k,m}^h = (\hat{u}_{k,m-p+1}^h, \dots, \hat{u}_{k,m}^h)^T$, $\vec{f}_{k,m} = (0, \dots, 0, \hat{f}_{k,m})^T$, and $\vec{\eta}_{h,k,m} = (0, \dots, 0, \eta_{h,k,m})^T$ are p -dimensional complex valued column vectors. In (2.13), the dynamical operator $F_{h,k}$ is defined as follows,

$$F_{h,k} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_{k,1} & a_{k,2} & a_{k,3} & \dots & a_{k,p-1} & 1 + a_{k,p} \end{pmatrix}, \quad (2.14)$$

and system noise $\eta_{h,k,m}$ is Gaussian with variance $r_{h,k} = \sigma_k^2 \Delta t$ (through standard Euler approximation). We define the characteristic polynomial of $F_{h,k}$ as follows:

Definition 1. (Characteristic Polynomial) *The characteristic polynomial of matrix $F_{h,k}$ is given by*

$$\Pi_k(x) = (-1)^{p-1} \left(\sum_{j=1}^p a_{k,j} x^{j-1} + x^{p-1} - x^p \right). \quad (2.15)$$

It is a well known fact that the zeros of this characteristic polynomial are the eigenvalues of $F_{h,k}$.

Definition 2. (Consistency of order- q) *The p th order multistep method in (2.10) is consistent of order- q , for $1 \leq q \leq p$, if the following algebraic constraints are satisfied:*

$$\ell \sum_{j=1}^p (j-p)^{\ell-1} a_{k,j} = (-\gamma_k + i\omega_k) \Delta t, \quad \ell = 1, \dots, q. \quad (2.16)$$

This condition is equivalent to Theorem 11.3 in [39] for the multistep method in (2.10).

From these two definitions, we deduce the following:

Lemma 1. *The consistency of order-2 in the multistep scheme in (2.13) implies:*

$$\Pi_k(1) = (-1)^{p-1} \left(\sum_{j=1}^p a_{k,j} \right) = (-1)^{p-1} ((-\gamma_k + i\omega_k)\Delta t), \quad (2.17)$$

$$\begin{aligned} \Pi'_k(1) &= (-1)^{p-1} \left(-1 + \sum_{j=1}^p (j-1)a_{k,j} \right) \\ &= (-1)^{p-1} \left(-1 + \frac{2p-1}{2}(-\gamma_k + i\omega_k)\Delta t \right). \end{aligned} \quad (2.18)$$

Proof. See Appendix. □

For consistency of order-1, coefficients $\{a_{k,j}\}_{j=1}^p$ satisfy only one algebraic condition, (2.17). In general, consistency of order- q implies q linear algebraic conditions for coefficients $\{a_{k,j}\}_{j=1}^p$. So, if $q = p$, we have a unique choice for $\{a_{k,j}\}_{j=1}^p$ that corresponds to the explicit p th-ordered Adams-Bashford method. In Section 3, we will show that optimal filtering with multistep model of order $p > 2$ can be achieved even with only consistency of order-2. Thus, when coefficients $\{a_{k,j}\}_{j=1}^p$ are determined from turbulent signals, we can use the algebraic conditions in Lemma 1 to determine the filtering performance without actually simulating the filter (i.e., offline optimal filtering conditions) since we can typically extract the damping and frequency phase from measuring the energy spectrum and decorrelation time as we discussed in Section 2.1.

2.3 The discrete approximate filter with model errors

We consider the plentiful observations [5], that is, physical space observations are available at every model grid point x_j and discrete time step $T_{obs} = n\Delta t = t_{mn} - t_{(m-1)n}$ for fixed $n \geq 1$,

$$v_{j,mn} = u(x_j, t_{mn}) + \varepsilon_{j,mn}, \quad \varepsilon_{j,mn} \sim \mathcal{N}(0, r^o), \quad (2.19)$$

where r^o denotes the physical space observation noise variance. In (2.19), we implicitly assume that the observation errors are spatially and temporally uncorrelated and u is the ‘true’ solution of (2.1) up to $2N + 1$ spatial discretization.

In Fourier space, the **true discrete-time filtering problem** in (2.1), (2.19) can be ex-

pressed as follows

$$\begin{aligned}\hat{u}_{k,m} &= F_k \hat{u}_{k,m-1} + \hat{f}_{k,m} + \eta_{k,m}, \quad |k| \leq N, \\ \hat{v}_{k,mn} &= \hat{u}_{k,mn} + \hat{\varepsilon}_{k,mn}, \quad \hat{\varepsilon}_{k,mn} \sim \mathcal{N}(0, \hat{r}^o),\end{aligned}\tag{2.20}$$

where $\hat{u}_{k,m}$ are the exact solutions of SDE in (2.6) at timestep $m\Delta t$ with exact dynamical operator F_k , known forcing terms $\hat{f}_{k,m}$, and ‘true’ system noise variance r_k given respectively as follows:

$$F_k = e^{-(\gamma_k + i\omega_k)\Delta t},\tag{2.21}$$

$$\hat{f}_{k,m} = \int_0^{m\Delta t} \hat{f}(s) e^{-(\gamma_k + i\omega_k)(m\Delta t - s)} ds,\tag{2.22}$$

$$r_k = \frac{\sigma_k^2}{2\gamma_k} (1 - e^{-2\gamma\Delta t}) = E_k(1 - |F_k|^2).\tag{2.23}$$

In (2.20), the observation noises are Gaussian with mean zero and variance

$$\hat{r}^o = \langle \hat{\varepsilon}_{k,m} \hat{\varepsilon}_{\ell,m}^* \rangle = \frac{r^o}{2N+1} \delta_{k+\ell}.\tag{2.24}$$

In this thesis, we consider approximating the filtering problem in (2.20) with the multistep discretized forward model in (2.13). In particular, we define the **approximate multistep filter** as follows:

$$\begin{aligned}\vec{u}_{k,m}^h &= F_{h,k} \vec{u}_{k,m-1}^h + \vec{f}_{k,m} + \vec{\eta}_{h,k,m}, \quad |k| \leq N, \\ \hat{v}_{k,mn} &= \hat{u}_{k,mn} + \hat{\varepsilon}_{k,mn} \approx G \vec{u}_{k,mn}^h + \hat{\varepsilon}_{k,mn}, \quad \hat{\varepsilon}_{k,mn} \sim \mathcal{N}(0, \hat{r}^o).\end{aligned}\tag{2.25}$$

The observation operator $G = (0, \dots, 0, 1) \in \mathbb{R}^{1 \times p}$ in (2.25) maps the concatenated model state vector $\vec{u}_{k,mn}^h$ to the Fourier coefficient of the observation $\hat{v}_{k,mn}$. Note that with this approximate observation model, we commit model errors through p th-ordered multistep temporal discretization. For each Fourier coefficient, we have a p -dimensional filtering problem with a complex valued scalar observation.

2.4 The multistep Kalman filtering algorithm

The classical Kalman filter formula for the approximate multistep filtering problem in (2.25) consists of the following two steps: First, given the best estimate at time $t_{(m-1)n}$, we find the prior statistical estimates at the future observation time t_{mn} (assuming observations are

available at every $n\Delta t$ time units), through the following update formulas:

$$\langle \vec{u}_{k,mn|(m-1)n}^h \rangle = F_{h,k}^n \langle \vec{u}_{k,(m-1)n|(m-1)n}^h \rangle + \sum_{j=0}^{n-1} F_{h,k}^j \vec{f}_{k,(m-1)n+j+1}, \quad (2.26)$$

$$R_{k,mn|(m-1)n} = F_{h,k}^n R_{k,(m-1)n|(m-1)n} (F_{h,k}^*)^n + \sum_{j=0}^{n-1} F_{h,k}^j Q_{h,k} (F_{h,k}^*)^j, \quad (2.27)$$

where $\langle \vec{u}_{k,mn|(m-1)n} \rangle$ denotes the prior mean estimate and $R_{k,mn|(m-1)n} \equiv \langle |\vec{u}_{k,mn|(m-1)n} - \langle \vec{u}_{k,mn|(m-1)n} \rangle|^2 \rangle$ defines the prior error covariance matrix. In the covariance prior update in (2.27), the $p \times p$ system covariance matrix $Q_{h,k} = \langle \vec{\eta}_{h,k,m} \vec{\eta}_{h,k,m}^* \rangle$ equals to $r_{h,k}$ on the (p, p) -component and zero otherwise.

The posterior statistical estimates at time t_{mn} are found through the following formula

$$\langle \vec{u}_{k,mn|mn}^h \rangle = \langle \vec{u}_{k,mn|(m-1)n}^h \rangle + K_{k,mn} (\hat{v}_{k,mn} - G \langle \vec{u}_{k,mn|(m-1)n}^h \rangle), \quad (2.28)$$

$$R_{k,mn|mn} = (\mathcal{I} - K_{k,mn} G) R_{k,mn|(m-1)n}, \quad (2.29)$$

where

$$K_{k,mn} = R_{k,mn|(m-1)n} G^T (G R_{k,mn|(m-1)n} G^T + \hat{r}^o)^{-1} \quad (2.30)$$

is the p -dimensional Kalman gain vector and $R_{k,mn|mn} \equiv \langle |\vec{u}_{k,mn|mn} - \langle \vec{u}_{k,mn|mn} \rangle|^2 \rangle$ denotes the posterior error covariance matrix. The posterior mean update in (2.28) effectively is a smoother since the observation at time t_{mn} affects the mean vector $\langle \vec{u}_{k,mn|mn}^h \rangle$ which components consist of mean state at lags $t_{mn}, t_{mn-1}, \dots, t_{mn-p+1}$. In our numerical implementation in Chapter 4 below, we follow [21] to perform an “honest” filter by updating only the state at the current lag t_{mn} ; that is, we set $\langle \vec{u}_{k,mn|mn}^h \rangle = G \langle \vec{u}_{k,mn|mn}^h \rangle$.

Chapter 3

Optimal multistep filters

The classical theoretical result for convergence of the Kalman filter formulation [20] states that a necessary condition for linear filtering stability is the observability condition. Recently, Majda and Grote [29] showed that observability condition is not a sufficient condition for accurate filter solutions with unstable dynamical operator. The results in [5] confirmed that when the filter model dynamical operator is unstable, the observability condition produces convergent filtered solutions but the accuracy of the posterior estimate is nothing more than simply trusting the observations as we will confirm in the next numerical example. The goal of this chapter is to find an offline condition for accurate practical filtered solutions in the presence of model errors through multistep discretization.

3.1 One-mode numerical example

Before, we describe the mathematical offline conditions, let's build an intuition through the following numerical simulations. We consider a single-mode of the filtering problem in (2.20) with the following parameters (ignoring the subscript k for the remainder of this chapter) to mimic the fifth-mode ($k = 5$) of barotropic Rossby waves in a one-dimensional periodic geometry (see Chapter 5 of [30]) with damping coefficient $\gamma = 1.5$, frequency $\omega = 8.91/k$, equilibrium energy $E = k^{-3}$, and periodic forcing $f(t) = 0.1e^{i0.05t}$. Synthetic observations are generated through the exact formula in (2.20) with (2.21)-(2.23) at every observation timestep $n\Delta t = \gamma^{-1}$ and observation noise variance $\hat{r}^o = .5E = 0.004$. In Fig. 3.1, we show the relative improvements of the multistep filter with fixed $p = 6$ and $n = 24$ over trusting the observations as functions of the largest eigenvalues of the 6×6 matrix F_h as defined in (2.14) for various sets of randomly chosen $\{a_j\}_{j=1}^6$. The relative improvement is defined through a ratio between the time average

RMS difference between the posterior mean error and the theoretical observation error,

$$\text{relative improvement} \equiv 1 - \frac{\|\langle \hat{u}^h \rangle - \hat{u}\|_{RMS}}{\sqrt{r^o}}, \quad (3.1)$$

where $\|\vec{x}\|_{RMS}^2 = \frac{1}{T} \sum_{j=1}^T x_j^2$ is the root-mean-square value of vector $\vec{x}$. In our computation, we set $T = 10,000$. Each circle in Fig. 3.1 indicates the relative improvement for a specific randomly chosen $\{a_j\}_{j=1}^6$ with no consistency constraint (left panel) and consistency of order-2 constraint (right panel). See Lemma 1 for the consistency constraint of order-2.

Figure 3.1: Percentage of improvement in the accuracy of the filtered solutions relative to trusting the observations without the consistency condition (left) and with the consistency condition of order-2 (right). Each circle denotes relative improvement that is estimated from a long simulation with a random choice of parameter set $a_{k,j}$ with $p = 6$ and $n = 24$. The consistency of order-2 and absolute stability guarantee the filter accuracy to improve by at least 15%.

For $p = 6$, if we impose consistency constraints of order-6 in our numerical algorithm, we simply implement the 6th-ordered Adams-Bashford scheme and all of the circles in Fig 3.1 collapse to a unique quantity (denoted by ‘triangle’). We include the true filter (denoted by ‘x’) for diagnostic purposes; notice that the true filter is about 5% more skillful relative to the optimal approximate multistep filtered solutions. In [29, 5], they discuss a strategy to reduce this error gap by choosing the system noise covariance matrix, Q_h , based on information criteria for filtering with forward/backward Euler and the semi-implicit trapezoidal schemes. Here, our focus is to understand the effect of changing the multistep coefficients a_j with fixed noise covariance matrix, Q_h . The circles to the left of the unit vertical line in Fig 3.1 satisfy the absolute stability condition ($\max_{j=1}^p |\lambda_j| < 1$ where λ_j are eigenvalues of F_h) of the multistep discretization scheme; the percentage of improvement relative to trusting the observation of these stable filter models vary from about 8-18% (see the left and middle panels in Fig 3.1). When the consistency condition of order-2 is imposed, the improvement is guaranteed to be always greater than 15% whenever the autoregressive model is stable.

On the other hand, whenever the filter is unstable (to the right of the vertical unit line), we also find that the filter observability condition is fully satisfied yet the filter simply trusts the observations (0% improvement). In our application, the observability condition,

$$\det \begin{pmatrix} G^T & (GF_h^n)^T \dots & (GF_h^{n(p-1)})^T \end{pmatrix} \neq 0,$$

is satisfied but the determinant is very small as n is large. This result reconfirms that observability condition is not sufficient to guarantee accurate filtered solutions beyond trusting observations [5].

The fact that largest accuracy improvement is attainable with only consistency condition of order-2 in the multistep filter of order $p = 6$ suggests there are non-unique parameter sets $\{a_j\}$ that produce nearly optimal result. In contrast, if consistency condition of order $p = 6$ is required to guarantee optimal filtering then there is only one parameter set that satisfies all p consistency condition which is the Adam-Bashford order-6 parameter values. Fortunately, we don’t need the rigid latter condition. This result is very interesting because it implies that we can obtain optimal filtering even with model errors through multistep discretization as long as we choose coefficients $\{a_j\}$ that satisfy the absolute stability and consistency condition of only order-2. In the next section, we will prove this statement rigorously.

3.2 Consistency conditions for optimal multistep filters

In the presence of model errors, we define the mean model errors as in [1, 30]:

$$\mathcal{E}_{k,mn} \equiv \langle \vec{u}_{k,mn| (m-1)n}^h \rangle - \vec{u}_{k,mn}, \quad (3.2)$$

where components $\vec{u}_{k,mn} = (\hat{u}_{k,mn-p+1}, \dots, \hat{u}_{k,mn-1}, \hat{u}_{k,mn})^T$ are the true signals. Substituting the mean update formulas in (2.26), (2.28) to (3.2), we deduce the following recursive equation for the mean model errors

$$\mathcal{E}_{k,mn} = F_{h,k}^n (\mathcal{I} - K_{k,mn} G) \mathcal{E}_{k,(m-1)n} + (F_{h,k}^n - F_{t,k}^n) \langle \vec{u}_{k,(m-1)n} \rangle, \quad (3.3)$$

where for $n > 2$, $F_{t,k}^n$ denotes a matrix with the last column given by $(F_k^{n-p+1}, \dots, F_k^{n-1}, F_k^n)^T$, where F_k is defined in (2.21), and zero otherwise. The first term in (3.3) measures the effect of filtering through the stability operator, $F_{h,k}^n (\mathcal{I} - K_{k,mn} G)$, while the second term measures the effect of model errors through the numerical discretization for a nonzero initial mean.

The main result in this chapter is given as follows:

Theorem 1. *Consider model errors through the approximate multistep filter in (2.25) with $p > 2$ on solving the true filtering problem in (2.20). Suppose that the characteristic matrix $F_{h,k}$ has eigenvalues λ_j such that $|\lambda_p| \leq \dots \leq |\lambda_1| < 1$ (this is also called absolute root condition), observation matrix $G = (0, \dots, 0, 1)$, and there is an $\varepsilon > 0$ such that $|\lambda_1 - 1| \leq \varepsilon$. Then, as $n \rightarrow \infty$, the stability factor, $\|F_{h,k}^n (\mathcal{I} - K_{k,mn} G)\|_2 \rightarrow 0$, with convergence rate*

$$\frac{|\lambda_1|^n \|S\|_2}{|\Pi'_k(\lambda_1)|} \leq \frac{|\lambda_1|^n \|S\|_2}{|\Pi'_k(1)| - \varepsilon |\Pi''_k(1)|}. \quad (3.4)$$

The upper bound in (3.4) is finite since $\|S\|_2$ depends only on p and $\Pi_k(x)$ is the characteristic polynomial of $F_{h,k}$ as defined in (2.15).

Proof. See Appendix. □

Practically, Theorem 1 suggests that the stability factor of the p th-ordered multistep filter for large n is bounded above by a constant that is inversely proportional to $|\Pi'_k(1)|$, which is nothing but the consistency of order-2 (see Lemma 1). From the algebraic constraint in (2.18), $|\Pi'_k(1)| = |-1 + \frac{2p-1}{2}(-\gamma_k + i\omega_k)\Delta t|$ is almost always smaller than that when the consistency of order-2 is not satisfied. In Figure 3.2, we confirm the convergence rate in (3.4) for finite n and the tightness of its upper bound.

This theorem is reminiscent of the Lax-equivalence theorem, which is the fundamental theorem in the analysis of finite difference discretization scheme for the numerical solutions of

Figure 3.2: When $n \gg 1$, the stability factor $\|F_{h,k}^n(\mathcal{I} - K_{k,mn}G)\|_2$ converges to $\frac{|\lambda_1|^n \|S\|_2}{\Pi'(\lambda_1)}$ (labeled by asterisk sign), confirming (3.4). The upper bound in (3.4) (labeled by '+' sign) is tight.

partial differential equations [24]. Here, we have an analogous theorem for optimal filtering of the stochastic PDE test model in (2.1) with model errors through the p th-ordered multistep scheme in which we show that absolute stability of $F_{h,k}$ for $p > 2$ plus consistency of order-2 yields optimal filtered solutions. In particular, we have:

Corollary 1. *Consider the approximate multistep filter in (2.25) with $F_{h,k}$ satisfying the con-*

sistency of order-2 and the absolute root condition. Then, as $n \gg 1$, we have

$$|G\mathcal{E}_{k,mn}| \leq \frac{|\lambda_1|^n \|S\|_2 \|\mathcal{E}_{k,(m-1)n}\|_2}{|\Pi'_k(1)| - \varepsilon |\Pi''_k(1)|} + 2MT_{obs}e^{QT_{obs}}\mathcal{O}(\Delta t^2). \quad (3.5)$$

for observation time interval $[(m-1)T_{obs}, mT_{obs}]$, where $T_{obs} = n\Delta t$ is fixed, and some constant $M, Q > 0$.

Proof. We consider only the last component of $\mathcal{E}_{k,mn}$ by premultiplying (3.3) with G from the left and take the ℓ_2 -norm. The first term in the right hand side of (3.5) follows directly from Theorem 1 and $\|G\|_2 = 1$. The second term in the right hand side of (3.5) follows from the standard convergence results of linear multistep method (see Theorem 11.5 in [39]) in which we implement the upper bound with no initial errors since both matrices $GF_{h,k}^n$ and $GF_{t,k}^n$ operate on the mean vector state $\langle \vec{u}_{k,(m-1)n} \rangle$ as shown in (3.3), absolute root conditions, and consistency of order-2 that yields local truncation errors with $\mathcal{O}(\Delta t^2)$. $\square$

Chapter 4

Improved autoregressive filtering with consistency constraint

In this chapter, we apply the two consistency constraints in Lemma 1 to improve filtered solutions of signals with long memory depth. First, we review the standard least squares minimization problem to obtain the autoregressive model parameters [21] and then we discuss the new regression strategy that respects the two consistency constraints in Lemma 1. Then, we apply the filtering scheme discussed in (2.26)-(2.30) on two turbulent signals with long memory depth: the first Fourier coefficient of the truncated Burgers-Hopf (TBH) model [31, 26, 22] and the wavenumber-8 of the Lorenz-96 model in weakly chaotic regime [13, 30, 21]. In particular, we will compare the accuracy of the filtered solutions from these two autoregressive models, one that respects the two consistency constraints and another one that does not, for various values of $n = T_{obs}/\Delta t$ where Δt is the model discretization time step and T_{obs} is the observation time interval.

4.1 Linear Regression Fitting

Given timeseries of each Fourier mode, $\{\hat{u}_{k,n}, n = 1, \dots, N\}$, we consider to fit an autoregressive model of order p in (2.10); that is, we are looking for coefficients, $\{a_{k,j}\}_{j=1}^p$, the constant forcing $\hat{f}_{k,m}$, noise variance $r_{h,k}$, and the autoregressive model order p . Following [21], we take the constant forcing term $\hat{f}_{k,m}$ to be the empirical mean state $\bar{\hat{u}}_k$. Then we subtract the forcing (empirical mean) and define an $(N - p)$ -dimensional vector $Y \equiv (X_{k,p+1}, \dots, X_{k,N})^T$ with

$X_{k,n} = \hat{u}_{k,n} - \hat{f}_{k,m}$, and an $(N-p) \times p$ matrix X

$$X \equiv \begin{pmatrix} X_{k,1} & X_{k,2} & \cdots & X_{k,p} \\ X_{k,2} & X_{k,3} & \cdots & X_{k,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ X_{k,N-p} & X_{k,N-2} & \cdots & X_{k,N-p} \end{pmatrix}.$$

Then, the AR(p) model in (2.13) is equivalent to

$$Y = X(\vec{a} + \vec{e}_p) + \vec{\eta},$$

where $\vec{a} = (a_{k,1}, \dots, a_{k,p})^T$, $\vec{e}_p = (0, \dots, 0, 1)^T \in \mathbb{R}^p$ and $\vec{\eta} \equiv (\eta_{k,p+1}, \dots, \eta_{k,N}) \sim \mathcal{N}(0, r_{h,k}\mathcal{I})$. The standard classical linear regression without constraints solve the linear regression problem,

$$\min_{\vec{a}} \|X(\vec{a} + \vec{e}_p) - Y\|_2^2, \quad (4.1)$$

with solutions given as follows:

$$\begin{aligned} \hat{a} &= (X^*X)^{-1}X^*(Y - X\vec{e}_p), \\ \hat{r}_{h,k} &= \frac{|Y - X(\hat{a} + \vec{e}_p)|^2}{N-p}. \end{aligned} \quad (4.2)$$

The new autoregressive model in this chapter is obtained by solving the regression problem in (4.1) subject to two consistency constraints in Lemma 1. In this case, the parameters $\hat{a}$ can be obtained by solving a standard regression algorithm with linear equality constraints (see Chapter 12 of [11]) and the system covariance $r_{h,k}$ is estimated as in (4.2). However, recall that the constraints in Lemma 1 depend on the damping coefficient γ_k and the frequency ω_k as follows:

$$\sum_{j=1}^p a_{k,j} = (-\gamma_k + i\omega_k)\Delta t, \quad (4.3)$$

$$\sum_{j=1}^p (j-1)a_{k,j} = \frac{2p-1}{2}(-\gamma_k + i\omega_k)\Delta t. \quad (4.4)$$

We propose to estimate γ_k and ω_k based on the Mean Stochastic Model (MSM) fitting as in [28, 15, 30]. That is, we simply match the integral of the empirical autocorrelation function,

$$\hat{T}_{corr} = \lim_{t \rightarrow \infty} \int_0^\infty \frac{(\hat{u}_k(t) - \bar{\hat{u}}_k)(\hat{u}_k(t+\tau) - \bar{\hat{u}}_k)^*}{(\hat{u}_k(t) - \bar{\hat{u}}_k)(\hat{u}_k(t) - \bar{\hat{u}}_k)^*} d\tau, \quad (4.5)$$

to the explicit correlation time for the Ornstein-Uhlenbeck process governed by the Langevin equation in (2.6), $T_{corr} = (\gamma_k + i\omega_k)^{-1}$. This strategy tacitly assumes that the equilibrium statistics of the true signal are approximately those of the Ornstein-Uhlenbeck process, Gaussian statistics. In the numerical examples below, we will check the performance when the true signals are solutions of nonlinear models with Gaussian and slightly skewed from Gaussian equilibrium statistics. Note that with these approximate constraints, the regression strategy does not necessarily fit to the equilibrium statistics of the true signals; quite often, we find the autocorrelation function of the new autoregressive model does not match the true autocorrelation function.

4.2 Numerical Examples

In our first numerical example, we consider the first mode of the truncated Burgers-Hopf (TBH) model, which has the longest memory depth (slowest correlation decay), see [31, 26, 22]. The truncated Burgers-Hopf model (TBH) is a Fourier Galerkin approximation to the inviscid Burgers equation which has intrinsic stochastic dynamics with strong numerical evidence of ergodic and mixing for large enough degrees of freedom with Gaussian equilibrium distribution. The governing equations of its Fourier coefficients are as follows

$$\frac{d\hat{u}_k}{dt} = -\frac{ik}{2} \sum_{\substack{k+p+q=0 \\ |p|, |q| < N/2}} \hat{u}_p^* \hat{u}_q^*, \quad \hat{u}_{-k} = \hat{u}_k^*, \quad |k| \leq 20. \quad (4.6)$$

Following [31], we numerically integrate the TBH model with a pseudo-spectral method combined with the fourth order Runge-Kutta time integrator with small enough time step $\delta t = 2.5 \times 10^{-4}$ to conserve the energy with small enough relative error. Initial conditions are randomly drawn from the uniform distribution on a constant energy surface $E = 0.1$. We parameterize the MSM model with solutions of the TBH model in (4.6) for total time 10^5 as the training dataset. For the AR(p) model, we estimate the parameters with discrete time step $\Delta t = 0.1$ (we found worse solutions with smaller Δt , [22]).

In our numerical example, we apply the autoregressive filter for observations using $n = 5$ with observation noise errors corresponding to signal-to-noise ratio that is inversely proportional to 10% of the truth signal variability, $r^o = 0.01$. We find that the relative improvement increases from 79% to 92% when the two consistency constraints in (4.3)-(4.4) are utilized. In Figure 4.1, we see that the estimated trajectory with the consistency constraints is much better than the trajectory without consistency constraints. We also check the case in which the variability of the observation errors is 25% of the truth signal variability; in this case, we find that for $n = 10$ the relative improvement increases from 81% to 92% when the two consistency constraints in

(4.3)-(4.4) are utilized.

Figure 4.1: Trajectories of filtered solutions estimated with AR(p)-filter with $p = 63$ for $n = 5$, $\Delta t = 0.1$, and $r^o = 10\%E_k$ without (top) and with (bottom) consistency constraints.

In our second numerical example, we test it on the Lorenz-96 model,

$$\frac{du_j}{dt} = (u_{j+1} - u_{j-2})u_{j-1} - u_j + F,$$

for $j = 1, \dots, 40$, with periodic boundary condition, $u_{j+J} = u_j$, in weakly chaotic dynamical regime with $F = 6$ [26, 13, 21]. This model was first introduced with stronger chaotic regime with $F = 8$ [25] as a toy model for an “atmospheric variability” of u_j at equally spaced grid points on a constant latitude. In this example, we consider to recover the wavenumber-8 which is the most energetic mode with longest memory depth when $F = 6$. Here, the probability distribution of this mode is slightly skewed from Gaussian distribution (see Chapter 11 of [30]). The FPE criterion produces $p = 15$ [21].

Unlike in the first numerical example, the set of parameters $a_{k,j}$ we obtain does not satisfy the absolute stability condition. In other words, one of the eigenvalues of $F_{h,k}$ has an absolute value greater than 1. Since absolute stability is necessary for a convergent solution (and necessary for the Theorem 1 to apply), it is better to omit consistency constraints as in [30].

We purposely report this negative result in addition to the successful case with the TBH time series to emphasize that in nature sometimes it is not possible to satisfy both the consistency constraints up to order two and the absolute stability condition; this result, however, does not contradict Theorem 1.

Chapter 5

Summary

In this thesis, we find offline criteria for optimal filtering with the AR(p) model. Specifically, we show rigorously and numerically that autoregressive parameters satisfying absolute stability and consistency constraints in (2.17)-(2.18) guarantee optimal filtering.

To facilitate the optimal filtering, we use a test model for complex turbulent system in (2.1). We have developed a new autoregressive model that respects the two consistency constraints; we apply the linear regression scheme subject to linear equality constraints [11] to timeseries with long memory depth. If the absolute stability is satisfied, as we found in the TBH case, the filtering skill with consistency constraints in (2.17)-(2.18) significantly improves over those without consistency constraints. When the absolute stability is not satisfied, as we found in the L-96 case, we suggest to omit these consistency constraints and proceed as in [21].

In the future, we will study the robustness of multistep filters for various turbulent nature, as well as their application to sparse observation networks.

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APPENDIX

Appendix A

Lemmas and Proofs

In this appendix, we prove Lemma 1 and the main Theorem 1. To avoid cumbersome notation, we ignore subscript k in the exposition throughout this appendix. To assist the reader, we begin by noting a few Lemmas that will be used in proving the main theorem.

Proof of Lemma 1. Recall from Definition 2 that a method of order q satisfies the following algebraic constraints:

$$\ell \sum_{j=1}^p (j-p)^{\ell-1} a_j = (-\gamma + i\omega) \Delta t, \quad \ell = 1, \dots, q.$$

When $q = 2$, the following two constraints are satisfied:

$$\sum_{j=1}^p a_j = (-\gamma + i\omega) \Delta t, \tag{A.1}$$

$$\sum_{j=1}^p (j-p) a_j = \frac{1}{2} (-\gamma + i\omega) \Delta t. \tag{A.2}$$

The characteristic polynomial from (2.15) and its first derivative are

$$\begin{aligned} \Pi(x) &= (-1)^{p-1} \left(\sum_{j=1}^p a_j x^{j-1} + x^{p-1} - x^p \right), \\ \Pi'(x) &= (-1)^{p-1} \left(\sum_{j=1}^p (j-1) a_j x^{j-2} + (p-1) x^{p-2} - p x^{p-1} \right). \end{aligned}$$

We can use consistency conditions (A.1) and (A.2) to evaluate $\Pi(1)$ and $\Pi'(1)$.

$$\begin{aligned}
\Pi(1) &= (-1)^{p-1} \left(\sum_{j=1}^p a_j \right) \\
&= (-1)^{p-1} (-\gamma + i\omega) \Delta t, \\
\Pi'(1) &= (-1)^{p-1} \left(\sum_{j=1}^p (j-1) a_j + (p-1) - p \right) \\
&= (-1)^{p-1} \left(\sum_{j=1}^p (j-1) a_j - 1 - \sum_{j=1}^p (j-p) a_j + \left(\frac{1}{2} (-\gamma + i\omega) \Delta t \right) \right) \\
&= (-1)^{p-1} \left((p-1) \sum_{j=1}^p a_j - 1 + \frac{1}{2} (-\gamma + i\omega) \Delta t \right) \\
&= (-1)^{p-1} \left(\frac{2p-1}{2} (-\gamma + i\omega) \Delta t - 1 \right).
\end{aligned}$$

□

Lemma 2. *Given a matrix F_h associated with characteristic polynomial,*

$$\Pi(x) = (-1)^{p-1} \left(\sum_{j=1}^p a_j x^{j-1} + x^{p-1} - x^p \right),$$

eigenvectors of F_h form a Vandermonde matrix Q . Furthermore, the first row of the inverse of Q is

$$\vec{e}_1^T Q^{-1} = \frac{1}{\prod_{i=2}^p (\lambda_1 - \lambda_i)} \left((-1)^{p-1} \sigma_{p-1}, (-1)^{p-2} \sigma_{p-2}, \dots, (-1)^1 \sigma_1, (-1)^0 \sigma_0 \right),$$

where elementary symmetric polynomials

$$\sigma_i(\lambda_2, \dots, \lambda_p) = \sum_{2 \leq k_1 < \dots < k_i \leq p} \lambda_{k_1} \cdots \lambda_{k_i}.$$

Proof. For every eigenpair $(\vec{v}, \lambda)$ of F_h ,

$$F_h \vec{v} = \begin{pmatrix} 0 & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \\ a_1 & a_2 & \cdots & a_{p-1} & 1 + a_p \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{p-1} \\ v_p \end{pmatrix} = \begin{pmatrix} v_2 \\ v_3 \\ \vdots \\ v_p \\ v_p + \sum v_p a_p \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{p-1} \\ v_p \end{pmatrix}.$$

For all $i \in \{1, \dots, p\}$, $v_i = \lambda v_{i-1} = \dots = \lambda^{i-1} v_1$. Eigenvector $\vec{v}$ is a constant multiple of a Vandermonde vector $(1, \lambda, \dots, \lambda^{p-1})^T$. The Vandermonde matrix Q of eigenvectors of F_h has an explicit inverse whose first row is $\vec{e}_1^T Q$ [37]. $\square$

Lemma 3. Suppose that λ_1 is the largest eigenvalue of the characteristic matrix F_h , which coefficients $\{a_j\}_{j=1}^p$ satisfy consistency of order ≥ 1 . Then, $\lambda_1 \neq 1$.

Proof. From Lemma 2, it is obvious that the eigenvector associated with $\lambda_1 = 1$ is $\vec{v} = \vec{1}$. If matrix F_h has $\vec{1}$ as an eigenvector, then the last row sum forces $\sum_{j=1}^p a_j = 0$ but the consistency of order ≥ 1 guarantees that $\sum_{j=1}^p a_j = (-\gamma + i\omega) \Delta t$ (see Lemma 1). Contradiction, and so $\lambda_1 \neq 1$. $\square$

Lemma 4. Given the characteristic polynomial $\Pi(x)$ of F_h with coefficients a_j and eigenvalues λ_j such that $|\lambda_p| \leq \dots \leq |\lambda_1| < 1$ and the elementary symmetric polynomials σ_i defined as in Lemma 2, we can rewrite σ_i as a function of the largest eigenvalue and the polynomial coefficients as follows:

$$\sigma_i = \begin{cases} (-1)^i \sum_{j=1}^{p-i} \frac{a_{p-j}}{\lambda_1^j}, & \text{if } i \geq 1, \\ 1, & \text{if } i = 0. \end{cases}$$

Proof. The elementary symmetric polynomials σ_i are defined in Lemma 2 as

$$\sigma_i(\lambda_2, \dots, \lambda_p) = \sum_{2 \leq k_1 < \dots < k_i \leq p} \lambda_{k_1} \cdots \lambda_{k_i}.$$

For example, for $p = 5$,

$$\begin{aligned}
\sigma_0 &= 1, \\
\sigma_1 &= \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5, \\
\sigma_2 &= \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_2\lambda_5 + \lambda_3\lambda_4 + \lambda_3\lambda_5 + \lambda_4\lambda_5, \\
\sigma_3 &= \lambda_2\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_5 + \lambda_2\lambda_4\lambda_5 + \lambda_3\lambda_4\lambda_5, \\
\sigma_4 &= \lambda_2\lambda_3\lambda_4\lambda_5.
\end{aligned}$$

Recall that the characteristic polynomial of F_h can be written in two ways

$$\Pi(x) = (-1)^{p-1} \left(\sum_{j=1}^p a_j x^{j-1} + x^{p-1} - x^p \right), \quad (\text{A.3})$$

$$\Pi(x) = (\lambda_1 - x)(\lambda_2 - x) \cdots (\lambda_p - x), \quad (\text{A.4})$$

because the eigenvalues of F_h are roots of the characteristic polynomial.

Expanding (A.4), we find that

$$\Pi(x) = (-x)^p + \lambda_1 \sigma_{p-1} + \sum_{i=1}^{p-1} (-1)^{p-i} [\sigma_i + \lambda_1 \sigma_{i-1}] x^{p-i}. \quad (\text{A.5})$$

By matching the powers of x in (A.5) with (A.3), we see that

$$\begin{pmatrix}
(-1)^{p-1} \lambda_1 & & & & \\
(-1)^{p-2} & (-1)^{p-2} \lambda_1 & & & \\
& (-1)^{p-3} & (-1)^{p-3} \lambda_1 & & \\
& & \ddots & \ddots & \\
& & & (-1)^1 & (-1)^1 \lambda_1
\end{pmatrix}
\begin{pmatrix}
\sigma_{p-1} \\
\sigma_{p-2} \\
\vdots \\
\sigma_2 \\
\sigma_1
\end{pmatrix}
=
\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_{p-2} \\
a_{p-1}
\end{pmatrix}.$$

Inverting the matrix, we find that

$$\sigma_i = \begin{cases} (-1)^i \sum_{j=1}^{p-i} \frac{a_{p-j}}{\lambda_1^j}, & \text{if } i \geq 1, \\
1, & \text{if } i = 0. \end{cases}$$

□

Lemma 5. *If $|\lambda_p| \leq \dots \leq |\lambda_2| < |\lambda_1| < 1$, then as the number n of prediction steps between*

every two analysis steps increases, there exists a matrix S such that

$$F_h^n(\mathcal{I} - K_\infty G) \rightarrow \frac{\lambda_1^n S}{\prod_{j=2}^p (\lambda_j - \lambda_1)}. \quad (\text{A.6})$$

This matrix S is explicit and does not depend on λ_1^n .

Proof. In Lemma 2, F was eigendecomposed into $F = Q\Lambda Q^{-1}$. As n increases, $F = Q\Lambda^n Q^{-1}$ and $\Lambda^n \rightarrow \begin{pmatrix} \lambda_1^n & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$F^n \rightarrow \lambda_1^n Q \begin{pmatrix} \bar{e}_1^T \\ 0 \end{pmatrix} Q^{-1} = \lambda_1^n Q \begin{pmatrix} \bar{e}_1^T Q^{-1} \\ 0 \end{pmatrix}.$$

We can now use the first row of the inverse of Q , $\bar{e}_1^T Q^{-1}$ from Lemma 2, to obtain

$$Q \begin{pmatrix} \bar{e}_1^T Q^{-1} \\ 0 \end{pmatrix} = \frac{1}{\prod_{i=2}^p (\lambda_1 - \lambda_i)} \begin{pmatrix} (-1)^{p-1} \sigma_{p-1} & (-1)^{p-2} \sigma_{p-2} & \cdots & (-1)^0 \sigma_0 \\ (-1)^{p-1} \lambda_1 \sigma_{p-1} & (-1)^{p-2} \lambda_1 \sigma_{p-2} & \cdots & (-1)^0 \lambda_1 \sigma_0 \\ (-1)^{p-1} \lambda_1^2 \sigma_{p-1} & (-1)^{p-2} \lambda_1^2 \sigma_{p-2} & \cdots & (-1)^0 \lambda_1^2 \sigma_0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{p-1} \lambda_1^{p-1} \sigma_{p-1} & (-1)^{p-2} \lambda_1^{p-1} \sigma_{p-2} & \cdots & (-1)^0 \lambda_1^{p-1} \sigma_0 \end{pmatrix}.$$

Finally, we can multiply the above by $\mathcal{I} - K_\infty G$ and analytically find that

$$\begin{aligned} F_h^n(\mathcal{I} - K_\infty G) &\rightarrow \lambda_1^n Q \begin{pmatrix} \bar{e}_1^T Q^{-1} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & -K_{\infty,1} \\ 0 & 1 & \cdots & 0 & \vdots \\ \vdots & & \ddots & 1 & -K_{\infty,p-1} \\ 0 & \cdots & & 0 & 1 - K_{\infty,p} \end{pmatrix}, \\ &:= \frac{\lambda_1^n S}{\prod_{j=2}^p (\lambda_j - \lambda_1)}, \end{aligned} \quad (\text{A.7})$$

where

$$S = \begin{pmatrix} (-1)^{p-1} \sigma_{p-1} & \cdots & (-1)^1 \sigma_1 & 1 - \sum_{j=1}^p K_{\infty,j} (-1)^{p-j} \sigma_{p-j} \\ (-1)^{p-1} \lambda_1 \sigma_{p-1} & \cdots & (-1)^1 \lambda_1 \sigma_1 & \lambda_1 - \sum_{j=1}^p K_{\infty,j} (-1)^{p-j} \lambda_1 \sigma_{p-j} \\ \vdots & \ddots & \vdots & \vdots \\ (-1)^{p-1} \lambda_1^{p-1} \sigma_{p-1} & \cdots & (-1)^1 \lambda_1^{p-1} \sigma_1 & \lambda_1^{p-1} - \sum_{j=1}^p K_{\infty,j} (-1)^{p-j} \lambda_1^{p-1} \sigma_{p-j} \end{pmatrix}.$$

□

Lemma 6. Let λ_j be a root of characteristic polynomial $\Pi(x)$ such that λ_1 is simple, i.e. $\lambda_1 \neq \lambda_j$ for all $j \neq 1$. Then the scalar denominator term of (A.7) becomes

$$\prod_{j=2}^p (\lambda_j - \lambda_1) = -\Pi'(\lambda_1).$$

Proof. $\prod_{j=2}^p (\lambda_j - \lambda_1) = \lim_{x \rightarrow \lambda_1} \Pi(x)/(\lambda_1 - x) = -\Pi'(\lambda_1)$ by L'Hospital's rule. $\square$

Lemma 7. When $|\lambda_i| \leq 1$, the elementary symmetric polynomials in $\lambda_2, \dots, \lambda_p$ satisfy $|\sigma_i| \leq \binom{p-1}{i}$ and $\sum_{i=0}^{p-1} |\sigma_i| \leq 2^{p-1}$.

Proof. By definition, we have

$$|\sigma_i| \leq \sum_{2 \leq k_1 < \dots < k_i \leq p} |\lambda_{k_1} \cdots \lambda_{k_i}| \leq \sum_{2 \leq k_1 < \dots < k_i \leq p} 1 \leq \binom{p-1}{i},$$

where the second inequality is due to $|\lambda_i| \leq 1$ and the last inequality is from counting the number of terms in the summation. Then,

$$\sum_{i=0}^{p-1} |\sigma_i| \leq \sum_{i=0}^{p-1} \binom{p-1}{i} = 2^{p-1},$$

by the binomial theorem, $(1+1)^{p-1} = \sum_{i=0}^{p-1} \binom{p-1}{i} 1^i \cdot 1^{p-1-i}$. $\square$

Lemma 8. When each eigenvalue of F_h satisfies $|\lambda_i| < 1$, Lemma 7 guarantees that $\|S\|_2 < \infty$.

Proof. When $|\lambda_i| < 1$, Lemma 7 gives us a loose upper on the sum of elementary symmetric polynomials. We apply that upper bound to the careful expansion of (A.8), use triangle inequalities, rearrange terms, and set each term to its highest possible maximum given $|\lambda_i| < 1$. We obtain a very loose upper bound for $\|S\|_2$ that is nonetheless finite.

$$\|S\|_2 \leq \sqrt{\|S\|_1 \|S\|_\infty} < \infty, \quad (\text{A.8})$$

since $\|S\|_\infty \leq 2^p$ and $\|S\|_1 \leq (1 + 2^{p-1})(p-1)$. $\square$

Proof of Theorem 1. We can put an upper bound on the limit of $\|F^n(\mathcal{I} - K_{mn}G)\|_2$ when $n \gg 1$

$$\|F^n(\mathcal{I} - K_{mn}G)\|_2 \rightarrow \frac{|\lambda_1|^n \|S\|_2}{|\Pi'(\lambda_1)|} \quad (\text{A.9})$$

$$\leq \frac{|\lambda_1|^n \|S\|_2}{|\Pi'(1)| - \varepsilon |\Pi''(1)|} \quad (\text{A.10})$$

$$\rightarrow 0 \quad (\text{A.11})$$

Convergence in line (A.9) is derived step-by-step in Lemmas 2, 4, 5, and 6. If we expand $\Pi'(\lambda_1)$ as Taylor series centered around 1, then $\Pi'(\lambda_1) = \Pi'(1) + \varepsilon\Pi''(1) + \mathcal{O}(\varepsilon^2)$. Inequality (A.10) is due to the reverse triangle inequality applied to the Taylor expansion of $\Pi'(\lambda_1)$ around 1. The large denominator term in (A.10) is guaranteed by Lemma 1 when consistency is of order-2 or greater. The convergence to 0 in (A.10) is attributed to absolute stability $|\lambda_1| < 1$ and Lemma 8 that guarantees finite $\|S\|_2 < \infty$. $\square$