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ON A POWDER CONSOLIDATION PROBLEM*

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Abstract. The problem of the consolidation of an aerated fine powder under gravity is considered. The industrial relevance of the problem is discussed and a mathematical model is introduced. The mathematical structure is that of a coupled system for three unknowns, pressure, stress, and height of the powder in the (axisymmetric) bunker containing it. The system itself consists of a parabolic PDE, an ODE, and an integral equation determining a free boundary corresponding to the height of the powder. Existence and uniqueness of a solution is established. A numerical method based on a formulation of the semidiscretized problem as an index 1 DAE is proposed and implemented. The feasibility of the approach is illustrated by computational results.

Key words. consolidation, multiphase, parabolic, free boundary, DAE, integral equation

AMS subject classifications. 35K55, 65L80, 65M06, 76S05

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1. Introduction.

One important factor determining the mechanical properties of fine powders is the possible presence of an interstitial fluid, say air. Indeed, any pressure gradient clearly translates into an additional body force. In gravity flow, this force may be of the order of the weight density. One then speaks of *fluidization*, and the air-powder system essentially behaves as a liquid. This paper is concerned with the "opposite" phenomenon: *consolidation*. More precisely, consider storing some fine powder in a bunker or silo; see Figure 1.1. Inevitably, some air will get trapped during filling. The corresponding partial fluidization can have very serious and unwanted consequences in practice and may result, upon retrieval of the material, in uncontrollable flows and flooding. The excess air does diffuse through the powder and eventually escapes through the top surface. A natural question is then, How long does one have to wait for the air-powder system to settle and allow for safe handling? This is the motivation of this paper. We note that similar questions may apply in soil mechanics in general and the study of landfills in particular.

By a powder, we mean a material consisting of many individual solid particles of sizes roughly between 10^{-7}m and 10^{-5}m [13]. If the particles are much larger, then the gas can circulate nearly freely between them and thus is not an important factor. Although the present study could be generalized to any type of axisymmetric container, we consider for the sake of simplicity a vertical cylindrical bunker containing a granular material subject to gravity; see again Figure 1.1. We work under several, more restrictive, simplifying assumptions. The problem is made essentially one-dimensional by assuming all the physical variables to be uniform across horizontal

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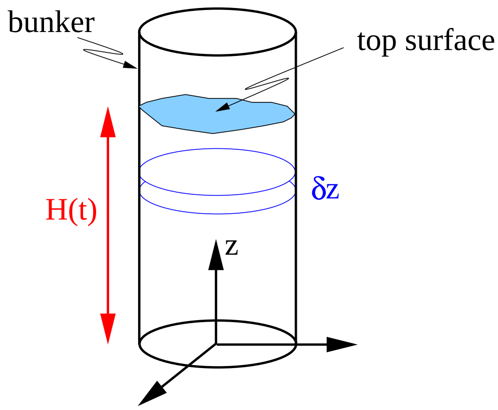

FIG. 1.1. *Geometry and coordinate systems for the vertical cylindrical bunker. The height of the column of powder of time t is denoted $H(t)$.*

sections. In general, such a simplification is clearly not fully justifiable. However, it has been shown to lead to meaningful asymptotic results in the context of a so-called Janssen analysis [6] of the behavior of columns of granular materials [13, pp. 84–90]. The present study can in fact be viewed as a generalization of Janssen's original approach to fine powders, where the presence of air cannot be neglected.

We focus here exclusively on slow flows for which the particles stay close together and grain to grain contact is sustained for long times, leading to a *frictional* model. Such an approach can be opposed to *collisional models* (see, for instance, [10], [14]), which are typically used to study fully fluidized granular materials and for which particles are spaced, deformations are rapid and the contacts are through collisions. In this latter type of models, the mechanical properties of the grains play a much reduced role.

The model directly results from classical conservation principles together with Darcy's law. The unknowns are the pressure of the gas p, the vertical stress σ, and the height of the column of powder H. The mathematical structure of the problem is nonstandard as it consists of a system of three equations: a parabolic PDE, an ODE, and an integral equation, which corresponds to a nonlocal equation for the top free boundary. The main physical assumptions roughly follow those of [8], where a Lagrangian formulation was used. This simplifies the derivation of the model but also leads to some numerical difficulties in complex situations, for instance if material is added or retrieved during the consolidation process. In [8], a rudimentary explicit numerical method was proposed and implemented. In [11], analytical solutions for consolidation problems were constructed under additional severe simplifying assumptions. A first attempt at obtaining a Eulerian formulation of the system based on assumptions similar to [8] can be found in [1]. The present model corrects some inconsistencies from [1], provides a full analysis of the problem and proposes and tests a robust and efficient numerical method.

The model is derived in section 2. In section 3, a much simpler auxiliary "toy problem" that roughly corresponds to heat conduction in an expanding rod is considered; see also [12] for a numerical study of a closely related thermal expansion model. That problem shares some, but not all, of the difficulties of the full problem, and its analysis is meant to provide a road map for what is covered in the rest of the paper. Section 4 is devoted to the analysis and numerical analysis of the full problem. Existence and uniqueness of a solution is established. A simple numerical method is proposed. Essentially, the height H is expressed as a function of the other variables. This expression is used to transform the problem into one defined in a fixed spatio-temporal domain. Then, the remaining system, now involving two unknowns, is semidiscretized in space. This results in a DAE which is solved through the use of a linearized implicit Euler method. The feasibility and efficiency of the method are illustrated in section 5, where computational experiments are presented. Finally, some closing remarks are offered in section 6.

2. The model. We denote by Γ the solid density, i.e., the actual density of the particle; Γ is assumed to be constant. The density ρ of the gas is an unknown function of the time t and the position z, i.e., the height (see Figure 1.1), under the above simplifying assumption. A most important quantity is the *bulk density* denoted by γ. The bulk density can be defined as the effective density of the granular material. More precisely, if f_s stands for the volume fraction occupied by the solid, we have

$$\gamma = f_s\Gamma + (1 - f_s)\rho. \tag{2.1}$$

Typically, the gas density ρ is at least three orders of magnitude less than Γ. Consequently, the solid fraction f_s can in fact by approximated by

$$f_s \approx \frac{\gamma}{\Gamma}.$$

The bulk density is found to increase with the application of increasing stresses. The following relation is well supported by experimental evidence [7]

$$\gamma = \gamma_m \left(1 + \frac{\sigma}{\sigma_m}\right)^{\beta}, \tag{2.2}$$

where $0 \leq \beta < 1$ is the coefficient of compressibility, $\gamma_m > 0$ and $\sigma_m > 0$ being two material constants and where σ stands for the vertical stress. Note that above formula makes sense only for "reasonable" values σ. Indeed, one does not expect γ to increase without bounds for increasingly large stresses. For instance γ should certainly not exceed Γ. Accordingly, the mathematical analysis of the problem was done under the following assumptions on γ (see Figure 2.1), which are consistent with (2.2) for σ not too large:

$$\gamma \in \mathcal{C}^{\infty}(R), \quad \gamma(0) = \gamma_m > 0, \quad 0 < \gamma(\sigma) \leq \tilde{\gamma} < \Gamma, \quad 0 < \gamma'(\sigma) \leq \gamma'_{\infty} \quad \forall\sigma. \tag{2.3}$$

In what follows, the cylindrical container is assumed to be of constant circular cross section A, but this is not essential; see section 6. The spatial coordinates are taken as the cylindrical coordinates r and z; see Figure 1.1. Under the assumption of horizontal uniformity, no angular variable is needed. We denote by $H(t)$ the height of the column of powder at time t. Since no powder escapes during the consolidation process, the total mass M of solid in the bunker is conserved. Under the above approximation of f_s, the global conservation of solid reads as

$$\int_0^{H(t)} \gamma(\sigma(z,t))\, dz = \frac{M}{\pi R^2} \equiv \tilde{M}, \qquad t \geq 0, \tag{2.4}$$

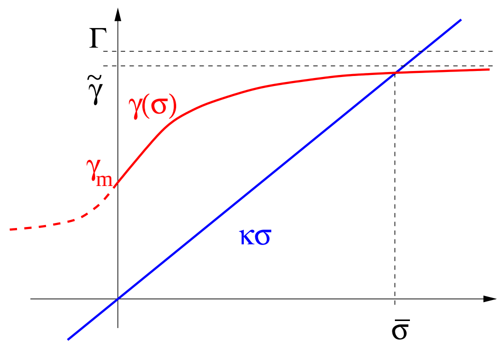

FIG. 2.1. *Graph of the bulk density γ as a function of the stress σ.*

where R denotes the radius of the container. The above equation can be considered as "the" equation for H, i.e., the relation defining the top free-boundary. Its nonlocal character should be noted. Local conservation of solid and gas yield

$$\partial_t \gamma + \partial_z(\gamma\, u_s) = 0, \tag{2.5}$$

$$\partial_t \left(\left(1 - \frac{\gamma}{\Gamma}\right) \rho \right) + \partial_z \left(\rho \left(1 - \frac{\gamma}{\Gamma}\right) u_g \right) = 0, \tag{2.6}$$

where u_s and u_g stand for the velocity of solid and gas, respectively.

The gas is assumed to be ideal and isothermal. In other words, if p denotes the pressure of the gas, we have

$$\frac{p}{\rho} = \frac{p_0}{\rho_0}, \tag{2.7}$$

where p_0 and ρ_0 are two constant reference values. Velocities and pressure are related to each other through Darcy's law, i.e.,

$$u_g - u_s = -K(\gamma)\partial_z p, \tag{2.8}$$

where $K = K(\gamma)$ is the permeability. The gas flow resulting from a pressure gradient is obviously also dependent on the bulk density. Here, we take

$$K(\gamma) = K_0 \left(\frac{\gamma}{\gamma_0}\right)^{-a}, \tag{2.9}$$

where K_0 and γ_0 are reference values and a is a positive constant. The parameters β, σ_m, γ_m, a, K_0, and γ_0 appearing in (2.2) and (2.9) can be determined experimentally.

Given the symmetry of the problem as considered, the stress tensor T has the form

$$T = \begin{bmatrix} \sigma_{rr} & \sigma_{rz} \\ \sigma_{rz} & \sigma_{zz} \end{bmatrix}.$$

In first approximation, we will assume that the vertical and horizontal stresses are principal stresses. This amounts to neglecting the contribution from σ_{rz} in T, again a questionable assumption in some cases [13, pp. 84–90].

Next, consider a balance of forces acting on an infinitesimal slice of radius R and height δz of the material; again see Figure 1.1. The various forces are as follows:

- weight of solid: $-\gamma \, \pi \, R^2 \, \delta z$;
- if τ_w is the wall shear stress, there is an upward force of $2 \, \pi \, R \, \delta z \, \tau_w$;
- pressure at bottom: $p(z)$, and top: $-(p(z)+\delta p)$; this creates a force $-\pi \, R^2 \, \delta p$;
- stress at bottom: $\sigma_{zz}(z)$, and top: $-(\sigma_{zz}(z) + \delta\sigma_{zz})$; (compressive stresses are taken as positive for granular material); this creates a force $-\pi \, R^2 \, \delta\sigma_{zz}$.

The resulting force balance gives

$$\partial_z \sigma_{zz} + \partial_z p - \frac{2}{R}\tau_w + \gamma = 0.$$

Further, by applying the law of sliding friction on the wall, one finds

$$\tau_w = \mu_w \sigma_{rr},$$

where μ_w is the coefficient of wall friction.

Finally, the two remaining components of the stress tensor T, σ_{rr} and σ_{zz}, are taken to be a constant multiple of each other:

$$\frac{\sigma_{rr}}{\sigma_{zz}} = J, \tag{2.10}$$

where J is known as the Janssen constant [13, p. 86]. If the powder is an ideal, cohesionless, Coulomb material, two extreme values of J are given by $J_{active} = \frac{1-\sin\delta}{1+\sin\delta}$ and $J_{passive} = \frac{1+\sin\delta}{1-\sin\delta}$, where δ is the angle of internal friction of the powder, i.e., the coefficient of friction is given by $\mu = \tan\delta$. The *active* state, for which $\sigma_{zz} > \sigma_{rr}$, and the *passive* state, where $\sigma_{zz} < \sigma_{rr}$, are realized if the material is at yield. In general, however, the coefficient J is determined through experimental considerations [8].

Using both (2.10) and the form of τ_w given above, we can eliminate σ_{rr} from the equations. Denoting by σ the remaining stress component σ_{zz}, we get

$$\partial_z \sigma + \partial_z p - \kappa \, \sigma + \gamma = 0, \tag{2.11}$$

where the constant κ is defined by $\kappa = 2\mu_w \, J/R$. In the absence of a pressure gradient, $\partial_z p = 0$, and in case the density γ is constant, the (elementary) resolution of (2.11) is usually referred to as Janssen's analysis [6], [13].

We now eliminate the velocities from the system. Relations (2.5) and (2.8) combine into

$$\partial_t \gamma + \partial_z (\gamma \, u_g + \gamma K \partial_z p) = 0.$$

After integration in space, and taking into account the boundary conditions $u_g(0,t) = 0$ and $\partial_z p(0,t) = 0$, we obtain

$$u_g = -\frac{1}{\gamma} \int_0^z \partial_t \gamma - K \, \partial_z p.$$

Next, using (2.7), one can rewrite (2.6) in terms of the pressure p, eliminating ρ from the problem. Plugging the expression for u_g in (2.6) then yields our last missing equation:

$$\left(1 - \frac{\gamma}{\Gamma}\right) \partial_t p - \frac{p}{\gamma} \partial_t \gamma - \partial_z \left(p \left(\frac{1}{\gamma} - \frac{1}{\Gamma} \right) \right) \int_0^z \partial_t \gamma - \left(1 - \frac{\gamma}{\Gamma}\right) \partial_z (p \, K \, \partial_z p)$$
$$+ \frac{Kp}{\Gamma} \partial_z \gamma \partial_z p = 0.$$

One last simplification is considered. The last term in the previous relation has very little influence on the problem. Although not obvious analytically, this was carefully verified numerically: for realistic values of the various parameters, the solution changes by less than .1% when switching this term on and off. Omitting this term simplifies somewhat the analysis on the one hand, and does not appear to influence the solution in any observable way on the other hand. In conclusion, and under the previous remarks and assumptions, the unknowns σ, p, and H are to be determined by the following system of three integrodifferential equations in $\{(z,t); 0 < t, 0 < z < H(t)\}$:

$$\left(1-\frac{\gamma}{\Gamma}\right)\partial_t p - \frac{p}{\gamma}\,\partial_t\gamma - \partial_z\left(p\left(\frac{1}{\gamma}-\frac{1}{\Gamma}\right)\right)\int_0^z \partial_t\gamma - \left(1-\frac{\gamma}{\Gamma}\right)\partial_z(p\,K\,\partial_z p) = 0, \tag{2.12}$$

$$\partial_z\sigma + \partial_z p - \kappa\,\sigma + \gamma = 0, \tag{2.13}$$

$$\int_0^{H(t)} \gamma(\sigma(\tilde{z},t))\,d\tilde{z} = \tilde{M}. \tag{2.14}$$

Those equations are combined with the initial and boundary conditions

$$p(\cdot,0) = p_0, \tag{2.15}$$

$$\sigma(H(t),t) = 0, \quad \partial_z p(0,t) = 0, \quad p(H(t),t) = p_{atm}, \tag{2.16}$$

where p_{atm} stands for the value of the atmospheric pressure. Note that the above condition at $z = 0$ corresponds to the bottom of the container being solid. This can be easily modified to include cases where some material exits the bunker through an outlet. This, as well as the generalization of the present approach to consolidation problems in silos of nonconstant cross section will be the object of future work.

3. An auxiliary problem. To guide us, as well as the reader, through the analysis of the above problem, an auxiliary problem that shares some of the nonstandard features encountered in the previous section is first analyzed. The problem below can be regarded as the description of the behavior of a one-dimensional rod subject to thermal expansion. Although such an interpretation is not strictly needed, one can think of the unknowns appearing below as the temperature θ, the linear density of the rod ρ, and its length at time t, $s(t)$. The problem is then

$$\begin{aligned}
\partial_t\theta - \partial_{xx}\theta &= 0, && 0 < x < s(t),\ t > 0,\\
\theta(0,t) = \theta(s(t),t) &= 0, && t > 0,\\
\theta(x,0) &= \theta_0(x), && 0 < x < s(0),\\
\int_0^{s(t)} \rho(\theta(x,t))\,dx = M &\equiv 1, && t > 0,
\end{aligned}$$

where θ_0 is a given initial condition. We start by mapping the problem into a fixed spatial domain by introducing the variables

$$y = x/s(t), \qquad u(y,t) = \theta(x,t).$$

The heat equation then turns into

$$\partial_t u - \frac{y\,s'(t)}{s(t)}\partial_y u - \frac{1}{s(t)^2}\partial_{yy} u = 0.$$

Now, $\int_0^{s(t)} \rho(\theta(x,t))\, dx = \int_0^1 \rho(u(y,t))\, s(t) dy = 1$ and thus

$$s(t) = \frac{1}{\int_0^1 \rho(u(y,t))\, dy} \quad \text{and} \quad s'(t) = \frac{-\int_0^1 \rho'(u)\partial_t u\, dy}{\left(\int_0^1 \rho(u)\, dy\right)^2}.$$

Finally, we obtain

$$\partial_t u + \frac{y}{\int_0^1 \rho(u)\, d\tilde{y}} \left(\int_0^1 \rho'(u)\, \partial_t u\, d\tilde{y}\right) \partial_y u - \left(\int_0^1 \rho(u)\, d\tilde{y}\right)^2 \partial_{yy} u = 0, \tag{3.1}$$

$$u(0,t) = u(1,t) = 0, \tag{3.2}$$

$$u(y,0) = u_0(y) \equiv \theta_0(y\, s(0)). \tag{3.3}$$

3.1. Analysis of the auxiliary problem. Let $\alpha(u) = (\int_0^1 \rho(u)\, dy)^2$ and let $\mathcal{H}(u)$ be the solution operator for the linear equation $\partial_t w - \alpha(u)\partial_{yy} w = f$. More precisely, $w = \mathcal{H}(u) f$ implies

$$\begin{aligned} \partial_t w - \alpha(u)\partial_{yy} w &= f, \\ w(0,t) = w(1,t) &= 0, \\ w(y,0) &= u_0. \end{aligned}$$

The following existence and regularity result, Lemma 3.1, is a direct consequence of [9, Theorem 5.2, p. 320]. For a nonintegral positive number α, the usual definitions and notations are used to denote the spaces of Hölder continuous functions of exponent α; see, e.g., [9, Chapter 1, section 1]. Further, $[\alpha]$ stands for the integral part of α, $(0,T)$ is the time interval over which the problem is solved, and we set $Q = [0,1] \times [0,T]$.

LEMMA 3.1. *Assume the density function ρ is of class $\mathcal{C}^1$ and is positive, uniformly bounded, and bounded away from zero. Let $\alpha > 0$ be a nonintegral number. Then if $f \in \mathcal{C}^{\alpha,\alpha/2}(\bar{Q})$, if $u_0 \in \mathcal{C}^{3+\alpha}([0,1])$ and satisfies the compatibility conditions of order $[\frac{\alpha}{2}] + 1$, there exists a unique solution w, $w \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\bar{Q})$. Moreover,*

$$\| w \|_Q^{(2+\alpha,1+\alpha/2)} \le C_T \left(\| u_0 \|_{(0,1)}^{(2+\alpha)} + \| f \|_Q^{(\alpha,\alpha/2)} \right).$$

Now, let L be defined by

$$L(u)w = w + \frac{y}{\int_0^1 \rho(u)\, d\tilde{y}} \left(\int_0^1 \rho'(u) w\, d\tilde{y} \right) \partial_y u.$$

L is a rank-one perturbation of I. It is easy to check directly that

$$L^{-1}(u) g = g - \frac{\left(\int_0^1 \rho'(u) g\, d\tilde{y}\right) y\, \partial_y u}{\int_0^1 \rho(u)\, d\tilde{y} + \int_0^1 \tilde{y}\, \partial_y u\, \rho'(u)\, d\tilde{y}} = g - \frac{\left(\int_0^1 \rho'(u) g\, d\tilde{y}\right) y\, \partial_y u}{\rho(u(1,t))}.$$

Therefore, by (3.2), $L(u)$ is nonsingular provided $\rho(u(1,t)) = \rho(0) \neq 0$. So, if u is the solution of (3.1), then

$$u = \mathcal{T}(u) \equiv \mathcal{H}(u) \left(-\frac{\alpha(u)}{\rho(0)}\, y\, \partial_y u \int_0^1 \rho'(u) \partial_{yy} u\, dy \right). \tag{3.4}$$

An existence theorem is then obtained by showing that $\mathcal{T}$ admits a fixed point.

THEOREM 3.2. *Let ρ and α be as in Lemma 3.1. Then if $u_0 \in \mathcal{C}^{3+\alpha}([0,1])$ and satisfies the compatibility conditions of order $[\frac{1+\alpha}{2}]+1$, the problem* (3.1), (3.2), (3.3) *admits a unique solution in $\mathcal{C}^{2+\alpha,1+\alpha/2}(\bar{Q})$ provided $|\,\rho'\,|_\infty \equiv \max_{x\in R}|\,\rho'(x)\,|$ and $\|\,u_0\,\|_{(0,1)}^{(2+\alpha)}$ are small enough.*

Proof. Let $X_0 = \mathcal{C}^{2+\alpha,1+\alpha/2}(\bar{Q})$ and $X_1 = \mathcal{C}^{3+\alpha,\frac{3+\alpha}{2}}(\bar{Q})$. By Lemma 3.1, one obtains $\mathcal{T} : X_0 \to X_1$, where $\mathcal{T}$ in defined in (3.4). Hence $\mathcal{T}$ is a compact mapping on X_0. Further, by Lemma 3.1

$$\|\,\mathcal{T}(u)\,\|_Q^{(2+\alpha,1+\alpha/2)} \le C_T\left(\|\,u_0\,\|_{(0,1)}^{(2+\alpha)} + c_\rho\,|\,\rho'\,|_\infty\big(\|\,u\,\|_Q^{(2+\alpha,1+\alpha/2)}\big)^2\right),$$

where c_ρ depends on ρ. For $|\,\rho'\,|_\infty$ and $\|\,u_0\,\|_{(0,1)}^{(2+\alpha)}$ small enough, we consider the set

$$\mathcal{S}_0 = \{u \in X_0; \|\,u\,\|_Q^{(2+\alpha,1+\alpha/2)} \le C_0\,\|\,u_0\,\|_{(0,1)}^{(2+\alpha)}\},$$

where $C_0 = (\frac{1}{C_T} + \sqrt{\frac{1}{C_T{}^2} - 4\,c_\rho|\,\rho'\,|_\infty\|\,u_0\,\|_{(0,1)}^{(2+\alpha)}})/\,(2\,c_\rho|\,\rho'\,|_\infty\|\,u_0\,\|_{(0,1)}^{(2+\alpha)})$. For $u \in \mathcal{S}_0$, we have

$$\|\,\mathcal{T}(u)\,\|_Q^{(2+\alpha,1+\alpha/2)} \le C_T\left(1 + c_\rho\,C_0^2\,|\,\rho'\,|_\infty\,\|\,u_0\,\|_{(0,1)}^{(2+\alpha)}\right)\,\|\,u_0\,\|_{(0,1)}^{(2+\alpha)} = C_0\,\|\,u_0\,\|_{(0,1)}^{(2+\alpha)}.$$

Therefore, $\mathcal{T}$ maps $\mathcal{S}_0$ into itself. Invoking the Schauder theorem, we get existence of a solution in X_0 for $|\,\rho'\,|_\infty$ and $\|\,u_0\,\|_{(0,1)}^{(2+\alpha)}$ small enough. Uniqueness is easily obtained through classical arguments thanks to the regularity of the above solution via an adaptation of the maximum principle; see [9, p. 22, Theorem 2.8]. ◻

3.2. Discretization of the auxiliary problem. Let $\Delta x = 1/(N+1)$ and let $x_i = i\Delta x$, $i = 0,1,\ldots,N+1$. A simple finite difference approximation of (3.1) is given by

$$\partial_t U_i + \frac{1}{S}\left(\sum_{j=1}^{N}\omega_j\rho'(U_j)\partial_t U_j\right)\frac{x_i}{2\Delta x}(U_{i+1}-U_{i-1}) + D(U)_i = 0, \qquad i = 1,\ldots,N,$$

where $S = \sum_{j=0}^{N+1}\omega_j\rho(U_j)$, ω_j's, $j = 0,1,\ldots,N+1$ are the weights of the numerical quadrature and where $U_0 = U_{N+1} = 0$. Further, in the previous relation

$$D(U) = \left(\sum_{j=0}^{N+1} w_j\rho(U_j)\right)^2 AU \in R^N,$$

where $A = (-\partial_{yy})_h \in R^{N\times N}$ is a discretization of the second order space derivative operator with homogeneous Dirichlet boundary conditions, obtained, for instance, by using the classical three-point formula, and U is the $N\times 1$-vector $[U_1\ldots U_N]$. Let $C = (\partial_y)_h \in R^{N\times N}$ be the matrix corresponding to the discretization of the convective term in (3.1); here $C_{ij} = (\delta_{i,i+1} - \delta_{i+1,i})/(2\Delta x)$, and let ω be the $N\times 1$-vector $\omega = [\omega_1\ldots\omega_N]$. One can define the discrete analog to $L(u)$ as

$$L_h(U) = I + \frac{1}{S}(xCU)(\omega\rho'(U))^T \in R^{N\times N}.$$

Note the outer product in the definition of $L_h(U)$. Using the new variable $Z = \partial_t U$, i.e., $Z_i = \partial_t U_i$, $i = 1, \dots, N$, the semidiscretized in space problem can be written as the following DAE:

$$\partial_t U = Z, \tag{3.5}$$

$$0 = L_h(U) Z + D(U). \tag{3.6}$$

By construction, $L_h(U)$ is equal to I plus a rank-one perturbation. Elementary linear algebra implies

$$L_h(U)^{-1} = I - \frac{(xCU)(\omega\rho'(U))^T}{S + (xCU)^T \omega\rho'(U)}.$$

In other words, $L_h(U)$ is nonsingular provided $S + (xCU)^T\omega\rho'(U) \neq 0$. As was the case above for the continuous problem, this nonsingularity condition can be considerably simplified and turns out to be very mild. Indeed after summation by parts and some rearrangements, one obtains, for $\omega_0 = \omega_{N+1} = \frac{\Delta x}{2}$ and $\omega_1 = \cdots = \omega_N = \Delta x$,

$$\begin{aligned} S + (xCU)^T\omega\rho'(U) &= \sum_{j=0}^{N+1} \omega_j \rho(U_j) + \sum_{j=1}^{N} x_j \frac{U_{j+1} - U_{j-1}}{2\Delta x} \omega_j \rho'(U_j) \\ &= \frac{1}{2}(\rho(U_N) + \rho(U_{N+1})) + \mathcal{O}(\Delta x \,|\, \rho'' \,|_\infty \,|\, \delta U \,|_\infty^2), \end{aligned}$$

where δU is the vector of all first divided differences. Therefore, the corresponding nonsingularity condition

$$\frac{1}{2}(\rho(U_N) + \rho(U_{N+1})) + \mathcal{O}(\Delta x \,|\, \rho'' \,|_\infty \,|\, \delta U \,|_\infty^2) \neq 0 \tag{3.7}$$

is clearly a discrete analogue to $\rho(0) \neq 0$ since $U_{N+1} = 0$.

In conclusion, our system is an index-1 DAE under condition (3.7). We apply a linearly implicit Euler discretization (see, e.g., [5, p. 427] or [2, Chapter 4]) and get

$$\begin{bmatrix} I & -\Delta t\, I \\ -\Delta t\, J^n & -\Delta t\, L_h(U^n) \end{bmatrix} \begin{bmatrix} U^{n+1} - U^n \\ Z^{n+1} - Z^n \end{bmatrix} = \Delta t \begin{bmatrix} Z^n \\ L_h(U^n) Z^n + D(U^n) \end{bmatrix}, \tag{3.8}$$

where Δt is the time step. To find the expression of J^n, we need to take the derivatives with respect to U of both $D(U)$ and $L_h(U)Z$, since $J^n = L_h'(U^n)Z^n + D'(U^n)$. This can be done by direct calculations; those expressions are omitted here.

The matrix to be inverted in (3.8) is nonsingular for Δt small enough. For the present type of index-1 DAEs, the usual convergence results turn out to be still satisfied. One easily obtains the following result.

THEOREM 3.3. *Let (U^0, Z^0) be consistent initial values, i.e., $U_i^0 = \theta_0(s(0)\, y_i)$, $i = 1, \dots, N$ and $Z^0 = -L_h(U^0)^{-1} D(U^0)$. Then, under condition* (3.7)*, one has*

$$\| \, U(t^n) - U^n \, \| + \| \, Z(t^n) - Z^n \, \| = \mathcal{O}(\Delta t).$$

Proof. See [3, Theorem 1, p. 504]. Note that the contractivity condition (1.4) in [3] is trivially satisfied here because the algebraic equation is linear in the algebraic variable Z. ∎

4. The full problem. Let us now turn to the powder problem. As in section 3, the problem is first rewritten in a constant domain. To keep the notational changes to a minimum, the original nontransformed variables from section 2 are denoted with a bar, and the new transformed variables are taken without one. The relations between them are

$$y = z/H(t), \qquad p(y,t) = \bar{p}(z,t), \qquad \sigma(y,t) = \bar{\sigma}(z,t).$$

When rewriting the pressure equation (2.12) in terms of the new variables, both $H(t)$ and $H'(t)$ appear in the terms involving time derivatives. Notice that $\tilde{M} = \int_0^{H(t)} \gamma(\bar{\sigma}(z,t))\,dz = \int_0^1 \gamma(\sigma(y,t))H(t)\,dy$, and thus

$$H(t) = \frac{\tilde{M}}{\int_0^1 \gamma(\sigma(y,t))\,dy} \quad \text{and} \quad H'(t) = -\tilde{M}\frac{\int_0^1 \gamma'(\sigma)\partial_t\sigma\,dy}{\left(\int_0^1 \gamma(\sigma(y,t))\,dy\right)^2}. \tag{4.1}$$

Having "solved" the equation for H, we substitute and get

$$\begin{aligned} &\left(1 - \frac{\gamma(\sigma)}{\Gamma}\right)\left(\partial_t p + \frac{A(\gamma'(\sigma)\partial_t\sigma)}{A(\gamma(\sigma))}\,y\,\partial_y p\right) \\ &-\frac{p}{\gamma(\sigma)}\left(\partial_t\gamma(\sigma) + \frac{A(\gamma'(\sigma)\partial_t\sigma)}{A(\gamma(\sigma))}\,y\,\partial_y\gamma(\sigma)\right) \\ &-\partial_y\left(p\left(\frac{1}{\gamma(\sigma)} - \frac{1}{\Gamma}\right)\right)\int_0^y\left(\partial_t\gamma(\sigma) + \frac{A(\gamma'(\sigma)\partial_t\sigma)}{A(\gamma(\sigma))}\,\tilde{y}\,\partial_y\gamma(\sigma)\right)d\tilde{y} \\ &-\frac{1}{\tilde{M}^2}(1 - \frac{\gamma(\sigma)}{\Gamma})\left(\int_0^1 \gamma(\sigma)\,dy\right)^2 \partial_y(pK\partial_y p) = 0, \end{aligned} \tag{4.2}$$

where $A(u) = \frac{1}{\tilde{M}}\int_0^1 u(y)\,dy$. The stress equation becomes

$$\partial_y\sigma + \partial_y p + \frac{\tilde{M}}{\int_0^1 \gamma(\sigma)\,dy}(-\kappa\sigma + \gamma) = 0. \tag{4.3}$$

4.1. Analysis of the full problem. Define the integral operators

$$B(u) = \int_y^1 u(z)\,dz \quad \text{and} \quad \tilde{B}(u) = \int_0^y u(z)\,dz.$$

The boundary conditions on the transformed variables are

$$\partial_y p(0,t) = 0, \quad p(1,t) = p_{atm}, \quad \text{and} \quad \sigma(1,t) = 0.$$

The initial condition on p is

$$p(y,0) = p_0(y), \qquad y \in (0,1).$$

Hereafter, the above problem is rewritten as a parabolic problem for the stress σ, and thus an initial condition σ_0 will have to be provided. It is found from (4.3). In other words, the initial stress σ_0 is taken as the solution to

$$\begin{aligned} \partial_y\sigma_0 + \partial_y p_0 + f(\sigma_0) = 0 &\qquad \text{for } 0 < y < 1, \\ \sigma_0(1) = 0, & \end{aligned}$$

where $f(\sigma) = \frac{\tilde{M}}{\int_0^1 \gamma(\sigma(\tilde{y}))\, d\tilde{y}}\big(\gamma(\sigma) - \kappa\sigma\big)$.

LEMMA 4.1. *Let $p_0 \in \mathcal{C}^1([0,1])$ with $|\, p_0'(y)\, | \le p_\infty'$ for any y, $0 < y < 1$. Then if γ satisfies* (2.3)*, there exists a unique function σ_0 verifying the above requirements.*

Proof. We construct a sequence $\{\sigma_0^n\}_{n\ge 0}$, $\sigma_0^0 \equiv 0$, by successively solving

$$\begin{aligned} \partial_y \sigma_0^n + \partial_y p_0 + f_n(\sigma_0^n) &= 0 \qquad \text{for } 0 < y < 1, \\ \sigma_0^n(1) &= 0, \end{aligned}$$

where $f_n(\sigma) = \frac{\tilde{M}}{\int_0^1 \gamma(\sigma_0^{n-1}(\tilde{y}))\, d\tilde{y}}\big(\gamma(\sigma) - \kappa\sigma\big)$. The function f_n is Lipschitz in σ with Lipschitz constant $L = \frac{\tilde{M}}{\gamma_m}(\gamma_\infty' - \kappa)$. Therefore, there exists a unique function σ_0^n, and the above sequence is well defined. By using the bounds

$$f^-(\sigma) \equiv -\frac{\kappa\tilde{M}}{\tilde{\gamma}}\sigma \le f_n(\sigma) \le \frac{\tilde{M}}{\gamma_m}(\tilde{\gamma} - \kappa\sigma) \equiv f^+(\sigma)$$

in place of $f_n(\sigma)$ in the above problem, one can explicitly construct uniformly bounded in n sub and supersolutions to the above problem. Hence, the sequence $\{\sigma_0^n\}$ is bounded uniformly in n, γ admitting upper and lower bounds by assumption; see (2.3). By construction, $\{\partial_y\sigma_0^n\}$ is consequently also uniformly bounded. Therefore, the sequence $\{\sigma_0^n\}$ is equicontinuous, and thus by Ascoli–Arzelà's theorem one can select from it a subsequence converging uniformly to a Lipschitz function σ_0 which solves the above problem.

Uniqueness follows from classical arguments due to the local Lipschitz property of f as a function of σ and to the regularity and boundedness of the solution to the above problem. $\square$

By integrating the stress equation (4.3) between y and 1, $0 \le y < 1$, we obtain

$$p = P(\sigma) \equiv p_{atm} - \sigma + \frac{1}{A(\gamma(\sigma))} B(-\kappa\sigma + \gamma(\sigma)). \tag{4.4}$$

Taking the derivative with respect to t yields

$$\partial_t p = -\partial_t\sigma - \frac{A(\gamma'(\sigma)\partial_t\sigma)}{A(\gamma(\sigma))^2} B(-\kappa\sigma + \gamma(\sigma)) + \frac{1}{A(\gamma(\sigma))} B(-\kappa\partial_t\sigma + \gamma'(\sigma)\partial_t\sigma). \tag{4.5}$$

The next step in the analysis consists of eliminating one of the two unknowns p or σ to obtain an integral equation similar to (3.1). Here, the pressure p will be eliminated to obtain a problem in terms of the stress σ only. This requires some manipulations of integral equations involving Volterra and rank-one operators.

By a Volterra operator, we mean an integral operator of the form

$$Vu(y) = \int_0^y v(y,z)u(z)\, dz \text{ or } Vu(y) = \int_y^1 v(y,z)u(z)\, dz,$$

where the kernel v is assumed to be in L^2. It is well known that the spectrum of a Volterra operator is the single point 0. Rank-one operators can be expressed as

$$Ru(y) = (r_l \otimes r_r)u(y) = r_l(y)(r_r, u) = r_l(y)\int_0^1 r_r(z)u(z)\, dz,$$

where $(\cdot,\cdot)$ denotes the $L^2(0,1)$ inner product.

LEMMA 4.2. *Let V be a Volterra operator, $R = r_l \otimes r_r$ be a rank-one operator, and $L = I + V + R$. Then L is nonsingular if and only if*

$$1 + (r_r, (I+V)^{-1} r_l) \neq 0, \tag{4.6}$$

in which case

$$L^{-1} = \left(I - \frac{((I+V)^{-1} r_l) \otimes r_r}{1 + (r_r, (I+V)^{-1} r_l)} \right) (I+V)^{-1}.$$

Proof. Under the above assumption, $(I+V)$ is clearly nonsingular; in fact, $(I+V)^{-1}$ is equal to its Neumann series $\sum_{n=0}^{\infty} (-1)^n V^n$. Therefore, the result is a simple consequence of the Sherman–Morrison–Woodbury formula; see, e.g., [4, p. 51]. ∎

Using the above result, one can thus write

$$(I + V + R)^{-1} - I = \bar{V} + \bar{R},$$

where $\bar{V}$ and $\bar{R}$ are themselves respectively Volterra and rank-one operators. We define the following Volterra and rank-one maps

$$V(u) = -\frac{1}{A(\gamma(\sigma))} B(-\kappa u + \gamma'(\sigma) u) \quad \text{and} \quad R(u) = \frac{A(\gamma'(\sigma) u)}{A(\gamma(\sigma))^2} B(-\kappa\sigma + \gamma).$$

By setting $L = I + V + R$, (4.5) takes the form $\partial_t p = L(-\partial_t \sigma)$. The rank-one map R can be written as $R = r_l \otimes r_r$ with

$$r_l = \frac{B(-\kappa\sigma + \gamma)}{\tilde{M}\, A(\gamma(\sigma))^2} \quad \text{and} \quad r_r = \gamma'(\sigma).$$

Since by (2.3), the kernel of V is bounded, one sees that the nonsingularity condition of L coming from (4.6) reads here as follows. If b denotes

$$(I+V)^{-1} r_l = \sum_{n=0}^{\infty} (-V)^n r_l,$$

i.e., the unique solution of

$$b - \frac{1}{A(\gamma(\sigma))} \int_y^1 (\gamma'(\sigma) - \kappa) b \, d\tilde{y} = \frac{1}{\tilde{M}\, A^2(\gamma(\sigma))} \int_y^1 (\gamma(\sigma) - \kappa\sigma) \, d\tilde{y},$$

then L is not singular provided

$$1 + \int_0^1 \gamma'(\sigma) b \, d\tilde{y} \neq 0. \tag{4.7}$$

The next step consists of rewriting the full problem (4.2), (4.3) in term of σ alone. The terms containing unintegrated time derivatives in the pressure equation (4.2) are

$$(1 - \gamma/\Gamma)\partial_t p - \frac{p}{\gamma} \partial_t \gamma = -\alpha(\sigma)\partial_t \sigma - \left(1 - \frac{\gamma}{\Gamma}\right) (V + R)\partial_t \sigma,$$

where $\alpha(\sigma) = (1 - \gamma/\Gamma) + \frac{P(\sigma)}{\gamma(\sigma)} \gamma'(\sigma)$ and where P is defined in (4.4). Using (4.3), the diffusive term takes the form

$$\begin{aligned} &\left(1 - \frac{\gamma(\sigma)}{\Gamma}\right) A(\gamma(\sigma))^2 \, \partial_y (pK\partial_y p) \\ &\quad = -\left(1 - \frac{\gamma(\sigma)}{\Gamma}\right) A(\gamma(\sigma))^2 \, \partial_y (P(\sigma) K \partial_y \sigma) + \alpha(\sigma)\, Q(\sigma) \end{aligned}$$

with

$$Q(\sigma) = -\frac{A(\gamma(\sigma))}{\alpha(\sigma)} \left(1 - \frac{\gamma(\sigma)}{\Gamma}\right) \partial_y (P(\sigma) K(\gamma(\sigma) - \kappa\sigma)).$$

The problem can then be rewritten as

$$\alpha(\sigma)\tilde{L}(\partial_t \sigma) = \left(1 - \frac{\gamma(\sigma)}{\Gamma}\right) A(\gamma(\sigma))^2 \, \partial_y (P(\sigma) K \partial_y \sigma) - \alpha(\sigma)\, Q(\sigma),$$

where $\tilde{L} = I + \tilde{V} + \tilde{R}$ with

$$\begin{aligned}
\tilde{V}(u) &= \frac{1}{\alpha(\sigma)} \left(1 - \frac{\gamma(\sigma)}{\Gamma}\right) V(u) \\
&\quad + \frac{1}{\alpha(\sigma)} \partial_y \left(P(\sigma) \left(\frac{1}{\gamma(\sigma)} - \frac{1}{\Gamma}\right)\right) \tilde{B}\left(\gamma'(\sigma) u + \frac{A(\gamma'(\sigma)u)}{A(\gamma(\sigma))} \, \tilde{y}\, \partial_y \gamma(\sigma)\right), \\
\tilde{R}(u) &= \frac{1}{\alpha(\sigma)} \left(1 - \frac{\gamma(\sigma)}{\Gamma}\right) R(u) \\
&\quad + \frac{1}{\alpha(\sigma)} \frac{P(\sigma)}{\gamma(\sigma)} \frac{A(\gamma'(\sigma)u)}{A(\gamma(\sigma))} \, y\, \partial_y \gamma(\sigma) - \frac{1}{\alpha(\sigma)} \left(1 - \frac{\gamma(\sigma)}{\Gamma}\right) \frac{A(\gamma'(\sigma)u)}{A(\gamma(\sigma))} \, y\, \partial_y P(\sigma).
\end{aligned}$$

We have implicitly assumed the coefficient $\alpha(\sigma)$ to be nonzero; this will be justified below where it will be shown under what condition $\alpha(\sigma)$ is positive. The rank-one map $\tilde{R}$ can be expressed as $\tilde{R} = \tilde{r}_l \otimes \tilde{r}_r$, where

$$\begin{aligned}
\tilde{r}_l &= \frac{1}{\alpha(\sigma)} \left(1 - \frac{\gamma(\sigma)}{\Gamma}\right) r_l + \frac{1}{\alpha(\sigma)} \frac{P(\sigma)}{\gamma(\sigma)} \frac{y\, \partial_y \gamma}{\tilde{M}\, A(\gamma(\sigma))} - \frac{1}{\alpha(\sigma)} \left(1 - \frac{\gamma(\sigma)}{\Gamma}\right) \frac{y\, \partial_y P(\sigma)}{\tilde{M}\, A(\gamma(\sigma))}, \\
\tilde{r}_r &= r_r = \gamma'(\sigma).
\end{aligned}$$

Assuming

$$1 + (\tilde{r}_r, (I + \tilde{V})^{-1} \tilde{r}_l) \neq 0, \tag{4.8}$$

the operator $\tilde{L}$ can be inverted using Lemma 4.2, leading to

$$\partial_t \sigma - \eta(\sigma) \partial_y (P(\sigma) K \partial_y \sigma) + Q(\sigma) = F(\sigma), \tag{4.9}$$

where

$$\begin{aligned}
\eta(\sigma) &= \frac{1}{\alpha(\sigma)} \left(1 - \frac{\gamma(\sigma)}{\Gamma}\right) A(\gamma(\sigma))^2, \\
F(\sigma) &= (\bar{V} + \bar{R}) \Big(\eta(\sigma) \partial_y (P(\sigma) K \partial_y \sigma) - Q(\sigma)\Big).
\end{aligned}$$

In the above equation (4.9) for σ, it is easily checked that the diffusive coefficient is positive provided $P(\sigma)$ stays positive. However, as can be seen from (4.4), $P(\sigma)$ is positive only for small enough values of σ, due to the growth condition on γ; see (2.2), (2.3). Accordingly, we first regularize (4.9) to ensure parabolicity, establish existence of a solution to the corresponding problem, and show under what conditions the result extends to the original nonregularized problem.

Considering P from (4.4) as an operator from $\mathcal{C}([0,1] \times [0,T])$ into itself, we define a "regularized" P_ϵ as follows. If $\bar{\sigma}$ stands for the unique solution of $\gamma(\sigma) = \kappa\sigma$ (see

Figure 2.1), then we set $\sigma^\star = \min(p_{atm}, \bar{\sigma})$. Further, for any $\sigma \in \mathcal{C}([0,1]\times[0,T])$ and for some $\epsilon > 0$, we define

$$S_\epsilon^\star = \{(y,t) \in [0,1]\times[0,T]; \sigma(y,t) < \sigma^\star - \epsilon\}.$$

The regularized function P_ϵ is taken as

$$\begin{aligned}
&P_\epsilon(\sigma)|_{S_\epsilon^\star} = P(\sigma)|_{S_\epsilon^\star},\\
&P_\epsilon(\sigma) \geq \epsilon \quad \forall \sigma \in \mathcal{C}([0,1]\times[0,T]),\\
&P_\epsilon \text{ is a smooth function of } \sigma.
\end{aligned}$$

The corresponding regularized problem is then

$$\partial_t \sigma - \eta P_\epsilon K \partial_{yy}\sigma - \eta \partial_y (P_\epsilon K)\partial_y \sigma + Q_\epsilon(\sigma) = F_\epsilon(\sigma), \tag{4.10}$$

$$\partial_y \sigma(0,t) + \frac{\tilde{M}}{\int_0^1 \gamma(\sigma(\tilde{y},t))\, d\tilde{y}} \Big(\gamma(\sigma(0,t)) - \kappa\sigma(0,t) \Big) = 0, \tag{4.11}$$

$$\sigma(1,t) = 0, \tag{4.12}$$

$$\sigma(y,0) = \sigma_0(y), \tag{4.13}$$

with Q_ϵ and F_ϵ defined in a natural way, P being replaced by P_ϵ. The condition (4.11) at $y=0$ is a direct consequence of (4.3) and $\partial_y p(0,t) = 0$.

LEMMA 4.3. *Let $0 < \alpha < 1$, and let the nonsingularity conditions* (4.7) *and* (4.8) *be satisfied. Let $p_0, \sigma_0 \in \mathcal{C}^{3+\alpha}([0,1])$, where σ_0 satisfies the compatibility condition of order* 1. *Then, for $T > 0$ small enough and under the assumptions of Lemma* 4.1, *the regularized problem* (4.10)–(4.13) *admits a solution σ_ϵ in $\mathcal{C}^{2+\alpha,1+\alpha/2}([0,1]\times[0,T])$ provided γ'_∞, K_0, Γ and $\|\sigma_0\|_{(0,1)}^{(2+\alpha)}$ are small enough.*

Proof. As was done for the auxiliary problem, a linearized solution operator corresponding to the above regularized problem is constructed. More precisely, for a given $u \in X_0 = \mathcal{C}^{2+\alpha,1+\alpha/2}(\bar{Q})$, consider $\mathcal{T}_\epsilon(u) = \sigma$, where σ satisfies

$$\begin{aligned}
\partial_t\sigma - \eta(u)P_\epsilon(u)K(u)\partial_{yy}\sigma - \eta(u)\partial_y(P_\epsilon(u)K(u))\partial_y u + Q_\epsilon(u) &= F_\epsilon(u),\\
\partial_y\sigma(0,t) + \frac{\tilde{M}}{\int_0^1 \gamma(u(\tilde{y},t))\, d\tilde{y}}\Big(\gamma(u(0,t)) - \kappa\sigma(0,t)\Big) &= 0,\\
\lambda\,\partial_y\sigma(1,t) + \sigma(1,t) - \lambda\,\partial_y u(1,t) &= 0,\\
\sigma(y,0) &= \sigma_0(y).
\end{aligned}$$

In order to avoid having to treat a mixed Dirichlet–Robin problem, the given condition at $y=1$, i.e., $\sigma(1,t) = 0$, has been rewritten; the coefficient $\lambda \neq 0$ is to be chosen below. Note that when $u = \sigma$ is a fixed point of $\mathcal{T}_\epsilon$, then $\sigma(1,t) = 0$ and thus the boundary value is correctly recovered.

Using classical regularity results (see, e.g., [9, p. 320, Theorem 5.3]), one obtains the existence of a unique solution $\sigma = \mathcal{T}_\epsilon(u)$ to the above linearized problem. Further, after relating the $(1+\alpha)/2$-norm of the boundary values $u(0,\cdot)$ and $u(1,\cdot)$ to $\|u\|_{(0,1)\times(0,T)}^{(2+\alpha,1+\alpha/2)}$ and absorbing a few constant in C_T, we obtain

$$\|\mathcal{T}_\epsilon(u)\|_{X_0} \leq C_T\left(\|\sigma_0\|_{(0,1)}^{(2+\alpha)} + K_0\left(\|u\|_{(0,1)\times(0,T)}^{(2+\alpha,1+\alpha/2)}\right)^2 + \gamma'_\infty\Gamma\|u\|_{(0,1)\times(0,T)}^{(2+\alpha,1+\alpha/2)}\right),$$

where λ was chosen so that the contributions from both boundary terms "match" in the above inequality. Therefore, $\mathcal{T}_\epsilon$ maps X_0 in $X_1 = \mathcal{C}^{3+\alpha,\frac{3+\alpha}{2}}(\bar{Q})$ and is thus a compact mapping on X_0. Proceeding along the lines of the proof of Theorem 3.2, one can again show, through the Schauder theorem, the existence of a fixed point $\sigma_\epsilon = \mathcal{T}_\epsilon(\sigma_\epsilon)$ for γ'_∞, K_0, Γ, and $\| \sigma_0 \|_{(0,1)}^{(2+\alpha)}$ small enough. The function σ_ϵ clearly satisfies (4.10)–(4.13). ▯

The restriction $\alpha < 1$ in the statement of Lemma 4.3 is not essential. Note that the local character of the existence results in this section is a consequence of the two nonsingularity assumptions (4.7) and (4.8). Unlike the corresponding relation in section 3, these do not easily lead to explicit conditions on the data.

Clearly now, using again the classical regularity result invoked in the latter proof, for small enough data the largest value of σ_ϵ will not exceed $\sigma^\star - \epsilon$, threshold above which the regularization of P takes place. Uniqueness of the solution follows from the same argument as in the proof of Theorem 3.2. We have thus established the following existence result for the main problem.

THEOREM 4.4. *If* $T > 0$ *is small enough and under the assumptions of Lemma* 4.3*, there exists* $\sigma \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\bar{\Xi})$*,* $p \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\bar{\Xi})$*, and* $H \in \mathcal{C}^{1+\alpha/2}([0,T])$ *which satisfy* (2.12)–(2.16)*, where* $\Xi = \{(z,t); 0 < t < T, 0 < z < H(t)\}$.

4.2. Discretization of the full problem. The problem is discretized under its formulation (4.2), (4.3), i.e., we work with the transformed variables in a fixed rectangular domain $(0,1) \times (0,T)$. Due to the different type of boundary conditions, the mesh related quantities are slightly modified from section 3. We set $\Delta x = 1/N$ and let $x_i = (i-1)\Delta x$, $i = 1, \dots, N+1$. The semidiscretized variables are the pressure $P(t) = [P_1(t), \dots, P_N(t)]$ and the stress $\Sigma(t) = [\Sigma_1(t), \dots, \Sigma_N(t)]$. The boundary conditions at $y = 1$ read $P_{N+1} = p_{atm}$ and $\Sigma_{N+1} = 0$. For both pressure and stress variables, the spatial discretization is taken to be second order. More precisely, we introduce the following discretized operators:

C_p: $N \times N$ matrix corresponding to a second order centered discretization of ∂_y with pressure boundary conditions (at $y = 0$ and $y = 1$),

D: $N \times N$ matrix corresponding to a second order centered discretization of ∂_{yy} with pressure boundary conditions (at $y = 0$ and $y = 1$),

C_σ: $N \times N$ matrix corresponding to a second order backward differentiation formula (BDF) [5] for ∂_y with stress boundary condition (at $y = 1$).

The construction of C_p and D is elementary. For C_σ, we take

$$C_\sigma = \frac{1}{2\Delta x} \begin{bmatrix} -3 & 4 & -1 & 0 & \dots & \dots & 0 \\ 0 & -3 & 4 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -3 & 4 & -1 & 0 \\ 0 & \dots & \dots & 0 & -3 & 4 & -1 \\ 0 & \dots & \dots & \dots & 0 & -3 & 4 \\ 0 & \dots & \dots & \dots & \dots & 0 & -2 \end{bmatrix}.$$

The good stability properties of the above BDF discretization used for the stress variable are especially important in complex situations for which κ (see (2.11) and (4.3)) may be discontinuous, as would be the case in a bunker with variable, nonsmoothly changing cross section (for instance, one consisting of both a conical part and a vertical standpipe). We also introduce discretized counterparts to the integral operators A and $\tilde{B}$,

$$A_\Delta(W) = \frac{1}{\tilde{M}} \sum_{j=1}^{N+1} \omega_j W_j \quad \text{and} \quad (\tilde{B}_\Delta(W))_i = \sum_{j=1}^{i} \omega_j W_j.$$

Finally, we introduce some notation for the time derivative of the two main unknowns P and Σ by setting $U = \partial_t P$ and $V = \partial_t \Sigma$. The discretized problem is then

$$\partial_t P = U \tag{4.14}$$

$$\begin{aligned} 0 = &\left(1 - \frac{\gamma}{\Gamma}\right)\left(U + \frac{A_\Delta(\gamma' V)}{A_\Delta(\gamma)} Y\, C_p P\right) - \frac{P}{\gamma}\left(\gamma' V + \frac{A_\Delta(\gamma' V)}{A_\Delta(\gamma)} \gamma'\, Y\, C_\sigma \Sigma\right) \\ &- \left(\left(\frac{1}{\gamma} - \frac{1}{\Gamma}\right) C_p P - \frac{P\gamma'}{\gamma^2} C_\sigma \Sigma\right)\left(\tilde{B}_\Delta(\gamma' V) + \frac{A_\Delta(\gamma' V)}{A_\Delta(\gamma)} \tilde{B}_\Delta(\gamma'\, Y\, C_\sigma \Sigma)\right) \\ &+ \left(1 - \frac{\gamma}{\Gamma}\right) A_\Delta(\gamma)^2 D(P, \Sigma) P + bc1 \equiv f(P, U, V, \Sigma), \end{aligned} \tag{4.15}$$

$$0 = C_\sigma \Sigma + C_p P + bc2 + \frac{1}{A_\Delta(\gamma)}\left(-\kappa \Sigma + \gamma\right) \equiv g(P, U, V, \Sigma), \tag{4.16}$$

$$\partial_t \Sigma = V, \tag{4.17}$$

where γ and γ' are to be understood as the vectors $\gamma(\Sigma)$ and $\gamma'(\Sigma)$ and where $Y = [x_1, \ldots, x_n]$. The two vectors $bc1$ and $bc2$ result from the presence of boundary conditions. Finally, the vector-vector multiplications in the above expressions are to be understood component by component.

The structure of (4.14)–(4.17) is slightly more complicated than that of the semidiscretized problem (3.5), (3.6). More precisely, (4.14)–(4.17) has the form

$$\begin{aligned} \partial_t P &= U, \\ 0 &= f(P, U, V, \Sigma), \\ 0 &= g(P, 0, 0, \Sigma), \\ \partial_t \Sigma &= V. \end{aligned}$$

The above system corresponds to a semiexplicit index 2 DAE or equivalently to a fully implicit index 1 DAE [5, sections VII.3, VII.4]. Indeed, it is clearly equivalent to

$$0 = f(P, \partial_t P, \partial_t \Sigma, \Sigma), \tag{4.18}$$

$$0 = g(P, 0, 0, \Sigma), \tag{4.19}$$

which is the discrete counterpart to (4.2), (4.3). Note that $g_1 = C_p$ being nonsingular, one can solve (4.19) into $P = \mathcal{P}(\Sigma)$, which corresponds to (4.4). Now, by differentiating (4.19), we get

$$g_1 \partial_t P + g_4 \partial_t \Sigma = 0.$$

This leads to

$$\partial_t P = -C_p^{-1} g_4 \partial_t \Sigma, \tag{4.20}$$

which is a discretized version of (4.5). Using again the Sherman–Morrison–Woodbury formula, one can check that g_4 is also nonsingular provided that

$$C_\sigma + \frac{1}{A_\Delta(\gamma)}\left(-\kappa I + \mathrm{diag}(\gamma'(\Sigma))\right) \quad \text{is not singular}, \tag{4.21}$$

$$1 - \frac{1}{\tilde{M} A_\Delta(\gamma)^2}(\gamma(\Sigma) - \kappa\Sigma)^T(\omega\gamma'(\Sigma)) \neq 0. \tag{4.22}$$

It is easy to verify directly from the matrix structures that for Δx small enough (4.21) is satisfied. Condition (4.22) is the counterpart here to (3.7) for the auxiliary problem and (4.7) for the nondiscretized full problem. Note that it is satisfied if $\max_i \omega_i \gamma'_\infty$ is small enough. Finally, plugging (4.20) into (4.18), we get

$$0 = f(\mathcal{P}(\Sigma), -C_p^{-1} g_4 \partial_t \Sigma, \partial_t \Sigma, \Sigma).$$

Under an involved nonsingularity condition that is the discrete counterpart to (4.8), the latter equation can be solved for $\partial_t \Sigma$. It is thus a fully implicit index 1 DAE. The time discretization of (4.14)–(4.17) is again taken as linearly implicit Euler, which reads here as

$$\begin{bmatrix} I & -\Delta t\, I & 0 & 0 \\ -\Delta t\, f_1^n & -\Delta t\, f_2^n & -\Delta t\, f_3^n & -\Delta t\, f_4^n \\ -\Delta t\, g_1^n & -\Delta t\, g_2^n & -\Delta t\, g_3^n & -\Delta t\, g_4^n \\ 0 & 0 & -\Delta t\, I & I \end{bmatrix} \begin{bmatrix} P^{n+1} - P^n \\ U^{n+1} - U^n \\ V^{n+1} - V^n \\ \Sigma^{n+1} - \Sigma^n \end{bmatrix} = \Delta t \begin{bmatrix} U^n \\ f^n \\ g^n \\ V^n \end{bmatrix}.$$

In the previous relation, the subscripts denote derivatives with respect to the corresponding variables. All the first partial derivatives of f and g are needed. For f, we note that $f_2^n = \mathrm{diag}(1 - \gamma(\Sigma^n)/\Gamma)$; the rest of the terms are evaluated numerically through finite differences. For g, we obtain

$$\begin{aligned} g_1^n &= C_p, \\ g_2^n &= g_3^n = 0, \\ g_4^n &= C_\sigma + \frac{1}{A_\Delta(\gamma)} \left(-\kappa\, I + \mathrm{diag}(\gamma'(\Sigma^n))\right) - \frac{1}{\tilde{M}\, A_\Delta(\gamma)^2} (-\kappa \Sigma + \gamma(\Sigma))(\omega \gamma'(\Sigma))^T. \end{aligned}$$

Note the outer product in g_4^n. The position of the free boundary can be determined a posteriori through (4.1), i.e., $H^n = \frac{1}{A_\Delta(\gamma(\Sigma^n))}$.

The above time discretization is simple, performs remarkably well for the problems at hand (see section 5), and is at least partially supported by theoretical results (see, e.g., [15], [5, Chap. VII]). (We do not pursue this issue further here.) More involved methods, such as higher order discretizations and/or time step adaption [2], [5] could also be considered but do not appear to be necessary for the present diffusion dominated phenomenon.

5. Computational results. We now illustrate the feasibility and efficiency of the above numerical method by discussing some computational results. The bulk density-stress relationship is taken as (2.2). The following values of the various parameters have been used and were chosen so as to correspond to a realistic situation:

$$\begin{aligned} &\beta_m = .25, && \Gamma = 200 \text{ lbs/ft}^3, \\ &\gamma_m = 60 \text{ lbs/ft}^3, && K_0 = 10^{-4} \text{ ft}^4\text{lbs}^{-1}\text{s}^{-1}, \\ &\sigma_m = 13 \text{ lbs/ft}^2, && a = 4, \\ &\gamma_0 = 80 \text{ lbs/ft}^3, && \kappa = 1 \text{ ft}^{-1}. \end{aligned}$$

The atmospheric pressure is $p_{atm} = 2116.8$ lbs/ft^2. The problem is initialized as follows. The initial height of the powder column is prescribed; here $H(0) = 5$ ft. The bunker is assumed to have a unit cross section ($R \approx .56$ ft). A initial pressure field p_0 which is consistent with the boundary conditions in (2.16) is given as

$$p_0(z) = p_{atm} \left(1 + \frac{H(0)}{100}\left(1 - \left(\frac{z}{H(0)}\right)^2\right)\right).$$

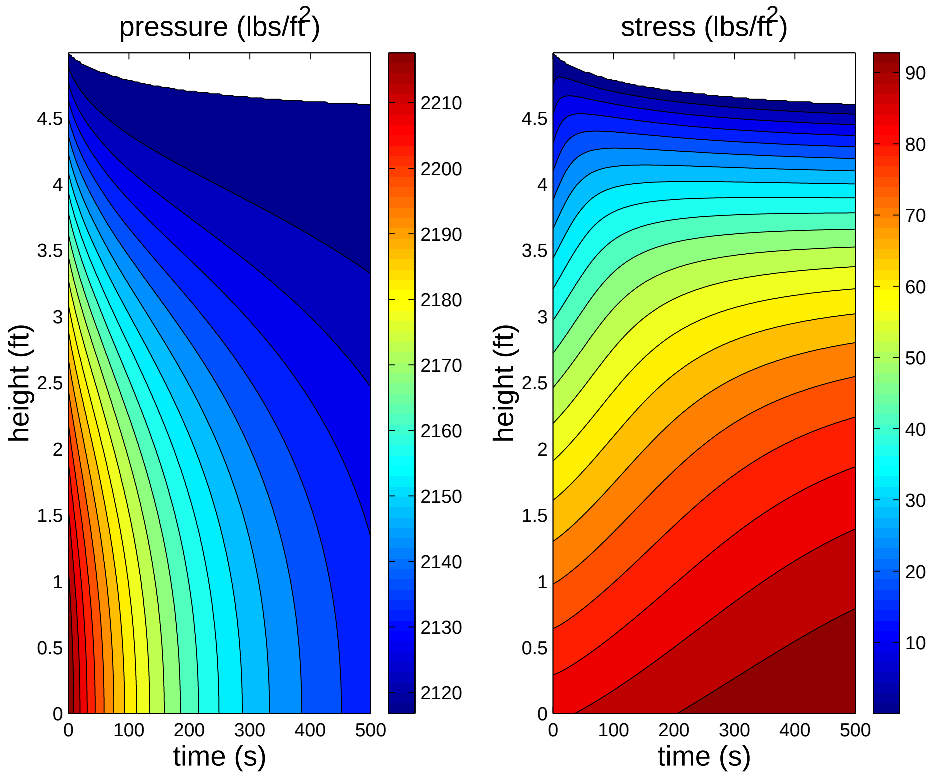

FIG. 5.1. *Calculated pressure and stress fields in the original geometry;* $N = 100$, $NT = 100$, $T = 500s$.

In the calculations below, we use $N = 100$ and $NT = T/\Delta t = 100$.

We are aware of very few reports of experimental results for this problem; see, e.g., [8], [11]. Typically, there is not sufficient data in those papers to determine the parameters for our model that would correspond to the reported experiments. Even so, with the choice of parameters given above, our computations agree qualitatively with the experimental data in [8], [11].

Figure 5.1 gives a typical illustration of the behavior of the problem. Note that Figure 5.1 is related to the original geometry of the problem, i.e., the variable is z, not y. As expected, one observes a settlement of the powder. For both graphs, the top line corresponds to the position of the top of the powder column, i.e., the free boundary. Further, the pressure field is found to asymptotically converge to a uniform pressure value corresponding to p_{atm}, the atmospheric pressure, i.e., equilibrium of the pressure is established. The stress field is also found to converge to a stationary distribution σ_∞ which, in terms of the original variable z, is the solution to

$$\begin{aligned} \partial_z \sigma + (-\kappa \sigma_\infty + \gamma(\sigma_\infty)) = 0, \qquad 0 < z < H_\infty, \\ \sigma_\infty(H_\infty) = 0, \end{aligned}$$

where H_∞ is the asymptotic value of the height of the powder column. From Figure 5.1, one can observe that the powder has not fully settled after 500s. Figure 5.2 illustrates the convergence to the asymptotic stationary state after 2000s.

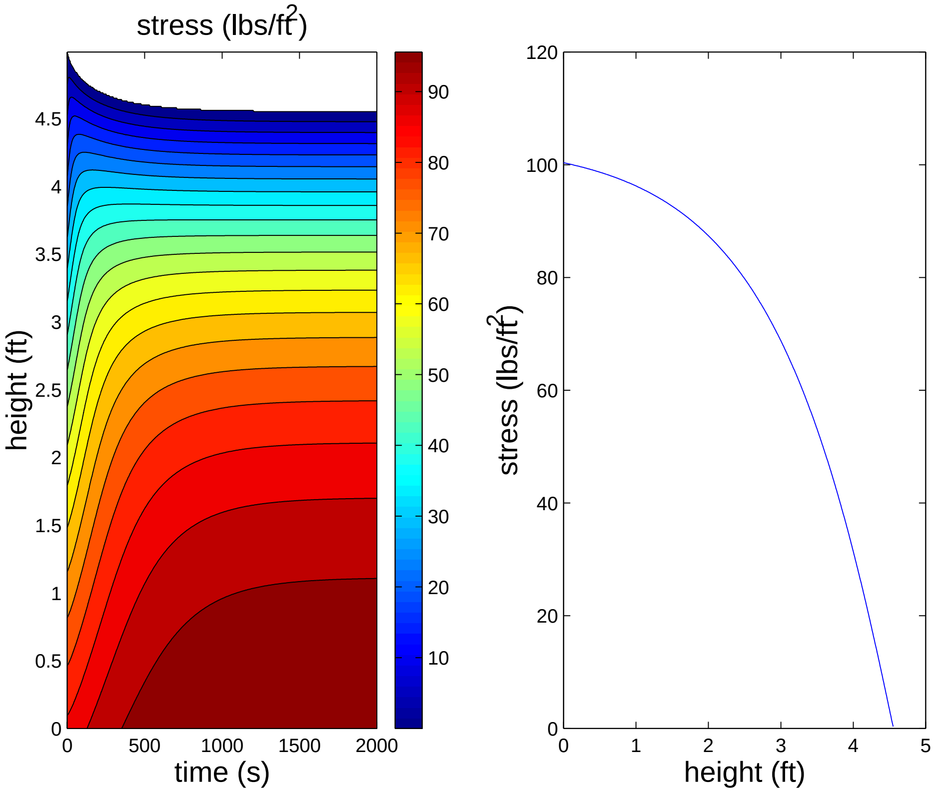

FIG. 5.2. *Calculated stress field in the original geometry; left: calculation from 0 to* $T = 2000s$*, right: asymptotic stress profile;* $N = 100$*,* $NT = 400$*.*

6. Conclusion. A mathematical model of the phenomenon of powder consolidation has been derived and analyzed. A robust numerical method has been proposed and successfully implemented. Several questions and generalizations deserve further study.

The model is a generalization of Janssen's approach, but it does rely on the same two crucial assumptions (quasi uniformity on horizontal cross sections, and horizontal/vertical alignment of the principal stresses) that may be questionable in some cases. How to bypass those assumptions will be the object of future work. From a more applied viewpoint, the bunker containing the powder may be axisymmetric but of variable cross section. This aspect may be easily included in the present model. Further, one may consider adding some powder from the top and/or retrieving some, typically through outlets at the bottom of the bunker. While the first case seems to bring in some modeling difficulties, the second one (retrieval) just amounts to changing one boundary condition and can thus be handled without difficulty in the present approach.

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