On the Poisson brackets of differentiable generators in classical field theory

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The canonical formulation of field theory on open spaces is considered. It is proved, under appropriate assumptions, that the Poisson bracket of two differentiable generators is also a differentiable generator.

I. INTRODUCTION

The following question naturally arises in the study of the Hamiltonian formulation of Einstein theory on open spacelike sections. Given the generators of two asymptotic symmetry transformations, supplemented by the appropriate surface terms at spatial infinity that make them differentiable (i.e., that make their functional derivatives well defined),^{1,2} can it be taken for granted³ that their Poisson bracket is also a differentiable generator? In this paper, we analyze this question and answer it affirmatively.

Because the theorem to be derived here can be applied to more general situations than just general relativity, we will adopt notations that do not explicitly refer to that case. The canonical variables will be denoted by $y^A(x)$ and the Poisson bracket of two differentiable functionals $F[y]$, $G[y]$ of the canonical variables will be denoted by

$$[F, G] = \int \frac{\delta F}{\delta y^A(x)} \frac{\delta G}{\delta y^B(x)} \sigma^{AB} d^n x, \quad (1)$$

where σ^{AB} is, for simplicity, a constant antisymmetric matrix. In (1), $\delta F / \delta y^A(x)$ is the functional derivative of F with respect to $y^A(x)$, defined by

$$\delta F = \int \frac{\delta F}{\delta y^A(x)} \delta y^A(x) d^n x \quad (2)$$

for any allowed variation δy^A of the fields. The functional F is said to be differentiable if its variation can indeed be brought to the form (2), with functional derivatives $\delta F / \delta y^A(x)$ that are regular functions of x .

The formula (1) can be rewritten in various useful ways as follows:

$$[F, G] = \delta_F G = -\delta_G F \quad (3a)$$

$$= \int \frac{\delta G}{\delta y^A(x)} \delta_F y^A(x) d^n x, \quad (3b)$$

with

$$\delta_F y^A(x) = \frac{\delta F}{\delta y^B(x)} \sigma^{BA}. \quad (4)$$

In the open case considered here, the field configurations $y^A(x)$ are restricted by appropriate boundary conditions at spatial infinity, which read

$$\chi^\alpha(y^A, y_i^A, \dots, y_{i_1 \dots i_k}^A, x^j) \rightarrow 0, \quad (5)$$

for some given functions χ^α of the fields and their derivatives $y_{i_1 \dots i_k}^A \equiv \partial y^A / \partial x^{i_1} \partial x^{i_2} \dots \partial x^{i_k}$, and for some specified rate of approach.

If the differentiable functional G is such that $y^A(x) + \delta_G y^A(x)$ obeys the boundary conditions whenever $y^A(x)$ does, G is said to be a differentiable generator. In that case the infinitesimal canonical transformation generated by G maps an allowed configuration on to another allowed configuration.

From now on, F and G will always stand for differentiable generators which are given by integrals of local densities,

$$F = \int f(y^A, y_k^A, y_{kl}^A, \dots, y_{k_1 \dots k_m}^A, x^i) d^n x, \quad (6a)$$

$$G = \int g(y^A, y_k^A, y_{kl}^A, \dots, y_{k_1 \dots k_p}^A, x^i) d^n x. \quad (6b)$$

Moreover, we assume that f and g are differentiable (as many times as needed) functions of their arguments. This case covers all field theoretical models of common interest.

From (6), it is easy to obtain⁴

$$\delta F = \int d^n x \frac{\delta f}{\delta y^A} \delta y^A + \oint d^{n-1} S_k V_f^k(\delta y), \quad (7)$$

where the following conventions have been adopted:

$$\frac{\delta f}{\delta y^A} = \frac{\partial f}{\partial y^A} - \partial_k \frac{\partial f}{\partial y_k^A} + \dots + (-)^m \partial_{k_1 \dots k_m} \frac{\partial f}{\partial y_{k_1 \dots k_m}^A}, \quad (8a)$$

$$V_f^k(\delta y) = \frac{\delta f}{\delta y_k^A} \delta y^A + \frac{\delta f}{\delta y_{kl}^A} \delta y_l^A + \dots + \frac{\delta f}{\delta y_{k l_2 \dots l_m}^A} \delta y_{l_2 \dots l_m}^A, \quad (8b)$$

$$\begin{aligned} \frac{\delta f}{\delta y_{k_1 \dots k_j}^A} &= \frac{\partial f}{\partial y_{k_1 \dots k_j}^A} - \partial_{k_{j+1}} \frac{\partial f}{\partial y_{k_1 \dots k_{j+1}}^A} \\ &+ \dots + (-)^{m-j} \partial_{k_{j+1} \dots k_m} \frac{\partial f}{\partial y_{k_1 \dots k_m}^A}. \end{aligned} \quad (8c)$$

The functions $\delta f / \delta y_{k_1 \dots k_j}^A$ are the "variational derivatives" of f with respect to $y_{k_1 \dots k_j}^A$.

Since the (already improved) functional F is differentiable by hypothesis, its variation (7) must reduce to the form written in (2) for any δy^A that preserves the boundary conditions (5). For the case in which the variations δy^A are of compact support but otherwise arbitrary, the surface inte-

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gral in (7) vanishes and the variation δF indeed reduces to the form (2). It then follows that at each point x ,

$$\frac{\delta F}{\delta y^A(x)} = \frac{\delta f}{\delta y^A}, \quad (9)$$

where the right-hand side of (9) is understood to be evaluated at x . Now by comparing (7) with (2) for arbitrary variations δy^A compatible with the boundary conditions, the surface integral is seen to vanish,

$$\oint d^{n-1} S_k V_f^k(\delta y) = 0, \quad (10)$$

for all allowed δy^A . Similar expressions hold for G .

We are now in a position to formulate the main theorem of our paper.

Theorem: The bracket $[F, G]$ of two differentiable generators is itself a differentiable generator.

This theorem contains two assertions. The first one is that $[F, G]$ is differentiable. The second one is that $\delta_{[F, G]} y^A$ has the correct asymptotic behavior, i.e., that $[F, G]$ generates a canonical transformation that preserves the boundary conditions.

II. PROOF OF THE THEOREM

In order to prove the theorem, we first demonstrate the following lemmas.

Lemma 1: Let $\mu^A(y^A, y_i^A, \dots, y_{i_1 \dots i_s}^A, x^j)$ and $\nu^A(y^A, y_i^A, \dots, y_{i_1 \dots i_s}^A, x^j)$ be two allowed, field-dependent, "infinitesimal" variations of y^A (i.e., let $y^A + \mu^A$ and $y^A + \nu^A$ obey the boundary conditions). Then, the bracket $[\mu, \nu]^A$ defined by

$$[\mu, \nu]^A = \delta_\mu \nu^A - \delta_\nu \mu^A \quad (11)$$

is also an allowed infinitesimal variation of y^A . In (11), $\delta_\mu \nu^A$ stands for the change in ν^A induced by the variation μ^A ,

$$\delta_\mu \nu^A = \sum_{k=0}^i \frac{\partial \nu^A}{\partial y_{i_1 \dots i_k}^c} \mu_{i_1 \dots i_k}^c. \quad (12)$$

The proof of this lemma is straightforward. Infinitesimal variations δy^A are indeed characterized by

$$\delta \chi^\alpha = \sum_{k=0}^q \frac{\partial \chi^\alpha}{\partial y_{i_1 \dots i_k}^A} \delta y_{i_1 \dots i_k}^A \rightarrow 0. \quad (13)$$

Also, note that $\delta_{[\mu, \nu]} \chi^\alpha$ is equal to $\delta_\mu \delta_\nu \chi^\alpha - \delta_\nu \delta_\mu \chi^\alpha$ [use $\delta_\mu h_{i_1 \dots i_k} = (\delta_\mu h)_{i_1 \dots i_k}$]. Then since $\delta_\nu \chi^\alpha$ and $\delta_\mu \chi^\alpha$ fall off for all admissible field configurations, it follows successively that $\delta_\mu \delta_\nu \chi^\alpha$, $\delta_\nu \delta_\mu \chi^\alpha$, and $\delta_{[\mu, \nu]} \chi^\alpha$ also go to zero at infinity and thus, that $[\mu, \nu]^A$ preserves the boundary conditions. $\square$

Lemma 2: Let μ^A and ν^A be as above. In performing the integrations by parts needed to transform

$$\int \delta_\nu \left(\frac{\delta f}{\delta y^B} \right) \cdot \mu^B d^n x$$

into a volume piece containing only undifferentiated ν^A plus a surface integral, that surface integral is simply given by

$$\oint d^{n-1} S_k A_f^k(\mu, \nu), \quad (14a)$$

with

$$A_f^k(\mu, \nu) = \delta_\mu V_f^k(\nu) - \delta_\nu V_f^k(\mu) + V_f^k([\nu, \mu]). \quad (14b)$$

This can be seen by using again the simple identity

$$\delta_\nu \delta_\mu f - \delta_\mu \delta_\nu f = \delta_{[\nu, \mu]} f. \quad (15)$$

By integrating (15) over x and taking into account (7) and the definition of the bracket $[\nu, \mu]^A$, this yields

$$\int \delta_\nu \left(\frac{\delta f}{\delta y^A} \right) \mu^A d^n x = \int \delta_\mu \left(\frac{\delta f}{\delta y^A} \right) \nu^A d^n x + \oint d^{n-1} S_k A_f^k(\mu, \nu), \quad (16)$$

where $A_f^k(\mu, \nu)$ is given by (14). Since the volume integral on the right-hand side of (16) contains no derivative of ν^A , the second lemma is proved. $\square$

Incidentally, (16) shows that the polynomials $\delta_\nu(\delta f / \delta y^A)$ are "self-adjoint," as indeed they should be (see, e.g., Ref. 4, pp. 202–204).

Lemma 3: Let μ^A and ν^A be given by

$$\mu^A = \delta_F y^A = \frac{\delta f}{\delta y^B} \sigma^{BA}, \quad \nu^A = \delta_G y^A = \frac{\delta g}{\delta y^B} \sigma^{BA}. \quad (17a)$$

The bracket $[\mu, \nu]^A$ is then equal to

$$[\mu, \nu]^A = \frac{\delta}{\delta y^B} \left(\frac{\delta f}{\delta y^C} \frac{\delta g}{\delta y^D} \sigma^{CD} \right) \sigma^{BA}. \quad (17b)$$

This is a straightforward consequence of the following identity of the calculus of variations:

$$\frac{\delta}{\delta y^A} \left(\frac{\delta f}{\delta y^B} \frac{\delta g}{\delta y^C} \sigma^{BC} \right) = \delta_F \frac{\delta g}{\delta y^A} - \delta_G \frac{\delta f}{\delta y^A} \quad (18)$$

(where $\delta_F = \delta_\mu$, $\delta_G = \delta_\nu$).

To prove this identity, integrate the right-hand side of (18), multiplied by an arbitrary function ϵ^A of compact support, over the whole space and make all necessary integrations by parts. The resulting surface integrals will all vanish since ϵ^A is identically zero sufficiently close to spatial infinity. This gives, using the equality (16),

$$\begin{aligned} \int \epsilon^A \left(\delta_F \frac{\delta g}{\delta y^A} - \delta_G \frac{\delta f}{\delta y^A} \right) d^n x \\ = \int \left(\delta_F y^A \delta_\epsilon \frac{\delta g}{\delta y^A} - \delta_G y^A \delta_\epsilon \frac{\delta f}{\delta y^A} \right) d^n x \\ = \int \delta_\epsilon \left(\frac{\delta f}{\delta y^A} \frac{\delta g}{\delta y^B} \sigma^{AB} \right) d^n x. \end{aligned} \quad (19)$$

Again making an integration by parts and taking the arbitrariness of ϵ^A into account, (18) is easily seen to hold true. $\square$

We now turn to the demonstration of the theorem itself. The first step in the demonstration amounts to showing that $[F, G]$ is differentiable. This is done by relating the surface term that appears in the variation of $[F, G]$ to the surface terms that arise in the variations of F and G , and that are known to vanish.

The definition (1) and the equality (9) may be used to write the variation of $[F, G]$ for an arbitrary (allowed) variation ν^A of y^A as

$$\delta_\nu [F, G] = \int \delta_\nu \left(\frac{\delta g}{\delta y^B} \right) \delta_F y^B d^n x - \int \delta_\nu \left(\frac{\delta f}{\delta y^B} \right) \delta_G y^B d^n x. \quad (20)$$

We want to prove that the surface integrals arising from the appropriate integrations by parts in each term of the right-hand side of (20) are separately zero. Since these terms have the same structure, it is enough to show this property for one of them.

From the second lemma, we know that the surface integral arising in $\int \delta_\nu (\delta f / \delta y^\mu) \delta_G y^\mu d^n x$ is given by $\oint A_f^k(\delta_G y, \nu) d^{n-1} S_k$ [formula (14)], where

$$A_f^k(\delta_G y, \nu) = \delta_G V_f^k(\nu) - \delta_\nu V_f^k(\delta_G y) + V_f^k([\nu, \delta_G y]). \quad (21)$$

But each of the terms in $A_f^k(\delta_G y, \nu)$ contribute zero, because (i) $\oint V_f^k(\nu) d^{n-1} S_k$ vanishes for any allowed variation performed about any allowed configuration, and so also does

$$\oint \delta_G V_f^k(\nu) d^{n-1} S_k = \delta_G \oint V_f^k(\nu) d^{n-1} S_k;$$

(ii) similarly, $\oint \delta_\nu V_f^k(\delta_G y) d^{n-1} S_k$ vanishes for an identical reason; and (iii) $[\nu, \delta_G y]^A$ is an allowed variation by virtue of the first lemma, and so its flux integral $\oint V_f^k([\nu, \delta_G y]) d^{n-1} S_k$ is also zero. Hence, there is no unwanted surface term in $\delta_\nu [F, G]$.

This proves that the Poisson bracket $[F, G]$ is differentiable; its functional derivatives are equal to [see (1) and (9)],

$$\frac{\delta [F, G]}{\delta y^A(x)} = \frac{\delta}{\delta y^A} \left(\frac{\delta f}{\delta y^B} \frac{\delta g}{\delta y^C} \right) \sigma^{BC}. \quad (22)$$

Then the result of the third lemma can be rewritten as

$$\delta_{[F, G]} y^A = [\delta_F, \delta_G] y^A, \quad (23)$$

and it follows from the first lemma that $\delta_{[F, G]} y^A$ is an allowed variation. This shows that the canonical transformation generated by $[F, G]$ preserves the boundary conditions and completes the demonstration of the theorem.

The conclusion of our paper is that as soon as the canonical generators of a field theory are supplemented, in the

spatially open case, by the appropriate surface terms needed to make them well defined, then their Poisson brackets may be taken as many times as desired (provided these generators are given by the integrals of C^∞ local densities). In that sense, the problem of the surface terms arises only once. In addition, the Poisson brackets possess the nice properties expected from classical mechanics with a finite number of degrees of freedom [cf. (23)].

As a final point, we note that the results of our paper can be easily extended to the case when some of the variables $y^A(x)$ are fermionic.

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¹The need to supplement the Dirac–A.D.M. Hamiltonian of general relativity by appropriate surface terms at spatial infinity has been pointed out, in the asymptotically flat case, by Regge and Teitelboim [T. Regge and C. Teitelboim, *Ann. Phys. N.Y.* **88**, 286 (1974)]. As shown by the authors, the surface terms yield the A.D.M. energy-momentum. These ideas have been further discussed in Y. Choquet-Bruhat, A. Fischer, and J. Marsden, “Maximal hypersurfaces and positivity of mass,” in *Proceedings of the Enrico Fermi Summer School of the Italian Physical Society*, Course LXVII, edited by J. Ehlers (North-Holland, Amsterdam, 1979).

²The extension of the results of Ref. 1 to the asymptotically anti-de Sitter case has been done in M. Henneaux and C. Teitelboim, *Phys. Lett. B* **142**, 355 (1984); M. Henneaux and C. Teitelboim, *Commun. Math. Phys.* **98**, 391 (1985).

³This question is important for deriving the Poisson bracket algebra of the canonical generators of asymptotic symmetries, and has been particularly emphasized in J. D. Brown and M. Henneaux, “Central charges in the canonical realization of asymptotic symmetries: An example from three-dimensional gravity,” to be published in *Commun. Math. Phys.* See also A. Ashtekar, “Asymptotic properties of isolated systems: Recent developments,” in *General Relativity and Gravitation*, edited by B. Bertotti, F. de Felice, and A. Pascolini (Reidel, Dordrecht, 1984).

⁴Th. De Donder, “*Thorie Invariantive du Calcul des Variations* (Gauthier–Villars, Paris, 1935), 2nd. ed.