Jacobi's action and the recovery of time in general relativity

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We argue that the usual action principle of general relativity, applied to spacetimes with closed spatial geometries, should be regarded as analogous to Jacobi's form of the principle of stationary action, in which the energy rather than a physical time is fixed. Following the paradigm of quantization based on Jacobi's action for a nonrelativistic particle, we show that the Wheeler-DeWitt equation corresponds to a time-independent Schrödinger equation. The relationship between Jacobi's and Hamilton's action principles then allows us to derive a time-dependent Wheeler-DeWitt equation of the Schrödinger type. In this equation, the role of energy is played by the cosmological constant and that of physical time by the four-volume of spacetime.

I. INTRODUCTION

One of the central features of general relativity is the absence of any explicit reference to physical time. Given initial and final three-geometries, their separation in proper time along any curve is only determined after solving the Einstein equations to obtain a spacetime that has these initial and final data. If there is a finite number of such solutions, one can say as a consequence that "three geometry is the carrier of information about time."^{1,2}

The indirect manner in which time arises in general relativity makes the process of quantization obscure, because the familiar prescription for quantization makes extensive use of time as a basic parameter entering into the description of the system. For example, in the quantization of a nonrelativistic particle, the usual starting point is Hamilton's principle

$$\bar{S}[x] = \int_{t_i}^{t_f} dt L(x, dx/dt), \quad (1)$$

in which L is the Lagrangian and the configuration-space paths are labeled by physical time t . Quantization can then proceed with t playing a central role in the formulation. The canonical form of the theory is obtained, defining the Hamiltonian and equal-time Poisson brackets. The latter become equal-time commutators and thereby determine the operator algebra. Also, a scalar product for the state vectors is defined such that the unitary evolution of the system implies that probabilities are conserved. Physical time t thus plays a primary role in the description of the quantum mechanics of nonrelativistic systems. In general relativity, however, physical time is only a secondary concept determined "on shell"—that is, by a classical solution for the spacetime metric.

But the key feature that time is not explicitly involved in the dynamical description of general activity is *not* an intrinsic property of general relativity. Rather, it is merely a consequence of the way in which it is usually described. Indeed, nonrelativistic mechanics can be described in a similar fashion by using Jacobi's formulation of the principle of least action as the basic starting point, rather than Hamilton's action (1). In Jacobi's formulation,^{3,4} the physical time interval between initial and final

configurations is not fixed, but instead the energy of the system is fixed. The action is an integral over the configuration-space path,

$$S[x] = \int dx \sqrt{2m[E - V(x)]}, \quad (2)$$

with E the fixed value of energy and $V(x)$ the potential energy. As we will show, the time interval between initial and final positions is determined through a metric relation, and only after the classical equations of motion are imposed. It is therefore Jacobi's action (2), not Hamilton's action (1), that is the appropriate analogue of general relativity, as it is usually described. For general relativity, the cosmological constant turns out to be the quantity that plays the role of the fixed energy in Jacobi's action. The relationship between nonrelativistic mechanics based on Jacobi's action and general relativity has also been discussed by Hartle.⁵

In Sec. II we investigate Jacobi's action in detail. A supplementary (metric) relation, which is just the energy equation on shell, is introduced to determine the motion of the system in time. The classical equations of motion that follow from extremizing Jacobi's action, along with the supplementary relation, are shown to be equivalent to Newton's equations with the energy fixed to the value E . We then use Dirac's procedure⁶ for passing to the canonical form of a constrained theory to obtain the canonical form of Jacobi's action. In Sec. III Jacobi's action is derived from the parametrized form of Hamilton's action, by eliminating time as a kinosthenic (cyclic) variable. By reversing this procedure, we then show how time and Hamilton's action can be recovered starting from Jacobi's action and the supplementary metric equation for time.

Section IV spells out in detail the formal analogy between nonrelativistic mechanics and general relativity, with particular attention given to the way in which time is incorporated into each theory. We show that the familiar action principle for general relativity is analogous to Jacobi's action for nonrelativistic mechanics. An action for general relativity that is analogous to Hamilton's principle is then obtained. A global time is fixed directly in this action, and can be identified as the total spacetime volume between initial and final surfaces. Remarkably,

this Hamilton-type action is equivalent to that recently obtained by Henneaux and Teitelboim.⁷

In Sec. V we spell out the formal analogy between non-relativistic quantum mechanics and quantum gravity. It is shown that quantization based on Jacobi's action for nonrelativistic mechanics leads to the time-independent Schrödinger equation. Likewise, quantization of general relativity using the (usual) Jacobi-type action gives the (usual) *time-independent* Wheeler-DeWitt equation. On the other hand, quantization of the Hamilton-type action for general relativity yields a *time-dependent* Wheeler-DeWitt equation, analogous to the time-dependent Schrödinger equation. We also discuss the possibility of introducing a positive-definite Schrödinger-type scalar product for quantum gravity. The time-dependent Wheeler-DeWitt equation, with the role of time played by the spacetime volume, has also been discussed recently in Refs. 7–10.

II. JACOBI'S ACTION FOR NONRELATIVISTIC MECHANICS

In this section we will investigate Jacobi's action for nonrelativistic mechanics in detail. Consider a system described by a D -dimensional configuration space with coordinates x^a , $a=1,2,\dots,D$. Jacobi's action is

$$S[x] = \int \sqrt{m_{ab}(x) dx^a dx^b} \sqrt{2[E - V(x)]}, \quad (3)$$

where m_{ab} is the Newtonian metric. (For example, for a single point particle of mass m in flat-space Cartesian coordinates, $m_{ab} = m \delta_{ab}$.) Also, $V(x)$ is the potential energy and E is a fixed constant. $S[x]$ is extremized by varying the configuration space path with fixed end points. Now introduce a parameter σ along the path, that increases monotonically from σ' at one end point to σ'' at the other end point. Denoting $\dot{x}^a = dx^a/d\sigma$, the action can be written

$$S[x] = \int_{\sigma'}^{\sigma''} d\sigma \sqrt{m_{ab}(x) \dot{x}^a \dot{x}^b} \sqrt{2[E - V(x)]}, \quad (4)$$

where $x(\sigma') = x'$ and $x(\sigma'') = x''$ are fixed. Jacobi's action is invariant under changes of the parametrization of the configuration-space path that preserve the end-point values of the parameter. That is, (4) is invariant under the replacement $x(\sigma) \rightarrow \bar{x}[f(\sigma)]$ with $f(\sigma') = \sigma'$ and $f(\sigma'') = \sigma''$. Because of this reparametrization invariance, the number of degrees of freedom of the system that enter Jacobi's action is $D-1$, one less than the dimension of the configuration space.

An extremum of the action (4) gives a classical path in the configuration space of the system, but says nothing about its motion in time. To determine the system's evolution in time requires the use of a supplementary condition³

$$\frac{1}{2} \frac{dx^a}{dt} \frac{dx_a}{dt} + V(x) = E \quad (5)$$

(where $dx_a \equiv m_{ab} dx^b$). This extra relation (5) is just the energy condition and once the configuration path $x(\sigma)$ is known, it can be integrated to give the physical time $t(\sigma)$. Although it may appear disturbing that the time

evolution is determined by an extra condition, and does not follow directly from the equations of motion, we will see later that this extra equation is precisely analogous to the metric of general relativity, which relates proper time increments to the dynamical variables, the metric coefficients.

Classically, the equations of motion following from Jacobi's action (4) and the supplementary condition (5) are equivalent to Newton's equations with the energy, a first integral of Newton's equations, fixed to the value E . This is shown explicitly by varying (4), which yields the equations of motion

$$\left[\nabla_{\dot{x}} \dot{x}_c + \frac{\dot{x}^2}{2(E-V)} \frac{\partial V}{\partial x^c} \right] \perp_a^c = 0. \quad (6)$$

Here, $\nabla_{\dot{x}}$ is a covariant derivative in the metric m_{ab} , and $\nabla_{\dot{x}} \dot{x}_c$ is the (mass times) acceleration of the configuration-space path with respect to the parameter σ . Also, $\perp_a^c \equiv \delta_a^c - \dot{x}^c \dot{x}_b / \dot{x}^2$ (with $\dot{x}^2 \equiv \dot{x}^a \dot{x}_a$) is the projection tensor onto the subspace of configuration space that is orthogonal to the configuration-space path. Using (5) to identify physical time, Eq. (6) just becomes

$$\left[\frac{1}{(i)^2} \nabla_{\dot{x}} \dot{x}_c + \frac{\partial V}{\partial x^c} \right] \perp_a^c = 0, \quad (7)$$

where $(\nabla_{\dot{x}} \dot{x}_c)/(i)^2$ is the (mass times) acceleration in physical time. Thus, (7) is the orthogonal projection of Newton's equations. The projection of Newton's equations parallel to the configuration path follows from Eq. (5) by differentiating with respect to t . Therefore, the equations of motion (6) obtained from the Jacobi action, supplemented by (5), are equivalent to Newton's equations with fixed energy E .

We will now construct the canonical version of Jacobi's action. Because the Lagrangian in (4) is homogeneous of degree one in the $\dot{x}$'s, the canonical Hamiltonian $\dot{x}^a \partial L / \partial \dot{x}^a - L$ vanishes identically. This is of course a well-known feature of reparametrization-invariant theories. Furthermore, the momenta conjugate to the configuration coordinates x^a , namely, $p_a = \dot{x}_a \sqrt{2(E-V)}/\dot{x}^2$, give rise to a constraint

$$\mathcal{H}(x, p) \equiv \frac{1}{2} m^{ab}(x) p_a p_b + V(x) - E \approx 0. \quad (8)$$

Because the canonical Hamiltonian is zero, there are no secondary constraints, and $\mathcal{H}$ is then trivially first class. The action in canonical form is just

$$S[x, p, N] = \int_{\sigma'}^{\sigma''} d\sigma [p_a \dot{x}^a - N \mathcal{H}(x, p)], \quad (9)$$

where N is a Lagrange multiplier, whose variation enforces the constraint (8). As before, the action is to be varied with $x(\sigma') = x'$ and $x(\sigma'') = x''$ held fixed. Equation (9), Jacobi's action in canonical form, will be the basis for our comparison of nonrelativistic mechanics with general relativity.

The equations of motion following from the action (9) are

$$\dot{x}^a = N[x^a, \mathcal{H}] = N p^a, \quad (10a)$$

$$\dot{p}_a = N[p_a, \mathcal{H}] = -N \left[\frac{1}{2} p_b p_c \frac{\partial m^{bc}}{\partial x^a} + \frac{\partial V}{\partial x^a} \right], \quad (10b)$$

$$\mathcal{H} \equiv \frac{1}{2} p^a p_a + V - E = 0, \quad (10c)$$

where $[,]$ denotes the Poisson brackets. Combining the first and third of these equations, the Lagrange multiplier N is given by

$$N = \left[\frac{\dot{x}^2}{2(E - V)} \right]^{1/2}. \quad (11)$$

According to the supplementary equation (5) for time t , the right-hand side of (11) is just $dt/d\sigma$, so that classically t is determined by

$$dt = N d\sigma. \quad (12)$$

On the other hand, Eq. (12) may be thought of as fundamental, providing a physical interpretation for N : the Lagrange multiplier N gives the lapse in physical time associated with an increment in the coordinate σ . So a complete description of the system, including its motion in time, is contained in the action principle (9) along with the metric condition (12). The energy equation (5) is then viewed as only holding true on shell, when the lapse N is eliminated by the classical equations of motion in favor of x . Equation (12) above plays a role analogous to the spacetime metric of general relativity.

Observe that the action (9) is invariant under the gauge transformation

$$\delta x = \epsilon[x, \mathcal{H}], \quad (13a)$$

$$\delta p = \epsilon[p, \mathcal{H}], \quad (13b)$$

$$\delta N = \dot{\epsilon}, \quad (13c)$$

with $\epsilon(\sigma)$ vanishing at the end points σ', σ'' . This transformation is just the canonical statement of reparametrization invariance.

Finally, note that the action (8), (9) can be put in Lagrangian form by integrating out the momenta, giving

$$S[x, N] = \int_{\sigma'}^{\sigma''} d\sigma \left[\frac{m_{ab} \dot{x}^a \dot{x}^b}{2N} + N[E - V(x)] \right]. \quad (14)$$

Going one step further, the lapse N can be integrated away yielding the original Jacobi action (4). In this process, the classical expression (11) for the lapse turns the metric back into the energy condition (5). This last step of eliminating the lapse cannot be done exactly in the path integral, since N does not enter the action (14) quadratically. Correspondingly, the action (9) or (14), and not (4), is an appropriate starting point for developing a quantum theory via the path integral.

III. JACOBI'S ACTION FROM HAMILTON'S PRINCIPLE

An alternative derivation of the action (9) can be given, based on Hamilton's principle in parametrized form.³ Start with the action in canonical form

$$\bar{S}[x, p] = \int_{t'}^{t''} dt \left[p_a \frac{dx^a}{dt} - h(x, p, t) \right], \quad (15)$$

with $h(x, p, t)$ the usual Hamiltonian. Now introduce the path parameter σ and promote t to the role of a dynamical variable:

$$\bar{S}[x, p, t] = \int_{\sigma'}^{\sigma''} d\sigma [p_a \dot{x}^a - i h(x, p, t)]. \quad (16)$$

Variation with respect to t in this action just gives the rate of change of the Hamiltonian, $\dot{h} = i \partial h / \partial t$. Passing to the canonical form of the action $\bar{S}$ with respect to the variable t leads once again to the expected results for a reparametrization-invariant theory: the new canonical Hamiltonian vanishes identically, and the definition of momentum leads to a constraint $\pi + h(x, p, t) \approx 0$, where π is the momentum conjugate to t . Thus the action becomes

$$\bar{S}[x, p, t, \pi, N] = \int_{\sigma'}^{\sigma''} d\sigma \{ p_a \dot{x}^a + \pi \dot{t} - N[\pi + h(x, p, t)] \}, \quad (17)$$

where N is a Lagrange multiplier. This action is to be varied with x and t fixed at the end points σ', σ'' .

The action $\bar{S}$ in (17) is the one used by Kuchař and Hartle^{11,12} as a simple example of a theory, like general relativity, that is reparametrization invariant. However, (17) fails in one way to be a close analogue of gravity (as it is usually formulated) because time appears explicitly in (17), and does not enter through an auxiliary (metric) condition as in (12). But time can be eliminated from the action and equations of motion whenever the original Hamiltonian h is independent of t . In that case, t is a kinosthenic (cyclic) variable and its conjugate π is conserved. We can apply the procedure due to Routh; in the canonical formalism, this amounts first to passing to a new action S in which π is fixed at the end points σ', σ'' rather than t . S is obtained from $\bar{S}$ in (17) by an integration by parts $\pi \dot{t} = -\dot{\pi} t + \partial(\pi t) / \partial \sigma$, and then discarding the boundary term $\int d\sigma \partial(\pi t) / \partial \sigma$. Using the equation of motion obtained by varying the action with respect to t , π can then be set equal to a constant which we call $-E$. The action becomes

$$S[x, p, N] = \int_{\sigma'}^{\sigma''} d\sigma \{ p_a \dot{x}^a - N[h(x, p) - E] \}, \quad (18)$$

which is precisely (9) with $\mathcal{H}(x, p) = h(x, p) - E$. Notice that in eliminating the pair t, π , one relationship has been lost: namely, the equation of motion generated by varying the action $\bar{S}$ with respect to π . This equation is just the metric $dt = N d\sigma$, which in the action (18) (as in general relativity) must be treated as a supplementary relationship that determines the evolution of the system in physical time.

Having derived Jacobi's action from Hamilton's action, how can the reasoning be reversed and Hamilton's action be recovered from Jacobi's action and the metric $dt = N d\sigma$? To answer this question, consider the Jacobi action (18) modified by fixing the lapse $N(\sigma)$ as a prescribed function, not to be varied. With N fixed, the total time interval $\int_{\sigma'}^{\sigma''} N d\sigma$, which is gauge invariant ac-

cording to (13c), is fixed as in Hamilton's principle. Also, since N is not varied, the Hamiltonian constraint $\mathcal{H} \approx 0$ is not imposed and the original Hamiltonian $h(x, p)$ is not fixed to the particular value E . Indeed, the supplementary metric condition (12) suggests that N be written as $\dot{t}$ in (18). Then discarding the constant $E(t'' - t')$, Jacobi's action is converted into Hamilton's action (16), but with $t(\sigma)$ now a prescribed function. Because $t(\sigma)$ is not a dynamical variable, this version of Hamilton's principle is not reparametrization invariant, and is in fact equivalent to (15).

Of course, t in (16) can be varied, since the resulting equation $\dot{h} = 0$ (where h has no explicit t dependence) gives no new information. This is because the Hamiltonian $h(x, p)$ is already determined to be a constant of motion by the equations of motion for x and p . The result is that Hamilton's action in (16) with t dynamical can be obtained from Jacobi's action (18) by simply letting $\dot{t}$ replace N and discarding the boundary term $E(t'' - t')$. The fully canonical form (17) of Hamilton's principle is then found from (16) as before.

In the next section we will identify the usual canonical action for general relativity as a Jacobi-type action, with the cosmological constant Λ the fixed value of a constant of motion. By following the above reasoning we will then be able to write down a Hamilton-type action, in which a cosmological time is fixed as a boundary condition, whereas the constant of motion is not fixed.

IV. ANALOGY WITH GENERAL RELATIVITY

Consider general relativity as applied to closed cosmologies; that is, assume the spacetime manifold admits closed spatial sections. Then let σ^0, σ^i denote the four spacetime coordinates, and g_{ij} the spatial metric with curvature R . Also let π^{ij} denote the momentum conjugate to the metric. The action in canonical form is

$$S[g, \pi, N, N^i] = \int d^4\sigma (\pi^{ij} \dot{g}_{ij} - N\mathcal{H} - N^i \mathcal{H}_i), \quad (19a)$$

$$\mathcal{H} \equiv (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{1}{2\kappa} \sqrt{g} (R - 2\Lambda), \quad (19b)$$

$$\mathcal{H}_i \equiv -2\pi_i^j{}_{|j}, \quad (19c)$$

where 2κ is 16π times Newton's constant and the overdot denotes differentiation with respect to σ^0 . In (19) the superspace metric G^{ijkl} and its inverse G_{ijkl} (in four spacetime dimensions) are given by

$$G^{ijkl} \equiv \frac{\sqrt{g}}{2} (g^{ik} g^{jl} + g^{il} g^{jk} - 2g^{ij} g^{kl}), \quad (20a)$$

$$G_{ijkl} \equiv \frac{1}{2\sqrt{g}} (g_{ik} g_{jl} + g_{il} g_{jk} - g_{ij} g_{kl}). \quad (20b)$$

The spacetime metric, the supplementary condition analogous to $dt^2 = N^2 d\sigma^2$ in nonrelativistic mechanics, is

$$d\tau^2 = N^2 (d\sigma^0)^2 - (N^i d\sigma^0 + d\sigma^i) g_{ij} (N^j d\sigma^0 + d\sigma^j). \quad (21)$$

Note that the analysis of this section is unchanged when matter fields are included in the system.

We will now show that this action is analogous to the Jacobi action (18) for nonrelativistic mechanics in that Λ ,

like E , is the fixed value of a constant of motion. First note that in the usual Hamiltonian constraint $\mathcal{H}$, which is a density of weight 1, Λ appears multiplied by $\sqrt{g}$. In order to isolate Λ as an additive constant in the constraint write $N\mathcal{H} = \tilde{N}\tilde{\mathcal{H}}$ where

$$\tilde{N} = N\sqrt{g}, \quad (22a)$$

$$\tilde{\mathcal{H}} = \left[\frac{2\kappa}{\sqrt{g}} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{1}{2\kappa} (R - 2\Lambda) \right]. \quad (22b)$$

The "lapse density" $\tilde{N}$ is just the square root of (minus) the determinant of the four-metric, and the new Hamiltonian constraint $\tilde{\mathcal{H}}$ has scalar weight. Furthermore, recognize that for each $\sigma^0 = \text{const}$ surface, the constraint $\tilde{\mathcal{H}} \approx 0$ is an infinity of conditions, one for each space point. The constant Λ occurs effectively in only one of these, which we shall refer to as the spatially constant or "zero-mode" part of $\tilde{\mathcal{H}}$. The spatially constant part of the Hamiltonian constraint $\tilde{\mathcal{H}} \approx 0$ is obtained as a classical equation by varying the action (19) with respect to $\int d^3\sigma \tilde{N}$, which is the zero-mode part of $\tilde{N}$.

The analogue here of the original Hamiltonian $h(x, p)$ that occurs in Jacobi's action (18) is the spatially constant part of the quantity

$$h(g, \pi) \equiv \left[\frac{2\kappa}{\sqrt{g}} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{1}{2\kappa} R \right]. \quad (23)$$

This is demonstrated by showing that $h(g, \pi)$ is a constant of motion by virtue of the equations of motion, without making use of the spatially constant part of the Hamiltonian constraint $\tilde{\mathcal{H}} \approx 0$. Indeed, the evolution of $h(g, \pi)$ is generated through the Poisson brackets by $H = \int d^3\sigma (\tilde{N}\tilde{\mathcal{H}} + N^i \mathcal{H}_i)$, and explicit calculation shows that $\dot{h}(g, \pi)$ equals a sum of terms proportional to $\mathcal{H}_i$ and $\tilde{\mathcal{H}}_{,i}$. These terms all vanish as long as the momentum constraint and the spatially variable part of the Hamiltonian constraint hold. Then in the action (19), variation of $\int d^3\sigma \tilde{N}$ just fixes the zero mode of this constant of motion $h(g, \pi)$ to the particular value $-\Lambda/\kappa$.

Having recognized the usual action for general relativity as analogous to the Jacobi action for nonrelativistic mechanics, we now seek the analogue of Hamilton's action principle for gravity. Recalling the logic of Sec. III, Hamilton's action is obtained from Jacobi's action either by fixing the lapse $N(\sigma)$ as a prescribed function, or by simply replacing N by $\dot{t}$. In both cases, the total time interval $t'' - t'$ becomes fixed in the resulting variational principle, and the constant term $E(t'' - t')$ is discarded. Furthermore, the equations of motion following from the new (Hamilton's) action do not fix the constant of motion $h(x, p)$ to the particular value E , because the Hamiltonian constraint is not imposed.

For gravity then, first consider the modified variational principle obtained from the action (19) by fixing the zero mode of the lapse density $\int d^3\sigma \tilde{N}$ as a prescribed function of σ^0 . Thus, the integral $\int d\sigma^0 (\int d^3\sigma \tilde{N})$ is fixed; by the supplementary condition (21), this is just the total spacetime volume between initial and final three-surfaces and is therefore an invariant measure of the cosmological

time interval. Further notice that the term in the action $-\int d^4\sigma \tilde{N}\Lambda/\kappa$ is just a constant [the analogue of $E(t''-t')$] and can be discarded. The resulting action principle is indeed a Hamilton-type action, in that a global, cosmological time is fixed and the constant of motion—namely, the zero mode of $h(g, \pi)$ —is not restricted to a particular value.

It has been known for some time in the context of the Lagrangian action that by fixing the spatially constant part of $\tilde{N}$, the cosmological constant enters the equations of motion as an integration constant. In the Lagrangian case, however, $\tilde{N} = \sqrt{-(4)g}$ can be fixed, and not varied, for all spacetime points as opposed to just the zero-mode part.¹³ Then the equations of motion are the trace-free part of the Einstein equations and allow for a Λ term that is a spacetime function. But the contracted Bianchi identities then restrict this function to be a constant. In the canonical case, the spatially variable part of $\tilde{N}$ must be varied, because the spatially variable part of the Hamiltonian constraint must hold in order for the momentum constraint to remain zero under evolution by H (see Refs. 7 and 8). This is not surprising, because the first-class property of the constraints, that is, the property that they remain zero under evolution, is the canonical counterpart of the Bianchi identities. Thus, in the canonical case, the contracted Bianchi identities must effectively be imposed in formulating the modified variational principle, whereas in the Lagrangian case, there is no such restriction on the variational principle.

In obtaining the form (16) of Hamilton's action in which t is dynamical, the lapse N in Jacobi's action (18) was replaced by i and the boundary term $E(t''-t')$ dropped. Variation of t gives $\dot{h}=0$, an equation that is already true by virtue of the other equations of motion. For gravity the analogue is to replace the lapse density by the divergence of a spacetime vector density, $\tilde{N} = \tau^\mu_{,\mu}$. The resulting action is

$$\bar{S}[g, \pi, N^i, \tau^\mu] = \int d^4\sigma [\pi^{ij}\dot{g}_{ij} - N^i \mathcal{H}_i - \tau^\mu_{,\mu} h(g, \pi)], \quad (24)$$

where the term in the action (19) involving the cosmological constant is now a complete divergence $-(\Lambda/\kappa)\int d^4\sigma \tau^\mu_{,\mu}$ and has been discarded. Variation of τ^μ gives $h_{,\mu}=0$; the equations $h_{,i}=0$ are equivalent to the nonzero-mode part of the Hamiltonian constraint, while the equation $\dot{h}=0$ is already true by virtue of the other equations of motion. The action (24) is the analogue of Hamilton's action (16) for nonrelativistic mechanics.

In the action (24) all the original constraints hold except for the zero mode of the Hamiltonian constraint $\mathcal{H} \approx 0$. Thus, the constant of motion $h(g, \pi)$ is not constrained to equal $-\Lambda/\kappa$. Alternatively τ^0 , the zero component of τ^μ , must be fixed on the initial and final three-surfaces in this action principle. The interpretation of the field τ^μ is obtained from its definition $\tilde{N} = \tau^\mu_{,\mu}$ using the supplementary metric condition (21). Defining the spatial average $T = \int d^3\sigma \tau^0$, the total spacetime volume is $\int d^4\sigma \tilde{N} = \int d^4\sigma \tau^\mu_{,\mu} = T'' - T'$, where T' and T'' are the values of $T(\sigma^0)$ on the initial and final three-surfaces.

Thus, T is an invariant, global measure of cosmological time. Notice that the action (24) is invariant under the gauge transformation $\tau^\mu \rightarrow \tau^\mu + \epsilon^\mu$ where ϵ^μ is a spacetime vector density obeying $\epsilon^\mu_{,\mu} = 0$. The only invariant parts of τ^μ correspond to changes in the cosmological time T , such as $T'' - T'$.

The Hamilton-type action (24) for general relativity can be put in a canonical form with respect to the variables τ^μ , in the same way that the action (17) for nonrelativistic mechanics follows from (16). The resulting action has been obtained previously by Henneaux and Teitelboim,⁷ and reads

$$\begin{aligned} \bar{S}[g, \pi, \tau^0, \lambda, \tilde{N}, N^i, \tau^i] \\ = \int d^4\sigma \left[\pi^{ij}\dot{g}_{ij} + \lambda \dot{\tau}^0 - \tau^i[\lambda_{,i}] - N^i[-2\pi_i{}^j{}_{|j}] \right. \\ \left. - \tilde{N} \left[\frac{2\kappa}{\sqrt{g}} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{1}{2\kappa} R + \lambda \right] \right]. \quad (25) \end{aligned}$$

Here, λ is the momentum conjugate to τ^0 , and τ^i are seen to be Lagrange multipliers that enforce the spatial constancy of λ . The equation of motion for τ^0 implies that λ is a constant in σ^0 , and the equation of motion for λ gives back the relationship $\tau^\mu_{,\mu} = \tilde{N}$. This action (25) can also be put in Lagrangian form with respect to the variables g_{ij}, π^{ij} by integrating out the momenta π^{ij} . The result is⁷

$$\begin{aligned} \bar{S}[(4)g, \tau, \lambda] = \frac{1}{2\kappa} \int_M d^4\sigma [\sqrt{-(4)g} ({}^{(4)}R - 2\kappa\lambda) \\ + 2\kappa\lambda\tau^\mu_{,\mu}] \\ + \frac{1}{\kappa} \int_{\partial M} d^3\sigma \sqrt{g} K, \quad (26) \end{aligned}$$

where $(4)g_{\mu\nu}$ is the four-metric with scalar curvature $(4)R$, and K is the trace of the extrinsic curvature on the boundaries ∂M .

V. QUANTIZATION

In the previous sections we have developed nonrelativistic mechanics and general relativity in such a way that the two systems bear a close formal relationship. We will now use the accepted quantum description of mechanics to suggest the correct procedure for quantizing general relativity.

With nonrelativistic mechanics described according to Jacobi's action (8), (9), the canonical Hamiltonian is zero but the system is constrained, $\frac{1}{2}p^a p_a + V - E \approx 0$. Then Dirac's procedure⁶ for quantizing a constrained Hamiltonian system is applicable. In this approach the classical constraint is imposed as a quantum operator equation acting on the wave function. So let the canonical variables become operators satisfying the usual commutator algebra; in the coordinate representation, $\hat{x} = x$, $\hat{p} = (\hbar/i)\partial/\partial x$, and the wave function ψ_E depends on x . The resulting operator-constraint equation is just the time-independent Schrödinger equation

$$\left[-\frac{\hbar^2}{2} \nabla^2 + V(x) - E \right] \psi_E(x) = 0, \quad (27)$$

where ∇^2 is the Laplacian in the configuration-space metric m_{ab} . Notice that in (27) we have ignored the possible addition of a term proportional to $\hbar^2$ times the configuration-space scalar curvature.¹⁶ The time-independent equation (27) is obviously obtained as a result of starting with Jacobi's action, in which time nowhere appears but the energy is fixed.

The time-dependent Schrödinger equation can be recovered by following the analysis of Sec. III to obtain Hamilton's action (17) from Jacobi's action. In this case the canonical variables include the pair t, π and, as quantum operators in the coordinate representation, they become $\hat{t}=t$, $\hat{\pi}=(\hbar/i)\partial/\partial t$. The wave functions ψ depend on x and t , and the operator constraint equation is the time-dependent Schrödinger equation

$$\left[-\frac{\hbar^2}{2}\nabla^2 + V(x) + \frac{\hbar}{i}\frac{\partial}{\partial t} \right] \psi(x, t) = 0. \quad (28)$$

Thus we see that the quantum theory (27) based on Jacobi's action is equivalent to the theory obtained by restricting the states $\psi(x, t)$ to a particular energy eigenvalue. This is made explicit by writing the time-dependent wave function in (28) as

$$\psi(x, t) = \psi_E(x) e^{-iEt/\hbar}, \quad (29)$$

giving (27).

In general relativity the analogue of Jacobi's action (17) is the action (19), in which time does not appear and Λ is a fixed constant. Quantization of this theory makes the dynamical variables become quantum operators; in the coordinate representation, the coordinate operators correspond to multiplication $\hat{g}_{ij}=g_{ij}$, the momentum operators become functional derivatives, $\hat{\pi}^{ij}=(\hbar/i)\delta/\delta g_{ij}$, and the wave functional Ψ_Λ depends on g_{ij} . The quantum constraints obtained from the action (19) become operator equations

$$\left[-(2\kappa\hbar^2)\frac{1}{\sqrt{g}}G_{ijkl}\frac{\delta}{\delta g_{ij}}\frac{\delta}{\delta g_{kl}} - \frac{1}{2\kappa}(R-2\Lambda) \right] \Psi_\Lambda[g] = 0, \quad (30a)$$

$$\left[-2D_j\frac{\delta}{\delta g_{ij}} \right] \Psi_\Lambda[g] = 0, \quad (30b)$$

where D_i is a covariant derivative in the metric g_{ij} . Equation (30a) is the usual *time-independent* Wheeler-DeWitt equation, the analogue of the time-independent Schrödinger equation (27). Notice that we have made no attempt to address the factor-ordering issue in writing Eqs. (30).

The *time-dependent* Wheeler-DeWitt equation is obtained by quantizing general relativity based on the Hamilton-type action (25). In that case the canonical variables include τ and λ , which become quantum operators $\hat{\tau}=\tau$, $\hat{\lambda}=(\hbar/i)\delta/\delta\tau$. (We have dropped the index on τ^0 , which will be denoted τ hereafter.) The wave functional Ψ depends on g_{ij} and τ , and satisfies the operator-constraint equations

$$\left[-(2\kappa\hbar^2)\frac{1}{\sqrt{g}}G_{ijkl}\frac{\delta}{\delta g_{ij}}\frac{\delta}{\delta g_{kl}} - \frac{1}{2\kappa}R + \frac{\hbar}{i}\frac{\delta}{\delta\tau} \right] \Psi[g, \tau] = 0, \quad (31a)$$

$$\left[-2D_j\frac{\delta}{\delta g_{ij}} \right] \Psi[g, \tau] = 0, \quad (31b)$$

$$\left[D_i\frac{\delta}{\delta\tau} \right] \Psi[g, \tau] = 0. \quad (31c)$$

Equation (31a) is the *time-dependent* Wheeler-DeWitt equation.⁷⁻¹⁰ Equations (31b) and (31c) constrain the wave functional to depend only on the gauge-invariant parts of g_{ij} and τ : namely, the three-geometry and the global time T . Then the time-dependent Wheeler-DeWitt equation (31a) actually represents a single equation describing the evolution of Ψ in the spacetime-volume time T , along with an infinity of operator constraints. These constraints are just the conditions that the wave functional Ψ be annihilated by the spatially variable part of the operator $\hat{\mathcal{H}}$. Thus, (31a) and (31c) imply

$$(D_i\hat{\mathcal{H}})\Psi[g, \tau] = 0. \quad (32)$$

On the other hand, the time-independent Wheeler-DeWitt equation (30a) states that the full operator $\hat{\mathcal{H}}$ annihilates Ψ_Λ . The time-independent equations (30) are obtained from the time-dependent ones (31) by choosing the wave functional Ψ to be in a Λ eigenstate; that is, by writing

$$\Psi[g, \tau] = \Psi_\Lambda[g] \exp \left[\frac{i}{\hbar} \int d^3\sigma \tau \Lambda / \kappa \right]. \quad (33)$$

(The integral in the exponent above is well defined because τ transforms as a spatial scalar density of weight 1.)

The close relationship between general relativity and nonrelativistic mechanics suggests that a scalar product for quantum gravity might be chosen in analogy with the scalar product for ordinary quantum mechanics. That is, define the positive-definite expression

$$dP = |\Psi[g, \tau]|^2 d\mu[g] \quad (34)$$

and interpret dP as the probability for the three-geometry at the global time T to be found in the region of superspace with volume element $d\mu[g]$. This definition for probability can only make sense if there exist wave functionals satisfying (31) that can be normalized. But in fact, because the time-dependent Wheeler-DeWitt equation still imposes an infinity of constraints, it seems likely that there are no normalizable wave functionals. Whether or not this is the case ultimately depends also on the measure $d\mu[g]$ on superspace. Note however, that if wave functionals do exist for which the total probability (34) is finite, then the probability could be shown to be conserved in time T , by using arguments similar to those used in nonrelativistic quantum mechanics to show that probability is conserved. (In the case of quantum gravity, such an argument would also rely on the existence of an ordering of the operators in (31) that renders them self-adjoint, in a functional sense, with respect to the

measure $d\mu[g]$.) Finally, notice that the definition (34) is essentially the same as that in Refs. 17 and 18, with the exception that there, the wave functionals only depend on the metric g_{ij} , and are therefore Λ eigenstates. Of course, besides the issue of the correct formal expression for probability, there is the fundamental question of the meaning of probability when applied to the Universe as a whole, which we will not address here.

An alternative definition of probability that is often advocated^{19,20} arises from interpreting the time-independent Wheeler-DeWitt equation (30a) as analogous to the Klein-Gordon equation, rather than the time-independent Schrödinger equation. This approach is plausible, because the superspace metric G_{ijkl} is not positive definite—the “conformal mode-conformal mode component” of G_{ijkl} is negative, suggesting that the conformally variable part of the three-metric should play the role of time. This leads to the definition of a Klein-Gordon-type probability current, and the familiar problem that the probability is not positive definite. It would then appear necessary to second (or, as is sometimes said, third) quantize the theory to obtain a valid probabalistic interpretation.

However, it is worth noting that the appearance of a negative component in G_{ijkl} does not by itself demand that the scalar product must be of the Klein-Gordon type. In fact, even the relativistic particle itself can be quantized using a Schrödinger-type scalar product.^{21–23} Such a quantization scheme is suggested by treating the usual canonical action for the relativistic particle as a Jacobi-type action, with the mass squared, m^2 , the fixed value of a constant of motion. Following the analysis of Sec. III, a Hamilton-type action is obtained in which “time” is related to the proper time of the particle and is conjugate to m^2 . Quantization based on the Jacobi-type action yields the usual “time-independent” Klein-Gordon

equation, whereas quantization based on the Hamilton-type action yields a “time-dependent” Klein-Gordon equation, analogous to the time-dependent Schrödinger equation. A positive-definite Schrödinger-type scalar product can then be introduced, resulting in a consistent first-quantized theory of the relativistic particle.²³ Also note that for nonrelativistic quantum mechanics, Kandrup²⁴ has explored the possibility of treating the time-independent Schrödinger equation as a Klein-Gordon equation, by selecting one of the coordinates to play the role of time. In the case of gravity, the question of interpretation is still open.

Finally, let us mention that the path integral based on the Jacobi action (8) and (9) is just the retarded Green function for the time-independent Schrödinger equation (27), assuming the histories integrated are those with a positive time interval, $t'' - t' = \int_{\sigma'}^{\sigma''} d\sigma N > 0$. The path integral based on Hamilton’s action is of course just the transition amplitude. The two functional integrals differ in that the one for Jacobi’s action includes an integral over the total time interval, and also contains a term $\int_{\sigma'}^{\sigma''} d\sigma NE = E(t'' - t')$ in the exponent. As expected then, the Green function and amplitude are related by an integral transform. The path-integral quantization of nonrelativistic mechanics based on Jacobi’s action will be developed more carefully elsewhere.²⁵

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