The Effect of a Stiffener on a Cracked Plate Under Bending

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Abstract

In this study the problem of a stiffened plate containing a through crack under uniform bending load is analyzed. The problem is formulated for a specially orthotropic material by using Reissner's plate theory. By using the Fourier integral transform technique the problem is reduced to a singular integral equation.

Asymptotic stress states around the crack tip are investigated and some sample results showing the effect of the crack geometry and the stiffener are also presented.

1. Introduction

A stiffened plate is a very common type of structure and the interaction of cracks with stiffener is an important problem. The original solution of the problem was given by Greif and Sanders in [1]. The special case of the problem in which the crack tip terminates at the stiffener was given by Yahsi and Erdoğan [2]. In both of these two solutions the cracked plate is under the effect of membrane loading only. However, from a practical point of view, a solution for the stiffened plate containing through crack under bending may be extremely important and in this paper, the problem of stiffened plate containing through crack under bending is analyzed by using the Reissner's plate theory. The special case of the problem in which the crack tip terminates at the stiffener is also considered in order to assess the crack arrest effectiveness of the stiffener.

2. Integral Equations

The problem under consideration is described in Fig.1. It will be assumed that by a proper superposition the problem is reduced to a stress disturbance problem in which self-equilibrating bending moment resultant applied to the crack surface is the only nonzero external load. It will also be assumed that the material is specially orthotropic and in this paper the plate equations developed by Delale and Erdoğan [3] for specially orthotropic material has been used.

From Fig.1 one may note that if the loading is symmetric, $x_1=0$ is a plane of symmetry and the solution of the stiffened plate problem may be expressed as follows:

$$w(x,y) = \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{\infty} [R_1(\alpha) + xR_2(\alpha)] e^{-|\alpha|x} e^{-iy\alpha} d\alpha \\ + \frac{2}{\pi} \int_0^{\infty} [S_1(\beta) + yS_2(\beta)] e^{-\beta y} \cos\beta x d\beta, & 0 < x < \infty, 0 < y < \infty, \\ \frac{1}{2\pi} \int_{-\infty}^{\infty} [R_1(\alpha) + xR_2(\alpha)] e^{-|\alpha|x} e^{-iy\alpha} d\alpha \\ + \frac{2}{\pi} \int_0^{\infty} [T_3(\beta) + yT_4(\beta)] e^{\beta y} \cos\beta x d\beta, & (-\infty < y < 0), \end{cases} \quad (1)$$

$$\psi(x,y) = \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{\infty} [(2b|\alpha| - x)R_2(\alpha) - R_1(\alpha)] e^{-|\alpha|x} e^{-iy\alpha} d\alpha \\ + \frac{2}{\pi} \int_0^{\infty} [(2b\beta - y)S_2(\beta) - S_1(\beta)] e^{-\beta y} \cos\beta x d\beta, & (0 < y < \infty), \\ \frac{1}{2\pi} \int_{-\infty}^{\infty} [(2b|\alpha| - x)R_2(\alpha) - R_1(\alpha)] e^{-|\alpha|x} e^{-iy\alpha} d\alpha \\ + \frac{2}{\pi} \int_0^{\infty} [(-2b\beta - y)T_4(\beta) - T_3(\beta)] e^{\beta y} \cos\beta x d\beta, & (-\infty < y < 0), \end{cases} \quad (2)$$

$$\Omega(x,y) = \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\alpha) e^{r_1 x} e^{-i\alpha y} d\alpha + \frac{2}{\pi} \int_0^{\infty} B(\beta) e^{s_1 y} \sin\beta x d\beta, & 0 < x < \infty, 0 < y < \infty, \\ \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\alpha) e^{r_1 x} e^{-i\alpha y} d\alpha + \frac{2}{\pi} \int_0^{\infty} C(\beta) e^{t_2 y} \sin\beta x d\beta, & (-\infty < y < 0), \end{cases} \quad (3)$$

where $r_1 = -[\alpha^2 + \frac{2}{b(1-\nu)}]^{1/2}$, $s_1 = -[\beta^2 + \frac{2}{b(1-\nu)}]^{1/2}$, $t_2 = [\beta^2 + \frac{2}{b(1-\nu)}]^{1/2}$ and $R_1, R_2, S_1, S_2,$

T_3, T_4, A, B and C are unknown functions. Considering symmetry the nine unknowns of the problem may be determined from the following continuity, equilibrium and boundary conditions:

$$w(x,0^+) = w(x,0^-), \quad \beta_x(x,0^+) = \beta_x(x,0^-), \quad \beta_y(x,0^+) = \beta_y(x,0^-), \quad 0 < x < \infty \quad (4a,b,c)$$

$$M_{yy}(x,0^+) = M_{yy}(x,0^-), \quad V_y(x,0^+) = V_y(x,0^-), \quad M_{xy}(x,0^+) - M_{xy}(x,0^-) = -A\epsilon \frac{\partial^2 \beta_x}{\partial x^2} \Big|_{(x,0^-)},$$

$$0 < x < \infty, \quad (5a,b,c)$$

$$V_x(0,y) = 0, \quad M_{xy}(0,y) = 0, \quad -\infty < y < \infty \quad (6a,b)$$

$$M_{xx}(0,y) = F(y) \quad -\sqrt{cd}/a < y < -\sqrt{cb}/a$$

$$\beta_x(0,y) = 0 \quad -\infty < y < -\sqrt{cd}/a, \quad -\sqrt{cb}/a < y < \infty \quad (7a,b)$$

where $F(y)$ is a known function. Equation (5c) is obtained from the equilibrium of the stiffener (of length dx_1) in x_1 direction by observing the bending moment in the stiffener is $E_s I_s \partial \beta_1 / \partial x_1$. Therefore, by using normalized quantities given in Appendix of [3] the stiffness constants A and ϵ are given by $A = I_s / (h^2 a^2 c^{3/2})$, $\epsilon = E_s / E$, where I_s is the moment of inertia of the cross sectional area of the stiffener, E_s and E are the Young's moduli of the stiffener and the plate, respectively.

Defining a new unknown function $G(y)$ as $G(y) = \partial u(x,0^+) / \partial y$, $-\infty < y < \infty$, and by using (4a,b,c), (5a,b,c) and (6a,b) all the unknowns of the problem can be expressed in terms of

$G(y)$ which may be determined from the mixed boundary conditions (7a,b).

From (7b) by observing that $G(y) = 0$ for $-\infty < y < -\sqrt{cd}/a$ and $-\sqrt{cb}/a < y < \infty$, after some simple manipulations (7a) may be reduced to

$$\frac{\gamma(1-\nu^2)}{2\pi} \int_{-\sqrt{cd}/a}^{-\sqrt{cb}/a} \frac{G(t)dt}{(t-y)} + \frac{\gamma(1-\nu)}{\pi} \int_{-\sqrt{cb}/a}^{-\sqrt{cb}/a} G(t) K(y,t) dt = F_1(y), \quad (8)$$

where

$$K(y,t) = K_S(y,t) + K_F(y,t), \quad (9)$$

$$K_S(y,t) = -\frac{(1+\nu)(3+\nu)}{2(3-\nu)} \frac{1}{(t+y)} + \frac{3\nu^2+4\nu-3}{2(3-\nu)} \frac{y}{(t+y)^2} + \frac{(1-\nu^2)}{(3-\nu)} \frac{y^2}{(t+y)^3}, \quad (10)$$

$$K_F(y,t) = \int_0^\infty [b(1-\nu)\alpha^2 + b(1-\nu)\alpha r_1 + 1] \sin \alpha(t-y) d\alpha \\ + \int_0^\infty \frac{\epsilon A [\beta + \frac{t\beta^2(1+\nu)}{2}]}{[4\gamma - \epsilon A(2b\beta^3 - 2b\beta^2 t_2 - \beta)]} [2b\beta t_2 e^{(t_2-\beta)y} - 2b\beta^2 - \beta y \\ + \frac{(1+\nu)}{1-\nu}] - \frac{(1+\nu)}{2} \left[\frac{2+(1-\nu)\beta t}{3-\nu} \right] \left[y\beta + \frac{(3+\nu)}{(1+\nu)} \right] e^{\beta(t+y)} d\beta. \quad (11)$$

If the crack is embedded in the left-hand plane (i.e. if $-d < -b < 0$), the Kernel $K(y,t)$ is bounded for all values of y and t in the closed interval $[-d, -b]$. However, if the crack terminates at the stiffener (i.e. if $b=0$, or $c=a$, Fig.1), then due to $K_S(y,t)$ term, $K(y,t)$ becomes unbounded as y and t go to the end point $b=0$ simultaneously. In this limiting case of $b=0$, $K_S(y,t)$ and the Cauchy Kernel in (8) constitute a generalized Cauchy Kernel.

To complete the formulation of the problem, one must also require the single-valuedness of the rotation. From equation (7b), it follows that for β_x to be single-valued G must satisfy the following condition:

$$\int_{-\sqrt{cb}/a}^{-\sqrt{cd}/a} G(t) dt = 0 \quad (12)$$

Thus, by solving the singular integral equation (8) with the single-valuedness condition (12), the unknown function $G(y)$ can be found.

For uniform bending moment M_{11} , the external load and related input function are:

$$M_{11}(0, x_2) = -M_{11} = -h^2 \sigma_b / 6, \quad -d < x < b; \quad F_1(y) = -\sigma_b / 6cE, \quad (13)$$

where σ_b is the maximum bending stress.

For $-b < 0$ the solution of (8) may be obtained by introducing the following dimensionless quantities:

$$\eta = \frac{y}{\sqrt{c}} + \frac{b+d}{2a}, \quad \tau = \frac{t}{\sqrt{c}} + \frac{b+d}{2a}, \quad H(\tau) = G(t), \quad (-1 < \eta, \tau < 1), \quad (14)$$

and by using a standard Gauss-Chebyshev integration procedure [4]. In this case the unknown function $H(\tau)$ is of the following form

$$H(\tau) = h(\tau) / \sqrt{1-\tau^2}, \quad (15)$$

where $h(\tau)$ is a bounded function. After determining $h(\tau)$, the Mode I stress intensity factors at the crack tips may be defined as

$$k_1(-b) = \lim_{x_2 \rightarrow -b} \sqrt{2(X_2+b)} \sigma_{11}(0, X_2, X_3). \quad (16)$$

By using results given in [3] and normalizing the stress intensity factor with respect to $\sigma_b \sqrt{a}$ one can obtain the following expression

$$k_b(-b) = -h(1) h/a c^{3/4} E/4\sigma_b \quad (17)$$

Similarly, the normalized Mode I stress intensity factor for the other end of the crack may be obtained as follows

$$k_b(-d) = h(-1) h/a c^{3/4} E/4\sigma_b. \quad (18)$$

In the case $b=0$, because of the Kernel K_s given by (10) the solution of integral equation (8) has no longer square root singularities. By introducing the new normalized quantities

$$\eta = y/\sqrt{C}+1, \quad \tau = t/\sqrt{C}+1, \quad H(\tau) = G(t), \quad -1 < \eta, \tau < 1,$$

assuming $H(\tau)$ of the form

$$H(\tau) = h(\tau)(1-\tau)^\alpha(1+\tau)^\beta \quad (-1 < \operatorname{Re}(\alpha, \beta) < 0), \quad (19)$$

and by following the function theoretical method outlined in 4 the characteristic equations of the problem giving α and β may be obtained as follows:

$$\cos \pi \beta = 0, \quad \cos \pi \alpha - \frac{(1+\nu)(3+\nu)}{2(3-\nu)} - \frac{(3\nu^2+4\nu-3)}{2(3-\nu)} \alpha + \frac{(\alpha^2-\alpha)(1-\nu^2)}{2(3-\nu)} = 0 \quad (20a,b)$$

Equation (20a) gives the expected result of $\beta = -1/2$ whereas (20b) shows that α depends on the Poisson's ratio of the plate only. With α and β known, (8) may be solved by using a Gauss-Jacobi integration formula [4]. For this case Mode I stress intensity factor normalized by $\sigma_b a^{-\alpha}$ can be obtained as follows:

$$k_b(0) = -h(1) \frac{cEh/a}{4\sigma_b \sin \pi \alpha} \left[-\frac{2\nu}{3-\nu} - \frac{2\nu^2+4\nu-2}{(3-\nu)(1+\nu)} \alpha + \frac{1-\nu}{3-\nu} \alpha^2 \right] \quad (21)$$

Similarly, the normalized Mode I stress intensity factor for the other end of the crack may be obtained as

$$k_b(-d) = h(-1) h/a 2^{\alpha+1/2} c^{3/4} E/4\sigma_b. \quad (22)$$

3. Asymptotic Stress Fields Around the Crack Tip

For the plane with a crack terminating at the stiffener, in order to study the initiation of various modes of fracture growth, the asymptotic behavior of the stress state around the crack tip $x=0, y=0$ has to be investigated. Going back to the basic formulation of the problem various moments, leading to respective stress components around the crack tip, may easily be

expressed in terms of bounded integrals with $G(y)$. By observing (see (19))
 $G(t) = h(\tau)(1-\tau)^\alpha(1+\tau)^{-1/2} = g(t)(-t)^\alpha(t+2\sqrt{c})^{-1/2}$ and by using the method explained in [2]
the asymptotic values of critical stress components near the crack tip $x=0, y=0$ may be
determined as follows:

$$\sigma_{xx}(0,y) = \frac{a3g(0)y^\alpha\gamma(1-v^2)}{\sqrt{2}c^{1/4}\sin\pi\alpha} \left[\frac{2v}{(v-3)} - \frac{(2v^2+4v-2)}{(3-v)(1+v)}\alpha + \frac{(1-v)}{(3-v)}\alpha^2 \right], \quad y > 0 \quad (23)$$

$$\sigma_{yy}(0,y) = \frac{a3g(0)y^\alpha\gamma(1-v^2)}{\sqrt{2}c^{1/4}\sin\pi\alpha} \left[\frac{2(2-v)}{(3-v)} + \frac{(2v^2+4v-2)}{(3-v)(1+v)}\alpha - \frac{(1-v)}{(3-v)}\alpha^2 \right], \quad y > 0 \quad (24)$$

$$\sigma_{yy}(x,-0) = \frac{a3g(0)x^\alpha\gamma(1-v)^2}{4\sqrt{2}c^{1/4}\sin\pi\alpha/2} \left[\frac{(2+v)(2+\alpha)}{(v-3)} \right], \quad x > 0 \quad (25)$$

$$\sigma_{xy}(x,-0) = \frac{a3g(0)x^\alpha\gamma(1-v)^2}{\sqrt{2}c^{1/4}} \left[\frac{(2-v+v^2)}{(3-v)(1-v)} \frac{1}{\cos\pi\alpha/2} - \frac{(2-v)}{(3-v)} \frac{\alpha}{\sin\pi\alpha/2} \right], \quad x > 0 \quad (26)$$

4. Results and Discussion

The main interest in this study is in evaluating the stress intensity factors in stiffened plates for various crack geometries and various values of the stiffness of the stiffener.

For an isotropic stiffened plate with a Poisson's ratio $\nu=0.3$ the calculated results are shown in Fig.2-4. From the analysis of these figures it can be seen that, as expected, the stress intensity factors decrease with the decrease in c/a value, i.e. as the crack moves closer to the stiffener. It is also important to observe that as c/a increases or as the crack tip moves away from the stiffener the effect of the stiffener decreases and the results approach to the corresponding unstiffened plate values. In agreement with this observation, it can be seen that the stress intensity factor at the crack tip near the stiffener is smaller than the stress intensity factor at the other end of the crack.

From Fig. 2 and 4 it can be seen that existence of stiffener has considerable effect on the stress intensity factor but this effect generally does not depend on the stiffness of the stiffener especially after a certain value of the stiffness constant S . From Fig. 3 and 4 it may also be seen that under uniform bending moment if the crack does not touch the stiffener, the stress intensity factors decrease as a/h increases.

5. References

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Fig.1. The Geometry of a Stiffened Plate Containing a Crack.

Fig.2. Stress Intensity Factor Ratio k_b in a Stiffened Plate Containing a Crack Under Uniform Bending Moment M_{11} , $\nu=0.3$, $a/h=5$.

Fig.3. Stress Intensity Factor Ratio k_b in a Stiffened Plate Containing a Crack Under Uniform Bending Moment M_{11} , $\nu=0.3$, $S=2$.

Fig.4. Stress Intensity Factor Ratio k_b in a Stiffened Plate Containing a Crack Under Uniform Bending Moment M_{11} , $\nu=0.3$, $c/a=1.5$.