Dynamic-Stiffness Matrix of Embedded and Pile Foundations by Indirect Boundary Element Method

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SUMMARY

The boundary-integral equation method is well suited for the calculation of the dynamic-stiffness matrix of foundations embedded in a layered visco-elastic halfspace (or a transmitting boundary of arbitrary shape), which represents an unbounded domain. It also allows pile groups to be analyzed, taking pile-soil-pile interaction into account. The discretization of this boundary-element method is restricted to the structure-soil interface. All trial functions satisfy exactly the field equations and the radiation condition at infinity. In the indirect boundary-element method distributed source loads of initially unknown intensities act on a source line located in the excavated part of the soil and are determined such that the prescribed boundary conditions on the structure-soil interface are satisfied in an average sense. In the two-dimensional case the variables are expanded in a Fourier integral in the wave number domain, while in three dimensions, Fourier series in the circumferential direction and Bessel functions of the wave number in the radial direction are selected. Accurate results arise with a small number of parameters of the loads acting on a source line which should coincide with the structure-soil interface. In a parametric study the dynamic-stiffness matrices of rectangular foundations of various aspect ratios embedded in a halfplane and in a layer built-in at its base are calculated. For the halfplane, the spring coefficients for the translational directions hardly depend on the embedment, while the corresponding damping coefficients increase for larger embedments, this tendency being more pronounced in the horizontal direction. For the rocking direction, both coefficients increase. The frequency dependency is more marked for increasing embedment. For the layer large oscillations of the dynamic-stiffness coefficients arise. For an undamped layer the damping coefficients vanish for frequencies below the fundamental frequency of the layer. It is demonstrated that for the foundation embedded either in a halfplane or in a layer, a two-dimensional model cannot be used to represent a truly three-dimensional situation.

1. INTRODUCTION

The dynamic-stiffness matrix of the soil $[S]$ appears in the basic equation of motion of the analysis of soil-structure interaction using the substructure method. This matrix specifies the amplitudes of the forces $\{P_b\}$ due to displacements of unit amplitude $\{u_b\}$ that are applied at the nodes b of the structure-soil interface for harmonic motion (Fig. 1a).

$$\{P_b\} = [S] \{u_b\} \quad (1)$$

The interface of arbitrary shape between the embedded structure and the soil can be flexible or rigid. The soil consists of horizontal layers of visco-elastic material with hysteretic damping which rest on a halfspace with the same material characteristics.

The boundary-integral equation method is well suited to model the unbounded soil domain. As analytical solutions satisfying exactly all field equations and the radiation condition at infinity are used as trial functions, only the boundary needs to be discretized. Well established procedures exist to calculate these analytical solutions for a layered halfspace, which will rigorously enforce the conditions at the interface of two adjacent layers and at the free surface. The latter results in only the structure-soil interface having to be modelled in this boundary-element method. Two approaches can be identified. In the first, called the direct method, both displacements and surface tractions are discretized in the boundary-integral equation which represents the exact solution. In the second, the indirect method, fictitious sources (loads) with unknown intensities are assumed to act in the part of the soil which later will be excavated. Their intensities are calculated such that the prescribed boundary conditions on the structure-soil interface are satisfied in a weighted integral sense.

To calculate the frequency-dependent dynamic-stiffness matrix, the direct method is applied in Refs. [1] and [2] for a structure embedded in a homogeneous halfspace and halfplane, respectively. Pile foundations, including pile-soil-pile interaction, are discussed in Ref. [3]. The indirect method is used in Ref. [4] to calculate the dynamic-stiffness matrix of cylindrical foundations embedded in a layered halfspace. The substructure deletion method of Ref. [5] could be interpreted as an indirect boundary-element method for embedded foundations, selecting the sources on the free surface. In Ref. [6], the authors calculate the dynamic-stiffness matrix of surface foundations on layered halfspaces and halfplanes. This same procedure, using the indirect boundary-element method is extended in this paper to embedded, two-dimensional foundations.

2. MATHEMATICAL FORMULATION

To calculate the dynamic-stiffness matrix $[S]$ of the foundation embedded in a layered halfplane, referred to the nodes b (Fig. 1a) located on the structure-soil interface S , a distribution of loads with initially unknown intensities is assumed to act on a source line S' (Fig. 1b). S' can coincide with S or be offset towards the soil region to be excavated, as illustrated in Fig. 1b. The loads act on the dynamic system consisting of the continuous soil, i.e. on the layered halfplane without excavation. Selecting an interpolation function $[L(s')]$, the load amplitudes $\{p(s')\}$ are expressed as a function of the source parameters $\{p\}$ associated with values at all nodes

$$\{p(s')\} = [L(s')] \{p\} \quad (2)$$

Here, s' denotes symbolically a point on the source line. For the two-dimensional case the source loads $p_x(s')$ and $p_z(s')$ will act in the horizontal and vertical directions. Discontinuities can be introduced, as shown for the horizontal direction in Fig. 1d. The amplitudes of the displacements $\{u_p(s)\}$ and of the line tractions $\{t_p(s)\}$ on the line S which subsequently will form the structure-soil interface are formulated as

$$\{u_p(s)\} = [g_u(s)] \{p\} \quad (3)$$

$$\{t_p(s)\} = [g_t(s)] \{p\} \quad (4)$$

(One column of $[g_u(s)]$ consisting of $u_p(s)$ and $w_p(s)$ is shown in Figs. 3a and 3b, and the corresponding column of $[g_t(s)]$ made up of $(t_x(s))_p$ and $(t_z(s))_p$ in Figs. 3c and 3d. The directions are illustrated in Fig. 1c, whereby the subscript p is omitted). The calculation of the Green's functions $[g_u(s)]$ and $[g_t(s)]$ of the continuous layered halfplane is outlined at the end of this section.

To calculate the dynamic-stiffness matrix of the soil, shape functions $[N(s)]$ are chosen which relate the amplitudes of the displacements at the nodes b , $\{u_b\}$, to those of the displacements on S , $\{u(s)\}$

$$\{u(s)\} = [N(s)] \{u_b\} \quad (5)$$

In Fig. 1c a linear expansion is shown for the horizontal direction.

The displacement-boundary condition on the structure-soil interface S can be satisfied only in an average sense as

$$\int_S [W(s)] (\{u_p(s)\} - \{u(s)\}) ds = 0 \quad (6)$$

which is used to determine the source parameters $\{p\}$. $[W(s)]$ denotes the weighting function, for which various choices are possible. In the following, $[W(s)] = [g_t(s)]^t$ is selected. Substituting eqs. 3 and 5 in eq. 6 results in

$$[G] \{p\} = [T] \{u_b\} \quad (7)$$

where

$$[G] = \int_S [g_t(s)]^t [g_u(s)] ds \quad (8a)$$

$$[T] = \int_S [g_t(s)]^t [N(s)] ds \quad (8b)$$

The flexibility matrix $[G]$ is symmetric, as is easily verified using Maxwell-Betti's reciprocity law. Formulating the latter for the two loading states corresponding to p_i and p_j results in

$$\int_S \{t_p(s)\}_i^t \{u_p(s)\}_j ds = \int_S \{t_p(s)\}_j^t \{u_p(s)\}_i ds \quad (9)$$

Substituting eqs. 3 and 4 formulated for p_i and p_j into eq. 9 leads to

$$p_i \int_S \{g_t(s)\}_i^t \{g_u(s)\}_j ds p_j = p_j \int_S \{g_t(s)\}_j^t \{g_u(s)\}_i ds p_i \quad (10a)$$

or, using eq. 8a

$$G_{ij} = G_{ji} \quad (10b)$$

The concentrated nodal forces $\{P_b\}$ are obtained as

$$\{P_b\} = \int_S [N(s)]^t \{t_p(s)\} ds \quad (11)$$

Now, solving eq. 7 for $\{p\}$ and substituting eq. 4 in eq. 11 leads to

$$\{P_b\} = [T]^t [G]^{-1} [T] \{u_b\} \quad (12)$$

This defines the dynamic-stiffness matrix $[S]$ (eq. 1) as

$$[S] = [T]^t [G]^{-1} [T] \quad (13)$$

For a rigid foundation, $[N(s)]$ in eq. 5 expresses rigid-body kinematics. $\{u_b\}$ is replaced by $\{u_0\}$, which denotes the rigid-body degrees of freedom.

The calculational procedure to determine the Green's functions $[g_u(s)]$ and $[g_t(s)]$ is outlined next. The load amplitudes $\{p(s')\}$ (arising from the source parameters $\{p\}$) can be expanded in the horizontal direction into a Fourier integral with terms e^{-ikx} , k being the wave number. As k runs from $-\infty$ to $+\infty$, all types of waves are treated. The following transforms apply

$$\begin{aligned} \{p(k,z)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{p(x,z)\} e^{ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [L(x,z)] e^{ikx} dx \{p\} = [L(k,z)] \{p\} \end{aligned} \quad (14a)$$

$$\begin{aligned} \{p(x,z)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{p(k,z)\} e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [L(k,z)] e^{-ikx} dx \{p\} = [L(x,z)] \{p\} \end{aligned} \quad (14b)$$

$\{p(x,z)\}$ denotes the load amplitudes $\{p(s')\}$ acting on the source line. Considering each layer loaded by $\{p(k,z)\}$ to be built-in at its interfaces, results in the local displacement and stress variations affecting $[g_u(k,z)]$ and $[g_t(k,z)]$, respectively. In addition, the external loads acting at the interfaces follow. These are then used in the direct stiffness approach for the analysis of the site (Refs. 7, 8 and 6). The global components of $[g_u(k,z)]$ and $[g_t(k,z)]$ are thus obtained. Applying the inverse transformation of eq. 14b to $[g_u(k,z)]$ and $[g_t(k,z)]$ results in $[g_u(x,z)]$ and $[g_t(x,z)]$.

3. APPLICATIONS

To evaluate the influence of embedment on the dynamic-stiffness matrix of the soil, the two extreme cases of a two-dimensional site, the halfplane and the homogeneous layer with depth d built-in at its base, are examined (Fig. 2). The rectangular soil-structure interface, along which welded contact is assumed, is regarded as rigid. The ratio of embedment to half-width h/b is varied parametrically. Poisson's ratio equals .33. The damping ratio equals 0.001, which represents the undamped case. For the layer a damping ratio of 0.05 is also investigated. The dynamic-stiffness matrix $[S]$ in eq. 13 is decomposed as follows

$$[S] = [S_0] ([k] + ia_0 [c]) \quad (15)$$

The diagonal matrix $[S_0]$ contains the values πG and $\pi G b^2$ for the translational and rotational degrees of freedom, respectively, where G is the shear modulus. $[k]$ and $[c]$ are the matrices of the (non-dimensionalized) spring and damping coefficients, respectively. k_x denotes the spring coefficient in the horizontal direction, k_z in the vertical direction, $k_{\phi y}$ is used for rocking (referred to the center of the rigid basemat, Fig. 2a) and $k_{x\phi y}$ represents the coupling between the horizontal direction and rocking. a_0 represents the dimensionless frequency

$$a_0 = \omega b / c_s \quad (16)$$

where c_s is the shear-wave velocity.

At first the various factors affecting the accuracy of the results are discussed for the foundation with $h/b = 0.5$ embedded in the halfplane for $a_0 = 1$. The Green's functions for a horizontal load whose amplitude varies linearly from unity at the free surface to zero at one third of the embedment and which acts on a line in the continuous soil, which will subsequently form the vertical part of the structure-soil interface shown in Fig. 3a, are investigated. This loading distribution corresponds to a source parameter of unity at the node on the free surface. In this case the source line is selected to coincide with the structure-soil interface. The corresponding amplitudes of the displacements in the horizontal and vertical directions u_p and w_p which are the two elements of the column of $[g_u(s)]$ associated with the source parameter are shown in Figs. 3a and 3b. Figs. 3c and 3d show the corresponding line tractions $(t_x)_p$ and $(t_z)_p$ which are the elements of a column of $[g_t(s)]$. On the vertical part of the source line three elements are selected and four on the horizontal part corresponding to the half-width. The load of the source varies linearly along an element with continuity generally enforced at all nodes except the corner one. This interpolation of the source loads results in a total of 18 source parameters for half of the embedded foundation. To evaluate the integrations in $[G]$ and $[T]$ in eq. 8, two Gaussian integration points are chosen on each element. For the prescribed displacement amplitudes corresponding to the three degrees of freedom u_0 , w_0 and ϕ_y (Fig. 2a), the variation of the source loads $\{p(s')\}$ determined from eqs. 7 and 2, of the line tractions $\{t_p(s)\}$ acting on the structure-soil interface calculated from eq. 4 and of the displacements $\{u_p(s)\}$ resulting from eq. 3 are presented in Figs. 4a to 4c.

To study the influence of selecting different locations for the source line with respect to the structure-soil interface, the offset e shown in Fig. 2a is varied, keeping all other factors constant. It follows from the variation of the sources, tractions and displacements for the prescribed vertical displacement shown in Figs. 5a and 5b for $e/b = 0.05$ and 0.2 respectively, when compared to Fig. 4b, that the more the offset is increased, the less accurate become the results, which is apparent from the unrealistic oscillations. This is also confirmed from the values of dynamic-stiffness coefficients shown for all degrees of freedom in Figs 5c and 5d, although to a lesser extent, since these are "global" values determined by integration along the structure-soil interface. In the following the source line is always selected to coincide with the structure-soil interface.

The number of source parameters is addressed next. Reducing their number to 8 for half of the foundation results in a linear variation of the source loads along the embedment and the half-width. This leads to a variation of the sources, tractions and displacements shown in Fig. 6a for the prescribed vertical displacement, which bears comparison to that of 18 source parameters (Fig. 4b). The agreement of the dynamic-stiffness coefficients presented in Figs. 6c and 6d is also quite good. As expected, choosing 26 source parameters hardly affects the results (Figs. 6b, 6c and 6d). 18 source parameters are sufficient for this ratio of h/b .

The parametric study is performed selecting the source line to coincide with the structure-soil interface ($e = 0$ in Fig. 2a). On the horizontal part of the source line 4 elements are used per half width. In the vertical direction 3 elements are chosen for $h/b = 0.5$, 4 elements for $h/b = 1$ and 6 elements for $h/b = 2$. This results in a total of 18, 20 and 24 source parameters per half width, respectively, for the three cases. The results for the halfplane (Fig. 2a) are discussed first. For comparison the dynamic-stiffness coefficients are presented for the surface foundation ($h/b = 0$) in Fig. 7. For the embedded foundation the results are shown in Fig. 8 for $h/b = 0.5$, in Fig. 9 for $h/b = 1$ and in Fig. 10 for $h/b = 2$. Quite surprisingly, the two translational spring coefficients are not strongly affected by embedment (in contrast to the cylinder embedded in a halfspace (Ref. [4])) while the increase of the corresponding damping coefficients is approximately proportional to the embedment, the increase in the horizontal direction being more marked. In the rocking direction both the spring and damping coefficients increase with embedment. Augmenting the embedment results in a stronger frequency dependence of the dynamic-stiffness coefficients.

Turning to the foundation embedded in a homogeneous layer (Fig. 2b), the dynamic-stiffness matrix for $h/b = 1$ and for $d/b = 2$ is examined. For the undamped case the results are presented in Fig. 11. Especially in the high frequency range, large oscillations exist. Even negative spring coefficients arise. In contrast to the elastic halfplane the built-in layer exhibits a cut-off frequency below which, for the case of no damping, no radiation of energy occurs. c_x is zero below the horizontal fundamental frequency of the layer which corresponds to $a_0 = \pi/4$. For larger frequencies significant values arise. c_z and c_{ϕ_y} are almost zero below the vertical fundamental frequency $a_0 = \pi/2$. For the damped case (Fig. 12) no cut-off frequency exists any more. Below the fundamental frequencies the imaginary part of the dynamic-stiffness coefficients $a_0 c$ remains small. In addition, the dependency on frequency is reduced. The dynamic-stiffness coefficients for a surface foundation, $h/b = 0$, resting on the same layer are presented in Fig. 13 as comparison. As expected, the translational spring

coefficients depend strongly on the embedment for the layer built-in at its base (Figs. 12a and 13a).

Finally, the controversial matter of modelling a three-dimensional problem with a two-dimensional model is briefly addressed. Besides the strip's depth in the y -direction (transverse direction), which enters only into the stiffness matrix $[S_0]$, the half-width b of the equivalent two-dimensional model has to be selected. Arbitrarily choosing b equal to the radius of the embedded cylinder, it is instructive to examine the ratio of the damping coefficient and of the spring coefficient for the two- and three-dimensional cases. In Fig. 14a $(c/k)_{2-D}/(c/k)_{3-D}$ is plotted for the various degrees of freedom as a function of a_0 for the foundation with $h/b = 1$ embedded in the damped layer with $d/b = 2$. The values for the two-dimensional case follow from Fig. 12, those for the three-dimensional case are taken from Ref. [9]. For the rocking motion this ratio is larger than 1 for a significant range of frequency, suggesting that the two-dimensional model will significantly underestimate the true three-dimensional dynamic response. For the translational degrees of freedom the tendency is nonuniform. In Fig. 14b the same information is shown for the foundation with $h/b = 1$, embedded in the undamped halfplane/halfspace. The results for the three-dimensional case originate from Ref. [4], where somewhat different values of the Poisson's ratio (0.25) and the damping ratios (0.01 for shear waves and 0.005 for compressional waves) are used. The two-dimensional approximation vastly overestimates the damping for the translational degrees of freedom, which leads to a significant underestimation of the response. The same applies, to a lesser degree, to the rocking direction.

Three dimensional loads acting in a layered halfspace (Fig. 15) can also be processed by choosing a Fourier series in the circumferential direction θ and Bessel functions involving the wave number k in the radial direction r (Refs. [7] and [6]). This allows the dynamic-stiffness matrix of pile groups, including pile-soil-pile interaction, to be calculated. The source loads can be selected to act along the axes of the piles. The computational procedure is outlined schematically in Fig. 15.

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Fig. 1 - Nomenclature for Embedded Foundation

a) Discretized Structure-Soil Interface b) Source Line and Structure-Soil Interface

c) Prescribed Displacement d) Selected Source Load Distribution

Fig. 2 - Investigated Sites

a) Halfplane b) Layer Built-In at its Base

Fig. 3 - Green's Functions for Source Load of Unit Amplitude

Fig. 4 - Variation of Sources, Tractions and Displacements for
a) Prescribed Horizontal Displacement
b) Prescribed Vertical Displacement
c) Prescribed Rocking Displacement

Fig. 5 - Influence of Offset of Source Line from Structure-Soil Interface

a) Variation of Sources, Tractions and Displacements for Prescribed Vertical Displacement for $e/b = 0.05$

b) Variation of Sources, Tractions and Displacements for Prescribed Vertical Displacement for $e/b = 0.2$

c) Spring Coefficients

d) Damping Coefficients

Fig. 6 - Influence of Number of Source Parameters

a) Variation of Sources, Tractions and Displacements for Prescribed Vertical Displacement for 8 Source Parameters

b) Variation of Sources, Tractions and Displacements for Prescribed Vertical Displacement for 26 Source Parameters

c) Spring Coefficients

d) Damping Coefficients

Fig. 7 - Dynamic-Stiffness, Halfplane, $h/b = 0$, no Damping

Fig. 8 - Dynamic-Stiffness, Halfplane, $h/b = 0.5$, no Damping

Fig. 9 - Dynamic-Stiffness, Halfplane, $h/b = 1$, no Damping

Fig. 10 - Dynamic-Stiffness, Halfplane, $h/b = 2$, no Damping

Fig. 11 - Dynamic-Stiffness, Layer Built-in at its Base, $d/b = 2$, $h/b = 1$, no Damping

Fig. 12 - Dynamic-Stiffness, Layer Built-in at its Base, $d/b = 2$, $h/b = 1$, with Damping

Fig. 13 - Dynamic-Stiffness, Layer Built-in at its Base, $d/b = 2$, $h/b = 0$, with Damping

Fig. 14 - Ratio of Damping to Spring Coefficient, 2-Dimensional versus 3-Dimensional
a) Layer Built-in at its Base, $d/b = 2$, $h/b = 1$, with Damping
b) Halfplane/Halfspace, $h/b = 1$, no Damping

Fig. 15 - Computational Procedure for Dynamic Stiffness Matrix of Pile Groups: Prescribed Displacement, Source Load with Corresponding Surface Traction and Displacement