Relaxation of a Rectangular Beam and Circular Shaft

N.S. Ottosen

Engineering Department, Risø National Laboratory, DK-4000 Roskilde, Denmark

This paper provides exact and approximate solutions to the title problems. Time-hardening creep is adopted and the stress dependence is assumed to follow the exponential expression proposed by Soderberg. Previous investigations have demonstrated the predictions of this expression to be in close agreement with experimental data. For small stresses the expression results in a linear stress dependence in evident contrast to the predictions of Norton's power law, but in agreement with recent experimental data.

The derived exact solutions involve infinite series expressions which, however, converge very quickly. As an example, the maximum error is around 6% if the series expressions are completely ignored and around 1% if just the first term in the series is considered. This accuracy is clearly superior to those of the approximations currently in use, namely the Kachanov approximation and the reference stress concept. Moreover, error bounds, which are directly and easily available, are related to the present approximations. In fact, the maximum error is less than the magnitude of the first series term not considered.

The exact and approximate solutions are applied to a rectangular beam and circular shafts. Different physical aspects are discussed and a comparison is performed with the predictions following the use of Norton's creep law.

INTRODUCTION

The behaviour of structural elements under high temperature where creep deformations occur is of great importance for the design of power plant systems. For such systems the creep strains depend nonlinearly on the stresses. As a result the solutions become complex and numerical solutions are most often necessary. This situation arises both when the boundary stresses are known, as in the case of the relaxation problem where the known boundary displacements are held constant. Often, however, relaxation problems are somewhat simpler as the constitutive conditions become differential equations in which all terms are known explicitly. This is the case for rectangular beams and circular shafts.

However, only a few closed-form solutions are known for such structures. For linear viscoelasticity the solution becomes trivial. For nonlinear creep, on the other hand, exact solutions have been derived apparently only when the stress exponent in Norton's creep law equals 4 and 5 for the rectangular beam and circular shaft, respectively, cf. Kachanov [1]. Otherwise, we must resort to tedious numerical integrations even though some approximate solutions have been discussed by Spence and Hult [2] for the beam problem.

In the present paper closed-form solutions are derived using the nonlinear exponential creep law proposed by Soderberg [3]. Even though these solutions are very easy to work with, some approximate solutions are discussed. These approximate solutions are obtained from the exact ones by simply ignoring certain terms; error bounds are then directly available. The exact and approximate solutions are applied to specific problems and some physical aspects are discussed.

CONSTITUTIVE EQUATIONS

Assuming small strains and deformations, the total strains are composed of the elastic and the creep strains, i.e.

$$\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^c \quad (1)$$

where a dot indicates the time rate. The elastic strains follow from Hookes law. The creep strain rates are assumed to be of the usual type derived from a von Mises potential. Therefore, using standard tensor notation we have

$$\dot{\epsilon}_{ij}^c = \frac{3\dot{\epsilon}_e}{2\sigma_e} s_{ij} \quad (2)$$

where $\dot{\epsilon}_e = (2\dot{\epsilon}_{ij}^c \dot{\epsilon}_{ij}^c/3)^{1/2}$ and $\sigma_e = (3s_{ij}s_{ij}/2)^{1/2}$ are the effective creep strain rate and effective stress, respectively, and $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{kk}/3$ is the deviatoric stress tensor.

Time-hardening is assumed and the stress dependence is assumed to follow the exponential form proposed by Soderberg [3], i.e.

$$\dot{\epsilon}_e = mA(e^{B\sigma_e} - 1)t^{m-1} \quad (3)$$

where m , A and B are positive parameters and t is the time. According to

both Soderberg [3] and Popov [4], this expression provides close fits to experimental data. Moreover, for practical purposes, it gives almost identical results as the hyperbolic sine relation suggested, for instance, by Nadai [5]. Nadai and McVetty [6] and McVetty [7] found that the hyperbolic sine relation provides a better fit to experimental data than Norton's power law, whereas Pickel et al. [8] found almost the same accuracy.

In the past, most papers have favoured Norton's power law due to its apparent simplicity. However, as shown here and in [9] the differential equations involved in creep problems are sometimes more amenable to analytical solutions when the Soderberg expression (or alternatively the hyperbolic sine relation) is adopted. Moreover, for small stresses these expressions provide a linear stress dependence in evident contrast to the predictions of the power law. This linear behaviour is confirmed by recent experimental data for metals and steels [10,11] and it is in accordance with the general appearance of deformation maps.

BENDING OF A RECTANGULAR BEAM

Consider a rectangular beam made of a homogeneous, isotropic material and loaded by a bending moment M , cf. fig.1. The relaxation condition infers that $\dot{\epsilon}_z = 0$; as the stress state is uniaxial then, using eqs. (1)-(3), the constitutive equation reads

$$\dot{\sigma}_e + mAE(e^{B\sigma_e} - 1)t^{m-1} = 0$$

where E is Young's modulus. Using the transformation $\phi = e^{-B\sigma_e}$, the solution to this equation is obtained as

$$\sigma_e = -\frac{1}{B} \ln[1 + C(y)e^{-Nt^m}] \quad (4)$$

where the positive parameter N is defined by

$$N = ABE$$

The arbitrary function $C(y)$ is determined so that (4) for $t = 0$ yields the initial elastic solution; thus,

$$\sigma_e = -\frac{1}{B} \ln(a + b e^{-H|y|}) \quad (5)$$

where the two time-dependent functions a and b are defined by

$$a(t) = 1 - b(t); \quad 0 \leq a < 1, \text{ and } b(t) = e^{-Nt^m}; \quad 0 < b \leq 1$$

and where the positive parameter H is defined by

$$H = B \frac{|M_i|}{I}$$

In this expression, M_i denotes the initial bending moment and $I = wh^3/12$ is the moment of inertia about the x -axis.

The time-dependent moment M is given by

$$M = 2wT \int_0^{h/2} \sigma_e y dy$$

where $T = 1$ if $M_i > 0$ and $T = -1$ if $M_i < 0$. Use of eq. (5) yields

$$M = -\frac{2wT}{B} \int_0^{h/2} y \ln(a + b e^{-Hy}) dy \quad (6)$$

Integration of this expression is performed after the logarithmic function is expanded into a Taylor series. However, this expansion depends on the magnitude of the positive term $\frac{b}{a} e^{-Hy}$, and it becomes convenient to define the time-dependent distance y_0 so that $\frac{b}{a} e^{-Hy_0} = 1$, i.e.

$$y_0 = -\frac{1}{H} \ln \frac{a}{b}$$

Thus y_0 decreases with time. If $y_0 \geq y$ then $\frac{b}{a} e^{-Hy} \geq 1$, whereas if $y_0 \leq y$ then $\frac{b}{a} e^{-Hy} \leq 1$. Depending on the value of y_0 , we are now in a position to determine the bending moment using (6).

If $y_0 \leq 0$ then

$$\ln(a + b e^{-Hy}) = \ln a - \sum_{n=1}^{\infty} \frac{1}{n} \left(-\frac{b}{a} e^{-Hy}\right)^n$$

By straightforward integrations, eq. (6) then becomes

$$M = -\frac{2wT}{B} \left\{ \frac{h^2}{8} \ln a + \frac{1}{H^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n \left[\left(\frac{b}{a} e^{-Hh/2} \right)^n \left(\frac{Hh}{2} + \frac{1}{n} \right) - \left(\frac{b}{a} \right)^n \frac{1}{n} \right] \right\} \quad (7)$$

If $y_0 \geq \frac{h}{2}$ then

$$\ln(a + b e^{-Hy}) = \ln b - Hy - \sum_{n=1}^{\infty} \frac{1}{n} \left(-\frac{a}{b} e^{Hy} \right)^n$$

Use of this expression in eq. (6) yields

$$M = -\frac{2wT}{B} \left\{ \frac{h^2}{8} \ln b - \frac{h^3}{24} H - \frac{1}{H^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n \left[\left(\frac{a}{b} e^{Hh/2} \right)^n \left(\frac{Hh}{2} - \frac{1}{n} \right) + \left(\frac{a}{b} \right)^n \frac{1}{n} \right] \right\} \quad (8)$$

For $t = 0$, the initial elastic solution follows from (8) with $a = 0$ and $b = 1$. Finally, if $0 \leq y_0 \leq \frac{h}{2}$ then by completely analogous calculations we obtain

$$M = -\frac{2wT}{B} \left\{ \frac{y_0^2}{2} \ln \frac{b}{a} + \frac{h^2}{8} \ln a - \frac{y_0^3}{3H} + \frac{1}{H^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n \left[\left(\frac{b}{a} e^{-Hh/2} \right)^n \left(\frac{Hh}{2} + \frac{1}{n} \right) - \left(\frac{a}{b} \right)^n \frac{1}{n} - 2Hy_0 \right] \right\} \quad (9)$$

The series present in (7) - (9) are alternating, i.e., they converge if the absolute value of the term, n , decreases, and approaches zero as $n \rightarrow \infty$. Noting that eqs. (7) - (9) have been obtained under recognition of the magnitude of the term $\frac{b}{a} e^{-Hy}$ and that $\frac{a}{b} > 1$ only when $y_0 < 0$ then the convergence of the series in (7) - (9) follows directly.

Thus, depending on the value of y_0 the bending moment M is determined using either (7), (8) or (9). As only a , b and y_0 depend on time, these closed-form solutions are quite easy to work with. Moreover, the series present in the solutions converge very quickly. In addition to that these terms are small compared with the other terms. This means that a close estimate to the bending moment can be achieved by ignoring the series expressions completely and considering just the first term in the series will raise the accuracy to a very high

level. These important aspects are demonstrated in the section on applications. It is very essential to note that as the series alternate the maximum error of such approximations is less than the magnitude of the first series term not considered.

TORSION OF A CIRCULAR SHAFT

Consider a circular shaft made of a homogeneous, isotropic material and loaded by a twisting moment M_t , cf. fig. 2. Due to symmetry the torsional moment implies shear stresses only. Therefore $\sigma_e = T\sqrt{3}\tau$ where τ is the shear stress and $T = 1$ if $M_t > 0$ and $T = -1$ if $M_t < 0$.

The determination of the shear stress is completely similar to the derivation of eq.(5). Therefore, we shall just state the result which is

$$\tau = -\frac{T}{B\sqrt{3}} \ln(c + de^{-Qr}) \quad (10)$$

where the two time-dependent functions c and d are defined by

$$c(t) = 1 - d(t); \quad 0 \leq c < 1, \text{ and } d(t) = e^{-St^m}; \quad 0 < d \leq 1$$

and where the two positive parameters S and Q are defined by

$$S = 3 ABG; \quad Q = T \frac{\sqrt{3} B M_{t,i}}{I_p}$$

In these expressions, G is the shear modulus, $M_{t,i}$ denotes the initial torque, and $I_p = \pi(r_2^4 - r_1^4)/2$ the polar moment of inertia.

Using (10) the time-dependent torque M_t is given by

$$M_t = -\frac{2\pi T}{B\sqrt{3}} \int_{r_1}^{r_2} r^2 \ln(c + de^{-Qr}) dr \quad (11)$$

As was done previously, integration of this expression is performed after the logarithmic function is expanded into a Taylor series. Here, this expansion depends on the magnitude of the positive term $\frac{d}{c}e^{-Qr}$, and it becomes convenient to define the time-dependent radius r_0 so that $\frac{d}{c}e^{-Qr_0} = 1$, i.e.

$$r_0 = -\frac{1}{Q} \ln \frac{c}{d}$$

Thus r_0 decreases with time. If $r_0 \geq r$ then $\frac{d}{c}e^{-Qr} \geq 1$, whereas if $r_0 \leq r$ then $\frac{d}{c}e^{-Qr} \leq 1$. Depending on the value of r_0 we are now in a position to determine the twisting moment using (11). The derivations are completely similar to the previous ones and we shall therefore just state the results.

If $r_0 \leq r_1$, then

$$M_t = -\frac{2\pi T}{B\sqrt{3}} \left\{ \frac{1}{3}(r_2^3 - r_1^3) \ln c + \right. \quad (12)$$

$$\left. \frac{1}{Q} \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n \left[\left(\frac{d}{c} e^{-Qr_2} \right)^n \left(r_2^2 + \frac{2}{Qn} r_2 + \frac{2}{(Qn)^2} \right) - \left(\frac{d}{c} e^{-Qr_1} \right)^n \left(r_1^2 + \frac{2}{Qn} r_1 + \frac{2}{(Qn)^2} \right) \right] \right\}$$

If $r_0 \geq r_2$, then

$$M_t = -\frac{2\pi T}{B\sqrt{3}} \left\{ \frac{1}{3}(r_2^3 - r_1^3) \ln d - \frac{1}{4}(r_2^4 - r_1^4) Q - \right.$$

$$\frac{1}{Q} \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n \left[\left(\frac{c}{d} e^{Qr_2} \right)^n \left(r_2^2 - \frac{2}{Qn} r_2 + \frac{2}{(Qn)^2} \right) - \left(\frac{c}{d} e^{Qr_1} \right)^n \left(r_1^2 - \frac{2}{Qn} r_1 + \frac{2}{(Qn)^2} \right) \right] \quad (13)$$

For $t = 0$, the initial elastic solution follows from (13) with $c=0$ and $d=1$.

If $r_1 \leq r_0 \leq r_2$, then

$$M_t = -\frac{2\pi T}{BV^3} \left\{ \frac{1}{3}(r_0^3 - r_1^3) \ln d + \frac{1}{3}(r_2^3 - r_0^3) \ln c - \frac{1}{4}(r_0^4 - r_1^4) Q + \frac{1}{Q} \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n \left[\left(\frac{d}{c} e^{-Qr_2} \right)^n \left(r_2^2 + \frac{2}{Qn} r_2 + \frac{2}{(Qn)^2} \right) + \left(\frac{c}{d} e^{Qr_1} \right)^n \left(r_1^2 - \frac{2}{Qn} r_1 + \frac{2}{(Qn)^2} \right) - 2 \left(r_0^2 + \frac{2}{(Qn)^2} \right) \right] \right\} \quad (14)$$

Identical comments apply to these closed-form expressions as to the solutions of the beam problem.

APPLICATIONS AND APPROXIMATIONS

For an austenitic stainless steel at high temperature assume the following typical creep parameters of eq. (3): $m=1$, $A=7 \cdot 10^{-9} [1/h]$ and $B = 5.4 \cdot 10^{-2} [1/MPa]$.

The first problem is that of relaxation of a rectangular beam with $h = 2 \cdot 10^{-2} m$ and $w = 10^{-2} m$. Moreover, $E = 1.5 \cdot 10^5 MPa$ and $M_i = 100 Nm$.

Figure 3 shows the development with time of the bending moment. The exact solution follows directly from (7)-(9) using a sufficient number of terms in the series expressions. The approximate solution corresponds to the very simple expressions which follow if all series terms are completely ignored. A close estimate is obtained.

For Norton's power law, Spence and Hult [2] made a comparison between the exact numerical solutions and the approximations currently in use, namely the Kachanov approximation [1] and the reference stress concept. The relative errors of the simple approximate solution suggested here are much smaller than those related to the Kachanov approximation and at least as small as those of the reference stress concept. Apart from that, the accuracy of the approximate solution shown here demonstrates that the exact solution is very easy to work with, as the series present in (7)-(9) converge very quickly. As an illustration of this important aspect, the maximum error of the simple approximate solution is less than 6% and if just the first term in the series is considered, the accuracy increases to around 1%, which is close to the drawing accuracy in fig. 3. If the first four terms in the series are considered the accuracy is better than 0.2%. Consequently, even hand-calculations suffice to achieve a very high accuracy. Moreover, as discussed previously, simple error bounds are directly available for these approximate solutions.

It might be of interest to compare the exact solution using the Soderberg creep law with one using the Norton creep law. Noting that the maximum value of the effective stress is 150 MPa for the beam considered, the effective creep strain rates predicted by the Soderberg expression are shown in fig. 4 for the stress range of interest. If Norton's creep law is to be calibrated to the same stress range several possibilities exist. One possibility infers that the stress exponent $n = 5$, cf. fig. 4. For this exponent the behaviour shown in fig. 3 was determined by Spence and Hult [2] by numerical integrations. Another reasonable

calibration of Norton's creep law provides $n = 4$, cf. fig.4. In that case it is possible to obtain simple closed-form solutions, cf. Kachanov [1]. The resulting development is shown in fig.3. Compared with the Soderberg creep law this figure shows that initially Norton's law will predict a more rapid decrease of the bending moment whereas later it will predict a larger bending moment. These differences are readily explained by fig.4 and are a result of the large range of the effective stress which includes zero stress. It is of interest that the choice between the Soderberg and Norton expressions results in larger differences than the differences between the exact and the simplest approximate solution of the Soderberg relaxation problem.

Consider now a solid circular shaft in relaxation. The creep parameters are as before. Moreover, $G = 5.8 \cdot 10^4$ MPa, $r_1 = 0$, $r_2 = 9 \cdot 10^{-3}$ m and $M_{t,i} = 100$ Nm.

The development with time of the torque is shown in fig.5. As above, the approximate solution corresponds to the very simple expressions which follow from eqs.(12)-(14), if all series terms are completely ignored. The accuracy of this approximate solution as well as the accuracies of those approximate solutions, in which only the first term or the first four terms in the series are considered, are equal to the accuracies obtained for the beam. Therefore, the appropriateness of the use of eqs.(12)-(14) follow directly. As before, error bounds are readily available for approximate solutions.

The maximum effective stress is again around 150 MPa and if Norton's creep law is used, cf. fig.4, it is possible to obtain simple closed-form solutions when $n = 5$, cf. Kachanov [1]. The resulting development of the torque is shown in fig.5. Similar conclusions as those related to fig.3 can be drawn.

The final examples refer to circular shafts with different r_1/r_2 -ratios. Apart from the r_1 -values all data are as previously. The development with time of the torque is shown in fig.6. It appears that the thinner the circular shaft the more accurate the simple approximate solution which ignores all series terms completely. Naturally, the thinner shafts relax faster because they are initially more highly stressed.

CONCLUSIONS

Assuming that creep is determined by the exponential expression proposed by Soderberg [3], exact and approximate closed-form solutions have been derived for the relaxation of rectangular beams and circular shafts. The exact solutions involve infinite series which converge very quickly, however. Moreover, the contributions from these series are small. Therefore, by ignoring these series completely very simple approximate solutions can be obtained having an accuracy of around 6% and considering just the first term in the series raises the accuracy to around 1%. Simple error bounds for such approximate solutions are directly available. These aspects are clearly superior to the approximations currently in use. Finally, specific examples are considered and compared with the predictions following the use of Norton's power law.

REFERENCES

[1] Kachanov, L.M., "The Theory of Creep", (Transl.), National Lending Library for Science and Technology, UK (1967).

[2] Spence, J., Hult, J., "Simple Approximations for Creep Relaxation", Int. J. Mech. Sci. 15, 741-755 (1973).

[3] Soderberg, C.R., "The Interpretation of Creep Tests for Machine Design", Trans. ASME 58, 733-743 (1936).

[4] Popov, E.P., "Correlation of Tension Creep Tests with Relaxation Tests", Trans. ASME 69, A 135-A 142 (1947).

[5] Nadai, A., "The Influence of Time upon Creep. The Hyperbolic Sine Creep Law", 155-170 in Stephen Timoshenko Anniversary Volume, The Macmillan Co., New York (1938).

[6] Nadai, A., McVetty, P.G., "Hyperbolic sine chart for estimating working stresses of alloys at elevated temperatures", Am. Soc. Testing Mats. 43, 735-745 (1943).

[7] McVetty, P.G., "Creep of Metals at Elevated Temperatures-the Hyperbolic-Sine Relation Between Stress and Creep Rate", Trans. ASME 65, 761-767 (1943).

[8] Pickel, T.W., Sidebottom, O.M. Boresi, A.P., "Evaluation of Creep Laws and Flow Criteria for Two Metals Subjected to Stepped Load and Temperature Changes", Exp.Mech.11, 202-209 (1971).

[9] Ottosen, N.S., "Relaxation of Thick-Walled Cylinders and Spheres", J.Appl.Mech. 49, 487-491 (1982).

[10] Yavari, P., Langdon, T.G., "A Constant Stress Tensile Creep Machine for Very Low Stresses", J. Test. Eval. 10, 174-178 (1982).

[11] Cho, U.W., Findley, W.N., "Creep and Creep Recovery of 304 Stainless Steel at Low Stresses with Effects of Aging on Creep and Plastic Strains", J.Appl.Mech.48, 785-790 (1981).

Fig. 1 - Beam configuration

Fig. 2 - Shaft configuration

Fig. 3 - Development of bending moment

Fig. 4 - Effective creep strain rates

Fig. 5 - Development of torque

Fig. 6 - Development of torques