Interaction Buckling — Progressive Deformation

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SUMMARY :

Progressive deformation due to ratchetting may lead to the failure of structure in two different manners : by local fracture of the material when the strain is greater than an admissible limit or by a global destabilization of the loaded structure. This second mechanism will be considered here and corresponds to the principal interaction between buckling and progressive deformation.

Under additional loading over a given dead load P , a structure initially in equilibrium will have an associated displacement $u(t)$. Its failure by instability may occur if this displacement is great enough so that the response attains an instable branch of the load - displacement curve. Under cyclic additional loading of thermal or mechanical nature, this situation happens if there is ratchetting. The progressive plastic deformation which increases after each cycle, will produce greater and greater displacement and then perhaps failure by instability before local fracture by excessive local strain.

In this paper, we present a theoretical and numerical analysis of this phenomena in function of the material characteristics and of the amplitude of the loads.

In the first part, numerical calculations of the response of simple structures such as Shanley model or beam models under axial compression and cyclic thermal condition are performed to obtain the failure mechanism. It is shown that the interaction between buckling and progressive deformation is strong for weak material hardening. As a consequence, perfect plastic structure are particularly sensitive to this interaction.

In the second part, the theoretical aspect of the problem is investigated when the response may be considered as small motions superposed on the equilibrium configuration. The stabilization of the cyclic response is discussed. Elastic shake-down conditions are obtained following a result from Maier. Plastic shake-down theorem is established for the kinematic hardening model, following a result from Halphen. Our principal result is the definition of a critical load P_{CS} in the sense of the security such that if $P < P_{CS}$ then plastic or elastic shake-down theorems are available and one can define a bound for the additional load to prevent the loaded structure from progressive deformation. If $P > P_{CS}$ our examples showed that the destabilization is practically certain. The critical load P_{CS} may be identified as the smallest tangent modulus critical load.

1 - INTRODUCTION

It is well known that the study of the equilibrium of a thin structure, like a beam or a shell, under a given load P has to take into account the risk of buckling. The behaviour of this structure in that case depends on many parameters. For usual structures, the maximum load P_m depends very much on the difference between the real geometry and the one of the associated perfect structure, what is called geometric imperfections, cf. Koiter [8]. Many studies have been developed in order to determine this maximal load P_m , specially for elastic and plastic materials.

Let us consider a stable equilibrium of a thin structure under the dead loading $P < P_m$. If additional thermal or mechanical cyclic loads are applied, the structure response can be described by the displacement field $u(t)$ with respect to the perfect structure. This displacement appears to show how the initial imperfections $\bar{u} = u(0)$ grow with the action of the additional loads.

If this displacement becomes big enough, then buckling can occur, because the stability depends on the actual value of both the load P and the displacement u as it is suggested by the load-displacement curves obtained in several studies in elasticity and mainly in plasticity. We understand thus how the progressive deformation can have a destabilization effect.

It is interesting to underline that, even if $P = 0$, the growing residual stress due to the accumulated plastic strain can be strong enough to be a critical load of buckling. This is an another destabilization aspect.

To our knowledge, all these destabilization mechanisms have not yet been discussed sufficiently in the literature, perhaps because of their complexity. Some authors, like Neale and Schroeder [11] have studied the modification on the critical bifurcation load for cylindrical shells when the additional strain due to cyclic additional loading induce some variations of the geometric characteristics of the shell. But the real problem of destabilization has been only recently considered by the DMT (CEA) [1] on the base of experiments and numerical simulations. The phenomenon of destabilization is observed in the case of cyclic torsion tests of cylindrical tubes under axial compression [1]. Buckling of plates under residual stress due to progressive plastic strain under thermal cyclic loading has also being discussed very recently.

In this paper, we wish to examine this interaction in terms of the loading force P , of the material properties of the structure itself and of the cyclic additional loading.

2 - INTERACTION BUCKLING-PROGRESSIVE DEFORMATION

In order to illustrate the interaction, let us consider first a simple example. The behaviour of an axially compressed beam under the load P and subjected to thermal cyclic loading can be schematically described by an improved Shanley model with more than two elements (figure 1).

Each element is an elastic-plastic bar with a kinematic strain hardening and constant tangent modulus E_t . The limit value $E_t = 0$ will describe the case of perfect plasticity. The material properties allow the introduction of different critical loads : the Euler's

critical load P_E which is associated with the loss of elastic stability and various critical loads of tangent modulus in the classical sense. The smallest critical load P_T is obtained when all the elements are in the plastic range. This Shanley model is quite familiar in the studies of elastic plastic buckling. The characteristics of the chosen model correspond to a beam of slenderness $L/e = 18$. Figure 2 shows the Force-Displacement curve (P, u) when the initial imperfection is small ($\bar{u}/e = 2 \cdot 10^{-3}$) obtained in a displacement control test for different values of E_t . They give an idea of the load capacity of the beam. For small values of E_t , it is well known that the force is decreasing for sufficient displacement u . From the figure 2, we can guess that instability will appear if the displacement exceeds 4% L even with a small load ($\sigma/\sigma_y = 0.5$) when $E/E_t = 1000$.

We have studied the response of our model when the prestress due to the dead load P equals $\sigma^0 = 0.8 \sigma_y$. In the first step, the model with a given initial imperfection is loaded from 0 to P . The elements of the model are then submitted to a cyclic thermal loading.

Two different distributions of the temperature have been considered. In the first one we have a constant gradient through the thickness. In the second, the temperature is constant on the half-thickness. The maximum value of the temperature Θ is a cyclic function of time (figure 3).

When the gradient is constant, the induced flexion stresses are weak because the beam is free-ended. The response of the beam is very simple and depends on the load magnitude $\Delta\Theta$. The actual tangent modulus P_t depends on the number and the location of plastic elements. If $P < P_t$ (and this is always the case if $P < P_T$) for all values of P_t we observed that elastic shake-down appears after one single cycle. On the other hand, if P exceeds P_t for a certain distribution of plastic elements, quasi-static response to a temperature increment is not possible, the actual response is dynamic and instable. We are on the negative slope part of the curve (P, u) and the failure is due to buckling.

When the temperature is uniform on a half thickness, thermal stresses become more complex. Figure 4 shows how the displacement increases with time when $P > P_T$. After a few cycles, again there is failure by loss of stability. But, a different behaviour is observed when strain hardening modulus increases and P_T becomes bigger than P . After several cycles, elastic shake-down (figure 5) or plastic shake-down (figure 6) can be obtained, depending on the magnitude of thermal loading.

The fact that the risk of instability increases when the strain hardening decreases, already shown in figure 2 for a monotonic displacement control test, is again confirmed here by our numerical results. If the material is perfectly plastic, $P_T = 0$ thus the structure will be particularly sensitive to the buckling-progressive deformation interaction.

It seems to be clear that if $P < P_T$, there is always shake-down and that if $P > P_T$ the risk of buckling due to progressive deformation is important. Since such a conclusion may be interesting for practical applications, we give in the next section a theoretical justification basing upon the assumption of small motions superposed on a prestressed state.

3 - THEORETICAL ANALYSIS

The evolution of the structure under additional loading is governed by the equilibrium equation and the constitutive laws. We are going to study those equations in the quasi-static situations assuming the strains remain small ($u/L \approx 0.1$ in our example).

3.1- Evolution equations :

Let Ω be the position of structure under the dead load P , σ_0 the associated stress state. The additional loading is defined by the given forces $F(t) = (g(t), R(t))$ and the given thermal strain $\epsilon^\theta(t)$, g denotes the volume force and R the surfacic ones. For clarity, all the implied displacements are assumed to be zero. The evolution of the structure is described by the additional displacement u , the additional strain ϵ and the actual stress σ (Green strain, Kirchhoff stress with respect to the configuration Ω). The evolution is defined by the following equations :

$$\sigma_{ij,j} + [\sigma_{jk} u_{i,k}]_{,j} + g_i = 0 \quad \left. \begin{array}{l} \text{(Equilibrium)} \end{array} \right\} \quad (1)$$

$$2 \epsilon_{ij} = u_{i,j} + u_{j,i} \quad \left. \begin{array}{l} \text{(Compatibility)} \end{array} \right\} \quad (2)$$

$$\begin{aligned} \sigma &= \sigma_0 + D \cdot \epsilon^e \\ \epsilon &= \epsilon^e + \epsilon^\theta + \epsilon^p \\ \dot{\epsilon}^p &= (\text{Normality laws, to be explicited}) \end{aligned} \quad \left. \begin{array}{l} \text{Elastic-plastic constitutive} \\ \text{equations.} \end{array} \right\} \quad (3)$$

- Boundary conditions (Force and Displacement) and Initial Conditions.

Even in the case of small additional strains, the evolution problem is non linear because (1) and (3) are non linear equations.

In the following sections, we shall restrict ourselves to particular situations such that the equilibrium (3) can be approximately replaced by the linear one :

$$\sigma_{ij,j} + (\sigma_{jk}^0 u_{i,k})_{,j} + g_i = 0 \quad (4)$$

which means that the additional stress $\sigma - \sigma_0$ is not essential in the equilibrium equations.

3.2- Small motions near a prestressed state :

The non linear equations (3) + (4) represent only small motions near a prestressed state. For simplicity, we now follow Maier [9] to discuss these evolutions in a matrix description.

Displacement, stress and strain fields will be represented by matrix u , σ and ϵ with tensor components, $\sigma = (\sigma_i)$, $\epsilon = (\epsilon_i)$, $i = 1, m$. We obtain then in finite element notations :

$$\begin{aligned} \epsilon &= B u & \text{compatibility} \\ \sigma - \sigma_0 &= D (\epsilon - \epsilon^p - \epsilon^\theta) & \text{elastic behaviour} \\ \tilde{B} \cdot (\sigma - \sigma_0) + K_{\sigma_0} \cdot u &= F & \text{equilibrium} \end{aligned} \quad \left. \right\} \quad (5)$$

From (5), u and σ can be expressed in terms of ϵ^p . In particular we obtain simply :

$$\sigma = \sigma_0 + \sigma^{E\theta} - Z \epsilon^p \quad (6)$$

where $Z = D - D B K^{-1} \tilde{B} D$, K denotes the stiffness matrix $K = \tilde{B} D B + K_{\sigma_0}$ and $\sigma^{E\theta}$ is the thermo-elastic response defined by :

$$\left. \begin{aligned} \epsilon^{E\theta} &= B \cdot u^{E\theta} \\ \sigma^{E\theta} &= D (\epsilon^{E\theta} - \epsilon^\theta) \\ \tilde{B} \cdot \sigma^{E\theta} + K_{\sigma_0} \cdot u^{E\theta} &= F \end{aligned} \right\} \quad (7)$$

For this reason, the plastic strain ϵ^P will be taken as the principal unknown of the evolution problem.

3.3- Evolution of the plastic strain ϵ^P :

The evolution of the plastic strain ϵ^P is given by the plastic behaviour that we examine now.

For the sake of clarity, we shall restrict our discussion here to the case of kinematic hardening (Ziegler-Prager's model). We recall that, locally, the plastic criterion may be written in that case as $\|A\| - k \leq 0$ with $A = V \sigma - C \epsilon^P$, V and C are symmetric matrices, V represents the deviator operator, $V \epsilon^P = \epsilon^P$. In terms of the global matrix σ and ϵ^P , we can introduce then the matrices V_{ij} and C_{ij} to write for the whole structure, the plastic condition as :

$$\left. \begin{aligned} \|A_i\| - k_i \leq 0 \quad \text{with} \quad A_i &= V_{ij} \sigma_j - C_{ij} \epsilon_j^P \\ V, C &\text{ sym.} \\ V \epsilon^P &= \epsilon^P \end{aligned} \right\} \quad (8)$$

The plastic strain rate $\dot{\epsilon}^P = (\dot{\epsilon}_i^P)$ is given by Hill's maximum principle :

$$\dot{\epsilon}_i^P (A_i - A_i^*) \geq 0 \quad \forall \quad \|A_i^*\| - k_i \leq 0 \quad (9)$$

Inequalities (8) represent in the force space $A = (A_1, \dots, A_m)$ a convex domain Q of admissible forces such that $\epsilon^P = (\epsilon_i^P, \dots, \epsilon_m^P)$ satisfy the normality law which follows from (9) :

$$\dot{\epsilon}^P = N_Q(A) \quad (10)$$

where $N_Q(A)$ denotes schematically the external normal operator at A to the convex domain Q . Finally, it follows that the evolution is defined formally by the following differential equation :

$$\left. \begin{aligned} \dot{\epsilon}^P &= N_Q \left\{ V \sigma_0 + V \sigma^{E\theta}(t) - (V Z V + C) \epsilon^P \right\} \\ \epsilon^P(0) &= \epsilon_0^P \end{aligned} \right\} \quad (11)$$

The asymptotic behaviour of the structural response is given by the asymptotic behaviour of the solution $\epsilon^P(t)$ of (11). All the data involved in the cyclic loading are condensed in the thermo-elastic response $\epsilon^{E\theta}(t)$.

The discussion of equation (11) can be done in the classical way if the following preliminary assumption is fulfilled :

$$\text{The matrix } V Z V + C \text{ is positive-definite in the plastic strain space } \left. \right\} \quad (12).$$

The physical significance of this preliminary assumption will be given later. We first accept it to derive from (11) two principal results concerning the asymptotic behaviour.

3.4- Elastic shake-down theorem :

In the prescence of prestress, elastic shake-down theorem has been given by Maier [9]. We give here a modified version which is based upon Koiter's paper [8].

Let us introduce first the Melan-Koiter's condition :

$$\left. \begin{aligned} &\text{It is assumed that there exist a plastic strain } \epsilon_*^P \text{ such that the associated} \\ &\text{force } A^* : \\ &A(t) = V \sigma_0 + V \sigma^{E\theta}(t) - (V Z V + C) \epsilon_*^P \\ &\text{verify strictly the plastic condition (8) in the following sense :} \\ &\exists m > 1, \forall t > t_0 \quad \text{we have} \quad \|A_i^*(t)\| - k_i \leq 0 \quad \forall i \end{aligned} \right\} \quad (13)$$

Then the following theorem is obtained :

. Elastic shake-down theorem :

$$\left. \begin{aligned} &\text{The asymptotic behaviour } \lim_{t \rightarrow \infty} \epsilon^P(t) = \epsilon_\infty^P \text{ holds,} \\ &\text{i.e. the response will be purely elastic if the Melan-Koiter} \\ &\text{condition is satisfied.} \end{aligned} \right\} \quad (14)$$

3.5- Plastic shake-down theorem :

Now the additional loading is assumed to be periodic, of period T ; $F(t + T) = F(t)$,
 $\epsilon^\theta(t + T) = \epsilon^\theta(t)$.

There is plastic shake-down if the response of the structure in stress and strain will become periodic and if the plastic strain $\epsilon^P(t)$ will become periodic. So elastic shake-down is the limiting case of plastic shake-down when the plastic strain will become constant which is a particular periodic function.

Plastic shake-down theorem has been discussed in small transformation by Halphen [5], in the case of kinematic hardening. His method can be again applied here, under the preliminary condition (12).

The following theorem is obtained :

. Plastic shake-down theorem :

$$\left. \begin{aligned} &\text{Kinematic hardening model (8) is assumed.} \\ &\text{If the preliminary condition (12) is satisfied, then :} \\ &- \text{there exists at least one periodic solution } \underline{\epsilon}^P(t) ; \\ &- \text{whatever be the initial condition } \epsilon_0^P, \text{ the plastic strain} \\ &\quad \epsilon^P(t) \text{ will tend to a periodic solution.} \end{aligned} \right\} \quad (15)$$

3.6- Interpretation and Consequence of the preliminary condition :

The preliminary condition has allowed us to generalize the different shake-down theorems. Classical methods and results are still available within some minor modifications. It is important to understand the physical meaning of this preliminary condition. Without loss of generality, we can for simplicity suppose that P depends on a single load factor λ ,
 $P = \lambda P_0$, $\lambda \geq 0$.

In perfect plasticity, $C = 0$, the preliminary condition then reduces to the positivity of matrix $\tilde{V} Z V$. Or, this is not always true. Indeed, in the case of no prestress or $\lambda = 0$, it is well-known that a compatible plastic strain ϵ^P does not induce residual stress, $\sigma^r \cdot \epsilon^P = -\epsilon^P \cdot Z \cdot \epsilon^P = 0$ if ϵ^P is a compatible strain. It then appears that Z is non negative

but cannot be strictly positive in the absence of prestress. If $\lambda > 0$, the matrix K^{-1} is more positive than K_0^{-1} when K_0 is negative definite, thus $\tilde{V} Z V$ will be often negative.

To ensure (12), it is then necessary that a positive strain hardening exists. In positive kinematic hardening, matrix C is positive and gives a positive contribution to matrix $\tilde{V} Z V + C$. When $\lambda = 0$, since $\tilde{V} Z V$ is non-negative, the preliminary condition is true as soon as the smallest eigen value of C is positive. When λ increases, the smallest eigen value of $\tilde{V} Z V + C$ decreases but remains positive until λ reaches a critical value λ_{cs} at which the matrix $\tilde{V} Z V + C$ is non-negative but no longer strictly positive. The preliminary condition defines then a critical load $P_{cs} = \lambda_{cs} \cdot P_0$.

The physical meaning of P_{cs} can be understood in the following sense. When $P = P_{cs}$, there exists at least one direction e^P such that :

$$\dot{\epsilon}^P \cdot (\tilde{V} Z V + C) \cdot e^P = 0 \quad (16)$$

Since e^P is plastically compatible (i.e. $V e^P = e^P$), it is always possible to associate this direction with a strain rate $\dot{\epsilon}^P$. In this case, equation (16) shows that the rate problem with homogenous condition ($\dot{F} = 0$, $\dot{\epsilon}^0 = 0$), has a non trivial solution $\dot{\epsilon}^P \neq 0$. This means that the equilibrium under dead load P_{cs} is critical in the sense of Hill [7]. More precisely, it can be said that the loss of positivity of the matrix $\tilde{V} Z V + C$ is associated with a possible bifurcation of the equilibrium, when the present plastic zone is large enough to allow $\dot{\epsilon}^P$ to be physically admissible. Critical load P_{cs} then appears to be the smallest bifurcation load of the structure under the given load P .

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Fig.1
Shanley model with n elements

Fig.3
Additional cyclic thermal load

Fig. 2
Load-displacement curve with imperfection
 2.10^{-3} and $E/E_T = 6, 10, 20, 1000$.

Fig. 4
Failure by interaction buckling -
progressive plastic deformation
($E/E_T = 20$, $P/P_T = 1.6$)

Fig. 5
Elastic shake-down (?)
($E/E_T = 10$, $P/P_T = .8$)

Fig. 6
Plastic shake-down
($E/E_T = 10$, $P/P_T = .8$)