SEISMIC RESPONSE OF A STRUCTURE SUBJECTED TO ROTATIONAL BASE EXCITATION

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SUMMARY

A modal superposition method which can perform the seismic analysis of a structure subjected to translational and rotational base excitation is presented. Discussed are two different approaches to derive the equations of motion of the structure. In the first approach, the reference axes are fixed in space and the equations of motion are derived with respect to these axes. In the second approach, the reference axes are rigidly fixed at the base of the structure. This approach is generally used when the structure is subjected to translational base excitation alone. For rotational base excitation, the equations of motion are shown to become nonlinear due to presence of the Coriolis acceleration term.

The first approach is used to derive the equations of motion. These equations may be integrated by the direct integration method or by the modal superposition method. For long time history analysis, the computer cost of using the direct integration method is significantly greater than that of using the modal superposition method. The latter method is given in this paper.

The equations of motion have time dependent displacement boundary conditions. The dependent variables are transformed so that the boundary conditions become homogeneous. The modal superposition method is applied to the transformed equations of motion. There are two conclusions:

1. To get the correct displacement response of a massless degree-of-freedom, at least one of the following two conditions must be satisfied.
 - The right hand side of the equations of motion has no force component acting on the massless degree-of-freedom.
 - The massless degree-of-freedom is rigidly connected to degrees-of-freedom with mass. That is, if all the degrees-of-freedom with mass are fixed, a force acting on the massless degree-of-freedom does not produce any displacement in the structure.
2. Almost all the modes have to be included in the time history analysis. This is explained as follows. The displacement response may be divided into two components: (a) displacement due to rigid body motion, and (b) displacement due to elastic deformation. The mode shapes are associated with the transformed homogeneous equations of motion which represent the structure fixed at the base. As these modes are not orthogonal to displacement response due to rigid body motion, almost all of the modes have to be included in the time history analysis.

The modal superposition method is applied to the seismic analysis of a building subjected to translational and rotational excitation. The displacement results and the computer cost of this analysis are compared with those of using the direct integration method. The computer cost associated with the modal superposition method is significantly lower than that associated with the direct integration method.

1. Introduction

Nuclear power plants are designed to withstand hypothetical accidents due to postulated seismic events. The evaluation of the design requires a time history analysis of a coupled building and reactor coolant loop system and its components subjected to base motions. Though the base motions consist of both translational and rotational excitations, only the translational excitation is generally included in the time history analysis. This paper presents a computationally economical method to analyze the dynamic response of a structure subjected to both translational and rotational excitations.

In the seismic analysis of a structure subjected to a translational excitation applied at a single point, it is common practice to fix the reference axes rigidly at the base of the structure. These axes are referred to as base axes and designated by the small letters x, y, z . The displacement response is relative to the base axes [1]. If the reference axes are fixed in space, then the displacement response is absolute [2]. In this case the reference axes are designated by the capital letters X, Y, Z . If, in addition to the single point translational excitation, the structure is also subjected to rotational excitation at the same point, the structure can be analyzed either with respect to the base axes x, y, z or with respect to the axes X, Y, Z . These two approaches are described in this paper.

The second approach is used to derive the equations of motion which may be integrated either by the direct integration method or by the modal superposition method. The computer cost of the seismic analysis of a large model (500-600 degrees-of-freedom) using the direct integration method is high. To reduce this computer cost, the modal superposition method was developed.

The modal superposition method is used [3,4] to analyze linear structures subjected to translational excitation due to ground motion. It has been shown that this method is computationally more economical as compared to direct integration method for linear and nonlinear analyses [5]. This method is extended to analyze a linear structure subjected to translational and rotational excitation applied at a single point. Two previously unreported observations regarding the modal superposition method are discussed.

To illustrate the application of the modal superposition method, the seismic analysis of a building subjected to translational and rotational excitation is given. In addition, comparisons of the displacement results and the computer costs of the analysis by modal superposition and direct integration methods are presented.

2. Theory

2.1 Equation of Motion

The equation of motion of a structure with respect to XYZ axes subjected to a single point seismic excitation is

$$[M] \left\{ \ddot{X} \right\} + [C] \left\{ \dot{X} \right\} + [K] \left\{ X \right\} = \left\{ F \right\} \quad (1a)$$

$$\text{where } \left\{ X \right\} = \left\{ \begin{matrix} X_1 \\ X_2 \end{matrix} \right\}$$

$$\left\{ X_2 \right\} = \left\{ e(t) \right\} = \left\{ \begin{matrix} e_t(t) \\ e_r(t) \end{matrix} \right\} \quad (1b)$$

$\left\{ X_2 \right\}$ corresponds to degrees-of-freedom at a node where the base displacement vector $\left\{ e(t) \right\}$ is applied and $\left\{ X_1 \right\}$ is the remaining degrees-of-freedom, $\left\{ e_t(t) \right\}$ and $\left\{ e_r(t) \right\}$ are translatory and rotatory components of excitation, respectively. $\left\{ \dot{X} \right\}$ and $\left\{ \ddot{X} \right\}$ are velocity and acceleration vectors, respectively. $[M]$, $[C]$, and $[K]$ are mass, damping, and stiffness matrices, respectively, and $\left\{ F \right\}$ is applied force vector. The equations (1a, 1b) are rewritten as

$$\left[\begin{matrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{matrix} \right] \left\{ \begin{matrix} \ddot{X}_1 \\ \ddot{X}_2 \end{matrix} \right\} + \left[\begin{matrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{matrix} \right] \left\{ \begin{matrix} \dot{X}_1 \\ \dot{X}_2 \end{matrix} \right\} + \left[\begin{matrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{matrix} \right] \left\{ \begin{matrix} X_1 \\ X_2 \end{matrix} \right\} = \left\{ \begin{matrix} F_1 \\ F_2 \end{matrix} \right\} \quad (2a)$$

and

$$\{X_2\} = \{e(t)\} \quad (2b)$$

The eqs. (2a and 2b) are rearranged so that the equation of motion for $\{X_1\}$ is

$$[M_{11}] \{\ddot{X}_1\} + [C_{11}] \{\dot{X}_1\} + [K_{11}] \{X_1\} = \{F_1\} - [M_{12}] \{\ddot{e}\} - [C_{12}] \{\dot{e}\} - [K_{12}] \{e\} \quad (3)$$

The eq. (3) may be integrated to find the absolute displacement response $\{X_1\}$

2.2 Equation of Motion with Respect to xyz Axes

The equation of motion of a structure with respect to xyz axes is

$$[M] \{\ddot{Y} + \ddot{E}\} + [C] \{\dot{Y} + \dot{E}\} + [K] \{Y + E\} = F \quad (4)$$

where $\{Y\} = \{Y_1, Y_2\}$, and $\{Y_2\} = \{X_2\} - \{e\} = \{0\}$

Vectors $\{Y\}$, $\{\dot{Y}\}$ and $\{\ddot{Y}\}$ are the displacement, velocity and acceleration vectors relative to xyz axes. The vectors $\{E\}$, $\{\dot{E}\}$ and $\{\ddot{E}\}$ are determined from the motion of the xyz axes and may or may not depend upon relative response vector $\{\dot{Y}\}$.

Let

$$\{E\} = \{E_1, E_2, \dots, E_i, \dots\} \quad (5)$$

and

$$\{E_i\} = \left\{ \begin{matrix} E_{it} \\ E_{ir} \end{matrix} \right\} \quad (6)$$

where $\{E_i\}$ is the displacement vector for node i, and $\{E_{it}\}$ and $\{E_{ir}\}$ are the translational and rotational displacement vectors, respectively.

The eq. (4) is analyzed for two types of excitations. The first one consists of single point translatory excitation and the second one consists of single point translatory and rotatory excitations.

2.2.1 Translatory excitation: If only translatory excitation is taken into account, then, the axes xyz have fixed orientation with respect to axes XYZ and they can be made parallel without any loss in generality.

Then, for the i^{th} node,

$$\{E_i\} = \left\{ \begin{matrix} e_t \\ 0 \end{matrix} \right\}, \{\dot{E}_i\} = \left\{ \begin{matrix} \dot{e}_t \\ 0 \end{matrix} \right\}, \text{ and } \{\ddot{E}_i\} = \left\{ \begin{matrix} \ddot{e}_t \\ 0 \end{matrix} \right\} \quad (7)$$

The displacement vector $\{E\}$ represents a rigid body motion of the structure. Thus, the elastic force vector $[K] \{E\}$ in eq. (4) is zero. If one assumes zero energy loss due to rigid body motion, the damping force vector $[C] \{\dot{E}\}$ is also zero. The resulting equation of motion, with the inertia force vector $[M] \{\ddot{E}\}$ shifted to the right hand side, is as follows:

$$[M] \{\ddot{Y}\} + [C] \{\dot{Y}\} + [K] \{Y\} = \{F\} - [M] \{\ddot{E}\}$$

and

$$\{Y_2\} = \{0\} \quad (8)$$

A comparison of eqs. (3) and (8) reveals the following. The right hand side of eq. (8) has only one term $[M] \{\ddot{E}\}$ due to base excitation $\{e\}$, while eq. (3) has three terms, i.e., $-[M_{12}] \{\ddot{e}\}$, $-[C_{12}] \{\dot{e}\}$, and $-[K_{12}] \{e\}$. The right hand side of eq. (3) has a higher order variation than the right hand side of eq. (8). For the direct integration method, the integration of eq. (3) will require a smaller time step size than that required for the integration of eq. (8).

For the modal superposition method where the uncoupled modal equations are integrated analytically, the same time step size can be used with eqs. (3) or (8). It is a general practice to analyze a structure subjected to a single point translatory motion due to seismic excitation by integrating eq. (8) instead of eq. (3).

2.2.2 Translatory and Rotatory Excitations: If the translatory and rotatory excitations are taken into account then the orientation of the axes xyz with respect to the axes XYZ changes continuously. The expressions of the vectors $\{E_i\}$, $\{\dot{E}_i\}$ and $\{\ddot{E}_i\}$ for the i^{th} node may be derived from kinematic consideration. Let

$$\bar{r} = x_i \bar{i} + y_i \bar{j} + z_i \bar{k} \quad (9)$$

be a position vector of node i with respect to xyz axes at time t . Let $\{\bar{e}\}$ be the excitation vector with respect to xyz axes. Let, the absolute displacement $\bar{X}_{it,x}$, velocity $\dot{\bar{X}}_{it,x}$, and acceleration $\ddot{\bar{X}}_{it,x}$ of node i along x direction (xyz axis) [6] be

$$\bar{X}_{it,x} = Y_{it,x} + E_{it,x}; \quad \dot{\bar{X}}_{it,x} = \dot{Y}_{it,x} + \dot{E}_{it,x}; \quad (10)$$

and

$$\ddot{\bar{X}}_{it,x} = \ddot{Y}_{it,x} + \ddot{E}_{it,x},$$

where

$$\begin{aligned} E_{it,x} &= \bar{e}_{t,x} \\ \dot{E}_{it,x} &= \dot{\bar{e}}_{t,x} + \bar{e}_{r,y} \dot{z}_i - \bar{e}_{r,z} \dot{y}_i \\ \ddot{E}_{it,x} &= \ddot{\bar{e}}_{t,x} + [\ddot{\bar{e}}_{r,y} z_i - \ddot{\bar{e}}_{r,z} y_i] + [\dot{\bar{e}}_{r,y} (\dot{\bar{e}}_{r,x} y_i - \dot{\bar{e}}_{r,z} x_i) - \dot{\bar{e}}_{r,z} (\dot{\bar{e}}_{r,x} x_i - \dot{\bar{e}}_{r,z} z_i)] \\ &\quad + [\ddot{\bar{e}}_{r,y} \dot{z}_i - \ddot{\bar{e}}_{r,z} \dot{y}_i] \end{aligned} \quad (11)$$

and

$$\dot{y}_i = \dot{Y}_{it,y}, \quad \dot{z}_i = \dot{Y}_{it,z}$$

also

$$\begin{aligned} \{E_{ir,x}\} &= \{\bar{e}_{r,x}\} \\ \{\dot{E}_{ir,x}\} &= \{\dot{\bar{e}}_{r,x}\} \end{aligned} \quad (12)$$

and

$$\{\ddot{E}_{ir,x}\} = \{\ddot{e}_{r,x}\}$$

Similarly, the other components of vectors $\{E_i\}$, $\{\dot{E}_i\}$ and $\{\ddot{E}_i\}$ may be derived. The four terms in the right hand side of eq. (11) are the x components of acceleration of the origin of xyz axes, tangential acceleration, centripetal acceleration and Coriolis acceleration, respectively.

The vectors $\{E\}$ and $\{\dot{E}\}$ represent rigid body motion of the structure. The elastic force vector $[K] \{E\}$ and the damping force vector $[C] \{\dot{E}\}$ on the left hand side of eq. (2) are equal to zero (Section 2.2.1). The resulting equation of motion with the inertia force vector $[M] \{\ddot{E}\}$ shifted to the right hand side is similar to eq. (8), but the vector $\{\ddot{E}\}$ depends on the relative velocity vector $\{\dot{Y}\}$, due to the presence of a Coriolis term in eq. (11). Even though the response of the structure is linear, its equation of motion becomes nonlinear. The integration of this nonlinear equation requires an iterative scheme or a small time step size, but the integration of eq. (3) does not require an iterative scheme and a larger time step may be employed. The approach of relative displacement, i.e., solution of eq. (4), loses its appeal when the structure is subjected to single point translatory and rotatory motion. Under these circumstances, the approach of absolute displacement response, i.e., solution of eq. (3), is more efficient.

3. Modal Superposition Method

3.1 The equation of motion, eqs. (1a, 1b) may be integrated by the direct integration method or by the modal superposition method. The direct integration scheme can be applied to eq. (3) without any modification, whereas, the application of the modal superposition method requires transformation of the dependent variable such that the boundary conditions become homogeneous [7,8]. For long time history analysis, the cost of using the modal superposition method is less than that of using the direct integration scheme.

The modal equations of motion are developed as follows.

Let

$$\{X_2(t)\} = \{Z_2(t)\} + \{e(t)\} \quad (13)$$

and

$$\{X_1(t)\} = \{Z_1(t)\}$$

be substituted in eqs. (2a-2b). The resulting equation of motion has homogeneous boundary conditions and is given below

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \begin{Bmatrix} \ddot{Z}_1 \\ \ddot{Z}_2 \end{Bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} \dot{Z}_1 \\ \dot{Z}_2 \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} Z_1 \\ Z_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} - \begin{bmatrix} M_{12} \\ M_{22} \end{bmatrix} \ddot{e} - \begin{bmatrix} C_{12} \\ C_{22} \end{bmatrix} \dot{e} - \begin{bmatrix} K_{12} \\ K_{22} \end{bmatrix} e \quad (14)$$

and

$$\{Z_2\} = \{0\}$$

The homogeneous, undamped equation of motion is:

$$[M] \{\ddot{Z}\} + [K] \{Z\} = \{0\} \quad (15)$$

Let $[\omega]$ and $[\phi]$ be the natural frequency and orthonormalized mode shape matrices associated with eq. (15). The following transformation,

$$\{Z\} = [\phi] \{q\} \quad (16)$$

is substituted in eq. (15) premultiplied by $[\phi]^T$ and employing the orthogonality relations expressed by

$$[\phi]^T [M] [\phi] = [I] \text{ and } [\phi]^T [K] [\phi] = [\omega_j^2],$$

and with damping represented by

$$[\phi]^T [C] [\phi] = [2\zeta_j \omega_j]$$

the resulting modal equations became

$$\{\ddot{q}\} + [2\zeta_j \omega_j] \{\dot{q}\} + [\omega_j^2] \{q\} = \{Q\} - \{Q_1\} \quad (17)$$

where

ζ_j = percentage of the critical damping for the j th mode

$$\{Q\} = [\phi]^T \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \text{generalized applied force vector}$$

and

$$\{Q_1\} = [\phi]^T \left\{ \begin{bmatrix} M_{12} \\ M_{22} \end{bmatrix} \ddot{e} + \begin{bmatrix} C_{12} \\ C_{22} \end{bmatrix} \dot{e} + \begin{bmatrix} K_{12} \\ K_{22} \end{bmatrix} e \right\}$$

= generalized force due to seismic excitation.

Arrays $\{q\}$, $\{\dot{q}\}$ and $\{\ddot{q}\}$ are the modal displacement, velocity, and acceleration vectors, respectively. The damping in the base elements may be assumed to be proportional to the stiffness of that element. Equation (17) represents a set of uncoupled equations. These equations are integrated analytically to avoid any numerical damping or frequency distortion during integration. Two observations regarding the modal superposition method follow.

3.2 Displacement Response of Slave Degree-of-Freedom

In the full modal analysis of a structure, all the degrees-of-freedom with inertia are called masters and the remaining degrees-of-freedom are called slaves. There is a necessary condition that has to be satisfied to get the correct displacement response of slaves. This is explained in the following paragraphs. Let the equation of motion of an undamped structure be

$$\begin{bmatrix} M_{mm} & O \\ O & O \end{bmatrix} \begin{Bmatrix} \ddot{U}_m \\ \ddot{U}_s \end{Bmatrix} + \begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix} \begin{Bmatrix} U_m \\ U_s \end{Bmatrix} = \begin{Bmatrix} P_m \\ P_s \end{Bmatrix} \quad (18)$$

Where $\{U_m\}$ are masters and $\{U_s\}$ are slaves $\{P_m\}$ and $\{P_s\}$ represent the forces acting on the masters and slaves, respectively.

The mode shape matrix associated with the corresponding eigenvalue problem is given by

$$[\phi] = \begin{bmatrix} \phi_m \\ \phi_s \end{bmatrix} \quad (19)$$

where $[\phi_m]$ corresponds to masters and $[\phi_s]$ corresponds to slaves. The matrix $[\phi_s]$ is related to the matrix $[\phi_m]$ by the following static relationship.

$$[\phi_s] = - [K_{ss}]^{-1} [K_{sm}] [\phi_m] \quad (20)$$

In the modal superposition analysis, the static relationship expressed in eq. (20) is also present between the transient responses of slaves and masters. That is,

$$\{U_s\} = - [K_{ss}]^{-1} [K_{sm}] \{U_m\} \quad (21)$$

From eq. (18), the relationship between slaves and masters is given by

$$\{U_s\} = [K_{ss}]^{-1} [\{P_s\} - [K_{sm}] \{U_m\}] \quad (22)$$

The comparison of eqs. (21) and (22) show that the right hand side of eq. (21) does not include the term $[K_{ss}]^{-1} \{P_s\}$. The omission of this term introduces an error in the displacement response of slaves in modal superposition analysis. This error is reduced to zero if $[K_{ss}]^{-1}$ is a null matrix and/or $\{P_s\}$ is a null vector. The slave, therefore, will have correct displacement response if at least one of the following two conditions are satisfied.

- The slave is rigidly connected to a master. That is, if all the masters are fixed, a force acting on the slave does not produce any displacement in the structure.
- The right-hand side of eq. (14) has a zero force component acting on the slave degree-of-freedom.

3.3 Number of Mode Shapes to be Included in Analysis

The orthonormalized mode shape matrix $[\phi]$ derived from eq. (15) represents the structure fixed at the base. No single mode shape in matrix $[\phi]$ is able to represent rigid body motion experienced by the structure. To represent the rigid body motion of the structure a linear combination of most of the mode shapes is required.

The solution of the eq. (14) gives the displacement response of the structure. This response represents the elastic deformation superimposed on the rigid body motion of the structure. Due to the presence of the rigid body motion, most of the mode shapes are continuously excited during the time history analysis, and they should be included. A mode shape, however, does not have to be included in the analysis if the subspace represented by the modeshape, during the entire time history analysis is orthogonal to the subspace containing the rigid body motion. It should be emphasized that a small number of mode shapes is required in the analysis of a structure relative to base axis xyz and subjected to translatory excitation alone.

4.0 Test Problem

The seismic analysis of a building subjected to translational and rotational excitation is performed to illustrate the application of the method developed in this paper. The line diagram of the simplified building model is shown in fig. 1. In this model, the lumped masses are associated with nodes 23, 26, and 29. Each lumped mass has four degrees-of-freedom: UX, UY, UZ, and RZ. The twelve natural frequencies of the model are 11.3, 14.7, 22.5, 32.9, 43.1, 49.1, 53.1, 85.9, 106.4, 110.0, 116.7, and 208.2 Hertz. The base motion is of fifteen second duration and has two components: (1) translational excitation along the Y-axis, and (2) rotational excitation about the Z-axis.

Various analyses are performed using the direct integration and the modal superposition methods to verify the latter method. The analyses are performed for a period of two and one half seconds; zero damping is assumed for all the mode shapes. The displacement responses of node 29 relative to node 20 along Y-direction are compared. Figures 2-a, 2-b, and 2-c represent the results from the analyses using the direct integration method, with $\Delta T = 0.005, 0.0025, \text{ and } 0.00125$ seconds, respectively. With the direct integration method [9], the displacement

response converges as the size of the time step becomes smaller. Because the modal superposition method employs the analytical integration scheme, the convergence of the displacement response is independent of the size of the time step. Fig. 2-d shows the results from the analysis using the modal superposition method with $\Delta T = 0.005$ seconds. The results given in fig. 2-c agree with those in fig. 2-d. The computer cost of using the direct integration method (fig. 2-c) is twelve times greater than the cost of the modal superposition method (fig. 2-d).

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Figure 1. Simplified Building Model

(a) DIRECT INTEGRATION,
 $\Delta T = .005$ SECONDS

(b) DIRECT INTEGRATION,
 $\Delta T = .0025$ SECONDS

(c) DIRECT INTEGRATION,
 $\Delta T = .00125$ SECONDS

(d) MODAL SUPERPOSITION
 $\Delta T = .005$ SECONDS

Figure 2. Displacement Response of Node 29 Relative to Node 20 Along Y-Direction