SHAKEDOWN ANALYSIS BY DISPLACEMENT METHOD AND EQUILIBRIUM FINITE ELEMENT

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SUMMARY

In practice, structural elements are often subjected to variable but repeated external actions P (loads, temperature charges, imposed displacements) such that one has a domain of loading : $P_{\min} < P < P_{\max}$.

The behaviour of such structure in elastic plastic range could be characterized by a shakedown limit beyond this, the structure presents two following alternative :

- a) Plastic deformations are accumulated with the increased cycles such that a mechanism may occur. This is the incremental collapse situation.
- b) Fracture occurs as the result of plastic deformations with alternating sign. This is called low-cycle fatigue or alternating plasticity.

The shakedown analysis could be summarized at the determination of the loading factor α

$\alpha \lambda^{\min} P < P < \alpha \lambda^{\max} P$ which give the shakedown limit of the loading domain.

The theoretical foundation of this formulation is based on the statical shakedown theorem proposed very early by MELAN in 1930. This theorem states that "Shakedown occurs if it is possible to find a field of fictitious residual stresses $\bar{\rho}_{ij}$, independant of time, such that for any variations of loads within the sum of this field with the elastic stresses σ_{ij}^E (corresponding to the same domain of loading) violates nowhere in the structure the condition of plasticity. In discrete formulation instead of the force method and displacement element, we adopt the displacement formulation using equilibrium finite element proposed in a previous paper. On the other hand to reduce the number of nonlinear inequalities we adopt the criterion of the mean proposed in a previous paper.

In the present finite element formulation the shakedown analysis is reduced to a simple case of non-linear programming problem

$$\begin{aligned} &\text{Maximize } \alpha \\ &\{c, \alpha\} \end{aligned}$$

$$\frac{1}{2} \left[\alpha \sum_{k=1}^n \lambda_k b_k^E + c \right]^T V \left[\alpha \sum_{k=1}^n \lambda_k b_k^E + c \right] \leq k^2$$

with c : system of unknown parameters

b^E : system of elastic responses $b^E = \sum_k \lambda_k b_k^E$

V : constraint matrix and the domain

of loading is : $\lambda_k^- \leq \lambda_k \leq \lambda_k^+$.

An equilibrium quadrangle composed of four constant stress triangles is used to discretize the structure. A strip with symmetric hole subjected to biaxial tensions is taken as example.

1. Introduction

In practice, structural elements are often subjected to variable but repeated external actions P (loads, temperature charges, imposed displacements), such that one has a domain of loading : $P_{\min} < P < P_{\max}$ (1)

The behaviour of such a structure in elastic-plastic range can be characterized by a shakedown limit beyond this, the structure presents two following alternatives :

- Plastic deformations are accumulated with the increased cycles, so that a mechanism may occur. This is the incremental collapse situation.
- Fracture occurs as the result of plastic deformations with alternating sign. This is called low-cycle fatigue or alternating plasticity.

The shakedown analysis can be summarized as the determination of the maximum loading factor α $\alpha P_{\min} < P < \alpha P_{\max}$ (2)

which gives the shakedown limit of the loading domain.

2. Finite element formulation

The theoretical foundation of this formulation is based on the statical shakedown theorem proposed very early by MELAN in 1938. This theorem states that "Shakedown occurs if it is possible to find a field of fictitious residual stresses $\bar{\rho}_{ij}$, independent of time, such that, for any variations of loads within (1), the sum of this field with the elastic stresses σ_{ij}^E (corresponding to the same domain of loading) violates nowhere in the structure the condition of plasticity". In discrete formulation, instead of the force method /1/ or displacement element /2/, we adopt the displacement formulation using equilibrium finite element proposed in a previous paper /3/. On the other hand, in order to reduce the number of nonlinear inequalities, we adopt the criterion of the mean proposed in /3/.

2.1 Criterion of the mean

Instead of satisfying the von MISES yield criterion everywhere in the body, we use a mean yield criterion performed in each finite element :

$$\frac{1}{2\Omega_e} \int_{\Omega_e} \sigma^T D \sigma d\Omega - k^2 \leq 0 \quad (3)$$

$$\text{In the case of plane stresses : } \sigma^T = (\sigma_x, \sigma_y, \tau_{xy}) \quad (4)$$

$$D = \begin{vmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ 0 & 0 & 2 \end{vmatrix} \quad k \text{ is the yield stress in shear.} \quad (5)$$

2.2 Elastic analysis - equilibrium formulation

Let us assume there are nobody forces; the equilibrated stress field of element e is usually taken as a set of polynomials depending on a system of independent parameters b_e :

$$\sigma_e = S_e b_e \quad (6)$$

$$S_e \text{ is such that the local equilibrium } \sigma_{ij,j} = 0 \text{ is satisfied.} \quad (7)$$

An appropriate system of generalized forces g_e must be defined on the boundary of the element, such that the stress transmission is perfectly continuous : $g_e = C_e b_e$ (8)

Let us introduce the flexibility matrix F_e , connected with the HOOKIAN matrix H by :

$$F_e = \int_{\Omega_e} S_e^T H S_e d\Omega \quad (9)$$

$$\text{and the stiffness matrix of element } e : K_e = C_e F_e^{-1} C_e^T \quad (10)$$

If q_e is the vector of the generalized displacements associated with the generalized

forces g_e , one may show that : $g_e = K_e q_e$ (11)

which results in the following structural linear system : $g = K q$ (12)

where $q_e = L_e q$ $g = \sum_e L_e^T g_e$ (13) (14)
 L_e being a boolean matrix.

(12) is the classical system of the displacement finite element method.

2.3 Discrete shakedown equilibrium formulation

Now consider a body subjected to n modes of loading P_k , $k = 1, \dots, n$. Any arbitrary loading in the loading domain may be considered as a combination of these modes :

$$P = \sum_{k=1}^n \mu_k b_k^E \quad (15)$$

with the following domain of loading : $\mu_k^- < \mu_k < \mu_k^+$ (16)

The elastic solution corresponding to the load P is : $b^E = \sum_{k=1}^n \mu_k b_k^E$ (17)

Like in /3/, we make here the basic assumption that the stress modes of the residual stress field are the one of the elastic stress field (6) : $p_e = S_e c_e$ (18)

The structural vector of unknown stress parameters c is such that :

$$c_e = M_e c \quad M_e \text{ being again a boolean matrix.} \quad (19)$$

We must satisfy the global self equilibrated condition $p_{ij} n_j = 0$ (20)

on the part of the body surface where surface tractions are imposed. Practically, one has to assemble the generalized forces related to the residual stresses (18). So taking (8) and (14) into account, one obtains :

$$g = \sum_e L_e^T g_e = \sum_e L_e^T C_e c_e = C c \quad \text{with } C = \sum_e L_e^T C_e M_e \quad (21)(22)$$

Then removing the components of g associated with null (fixations) or imposed displacements, and eliminating the corresponding rows of C , the self equilibrated condition (20) may now be expressed with the aid of the new matrix C : $C c = 0$ (23)

A Gauss-Jordan elimination procedure is applied to the matrix C to get a system of independent parameters d directly related to c : $c = H d$ (24)

Let's now consider the yield condition (3). According to (6) and (18), it may be transformed into the following matrix form in terms of the elastic stresses already calculated and the unknown residual stresses : $\frac{1}{2} b_e^T v_e b_e \leq k^2$ (25)

where $b_e = \alpha \sum_{k=1}^n \mu_k b_{e,k}^E + c_e$ (26)

$$v_e = \frac{1}{\Omega_e} \int_{\Omega_e} S_e^T D S_e d\Omega \quad (27)$$

Finally, the shakedown analysis is reduced to a simple case of nonlinear programming problem : Maximize α subjected to :

$\frac{1}{2} b_e^T v_e b_e \leq k^2$ (25) for any element $\{d, \alpha\}$

Taking (19), (24) and (26) into account,

$$b_e = \alpha \sum_{k=1}^n \mu_k b_{e,k}^E + M_e H d \quad (28)$$

It must be observed that (25) has to be verified for any values of μ_k within the domain defined by (16). However, it is sufficient to verify (25) only for all combination of the extreme values of μ_k , i.e. either μ_k or μ_k^+ , which leads to 2^n constraints by element. Indeed, the matrix D , defined by (5) is positive definite, which insures the convexity of

the von Mises yield function. So, the matrix V_e defined by (27) is also positive definite, and the form $\frac{1}{2} b_e^T V_e b_e - k^2$ is a convex function of b_e and hence of μ_k .

3. Numerical results

An equilibrium quadrangle composed of four constant stress triangles is used to discretize the structure. Along each edge of the finite element mesh, the surface tractions are constant, so that only two parameters are needed to guarantee continuity of the two surface traction components between elements. These parameters are chosen as the components along the structural x and y axis of the surface traction resultant and are denoted by g_x and g_y (fig. 1).

3.1 Thick cylinder subjected to internal pressure

Let us consider the plane stress problem of a thick cylinder subjected to variable internal pressure : $0 < p < p_{\max}$ (29)

with internal and external radii respectively denoted by a and b . Let $\frac{a}{b} = 0,4$ (30)

Using the symmetry of revolution, only a 15° opening sector, idealized with 4 elements, is considered (fig. 2).

The program leads to incremental collapse as the ruin mechanism. This is in fact the collapse mechanism of limit analysis, as a consequence of the following statement : "In the case of a loading consisting in one variable load, shakedown and limit analysis coincide if shakedown foresees incremental collapse". The shakedown pressure found by the program is

$$p_s = 0,959 \sigma_p \quad (31)$$

σ_p being the yield stress in traction, whereas the exact solution of the limit analysis problem is known of TRESCA's criterion is used :

$$p_e = \sigma_p \ln \frac{b}{a} = 0,917 \sigma_p \quad (32)$$

The corresponding residual stress fields are given on fig. 3.

3.2 Square slab with a circular opening (fig. 4)

A strip with a circular hole is subjected to biaxial uniform tractions which vary independently. The ratio between the diameter of the opening and the length of the slab is 0.2. Three loading cases are considered (fig. 5) and the results are illustrated by fig. 6. We give there also the elastic and rigid-plastic limit domain obtained in the same program. The ACDPAC nonlinear programming algorithm performed by BEST /4/ was utilized to get these results. More details may be found in reference /5/.

It is useful to mention that there are 24 elements so 24 inequalities and 116 degrees of freedom. The following table indicates the comparison with the other results existing in the literature.

Table : Comparison between numerical results.

Authors	Loading cases		
	$n_x = n_y$	$n_x = n_y/2$	$n_x = 0$
Belytschko	$n_y = 0.431$	$n_y = 0.501$	
ref. /2/	$n_y = 0.504$		$n_y = 0.654$
Present	$n_y = 0.431$	$n_y = 0.532$	$n_y = 0.557$

4. Conclusion

Shakedown analysis may be realized by displacement method using equilibrium finite element. The mean yield criterion allows to reduce considerably the nonlinear inequalities. Although this important approximation, the present numerical results are still good comparing with the other analytical or numerical ones existing in the literature. Because of the nonlinear programming aspect of the formulation number of degrees of freedom have to be restricted to avoid important expense in term of computing time, unless more sophisticated nonlinear programming algorithm would be constructed, specially for the shakedown problem.

References

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Figure 1 Used equilibrium finite element

Figure 2 Idealization of a thick cylinder sector subjected to pressure p

Figure 3 Comparison between the two residual stress fields

Figure 4 Square slab with a hole in biaxial traction

Figure 5 Loading cases

Figure 6 Numerical results of the square slab in biaxial traction