FINITE ELEMENT ANALYSIS OF CRACK PROPAGATION IN THREE-DIMENSIONAL SOLIDS UNDER CYCLIC LOADING

P.G. BERGAN, B. AAMODT*

*Institutt for Statikk, The Norwegian Institute of Technology,
The University of Trondheim, N-7034 Trondheim-NTH, Norway
(*on Leave from Det Norske Veritas, Oslo)*

SUMMARY

The paper is concerned with an efficient computational procedure for analyzing crack propagation in solids. The method is general; however, its application to semi-elliptical surface cracks in thick plates is discussed in particular.

The strain energy release rate G for a crack in mode I is a function of the crack geometry, the direction of crack propagation and the state of loading. When G is known, the stress intensity factor K_I can easily be obtained.

In this paper the strain energy of the plate is computed numerically for a wide range of crack geometries using the finite element method. A 20-node isoparametric solid element is employed in modelling the structure. Certain special techniques for increasing the computational efficiency of the method, such as multilevel subdivision of the structure (substructuring) and condensation of degrees of freedom that are not needed in the crack propagation analysis, are emphasized. In fact, analysis of a large number of crack geometries requires only insignificantly more computational efforts than treating a single crack. Certain other aspects of the finite element modelling are also discussed.

Two methods for replacing the computed discrete values of strain energy by continuous functions are presented. These functions are expressed in terms of the two half-axes defining the geometry of the elliptical crack and they are determined using a least square technique. G and K_I are easily deduced from these functions.

As an example, a semi-elliptical, part-through, surface crack in a thick Ni-steel plate is analyzed. The crack is subjected to a combination of axial and bending loading. The load is applied cyclically. From the finite element calculations of the strain energy and the stress intensity factors which are computed accordingly, crack propagation along the two half-axes of the ellipse is calculated by utilization of a formula suggested by Paris. The results are checked against laboratory fatigue tests.

The method has proven to be very efficient and accurate. Due to its generality it can also be applied to complicated geometries and complex states of loading.

1. Introduction

Considerable efforts have been made during recent years to develop efficient computational techniques for solving practical problems in fracture mechanics. In particular, the finite element method has proven to be very useful in finding displacements, stresses and strain energy in structures containing cracks. This method can, in theory, be utilized in connection with structures of great complexity. However, analysis of cracks in three-dimensional bodies is generally very expensive and may be prohibitive for economical reasons. A highly efficient formulation of the finite element method is therefore essential in this connection.

Analysis of semi-elliptical surface flaws in thick plates is of particular interest in connection with design of various types of pressure vessels, tanks, pipe lines etc. Approximate methods of analysis for such cracks have been suggested by Irwin (1), Rice and Levy (2) and Thresher and Smith (3). Analyses by three-dimensional finite elements have been carried out by Levy et al. (4) and Marcal (5). Also, finite elements containing special singularity functions to be used near the tip of the crack have been developed by Byskov (6), Pian et al. (7), Tracey (8), Marcal (5) and others.

Unfortunately, the use of finite element idealization adapted to specific crack geometries and utilization of singularity elements is not economical in crack propagation problems which require a large number of different crack geometries be analyzed. A special technique by which the same finite element idealization can be utilized for analysis of cracks of arbitrary geometry will be presented in the following. It will be demonstrated that a large number of crack geometries can be analyzed at a cost which is only slightly higher than the cost of analysis of a single crack.

2. Calculation of stress intensity factors by the finite element method

The stress intensity factor is a quantity which is used in predicting the stability of cracks in a structure subjected to a specific type of loading. Rowe (9) has given a review of techniques suited to calculating stress intensity factors by use of the finite element method. A comparative example will now be presented in order that some general trends concerning the relative accuracy of four different techniques of this type may be established.

The first two of these techniques employ the computed displacement field or the stress field at the vicinity of the crack tip. By comparing the results with the corresponding analytical solutions for the two-dimensional crack problem (see for instance Irwin (10)), the stress intensity factor can be established. The third method employs the path-independent J-integral as suggested by Rice (11). Finally, the fourth technique is based on computation of the so-called strain energy release rate $\dot{G}$. This quantity is found by calculating the strain energy of the cracked body for different crack areas and taking the differential

$$G = \frac{dU}{dA} \quad (1)$$

where U is the strain energy and A is the area of the crack. During this computation the external loads should be kept constant. Assuming plane strain conditions the stress intensity factor for the opening mode is given by

$$K_I = \left(\frac{E}{1 - \nu^2} G \right)^{\frac{1}{2}} \quad (2)$$

where E is the modulus of elasticity and ν is Poisson's ratio. The strain energy is found from finite element analysis using the equation

$$U = \frac{1}{2} R^T r \quad (3)$$

where R and r are the load and displacement vector of the system.

Fig. 1 shows a plate with a center-crack subjected to pure axial loading. One quarter of the plate is modelled using 8 degrees of freedom, rectangular, plane strain elements with constant shear stress. Cracks of different lengths are introduced by imposing appropriate boundary conditions in the plane of crack. The results obtained using the four techniques mentioned above are plotted in Fig. 2. Results given by Isida and by Irwin-Westergaard (12) are also shown. As explained by Paris and Sih (13) the curve by Isida can be expected to be the more accurate of the two. The graph shows that the stress intensity factors computed by the different techniques are generally very good. However, the strain energy method based on eq. (1) seems to give the best over-all results compared with the values given by Isida.

Based on the results of this comparative study and the fact that the strain energy approach has proven to be well suited for crack propagation analysis further investigations have been limited to use of this technique only.

A prerequisite for successful application of this method to three-dimensional crack propagation problems is the use of a substructure or super-element formulation of the finite element method. Thereby an elimination (condensation) can be carried out for all degrees of freedom of the total system except for those that are essential for application of the external loads and those needed for defining the various crack shapes. This technique has been described in more detail by Aamodt et al. (14).

3. Crack propagation analysis of semi-elliptical surface cracks

The analysis of fatigue propagation of semi-elliptical surface cracks will be based on the formula suggested by Paris (15). Denoting the stress intensity factor for propagation of the crack along the x -axis of Fig. 3 by K_{Ix} and along the y -axis by K_{Iy} , the propagation rate due to cyclic loading can be found from

$$\frac{dc}{dN} = C_1 (\Delta K_{Ix})^{m_1} \quad (4)$$

and

$$\frac{da}{dN} = C_2 (\Delta K_{Iy})^{m_2} \quad (5)$$

in which C_1 , C_2 , m_1 and m_2 are material constants and N is the number of loading cycles. The variations of the stress intensity factors are defined by

$$\Delta K_{Ix} = (K_{Ix})_{\max} - (K_{Ix})_{\min} \quad (6)$$

and

$$\Delta K_{Iy} = (K_{Iy})_{\max} - (K_{Iy})_{\min} \quad (7)$$

The stress intensity factors K_{Ix} and K_{Iy} are obtained by the energy approach discussed above. Due to symmetry about the y - z plane of Fig. 3, only the right half of the crack need be considered. Thus, the differentiation of the strain energy with respect to the area of the crack as defined in eq. (1) can be carried out by parts as follows

$$\frac{dU}{dA} = \frac{\partial U}{\partial c} \frac{\partial c}{\partial A} + \frac{\partial U}{\partial a} \frac{\partial a}{\partial A} = \frac{4}{\pi} \left(\frac{1}{a} \frac{\partial U}{\partial c} + \frac{1}{c} \frac{\partial U}{\partial a} \right) \quad (8)$$

It is assumed that plane strain conditions prevail along the entire periphery of the crack. Obviously, this is not quite true for the conditions at the free surface of the plate. However, the error introduced by this assumption is believed to be negligible. The partial differentials $\partial U/\partial c$ and $\partial U/\partial a$ of eq. (8) can be found by variation of one of the half-axes c or a , while the other is kept constant. Keeping a constant, the stress intensity factor for crack propagation in the x -direction is given by

$$K_{Ix} = \left(\frac{4E}{\pi(1-\nu^2)} \frac{1}{a} \frac{\partial U}{\partial c} \right)^{\frac{1}{2}} \quad (9)$$

Similarly, for propagation in the y -direction

$$K_{Iy} = \left(\frac{4E}{\pi(1-\nu^2)} \frac{1}{c} \frac{\partial U}{\partial a} \right)^{\frac{1}{2}} \quad (10)$$

Using the finite element method to compute the strain energy of eq. (3), the stress intensity factors, K_{Ix} and K_{Iy} , can now be found for arbitrary crack shapes.

The integration of the two differential equations (4) and (5) for the crack propagation analysis has to be carried out simultaneously because they are not independent. Simple Euler-integration with variable step-length is used in the present procedure.

4. Functional fits to energy surfaces

For a given type of loading and specific values of the elliptical half-axes c and a the finite element analysis yields a discrete value of the strain energy of the system. Thus, when the analysis is carried out for all possible combinations between a set of c -values and a set of a -values, the strain energy is obtained for a rectangular pattern of points in the c - a or x - y coordinate system, see Fig. 4. The crack propagation analysis requires that the slopes of the energy surface $U(c,a)$ be computed at arbitrary values

of c and a , see eqs. (4), (5), (9) and (10). This can be done by replacing the discrete values of U by an approximate, differentiable function.

Such approximate energy surfaces may be computed efficiently by a least square fit of a polynomial in two variables. Since some scatter occurs in the computed values of U it is important to balance the accuracy of the fit against the smoothness of this function. High order polynomials tend to exhibit undesirable oscillations. Good performance has been experienced with the following polynomial

$$U \approx B_1 a^2 c^2 + B_2 a^2 c + B_3 a^2 + B_4 a c^2 + B_5 a c + B_6 a + B_7 c^2 + B_8 c + B_9 \quad (11)$$

where the coefficients B_i are determined by least squares.

When the ranges of variation of the two half-axes are large, it is necessary to divide the total domain into overlapping subdomains in order that sufficient accuracy be obtained. Weighted averages between the overlapping polynomials are used in the transition zones.

A more general method of polynomial fitting based on a collocation formulation of the finite element method was also developed. By this technique the total domain defined in the x - y (c - a) plane is divided into finite element subdomains. Within every such subdomain the approximate function f_i is determined by a set of nodal point values q_i and a corresponding interpolation polynomial N_i

$$f_i = N_i q_i \quad (12)$$

Assuming that there are n_i function values belonging to the element domain, the sum of the least squares is

$$W_i = \sum_{\alpha=1}^{n_i} (f_{i\alpha} - F_{i\alpha})^2 = \sum_{\alpha=1}^{n_i} (N_{i\alpha} q_i - F_{i\alpha})^2 \quad (13)$$

where $F_{i\alpha}$ is the computed function value at point α in element " i ".

Denoting the vector of nodal point values of the total domain by q , topological considerations give the relation

$$q_i = a_i q \quad (14)$$

By taking the sum of least squares from a total of N elements and setting its variation equal to zero, the following system of equations arises

$$\left(\sum_{i=1}^N a_i^T \left(\sum_{\alpha=1}^{n_i} N_{i\alpha}^T N_{i\alpha} \right) a_i \right) q = \sum_{i=1}^N a_i^T \left(\sum_{\alpha=1}^{n_i} N_{i\alpha}^T F_{i\alpha} \right) \quad (15)$$

This result is analogous to a standard finite element technique. Here, the element "stiffness" or coefficient matrix is given by

$$k_i = \sum_{\alpha=1}^{n_i} N_{i\alpha}^T N_{i\alpha} \quad (16)$$

and the "load" or constant vector is

$$P_i = \sum_{\alpha=1}^{n_i} N_{i\alpha}^T F_{i\alpha} \quad (17)$$

Similar matrices for the total system are obtained by standard element assemblage procedures which correspond to the topology transformations of eq. (15). Having solved the resulting equations, the fitted function and its derivatives may be found for any value of x and y using eqs. (12) and (14).

Fig. 4 shows an example where the total domain is divided into a 2 by 2 finite element mesh. The nodal points located at the corners of the rectangular elements have the approximate function and its two partial derivatives as degrees of freedom. Using the Hermitian polynomials (16) within each element, complete slope continuity (conformity) is obtained between adjacent elements. This fact implies continuity of the stress intensity factors when F is taken as the strain energy U .

Although this method works satisfactorily when the discrete function values possess sufficient smoothness, the third degree terms in x and y (c and a) of the interpolation polynomials may in many cases lead to undesirable oscillations of the fitted function. It is expected that these problems can be overcome by requiring the functions to vary piecewise as parabolas along the element boundaries. These requirements imply introduction of linear dependencies between certain nodal point degrees of freedom. This last feature has not been tested yet.

5. Example problem

As an example of the use of the present method a propagating surface crack in a 9 percent Ni-steel plate with thickness 1.1 in (28 mm) will now be analyzed. The numerical results will be compared with experimental data obtained by Det norske Veritas (17). The cyclic loading applied to the plate consisted of a combination of axial loading and bending. The test was carried out at a temperature of -162°C .

The plate with the surface crack is shown in Fig. 3. Only one quarter of the total plate was considered in the analysis due to double symmetry. The finite element idealization of the plate is shown in Fig. 5. Three levels of superelements involving 163 isoparametric hexahedral elements and 3360 nodal degrees of freedom were used. All reductions (generation of reduced stiffness matrices) were carried out using the program system SESAM-69 (18). A total of approximately 2 hours of computer time was required for this reduction on a UNIVAC-1108 computer.

All degrees of freedom that were not essential to the crack propagation analysis were eliminated (condensed) during the reduction of the superelements. Thus, only 4 nodal points for load application, see Fig. 5, and 161 nodal points in the plane of the crack, see Fig. 6, were retained. It should also be noted that only the displacement component in the z -direction was kept for the nodes in the crack plane.

A great variety of different crack shapes were simulated by introduction of proper boundary constraints at the nodes in the plane of the crack. For instance, the elliptical crack in Fig. 6 was approximated by restraining the nodes on and outside the dashed boundary line from moving in the z -direction while the nodes inside this line were permitted to move. Since only very few degrees of freedom were retained in the final model, the analysis of every new crack shape (three loading cases) required only 1.6 secs. CPU-time.

The discrete values of the strain energy by the finite element solution were replaced by polynomial fits based on eq. (11). The range of variation of each of the half-axes c and a was divided into two overlapping subintervals.

The finite element idealization deserves some comments. A major problem is the transition from a fine mesh near the crack zone to a relatively coarse mesh away from the crack. This can be accomplished by distorting the hexahedral elements drastically (for instance into triangular prisms) or by introducing linear constraints between nodes. Fig. 7 shows a mesh with extremely distorted elements quite close to the crack zone. The results obtained with this idealization for superelement (1.1) agreed very well with K_{Iy} for pure axial loading as given by Rice and Levy (2), however, the discrepancy for bending load was considerable. Therefore, it was decided to try a new mesh as shown by superelement (1.1) in Fig. 5. This element idealization is finer and it has a smoother transition from fine to coarse zone. The remarkable fact, however, was that results for K_{Iy} by this new analysis deviated less than 2 percent from the first results.

Figs. 8a-b and 9a-d show the results obtained by the finest element mesh. σ_0 represents the nominal tensile stress in the plate for axial loading and the nominal maximum stress for bending. Only values for K_{Iy} for axial and bending loading as given in Ref. (2) are available in the literature and will be used for comparison. From Fig. 9c it is seen that the agreement between the two solutions is very good for pure axial loading whereas the discrepancies between the K_{Iy} -values for pure bending are quite large, see Fig. 9d. The reasons for this are not clear.

Possible error sources in the present method are: (i) errors introduced by modelling elliptical cracks by restraining displacements in a rectangular nodal point mesh, and (ii) constraints in the displacement field as introduced by the finite element method. Errors of the first category are believed to be small since the smoothing effect of the polynomial fit for the energy tends to cancel these out. The good results obtained for the axial loading also support this theory. Errors of category (ii) are hard to predict; however, the fact that two different element meshes gave almost identical results suggests that these errors are quite small also.

On the other hand, the results given in Ref. (2) were based on a simplified two-dimensional analysis in which membrane theory and Kirchhoff plate bending theory were employed. These assumptions are questionable for small cracks. This is particularly the case for bending loads. Further detailed

studies will hopefully clarify this matter.

A crack propagation analysis was carried out using eqs. (4) and (5) and the following material constants

$$C_1' = C_2 = 2.05 \cdot 10^{-17} \text{ (in, lbs)} = 1.18 \cdot 10^{-12} \text{ (mm, N)}$$

$$m_1 = m_2 = 2.67$$

These constants were determined from a large number of test specimens (17). The propagation analysis was carried out for a combined state of loading in which the resulting nominal stress varied linearly through the plate thickness. Maximum nominal tensile stress was at the surface containing the crack opening and the stress at the opposite surface was zero. The mean value of the maximum stress was maintained at 22750 psi (157 N/mm²) for the entire load history whereas the amplitudes of the maximum stress were kept at 15650 psi (108 N/mm²). The stress intensity factors for the combined state of loading were determined using the third loading case in the analysis discussed above. It should be noted that stress intensity factors for different types of loading are not additive when one load tends to close the crack.

Figs. 10a and 10b show comparisons between experimental results for the crack size and analytical results obtained by the present method. The dashed lines indicate upper and lower bounds arising when the standard deviation obtained by the material testing is added (subtracted) to the material constants. It is seen that the computed results are quite close to the observed crack sizes and they are well within the bounds of uncertainty of the material constants.

The correspondence between computed and experimentally determined shapes of the elliptical crack during propagation is illustrated in Fig. 11. The good agreement is encouraging.

6. Conclusions

A general approach for analysis of cracks in three-dimensional bodies has been presented. The present technique is based on the finite element method and it utilizes an energy formulation for computing stress intensity factors. It is particularly well suited for a crack propagation analysis requiring a large number of different cracks to be analyzed. The same finite element mesh is kept regardless of the shape of the crack. Errors introduced by the element idealization of the crack geometry are to a large extent eliminated by smoothing techniques for the strain energy surface.

Although very few analytical results can be found in the literature for comparison, the results obtained by the present technique agree very well with experimental evidence.

It is believed that the superelement technique suggested here is also well suited for use in connection with nonlinear fracture mechanics involving large scale yielding near the crack.

References

- (1) IRWIN, G.R., "Crack-Extension Force for a Part-Through Crack in a Plate", Journal of Applied Mechanics, Vol. 29, Trans. ASME, Vol. 84, Series E, 1962, pp. 651-654.
- (2) RICE, J.R., LEVY, N., "The Part-Through Surface Crack in an Elastic Plate", Journal of Applied Mechanics, Vol. 39, Trans. ASME, Vol. 94, Series E, 1972, pp. 185-194.
- (3) THRESHER, R.W., SMITH, F.W., "Stress-Intensity Factors for a Surface Crack in a Finite Solid", Journal of Applied Mechanics, Vol. 39, Trans. ASME, Vol. 94, Series E, 1972, pp. 195-200.
- (4) LEVY, N., MARCAL, P.V., RICE, J.R., "Progress in Three-Dimensional Elastic-Plastic Stress Analysis for Fracture Mechanics", Nuclear Engineering and Design, Vol. 17, No. 1, 1971, pp. 64-75.
- (5) MARCAL, P.V., "Three-Dimensional Finite Element Analysis for Fracture Mechanics", Surface Crack, Physical Problems and Computational Solutions, ASME N.Y., 1972, pp. 187-202.
- (6) BYSKOV, E., "The Calculation of Stress Intensity Factors using the Finite Element Method with Cracked Elements", International Journal of Fracture Mechanics, Vol. 6, No. 2, June, 1970, pp. 159-167.
- (7) PIAN, T.H.H., TONG, P., LUK, C.H., "Elastic Crack Analysis by a Finite Element Hybrid Method", 3rd Conference on Matrix Methods in Structural Mechanics, Wright-Patterson AFB, Ohio, Oct. 1971.
- (8) TRACEY, D.M., "Finite Elements for Determination of Crack Tip Elastic Stress Intensity Factors", Engineering Fracture Mechanics, Vol.3, No.3, 1971, pp. 255-265.
- (9) ROWE, G.H., "Matrix Displacement Methods in Fracture Mechanics Analysis of Reactor Vessels", Nuclear Engineering and Design, Vol. 20, No. 2, 1972, pp. 251-263.
- (10) IRWIN, G.R., "Analysis of Stresses and Strain Near the End of a Crack Traversing a Plate", Journal of Applied Mechanics, Vol.24, Trans. ASME, Vol. 79, 1957, pp. 361-364.
- (11) RICE, J.R., Fracture, edited by LIEBOWITZ, H., Vol. 2, Academic Press, New York, 1968, pp. 191-311.
- (12) BROWN, W.F., SRAWLEY, J.E., ASTM STP 410, American Society for Testing and Materials, Philadelphia, 1966.
- (13) PARIS, P.C., SIH, G.C., "Stress Analysis of Cracks", Fracture Toughness Testing and Its Applications, ASTM STP 381, American Society for Testing and Materials, Philadelphia, 1965.
- (14) AAMODT, B., BERGAN, P.G., KLEM, H.F., "Calculation of Stress Intensity Factors and Fatigue Crack Propagation of Semi-Elliptical, Part-Through Surface Cracks", to be presented at Second International Pressure Vessel Conference, San Antonio, Texas, U.S.A., 1-4th Oct. 1973.
- (15) PARIS, P.C., GOMEZ, H.P., ANDERSON, W.E., "A Rational Analytic Theory of Fatigue", The Trend in Engineering, Vol. 13, No. 9, 1961.
- (16) ZIENKIEWICZ, O.C., The Finite Element Method in Engineering Science, McGraw-Hill, London (1971).
- (17) TENGE, P., SOLLI, O., "Fracture Mechanics in Design of Large Spherical Tanks for Ship Transport of LNG", Norwegian Journal of Maritime Research, No. 2, 1973.
- (18) ARALDSEN, P.O., "SESAM-69. A General Purpose Finite Element Method Program", Application of Computerized Methods in Analysis and Design of Ship Structures, Marin Structures, and Machinery, a course arranged at Det norske Veritas, Oslo, Nov. 1972.

Fig. 1 Finite element idealization of center-crack plate

Fig. 2 Stress intensity factor for a center-crack plate computed by different methods

Fig. 3 Semi-elliptical, part-through surface crack in a plate

Fig. 4 Functional fit by four rectangular finite element sub-domains

Fig. 5 Finite element idealization of one quarter of the plate

Fig. 6 Remaining nodes in the x-y plane of the final model

SUPERELEMENT
LEVEL 1, TYPE 1

Fig. 7 Alternative element idealization of the near crack super-element

Fig. 8a Strain energy as function of a for pure axial loading

Fig. 8b Strain energy as function of a for pure bending

Fig.9a Stress intensity factor K_{I_x} as function of c for pure axial loading

Fig.9b Stress intensity factor K_{I_x} as function of c for pure bending

Fig.9c Stress intensity factor K_{I_y} as function of a for pure axial loading

Fig.9d Stress intensity factor K_{I_y} as function of a for pure bending