SOME CONSIDERATIONS FOR THE EVALUATION OF FINITE ELEMENT MODELS

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SUMMARY

The object of this paper is an assessment of the present state of the art of finite element discretisation techniques.

There are a number of factors which affect the performance of individual elements in stress-analysis:

- (1) Variational Formulation (Basis of displacement, stress and mixed models).
- (2) Modal Functions (Element interpolation of independent field variables).
- (3) Geometric Mapping of Element Functionspace (Transformation between homogeneous and global coordinates).
- (4) Numerical Integration of Energy Density (Evaluation of element properties).
- (5) Inter-element Continuity (Compatibility of independent field variables across element boundaries).

After a critical appraisal of these effects on the element behaviour, we concentrate our investigation on compatible displacement models.

First, some concepts of bounding the discretisation error are critically reviewed and assessed in regard to their practical usage. Then, appropriate numerical methods are developed to measure energy contributions of different finite element models with eigenvalue techniques for both, the element level as well as the structural level. In conclusion, several examples are given in which these concepts are applied to investigate effects such as the choice of modal functions, the distortion due to non-linear mappings, and the order of numerical integration schemes.

1. Introduction

The following discussion is concerned with two aspects of the finite element method, the evaluation and assessment of stiffness properties and the computation of stress fields. Both topics are studied within the context of planar and axisymmetric analyses, the conclusions of which readily extend to three-dimensional problems as well as plates and shells.

Basically there are three factors influencing the stiffness properties of the idealised structure :

1. Virtual work or energy formulation of the boundary value problem.
2. Approximation of geometry and field variables.
3. Numerical evaluation of stiffness properties (numerical integration and differentiation).

In the first part of the paper appropriate energy measures are developed based on eigenvalue techniques for comparison of different stiffness matrices. The stiffness properties of a conventional axisymmetric displacement model are then compared with those of an element with identical nodal configuration, in which the strain field is approximated directly in analogy to the natural mode technique [1],[2],[3]. The question of the relative order of approximation between strain and displacement expansions are explored from a theoretical point of view and are illustrated on a number of examples. In particular, for an axisymmetric element the polynomial interpolation of the circumferential strain components is considered as opposed to that for the conventional displacement model. Subsequently, the numerical integration of the strain energy expressions is reviewed and compared with the direct strain approximation techniques mentioned above. An element study demonstrates some of the effects due to the reduction of strain expansions and integration rules. In conclusion, the results of the element investigations are verified on a structural example, the axisymmetric analysis of a PCR.V.

In the second part of the paper various methods are discussed for the evaluation of representative stress fields in displacement models. A brief review of "local" element stress computations, which are normally combined with averaging techniques is followed by a presentation of "integral" stress calculations, in which the element stress field is approximated by polynomial expansions. The integral formulation may be regarded "consistent" in the energy sense as it represents a "stress error" minimisation technique within the element domain, according to the principle of virtual work or the equivalent Galerkin procedure. This method yields stress distributions which are discontinuous along element interfaces and hence are amenable to averaging techniques when nodal stresses are to be determined. In conclusion the conjugate stress approximation [11] is briefly discussed and illustrated on a pathological example for which results are also obtained by the formulations presented previously.

2. Stiffness Evaluation

In what follows a number of methods are reviewed for the selection of elements which are optimal with regard to accuracy and computational effort. These techniques are applied to examples on the element as well as the structural level, in order to illustrate the effects of direct strain approximation, as opposed to conventional displacement models, and to discuss the influence of the numerical integration scheme.

2.1 Energy Measures

The finite element method is based on virtual work or equivalent energy principles which are discretised by elementwise approximation of the energy distribution. Convergence in the energy norm can be assured from a theoretical point of view, as long as the energy expansion satisfies certain completeness and continuity requirements, while boundedness of the solution is ascertained in the case that minimum principles are used. The asymptotic rate of the discretisation error is related to the order of strain approximation in the energy expression and can be measured by the following norm [9]

$$\|u - \hat{u}\|_m = c_1 h^{2(p+1-m)} \quad (1)$$

in which c_1 denotes a proportionality factor, h a representative measure of the element dimension, p the order of polynomial approximation and m the order of differentiation in the strain energy density. This theoretical observation is of little practical interest since no information is available on c_1 for finite mesh lay-outs in which h might become larger than one.

A more practical approach to estimate the discretisation error is based on direct convergence studies of the strain energy for different mesh lay-outs

$$U = \frac{1}{2} r^t K r \quad \text{where} \quad K r = R \quad (2)$$

We observe that the strain energy U depends on both the structural stiffness property K and the nodal displacements r . In the case of displacement models the resulting strain energy forms an upper bound for displacement boundary conditions and a lower for force boundary conditions according to the principle of minimum potential energy. The eigenvalue decomposition of K provides an orthogonal basis X in which the displacement vector r can be expanded as a linear combination of the individual eigenvector x_i

$$\begin{aligned} X^t K X &= \Lambda & \text{where} & \quad \Lambda = [\lambda_i] \\ r &= A X & & \quad A = [\alpha_i] \end{aligned} \quad (3)$$

the undetermined coefficients α_i form weighting functions of the eigenvalues λ_i which contain the basic information on the strain energy content. These weighting coefficients are easily determined using the orthogonality properties of X

$$\begin{aligned} \alpha_i &= x_i^t r & \text{for displacement boundary conditions} \\ \alpha_i &= \frac{1}{\lambda_i} x_i^t R & \text{for force boundary conditions} \\ U &= \frac{1}{2} A \Lambda A \end{aligned} \quad (4)$$

An assessment of the stiffness properties for specific boundary value problems should be based on the direct evaluation of the strain energy expression in eq.(2). The direct approach forms a standard procedure available in most finite element software packages designed for multiple load cases, while the eigenvalue method poses a formidable computational task offering no advantage in contrast to the opinion expressed in [15]. However, the latter technique yields additional information if no specific boundary values are to be considered. If individual elements or structural idealisations are to be compared having the same nodal configuration and displacements along boundaries, the eigenvalues of the stiffness or flexibility difference matrix provide valuable information on their relative behaviour [10]. The difference of the strain energy U_{Δ} is once again given by

$$U_{\Delta} = U_r - U_t = \frac{1}{2} \mathbf{r}^t (\mathbf{K}_r - \mathbf{K}_t) \mathbf{r} = \frac{1}{2} \mathbf{r}^t \mathbf{D}_k \mathbf{r} \quad (5)$$

where $\mathbf{D}_k$ denotes the stiffness difference matrix. The definiteness of this quadratic form assures that one stiffness matrix is more flexible than the other for all conceivable boundary conditions. In the case that $\mathbf{D}_k$ is indefinite then no conclusion can be drawn without prior knowledge of some specific boundary conditions. The direction of a displacement vector $\mathbf{r}$ corresponds then to an eigenvector $\mathbf{Y}_i$ whose associated eigenvalue ω_i determines which of the two matrices will provide optimal strain energies

$$\mathbf{Y}^t \mathbf{D}_k \mathbf{Y} = \Omega \quad \text{where} \quad \Omega = \mathbf{r}^t \omega_i \mathbf{r} \quad (6)$$

$$\mathbf{D}_k = \mathbf{K}_r - \mathbf{K}_t$$

The spectral radius of the difference matrix forms a suitable norm to measure the "distance" between a reference stiffness $\mathbf{K}_r$ and the stiffness under investigation. The spread of eigenvalues indicates the variation of the difference in strain energy for all possible deformation states. We emphasize that only positive or negative definite difference matrices permit a unique conclusion about the relative stiffness of the two matrices under consideration. On the other hand, only the eigenvalues of the difference matrix $\mathbf{D}_k$ form a proper measure of the strain energy difference in contradistinction to the difference of eigenvalues of each individual stiffness matrix, which in general do not correspond to the same eigenvectors. Exactly for this reason it is not permissible to compare simply the trace or other matrix norms of the individual matrices separately [16].

2.2 Axisymmetric Solid Element

The evaluation of optimal axisymmetric stiffness properties is of particular interest to stress analysts in reactor technology. For this reason we concentrate on an assessment of the stiffness representation in a 9-point axisymmetric quadrilateral element, whose geometry is based on second order Lagrangean interpolation [5]. Similar phenomena can be observed at the 6-point axisymmetric triangular element TRIAX 6 which was introduced first in [4].

The derivation of the elastic stiffness properties has been documented extensively elsewhere [1],[2], here we only recall that the displacement model leads to the following integral expression for the element strain energy U_i

$$U_i = \frac{1}{2} U_i^T k_{uu} U_i = \frac{1}{2} \int_A \epsilon^T E \epsilon r dA \quad (7)$$

for which the linearized strain-displacement relationships are assumed to hold everywhere within the element domain

$$\epsilon = D^T u \quad (8)$$

with

$$D = \begin{Bmatrix} \partial/\partial r & 0 & 1/r & \partial/\partial z \\ 0 & \partial/\partial z & 0 & \partial/\partial r \end{Bmatrix}$$

and

$$u = \{ u_r \ u_z \} \quad (9)$$

$$\epsilon = \{ \epsilon_{rr} \ \epsilon_{zz} \ \epsilon_{\theta\theta} \ \epsilon_{rz} \}$$

Introducing appropriate approximations for the element geometry and the displacement field (e.g. biquadratic Lagrangean interpolation for the QUAXC9 element), we obtain the stiffness of the conventional displacement model

$$x = \omega_x x_i$$

$$u = \omega_u u_i \quad (10)$$

$$k_{uu} = \int_A (D^T \omega_u)^T E (D^T \omega_u) r dA$$

The element stiffness properties and the separate contribution of individual strain components are shown in fig. 1 in form of eigenvalue spectra for four typical element configurations. The modes corresponding to zero eigenvalues are omitted in the illustration. We observe that in all cases the circumferential components contribute less to the stiffness than the other components, hence they play the most dominant role for the flexibility matrix, whose behaviour is governed by the reciprocals of the stiffness eigenvalues. Similarly the radial components add a very high stiffness content to the cylindrical "shell element" while the flexibility action is governed again by circumferential and longitudinal straining. A comparison of the square and curved quadrilateral shows the effect of geometric distortion. The spectrum of the radial strains, which are mostly concerned, is magnified by orders of magnitude while the other strain components seem hardly influenced.

We recall that the variation of the field variables is normally defined in the homogeneous ξ coordinates which are mapped into the global coordinates using the parametric representation of the element geometry. It is clear that the circumferential strain expression in eq. (10) yields fractional rational functions although the global displacements and displacement gradients are polynomials. In analogy the global displacement gradients do also change their character generating fractional rational functions if higher order mappings are used for the geometric representation.

It is recognised that these geometric distortions may cause a reduction of the displacement approximation in the global analysis space. For example, let us consider a simple one-dimensional line element whose geometry and displacement field are defined by polynomial expansions in the homogeneous ξ - coordinate.

$$\begin{aligned} x &= x(\xi^q) & q &= \text{order of geometric expansion} \\ u &= u(\xi^p) & p &= \text{order of displacement expansion} \end{aligned} \quad (11)$$

Inversion of the mapping between the global and the parameter space yields the following approximation for the global variation of the displacements and strains

$$\begin{aligned} \xi &\rightarrow \xi(x^{1/q}) \\ u &\sim u(x^{p/q}) \\ \varepsilon &= \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} \sim \frac{p}{q} u(x^{p/q-1}) \end{aligned} \quad (12)$$

This clearly indicates that the geometric representation is primarily improved at the expense of the displacement approximation in the global analysis space. Therefore, we investigate in the following an alternative formulation in which the integration of the strain energy is carried out over the actual curved element domain, while the strain field is approximated directly by polynomial expansions in analogy to the natural mode technique [1],[2],[3]. This procedure provides a mechanism to replace the fractional rational functions arising in the displacement gradient field in order to improve hopefully the energy approximation within the element.

Generalizing the principle of minimum potential energy [14] to include the strains as independent field variables we arrive at the following functional for the isothermal case

$$I(u, \varepsilon) = \int_A \left(-\frac{1}{2} \varepsilon^t E \varepsilon + \varepsilon^t E D^t u - u^t p_v \right) r dA - \int_{S_\sigma} u^t p dS \quad (13)$$

Tacitly we assume that the stress-strain relations and the displacement boundary conditions are satisfied locally

$$\begin{aligned} \sigma &= E \varepsilon \\ u &= u, \quad \text{auf } S_u \end{aligned} \quad (14)$$

The independent field variables u and ε are now approximated by finite element expansions

$$\begin{aligned} u &= \omega_u u_i \\ \varepsilon &= \omega_\varepsilon \varepsilon_i \end{aligned} \quad (15)$$

The stationary value of the functional yields then a set of discrete equilibrium and strain-displacement equations

$$\begin{bmatrix} 0 & k_{ue} \\ k_{eu} & k_{ee} \end{bmatrix} \begin{bmatrix} u_I \\ e_I \end{bmatrix} = \begin{bmatrix} P_I \\ 0 \end{bmatrix} \quad (16)$$

where the individual submatrices are

$$\begin{aligned} k_{ee} &= \int_A \omega_e^t E \omega_e r dA \\ k_{eu} &= \int_A \omega_e^t E (D^t \omega_u) r dA \\ P_I &= \int_A \omega_u^t p_v r dA + \int_{S_g} \omega_u^t p_s ds \end{aligned} \quad (17)$$

The strain field may be discontinuous across element interfaces without generating unbounded energy contributions. Therefore, the strain degrees of freedom e_I can be eliminated by static condensation prior to the assembly of elements

$$S_{uu} = k_{eu}^t k_{ee}^{-1} k_{eu} \quad (18)$$

The resulting equilibrium conditions are formally identical with those of the pure displacement model

$$S_{uu} u_I = P_I \quad (19)$$

The "stiffness" matrix S_{uu} is symmetric and positive definite if the rank of the submatrix k_{ee} is greater than the number of displacement degrees of freedom. This implies that more strain degrees of freedom have to be used than displacement degrees of freedom if singularities are to be avoided. In the case of the QUAXC 9 element, the constant strain approximation generates a "stiffness" matrix of rank four. Only the higher order biquadratic expansion of the strain field assures positive definiteness after removal of the rigid body translation in the z-direction. Therefore, in praxis, only selected strain components ought to be approximated directly, while the others are still derived from the displacement gradient field.

It was proven in [16] that the direct strain approximation yields "stiffness" matrices which are always more flexible than the associated displacement formulations as long as the strain expansion is of lower order than the variation of the displacement gradient field.

$$\begin{aligned} \text{If} \quad & \omega_e \subset (D^t \omega_u) \\ \text{then} \quad & u_I^t S_{uu} u_I < u_I^t k_{uu} u_I \end{aligned} \quad (20)$$

On the other hand, the upper bound properties are lost in the mixed model, but for higher order strain expansions, $\omega_{\epsilon} \supset (\mathbf{D}^t \omega_u)$ the stiffness properties of the displacement model are recovered. The behaviour changes only if the strain degrees of freedom are not eliminated by static condensation, but are retained and used for the element assembly. The latter approach, based on the Hellinger-Reissner theorem, has been utilized elsewhere [12], [8], but rather discouraging results were reported.

In conclusion these theoretical observations are illustrated with some element studies. To this end we construct the stiffness difference matrix $\mathbf{D}_k$ between the displacement model and various mixed models, in which either all strain components ϵ are approximated by constant, bilinear and biquadratic expansions, or only the circumferential or shear strain components $\epsilon_{\theta\theta}$ and ϵ_{rz}

$$\mathbf{D}_k = (\mathbf{k}_u - \mathbf{k}_\omega) \quad (21)$$

with

$$\omega_{\epsilon} = \omega^0, \omega^1, \omega^2$$

The stiffness of the displacement model acts as a reference matrix which is compared with the stiffness properties obtained from various direct strain approximations. Figures 2, 3, 4 and 5 illustrate the eigenvalue spectra for four representative element geometries. The results clearly demonstrate the positive-semi definiteness of all stiffness difference matrices, supporting the statement that mixed models are always more flexible or equal to the associated displacement model. Moreover, the circumferential strain approximation affects the stiffness properties to a much smaller extent than e.g. the shear strain expansion and appears to be negligible altogether in the case of the thin-walled cylinder and the circular plate.

A check of the eigenvalues of the various "stiffness" matrices verifies the comments with regard to the reduction of the rank. In the case of $\epsilon = \omega_{\epsilon} \epsilon_z$ only the quadratic expansion, $\omega_{\epsilon} = \omega^2$, ensures the retention of the full rank (17 positive eigenvalues), while all other interpolation schemes lead to zero straining for some element deformation modes. The circumferential strain expansion maintains the full rank of 17 only for the quadratic interpolation, while the lower order expansions lead to additional singularities besides the rigid body translation. In the case of the shear strain approximation both the linear and the quadratic expansions cause no reduction of the rank of 17, in contrast to the constant interpolation scheme $\epsilon_{rz} = \omega^0 \epsilon_z$. From this element study it can be concluded that the quadratic strain expansion does not reduce the stiffness properties sufficiently to form an alternative to the displacement model. On the other hand, from the lower order expansion only the linear shear strain approximation provides element "stiffness" matrices which are "safe", that means they can not create undesirable singularities on the structural level.

2.3 Quadrature Effects

In this section the effects of numerical integration on the element stiffness are investigated. We recall that the strain energy density contains fractional rational functions which have to be integrated numerically. Therefore, the quadrature introduces an additional approximation for the variation of the strain energy since only polynomials are integrated exactly; in other words, the order of the numerical integration corresponds to a polynomial variation of the specific energy within the element domain, a phenomenon which is comparable with that of the direct strain approximation by polynomials.

The concept of stiffness difference matrices is ideally suited to study the order of the integration rule numerically for a displacement model. The quadratic displacement model with a 6×6 Gaussian quadrature rule provides the reference properties with which other stiffness matrices are compared using lower order integration rules $n = 1, 2 \dots 5$. The stiffness difference matrix is then

$$D_k = (k_c - k_n) \quad \text{with} \quad n = 1, 2 \dots 5 \quad (22)$$

Figure 6 shows the resulting eigenvalue spectra of D_k for four element geometries. We observe that in all cases the stiffness difference matrix is indefinite, which implies that no single valued conclusion can be drawn in regard to the effect of the reduced quadrature rule. This fact asserts that contrary to the opinion of many investigators there exist boundary conditions under which the lower order integration increases the stiffness properties. From the spread of eigenvalue spectra the opposite phenomenon will be in general more probable, a fact which was observed and utilized in [13], [18] to simulate the shell behaviour by degenerate solid elements. To this end a word of caution seems appropriate, as there is no guarantee for the reduction of stiffness, as shown above. Moreover, the reduction of integration rule does also affect the rank of the resulting element stiffness matrix in analogy to the various orders of strain approximations. For example, a one-point quadrature rule accommodates the constant strain states only, reducing the rank of 17 to the order four. In analogy the 2×2 Gaussian quadrature rule, which was recommended in [18], causes a reduction of the rank which may lead to undesirable singularities of the stiffness even after the element assembly.

2.4 Case Study on a Prestressed Concrete Reactor Vessel

The previous observations on the element level are now exemplified on a more realistic example, the THTR 1 : 5 scale model of a PCRV. Figure 7 shows the finite element idealisation of the structural components in the upper half of the vessel. The specifications of the PCRV are presented in detail in [6], [17].

The reactor example was chosen to investigate three effects on the strain energy and the eigenvalue spectrum of K , the influence of the mesh lay-out, the influence of numerical integration and the influence of direct strain approximation. Three load cases were considered, internal pressure, prestress and temperature, from which only a few results are shown due to space limitations.

Figure 8 shows the strain energy due to internal pressure for various rules of numerical integration and different orders of polynomial strain approximation. We observe that there is virtually no effect of quadrature rules higher than three, while the 2×2 rule leads to a considerable increase of the strain energy content without causing a singularity in the stiffness. The positive definiteness of K can be due to the stabilizing effect of the other structural components, such as the liner and the prestressing cables. On the other hand the 2×2 Gaussian integration rule reduces the rank of the element stiffness matrix and led in other examples to singularities of the assembled stiffness which could not be removed with the rigid body modes of the structure. The direct strain approximation yields identical results for the higher order expansions of $\epsilon_{\theta\theta}$ as well as ϵ_{rz} , only the constant circumferential strain expansion causes a considerable increase of the strain energy content. We also observe that the constant shear strain and of course the

constant interpolation of all strain components introduce additional singularities in the structural stiffness.

Figure 9 illustrates the variation of the highest and lowest eigenvalues of $\mathbf{K}$ for various rules of integration and different orders of strain approximation. In addition the distribution of the conditioning number is shown, which is normally used for estimations of the numerical error. Again, there is no effect of higher order quadrature rules, only the 2×2 rule causes a considerable increase of both $\lambda_{\max}$ and especially $\lambda_{\min}$, while the conditioning number is reduced accordingly. The low order strain expansions for ϵ_{xx} and ϵ_{yz} introduce also slight changes in the eigenvalue spectrum and the conditioning as long as they don't cause singularities altogether.

3. Stress Evaluation

In engineering practice the derivation of the stress distribution forms, in general, the main objective of the analysis. Within this context the difficulties of computing representative stress fields from displacement approximations have long been a point of major interest in stress analysis. In the following, various techniques are discussed for the element stress evaluation and the development of representative nodal stresses, e.g. by averaging techniques. In principle, the local stress calculation based on the displacement gradient field is compared with direct polynomial approximations of the stress field constructed with a "stress-error" minimisation procedure within the element domain. The latter is "consistent" in an integral sense and corresponds to the energy approximation in the stiffness relations.

3.1 Local Stress Calculation

The standard approach for the evaluation of element stresses for displacement models is based on the displacement gradient field obtained by appropriate differentiation of the displacement expansion. This local point of view yields the following expression for the element nodal stresses

$$\sigma = E \epsilon = E_x (D_x^t \omega) u_x \quad (23)$$

The local element stresses violate both, the interior differential equilibrium and the stress boundary conditions and generate considerable discontinuities at the element nodes, forming some sort of measure for the discretisation error. In engineering practice it is customary to average the element contributions at common nodes of contiguous elements. Refined weighting schemes have been proposed to improve the resulting stress distribution, but the basic source of difficulties, i.e. the differentiation of the displacement field, could not be eliminated. In analogy to the arguments for a "better" representation of the strain energy by direct polynomial approximation of the strains we concentrate in the following on alternative techniques for direct interpolation of the stress field.

3.2 Direct Stress Approximation

We recall that the principle of virtual displacements yields element nodal forces P_x which correspond in the energy-sense to the stress field σ

$$\int_V u_x^t P_x = \int_V \epsilon^t \sigma \, dV \quad (24)$$

The stress field σ could be the exact one or any other field which satisfies equilibrium with the external body forces and surface tractions in an integral sense. In the displacement formulation these are the local stresses which correspond to the displacement gradient field $D^t u$

$$\sigma = E (D^t \omega_u) u_x \quad (25)$$

In the case of beams and flanges the nodal forces P_x are exactly the stress resultants of interest to the stress analyst. Therefore, we consider now the question if we can't develop an analogous procedure to obtain nodal element stress resultants σ_x for elasticity problems, which are equivalent to the local stress distribution σ in some integral sense.

Invoking Galerkin's method we approximate the stress distribution by e.g. polynomial expansions such that the stress error function $(\sigma_A - \sigma)$ is minimised by the orthogonality condition

$$\int_V \omega^t (\sigma_A - \sigma) dV = 0 \quad (26)$$

where

$$\sigma_A = \omega \sigma_x$$

Alternatively, the principle of virtual work yields the same definition of the element nodal stresses σ_x considering that

$$\int_V \delta \epsilon_A^t (\sigma_A - \sigma) dV = 0 \quad (27)$$

In the case of homogeneous material properties we can use the following approximation

$$\begin{aligned} \epsilon_A &= \omega \epsilon_x \\ \sigma_A &= \omega \sigma_x \end{aligned} \quad (28)$$

which yields after substitution

$$\delta \epsilon_x^t \int_V \omega^t \omega dV \sigma_x = \delta \epsilon_x^t \int_V \omega^t \sigma dV \quad (29)$$

where

$$\sigma_x = C^{-1} \int_V \omega^t \sigma dV$$

The fundamental matrix of the expansion is defined by

$$C = \int_V \omega^t \omega dV \quad (30)$$

in which the quadratic form of weighting functions ω forms the kernel. We observe the analogy to the consistent mass matrix which tends to smoothen out discontinuities and stress concentration effects in the case of dynamic response analysis. Similarly the fundamental matrix forms a weighting mechanism to even

out local concentrations in the stress gradient field and yields a stress field which is a best fit in a "least square" sense.

The asymptotic rate of convergence is equal for both the local and the integral stress evaluation technique. If the exact nodal stresses are

$$\hat{\sigma}_I = C^{-1} \int_V \omega^t \hat{\sigma} dV \quad (31)$$

then the stress error is measured by the L_2 norm

$$\begin{aligned} \|\sigma_x - \hat{\sigma}_I\| &= \|C^{-1} \int_V \omega^t (\sigma - \hat{\sigma}) dV\| \\ &\leq (C^{-2} \int_V \omega^t \omega dV \cdot \int_V (\sigma - \hat{\sigma})^t (\sigma - \hat{\sigma}) dV)^{1/2} \\ &\leq (C^{-1} \int_V (\sigma - \hat{\sigma})^t (\sigma - \hat{\sigma}) dV)^{1/2} \end{aligned} \quad (32)$$

which was to be shown. On the other hand the direct stress approximation is based on an integral technique in analogy to representation stiffness of the strain energy and may be termed in this sense "consistent".

The direct stress approximation technique involves three steps of calculation on the element level

1. Computation of the fundamental matrix

$$C = \int_V \omega^t \omega dV \quad (33)$$

2. Computation of the weighted load vector

$$P = \int_V \omega^t \sigma dV \quad (34)$$

3. Solution of the unknown element nodal stresses

$$C \sigma_I = P \quad (35)$$

The computational effort of this technique exceeds that of the element stiffness evaluation since it requires an additional solution of equations on the element level. For this reason an extrapolation technique was developed in [6], [7], which corresponds to a multilinear expansion of ω within the element domain. There, the stresses are first evaluated at the pivot points of numerical integration (3 x 3 rule for the quadrilateral, 7-point rule for the triangle) which are subsequently extrapolated towards the nodes. This extrapolation technique requires hardly any additional effort for the stress computation when compared with the conventional local stress evaluation of displacement models.

In conclusion we direct our attention to the stress computation technique developed in [11] which utilizes the "conjugate" function space for direct stress approximation. This basis is biorthogonal to the function space used for displacement approximations and thus maintains the same continuity properties

along element boundaries. For this reason a large system of equations has to be solved for the construction of the conjugate basis, which compares with the computational effort required for the displacement solution. Moreover, the continuous conjugate base functions can not accommodate discontinuous stress fields arising in the case of inhomogeneous and composite structures, i.e. problems, for which the finite element method found the widest area of application.

3.3 Case Study on a Square Sheet

In this example a square sheet is subjected to a bilinear temperature gradient. Figure 10 defines the input data of the problem which should yield zero stresses for the temperature conditions shown. Figure 11 describes the idealisation with the highly distorted plane stress elements TRIMC 6 and QUAMC 9 and gives the strain energy results. Due to the non-linear mappings both elements are incapable of reproducing linear strain states, whereby the quadrilateral seems more affected than the triangle when the strain energy contents are compared.

Figure 12 summarizes the stress distributions of both elements. The results are presented on the element level as well as on the structural level for which all element contributions are averaged to provide nodal stresses. The figure shows the variation of the σ_{xx} stress components along the curved element boundary 3, 8, 13, 18, 23, which should be theoretically zero. The results of the local stress evaluation illustrate the disastrous effects of fractional rational functions generated by the displacement gradient field. We observe that the geometric distortion affects the quadrilateral more than the triangle, especially if we consider the discontinuities of element stresses at the center node 13. A considerable improvement is achieved with direct approximation of the stress field by quadratic, linear and multilinear polynomials, while additional averaging eliminates the discontinuities of the nodal stresses. The figure clearly demonstrates the minimisation of the stress-error function in an integral sense. Note the closeness of the results between the quadratic and the multilinear stress expansions, the latter of which yields identical results as the extrapolation technique presented in [6],[7]. It should be understood that the direct stress approximation provides no alternative to the local stress evaluation in the case of polynomial displacement gradient fields.

In conclusion we compare the results of the conjugate stress approximation [11] with the polynomial stress expansion within each element. We observe that the quadratic element stress expansion yields virtually identical results if the element stresses are averaged at the nodes. Therefore, the elementwise stress approximation proposed in this paper furnishes an optimal procedure in regard to both accuracy and computational effort.

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Fig. 1 Strain Energy Contributions to QUAXC9 Stiffness
Comparison of Eigenvalue Spectra

Fig. 2 Direct Approximation of Strainfield
Eigenvalue Spectra of Stiffness - Difference Matrices D_s

Fig. 3 Direct Approximation of Strainfield
Eigenvalue Spectra of Stiffness-Difference Matrices D_2

Fig. 4 Direct Approximation of Strainfield
Eigenvalue Spectra of Stiffness-Difference Matrices D_2

Fig. 5 Direct Approximation of Strainfield
Eigenvalue Spectra of Stiffness-Difference Matrices D_0

Fig. 6 Numerical Integration of QUAXC9 Stiffness
Eigenvalue Spectra of Stiffness-Difference Matrices D_0

Fig. 7 THTR 1:5 Scale Model
Axisymmetric Idealisation

Fig. 8 Strain Energy of PCRV due to Internal Pressure
Effects of Integration and Strain Expansions

Fig. 9 Eigenvalue Spectrum and Conditioning of PCRV
Influence of Quadrature and Strain Expansions

Fig. 10 Square Sheet
Input Data

Fig. 11 Finite Element Idealisation
Internal Energy Results

Fig. 12 Stress Variation along Element Boundaries
Comparison of Various Stress Evaluation Techniques