A SIMPLE 'MIXED' METHOD FOR PLATE- AND SHELL PROBLEMS

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SUMMARY

Plates and shells normally will be analyzed by the finite element method specially by the deformation method. In this case the most interesting interior forces and moments will be obtained as differences of the unknown deformations, which yield to a great loss in accuracy. To avoid such a loss in accuracy, the interior forces and the strains will be introduced as unknowns. That are the stretching forces n_x , n_y , n_{xy} and the stretching strains ϵ_x , ϵ_y , ϵ_{xy} furthermore the moments m_x , m_y , m_{xy} and the curvatures κ_x , κ_y , κ_{xy} . These forces and strains as well as the geometrical curvatures of the shell and the loads will be approximated linearly from point to point. The structure can be divided into triangular and rectangular elements.

The interior forces and the strains do not satisfy the equilibrium- and compatibility conditions, therefore equilibrium- and compatibility equations have to be formulated as in a mixed method. This will be done by means of the principles of virtual work, but in a very simple manner, for example as virtual displacements rigid body motions of the elements will be chosen. In this case the interior forces work only along the gridlines as in a grid of beams. Thus it is quite simplified to establish the equations.

By using the duality between plate-bending and -stretching the effort of computer programming is reduced. Both problems are treated with the same program, which only needs to be slightly extended for the application to shallow shells. In the case of the deformation method the stiffness matrices for plate-stretching and -bending have unfortunately a quite different structure and it is difficult to develop for double curved shells simple stiffness matrices with unrestrained rigid body motions of the elements. Furthermore the stretching- and bending-stresses have different approximations and they are discontinuous at the boundaries of the elements. Here each stress and strain has the same approximation with no discontinuities at the boundaries of the elements. Results will be shown for stress concentration problems in plates and shells with holes and cut-outs.

1. Introduction

Plates and shells normally will be analyzed by the finite element method, specially by the deformation method [1]. In this case the most interesting normal forces and bending moments will be obtained as differences of the unknown deformations, which yields to a great loss in accuracy. To avoid such a loss we introduce the interior forces and the strains as unknowns [2,3]. That are the stretching forces n_x, n_y, n_{xy} and the stretching strains $\epsilon_x, \epsilon_y, \epsilon_{xy}$ furthermore the moments m_x, m_y, m_{xy} and the curvatures $\kappa_x, \kappa_y, \kappa_{xy}$. These forces and strains as well as the geometrical curvatures a_x, a_y, a_{xy} of the shell and the loads p_x, p_y, p_z will be approximated linearly from point to point. The structure can be divided into triangular and rectangular elements. In this compact I will restrict myself essentially to show the background of the theory for triangular elements and to solve some simple examples [4,5]. More complicated structures will be treated in the fully detailed publication.

The interior forces and the strains do not satisfy the equilibrium- and compatibility conditions, therefore equilibrium- and compatibility equations have to be formulated as in a mixed method. This will be done by means of the principles of virtual work, but in a very simple manner, for example as virtual displacements rigid body motions of the elements will be chosen. In this case the interior forces work only along the gridlines as in a grid of beams and not in the interior of the elements. Thus it is quite simplified to establish the equations because only line integrals must be determined instead of surface integrals.

By introducing the interior forces and the strains in the nodal points as unknowns the linear approximation of the interior forces and strains within the elements and their continuity from element to element is stated. Thus we get at first twelve unknowns per point. However by means of the stress-strain-relationships

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{bmatrix} = \begin{bmatrix} f_x & f_{xy} & . \\ f_{xy} & f_y & . \\ . & . & f \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_{xy} \end{bmatrix} \quad \begin{bmatrix} m_x \\ m_y \\ m_{xy} \end{bmatrix} = \begin{bmatrix} b_x & b_{xy} & . \\ b_{xy} & b_y & . \\ . & . & b \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad (1)$$

the stretching strains can be substituted by the stretching forces and the moments by the curvatures, thus only six unknowns per point remain. For these six unknowns we are able to formulate (in the infinitesimal range) one stretching

-compatibility equation and one bending-equilibrium equation

$$\begin{aligned} \epsilon_y'' &= 2\epsilon_{xy}' + \epsilon_x'' & m_x'' + 2m_{xy}' + m_y'' \\ -a_x \kappa_y - 2a_{xy} \kappa_{xy} - a_y \kappa_x &= 0 & +a_x n_x + 2a_{xy} n_{xy} + a_y n_y + p_z = 0 \end{aligned} \quad (2)$$

as well as two stretching-equilibrium equations and two bending-compatibility equations

$$\begin{aligned} n_x' + n_{yx}' + p_x &= 0 & \kappa_y' - \kappa_{yx}' &= 0 \\ n_{xy}' + n_y' + p_y &= 0 & \kappa_{xy}' - \kappa_x' &= 0 \end{aligned} \quad (3)$$

Only the first equations (2) are coupled for shallow shells. From these equations follows immediately the duality between the stretching forces n and curvatures κ as well as between the stretching strains ϵ and the moments m . By using this duality between bending and stretching the effort of computers programming is reduced. Both problems are treated with the same program, which only needs to be slightly extended for the application to shallow shells.

In the case of the deformation method the stiffness matrices for stretching and bending have unfortunately a quite different structure and it is difficult to develop for double curved shells simple stiffness matrices with unrestrained rigid body motions of the elements. Furthermore the stretching- and bending-stresses have different approximations and are discontinuous at the boundaries of the elements. In this method each stress and strain has the same approximation and has no discontinuity at the boundaries of the elements, that is essentially for shell problems. Finally every state of virtual forces and displacements can be derived from only one state and the formulation of the equations becomes very easy, therefore a matrix-formulation is not necessary. Because the structure will be divided into triangles we need besides the global system x, y, z the s, n -system - parallel and normal to the boundaries of the triangular (figure 1).

2. Finite equations

The finite equations corresponding to (2) and (3) can be derived reasonably by the principles of virtual work. Let us begin with the equilibrium equation for bending. For that purpose we introduce the virtual displacement δw at the interesting nodalpoint (figure 2a, b). Each neighbouring element will be rotated about the basic line as a rigid body and a constant virtual kink $\delta\beta$ arises at each boundary (figure 2b).

$$\begin{aligned} \delta\beta_{23} &= \frac{1}{h_{23}} = -\delta N_{23} \quad , \\ \delta\beta_{31} &= -\delta\beta_{23} \cos\gamma_{12} - \delta\beta_{34} \cos\gamma_{41} = -\frac{\cos\gamma_{12}}{h_{23}} - \frac{\cos\gamma_{41}}{h_{34}} = -\delta N_{13} \quad (4) \end{aligned}$$

At these kinks $\delta\beta$ the real bending-moments m_n (normal to the edges) work as external edge moments, furthermore the vertical load p_z works at the virtual displacement δw , that the whole virtual work will be

$$\delta A_a = \int m_n(s) \delta\beta ds + \iint p_z \delta w \, dx dy. \quad (5)$$

If the distribution of the bending moments and of the load will be approximated linearly, we get the following work for the element 123

$$\begin{aligned} \delta A_{a,123} = & \frac{l_{23}}{2h_{23}} (m_{n,2} + m_{n,3})_{23} - \frac{l_{31} \cos \gamma_{12}}{2h_{23}} (m_{n,3} + m_{n,1})_{31} - \frac{l_{12} \cos \gamma_{31}}{2h_{23}} (m_{n,1} + m_{n,2})_{12} \\ & + \frac{F_{123}}{12} (2p_1 + p_2 + p_3)_{123}. \end{aligned} \quad (6)$$

Along the lines 12 and 13 only work due to the components of $\delta\beta_{23}$ is included in (6). Furthermore it must be observed, that the moments $m_{n,3,23}$ and $m_{n,3,31}$ are not the same, because they act in different sections.

In a shell due to the virtual displacement δw also extensions $\delta\epsilon$ of the middle surface do arise, which are proportional to δw

$$\delta\epsilon_x = -a_x \delta w, \quad \delta\epsilon_y = -a_y \delta w, \quad \delta\epsilon_{xy} = -a_{xy} \delta w. \quad (7)$$

At these virtual strains work the real stretching forces

$$\begin{aligned} \delta A_i = & \iint (n_x \delta\epsilon_x + 2n_{xy} \delta\epsilon_{xy} + n_y \delta\epsilon_y) \, dx dy \\ = & - \iint (a_x n_x + 2a_{xy} n_{xy} + a_y n_y) \delta w \, dx dy. \end{aligned} \quad (8)$$

If the real stretching forces n are approximated linearly and the geometrical curvatures a are constant within each element, the virtual work (8) has the same force as the work of the load p . Therefore we must add to each load value p the shell term

$$a_x n_x + 2a_{xy} n_{xy} + a_y n_y = \sum_k a_k n_k \quad \text{with } k = x, y, xy. \quad (9)$$

Because of the duality between stretching and bending the equations (6) and (9) are also valid as stretching-compatibility equation, only m_n must be substituted by ϵ_s , n_k by κ_k and p_z by 0. In order to derive this compatibility equation we have to utilize the principle of virtual forces. The most simple virtual force system, restricted to the neighbouring elements of the nodal-point 1 is the normal force group δN_s in figure 2b. These normal forces are concentrated at the gridlines and have the same values as the kinks $\delta\beta$ due to δw but with the opposite sign (4), because they can be derived from an diry-stress function $\delta\phi = -\delta w$.

For a shell the gridlines are curved and the virtual forces δN_s themselves do not satisfy the equilibrium condition, in addition moments are necessary

$$\delta m_y = a_x \delta \phi, \quad \delta m_x = a_y \delta \phi, \quad \delta m_{xy} = a_{xy} \delta \phi. \quad (10)$$

Thus the whole virtual work is the interior work

$$\delta A_i^* = \int \delta N_s ds + \iint (\delta m_x \kappa_x + 2\delta m_{xy} \kappa_{xy} + \delta m_y \kappa_y) dx dy, \quad (11)$$

which is analogous to equation (5) and (9).

Next we have to formulate according to (3) the two bending-compatibility equations also by the principle of virtual forces. Therefore we introduce two self equilibrium virtual moment groups, that is a group of bending moments δM_s (figure 2c) and a group of torsional moments δM_{sn} (figure 2d). These bending and torsion moments, which act only on the gridlines have the same value as the normal forces δN_s in figure 2b, because the force diagrams in the figures 2b, c, d are the same, only in figure 2c it is rotated about 90° . Virtual work arises only along the gridlines, it is the work of the virtual bending moments δM_s on the real curvatures κ_s and in the second case the work of the virtual torsion moments δM_{sn} on the real twisting curvatures κ_{sn}

$$\delta A_i^* = \int \delta M_s \kappa_s ds = 0 \quad \text{and} \quad \delta A_i^* = \int \delta M_{sn} \kappa_{sn} ds = 0. \quad (12)$$

External work do not exist. The corresponding equations have the same form as the equilibrium equation (6), we have derived before, only m_n must be substituted by κ_s or κ_{sn} and the virtual work of the load must be cancelled.

Finally two equilibrium conditions for the stretching forces must be formulated by the principle of virtual displacements. The virtual displacements are given in figure 2c and d. The elements will be shifted as rigid bodies in such a manner, that in the first case only discontinuities δs_n in the deformations normal to the gridlines and in the second case only discontinuities δs_{sn} parallel to the gridlines result. The normal-forces n_n respectively the shearing-forces n_{sn} work at these gaps.

Someone might object that the virtual displacements are not geometrically compatible, because there are gaps in the displacements. But the principle of virtual displacements only demands, that the displacements are compatible within the elements. If there are discontinuities at their boundaries, then the real interior forces there act as external loads.

We have seen: In each virtual state there exist only one kind of virtual forces or virtual strains, which are concentrated on the gridlines. In figure 2b we have normal-

forces δN_s and kinks $\delta\beta$, in figure 2c gaps δs_n normal to the gridlines and bending moments δM_s and finally in figure 2d gaps δs_{sn} parallel to the gridlines and twisting moments δM_{sn} . They have throughout the same values. At these virtual states only one kind of real strains or forces, which will be summarized in the vector

$$\xi^t = (\epsilon_s, n_n, n_{sn}, m_n, \kappa_s, \kappa_n). \quad (13)$$

Therefore we get one typ of equation like (6) with throughout the same coefficients

$$\frac{l_{23}}{h_{23}} = \frac{l_{23}^2}{2F_{123}} = \frac{x_{23}^2 + y_{23}^2 + z_{23}^2}{2F_{123}}$$

$$\frac{l_{31} \cos \gamma_{12}}{h_{23}} = \frac{l_{23}^2 + l_{31}^2 - l_{12}^2}{4F_{123}} = \frac{x_{31}x_{23} + y_{31}y_{23} + z_{31}z_{23}}{2F_{123}}$$

$$(\text{with } x_{ik} = x_i - x_k, \quad y_{ik} = y_i - y_k, \quad z_{ik} = z_i - z_k)$$

the only difference are in the load- and shell terms:

$$\sum_{\Delta} \frac{1}{F_{123}} \{ (x_{12}x_{23} + y_{12}y_{23} + z_{12}z_{23})(\xi_1 + \xi_2)_{12} + (x_{23}^2 + y_{23}^2 + z_{23}^2)(\xi_2 + \xi_3)_{23} + (x_{31}x_{23} + y_{31}y_{23} + z_{31}z_{23})(\xi_3 + \xi_1)_{31}$$

$$= \sum_{\Delta} \frac{F_{123}}{3} \left\{ \begin{array}{l} - \sum_k a_k (2\kappa_{k,1} + \kappa_{k,2} + \kappa_{k,3}) \\ 0 \\ 0 \\ \sum_k a_k (2n_{k,1} + n_{k,2} + n_{k,3}) + (2p_{z,1} + p_{z,2} + p_{z,3}) \\ 0 \\ 0 \end{array} \right\} \quad (14)$$

The sum $\sum_{\Delta}$ has to take about all elements adjoining the investigated Δ point, here point 1.

The unknowns in equation (14) are forces and strains normal or parallel to the edges, they must transformed into the global x, y, z system. This transformation costs often a lot of computer time. It can be avoided in the important case of isotopic plates, when utilizing both principles of virtual work simultaneously by introducing the virtual displacement δw (figure 2b) and the virtual moments δM_s (figure 2c). The entire virtual work is the work of the real bending moments m_n on the appropriated kinks $\delta\beta$, the work of the virtual bending moments δM_s on the real curvatures κ_s and finally the work of the real normal load p at the virtual displacement δw .

$$\delta A_a - \delta A_i^* = \int m_n \delta \beta ds + \int p \delta w dF - \int \delta M \kappa_s ds = 0 \quad (15)$$

Choosing

$$\delta M_s = -(1-\nu)b\delta\beta$$

the virtual work (15) might be summarized

$$\int \underbrace{\left[m_n + (1-\nu)b\kappa_s \right]}_{b(\kappa_n + \kappa_s)} \delta \beta ds + \int p \delta w dF = 0, \quad (15a)$$

where the sum of the curvatures $\kappa_n + \kappa_s = \kappa_x + \kappa_y$ is invariant.

3. Examples

The simple handling of the equations will be shown at two examples, first a simply supported and uniform loaded triangular plate (figure 3) The coordinate-system will be orientated along one of the middle-lines. Because of the simply supported boundaries the sum of the moments at each boundary vanished. In the center-point the two curvatures κ_x and κ_y are the same. For the two unknowns $m=b(\kappa_x + \kappa_y)$ in point 1 and point 2 we have to formulate two equations (15a) by means of the two virtual displacements in figure 3:

$$2m_1 - m_2 = pa^2/24$$

$$-m_1 + 8m_2 = pa^2/12$$

The solution

$$m_1 = 2m_2 = pa^2/36$$

is exact [6].

The second example is a perforated plate (figure 4), like a tube sheet in a heat exchanger, which will be extended uniformly $n_x = n_y = 1$. Because of the symmetries to 1'-1', 1-1, 3-4 and 2-3' it is reasonable to approximate the stress distribution at the grid lines parabolically instead of linearly. In the case of the coarse grid 11, 23' (without point 3 and 4) we have only three unknowns $n_{y,1} = n_{x,3'}$, $n_{x,2'}$, $n_{y,2}$. At the contour of the hole the stress $n_s(s)$ must be constant ($n_s = n_{y,1}$). The calculated value is $n_{y,1} = 2,55$ for $h/a = 1$. Introducing furthermore the points 3 and 4 a variable stress distribution $n_s(s)$ at the contour of the hole can be considered, because of the symmetry it must be

$$n_s = \frac{1}{2} \{ n_1 + n_3 + (n_1 - n_3) \cos 6\phi \}.$$

Furthermore the curvature $\kappa_z = \kappa_1 \cos 6\phi$ about the z-axis must be introduced, which take into account the deformation of the

contour, since bending moments δM_z belonging to the virtual forces δN_s in figure 4c. For this finer mesh we get $n_1 = 2,62$ and $n_3 = 2,55$, which coincide with the exact values in [7]. Further examples will be shown in the lecture and in the complete paper.

References

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Figure 1: Interior forces and loads

Figure 2: Virtual states

Figure 3: Simply supported triangular plate

Figure 4: Perforated plate