REVIEW OF CONTINUUM, FINITE ELEMENT AND HYBRID TECHNIQUES IN THE ANALYSIS OF STRESS CONCENTRATIONS IN STRUCTURES

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SUMMARY

When a uniform flow of any nature is interrupted in any way, the readjustment of the flow results in concentrations and rarefactions, so that the peak value of the flow parameter will be higher than what an elementary computation would suggest. When stress flow in a structure is interrupted, there are stress concentrations. These are generally localised, and often large, in relation to the values indicated by simple equilibrium calculations. With the advent of the industrial revolution, dynamic and repeated loading of materials had become common place in engine parts and fast moving vehicles of locomotion and this led to serious fatigue failures arising from stress concentrations. Also, many metal forming processes, fabrication techniques and weak-link type of safety systems, benefit substantially from intelligent use or avoidance, as appropriate, of stress concentrations. As a result, in the last eighty years, the study and evolution of stress concentrations has been a premier objective in the study of solid mechanics.

Exact mathematical analysis of stress concentrations in finite bodies presents considerable difficulty for all but a few problems of infinite fields, concentric annuli and the like, treated under the presumption of small deformation, linear elasticity. A whole series of techniques have been developed to deal with different classes of shapes and domains, causes and sources of concentration, material behaviour, phenomenological formulation, etc. These include real and complex functions, conformal mapping, transform techniques, integral equations, finite differences and relaxation, and, more recently, the finite element methods. With the advent of large high speed computers, development of finite element concepts and a good understanding of functional analysis, it is now, in principle, possible to obtain with economy satisfactory solutions to a whole range of concentration problems by intelligently combining, theory and computer application. An example is the hybridisation of continuum concepts with computer based finite element formulations. This new situation also makes possible a more direct approach to the problem of design which is the primary purpose of most engineering analyses. The trend would appear to be clear: the computer will shape the theory, analysis and design.

INTRODUCTION

When a uniform flow of any nature is interrupted in any way, the readjustment of the flow results in concentrations and rarefactions, so that the peak value of the flow parameter will be higher than what an elementary computation would suggest. When stress flow in a structure is interrupted, there are stress concentrations. These are generally localised, and often large in relation to the values indicated by simple equilibrium calculations. Nature and art abound in examples of this phenomenon. For centuries man had lived with it, often enough avoided it by evolutionary and intuitive processes, and, occasionally used it to profit, without being particularly concerned with clear understanding of its basis or control. This is because under static or slow loading conditions, the strength of non-brittle materials is not significantly reduced by stress concentrations. But with the advent of the industrial revolution, dynamic and repeated loading of materials had become common place in engine parts and fast moving vehicles of locomotion and this led to serious fatigue failures arising from stress concentrations. Also, many metal forming processes, fabrication techniques and weak-link type of safety systems, benefit substantially from intelligent use or avoidance, as appropriate, of stress concentrations. As a result, in the last eighty years, the study and evolution of stress concentrations has been a premier objective in the study of solid mechanics by analysis and experiment.

In this period the problems considered were many, the techniques used are varied and the literature is extensive. As such the subject has periodically attracted excellent compilations and reviews from outstanding scientists. The first compilation of the knowledge on stress concentration by Lehr [10] in 1934 was soon followed by Neuber's Kerbspannungslehre [15]. Timoshenko's books, in particular Theory of Elasticity [28], make a lucid presentation of many analytical techniques developed by numerous authors and list their original papers. Muskhelishvili [14] gives the Kolosov-Muskhelishvili theory of complex variables and conformal mapping for two-dimensional elasticity and cogently presents the extensive literature that had grown up till 1954. Savin and his collaborators have written a series of monographs compiling solutions to different classes of elastic stress concentrations around holes in plates and shells. His NASA publication [24] is an extensive statement of solutions mainly using the Muskhelishvili technique in a form suitable for application. He has also compared many results with experimental data available in international literature. Sneddon and Lowengrub [25] have written a monograph on crack problems in the classical theory of elasticity; they cover two and three dimensional problems and the use of different mathematical techniques. Sneddon has also reviewed the use of transform and other methods in a number of papers and reports (vide [25]). Mindlin [13] reviewed the case

for Cosserat formulation of the medium for stress concentrations around small circular holes in plates under extension and pure shear loading. Lipson et al (128 of [16]) and Peterson (179 of [16]) have compiled definitive information on stress concentrations for design applications. In 1953 Timoshenko, in his Murray lecture, dealt with stress concentration in the history of strength of materials [29]. In 1958 Sternberg surveyed three dimensional stress concentrations in the theory of elasticity [27]. In 1966 Neuber and Hahn wrote an extensive review of stress concentrations in scientific research and engineering [16]. In 1971, Gallagher made an up to date survey and evaluation of finite element methods being developed for purposes of fracture mechanics. In a brief text like the present, it is difficult to do justice to all the work in the field or even to all the important compilations, reviews and surveys. In the present paper we will only attempt a broad review of the trends in the analytical methods and techniques with particular reference to the obtaining of accurate results for practical problems.

The simplest problem of elastic stress concentration is obviously that of uniform radial pressure 'p' around a circular hole of radius 'a' in an infinite medium. One can by inspection write down the closed form solution $\sigma_r = pa^2/r^2$, $\sigma_\theta = -pa^2/r^2$, $\sigma_{r\theta} = 0$. The axisymmetric Lamé solution of 1852 (p 59 of [28]) for thick cylinders is almost as simple to visualise. Thereafter any complication in the problem needs disproportionately greater mathematical sophistication and computational effort.

Possible sources of stress concentration in a finite or an infinite field can all be described as disturbances or discontinuities. In Fig. 1 we attempt to classify different types of discontinuities in some detail. The complexity in the analysis of the concentration depends upon the extent and shape of the domain, and the nature and shape of the discontinuity. It is further complicated by various influencing factors such as listed in Fig. 2. The order of analytical and computational effort increases stupendously as one moves from a regular to irregular domain or a two-dimensional to three a three-dimensional formulation, or from isotropic to even orthotropic medium or from small to large deformation or from elastic to any inelastic constitutive laws. Fig. 3 introduces us to a different dimension in the picture. There are many possible purposes and view points for the analysis. They are related to the phenomenological aspects that should be introduced into the study to ensure proper functioning of the structure of component under its working environment. Against these complexities of need, one has to pit the power of analysis and computation available to him. The analytical techniques developed over the years are broadly classified in Fig. 4. The growth of analytical capability since the days of Lamé, St. Venant and

Poisson is phenomenal. But it is more than matched by the development of computational capability in the last 25 years. We are not including a pictorial representation of this because most of us have personally experienced its impact. With the extraordinary rapid development of the electronic digital computer; many hitherto intractable problems should now be quite amenable to solution. So one should attempt to match the technique to the nature and complexity of the problem and the inputs that can be justified by the need. In fact a few like Argyris [1] have had the vision to anticipate this development and to create new theory to exploit its tremendous potential. In Fig. 5, we have a schematic diagram for decision making, for deciding on the best technique for a given situation.

FUNCTIONAL METHODS:

Most practical problems are those of a finite domain with one or more sources of concentration and an external boundary. The solution should satisfy the governing field equation (taking into account the constitutive laws), the local conditions at the source (some or all of which have the character of boundary conditions) and the external boundary conditions.

The basic procedures can be classified into three types, Collatz [3]. In the direct method, the differential equation is identically satisfied, but the boundary conditions are either all or in part only approximately satisfied. In the interior method (typified by the Galerkin type of variational solution) the boundary conditions are all identically satisfied, but the differential equation is satisfied only in the average. In the mixed methods (such as the Rayleigh-Ritz method applied to problems with some natural boundary conditions) the differential equation and some or all of the boundary conditions are only approximately satisfied.

We may also classify the solutions from another point of view. The simplest solution would be an explicit closed form function which can be accurately evaluated. One can also obtain a closed form solution in the form of an integral which is to be numerically evaluated and as such is subject to numerical errors. The next is a notionally closed form solution in the form of an explicit series whose numerical evaluation implies truncation errors. Then, one has series solutions with arbitrary coefficients which are to be determined by the approximate satisfaction of one or more of the field equation, the source conditions and the boundary conditions. These include errors due to truncation and due to the approximate satisfaction of relevant conditions.

In general the procedures that depend on approximate satisfaction of

the differential equation by averaging the errors in the entire field and procedures that violate significant conditions at or near the source cannot yield accurate values for the stress concentration. Such methods are particularly inappropriate for stress singularities within the domain (e.g. at sharp corners). As such, solutions by variational methods which were popular at one time have been losing favour [27].

In applying the direct or boundary method for finite domains, one often encounters difficulty in satisfying both the source conditions and the boundary conditions. So, much of the early work e.g. Kirsch (p 80 of [28]) & Inglis (p 198 of [28]) took advantage of St. Venant's principle and pushed the boundary to infinity. Another way of overcoming this problem is to consider a repeating array of disturbances and using functions exhibiting periodicity (e.g. Fourier series). The work of Howland on perforated strips (p 83 of [28]) is an example. The Muskhelishvili transform & integral equation techniques are also well suited for infinite strips, semi-infinite and infinite domains and repeating element formulations [2, 6, 7]. However the value of these solutions suffers when the true boundary is very close to the disturbance or when the problem has elastic stress singularities. In either case, St. Venant's principle is in general inapplicable. Further, conformal mapping when applied to sharp cornered holes, steps, etc. distorts the corner to a smooth fillet. However these techniques of analysis of infinitely stretched domains have an advantage in that they can often isolate the nature of the elastic singularity in the problem, that is they determine the "k" in the singularity Ar^{-k} . There is a much simpler and more direct method for identifying this singularity for many problems. Klein, Westergaard & Brahtz used it for relatively simple problems. It was extended and systematized by Williams [30] for homogeneous fields and Rao [22] for composite fields. Once the appropriate singularities and constant stress field [11] are identified, they can be utilised for a full fledged accurate (though not formally exact) stress analysis for the finite field containing the corner(s) [4, 11, 19, 23]. This procedure too has limitations placed on it by irregularity of boundary and/or boundary conditions (leading to poor convergence) and by the difficulty of handling multiple concentrations. These difficulties can be handled, the first by using more sophisticated methods of satisfying the boundary condition [30] and the second by a method of multiple solutions [4].

The most spectacular achievement of the functional method has been the development of the concept of neutral holes by Mansfield [12]. This view point of developing design concepts and configurations directly by analytical methods deserves special attention. The Muskhelishvili method and its variants in particular appear to hold promise in this direction.

NUMERICAL METHODS:

In order to overcome the convergence and computational problems of functional methods, effort was put into the development of powerful numerical methods. The most widely used methods are the grid methods of finite difference and relaxation [26]. When there are stress singularities or large stress gradients one runs into numerical instability and large errors at locations where high accuracy is desired. Often enough, one can obtain spurious local results that hide the presence of the concentration or singularity. The treatment of irregular boundaries, in particular curved boundaries, needs sophistication of technique. One interesting method of overcoming the first difficulty is to "subtract the concentration its nature being obtained from a functional method and its strength, if unknown a priori, being determined iteratively till the solution stabilises.

FINITE ELEMENT METHODS:

We have seen that the direct functional and numerical methods pose significant difficulties in achieving accurate predictions for the stress concentrations and singularities. The difficulties multiply with introduction of anisotropy, three dimensionality, etc. (Fig. 3) and many problems have been virtually intractable. The advent of the electronic digital computer has changed the picture radically. It is now possible to satisfactorily compute solutions (functional and numerical) for many of these problems. This is particularly so, when the singular (or rapidly varying part) can be separated from the regular part. However to follow only this logic would be to misuse the computer as a brute, ignoring its potentialities in developing new theory. There are trends that indicate a healthy development in formulating computer based finite difference methods to problems of stress concentrations [5]. But the finite element method is one development directly geared to the capabilities of the computer [33]. In this method we discretize the structure into a large number of small elements, whose stress and or kinematics are expressed in very simple terms. The solution is finalised by ensuring equilibrium and continuity of displacements between elements in an average sense. The formulation overcomes all difficulties associated with irregular finite boundaries, and boundary conditions. Inhomogeneous (composite) domains, large deformations, inelasticity, time dependent phenomena can all be handled, generally by iterative procedures. Convergence and improvement of solution is achieved by using successively finer meshes of smaller sized elements. The refinement of the mesh can be selective, emphasising regions of concentration. But in practice, where there are singularities or very steep stress gradients such refinement soon reaches a stage of defeating its purpose. The large number of elements lead to large order matrix operations which in turn can lead to computer overflow

and numerical instability before reaching necessary accuracies. This has led to a spate of papers by Pian, Tong, Carmignani, Morley, Rao, Yamamoto and others [9,21,32] describing various hybrid techniques, Fig.4, in which the local singularity or concentration is separated from the regular part in one way or the other. This results in a collapse of the number of elements required for the analysis simultaneously with the achievement of accuracy. Prima facie, the singularity element method, in which the continuum functions are applied in a primary element only, [17,19] appears to have the advantage of direct application in standard finite element procedures[9]. The procedure directly determines the strength of the singularity. But the relative merits of all the techniques and the possibility of their further refinement are yet to be carefully established.

Another defect in the original finite element formulation was the difficulty of reproducing curved boundaries to ensure accuracy of solution. This is now successfully overcome by curvilinear elements of special shapes like sectors [18], rings, etc. or of versatile profiles like isoparametric elements [8] and the curved elements (TEC,TRIC,etc.) of ASKA.

The design modification procedure[1] of the finite element method is a powerful tool that can be applied to control stress concentration through design. It is suggested that this aspect deserves substantial attention.

CONCLUDING REMARKS:

In this brief compact we have presented a broad review of techniques at the expense of detail. The review leads us to the conclusion that in future it would be most effective to use functional or continuum methods to identify the nature of the concentration or singularity and apply a hybrid finite difference or finite element procedures for dealing with concentrations. The value of this procedure will increase with the complexity of the problem. One cannot do better than conclude with a quotation from Argyris & Prager's inaugural editorial in the Journal of Computer Methods in Applied Mechanics & Engineering:

"It is more than twenty years that Matrix Methods (or finite element methods as they are now predominantly called) were first developed. This development originated not only from the stringent requirements of Aeronautical Design and of sciences but also from the feeling that theory and application of the emerging digital computers should be amalgamated in a powerful symbiosis. Atomic energy developments and large scale meteorological problems contributed substantially to this view even though finite difference formulations were predominant in these fields. Whichever approach one may choose, it is imperative to realise that it is a cardinal error to separate fundamental concepts from computer applications. Instead the computer must shape the theory".

SHORT LIST OF REFERENCES:

1. ARGYRIS, J.H. & KELESY, S. "Energy Theorems & Structural Analysis", Butterworths, Lond. 1960.
2. BOGY, D.B. & WANG, K.C., "Stress singularities at interface corners in bonded dissimilar isotropic elastic materials", Int.J.Solids & Structures, Vol. 7, No. 8, 993-1005, August 71.
3. COLLATZ, L., "The numerical treatment of differential equations", Springer-Verlag, Third edition, 1960.
4. DATTAGURU, B., "Some problems of diffusion & concentration in two-dimensional elasticity", Ph.D.Thesis, I.I.Sc., Bangalore 12, India 1972
5. EMERY, A.F. & SEGEDDIN, C.M., "Singularity programming: A numerical technique for determining the effect of singularities in finite difference solutions illustrated by application to plane elastic problems", Int.J. for Num.Meth. in Engg., Vol.6, 367-380, 1973.
6. ERDOGAN, F. & GUPTA, G., "The stress analysis of multilayered composites with a flaw", Int.J. Solid & Structures, Vol.7, 39-61, Jan. 1971.
7. ERDOGAN, F., "Stress in bonded dissimilar material with cracks", J.Appl. Mech., Vol. 32, 403-410, 1965.
8. ERGATOUDIS, I., IRONS, B.M. & ZIENKIEWICZ, O.C., "Curved isoparametric quadrilateral elements for finite element analysis", Int.J.of Solids & Structures, Vol. 4, 31-42, 1968.
9. GALLAGHER, R.H., "Survey & evaluation of the finite element method in fracture mechanics analysis", L6/9, First SMIRT Conference, 1971.
10. LEHR, E., "Spannungsverteilung in Konstruktionselementen" Berlin 1934.
11. MANSFIELD, E.H., "Stress concentration at the cutouts in a swept wing", Rep. & Mem.No. 2823 of ARC (UK) 1951.
12. MANSFIELD, E.H., "Neutral holes in plane sheet - Reinforced holes which are elastically equivalent to the uncut sheet", Quart.J. Mech. Appl. Math., Vol. 6, Part 3, 1953.
13. MINDLIN, R.D., "Influence of couple stress on stress concentrations", Experimental Mechanics, Vol.3, No. 1, 1-7, Jan. 1963.
14. MUSKHELISHVILI, N.I., "Some basic problems of the mathematical theory of elasticity", P.Noordhoff, Groningen Neth., 1963.
15. NEUBER, H., "Kerbspannungslehre, Grundlagen für genaue Spannungsberechnung 1st ed., Berlin 1937.
16. NEUBER, H. & HAHN, H.G., "Stress concentration in scientific research & engineering" AMR 19, No.3, 187-199, March 1966.
17. PATIL, S.S., RAO, A.K. & RAJU, I.S., "Special finite elements for flaw & discontinuity stresses in composites", M3/6 in Second Int.Conf. on SMIRT, Berlin, September 1973.
18. RAJU, I.S., "Finite element analysis of stress concentrations and singularities", Ph.D.Thesis, I.I.Sc., Bangalore 12, India, 1972.

19. RAMAMURTHY, T.S. & RAO, A.K., "Stresses in bonded joints" (C/3), presented at 4th CANCAM, Montreal, Canada, 1973.
20. RAO, A.K., "A study into problem of diffusion of loads", Ph.D. Thesis, University of London, 1958.
21. RAO, A.K., RAJU, I.S., & MURTY, A.V.K., "A powerful hybrid method in finite element analysis", Int. J. Num. Meth. Engg., Vol. 3, No. 3, 389-403, 1971.
22. RAO, A.K., "Stress concentrations & singularities at interface corners", ZAMM 51, 395-406, 1971.
23. RAO, A.K., PATIL, S.S. & RAJIAH, K., "Plane elastic analysis of fibre-matrix composites" (C/4) presented at 4th CANCAM, Montreal, Canada, 1973.
24. SAVIN, G.N., "Stress distribution around holes", NASA TTF-607, Washington, D.C. 1970.
25. SNEDDON, I.N. & LOWENGRUB, N., "Crack problems in the mathematical theory of elasticity", Wiley, New York, 1969.
26. SOUTHWELL, R.V., "Relaxation methods in engineering science", London, 1951.
27. STERNBERG, E., A review of stress concentrations in three dimensional elasticity, AMR 11, No. 1, 1-4, Jan. 1958.
28. TIMOSHENKO, S., & GOODIER, J.N., "Theory of elasticity" McGraw Hill-Book Co., New York, 2nd edition, 1951.
29. TIMOSHENKO, S., "W.M. Murray lecture" SESA Proceedings, Vol. XII, No. 1 1954.
30. VENKATARAMAN, N.S., "A study into analysis of interference fits & related problems", Ph.D. Thesis, I.I.Sc., Bangalore 12, India, 1966.
31. WILLIAMS, M.L., "Stress singularities resulting from various boundary conditions in angular corners of plates under extension", J. Appl. Mech. Vol. 19, No. 4, 526-528, 1952.
32. YAMAMOTO, Y., & TOKUDA, N., "Determination of stress intensity factors in cracked plates by the finite element method", Int. J. Num. Meth. in Engg., Vol. 6, 427-439, 1973.
33. "Proceedings of the Conferences on Matrix Methods in Structural Mechanics", WPAFB, Ohio, USA, 1965, 1968, 1971.

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FIG. 1 CAUSES AND SOURCES OF STRESS CONCENTRATION

FIG. 2 INFLUENCING FACTORS

STRESS FIELD
DEFORMATION FIELD
FUNCTIONAL STRENGTH
ULTIMATE STRENGTH
STIFFNESS
DEFORMATION LIMITS
SHOCK RESISTANCE
VIBRATIONS
STABILITY
FRACTURE
FATIGUE
CRACK PROPOGATION

FIG. 3 PURPOSE AND VIEW POINT OF ANALYSIS

FIG. 4 TECHNIQUES FOR STUDY OF STRESS CONCENTRATIONS

FIG. 5 FACTORS AFFECTING CHOICE OF TECHNIQUE

FIG 6 A CONTINUUM FINITE ELEMENT SCHEME
FOR FIBRE-MATRIX COMPOSITE