Quadrilateral with High Coarse-Mesh Accuracy for Solid Mechanics in Axisymmetric Geometry

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INTRODUCTION

The purpose of this paper is to present variationally consistent 4-node axisymmetric isoparametric element for solid mechanics with high accuracy for coarse meshes. The aim was to develop an element that provides for bending problems results comparable with those obtained by high-order solid elements or plate elements using only a single layer of elements through the depth. Although (Belytschko, 1986) presents the development of the stabilization matrix in plane geometry through mixed variational principle so as to enhance the accuracy of the element in addition to suppressing the spurious singular (kinematic) modes, the extension to axisymmetric element of this concept presents some unique difficulties. Above all, it is the necessity to resist both, communicable and non-communicable kinematic modes because they are both involved in bending mode.

THE HU-WASHIZU VARIATIONAL PRINCIPLE

The Hu-Washizu functional (stationary case) for a single axisymmetric element can be written as

$$\pi(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}) = \int_A \left[\frac{1}{2} \boldsymbol{\varepsilon}^t \mathbf{D} \boldsymbol{\varepsilon} - \boldsymbol{\sigma}^t (\boldsymbol{\varepsilon} - \nabla^S \mathbf{u}) \right] r dA - \mathbf{d}^t \mathbf{f}, \quad (1)$$

where A is the element area in r - z plane; $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are the strain and stress fields; $\mathbf{D}$ is the matrix of material properties; $\mathbf{u}^t = (u_r(r,z), u_z(r,z))$ is the displacement field; $\mathbf{d}^t = (d_r^t, d_z^t)$ are the nodal displacements in both directions; $\mathbf{f}^t = (f_r^t, f_z^t)$ are the nodal forces and ∇^S is the symmetric part of the gradient operator. The column arrays for the stresses, strains and $\nabla^S \mathbf{u}$ are given by

$$\boldsymbol{\sigma}^t = [\sigma_r, \sigma_z, \sigma_\varphi, \sigma_{rz}], \quad \boldsymbol{\varepsilon}^t = [\varepsilon_r, \varepsilon_z, \varepsilon_\varphi, 2\varepsilon_{rz}], \quad (2)$$

$$\nabla^S \mathbf{u} = \left[u_{r,r}, u_{z,z}, \frac{1}{r} u_r, u_{r,z} + u_{z,r} \right]. \quad (3)$$

Considering isotropic linear case, the matrix $\mathbf{D}$ can be written as

$$D = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix}, \quad (4)$$

where λ and μ are Lamé constants. The weak form is obtained by finding the stationary point of (1). In this principle u must be at least C^0 , while ε and σ may be C^{-1} functions.

Instead of standard C^0 isoparametric interpolation $u_i = N^t(\xi, \eta) d_i$ for the 4-node element we use following, more useful expression of the displacement field:

$$u_i(r, z) = \left[c + \psi_j b_j + w\beta \right]^t d_i = N^t d_i, \quad (5)$$

where the vectors b_i and c are the components of the discrete gradient operator (Matejović, 1988, 1989)

$$b_i = \frac{1}{V} \int_A N_{,i} r dA, \quad c = \frac{1}{A} \int_A N dA, \quad (6)$$

and β is cylindrical hourglass (HG) projection associated with cylindrical HG mode vector. The functions ψ_i and w are given by

$$\psi_i = x_i - (c^t x_i), \quad w = 4\xi\eta - (b_i^t h) x_i, \quad (7)$$

where x_i is the vector of nodal coordinates in i -th direction and h is plane HG mode vector. These functions have the following important properties:

$$\begin{aligned} \psi_{i,j} &= \delta_{ij}, \quad \int_A \psi_i dA = 0, \\ \int_A w dA &= \int_A w_{,i} r dA = 0, \end{aligned} \quad (8)$$

where δ_{ij} is the Kronecker delta. Finally, using (5) and (8), $\nabla^S u$ is given by

$$\nabla^S u = \begin{bmatrix} u_{r,r} \\ u_{z,z} \\ \frac{1}{r} u_r \\ u_{r,z} + u_{z,r} \end{bmatrix} = \begin{bmatrix} b_r^t + w_{,r} \beta^t & 0^t \\ 0^t & b_z^t + w_{,z} \beta^t \\ \frac{1}{r} [c + \psi_i b_i + w\beta]^t & 0^t \\ b_z^t + w_{,z} \beta^t & b_r^t + w_{,r} \beta^t \end{bmatrix} \begin{bmatrix} d_r \\ d_z \end{bmatrix} = B d, \quad (9)$$

where B is continual gradient operator.

To approximate the strain and stress fields within the element, we use notation taken from (Belytschko, 1986):

$$\varepsilon = E e, \quad \sigma = S s, \quad (10)$$

where E , S are interpolation matrices and e , s are the vectors of strain and stress parameters. The interpolants in E and S must be square integrable over element volume (Stolarski, 1987). Furthermore, if $\dim s = \dim e$ and only constants and functions orthogonal to constant fields are in interpolation matrices, the

resulting stiffness can be decomposed into two parts

$$K = K^0 + K^{\text{stab}}, \quad (11)$$

where K^0 is the stiffness obtained by one-point integration. The second, stabilization matrix depends on the selection of stress and strain interpolants.

UNDERINTEGRATED SOLUTION AND MODE DECOMPOSITION

Let ϵ, σ are C^{-1} functions and u is given by (5). In this case $E = S = I_{4 \times 4}$ (unit matrix) and

$$e^t = [\bar{\epsilon}_r, \bar{\epsilon}_z, \bar{\epsilon}_\varphi, 2\bar{\epsilon}_{rz}]^t, \quad s^t = [\bar{\sigma}_r, \bar{\sigma}_z, \bar{\sigma}_\varphi, \bar{\sigma}_{rz}]^t, \quad (12)$$

where superposed bars designate the constant parts of all the fields. Following (Belytschko, 1986) we obtain element stiffness K^0 equal to this one obtained via mean value approach from displacement principle in (Matejovič, 1988).

The equivalent result obtained from various principles includes, of course, similar difficulties. Among those are kinematic modes, that surely occur if $\dim d - n > \dim s$, where n is the number of rigid body modes. Since in the case considered there is only one rigid body mode - the uniform displacement d_5 in z direction - there must be 3 kinematic modes in kernel of K^0 and K^0 is therefore of rank 4. First kinematic mode d_6 corresponds to "rigid" element rotation around geometrical centre of the element in r - z plane and will be therefore called rotational mode. This mode can be described as noncommunicable kinematic mode because in an assembly of two or more elements it cannot occur. Second d_7 and third d_8 kinematic modes correspond to communicable HG modes in r and z direction which, contrary to plane HG modes, depend on element geometry and position in r - z plane. All possible uniform strain $d_1 \div d_4$ and volumetric modes $d_5 \div d_8$ shown in Fig. 1 are linearly independent and span therefore the displacement space R^8 .

VARIATIONALLY CONSISTENT SUPPRESSION OF KINEMATIC MODES

Suppression of HG modes

To suppress the communicable HG modes d_7 and d_8 we use the following approximations for strain and stress fields:

$$E = \begin{bmatrix} & w_{,r} & 0 \\ & 0 & w_{,z} \\ I_{4 \times 4} & 0 & 0 \\ & 0 & 0 \end{bmatrix}, \quad S = E. \quad (13)$$

The resulting stiffness K is in this case composed of two parts

$$K = K^0 + K^{\text{HG}}, \quad (14)$$

where

$$\mathbf{K}^{HG} = \begin{bmatrix} (\lambda + 2\mu) w_{rr} \mathbf{BB}^t & \lambda w_{rz} \mathbf{BB}^t \\ \lambda w_{rz} \mathbf{BB}^t & (\lambda + 2\mu) w_{zz} \mathbf{BB}^t \end{bmatrix}, \quad (15)$$

and

$$w_{ij} = \int_A w_{,i} w_{,j} r dA. \quad (16)$$

This form of $\mathbf{K}$ is an analogy of plane development from (Belytschko, 1986). $\mathbf{K}^0$ acts only on uniform strain modes $\mathbf{d}_1 \div \mathbf{d}_4$, that means only on linear part of displacement field, whereas $\mathbf{K}^{HG}$ acts only on HG modes $\mathbf{d}_7, \mathbf{d}_8$, that means on nonlinear part of displacement field and the rigid body mode $\mathbf{d}_5$ remains strain-free. However, this solution still remains rank deficient 1 because rotational mode $\mathbf{d}_6$ is in the kernel of both matrices. Since the bending mode involves both, HG and rotational modes, the performance of this approach for bending problems computed with only one element through the thickness is poor. The solution obtained is too soft.

Suppression of all kinematic modes

A necessary condition for the stiffness to be of sufficient rank is that the number of stress parameters (i.e. $\dim \mathbf{s}$) should be greater than or equal to the total number of displacement degrees of the element minus number of possible rigid-body modes. To find the missing rank of (14) it is therefore necessary to increase the number of stress parameters at least to seven. We choose therefore $\epsilon\varphi$ and $\sigma\varphi$ as a C^0 functions. Note that the strain $\epsilon\varphi$ derivable its origin from the expression (5) for $\mathbf{u}$ via (9) contains interpolants which are not square integrable for the elements on the axis of symmetry. We use therefore the simplest quasilinear expression using the functions $w_{,i}$:

$$\mathbf{E}^{OB} = \begin{bmatrix} w_{,r} & 0 & 0 & 0 \\ 0 & w_{,z} & 0 & 0 \\ 0 & 0 & w_{,r} & w_{,z} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (17)$$

The subscript OB denote an cylindrical analogy of plane development given in (Belytschko, 1986). QBI element of this concept ist also possible to develop. In both cases $\mathbf{S} = \mathbf{E}^{OB}$ and the final expression for $\mathbf{K}$ is composed of three parts

$$\mathbf{K} = \mathbf{K}^0 + \mathbf{K}^{HG} + \mathbf{K}^*, \quad (18)$$

where $\mathbf{K}^*$ is in coupled form and acts therefore on both, linear and nonlinear fields, including rotational mode $\mathbf{d}_6$. Rigid body mode remains strain free and $\mathbf{K}$ is therefore rank of 7.

All presented elements are not frame-invariant. The extension to dynamic case is straightforward.

NUMERICAL RESULTS

Two examples are reported to demonstrate the performance of development elements for bending problems.

Example 1. A thick cylindrical plate, simply supported on the outer radius, with the meshes shown in Fig. 2 was considered. The linear Hooke's law with density $\rho = 1000 \text{ kgm}^{-3}$, $\lambda = 0$, $\mu = 5.10^5 \text{ Pa}$ (i.e. $\nu = 0$) was used. A uniform pressure $p = -1.5 \times 10^6 \text{ Pa}$ was applied as a step function on the upper plate surface at time $t = 0$. The results for the maximal center-deflection $u_{\max}$ (point A) were compared with those obtained with 8×4 one point integration quadrilaterals with elastic HG-control (Matejovič, 1988). In our case, $u_{\max} = 6.21 \times 10^{-2} \text{ m}$ for rectangular and $u_{\max} = 6.19 \times 10^{-2} \text{ m}$ for irregular mesh. $u_{\max}$ taken from (Matejovič, 1988) lies between 6.1 and $6.5 \times 10^{-2} \text{ m}$ in dependence of HG-control parameter used. Note that if we use approximation $(15 \div 18)$, $u_{\max} = 9 \times 10^{-2} \text{ m}$.

Example 2. Spherical cap loaded by uniform pressure. The problem description is shown in Fig. 4. The parameters and material properties are given in Table 1. The results for the center-deflection time history are compared to the results obtained using eightnode, axisymmetric isoparametric elements and quadrilateral plate elements (Kennedy, 1987) in Fig. 5. Note that in this case it is necessary to use QBI element.

Table 1. Material properties and parameters for spherical cap

Radius	22.27 in	Density	$2.45 \times 10^{-4} \text{ lb} - \text{sec}^2/\text{in}^4$
Thickness	0.41 in	Young's modulus	$1.05 \times 10^7 \text{ psi}$
Angle	26.67°	Poisson's ratio	0.3
Pressure	600 psi		

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Fig. 1 Mode decomposition

Fig. 2 Example 1. Computational meshes

Fig. 3 Example 1. Mesh evolution in case a)

Fig. 4 Example 2. Problem definition

Fig. 5 Example 2. Center-deflection time history
--- 8-node elements; -.-.- plate elements;
— 4-node QBI elements