Continuum Damage Mechanics Analysis of Crack Tip Zone

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1. INTRODUCTION

The crack tip field and its intensity factor play an important role in fracture mechanics. Generally, the damage such as microcracks, microvoids etc. will initiate and grow in materials as the cracked body is subjected to external loadings, especially in the crack tip zone. The damage evolution will lead to the crack tip damage field and the change of the stress, strain and displacement fields of crack tip zone. In this paper, on the basis of continuum damage mechanics, we derived the equations which the crack tip field and its intensity factor must satisfy in a loading process, calculated the angle distribution curves of stress, strain and displacement fields in a crack tip zone and compared them with the corresponding curves of HRR field (Hutchinson, 1968, Rice and Rosengren, 1968) and linear elastic field in undamaged materials. The equations of crack tip field intensity factors have been solved and its solutions give the variation of the field intensity factors with the loading parameter.

2. THE DAMAGE THEORY

The material damage is usually considered as the microcracks or microvoids distributed in materials. For isotropic materials, the young's modulus $\tilde{E}$ and shear modulus $\tilde{\mu}$ of the damaged materials can be written as

$$\tilde{E} = E/(1+\bar{D}_1), \quad \tilde{\mu} = \mu/(1+\bar{D}_2) \quad (1)_{a,b}$$

Where E and μ are young's modulus and shear modulus of the undamaged materials respectively. $\bar{D}_1$ and $\bar{D}_2$ are the functions of microcrack or microvoid size a and density n . From eqs.(1), the damage variables $\bar{D}_1$ and $\bar{D}_2$ are defined as

$$\bar{D}_1 = (E-\tilde{E})/\tilde{E}, \quad \bar{D}_2 = (\mu-\tilde{\mu})/\tilde{\mu} \quad (2)_{a,b}$$

It can be seen that $\bar{D}_1$ is opening mode damage which will contribute to the opening mode cracking and $\bar{D}_2$ is shear mode damage which will contribute to the shear mode cracking.

The poisson's ratio of the damaged materials $\tilde{\nu}$ is

$$\tilde{\nu} = \tilde{E}/2\tilde{\mu} - 1 = [\nu(1+\bar{D}_2) + (\bar{D}_2-\bar{D}_1)]/(1+\bar{D}_1) \quad (3)$$

where ν is poisson's ratio of the undamaged materials.

Hereinafter it is assumed that $\tilde{\nu}$ is independent of damage variables. We have $\bar{D}_1 = \bar{D}_2 = \bar{D}$ and $\tilde{\nu} = \nu$. It implies that the probabilities of opening and shear mode cracking have are same. Without regarding to temperature effects, the

Helmholtz free energy ψ can be written as

$$\psi = \frac{1}{2} \tilde{E}_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \tilde{C} = \frac{1}{2} \tilde{A}_{ijkl} \sigma_{ij} \sigma_{kl} + \tilde{C} \quad (4)$$

where $\tilde{E}_{ijkl} = \frac{E}{(1+\nu)(1+\bar{D})} \left[\frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj}) + \frac{\nu}{1-2\nu} \delta_{ij} \delta_{kl} \right]$, δ_{ij} is kronecker delta and $\tilde{A}_{ijkl} = \frac{1+\bar{D}}{E} \left[\frac{1+\nu}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj}) - \nu \delta_{ij} \delta_{kl} \right]$.

Hence, the stress-strain relation is

$$\sigma_{ij} = \partial \psi / \partial \varepsilon_{ij} = \tilde{E}_{ijkl} \varepsilon_{kl} \quad (5)$$

According to the principle of minimum potential energy, the static equilibrium equation is

$$\sigma_{ij,j} = 0 \quad (6)$$

and the damage force equilibrium equation is

$$\frac{1}{2} \frac{\partial \tilde{A}_{ijkl}}{\partial \bar{D}} \sigma_{ij} \sigma_{kl} + \frac{\partial \tilde{C}}{\partial \bar{D}} = 0 \quad (7)$$

Making $\sigma_e^2 = E \frac{\partial \tilde{A}_{ijkl}}{\partial \bar{D}} \sigma_{ij} \sigma_{kl} = E \tilde{A}_{ijkl} \sigma_{ij} \sigma_{kl}$, we have

$$\frac{d\tilde{C}}{d\bar{D}} = - \frac{\sigma_e^2}{2E} \quad (8)$$

Assuming $\tilde{C} = - \frac{C_0}{\alpha+1} (1+\bar{D})^{\alpha+1}$ and $\alpha = \frac{2}{a}(1+b)$, it follows from eq.(8) that

$$\frac{d\bar{D}}{d\sigma_e^2} = C_a \frac{\sigma_e^a}{(1+\bar{D})^b} \quad (9)$$

where C_a , a and b are material constants.

3. THE CRACK TIP FIELD ANALYSIS

In the plane strain condition, the polar coordinate stress strain relations are

$$\varepsilon_r = \frac{1-\nu^2}{E} \left(\sigma_r - \frac{\nu}{1-\nu} \sigma_\theta \right) = \frac{1-\nu^2}{E} \left(\sigma_r - \frac{\nu}{1-\nu} \sigma_\theta \right) (1+\bar{D}) \quad (10)_a$$

$$\varepsilon_\theta = \frac{1-\nu^2}{E} \left(\sigma_\theta - \frac{\nu}{1-\nu} \sigma_r \right) = \frac{1-\nu^2}{E} \left(\sigma_\theta - \frac{\nu}{1-\nu} \sigma_r \right) (1+\bar{D}) \quad (10)_b$$

$$\gamma_{r\theta} = \frac{2(1+\nu)}{E} \tau_{r\theta} = \frac{2(1+\nu)}{E} \tau_{r\theta} (1+\bar{D}) \quad (10)_c$$

The crack-tip stress and damage fields are expressed as

$$\sigma_{ij} = A_1 r^{p_1} \hat{\sigma}_{ij}(\theta) \quad (11)$$

$$\bar{D} = B_1 r^{p_2} \hat{D}(\theta) \quad (12)$$

where p_1 and p_2 are less than zero.

Substituting eqs.(11) and (12) into eq.(9) and then integrating it, we obtain

$$r^{p_1(p_2+b)} \hat{D}^{p_2+b}(\theta) \int_0^{B_1} B_1^b dB_1 = C_a \int_0^{A_1} A_1^{a-1} dA_1 r^{p_1 a} \hat{\sigma}_e^a(\theta) \quad (13)$$

Again considering eqs.(11) and (12), the eq.(13) becomes

$$\bar{D} = \left(\frac{C_a}{\gamma-1} \right)^{\frac{1}{1+b}} \sigma_e^{\gamma-1} \quad (14)$$

where $\gamma = 1 + \frac{a}{1+b}$.

From eq.(13), we can also get

$$p_2 = (\gamma-1)p_1, \quad \hat{D}(\theta) = \hat{\sigma}_e^{\gamma-1}(\theta), \quad B_1 = \left(\frac{C_a}{\gamma-1} \right)^{\frac{1}{1+b}} A_1^{\gamma-1} \quad (15)_{a,b,c}$$

Taking the stress function ϕ as

$$\phi = \frac{A_1}{(p_1+2)(p_1+1)} r^{p_1+2} f(\theta)$$

the stress components are given by

$$\sigma_{ij} = A_1 r^{p_1} \hat{\sigma}_{ij}(\theta) \quad (16)$$

$$\text{and } \hat{\sigma}_r(\theta) = \frac{1}{p_1+1} \left[f(\theta) + \frac{1}{p_1+2} f''(\theta) \right], \quad \hat{\sigma}_\theta(\theta) = f(\theta), \quad \hat{\tau}_{r\theta}(\theta) = -\frac{1}{p_1+2} f'(\theta) \quad (17)_{a,b,c}$$

Substituting eq.(16) into eqs.(10), we get

$$\varepsilon_{ij} = \frac{A_1 B_1}{E} r^{\gamma p_1} \hat{\varepsilon}_{ij}(\theta) \quad (18)$$

$$\text{and } \hat{\varepsilon}_r(\theta) = (1-\gamma^2) \hat{\sigma}_e \left(\hat{\sigma}_r - \frac{\gamma}{1-\gamma} \hat{\sigma}_\theta \right), \quad \hat{\varepsilon}_\theta(\theta) = (1-\gamma^2) \hat{\sigma}_e \left(\hat{\sigma}_\theta - \frac{\gamma}{1-\gamma} \hat{\sigma}_r \right),$$

$$\hat{\gamma}_{r\theta}(\theta) = 2(1+\gamma) \hat{\sigma}_e^{\gamma-1} \hat{\tau}_{r\theta} \quad (19)_{a,b,c}$$

with the help of the geometric equations $\varepsilon_r = \frac{\partial u_r}{\partial r}$ and $\varepsilon_\theta = \frac{\partial u_\theta}{r \partial \theta} + \frac{u_r}{r}$, we have

$$u_i = \frac{A_1 B_1}{E} r^{\gamma p_1+1} \hat{u}_i(\theta) \quad (20)$$

$$\text{and } \hat{u}_r(\theta) = \frac{1}{\gamma p_1+1} \hat{\varepsilon}_r, \quad \hat{u}_\theta(\theta) = \frac{1}{\gamma p_1+1} \int_0^\theta \left[(\gamma p_1+1) \hat{\varepsilon}_\theta - \hat{\varepsilon}_r \right] d\theta \quad (21)_{a,b}$$

Applying the J-integral (Rice, 1968) to the damaged materials and designating it as J_{Dc}

$$J_{Dc} = \int_\Gamma \left[W dy - T_i \frac{\partial u_i}{\partial x} ds \right] \quad (22)$$

Because of the eq.(14), we have $\sigma_{ij} = \frac{E}{1+\gamma} \left(\varepsilon_{ij} + \frac{\gamma}{1-2\gamma} \varepsilon_{kk} \delta_{ij} \right) \frac{1}{1+\bar{D}} = F(\varepsilon_{ij})$ and hence $W = \int_0^{\varepsilon_{ij}} \sigma_{kl} d\varepsilon_{kl} = W(\varepsilon_{ij})$. In analogy to the usual way of proving the path independence of J-integral, J_{Dc} can also be proved to be independent of integral path. Hence we have $\sigma_{ij} \varepsilon_{ij} \sim r^{-1}$ and $p_1 = -1/(\gamma+1)$.

Substituting eqs. (18), (19) and (17) into the compatibility equation, we get

$$\frac{d^2}{d\theta^2} \left[\hat{\sigma}_e^{\gamma-1} \left(f'' + (1 - \frac{\gamma}{1-\gamma} \lambda) (1+\lambda) f \right) \right] + \frac{2\lambda}{1-\gamma} \left[\gamma(\lambda-1)+1 \right] \frac{d}{d\theta} \left[\hat{\sigma}_e^{\gamma-1} f' \right] + \gamma(\lambda-1) \hat{\sigma}_e^{\gamma-1} \left[\left((\lambda - \frac{\gamma}{1-\gamma}) \gamma + \frac{1}{1-\gamma} \right) (\lambda^2-1) f - \left(\frac{1}{1-\gamma} + \gamma(\lambda-1) \frac{\gamma}{1-\gamma} \right) f'' \right] = 0 \quad (23)$$

where $\lambda = \frac{\gamma}{\gamma+1}$. The eq.(23) is a fourth order nonlinear ordinary differential

equation of $f(\theta)$. We need to solve it with its solution satisfying the stress free boundary conditions on the crack faces.

Hereon the crack-tip fields can be expressed as

$$\bar{D} = B_1 r^{-\frac{\gamma-1}{\gamma+1}} \hat{D}(\theta) \quad (24)_a$$

$$\sigma_{ij} = A_1 r^{-\frac{1}{\gamma+1}} \hat{\sigma}_{ij}(\theta) \quad (24)_b$$

$$\varepsilon_{ij} = \frac{A_1 B_1}{E} r^{-\frac{\gamma}{\gamma+1}} \hat{\varepsilon}_{ij}(\theta) \quad (24)_c$$

$$u_i = \frac{A_1 B_1}{E} r^{\frac{1}{\gamma+1}} \hat{u}_i(\theta) \quad (24)_d$$

4. THE FIELD INTENSITY FACTOR EQUATIONS AND THEIR SOLUTIONS

Taking the integral path as a circle, the eq.(22) is reduced to

$$J_{Dc} = r \int_{-\pi}^{\pi} \left[W \cos \theta - T_i \frac{\partial u_i}{\partial x} \right] d\theta \quad (25)$$

$$\text{in which } W = \int_0^{\varepsilon_e} \sigma_e d\varepsilon_e = \frac{1}{E} r^{-1} \hat{\sigma}_e^{\gamma+1} \int_{A_1} dA_1 B_1 = \frac{1}{2E} r^{-1} \hat{\sigma}_e^{\gamma+1} \left[A_1^2 B_1 + \int_0^{B_1} A_1^2 dB_1 \right] \quad \text{and}$$

$$T_i \frac{\partial u_i}{\partial x} = \frac{A_1^2 B_1}{E} r^{-1} \left[\frac{1}{1+\gamma} (\hat{\sigma}_r \hat{u}_r + \hat{\tau}_{r\theta} \hat{u}_\theta) \cos \theta + (\hat{\sigma}_r (\hat{u}_\theta - \hat{u}'_r) - \hat{\tau}_{r\theta} (\hat{u}_r + \hat{u}'_\theta)) \sin \theta \right]$$

Therefore, we have

$$J_{Dc} = \frac{1}{E} (A_1^2 B_1 I_1 + \int_0^{B_1} A_1^2 dB_1 I_2) \quad (26)$$

where $I_1 = I_2 = \int_{-\pi}^{\pi} \left[\frac{1}{1+\gamma} (\hat{\sigma}_r \hat{u}_r + \hat{\tau}_{r\theta} \hat{u}_\theta) \cos \theta + (\hat{\sigma}_r (\hat{u}_\theta - \hat{u}'_r) - \hat{\tau}_{r\theta} (\hat{u}_r + \hat{u}'_\theta)) \sin \theta \right] d\theta$ and $I_2 = \int_{-\pi}^{\pi} \frac{1}{2} \hat{\sigma}_e^{\gamma+1} \cos \theta d\theta$. The eqs.(26) and (15)_c constitute the governing equations for the field intensity factors A_1 and B_1 .

Because $\int_0^{B_1} A_1^2 dB_1 = \int_0^{A_1} A_1^2 \left(\frac{C_a}{\gamma-1} \right)^{\frac{1}{1+b}} (-1) A_1^{\gamma-2} dA_1 = \frac{\gamma-1}{\gamma+1} B_1 A_1^2$, we have

$$J_{Dc} = \frac{1}{E} A_1^2 B_1 I \quad (27)$$

where $I = I_1 + \frac{\gamma-1}{\gamma+1} I_2$.

Solving eqs.(27) and (15)_c for A_1 and B_1 , we obtain

$$A_1 = \left[\left(\frac{C_a}{\gamma-1} \right)^{-\frac{1}{1+b}} \left(\frac{E J_{Dc}}{I} \right) \right]^{\frac{1}{\gamma+1}} \quad (28)_a$$

$$B_1 = \left[\left(\frac{C_a}{\gamma-1} \right)^{\frac{2}{1+b}} \left(\frac{E J_{Dc}}{I} \right)^{\gamma-1} \right]^{\frac{1}{\gamma+1}} \quad (28)_b$$

$$\text{and} \quad A_1 B_1 = \left[\left(\frac{C_a}{\gamma-1} \right)^{\frac{1}{1+b}} \left(\frac{E J_{Dc}}{I} \right)^{\gamma} \right]^{\frac{1}{\gamma+1}} \quad (28)_c$$

where J_{Dc} is a load parameter or load factor.

Finally the crack-tip field expressions take the form

$$\bar{D} = \left[\left(\frac{C_a}{\gamma-1} \right)^{\frac{2}{1+b}} \left(\frac{E J_{Dc}}{I} \right)^{\gamma-1} \right]^{\frac{1}{\gamma+1}} r^{-\frac{\gamma-1}{\gamma+1}} \hat{D}(\theta) \quad (29)_a$$

$$\sigma_{ij} = \left[\left(\frac{C_a}{\gamma-1} \right)^{-\frac{1}{\gamma+1}} \left(\frac{E J_{pe}}{I} \right)^{\frac{1}{\gamma+1}} r^{-\frac{1}{\gamma+1}} \hat{\sigma}_{ij}(\theta) \right] \quad (29)_b$$

$$\varepsilon_{ij} = \frac{1}{E} \left[\left(\frac{C_a}{\gamma-1} \right)^{\frac{1}{\gamma+1}} \left(\frac{E J_{pe}}{I} \right)^{\frac{1}{\gamma+1}} r^{-\frac{\gamma}{\gamma+1}} \hat{\varepsilon}_{ij}(\theta) \right] \quad (29)_c$$

$$u_i = \frac{1}{E} \left[\left(\frac{C_a}{\gamma-1} \right)^{\frac{1}{\gamma+1}} \left(\frac{E J_{pe}}{I} \right)^{\frac{1}{\gamma+1}} r^{\frac{1}{\gamma+1}} \hat{u}_i(\theta) \right] \quad (29)_d$$

where the angle distribution functions $\hat{\sigma}_{ij}(\theta)$, $\hat{\varepsilon}_{ij}(\theta)$ and $\hat{u}_i(\theta)$ can be calculated from eqs.(15)_b, (17), (19) and (21), and $f(\theta)$ is obtained by solving the eq.(23).

5. CONCLUSIONS

The crack tip fields of damaged materials have been given by the eqs.(29). The calculations of angle distribution functions $\hat{\sigma}_{ij}(\theta)$, $\hat{\varepsilon}_{ij}(\theta)$ and $\hat{u}_i(\theta)$ hinge on solving eq.(23) for $f(\theta)$. The numerical results of angle distribution functions are given graphically in Fig.1 to 4 and the corresponding results of HRR field and linear elastic field for undamaged materials are also given in these figures. It can be seen from these figures that the effects of damage are important in crack tip zone. The damage, stress and strain singularities of the crack tip field are of the order $-(\gamma-1)(\gamma+1)$, $-1/(\gamma+1)$ and $-\gamma/(\gamma+1)$ respectively. There results will reduce to the linear elastic ones as $\gamma = 1$.

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Fig.1

Fig.2

Fig.3

Fig.4