ABSTRACT

BHAGWAT, RUTVIJ. Development of Stability Analysis Tools for High Speed Compressible Flows. (Under the direction of Pramod Subbareddy.)

Fluid flows experience transition to turbulence in ways which are not always easily explained through traditional ideas such as existence of a well defined flow instability. Experimentally, it can be seen that in many cases flows which are deemed stable by linear stability analysis experience transition. An idea that contributed towards bridging this gap was that of transient or nonmodal growth which was introduced in the early nineties. It proposed that for non-normal flow operators disturbances can grow over a short term before eventually decaying even if the system is stable and that this short term growth might be sufficient to trigger transition.

Often earlier computational work was done for simplified configurations or by just analyzing a subsection of the system at a time, such as analyzing a boundary layer profile at a certain streamwise station or the flow profile in a certain crossflow plane. These analyses often miss out on capturing interconnected flow mechanisms in complex systems. Such systems demand a 'global analysis' which analyzes the system in its entirety all at once. These methods are computationally expensive but developments in scientific computing have led to an increased availability of computational resources which in turn broadened the range of problems accessible for these kinds of analyses. One such area of interest is transition in high-speed flows and so there is demand for tools and solvers which cater specially to these flows. This work involves the development of a suite of stability analysis solvers in a 3D parallel unstructured framework. We have tried to follow a methodology of developing these stability analyses tools from standard compressible flow solvers which is general enough that it can be applied to other standard compressible solvers. These tools can also be used to analyze turbulent flows where the focus is not on predicting the onset of transition but rather on extracting structural and dynamical features of these turbulent flows. The work also demonstrates the broad usage of these tools for analyzing transitional and turbulent flow fields across a wide range of physical regimes from incompressible to hypersonic flows. A number of flowfields such as: hypersonic flows over flat plates, shock-boundary layer interactions, flow over a sharp wedge among others were investigated as a part of this work. A novel resolvent based inflow-I/O framework for predicting the response of the flow to freestream waves has also been proposed.

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Development of Stability Analysis Tools for
High Speed Compressible Flows

by
Rutvij Bhagwat

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APPROVED BY:

Jack Edwards

Venkat Narayanaswamy

Pierre Gremaud

Pramod Subbareddy
Chair of Advisory Committee

DEDICATION

To my parents.

BIOGRAPHY

Rutvij was born in the city of Pune in India and grew up there. He graduated from College of Engineering, Pune in 2015 with a degree in mechanical engineering. It was here that he developed an interest in computational fluid dynamics. He continued his studies by joining North Carolina State University for a master's degree. Shortly thereafter, he converted to a doctoral program under the direction of Dr. Pramod Subbareddy and joined his group to work on developing stability analysis tools for high-speed flows. In his time at NC State he developed a suite of stability analysis tools to explore the dynamics of transitional and turbulent flows with a special focus on high-speed flows. He hopes to continue to work on stability & transition, high-speed flows, complex dynamical systems and other areas of computational physics in the future. Outside of work his interests include classical music, opera, politics, and history.

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As I sit down to write this my mind goes to that oft-quoted line from Tennessee Williams' iconic play 'A Streetcar named Desire': "Whoever you are – I have always depended on the kindness of strangers". In my case, in the culmination of this work I have always depended greatly on the kindness and support of my advisor, my professors, my colleagues, my friends and my family.

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CHAPTER

1

INTRODUCTION

The stability of fluid flows has traditionally been a subject of interest in physics. Instabilities are at the heart of a wide range of physical processes ranging from the mixing of a drop of milk in coffee to large scale astrophysical phenomena. A deep fundamental understanding of instabilities and transition is critical in developing and supporting technological advances in a number of applications such as mixing in combustion processes, aeroacoustics and drag/surface heating reduction in hypersonic flows.

Initial mathematical developments in this field provided a robust framework for analyzing hydrodynamic instabilities but they proved to be inadequate to explain and predict transition in complex realistic flows. The laminar-turbulent transition is a highly complex physical process with sensitivity to a large number of variables and as a result predicting and even explaining some transition scenarios is a very involved task. Over the last few decades the scientific community has tried to address this gap in our knowledge of transition physics through number of different approaches such as development of more sophisticated instability analysis techniques, direct numerical simulations (DNS) and experimental investigations. A major breakthrough in terms of stability analysis was the concept of transient or nonmodal growth [92]. The transient growth is a phenomenon whereby disturbances can grow for a short period of time in flows deemed stable by modal analysis. In some cases, this is sufficient to trigger transition in modally stable flows. Many other techniques such as spatial optimal growth analysis, resolvent analysis, input-output analysis, parabolized stability equations (PSE) were developed and employed to analyze different flows and to build a more complete picture of transition physics.

1.1 Stability analysis approaches

1.1.1 Governing equations

We consider the three-dimensional compressible Navier Stokes equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0 \quad (1.1)$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_i u_j - \tau_{ij} + p \delta_{ij}) = 0 \quad (1.2)$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j} ((E + p) u_j - \tau_{ij} u_j + q_j) = 0, \quad (1.3)$$

where $E = p/(\gamma - 1) + \rho(u^2 + v^2 + w^2)/2$ is the total energy, the pressure is assumed to obey the ideal gas law,

$$p = \rho R T, \quad (1.4)$$

and the viscous stress and heat conduction terms are defined by

$$\tau_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \mu \theta \delta_{ij} \quad \theta = \frac{\partial u_k}{\partial x_k} \quad q_j = -\kappa \frac{\partial T}{\partial x_j} \quad \kappa = \frac{\mu C p}{Pr}.$$

The Navier-Stokes equations may be described in the operator form,

$$\frac{\partial q}{\partial t} = \mathcal{F}(q), \quad (1.5)$$

where q is a global state vector: $q = [\rho_i, \rho u_i, \rho v_i, \rho w_i, E_i]_{i=1,N}^T$, with N being essentially the total number of grid points in the system. The state vector is decomposed into a base flow component, $\bar{q} = [\bar{\rho}, \bar{\rho} \bar{u}, \bar{\rho} \bar{v}, \bar{\rho} \bar{w}, \bar{E}]^T$, and a disturbance component q' , given by:

$$q' = [\rho', \rho' \bar{u} + \bar{\rho} u', \rho' \bar{v} + \bar{\rho} v', \rho' \bar{w} + \bar{\rho} w', E']$$

A linear disturbance evolution equation can be formed upon neglecting higher-order disturbance terms,

$$\frac{\partial q'}{\partial t} = \underbrace{\frac{\partial \mathcal{F}(\bar{q})}{\partial \bar{q}}}_{\mathcal{A}} q' \quad (1.6)$$

More generally we can also consider an external forcing to this linear disturbance evolution equation and represent it as:

$$\boxed{\frac{\partial q'}{\partial t} = \mathcal{A}(\bar{q}) q' + f'} \quad (1.7)$$

1.1.2 Modal decomposition

The most general case in examining unsteady flows is one wherein the spatial field as well as the temporal evolution is fully resolved. In the linear limit we consider that the disturbance field consists of a superposition of waves at different frequencies and the evolution of these waves at each of these frequencies takes place in a manner independent of each other. This approach is called a triglobal approach, wherein 3D spatial functions are considered with a Fourier transform for time. Exploiting flow symmetries it might be possible to further reduce the size of the problem by applying a Fourier transform in directions in which the baseflow is homogeneous (typically spanwise and/or streamwise). These different approaches are typically referred as:

Table 1.1 Stability Analysis Approaches

Approach	Amplitude	Phase
Local	$\hat{q}(y)$	$e^{i(\alpha x + \beta z - \omega t)}$
Biglobal	$\hat{q}(x, y)$	$e^{i(\beta z - \omega t)}$
Triglobal	$\hat{q}(x, y, z)$	$e^{-i\omega t}$

1.1.3 Linearized Navier Stokes (LNS)

In absence of an external forcing f' , we can track the evolution of some initial disturbance field $q'(0) = q'_0$.

$$\frac{\partial q'}{\partial t} = \mathcal{A}(\bar{q})q' \quad q'(0) = q'_0 \quad (1.8)$$

The solution at a certain instant of time is given as:

$$q'(t) = e^{\mathcal{A}t} q'_0 \quad (1.9)$$

1.1.4 Global stability analysis

Without the loss of generality, for a triglobal problem, considering the modal decomposition $q'(\mathbf{x}, t) = \hat{q}(\mathbf{x})e^{i\omega t}$ and $f' = 0$, Eq.1.7 can be written as:

$$-i\omega \hat{q} = \mathcal{A} \hat{q} = V \Lambda V^{-1} \hat{q} \quad (1.10)$$

Thus if Λ are all the eigenvalues (λ) of the operator $\mathcal{A}$ the frequencies of the waves and the eigenvalues of the operator are related as: $-i\omega_j = \lambda_j$. For a temporal eigenvalue problem the eigenvalue ω is complex, with the real part indicating the oscillating frequency and the imaginary part indicating the growth rate of the disturbance. The eigenvalues with a positive growth rate indicate an exponentially

growing (with time) instability. If all eigenvalues have a negative growth rate, it indicates that all the disturbances will die out eventually and the flow is linearly stable in the time asymptotic limit.

1.1.5 Direct harmonic response

In the absence of any unstable eigenvalues, it might be useful to examine the response of the flow to external harmonic forcing. In this case, the frequency ω is purely real as the forcing is non-growing (and non-decaying). The forcing and response are of the form:

$$q' = \hat{q} e^{-i\omega_r t} \quad f' = \hat{f} e^{-i\omega_r t}$$

Substituting this in Eq.1.7

$$\hat{q} = (-i\omega - \mathcal{A})^{-1} \hat{f} \quad (1.11)$$

The operator $(-i\omega - \mathcal{A})^{-1}$ is known as the resolvent operator. In this case we prescribe a forcing of definite shape and frequency and then check the response of the flowfield to that forcing.

1.1.6 Resolvent analysis

The resolvent operator basically maps the forcing to the response of the flowfield. As a result, a lot of information about which kinds of forcings would the flow respond to or amplify the most can be obtained by examining the resolvent operator $\mathcal{H}(\omega) = (-i\omega - \mathcal{A})^{-1}$. A singular value decomposition of the resolvent operator can provide an orthonormal basis for the forcing and responses shapes ranked by the amplification factor which is given by the singular values:

$$\hat{q} = \mathcal{H} \hat{f} = U \Sigma V^* \hat{f} \quad (1.12)$$

The most amplified disturbance and response are $\hat{f}_1$ and $\hat{q}_1$ respectively. The factor of disturbance energy amplification is indicated by the gain given by σ_1^2 . The resolvent norm is given by:

$$\|\mathcal{H}(\omega)\| = \max_{\hat{f}} \frac{\|\hat{q}\|}{\|\hat{f}\|} = \sigma_1$$

A further detailed discussion on the choice of norms and how that fits into the analysis would be provided in a subsequent section (Sec.2.7.1). A more specialized form of the resolvent analysis is the input-output analysis (I/O). In I/O analysis we further restrict the types of inputs and outputs the system can admit. This can take the form of forcing being applied only in a certain area of the domain or only forcing a certain component of the flow (for example, forcing only in the u-momentum). This is achieved by introducing additional linear operators B and C which can filter potential inputs and outputs. In this case:

$$\hat{y} = C(-i\omega - \mathcal{A})^{-1} B \hat{f} = \mathcal{H} \hat{f} = U \Sigma V^* \hat{f}$$

1.1.7 Transient growth

In global stability analysis, we observed that in the case of flow with all decaying eigenmodes the flow is deemed stable in the long term as all disturbances die out eventually. In some cases however, these disturbances do exhibit growth in the short-term. This growth is known as transient or nonmodal growth [83]. The maximum possible growth over a certain time horizon is known as ‘optimal growth’[36]. In this case, time itself is a parameter. Optimal growth $G(t)$ is then given as:

$$G(t) = \max_{q'_0} \frac{\|q'(t)\|^2}{\|q'_0\|^2} = \|e^{\mathcal{A}t}\|^2$$

Whether a certain flow is susceptible to transient growth depends on the character of $\mathcal{A}$. If $\mathcal{A}$ is non-normal (it’s eigenvectors are not orthogonal to each other), transient growth is possible.

1.1.8 Spatial growth framework

The temporal view of stability has an advantage of being more intuitive. However, there are certain classes of flows where transition is dominated by growth of disturbance in space, rather than time. This is typically the case with high-speed flows without significant separation regions. In this scenario the streamwise wavenumber α is complex and frequency ω is purely real. Substituting $f' = 0$ and modal form $q' = \hat{q} e^{i(\alpha x - \omega t)}$ in Eq.1.7:

$$-i\omega_r q' = [\mathcal{A}_0 + \mathcal{A}_x + \mathcal{A}_{xx}] q' \quad (1.13)$$

Here the terms in $\mathcal{A}$ are split into three operators: $\mathcal{A}_0$, $\mathcal{A}_x$, $\mathcal{A}_{xx}$ containing terms without derivatives in the streamwise direction, single derivative in the streamwise direction and double derivative in the streamwise direction respectively.

$$-i\omega_r q' = [\mathcal{A}_0 + i\alpha \mathcal{B}_0 - \alpha^2 \mathcal{C}_0] q' \quad \mathcal{A}_x = \mathcal{B}_0 \frac{\partial}{\partial x} \quad \mathcal{A}_{xx} = \mathcal{C}_0 \frac{\partial^2}{\partial x^2} \quad (1.14)$$

In this case ω_r is prescribed and the problem is solved for α . To convert this quadratic eigenvalue problem (EVP) into a linear EVP two approaches are typically adopted. Either the α^2 terms are dropped or the state vector is modified $[q']^T \rightarrow [q', \alpha q']^T$

1.2 High speed stability and transition

Transition physics for high-speed flows is in some ways similar to low-speed flows and in some ways is quite dissimilar. The classical picture of transition for boundary layers involves[4]:

- Receptivity
- Modal growth of the instability
- Non-linear interactions

- Breakdown to turbulence

It can be seen in Fig. 1.1 that this very orderly ‘regular’ route of transition may be ‘bypassed’ and the process of transition may take some other avenue. The classical route does tend to explain transition scenarios in more simpler cases such as flat plate boundary layers and flows on sharp cones. But every level of complexity in the flowfield makes it that much more unlikely.

(a) Taken from Fedorov [37]; originally from Morkovin (1994)

Figure 1.1 Routes of transition

1.2.1 Linear instabilities

High-speed boundary layer transition is more conveniently viewed through the lens of spatial growth of disturbances.

1.2.1.1 Local spatial linear stability analysis

Consider a 2D flat plate boundary layer with a baseflow given by $\bar{q}(x, y)$ as shown in Fig.1.2. The premise of a local analysis is that because the boundary layer varies slowly in the streamwise direction it is reasonable to analyze the stability of the boundary layer profile locally at each streamwise station. For the purpose of this analysis a spatial analysis (ω : real, prescribed & α : complex, solved for) is

performed at some location x_0 with baseflow $\bar{q}(y)$ and the disturbance is of the form $\hat{q}(y)e^{i(\alpha x - \omega t)}$. The co-ordinates x and y are normalized by the Blasius length scale given by: $\mathcal{L} = \sqrt{\nu_e^* x^* / U_e^*}$ and the velocity is normalized by U_e^* . The time is non-dimensionalized by $\mathcal{L} / U_e^*$. The local Reynolds number $R = \sqrt{U_e^* x^* / \nu_e^*}$. In this case the subscript 'e' and superscript '*' indicate edge quantities and dimensional quantities respectively. The nomenclature is the same as Fedorov [37]. The non-dimensional frequency parameter is defined as:

Figure 1.2 Schematic: local spatial linear stability analysis

$$F = \frac{\omega^* \nu_e^*}{U_e^*} = \frac{2\pi f^* \nu_e^*}{U_e^*}$$

The frequency parameter F , the angular frequency ω and the Reynolds number R are related as:

$$\omega = F \cdot R$$

In casting the problem this way, Reynolds number can be used as a proxy for spatial location. So for a particular boundary layer as we travel downstream R increases due to an increase in x_e^* . A typical spatial spectrum (taken from Bitter [15]) at a certain spatial station is shown in Fig.1.3. $\alpha_i < 0$ indicates a growing mode and $\alpha_i > 0$ indicates a decaying one. Three distinct branches of modes can be observed: two branches of acoustic modes (fast and slow) and an almost vertical branch of vorticity/entropy modes. Additionally another mode labeled 'second mode' can be seen. This is a discrete unstable mode. Starting from the leading edge as we continue to probe downstream discrete modes break away or branch from the continuous branches. Some of these discrete modes may become unstable over a certain streamwise interval within which the amplitude of the disturbance increases.

The solution procedure in this case consists of solving the spatial eigenvalue problem at successive streamwise stations. At each station the spectrum changes slightly and the evolution of individual eigenmodes can be tracked from station to station. The evolution of two such discrete eigenmodes (F and S) is shown in Fig.1.4 (taken from- Nichols & Candler [70]) with the arrows

Figure 1.3 A typical spatial spectrum for a high-speed boundary layer: $\beta = 0$, $R = 2000$, $M_e = 4.5$, $T_e^* = 1500K$, $T_w^* = 300K$ - taken from Bitter [15]

indicating the direction of increasing R (also increasing x^*). The S and F originally belong to the slow and fast continuous acoustic branches respectively near the leading edge. The point at which the mode S crosses the dotted horizontal line it becomes unstable. The mode is ‘unstable’ over the interval which α_i continues to be negative. Over this interval a disturbance orthogonal to the mode -or the part of the disturbance which is orthogonal to the mode- will get amplified.

Figure 1.4 Spatial spectrum: the evolution of the discrete ‘F’ and ‘S’ mode as we go downstream from the leading edge- taken from Nichols & Candler [70]

As we trace these instabilities along the length of the boundary layer modes of different frequencies become unstable over different intervals. The unstable modes are often referred to as the ‘first mode’ and ‘second mode’. The terms ‘first mode’ and ‘second mode’ are somewhat of a misnomer

but their usage in this context has become ubiquitous in the hypersonic transition literature [38]. The second mode is the dominant instability at higher Mach numbers ($M > 4.5$). For the second mode the boundary layer acts as an acoustic waveguide and as a result the lower the frequency of the mode the further downstream it becomes unstable and/or peaks. This happens because short wavelength waves become attuned to the boundary layer further upstream when the boundary layer thickness is smaller. The first mode instability becomes unstable over a lower frequency range compared to the second mode instability.

The maximum spatial growth rate (α_i) a mode experiences is not the determining factor for transition onset- rather we must look at the integrated growth rate along the length of the flat plate to determine the energy amplification for a disturbance. This is usually expressed in terms of the ‘ N factor’. It is defined as:

$$N(x) = \ln \left(\frac{A(x)}{A(x_0)} \right) = \int_{x_0}^x -\alpha_i(\psi) d\psi \quad (1.15)$$

The choice of x_0 in this matter is somewhat arbitrary but the overwhelmingly popular choice is the neutral point when the mode becomes unstable. This procedure is done for multiple frequencies and the N -factor curves for each frequency are overlaid on one another to produce an envelope of N -factor curves. An N -factor envelope for a sharp 7° half-angle cone at $M_\infty = 10.04$ is shown in Fig.1.5 (taken from Grossir et al. [46]). The location where it is known that the transition occurs experimentally is shown with a vertical dotted line and maximum N -factor corresponding to this location is given by the horizontal dotted line. The N -factor method is widely used in hypersonics for estimating transition location. In order to do this one must have a sufficiently good idea of what the critical N -factor at which transition would occur (N_{tr}) should be. The N -factor in some ways represents a critical relative amplitude, but ultimately the possibility of transition onset at some location (x_1) is determined not by the degree to which the disturbance amplitude gets amplified in the interval from x_0 to x_1 but rather by the absolute level of disturbance energy at x_1 . So the other important piece of the puzzle is what disturbance energy level is available at the location x_0 . The critical N -factor N_{tr} is heavily influenced by the noise levels in flow-field it is typically in the range 5-10 for conventional windtunnels and over 15 in low-noise flight conditions. The knowledge about an appropriate N_{tr} is built using experimental correlations and other factors such as freestream noise, surface finish etc.

There are other limitations to the local spatial linear stability analysis or as it is generally referred to in this context: linear stability theory (LST). Firstly, it relies on the locally parallel assumption for the baseflow at each station and so doesn’t directly account for the spatial development effect. This is usually a decent assumption for simple flows over flat-plates and sharp cones and as long as the wavelengths of the disturbances being probed are smaller than the distance over which the baseflow develops appreciably. Secondly, the LST approach tracks the growth of an instability in a mode-by-mode basis and so any non-modal growth (spatial non-modal growth) will not be captured. High-speed stability theory involves a particularly influential phenomenon known as ‘synchronization’

(will be discussed in greater detail in the next section) is a specific example of an inter-modal interaction that will not be organically captured. Additionally, the points at which the discrete mode passes the continuous branch involve singularities and therefore demand special treatment. Fedorov [37] says: “*The LST and PSE methods should be supplemented by special procedures, which help to identify the singular regions and get through them in a proper way. This could be done using approximate theoretical models providing an intermodal exchange rule*”.

Figure 1.5 N -factor envelope for a 7° half-angle sharp cone- picture taken from Grossir et al. [46]

1.2.2 Receptivity mechanisms

A large number of studies in hypersonic transition deal with the identification and analysis of instabilities. For high-speed flows (especially boundary layers) the instability process is viewed through the lens of the spatial growth of a disturbance. The role of the instability is to amplify the amplitude of a certain disturbance as it travels downstream through the flow. To complete this picture it is essential to ascertain the initial amplitude of the disturbance available for the instability to amplify. Receptivity is the process by which this instability acquires an initial amplitude.

For low-speed flows the frequency-wavelength of the freestream waves does not synchronize with that of the instability and thus a special scale-conversion mechanism is required. In low-speed boundary layers the ability of acoustic waves interacting with surface irregularities to produce Tollmein-Schlichting (T-S) waves was shown by Ruban [80] and Goldstein [44]. In this case the frequency is determined by the freestream waves and the wavelength is imparted by the surface irregularity and this allows the disturbance to be synchronized with the T-S waves [24]. High-speed flows on the other hand do not require such a special scale-conversion mechanism as the freestream waves and the instability waves can naturally synchronize. In Fig.1.6 (a) (taken from Fedorov [37])

one can see the discrete modes F and S are synchronized with the fast and slow freestream acoustic waves near the leading edge. The phase speeds for the fast and slow acoustic waves are shown with thin solid lines labeled with $c = 1 + \frac{1}{M}$ and $c = 1 - \frac{1}{M}$ respectively. The dashed line labeled with $c = 1$ represents the vorticity/entropy waves. In the region near the leading edge (labelled 1) the discrete modes F and S are synchronized with the fast and slow acoustic waves and therefore are receptive to fast and slow waves respectively in this region. In this region these freestream waves can impart amplitude to these discrete modes. In the region labeled 2 the F mode crosses the dashed line representing the entropy/vorticity waves. At this point the mode F is receptive to freestream entropy or vorticity waves (freestream turbulence or temperature spots). Eventually the mode F and mode S merge. The point at which their wavenumbers (and frequencies) match is known as the synchronization point. This modal interaction is a very important phenomenon in hypersonic transition. As a part of this intermodal interaction the growth rates of both the modes are significantly affected as well. Almost simultaneously, the mode S is destabilized and mode F is stabilized. This particular interval of instability associated with mode S is commonly known as the ‘second mode’.

Figure 1.6 Dispersion curves for the F and the S mode -pictures taken from Fedorov [37]

Different scenarios by which transition may occur are shown in Fig.1.7 (taken from Fedorov [37]). For instance one path may involve freestream fast acoustic waves which trigger mode F near the leading edge which in turn imparts its amplitude to mode S through an intermodal interaction which then goes on to become unstable. Another scenario is through the action of entropy/vorticity waves to which the mode F is receptive in region 2 (refer Fig.1.6) which then imparts its amplitude to mode S . Yet another path involves slow acoustic wave directly triggering mode S near the leading edge which then goes on to become unstable.

In addition to the receptivity to freestream noise the receptivity of a flow to other factors or mechanisms is also important. These include: surface finish, distributed roughness, isolated roughness,

Figure 1.7 Paths of regular transition- taken from Fedorov [37]

suction/blowing etc.

1.3 Discrete and continuous spectra

The terms ‘discrete mode’ and ‘continuous branches’ often appear in the boundary layer transition literature. This particular character of the boundary layer spectrum is due to its unbounded domain (as against a channel which is bounded). This can be illustrated by a simple 1D example from Schmid et al. [83].

For a bounded domain a system is given by-

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad x \in [0, 1] \quad u(0, t) = u(1, t) = 0 \quad (1.16)$$

The solution is given by:

$$u(x, t) = f(x)e^{i\omega t} \quad \omega_n = n\pi \quad f_n(x) = \frac{1}{\sqrt{2}} \sin(n\pi x) \quad n = 1, 2, 3, \dots \quad (1.17)$$

For an unbounded domain a system is given by-

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad x \in [0, \infty) \quad u(0, t) = 0 \quad u(x, t) \text{ is bounded as } x \rightarrow \infty \quad (1.18)$$

The solution is given by:

$$u(x, t) = f(x)e^{i\omega t} \quad \omega \in \mathcal{R}_0^+ \quad f(x, \omega) = \frac{1}{\sqrt{2\pi}} \sin(\omega x) \quad n = 1, 2, 3, \dots \quad (1.19)$$

It can be seen that for the 1D example wherein the discrete modes could completely represent the bounded problem a continuous spectra was required by the unbounded flows. It was shown that for unbounded flows the spectrum consists of discrete modes and some portion of a continuous spectrum ([45], [81]).

It is important to distinguish between discrete and continuous modes when computing eigenvalues numerically. While the numerical spectrum can compute the discrete modes it only approximates the continuous spectrum [28]. Moreover, discrete mode amplitudes are expected to vanish as $y \rightarrow \infty$. The modes belonging to the continuous spectrum on the other hand are only expected to be bounded [5].

1.4 Outline

CHAPTER 2: Contains numerical details related to the development of a suite of stability analysis tools in a 3D unstructured parallel context. It also includes the details on using a novel dual number based strategy for obtaining accurate numerical jacobians. Additionally an analysis on shock capturing in the context of stability analysis tools is performed. Furthermore, an extension of the code to solve axisymmetric problems is also discussed.

CHAPTER 3: An extensive validation is performed to demonstrate the capabilities of the code in handling flow problems across a wide range of flow regimes from incompressible to hypersonic flows.

CHAPTER 4: Discusses the application of these stability analysis tools to turbulent mean flows with a view of extracting important structural and dynamical features. Compressible turbulent channel flows across a wide range of Mach numbers are analyzed.

CHAPTER 5: An input-output analysis of high speed boundary layer stability for boundary layers with thermal distortions such as temperature bumps and temperature steps is conducted.

CHAPTER 6: Discusses a novel inflow-I/O framework for freestream receptivity studies. Cases involving a $M_\infty = 6$ flat plate boundary layer and a $M_\infty = 6$, 7° half-angle sharp wedge are shown.

CHAPTER

2

NUMERICAL FRAMEWORK

In this chapter the different building blocks that make up the suite of stability solvers are discussed. The governing philosophy of this effort was to build the stability analysis tools in a way which makes it possible to convert standard compressible flow solvers into stability analysis tools in a seamless and straightforward manner.

2.1 LNS Operator

$\mathcal{A}$ is the linearized Navier-Stokes (LNS) operator and is essentially the jacobian of the global right hand-side with respect to the global state vector. A large number of tools and methods of analysis concerned with fluid stability use $\mathcal{A}$ in some form. In some algorithms it is not necessary to explicitly form $\mathcal{A}$, but the action of the matrix $\mathcal{A}$ on a vector is required. These approaches are known as ‘matrix free’ approaches. The ‘matrix forming’ approaches on the other hand require that the matrix be explicitly formed and stored. The matrix forming approaches provide a direct access to the linear operator and as a result provide more algorithmic flexibility. The matrix can be formed by deriving the linear operator based on the continuous form of nonlinear equations and then discretizing it (continuous approach) or by linearizing the already discretized non-linear equations (discrete approach)[43]. To compute the jacobian using the discrete approach, a brute-force way to do this would be to compute the entire global right hand-side each time a particular degree of freedom (an element of the state vector) is perturbed. Cook et al. [26] use an approach wherein one independently perturbs multiple DOFs simultaneously such that they affect mutually exclusive and independent domains in the system. This approach was earlier used by Nielsen & Kleb [73] to compute direct and adjoint operators using complex step derivative approach. In this work we use the discrete approach

along with dual number differentiation to compute the linearized operator and the details of the algorithm are mentioned in the next section.

2.2 Matrix formation algorithm

In order to efficiently form the matrix in a parallel framework we linearize fluxes at individual faces instead of the entire global right hand side. The Navier-Stokes equations in the standard finite-volume form for a cell i is given as:

$$\frac{\partial U_i}{\partial t} = \frac{1}{V} \sum_k \mathbf{F}_k^n S_k \quad (2.1)$$

$\mathbf{F}_k^n$ is the normal flux at a face. Our approach involves linearizing this individual entry. This procedure requires two ingredients:

- A local self-contained flux stencil
- A local flux routine to compute the flux at a face given all necessary information (through the self-contained flux stencil).

The algorithm is as follows:

1. Loop over all faces
2. Assemble a local stencil at each face
3. Perturb each DOF within the stencil and compute the derivative of the flux at that face with respect to that DOF
4. Assign entries to the global matrix (row determined by cells adjoining the face and column determined by DOF being perturbed)

This approach considers the fact that a particular entry in the global right hand side depends on a very limited DOFs in the system. This sparsity is exploited by assembling a local flux stencil which indicates that the value of a flux at a face is solely determined by the flow variables in the stencil. How large the local flux stencil needs to be or in other words what is the ‘extent’ of the local flux stencil is determined by flux numerics.

2.3 Flux numerics

The global matrix $\mathcal{A}$ is very sparse. Regardless of the size of the domain, the number of non-zeros in any row doesn’t exceed a fixed number which depends on the flux numerics used. This number is known a priori. The number of non-zeros depends on the extent of the local flux stencil. In our code the fluxes are composed of mainly two parts: inviscid and viscous. Additionally, a

dissipative flux complements the inviscid flux wherever necessary to ensure stability of solutions near discontinuities. The flux at a face is given as: $\vec{F} = \vec{F}^I + \vec{F}^V$. The inviscid flux is given as:

$$\vec{F}^I = \vec{F}_{\text{central}} + \alpha \vec{F}_{\text{diss}}$$

The central flux is a component of the flux designed to minimize dissipation but is prone to instabilities. Higher order central flux can be constructed by using gradients in the upstream and downstream cells. They are expressed as follows with the superscript indicating the order.

$$\begin{aligned}\vec{F}_{\text{central}}^{(2)} &= \vec{F}_{\text{central}}^{(2)}(U_i, U_{ii}) \\ \vec{F}_{\text{central}}^{(4)} &= \vec{F}_{\text{central}}^{(4)}(U_i, U_{ii}, \nabla U_i, \nabla U_{ii}) \\ \vec{F}_{\text{central}}^{(6)} &= \vec{F}_{\text{central}}^{(6)}(U_i, U_{ii}, \nabla U_i, \nabla U_{ii}, \nabla U_{iil}, \nabla U_{iir})\end{aligned}$$

The 2D version of the stencils representing fourth and the sixth order flux is shown in Fig.2.1.

Figure 2.1 Comparison of different numerical approximations

The viscous fluxes are based on central differencing. They are built using the primitives and gradients in the adjoining cells.

$$\vec{F}^V = \vec{F}^V(U_i, U_{ii}, \nabla U_i, \nabla U_{ii})$$

The dissipative flux is given as

$$\vec{F}_{\text{diss}} = -\frac{1}{2}|A|(U_i, U_{ii}) \cdot (U_R - U_L)$$

A is the jacobian of the normal flux through the face with respect to the conserved variables. It is built using an average of the conserved variables in the adjoining cells (here instead of a simple mean, a pressure weighted average is used). U_R and U_L are the right and the left states built using variable extrapolation. We use a TVD-MUSCL approach with appropriate limiters to build these states. These left and right states can be constructed as:

$$\begin{aligned} U_L &= U_i + \frac{1}{2}\phi(r_i) \cdot (U_{ii} - U_i) & U_R &= U_{ii} - \frac{1}{2}\phi(r_{ii}) \cdot (U_{iir} - U_{ii}) \\ r_i &= \frac{U_i - U_{iil}}{U_{ii} - U_i} & r_{ii} &= \frac{U_{ii} - U_i}{U_{iir} - U_{ii}} \end{aligned}$$

Thus the dissipative flux depends on the values in two upstream and the two downstream cells.

$$\vec{F}_{\text{diss}} = \vec{F}_{\text{diss}}(U_i, U_{ii}, U_{iil}, U_{iir})$$

2.4 Numerical perturbation:

In order to form the matrix, numerical differentiation may be used. The standard way to do this is by using a finite difference approximation based on a Taylor series expansion. For a certain function $f(q)$ the Taylor series expansion upon perturbing q with a small ϵ is given as:

$$f(q + \epsilon) = f(q) + \epsilon \frac{\partial f}{\partial q} + \frac{\epsilon^2}{2!} \frac{\partial^2 f}{\partial q^2} + \frac{\epsilon^3}{3!} \frac{\partial^3 f}{\partial q^3} + \dots$$

Using this expansion a first or second order finite difference approximation can be given as:

$$\frac{\partial f}{\partial q} = \frac{f(q + \epsilon) - f(q)}{\epsilon} + \mathcal{O}(\epsilon) \qquad \frac{\partial f}{\partial q} = \frac{f(q + \epsilon) - f(q - \epsilon)}{2\epsilon} + \mathcal{O}(\epsilon^2)$$

These approaches suffer from two kinds of errors: truncation error and error due to finite precision arithmetic. The truncation error can be reduced by selecting a very small value of ϵ . As the approximation involves calculating the difference between two numbers, its reliability suffers due to finite precision arithmetic if a very small ϵ is chosen. Thus there is always a competition between these two kinds of errors and a compromise must be struck to determine the optimal ϵ_{opt} . For large-dimensional systems this ϵ_{opt} could be different for different elements in the state vector and this cannot be determined apriori.

The error due to the finite precision arithmetic can be eliminated by not requiring the difference

operation to compute the derivative by using approaches such as complex step derivative or dual numbers. In this work we use dual numbers to compute high-precision numerical derivatives. Dual numbers are special numbers which are very similar to complex numbers in many ways but with the exception of one critical difference. These numbers have a special algebra wherein the higher orders of the dual unit ($\hat{\delta}$) are identically zero: $\hat{\delta} \cdot \hat{\delta} = 0$. For instance for a function $f(q)$, the argument q is replaced with a dual number: $q + 1\hat{\delta}$.

$$\begin{aligned} f(q + 1\hat{\delta}) &= f(q) + \hat{\delta} \frac{\partial f}{\partial q} + \hat{\delta}^2 \frac{1}{2!} \frac{\partial^2 f}{\partial q^2} + \hat{\delta}^3 \frac{1}{3!} \frac{\partial^3 f}{\partial q^3} + \dots \\ &= f(q) + \hat{\delta} \frac{\partial f}{\partial q} + 0 \\ &= (f(q), \frac{\partial f}{\partial q}) \end{aligned}$$

Thus the derivative of $f(q)$ with respect to q is exactly recovered in the dual part. It allows us to compute highly precise numerical derivatives which are exact to machine precision (as long as the underlying function is differentiable). An example of how these different approaches compare with each other for different functions can be seen in Fig.2.2. The error is calculated as ($\hat{f}_i$ refers to the exact derivative at the point x_i and f'_i is the numerical approximation for the same):

$$\text{Error} = \frac{\sqrt{\sum_{i=1,N} (f'_i - \hat{f}_i)^2}}{\sqrt{\sum_{i=1,N} (\hat{f}_i)^2}} = \frac{\|f' - \hat{f}\|_2}{\|\hat{f}\|_2} \quad (2.2)$$

For the $\text{abs}(x)$ function shown in Fig.2.2(a)&(b), which is a linear continuous and non-differentiable function, the error drops for both the first and the second order approximations at the same rate (as the function is linear in x), and then precipitously drops at a threshold level beyond which round-off errors take over. This threshold depends on how close a certain point is to $x = 0$, as this is where the function definition changes. If the $\epsilon > |dx|$, then the function will be perturbed into the other side of $x = 0$ and inaccurate derivatives will be obtained. The threshold drop occurs around 10^{-2} in Fig.2.2(a) and around 10^{-4} in Fig.2.2(b) and this reflects the value of dx in these two cases. Dual number approximation on the other hand gives exact derivatives at each point except $x = 0$ at which the function is not defined in any case. The error for the dual number approximation was zero and so doesn't show up on the log plot. For a more complex function such as $\frac{e^x}{\sqrt{\sin^3(x) + \cos^3(x)}}$ it can be seen in Fig.2.2(b) that the dual number approximation doesn't exactly show zero error but instead shows some level of machine precision level error.

Dual numbers have been used to compute high-precision derivatives in CFD applications before. Martins et al. [61] observed that the automatic differentiation and the complex step derivative approach are equivalent. Spall & Yu [88], implemented dual numbers in CFD codes in order to compute the sensitivity of the CFD solution to parameters such as fluid properties, geometric

parameters or turbulence model coefficients. Fike & Alonso [40] used a similar idea known as hyper-dual numbers to compute accurate first and second-derivatives of engineering parameters (C_L , C_D , C_M) with respect to flow and geometric parameters such as Mach number (M_∞) and angle of attack (α). D’Onofrio et al. [30], used dual numbers to form accurate jacobians for optimal control problems and found that compared to traditional methods they could get more accurate answers albeit at a higher computational cost.

(a) $f(x) = |x|, dx = 10.0^{-2}$

(b) $f(x) = |x|, dx = 10.0^{-4}$

(c) $f(x) = \frac{e^x}{\sqrt{\sin^3(x) + \cos^3(x)}}$

Figure 2.2 Comparison of different numerical approximations for computing first derivatives: variation of error with perturbation size (ϵ) for first-order finite difference, second-order finite difference and a dual number based approximation (ϵ not applicable)

2.5 Shock Capturing

Discontinuities in the form of shocks are routinely encountered in high-speed compressible flows. Low dissipation high-order schemes which are often used in computations to resolve fine structures better, tend to become unstable in the vicinity of shocks. The remedy usually consists of reducing the order of the scheme and/or introducing additional dissipation in the vicinity of shocks to ensure numerical stability. We use shock-capturing to deal with discontinuities. This approach contains two ingredients: a shock detector and a dissipative flux. The dissipative component can be given as:

$$\alpha \vec{F}_{\text{diss}} = -\alpha \frac{1}{2} |A| (U_i, U_{ii}) \cdot (U_R - U_L)$$

A Linearized version of this component can be given as:

$$(\alpha \vec{F}_{\text{diss}})' = -\frac{1}{2} \bar{\alpha} |\bar{A}| \cdot (U_R' - U_L') - \frac{1}{2} \bar{\alpha} |A'| \cdot (\bar{U}_R - \bar{U}_L) - \frac{1}{2} \alpha' |\bar{A}| \cdot (\bar{U}_R - \bar{U}_L) \quad (2.3)$$

Mathematically speaking, the full linearized version must be used, but each of the terms in Eq.2.3 represents different physical mechanisms. The additional dissipation through the shock-capturing method is added for numerical considerations not physical ones. Therefore, it is not obvious if all of these linear components should be retained. Moreover, it is important to note here that shock capturing is a remedy prescribed for a specific problem in non-linear computations. Whether the ‘linearization of the remedy’ an appropriate ‘remedy’ for the linearized problem remains to be seen. In our work we freeze the shock-detector switch by setting $\alpha' = 0$.

In the context of linearized analysis wherein we have fine instability waves traveling through a shock the desirable attributes of a shock-capturing method include:

- The amplitudes and wavelengths of the instability waves are correctly predicted downstream of the shock
- Disturbances do not generate aphysical oscillations when passing through a shock

2.5.1 Ducros switch: influence of ϵ_{duc}

The ducros switch is defined as:

$$\alpha_{duc} = \frac{\theta^2}{\theta^2 + \omega^2 + \epsilon_{duc}} \quad (2.4)$$

Here θ indicates divergence and ω indicates vorticity. The rationale is that in the regions of discontinuity $|\theta| \gg |\omega|$, $\alpha_{duc} \rightarrow 1$. In the regions away from the discontinuity $|\theta| \ll |\omega|$ and $\alpha_{duc} \rightarrow 0$. ϵ_{duc} is a small number to avoid division by zero. But this creates a problem in the freestream where $\omega \approx 0$, then if $\theta^2 \gg \epsilon_{duc}$, and $\alpha_{duc} \rightarrow 1$ despite being a smooth region of the flow. This causes the switch to oscillate erratically between 0 and 1, in a region where it should be 0. As a

remedy, instead of an arbitrarily chosen small ϵ_{duc} , a number scaled with the flow scales is used: $\epsilon_{duc} \sim \frac{\mathcal{U}^2}{\mathcal{L}^2}$. This is scaled as $\epsilon_{duc} = \epsilon_0 \frac{\mathcal{U}^2}{\mathcal{L}^2}$. To see how this fares, we will check α_{duc} for a 1D normal shock. This is an inviscid 1D flow and so $\omega = 0$. So the ducros switch becomes:

$$\alpha_{duc} = \frac{\theta^2}{\theta^2 + \epsilon_{duc}} \quad (2.5)$$

Figure 2.3 The effect of ϵ_{duc} on α_{duc} relative to the shock width

In Fig.2.3(a) and Fig.2.3(b) we can investigate how the switch would behave for different values of ϵ_{duc} . As a reference the value of the numerator (θ^2) is shown in Fig.2.3(a) and the baseflow pressure ($\bar{p}$) is shown in Fig.2.3(b). For this exercise we use a highly refined grid near the shock to explore the intricacies of the switch. It is reasonable to assume that the Ducros switch should be turned ON for a distance which is at least as large as the shock width in order to subsume the discontinuous region. Based on the pressure jump shown in Fig.2.3(b) it is clear that ϵ_{duc} must be less than 10^{-6} for the switch to be turned ON over the entire extent of the discontinuity. A very low value of ϵ_{duc} on the other hand might lead to an erratic behavior of the Ducros switch in the freestream leading to the dissipation of disturbances in the freestream. Thus, the Ducros switch needs to be carefully tuned to balance these competing interests. This is not however as burdensome as it may seem as the Ducros switch can be tuned for the stability analysis independent of how it was tuned for the baseflow. An analysis of the kind shown in Fig.2.3 can be performed almost readily based on the baseflow information and then ϵ_{duc} can be set accordingly.

Although the 1D inviscid cases are illustrative it is still an academic pursuit. The picture is more complicated for a realistic flowfield with $\theta, \omega \neq 0$ and where the magnitude of both the divergence and vorticity varies over many orders of magnitude over the flowfield.

(a) $\epsilon_{duc} = 10^3 \approx \frac{U_e^2}{\delta^2} \times 10^{-8}$

(b) $\epsilon_{duc} = 10^5 \approx \frac{U_e^2}{\delta^2} \times 10^{-6}$

(c) $\epsilon_{duc} = 10^7 \approx \frac{U_e^2}{\delta^2} \times 10^{-4}$

(d) $\epsilon_{duc} = 10^9 \approx \frac{U_e^2}{\delta^2} \times 10^{-2}$

Figure 2.4 The effect of ϵ_{duc} on the $\overline{\alpha_{duc}}$ for a converged baseflow ($M_\infty = 6$ flow over sharp wedge)

Figure 2.5 The effect of ϵ_{duc} on the $\overline{\alpha_{duc}}$ at $x = 0.4$ for a converged baseflow ($M_\infty = 6$ flow over sharp wedge) with reference to $\overline{P}$

In Fig.2.4 we can see the plot of the Ducros switch ($\bar{\alpha}_{duc}$) for different values of ϵ_{duc} for a $M_\infty = 6$ flow over a 7° wedge. The relevant length scale and velocity scale in this case are selected to be the boundary layer thickness at the outflow (δ) and the edge velocity (U_e). For $\epsilon_{duc} = 10^3$ the switch turns ON in most of the post-shock region and this is undesirable. For $\epsilon_{duc} = 10^5$ and $\epsilon_{duc} = 10^7$ the switch is mostly turned ON in the vicinity of the shock as is desirable. $\epsilon_{duc} = 10^9$, the switch is ON near the shock but the value is less than 1 downstream. To further investigate this we plot the values of $\bar{\alpha}_{duc}$ along a vertical line at $x=0.4$. It can be seen that $\epsilon_{duc} = 10^7$ seems to have no oscillations and is fully turned ON near the shock. For phenomena which involve the interaction of fine disturbance structures with shocks competing interests between stability and accuracy need to be balanced deftly.

To further illustrate how ϵ_{duc} plays a role in the transmission of waves through a shock we perform an unsteady computation using the linearized solver by looking at a fast acoustic wave ($\omega = 10$) passing through a $M_\infty = 3$ normal shock. We capture the wave at different instances of time specifically: before and after the wave hits the shock. We do this for three different values of $\epsilon_{duc} = 10^{-6}, 10^{-12}$ & 10^{-18} . In Fig.2.6 it can be seen that before the wave hits the shock ϵ_{duc} has little or no effect on the wave. However, once the wave hits the shock, it can be seen that for $\epsilon_{duc} = 10^{-6}$ numerical oscillations travel upstream and downstream from the shock and distort the wave structure. This effect is almost negligible for $\epsilon_{duc} = 10^{-12}$ and nonexistent for $\epsilon_{duc} = 10^{-18}$. However for $\epsilon_{duc} = 10^{-18}$ it can be seen that the wave dissipates more as it travels downstream (the horizontal lines indicate exact amplitude obtained from linear theory by McKenzie & Westphal [64]). Although the dissipation isn't significant in this case it indicates a potential drawback of choosing a very low ϵ_{duc} .

2.5.2 Grid resolution

Grid resolution is of critical importance to adequately resolve instability waves. This is usually measured in terms of X cells per wavelength of the instability wave. Broadly speaking about 8-12 cells are sufficient to resolve waves in smooth flows. We intend to investigate if this is still true in the context of shocks. We investigate for an acoustic wave passing through a $M_\infty = 3$ normal shock for a uniform grid distribution if an increased resolution is required due to the presence of a normal shock relative to a smooth flow. We introduce a fast acoustic wave ($\omega = 10$) from the inflow. The frequencies (and wavenumbers) are related by:

$$(u_1 + a_1)k_{1a} = (u_2 + a_2)k_{2a} = (u_2)k_{2e} = \omega$$

The wavelengths for the incident fast acoustic wave, the transmitted fast acoustic wave and the generated entropy wave are: $\lambda_{a1} = 0.8377$, $\lambda_{a2} = 0.5057$ and $\lambda_{e2} = 0.1629$. The smallest wavelength is the generated entropy wave and so we will use that as a metric in terms of what resolution is required. The basic grid is $N_x = 1501$ (8cells/ λ_{2e}). The other grids considered are $N_x = 3001$, $N_x = 6001$ and

Figure 2.6 The effect of ϵ_{duc} for a shock-acoustic wave interaction at $M_\infty = 3$; left panel ($t = 4.5s$, pre shock interaction), middle panel ($t = 45s$, close-up view)& right panel ($t = 45s$), full view

$N_x = 12001$.

It can be seen in the results shown in Fig.2.7 that although the acoustic wave is captured fairly well across all grid configurations, the grid configuration is much more critical in terms of capturing the much finer entropy wave (generated). In case of the entropy wave the effect of ϵ_{duc} can also be seen wherein the entropy wave dissipates far more aggressively for $\epsilon_{duc} = 10^{-18}$ than for $\epsilon_{duc} = 10^{-17}$.

2.5.3 2D shock-acoustic interaction

For an adequate resolution (16 cells / λ_{2e} for $\theta = 0^\circ$) we compare the performance for acoustic waves incident at different angles to the normal shock. The baseflow was obtained by positioning a normal shock ($M = 3$) at $x = 4$ as shown in Fig.2.8. The domain extends from $x = [0, 12]$ and $y = [-2.5, 7.5]$. Slow acoustic waves are introduced through a section of the inflow ($0 < y < 5$) as harmonic forcing. Sponge layers are applied for $x > 10.0$, $y > 5$ and $y < 0$ near the right, top and bottom outflow boundaries respectively to damp out outgoing waves. Grid stretching is also applied in this region for the same effect. As the orientation of the wave increases fine resolution is required in both x and y directions. It can be seen in Fig.2.9 that the two-dimensional interaction between the acoustic wave and the normal shock is captured well.

2.6 Handling of homogeneous directions

In stability analysis we are often concerned with problems where the baseflow might be invariant in one or two spatial directions but the disturbance field can be Fourier transformed in those directions. These directions are referred to as homogeneous directions. Thus the stability problem can be solved for disturbances which are resolved in some spatial direction/s and have a fixed wavenumber in the homogeneous directions.

Consider a case, where a particular direction, say x_j is homogeneous, then the derivatives in the direction x_j for the baseflow and the disturbances are given as: $\frac{\partial \bar{q}}{\partial x_j} = 0$, $\frac{\partial q'}{\partial x_j} = i\alpha_j q'$. The routine flux linearization described in §2.2 cannot handle this aspect. So the linearization must be conducted carefully to include the effect of this disturbance variation in the x_j direction. This is done in two steps. Firstly, the basic routine is modified to make sure that the flux in the x_j direction does not contribute to the linearized operator. Thereafter, the contributions in the x_j direction are replaced in the matrix using the correct form. The Navier-Stokes equations can be written as follows where the inviscid and the viscous fluxes are given by $\vec{F}^I$ and $\vec{F}^V$.

$$\frac{\partial \vec{U}}{\partial t} = -\frac{\partial}{\partial x_j} \vec{F}^I + \frac{\partial}{\partial x_j} \vec{F}^V \quad (2.6)$$

The inviscid and the viscous fluxes are implemented using different procedures as shown next.

Figure 2.7 Decomposing the density perturbations into **acoustic** (rows 1 & 3) and **entropy** (rows 2 & 4) components. The effect of grid resolution & ϵ_{duc} , (a)-(h): $\epsilon_{duc} = 10^{-18}$, (i)-(p): $\epsilon_{duc} = 10^{-17}$

Figure 2.8 Schematic of the 2D shock-acoustic interaction

Figure 2.9 2D interaction of a $M = 3$ normal shock with oblique acoustic waves; pressure disturbance (p')

2.6.1 Treatment of inviscid flux:

The inviscid fluxes can be handled in a straightforward way. Linearizing about mean flow $\bar{U}$ and writing the perturbed conserved vector as q' , the linearized disturbance equation can be written as,

$$\frac{\partial q'}{\partial t} = - \left[\frac{\partial}{\partial x_j} A_j^I \right] q' \quad (2.7)$$

If a direction is homogeneous it won't be handled through the usual Green-Gauss type transformation and is instead handled as cell-based sourceterm. Due to the absence of any spatial gradient terms in A_j^I and the fact that the baseflow is invariant in the homogeneous direction ' j ' the discrete form for cell i is given as (Here $\hat{A}$ indicates normal jacobians- $\hat{A} = A_x n_x + A_y n_y + A_z n_z$):

$$\frac{\partial q'_i}{\partial t} = - \underbrace{\frac{1}{V_i} \sum_k \hat{A}_k^I q'_k S_k}_{j \text{ is homogeneous}} - \underbrace{i \alpha_j [A_j^I] q'_i}_{j \text{ is not homogeneous}} \quad (2.8)$$

2.6.2 Treatment of viscous flux:

For the viscous part of the Navier-Stokes equations:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x_j} F_j^V \quad (2.9)$$

Each of the terms in the viscous fluxes can be represented in a generalized form: $\frac{\partial}{\partial x_l} \beta g \frac{\partial f}{\partial x_m}$, where $x_l, x_m = x, y, z$, $\beta = \mu, \frac{-2}{3}\mu, \frac{4}{3}\mu, \kappa$ and $g = u, v, w, 1$. $g = 1$, refers to the terms in the momentum equations and the heat flux terms in the energy equation. These terms can be classified into four types:

- Type 1: x_l is homogeneous; x_m is not
- Type 2: x_l is not; x_m is homogeneous
- Type 3: x_l is homogeneous; x_m is homogeneous
- Type 4: x_l is not; x_m is not

Then the viscous fluxes can be written as follows where the subscript indicates the type of terms:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x_j} F_{1,2,3}^V + \frac{\partial}{\partial x_j} F_4^V \quad (2.10)$$

For the generalized term the linearized form looks like:

$$\begin{aligned}
F^{\vec{\lambda}(V)} = & \left[\bar{\beta} \frac{\partial g'}{\partial x_l} + \bar{g} \frac{\partial \beta'}{\partial x_l} + \beta' \frac{\partial \bar{g}}{\partial x_l} + g' \frac{\partial \bar{\beta}}{\partial x_l} \right] \frac{\partial \bar{f}}{\partial x_m} \\
& + \left[\bar{\beta} \frac{\partial \bar{g}}{\partial x_l} + \bar{g} \frac{\partial \bar{\beta}}{\partial x_l} \right] \frac{\partial f'}{\partial x_m} \\
& + [\bar{\beta} g' + \beta' \bar{g}] \frac{\partial \bar{f}}{\partial x_l \partial x_m} \\
& + [\bar{\beta} \bar{g}] \frac{\partial f'}{\partial x_l \partial x_m}
\end{aligned}$$

The case 4 is handled in the code through the usual subroutine. For the specialized cases 1, 2 and 3, the simplified forms are given by:

$$\begin{aligned}
F_1^{\vec{\lambda}(V)} &= [i\alpha_l \bar{\beta} g' + i\alpha_l \bar{g} \beta'] \frac{\partial \bar{f}}{\partial x_m} + i\alpha_l [\bar{\beta} \bar{g}] \frac{\partial f'}{\partial x_m} \\
F_2^{\vec{\lambda}(V)} &= i\alpha_m \left[\bar{\beta} \frac{\partial \bar{g}}{\partial x_l} + \bar{g} \frac{\partial \bar{\beta}}{\partial x_l} \right] f' + i\alpha_m [\bar{\beta} \bar{g}] \frac{\partial f'}{\partial x_l} \\
F_3^{\vec{\lambda}(V)} &= i\alpha_l i\alpha_m [\bar{\beta} \bar{g}] f'
\end{aligned}$$

The terms which contain the derivative of a disturbance with respect to a non-homogeneous direction a Green-Gauss type approach is used to discretize the gradient operator. More details are provided in the Appendix.A.

2.7 Large scale eigenvalue problems: Implementation details

2.7.1 Norms

For analysis such as the resolvent or I/O analysis it is necessary to prescribe a ‘measure’ of disturbance energy. This choice is pretty straightforward for incompressible flows wherein the kinetic energy of the disturbance is chosen. For compressible flows this choice is not as obvious. A norm proposed by Chu [25] is quite popular and we use the same in this context. The inner product for a specific norm is defined as:

$$\langle f, g \rangle_{\mathcal{M}} = f^* \mathcal{M} g \quad (2.11)$$

For the Mack norm the matrix $\mathcal{M}$ is a diagonal matrix with the diagonal given by:

$$\text{diag}(\mathcal{M}) = \left(\frac{\bar{a}_0^2}{\bar{\rho}\gamma}, \bar{\rho}, \bar{\rho}, \bar{\rho}, \frac{\bar{\rho} c_v}{\bar{T}} \right)^T \quad (2.12)$$

This norm is defined for the primitive vector $v'_T = (\rho', u', v', w', T')^T$ whereas our system is based on the conserved vector $q' = (\rho', u', v', w', T')^T$. The equivalent matrix for a conserved vector $\mathcal{M}_U$ is given by:

$$\begin{aligned}
 q'^* \mathcal{M}_U q' &= v'^*_{T'} \mathcal{M} v'_{T'} \\
 \mathcal{M}_U &= \left(\frac{\partial V_T}{\partial U} \right)^* \mathcal{M} \left(\frac{\partial V_T}{\partial U} \right) \\
 \mathcal{M}_U^{\frac{1}{2}} &= \mathcal{M}^{\frac{1}{2}} \left(\frac{\partial V_T}{\partial U} \right) \quad \mathcal{M}_U^{-\frac{1}{2}} = \left(\frac{\partial U}{\partial V_T} \right) \mathcal{M}^{-\frac{1}{2}}
 \end{aligned} \tag{2.13}$$

2.7.2 Input-Output formulation

The input-output formulation as described earlier is expressed as:

$$\hat{y} = C(-i\omega - \mathcal{A})^{-1} B \hat{f} = \mathcal{H} \hat{f} = U \Sigma V^* \hat{f}$$

The goal is to find the shape of the inputs that will be amplified the most by the system. This energy amplification needs to be rooted in a particular definition of energy. This is achieved through the specification of a physically sensible norm. Mathematically this can be expressed as (the subscript E refers to some energy norm, it is also possible to use different energy norms for the input and outputs- say E and F):

$$\begin{aligned}
 G &= \max_{\hat{f}} \frac{\|\hat{y}\|_E^2}{\|\hat{f}\|_E^2} \\
 G &= \max_{\hat{f}} \frac{\hat{y} \mathcal{M}_U^{\frac{1}{2}*} \mathcal{M}_U^{\frac{1}{2}} \hat{y}}{\hat{f} \mathcal{M}_U^{\frac{1}{2}*} \mathcal{M}_U^{\frac{1}{2}} \hat{f}} \\
 G &= \max_{\hat{f}} \frac{\|\mathcal{M}_U^{\frac{1}{2}} \hat{y}\|_2^2}{\|\mathcal{M}_U^{\frac{1}{2}} \hat{f}\|_2^2} \\
 G &= \max_{\hat{f}} \frac{\|\tilde{y}\|_2^2}{\|\tilde{f}\|_2^2}
 \end{aligned}$$

This allows us to cast the maximization problem in terms of the 2-norm. The obvious advantage being the relationship of the 2-norm to the singular value decomposition. The modified vectors $\tilde{y}$ and $\tilde{f}$ are related by the modified transfer function $\tilde{\mathcal{H}}$:

$$\tilde{y} = \tilde{\mathcal{H}} \tilde{f} = \mathcal{M}_U^{\frac{1}{2}} \mathcal{H} \mathcal{M}_U^{-\frac{1}{2}} \tilde{f} \tag{2.14}$$

The singular value decomposition of $\tilde{\mathcal{H}}$ yields: $\tilde{\mathcal{H}} = U \Sigma V^*$. Here V and U provide bases for

inputs ($\tilde{f}$) and outputs ($\tilde{y}$) respectively. The basis in terms of the original inputs and outputs can be recovered by pre-multiplying with $\mathcal{M}_U^{-\frac{1}{2}}$.

2.7.3 Solution procedure for the SVD of the modified transfer function

Instead of solving the SVD directly we achieve that by solving an eigenvalue problem. If the SVD is expressed as: $\tilde{\mathcal{H}} = U\Sigma V^*$, the operator $\tilde{\mathcal{H}}^* \tilde{\mathcal{H}}$ can be expanded as:

$$\tilde{\mathcal{H}}^* \tilde{\mathcal{H}} = V\Sigma^* U^* U\Sigma V^* = \underbrace{V\Sigma^2 V^*}_{\text{Eigenvalue Decomposition}} \quad (2.15)$$

To solve the eigenvalue problem we use the implicitly restarted arnoldi method (IRAM) implemented in the open source package ARPACK (and PARPACK) [57], [62]. ARPACK is a matrix-free tool, in that it does not require the operator $\tilde{\mathcal{H}}^* \tilde{\mathcal{H}}$ explicitly. ARPACK instead requires the action of the operator on a vector. So if ARPACK gives us a vector v , we need to supply $w \leftarrow \tilde{\mathcal{H}}^* \tilde{\mathcal{H}} v$. We do this by solving: $\tilde{\mathcal{H}}^{-1} \tilde{\mathcal{H}}^{*-1} w = v$. This is done in two steps.

$$\begin{aligned} \text{Part A: Solve } \tilde{\mathcal{H}}^{-1} y = v \rightarrow \text{Get } y & \quad \tilde{\mathcal{H}}^{-1} = \mathcal{M}_U^{\frac{1}{2}} B^{-1} (-i\omega - \mathcal{A}) C^{-1} \mathcal{M}_U^{-\frac{1}{2}} \\ \text{Part B: Solve } \tilde{\mathcal{H}}^{*-1} w = y \rightarrow \text{Get } w & \quad \tilde{\mathcal{H}}^{*-1} = \mathcal{M}_U^{-\frac{1}{2}*} C^{*-1} (-i\omega - \mathcal{A})^* B^{*-1} \mathcal{M}_U^{\frac{1}{2}*} \end{aligned}$$

To solve these two linear systems in a parallel setup we use the open source package PETSC ([8],[9], [7]). The linear systems are solved directly by using a full LU pre-conditioner implemented in the MUMPS package ([2][3]) through PETSC.

2.8 Axisymmetric Formulation

As was discussed earlier the direct method for matrix inversion which forms the core of many of the stability analyses is extremely memory intensive. This step is the bottleneck and these high demands on memory often make realistic three-dimensional flowfields inaccessible. The only long-term solution to this problem is adopting suitable indirect methods for the linear system solve. Axisymmetric flowfields offer a unique opportunity for us to access realistic three-dimensional flowfields at the cost of a 2D analysis. Most common solvers in industry and academia are developed using a Cartesian framework. This section deals with the modifications needed to be able to equip the solver to handle axisymmetric configurations. This is done in a way that retains as much of the original code as possible. This section should be read in reference to Sec.2.6 as that will serve as a baseline.

2.8.1 Treatment of the inviscid flux

We start with the compressible Navier-Stokes equation in cylindrical co-ordinates. x is the axial direction, r is the radial direction and θ indicates the azimuthal co-ordinate.

Vectorial form (Inviscid):

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho u_x \\ \rho u_r \\ \rho u_\theta \\ E \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} \rho u_x \\ \rho u_x^2 + p \\ \rho u_x u_r \\ \rho u_x u_\theta \\ (E+p)u_x \end{pmatrix} + \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \begin{pmatrix} \rho u_r \\ \rho u_r u_x \\ \rho u_r^2 + p \\ \rho u_r u_\theta \\ (E+p)u_r \end{pmatrix} \right\} + \frac{1}{r} \frac{\partial}{\partial \theta} \begin{pmatrix} \rho u_\theta \\ \rho u_\theta u_x \\ \rho u_\theta u_r \\ \rho u_\theta^2 + p \\ (E+p)u_\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{p+\rho u_\theta^2}{r} \\ -\frac{\rho u_\theta u_r}{r} \\ 0 \end{pmatrix}$$

Multiplying by r through out:

$$\frac{\partial \mathbf{r} \vec{U}}{\partial t} + \frac{\partial}{\partial x} \mathbf{r} \vec{F}_x + \frac{\partial}{\partial r} r \vec{F}_r + \frac{\partial}{\partial \theta} \vec{F}_\theta = \mathbf{r} \mathcal{Q} \quad (2.16)$$

On applying the Green-Gauss transformation and upon discretization we get:

$$\mathbf{r}_i \Psi_i \frac{\partial U_i}{\partial t} + \sum_k (F_x \hat{n}_x + F_r \hat{n}_r) \mathbf{r}_k S_k + \Psi_i \frac{\partial}{\partial \theta} \vec{F}_\theta = \Psi_i \mathbf{r}_i \mathcal{Q}_i \quad (2.17)$$

For this framework the faces k must exist purely in the $r-x$ plane. $\Psi'_i = r_i \Psi_i$ and $S'_k = r_k S_k$ are modified volumes.

$$\Psi'_i \frac{\partial U_i}{\partial t} + \sum_k (F_x \hat{n}_x + F_r \hat{n}_r) S'_k + \Psi_i \frac{\partial}{\partial \theta} \vec{F}_\theta = \Psi'_i \mathcal{Q}_i \quad (2.18)$$

On linearizing: $U_i = q_i = \bar{q}_i + q'_i$ and applying the modal form: $q'_i = \hat{q}(x, r) e^{i(m\theta - \omega t)}$

$$\boxed{-i\omega q'_i = -\frac{1}{\Psi'_i} \sum_k (A_x \hat{n}_x + A_r \hat{n}_r) q'_k S'_k - \frac{im}{r_i} A_\theta q'_i + \frac{\partial \mathcal{Q}}{\partial q} q'_i} \quad (2.19)$$

2.8.2 Treatment of the viscous flux

Viscous fluxes in the vector form are given as (on multiplication through out by r similar to the inviscid case):

$$\begin{aligned}
\frac{\partial}{\partial t} \mathbf{r} \begin{pmatrix} \rho \\ \rho u_x \\ \rho u_r \\ \rho u_\theta \\ E \end{pmatrix} &= \mathbf{r} \begin{pmatrix} 0 \\ 0 \\ -\frac{\tau_{\theta\theta}}{r} \\ \frac{\tau_{\theta r}}{r} \\ 0 \end{pmatrix} \\
&+ \frac{\partial}{\partial x} \mathbf{r} \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xr} \\ \tau_{x\theta} \\ u_x \tau_{xx} + u_r \tau_{rx} + u_\theta \tau_{\theta x} + q_x \end{pmatrix} + \frac{\mathbf{r}}{r} \frac{\partial}{\partial r} \left\{ r \begin{pmatrix} 0 \\ \tau_{rx} \\ \tau_{rr} \\ \tau_{r\theta} \\ u_x \tau_{xr} + u_r \tau_{rr} + u_\theta \tau_{\theta r} + q_r \end{pmatrix} \right\} \\
&+ \frac{\mathbf{r}}{r} \frac{\partial}{\partial \theta} \begin{pmatrix} 0 \\ \tau_{\theta x} \\ \tau_{\theta r} \\ \tau_{\theta\theta} \\ u_x \tau_{\theta x} + u_r \tau_{\theta r} + u_\theta \tau_{\theta\theta} + q_\theta \end{pmatrix}
\end{aligned}$$

This is very similar to the cartesian form, the only differences being:

1. Extra source terms exist
2. The flux in the x-direction and r-direction has an extra r (compared to x and y).
3. The conserved vector has an extra r
4. The divergence term is modified
5. Some τ terms have other extra terms

A $\tau_{ij} = \tau_{ij}^0 + d_{ij}$. d_{ij} denoting the differences. So we have:

$$\begin{aligned}
& \frac{\partial}{\partial t} r \begin{pmatrix} \rho \\ \rho u_x \\ \rho u_r \\ \rho u_\theta \\ E \end{pmatrix} = r \begin{pmatrix} 0 \\ 0 \\ -\frac{\tau_{\theta\theta}}{r} \\ \frac{\tau_{\theta r}}{r} \\ 0 \end{pmatrix} \\
& + \frac{\partial}{\partial x} r \begin{pmatrix} 0 \\ \tau_{xx}^0 \\ \tau_{xr} \\ \tau_{x\theta} \\ u_x \tau_{xx}^0 + u_r \tau_{rx} + u_\theta \tau_{\theta x} + q_x \end{pmatrix} + \frac{\partial}{\partial r} \left\{ r \begin{pmatrix} 0 \\ \tau_{rx} \\ \tau_{rr}^0 \\ \tau_{r\theta}^0 \\ u_x \tau_{xr} + u_r \tau_{rr}^0 + u_\theta \tau_{\theta r}^0 + q_r \end{pmatrix} \right\} \\
& + \frac{\partial}{\partial \theta} \begin{pmatrix} 0 \\ \tau_{\theta x} \\ \tau_{\theta r}^0 \\ \tau_{\theta\theta}^0 \\ u_x \tau_{\theta x} + u_r \tau_{\theta r}^0 + u_\theta \tau_{\theta\theta}^0 + q_\theta \end{pmatrix} \\
& + \frac{\partial}{\partial x} r \begin{pmatrix} 0 \\ \lambda \frac{u_r}{r} \\ 0 \\ 0 \\ u_x \lambda \frac{u_r}{r} \end{pmatrix} + \frac{\partial}{\partial r} \left\{ r \begin{pmatrix} 0 \\ 0 \\ \lambda \frac{u_r}{r} \\ -\mu \frac{u_\theta}{r} \\ u_r \lambda \frac{u_r}{r} - u_\theta \mu \frac{u_\theta}{r} \end{pmatrix} \right\} \\
& + \frac{\partial}{\partial \theta} \begin{pmatrix} 0 \\ 0 \\ -\mu \frac{u_\theta}{r} \\ \lambda \frac{u_r}{r} + 2\mu \frac{u_r}{r} \\ -u_r \mu \frac{u_\theta}{r} + u_\theta \lambda \frac{u_r}{r} + 2u_\theta \mu \frac{u_r}{r} \end{pmatrix}
\end{aligned}$$

In the following formulation, θ -direction is always homogeneous and x & r may or may not be- although one of them should be resolved. There are a number of double derivative terms and some single derivative terms. Some of the terms have analogues in the Cartesian form (regular viscous terms) and some which exist in the cylindrical form but do not have an equivalent Cartesian counterpart (special viscous terms). We treat all special terms as source terms and all the regular terms through the usual viscous source terms subroutine (similar to the Cartesian form and based on a generalized term). The details can be found in Appendix.B.2.

CHAPTER

3

CODE VALIDATION

This chapter contains a suite of cases, each of which tests several different aspects of the code in terms of numerics and physics. Together they indicate that the code is fairly robust, accurate and able to handle complex configurations over a wide range of flow regimes (incompressible to hypersonic).

3.1 1D incompressible channel flow

The first test case is for an incompressible laminar channel flow. The flow is invariant in the stream-wise and spanwise directions. The Mach number is set to be 0.025 to mimic incompressible behavior. The channel height is $2h$ and the domain in the wall-normal direction is given by $y = [-1, 1]$. The baseflow velocity is given by $[U(y), 0, 0]$. The density, temperature and viscosity are considered to be constant throughout the domain. The Reynolds number is based on the channel half-height and the maximum velocity U_{max} . The velocity profile is given as $U(y) = U_{max}(1 - y^2)$. We compute the spectrum for a case with $Re = 2000$ and spatial wavenumber $\alpha = 1$. We also compute the leading singular value (σ_1) of the resolvent operator $(i\omega - \mathcal{A})^{-1}$ for a range of frequencies. $N_y = 401$ points are used in the wall normal direction. We compare our results with those obtained using the code from Schmid & Brandt [84]. The results are shown in Fig.3.1. We see decent agreement for both the spectrum and the gain.

Figure 3.1 1D incompressible channel flow: $Re = 2000$, $\alpha = 1$, $\beta = 0$

3.2 1D compressible mixing layer

The second case concerns a compressible mixing layer. A one dimensional analytical profile is prescribed. The baseflow velocity is given as $[U(y), 0, 0]$. The domain extends from $y = [-50\delta, 50\delta]$ and the velocity profile is given as $U(y) = \tanh(y/\delta)$. The density, temperature and viscosity are considered to be constant throughout the domain. The problem is solved using two different setups. In the first (local analysis) the 1D profile is prescribed on a 1D grid and the streamwise wavenumber is explicitly prescribed as a 'sourceterm'. Another approach (global analysis) uses a 2D grid, where the same velocity profile is prescribed. Here, the streamwise wavenumber is prescribed indirectly by making the domain periodic in the x-direction and enforcing a maximum allowable disturbance wavelength by picking the length of the domain to be: $L_x = 2\pi/\alpha$. The 2D grid has $N_x/\lambda = 32$ points in the streamwise direction (across one wavelength). Both the 1D and the 2D grids have $N_y = 201$ points in the wall normal direction. The spectra from the local and global analysis are shown in the Fig.3.2. They are symmetric about imaginary axis. The comparison with literature can be seen in Table.3.1.

The dominant mode-shape for the shear layer instability can be seen in Fig.3.3. This was obtained by making the streamwise domain length twelve times that of the wavelength of the dominant instability: $L_x = 12\lambda = 12(2\pi/\alpha)$.

3.3 1D compressible couette flow

We perform a global stability analysis for compressible couette flow. The effect of compressibility in couette flows, in distinction to the incompressible case, is that the velocity profile is no longer linear. The baseflow can be obtained numerically using the governing equations given by Liepmann &

Figure 3.2 Compressible mixing layer: $Re_\delta = 100$, $M = 0.5$, $\alpha = 0.397$, $\beta = 0$.

Figure 3.3 Dominant mode shape (u')

Table 3.1 Value of the dominant eigenmode

	ω_i
Present (Local)	0.127292
Mack & Schmid, 2010 (Local)	0.12715
Present (Global)	0.127294
Ohmichi & Suzuki, 2016 (Global)	0.1276

Table 3.2 Compressible Couette Flow : Dominant Modes

Case	Mode	M	Re	α	$c_r + i c_i$	$c_r + i c_i[51]$
Case 1	I	2	2×10^5	0.1	1.2140 -0.01158i	1.2140 -0.01158i
Case 1	II	2	2×10^5	0.1	-0.2916 -0.01382i	-0.2916 -0.01382i
Case 2	I	5	5×10^6	2.1	0.9729 -0.003497i	0.9729 -0.003456i
Case 2	II	5	5×10^6	2.1	0.04073 + 0.0008833i	0.04072 + 0.0008855i

Roshko [58]. We consider, a one dimensional couette flow with the bottom wall being stationary (and adiabatic), and the top wall (which is held at a constant temperature) is moved at a constant velocity in the horizontal direction. A local analysis is conducted to compute spectrum for two values of Mach numbers: $M_\infty = 2$ & $M_\infty = 5$. The modal form for disturbances: $q'(x, y, z, t) = \hat{q}(y)e^{i(\alpha x + \beta z - \omega t)}$. The spectra for the two cases is shown in Fig.3.4. The spectrum of Case 1 has a ‘Y’-shaped structure whereas the spectrum of Case 2 has an ‘H’-shaped structure. The spectrum changes from a ‘Y’-shaped one to an ‘H’-shaped one with an increase in either Re or α [51]. Two families of modes are obtained- one related to the top wall and the other related to the bottom wall. We compare the eigenvalues of the dominant modes belonging to each of these families with Hu & Zhong [51] in Table.3.2 (shown in terms of $c = \omega/\alpha$, $N_y = 301$). Sutherland’s law of the form consistent with Hu & Zhong [51] is used for viscosity. We also study the convergence of the dominant eigenvalues and the results can be seen in Fig.3.5. We vary the order of the inviscid flux (second, fourth or sixth) and the number of points in the y-direction ($N_y = 201, 301, 401$ and 801). The reference is Hu & Zhong [51]’s result for $N_y = 300$. For the same resolution the error in the prediction of eigenvalue is less than 2 %. The error in the real part is far lower than the imaginary part.

Figure 3.4 1D compressible couette flow: spectrum

Figure 3.5 Grid convergence study for the unstable mode: $M = 5$ case

3.4 1D compressible boundary layer

We perform a global stability analysis for a 1D compressible boundary layer as described by [60]. The conditions are the same as Malik [60] and range from subsonic to hypersonic conditions. The details for the cases investigated are as shown in Table.3.3. The baseflow is obtained from a similarity solution. Additionally the estimates for the unstable eigenvalue from the current analysis as well as those provided by Malik are provided. We get very good agreement with the results in literature. A stretched grid with $N_y = 201$ is used for the computation. This case demonstrates that the code works well for one or two homogeneous directions across a broad range of conditions.

Table 3.3 Global Stability Analysis for a Compressible Boundary Layer

Ma	Re	T_w / T_{adb}	$T_0(K)$	α	β	$\omega_r + i\omega_i$ (Malik)	$\omega_r + i\omega_i$
0.5	2000	1	277.778	0.1	0.0	0.02908 + 0.002244i	0.02908 + 0.002242i
2.5	3000	1	333.33	0.06	0.1	0.03673 + 0.0005847i	0.03676 + 0.0005794i
10.0	2000	0.1	2333.33	0.105	0.0	0.09748 + 0.002030i	0.09737 + 0.002027i
10.0	1000	1	2333.33	0.12	0.0	0.1159 + 0.0001529i	0.1158 + 0.0001028i

3.5 2D compressible boundary layer: transient growth of streaks

In this case we investigate a streamwise developing two dimensional boundary layer. We are interested in steady vortical disturbances which experience amplification due to transient growth in the domain and produce elongated streaks. Such streaks are often observed when flows undergo sub-critical transition and therefore are considered to be an important aspect of transition physics. The case setup used is similar to the one used by Bugeat et al. [21]. The length of the domain (flat plate) is L . The Re_x at the outflow is $Re_{x-out} = 1.2996 \times 10^6$. The location of the output is referred to as x_{opt} , is chosen such as $Re_{x-opt} = 1.0 \times 10^6$. The ratio $R = x_{in}/x_{opt}$ indicates the location of the input x_{in} relative to the output x_{opt} . The input and output locations are introduced through the operators B and C . B and C are spatial functions which have values of 1 in a region of width $0.04L$ and centered around x_{in} and x_{opt} respectively, and 0 everywhere else. The edges are made smooth using hyperbolic tangent profiles. Different norms are chosen for the inputs and outputs. The output norm is chosen to be the Chu norm [25] and this is a choice routinely employed for compressible flows. It has been observed in literature that the most optimal disturbances are streamwise vortices. With this in view the input norm contains the kinetic energy associated with the wall-normal and spanwise velocities. As a validation exercise we compare our results with [104], [21] for a case with $M_\infty = 3.0$ and $x_{in} = 0.2x_{out}$. There is a strong preference for a specific spanwise wavenumber (β_{opt}). The normalized resolvent gain ($G = \sigma_1^2$) over a range of spanwise wavenumbers is shown in Fig.3.7. As we compare with different methods of analyses we use normalized gain. The schematic and the comparison is shown in Fig.3.6 and Fig.3.7. Three different grids are tested and the results seem to be insensitive to grid resolution.

3.6 Shock dominated flows: Oblique shock boundary layer interaction

We compare our results with a case involving the interaction of an oblique shock with a boundary layer. This case was originally studied by Hildebrand et al. [49]. Consider a case of a temporal instability in a boundary layer due to shock-induced separation. This case contains an oblique shock wave at an angle $\theta = 13^\circ$ interacting with a $M_\infty = 5.92$ boundary layer. The flow admits a temporally growing instability over a range of shock angles (θ) and spanwise wavenumbers (β). The baseflow and shapes of the most unstable mode are also shown in Fig.3.8 (a) and Fig.3.8 (b),(c). The variation of the growth rate of the instability with spanwise wavenumber is given in Fig.3.8 (e). We

(a) Schematic

(b) Iso-surfaces for streamwise velocity and streamwise vorticity in the forcing plane

Figure 3.6 Transient growth of streaks

(a) Normalized gain

Figure 3.7 Transient growth of streaks

get the most unstable eigenvalue at $\beta = 0.20$. This is lower than the value of $\beta = 0.25$ of Hildebrand et al. [49]. The differences may be due to the fact that their linearization was based on $[p, \mathbf{u}, s]^T$ whereas ours is on conserved variables, the general trend however is consistent. The spectrum for $\beta = 0.20$ is shown in Fig.3.8 (d).

3.7 Shock dominated flows: Flow over a double wedge

This case was originally studied by GS et al. [47] and involves hypersonic ($M_\infty = 5$) flow over a double wedge. The first turn angle is 12° and the second turn angle is 20° . The baseflow and shapes of the most unstable mode are also shown in Fig.3.9 (a) and Fig.3.10 (b). The variation of the growth rate of the instability with spanwise wavenumber is given in Fig.3.10 (d). We get the most unstable eigenvalue at $\lambda_z = 8$. This is higher than the value at which GS et al. [47] get their most unstable eigenvalue. These trends are very sensitive to the size of the separation which in turn is very sensitive to grids and numerics. So small differences in grids between the two approaches can cause a difference in results. However, qualitatively the same trends are predicted. Another difference between the approach of GS et al. [47] and the present work is treatment of viscosity. In their work μ' has been set to zero, whereas we have linearized the viscosity and retained the μ' terms.

3.8 Trailing line vortex- inviscid instabilities

The baseflow velocity profile in cylindrical coordinates is given as:

Figure 3.8 Stability of an oblique shockwave interacting with a hypersonic boundary layer

Figure 3.9 Stability of hypersonic flow over a double wedge

$$\overline{U}_x = e^{-r^2} \quad \overline{U}_r = 0.0 \quad \overline{U}_\theta = \frac{q_s}{r}(1 - e^{-r^2})$$

The axial direction is aligned along the vortex core. q_s is the swirl parameter. In this inviscid case taken from [63]: $q_s = -0.5$, $m = 1$ and $\alpha = 0.5$. Three unstable modes are detected

Table 3.4 Trailing line vortex: inviscid analysis

Grid	Mode	ω_i	ω_r	%Error(ω_i)	%Error(ω_r)
201	I	0.20254400595987165	0.049760800049553390	0.0415	0.0848
401	I	0.20260445098301505	0.049729227330108275	0.0117	0.0213
801	I	0.20262074956678677	0.04972053093521056	0.0036	0.0038
Exact	I	0.2026281012942	0.0497186499174	-	-
201	II	0.10568439146131525	-0.016096947922266774	0.1312	0.6906
401	II	0.10578769281317513	-0.016180208901952181	0.0335	0.1769
801	II	0.10581387308760326	-0.016201669018182041	0.0088	0.0445
Exact	II	0.10582318954	-0.01620888889	-	-
201	III	0.059895849581601546	-0.029333413459780258	0.2628	0.6926
401	III	0.060013544879797598	-0.029486224372714264	0.0668	0.1753
801	III	0.060043388843209367	-0.029524944201077381	0.0171	0.0442
Exact	III	0.060053650	-0.029538003	-	-

It can be seen that in the case of each of the three unstable modes there is good agreement and the error for all three grids is significantly low (less than 1%). In general the errors for the frequency are higher than the errors for the growth rate. The errors also increase with mode number- the mode

(a) Unstable Mode ($\lambda_z = 8$) - $\hat{u}$ component

(b) Unstable Mode ($\lambda_z = 8$) - $\hat{w}$ component

(c) Spectrum ($\lambda_z = 8$)

(d) Unstable eigenvalue: effect of λ_z

Figure 3.10 Stability of hypersonic flow over a double wedge

structures are more complicated for the higher mode numbers. The mode structures for the three modes are shown in Fig.3.11.

Figure 3.11 Mode shapes for modes 1,2 and 3

3.8.1 Trailing line vortex- viscous instabilities

A similar viscous analysis is conducted here. This case is taken from Khorrami [55]. The baseflow is given as:

$$\overline{U}_x = 1 + e^{-r^2} \quad \overline{U}_r = 0.0 \quad \overline{U}_\theta = \frac{q_s}{r}(1 - e^{-r^2})$$

$$q_s = 0.8 \quad m = -2 \quad \alpha = 1.25$$

The analysis carried out for Reynolds numbers ranging from 10^2 to 10^5 . Two instabilities are tracked across different Re and their values are compared in Fig.3.12. The mode shapes for Re 10^3 and Re 10^5 are shown in Fig.3.13.

Figure 3.12 Mode shapes for modes 1 and 2: Viscous analysis

Figure 3.13 Mode shapes for modes 1 and 2: Viscous analysis

CHAPTER

4

APPLICATION TO TURBULENT FLOWS

4.1 Introduction

The traditional view of stability analysis is to assess the proclivity of a stable system to become unstable. In the context of fluid mechanics this stable state is the ‘steady laminar state’. Stability analysis techniques employ a linearization about this base state (laminar steady state) to analyze the stability of the system. A natural extension to this is the possibility of using a time averaged mean flow instead of a laminar base state to linearize about. While traditionally, linearizing about a laminar base state is well understood and well defined, the interpretation of linearizing about a time averaged mean flow (not a solution of the underlying equations) is not fully understood. However, these methods provide rich information about the underlying dynamical and structural features of the system. These methods and ideas have been applied to turbulent flows (time-averaged meanflow profile), albeit with different interpretations. The utility of this approach has been realized in a number of different contexts. McKeon & Sharma [66] used the linearization about the mean flow to interpret the self-sustaining turbulent processes as a closed loop forced dynamical system, where the forcing was produced by the non-linear term in the Navier-Stokes operator. Zare et al. [101] used the framework of LNS to compute second-order statistics if a subset of second-order statistics is available. While these works have addressed parallel and incompressible flows, the linearization about the mean flow has been used in the case of non-parallel compressible flows in several studies involving jets[52] [85], shock boundary layer interactions[72], cavity flows[59] and flows over airfoils[100].

Barkley [10], compared the effects of linearizing about the laminar steady state solution and the mean flow. They observed that when linearized about a stable steady state solution, they were

able to predict the critical value of the Reynolds number for the onset of unsteadiness. However, for Reynolds numbers beyond the critical limit there was a mismatch between the frequency of the dominant eigenvalue predicted by the stability analysis and the observed vortex shedding frequency. However if the operator was linearized about the mean flow, they were able to predict the frequency of vortex shedding far more accurately. Moreover, this eigenvalue was non-growing - which is consistent with the flow physics at those Reynolds numbers. Coherent turbulent structures in wall bounded turbulent flows have been a well studied phenomenon. Smits et al. [86] name four types of turbulent structures: near wall streaks, hairpin structures, large scale motions (LSMs) and very large scale motions (VLSMs). A relatively fewer number of studies have been focused specifically on the effects of compressibility on these structures. Wei & Pollard [98] showed that for compressible turbulent channel flow spanwise spacing and streamwise lengths of the near wall streaks both increase with an increase in Mach number. Brun et al. [20] say that although the streak size appears to change under the incompressible inner scaling, it is not an effect of Mach number but an effect of changing viscosity and would not be seen with an appropriate scaling. Morinishi et al. [68] showed that the spanwise scaling was the same as the incompressible case when semi-local scaling was used. For turbulent boundary layers Duan et al. [33] showed that the streak spacing slightly reduced with an increase in Mach number. Duan et al. [32] showed that with an increase in wall cooling the near wall streaks became more coherent in the streamwise direction and their spanwise spacing increases. They also showed that with an increase in wall cooling the hairpin packets are more organized and move slower.

We examined a canonical compressible turbulent flow (compressible turbulent channel flow) with the tools and methods of analysis more frequently used for incompressible flows in the past [14]. For compressible flows, two additional parameters emerge. One is compressibility which is quantified by the Mach number. It has been observed in many studies that turbulence statistics at different Mach numbers collapse when appropriately scaled. The other important parameter is the degree of wall cooling or the ratio of the wall temperature to the center-line temperature. A qualitative analysis of the coherent structures observed in compressible wall-bounded turbulent flows and the trends observed as a result of variation in the relevant parameters has been given here.

4.2 Mathematical Interpretations

In this section, the different interpretations of the linearized dynamics represented by the time-averaged mean flow of an unsteady flow proposed and adopted by different researchers are summarized and discussed. These different methods of analyses and interpretations are nevertheless closely connected to each other. The common themes that run through them and other closely related issues are discussed.

4.2.1 Resolvent Analysis

The resolvent operator can be used to describe the dynamics of statistically stationary turbulent processes by representing the dynamics as a forced closed loop linear dynamical system [66]. According to this view, a turbulent system can be described as forced linear dynamical system, linearized about the meanflow and with the forcing provided by the non-linear terms. The disturbance evolution equation with forcing (Eq.1.7) is given as:

$$q' = \mathcal{A}(\bar{q})q' + f'$$

The forcing in this case, f' is provided by the non-linear terms. For an incompressible flow this can be given as: $-\frac{\partial}{\partial x_j} (u'_i u'_j - \overline{u'_i u'_j})$, where $\bar{q}$ is the mean flow. This mean flow can only be 'maintained' by the 'correct' Reynolds' stress terms: $\overline{u'_i u'_j}$. Thus the disturbance evolution equation and the mean flow equation are coupled and must therefore be consistent with each other. The disturbance equation can be Fourier decomposed and then a single frequency component of the disturbance evolution equation may be written as:

$$\hat{q} = (-i\omega - \mathcal{A})^{-1} \hat{f} \quad (4.1)$$

The forcing $\hat{f}$ is an amplitude at a certain frequency ω . The evolution of $\hat{q}$ at the frequency ω , under the action of the forcing $\hat{f} e^{-i\omega t}$ can be represented by linear dynamics through the operator: $\mathcal{R}(\omega) = (-i\omega - \mathcal{A})^{-1}$. Although these dynamics are linear, the forcing itself is a non-linear term, given by the non-linearity: $u'_i u'_j$ (incompressible flows). Thus the forcing is built using a non-linear interaction of responses at different frequencies. Thus a combination of all possible non-linear interactions is responsible for a certain $\hat{f} e^{-i\omega t}$. An SVD of the resolvent operator yields an orthogonal basis for $\hat{f}$ and $\hat{q}$ ranked by the degree to which the resolvent operator amplifies the forcing along each of those directions. The full dynamical picture may be represented as a summation. Here, the index k indicates a particular frequency, and the index j indicates a particular singular vector obtained for a certain frequency.

$$q'(t) = \sum_{k=-\infty}^{\infty} \sum_{j=1}^N W_{k,j} \sigma_{k,j} \hat{q}_{k,j} \quad (4.2)$$

As long as the mean flow is available, $\sigma_{k,j}$ and $\hat{q}_{k,j}$ are easily accessible. The coefficients $W_{k,j}$, however are not. These coefficients can be given by: $W_{k,j} = \langle \mathbf{f}_k, \hat{f}_{k,j} \rangle$. $\mathbf{f}_k$ is the component of the forcing associated with the frequency ω_k . In cases where, $\sigma_{k,1}$ is much larger than other $\sigma_{k,j \neq 1}$, it can be argued that it will tend to dominate the system response at that frequency ω_k . But this is only true, if the coefficients $W_{k,j}$ are more or less equal. If one were to assume, that all (for k and j) $W_{k,j}$ are almost equal, or in other words the forcing does not have any preferential direction or

frequency, then the system response could be adequately represented as:

$$q'(t) \sim \sum_{k=-\infty}^{\infty} \sum_{j=1}^N \sigma_{k,j} \hat{q}_{k,j} \quad (4.3)$$

This, however is not true, and the missing element in this analysis is how to obtain $W_{k,j}$. This math is used to represent an unsteady signal associated with a statistically stationary turbulent process. For a statistically stationary signal the frequencies (ω) of these waves are real, thus ensuring that these individual waves are not energetically amplifying or decaying.

4.2.1.1 The relevance of the resolvent operator:

The incompressible Navier-Stokes equations are given by:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} [u_i u_j] + \frac{1}{\rho} \frac{\partial p}{\partial x_i} = \frac{\mu}{\rho} \frac{\partial^2 u_i}{\partial x_j^2} \quad (4.4)$$

In case of self-sustaining turbulent processes, it may be possible to decompose the full velocity and pressure field into it's Fourier modes(non-growing).

$$u_i = \sum_{m=-\infty}^{\infty} \tilde{u}_{im} e^{i\omega_m t} \quad u_j = \sum_{n=-\infty}^{\infty} \tilde{u}_{jn} e^{i\omega_n t} \quad p = \sum_{r=-\infty}^{\infty} \tilde{p}_r e^{i\omega_r t} \quad (4.5)$$

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{m=-\infty}^{\infty} \tilde{u}_{im} e^{i\omega_m t} + \frac{\partial}{\partial x_j} \left[\sum_{m=-\infty}^{\infty} \tilde{u}_{im} e^{i\omega_m t} \sum_{n=-\infty}^{\infty} \tilde{u}_{jn} e^{i\omega_n t} \right] \\ + \frac{1}{\rho} \frac{\partial}{\partial x_i} \sum_{r=-\infty}^{\infty} \tilde{p}_r e^{i\omega_r t} = \frac{\mu}{\rho} \frac{\partial^2}{\partial x_j^2} \sum_{m=-\infty}^{\infty} \tilde{u}_{im} e^{i\omega_m t} \end{aligned} \quad (4.6)$$

Isolating components for some value of angular frequency ω ,

$$\frac{\partial}{\partial t} \tilde{u}_i e^{i\omega t} + \frac{\partial}{\partial x_j} \left[\sum_{m=-\infty}^{\infty} \tilde{u}_{im} e^{i\omega_m t} \underbrace{\tilde{u}_{jn} e^{i\omega_n t}}_{\omega_n = \omega - \omega_m} \right] + \frac{1}{\rho} \frac{\partial}{\partial x_i} \tilde{p} e^{i\omega t} = \frac{\mu}{\rho} \frac{\partial^2}{\partial x_j^2} \tilde{u}_i e^{i\omega t} \quad (4.7)$$

The $\omega = 0$ terms in this Fourier expansion can now be seen as the 'DC component' which in our case would indicate that this is in fact the mean flow component.

$$\bar{u}_i = \tilde{u}_{i0} e^{i(0)t} \quad \bar{u}_j = \tilde{u}_{j0} e^{i(0)t} \quad \bar{p} = \tilde{p} e^{i(0)t} \quad (4.8)$$

Considering this we could expect to recover the mean flow equations by examining Eq.4.7 for $\omega = 0$.

$$\frac{\partial}{\partial t} \bar{u}_i + \frac{\partial}{\partial x_j} \underbrace{\left[\bar{u}_{i0} e^{i(0)t} \underbrace{\bar{u}_{j0} e^{i(0)t}}_{\omega_n=0=\omega_m} \right]}_{\text{Mean Flow terms: } \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j}} + \frac{\partial}{\partial x_j} \underbrace{\left[\sum_{m=-\infty}^{\infty} \bar{u}_{im} e^{i\omega_m t} \underbrace{\bar{u}_{jn} e^{i\omega_n t}}_{\omega_n=-\omega_m} \right]}_{\text{Reynolds' stress terms: } \frac{\partial \bar{u}'_i \bar{u}'_j}{\partial x_j}} + \frac{1}{\rho} \frac{\partial}{\partial x_i} \bar{p} = \frac{\mu}{\rho} \frac{\partial^2}{\partial x_j^2} \bar{u}_i \quad (4.9)$$

Similarly we can write the evolution equation for a single Fourier component for a non-zero frequency.

$$\frac{\partial}{\partial t} \bar{u}_i e^{i\omega t} + \frac{\partial}{\partial x_j} \left[\sum_{m=-\infty}^{\infty} \bar{u}_{im} e^{i\omega_m t} \underbrace{\bar{u}_{jn} e^{i\omega_n t}}_{\omega_n=\omega-\omega_m, \omega \neq 0} \right] + \frac{1}{\rho} \frac{\partial}{\partial x_i} \bar{p} e^{i\omega t} = \frac{\mu}{\rho} \frac{\partial^2}{\partial x_j^2} \bar{u}_i e^{i\omega t} \quad (4.10)$$

The Fourier expansion of the non-linear term contains physically distinct set of terms, and can be outlined as follows.

- Scenario 1 : $\omega_m = \omega, \omega_n = 0$

$$\frac{\partial}{\partial x_j} \left[\sum_{m=-\infty}^{\infty} \bar{u}_{im} e^{i\omega_m t} \underbrace{\bar{u}_{jn} e^{i\omega_n t}}_{\omega_n=\omega-\omega_m, \omega \neq 0} \right] = \frac{\partial}{\partial x_j} [\bar{u}_i e^{i\omega t} \bar{u}_j] \quad (4.11)$$

- Scenario 2 : $\omega_m = 0, \omega_n = \omega$

$$\frac{\partial}{\partial x_j} \left[\sum_{m=-\infty}^{\infty} \bar{u}_{im} e^{i\omega_m t} \underbrace{\bar{u}_{jn} e^{i\omega_n t}}_{\omega_n=\omega-\omega_m, \omega \neq 0} \right] = \frac{\partial}{\partial x_j} [\bar{u}_i \bar{u}_j e^{i\omega t}] \quad (4.12)$$

- Scenario 3 : $\omega_m \neq \omega \neq 0, \omega_n \neq \omega \neq 0$

$$\frac{\partial}{\partial x_j} \left[\sum_{m=-\infty}^{\infty} \bar{u}_{im} e^{i\omega_m t} \underbrace{\bar{u}_{jn} e^{i\omega_n t}}_{\omega_n=\omega-\omega_m, \omega \neq 0} \right] = \frac{\partial}{\partial x_j} \left[\underbrace{\sum_{m=-\infty}^{\infty} \bar{u}_{im} e^{i\omega_m t} \bar{u}_{jn} e^{i\omega_n t}}_{\omega_n=\omega-\omega_m, \omega \neq 0, \omega_m \neq 0, \omega_n \neq 0} \right] \quad (4.13)$$

Assembling these three scenarios, the equation for any disturbance evolution equation with a certain frequency $\omega \neq 0$ can be given as Eq.4.14.

$$\begin{aligned}
& \frac{\partial}{\partial t} \widetilde{u}_i e^{i\omega t} + \frac{\partial}{\partial x_j} [\widetilde{u}_i e^{i\omega t} \overline{u}_j] + \frac{\partial}{\partial x_j} [\overline{u}_i \widetilde{u}_j e^{i\omega t}] \\
& + \frac{\partial}{\partial x_j} \left[\underbrace{\sum_{m=-\infty}^{\infty} \widetilde{u}_{im} e^{i\omega_m t} \overline{u}_{jn} e^{i\omega_n t}}_{\omega_n = \omega - \omega_m, \omega \neq 0, \omega_m \neq 0, \omega_n \neq 0} \right] + \frac{1}{\rho} \frac{\partial}{\partial x_i} \widetilde{p} e^{i\omega t} = \frac{\mu}{\rho} \frac{\partial^2}{\partial x_j^2} \widetilde{u}_i e^{i\omega t}
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
& \frac{\partial}{\partial t} \widetilde{u}_i e^{i\omega t} + \frac{\partial}{\partial x_j} [\widetilde{u}_i e^{i\omega t} \overline{u}_j] + \frac{\partial}{\partial x_j} [\overline{u}_i \widetilde{u}_j e^{i\omega t}] + \frac{1}{\rho} \frac{\partial}{\partial x_i} \widetilde{p} e^{i\omega t} - \frac{\mu}{\rho} \frac{\partial^2}{\partial x_j^2} \widetilde{u}_i e^{i\omega t} \\
& = -\frac{\partial}{\partial x_j} \left[\underbrace{\sum_{m=-\infty}^{\infty} \widetilde{u}_{im} e^{i\omega_m t} \overline{u}_{jn} e^{i\omega_n t}}_{\omega_n = \omega - \omega_m, \omega \neq 0, \omega_m \neq 0, \omega_n \neq 0} \right]
\end{aligned} \tag{4.15}$$

Thus a coupled set of equations in the frequency domain can be written as shown in Eq.4.15. There is an additional consistency condition[65], wherein the dynamics of these perturbations must produce the ‘correct’ Reynolds’ stress such that the $\omega = 0$ equation(Eq.4.9) is satisfied. Thus collectively we have a set of equations which will admit a self-sustaining turbulent process as a solution. The terms on the right hand side of the disturbance equations are forcing terms which depend on the evolution of disturbances at other frequencies(or more specifically a triadically consistent subset of a combination of all other frequencies [65]. The set of all terms can be cast in terms of a forced dynamical system as indicated in Eq.4.16 and shown in Eq.4.17.

$$\begin{aligned}
& \overbrace{\frac{\partial}{\partial t} \widetilde{u}_i e^{i\omega t}}^{i\omega \hat{q} e^{i\omega t}} \\
& + \overbrace{\frac{\partial}{\partial x_j} [\widetilde{u}_i e^{i\omega t} \overline{u}_j] + \frac{\partial}{\partial x_j} [\overline{u}_i \widetilde{u}_j e^{i\omega t}] + \frac{1}{\rho} \frac{\partial}{\partial x_i} \widetilde{p} e^{i\omega t} - \frac{\mu}{\rho} \frac{\partial^2}{\partial x_j^2} \widetilde{u}_i e^{i\omega t}}^{-A(\overline{q}) \hat{q} e^{i\omega t}} \\
& = \overbrace{\left[\underbrace{\sum_{m=-\infty}^{\infty} \widetilde{u}_{im} e^{i\omega_m t} \overline{u}_{jn} e^{i\omega_n t}}_{\omega_n = \omega - \omega_m, \omega \neq 0, \omega_m \neq 0, \omega_n \neq 0} \right]}^{\hat{f} e^{i\omega t}} \\
& = -\frac{\partial}{\partial x_j} \left[\underbrace{\sum_{m=-\infty}^{\infty} \widetilde{u}_{im} e^{i\omega_m t} \overline{u}_{jn} e^{i\omega_n t}}_{\omega_n = \omega - \omega_m, \omega \neq 0, \omega_m \neq 0, \omega_n \neq 0} \right]
\end{aligned} \tag{4.16}$$

$$i\omega \hat{q} = A(\overline{q}) \hat{q} + \hat{f} \tag{4.17}$$

$$\hat{q} = (i\omega - A(\overline{q}))^{-1} \hat{f} \tag{4.18}$$

Thus a set of resolvent equations, such as the one given in Eq.4.18, across all frequencies collectively contains the description of dynamics of these self-sustaining turbulent processes (provided the consistency with Eq.4.9 is obeyed). Implicit in the statement that we are studying a subset of turbulent processes (the ones which have a time-invariant mean), is an assumption that the dynamics of these processes won't be affected by the initial state of a disturbance as it would eventually decay and not affect the dynamics (or more specifically the statistics thereof). Although the above equations describe incompressible Navier-Stokes equations, the same ideas can be carried over to compressible flows with the difference that, the non-linearities in that case won't be restricted to only the advective term, and would also be present in the viscous terms (due to the linearization of viscosity) and in the energy equations.

4.2.2 Stochastically excited Linearized Navier Stokes (LNS)

The resolvent approach examines the behavior of a system under harmonic forcing. It is also possible to examine the behavior of a system under stochastic forcing. Consider, the system:

$$q' = \mathcal{A}(\bar{q})q' + f' \quad (4.19)$$

The disturbance q' under the action of a stochastic forcing is also stochastic in nature. The second order statistics for the disturbance q' can be expressed through the co-variance matrix $\mathcal{Q}$.

$$\mathcal{Q} = \lim_{t \rightarrow \infty} \langle q' q'^* \rangle \quad (4.20)$$

Consider the stochastic forcing f' is zero-mean and white in time with co-variance $\mathcal{W} = \mathcal{W}^*$, and $\langle d(t_1) d^*(t_2) \rangle = \mathcal{W} \delta(t_1 - t_2)$. If the dynamics represented by $\mathcal{A}$ are stable then, given $\mathcal{W}$, the co-variance matrix $\mathcal{Q}$ can be computed as a solution to the Lyapunov equation:

$$\mathcal{A} \mathcal{Q} + \mathcal{Q} \mathcal{A}^* = -\mathcal{W} \quad (4.21)$$

This basically relates the forcing statistics to system statistics (second-order statistics). Zare et al. [101], in a more general case, consider the case where the stochastic forcing is colored in time. The problem essentially becomes to correctly identify the forcing statistics to obtain the correct second-order statistics (state co-variance matrix: $\mathcal{Q}$). The reader is referred to Zare et al. [101] for more details on this approach.

4.2.3 Assumptions underlying resolvent and Lyapunov analyses

The resolvent analysis relies on the assumption that the interaction coefficients or the inner products of forcing with singular forcing vectors are more or less equal. In other words, the forcing has no

particular preference in some direction. In the case of a Lyapunov-like analysis the forcing statistics are modeled and the response statistics are obtained as a function of the modeled forcing statistics and the linear dynamics of the system. Morra et al. [69] state that the assumptions underlying a white-in-time uncorrelated forcing (stochastic forcing) and resolvent analysis (deterministic forcing) are the same. They show this by constructing velocity power cross-spectral density from the average of Fourier transform of the velocity field (their notation is followed in this section):

$$S = \langle \tilde{u}(\alpha, y, \beta, \omega) \tilde{u}^*(\alpha, y', \beta, \omega) \rangle \quad P = \langle \tilde{f}(\alpha, y, \beta, \omega) \tilde{f}^*(\alpha, y', \beta, \omega) \rangle \quad (4.22)$$

$$S^{est} = \mathcal{H} P \mathcal{H}^* \quad (4.23)$$

If $\tilde{f}$ is considered to be δ -correlated in y with, decorrelated components: $P = p I$

$$S^{res} = p \mathcal{H} \mathcal{H}^* \quad (4.24)$$

Morra et al. [69] state that this assumption is the same as the one that underlies the resolvent analysis. However, if $P = p I$ (uncorrelated in y , white in time stochastic forcing), then the singular response vectors of the resolvent vector are the same as the eigenmodes of the covariance matrix.

$$S^{res} = p(U \Sigma V^*)(U \Sigma V)^* = p(U \Sigma V^* V \Sigma^* U^*) = p U \Sigma^2 U^* \quad (4.25)$$

Thus for this assumption of white noise forcing we have a special result:

$$q_k q_k^* = Q = p U \Sigma^2 U^* \quad (4.26)$$

$$q_k q_k^* = \mathcal{Q} = p \sum_{j=1}^N \sigma_j^2 \hat{q}_j \hat{q}_j^* \quad (4.27)$$

4.2.4 Eddy viscosity modification

Another approach that has been adopted by some researchers is to use an eddy-viscosity model to model a part of the non-linear forcing as a linear term with a modification of the linear operator [69]. This seems to yield better results as a part of the forcing is modeled with a physically reasonable model and so the demands on the forcing statistics model to ‘get it right’ are somewhat relaxed. To see which terms are modelled using eddy viscosity a triple decomposition is used[79]. The forcing is given as (the second term is modeled using eddy viscosity):

$$f = \frac{\partial}{\partial x_j} (\overline{\tilde{u}_i \tilde{u}_j} - \tilde{u}_i \tilde{u}_j) - \frac{\partial}{\partial x_j} (\langle u'_i u'_j \rangle - \overline{u'_i u'_j}) \quad (4.28)$$

4.2.5 Linearizing about a mean flow

Beneddine [13] refers to the analysis of the base flow and the mean flow in the context of two aspects of stability analysis- the ‘prediction of the onset of unsteadiness’ and ‘prediction of dynamics’. While the laminar base flow analysis is a more rigorous analysis, there is some evidence that the for the latter (prediction of dynamics) the mean flow analysis is in some cases the more appropriate choice. Barkley [10], compared the effects of linearizing about the laminar steady state solution and the mean flow for a flow over a cylinder. They observed that away from the critical point, if the operator was linearized about the meanflow, they were able to predict the frequency of vortex shedding far more accurately as shown in Fig.4.1. Moreover, this eigenvalue was a non-growing one- which is consistent with the observed behavior at these Reynolds numbers. The phenomenon of the existence of such eigenvalues whose real part is zero and the imaginary part is equal to a frequency in the non-linear flow is termed as RZIF (real part:zero, imaginary part :frequency) [95]. It is to be noted that in some literature the imaginary part is associated with amplitude and the real part with the frequency.

Figure 4.1 Linear Stability Analysis of a Flow over a cylinder (a) thick line: vortex shedding frequency, triangles: mean flow stability analysis, thin curve: base flow stability analysis (b) thin curve: baseflow stability analysis, triangles: mean flow stability analysis - picture from Barkley [10]

There is also a practical reason for using the mean flow analysis approach as it is not as restrictive as the baseflow analysis. The baseflow analysis is more restrictive as it is difficult to obtain a steady laminar state for all flows. In such cases a laminar base state may be obtained by suppressing some growing perturbations (for example: selective frequency damping proposed by: Akervik et al. [1]).

4.3 Formulation

The disturbance evolution equation (Eq.1.7) describes the dynamics of unsteady turbulent flows. This can also be recast as a forced dynamical system, with the resolvent operator $\mathcal{H} = (-i\omega - \mathcal{A}(\bar{q}))^{-1}$ converting the forcings into responses. The term f' is considered to be the forcing provided by the non-linear forcing. The system for a certain frequency can be written as:

$$q' = \hat{q} e^{-i\omega t} \quad f' = \hat{f} e^{-i\omega t}$$

$$\hat{q} = \mathcal{H} \hat{f} \quad (4.29)$$

A singular value decomposition of this resolvent operator yields the forcing directions which will experience high amplifications. The energy amplification is defined in the context of a particular definition of energy. For compressible flows, a measure of the disturbance energy contained in the state vector can be given by the norm proposed by Chu[25] as shown in Sec.2.7.1. The modified transfer function as a result is: $\tilde{\mathcal{H}} = \mathcal{M}_U^{\frac{1}{2}} \mathcal{H} \mathcal{M}_U^{-\frac{1}{2}}$. The singular value decomposition is expressed as:

$$\tilde{\mathcal{H}} \tilde{V} = \tilde{U} \Sigma \quad (4.30)$$

$$\hat{f} = \mathcal{M}_U^{-\frac{1}{2}} \tilde{f} \quad \hat{q} = \mathcal{M}_U^{-\frac{1}{2}} \tilde{q} \quad (4.31)$$

Thus if we consider a self-sustaining turbulent system to be a closed loop forced dynamical system, with forcing provided by the non-linearity in the system as proposed by McKeon & Sharma [66], then the singular vectors of the resolvent operator represent the basis of expansion for the forcing and response ordered by the degree of amplification given by the singular value. The idea is that in the absence of a strong bias in the forcing in favor of a particular direction the response vectors with high singular values will tend to dominate the total system response.

To analyze the resolvent operator a number of metrics could be used. For example,

1. $\Psi = \frac{\sigma_1^2 + \sigma_2^2}{\sum_j \sigma_j^2}$
2. $\|H\|_\infty(\alpha, \beta) = \max_\omega \sigma_1(\alpha, \beta, \omega)$
3. $\|H\|_2^2(\alpha, \beta) = \frac{1}{2\pi} \int \sum_j \sigma_j^2(\alpha, \beta, \omega) d\omega$
4. Visualization of mode shapes

The reason Ψ is significant is if the first (and second) singular values are extremely large compared to the others, it is likely that the system response for that frequency would be dominated by that leading response function. In fig.4.2 Moarref et al. [67] show that the contours of this ratio of singular values bears a striking resemblance with turbulent energy contours from DNS. The frequencies ω for each of these points is picked such that the phase speed ($c = \omega/\alpha$) matches the mean flow speed at some particular wall normal location $\bar{U}(y_0)$.

Jiménez [53] shows that at a particular wall normal location a large amount of structures travel with a phase speed equal to the local mean velocity at that height. It has been shown that the resolvent operator gives a high response in scenarios when the phase-speed ($c = \omega/\alpha$) is equal to the local mean velocity $U(y)$ [66]. Moarref et al. [67] showed that when they spanned the $\alpha - \beta$

Figure 4.2 The ratio of singular values ($\frac{\sigma_1^2 + \sigma_2^2}{\sum_{j=1}^{20} \sigma_j^2}$) along with turbulent kinetic energy contours (black lines) from DNS[50] -picture taken from Moarref et al. [67]

space by selecting frequencies such that the phase speed (c) is equal to mean velocity ($U(y)$) at certain locations such as $15y^+$ (near wall cycle), $100y^+$ and 0.2δ , the wavenumbers for which the the factor $\frac{\sigma_1^2 + \sigma_2^2}{\sum_j \sigma_j^2}$ was high, lined up with the contours of energetic wavenumbers from DNS. We wish to examine how this extends to compressible flows and what are the similarities and differences with incompressible flows. In addition to finding differences and trends it would be important to see if these also manifest in some form in other studies (DNS or experiments).

To analyze the resolvent operator one can examine the norms of the resolvent operator. Two such norms are: $\|H\|_\infty$ and $\|H\|_2$. The max gain ($\|H\|_\infty$) is important because it gives a ‘global’ picture—typically selecting the streamwise elongated ($\alpha \approx 0$) structures as the most amplified structures and can also be interpreted as the ‘worst case scenario’ with regard to maximum amplification[54]. $\|H\|_2^2$ is relevant in the context of stochastic forcing and can be interpreted as the ensemble average energy density[36],[54].

4.4 Effects of Compressibility

We wish to study the way in which compressibility affects the coherent structures in wall-bounded turbulent flows. It has been shown that the regions of the $\alpha - \beta$ space where the Ψ is high, lines up well with the turbulent kinetic energy spectrum from DNS in the $\alpha - \beta$ space[67]. Ψ along with the gain for each $\alpha - \beta$ pair will be used to illustrate the effects of compressibility. For this work we would define this ratio Ψ as the sum of the squares of the first two singular values to the sum of the first twenty singular values ($\Psi = \frac{\sigma_1^2 + \sigma_2^2}{\sum_{j=1}^{20} \sigma_j^2}$).

The mean flow profiles required for this analysis were obtained from the compressible channel flow data published by Ulerich [96]. Relevant parameters such as Re_b , Ma_b , Tc/Tw , and Re_τ are shown in Table.4.1.

Table 4.1 Turbulent base flows - DNS data from Ulerich [96]

Case	Ma_b	Re_τ	Re_b	T_c/T_w
A	0.1	191	3000	1.002
B	0.5	194	3000	1.043
C	1.5	222	3000	1.391
D	3.0	297	3000	2.666
E	0.1	298	5000	1.002
F	0.5	303	5000	1.042
G	1.5	349	5000	1.385
H	3.0	480	5000	2.598

4.4.1 Near Wall Region:

Fig.4.3 and Fig.4.4 represents the effect of Mach number on the low-rank factor for bulk Reynolds numbers 3000 and 5000 respectively. It can be seen that there is an appreciable difference in Ψ for the resolvent operator with a change in Mach number for a constant bulk Reynolds number. It is to be noted that the friction Reynolds number also increases with the Mach number. In order to attribute the differences to compressibility, the Reynolds number effects must be identified. We also perform a similar analysis of the four Mach numbers at a higher bulk Reynolds number (5000), such that at each of these Mach numbers we will have another result with a higher friction Reynolds number. Comparing Fig.4.3 and Fig.4.4 it is safe to say that in the near wall region, when appropriately scaled Ψ is fairly similar at different Reynolds numbers (bulk and friction) as long as the Mach number remains the same. Thus these large scale differences can be attributed to compressibility. We also observe that there is a local peak in the Ψ plots, corresponding to the near wall cycle ($\lambda_z^+ \approx 100-200$). It can be seen that along with this local maxima, Ψ is also high for highly elongated structures for a range of specific spanwise lengthscales. These seem to be related to the $\alpha \approx 0$ structures that have been found to exhibit a very high potential for growth/gain in a number of studies involving linear analysis of wall-bounded shear flows. Although these structures are found in many linear studies they haven't been observed in real turbulent flows. Very elongated and energetically important structures have been observed in turbulent flows, but nevertheless they were of a finite length-typically an order of magnitude higher than the boundary layer thickness (δ or h).

The sizes of the structures corresponding to the local peak associated with the near wall cycle in inner and outer lengthscales have been provided in Table.4.2. It can be seen that the spanwise spacing for the near wall cycle increases with Mach number and the trend is similar for both $Re_b = 3000$ and $Re_b = 5000$, with the streak spacing being slightly larger (in inner units) for the $Re_b = 3000$ case.

For channel flows an increase in Mach number is associated with an increase in wall cooling as well. Thus it is difficult to say if the increase in spanwise spacing is a result of wall cooling (quantified using the ratio of the centerline temperature to the wall temperature, T_c/T_w) or compressibility. The variation of the spanwise spacing with both can be seen in Fig.4.5. It can be seen that the relationship is almost linear with respect to T_c/T_w .

(a) $M = 0.1, Re_\tau = 191$

(b) $M = 0.5, Re_\tau = 194$

(c) $M = 1.5, Re_\tau = 222$

(d) $M = 3.0, Re_\tau = 297$

Figure 4.3 The effect of Mach number at $Re_b = 3000$, $c = U(15y+)$ on $\frac{\sigma_1^2 + \sigma_2^2}{\sum_{j=1}^{20} \sigma_j^2}$

(a) $M = 0.1, Re_\tau = 298$

(b) $M = 0.5, Re_\tau = 303$

(c) $M = 1.5, Re_\tau = 349$

(d) $M = 3.0, Re_\tau = 480$

Figure 4.4 The effect of Mach number at $Re_b = 5000$, $c = U(15y+)$ on $\frac{\sigma_1^2 + \sigma_2^2}{\sum_j^{20} \sigma_j^2}$

Table 4.2 The lengthscales associated with the local maxima (the same according to outer scaling presented in the parentheses)

Case	λ_{1x}^+	λ_{1z}^+
A	976.629 (5.11324)	127.44 (0.6672)
B	976.629 (5.0341)	127.44 (0.6569)
C	1071.35 (4.82589)	146.423 (0.6596)
D	1701.91 (5.73032)	212.038 (0.7140)
E	976.63 (3.28)	127.44 (0.4276)
F	976.63 (3.22)	127.44 (0.4206)
G	1022.89 (2.93)	139.8 (0.4006)
H	1551.44 (3.23)	202.45 (0.4218)

Figure 4.5 The variation in spanwise spacing for near-wall streaks

Another feature of this near wall region is the presence of vertical bands representing spanwise elongated structures restricted to a range of streamwise lengthscales ($30\lambda_x^+$ - $3000\lambda_x^+$). The band becomes more pronounced at higher Mach numbers. At Mach 3, the band takes the form of many narrow equispaced bands.

These bands appear not only in the plots for the Ψ but also in the absolute gain ($\sigma_1(\alpha, \beta)$) and this can be seen in Fig.4.6 where the gain has been plotted for the four Mach numbers at $Re_b = 3000$. The bands can be clearly seen in the Mach 3 case. Nevertheless, it must be noted that the gain plots are similar in many ways. They all have a strong preference for highly streamwise elongated structures, and mask other features of turbulence activity. This tendency to favor infinitely long structures is common to many linear analyses but such structures don't present themselves in real flows. One explanation that has been offered to explain this discrepancy is that such structures would take a very long time to develop and grow, and those timescales are not viable in real turbulent flows[22].

(a) $M = 0.1, Re_\tau = 191$

(b) $M = 0.5, Re_\tau = 194$

(c) $M = 1.5, Re_\tau = 222$

(d) $M = 3.0, Re_\tau = 297$

Figure 4.6 $\log(||H_\infty||(\alpha, \beta))$ at $Re_b = 3000$, $c = U(15y+)$

(a) $M = 0.1, Re_\tau = 298$

(b) $M = 0.5, Re_\tau = 303$

(c) $M = 1.5, Re_\tau = 349$

(d) $M = 3.0, Re_\tau = 480$

Figure 4.7 $\log(||H_\infty||((\alpha, \beta)))$ at $Re_b = 5000$, $c = U(15y+)$

The modes associated with the dominant forcing and response for two Mach numbers (0.1 and 3.0) can be seen in Fig.4.8. The streamwise velocity iso-surfaces (50 % of the maximum streamwise velocity have been plotted). The iso-surfaces have been colored with the wall normal component of the velocity. The response shapes for both the Mach numbers are fairly identical. There are slight differences with regard to the forcing shapes with the supersonic ones having a small protrusion near the nose of the mode with the incompressible one being more rounded. Moreover, the wall normal velocity component in the incompressible mode is more uniform over the structure. The forcing mode in the Mach 3 case on the other hand has relatively higher velocities (positive and negative) away from the wall and lower velocities closer to the wall. In general, the forcing shapes are upstream inclined structures and the response that they excite contains downstream inclined structures. The response shapes are related to the near wall streaks observed in turbulent flows.

4.4.2 Structures in the log layer

The structures in the log layer are explored by probing the flow at $y = 30y^+$ by prescribing the phase speed $c = U(30y^+)$. This is the region where the log-layer is supposed to begin. It can be seen in Fig.4.9 that except for the Mach 3 case, the effects of compressibility are not very noticeable. The vertical bands discussed before do seem to get pronounced with an increase in Mach number. Jiménez [53] showed band like structures in a similar region (in the region of the space where $\lambda_z > \lambda_x$) which represented the parameter space which showed the highest transient growth for the wall normal and spanwise components of velocity for the optimal amplification history of kinetic energy. To examine if these results are sensitive to Reynolds number we will compare the results for the same Mach numbers at a higher Reynolds number in Fig.4.10. It can be seen yet again, that when scaled appropriately the Reynolds number doesn't seem to affect the results to an appreciable degree (except the Mach 3 case).

4.4.3 Structures in the outer layer

In this case the structures moving with a phase velocity which is equal to the mean velocity at $y = 0.2\delta$. It can be seen in Fig.4.11 and Fig.4.12 that the effect of Mach number is not as pronounced as in the earlier cases which were closer to the wall. On the other hand it can be seen that between the two Reynolds numbers there is a visible difference across all Mach numbers. This is the only case where we are setting the phase velocities such that they are equal to a certain wall normal location in terms of the outer scalings (δ). In all other cases we set the phase velocity equal to the mean velocity at some constant wall normal location across cases in terms of wall units. In all cases, including these, it can be seen that the resolvent operator is more low-rank in the regions where $\lambda_x > \lambda_z$. A diagonal line, which is more obvious in some cases than others, persists through these graphs dividing the plots into two regions. In studies which investigated the turbulent channel flow by examining the wavelengths having the greatest potential for transient growth it was found that transient growth was seen for wavelength combinations where $\lambda_x > \lambda_z$ [105],[76].

(a) Forcing - $M = 0.1, Re_\tau = 191$

(b) Response - $M = 0.1, Re_\tau = 191$

(c) Forcing - $M = 3.0, Re_\tau = 297$

(d) Response - $M = 3.0, Re_\tau = 297$

Figure 4.8 Forcing and response shapes for the dominant mode (streamwise velocity)

(a) $M = 0.1, Re_\tau = 191$

(b) $M = 0.5, Re_\tau = 194$

(c) $M = 1.5, Re_\tau = 222$

(d) $M = 3.0, Re_\tau = 297$

Figure 4.9 The effect of Mach number at $Re_b = 3000$, $c = U(30y+)$ on $\Psi = \frac{\sigma_1^2 + \sigma_2^2}{\sum_{j=1}^{20} \sigma_j^2}$

(a) $M = 0.1, Re_\tau = 298$

(b) $M = 0.5, Re_\tau = 303$

(c) $M = 1.5, Re_\tau = 349$

(d) $M = 3.0, Re_\tau = 480$

Figure 4.10 The effect of Mach number at $Re_b = 5000$, $c = U(30y+)$ on $\Psi = \frac{\sigma_1^2 + \sigma_2^2}{\sum_j^{20} \sigma_j^2}$

(a) $M = 0.1, Re_\tau = 191$

(b) $M = 0.5, Re_\tau = 194$

(c) $M = 1.5, Re_\tau = 222$

(d) $M = 3.0, Re_\tau = 297$

Figure 4.11 The effect of Mach number at $Re_b = 3000$, $c = U(0.2\delta)$ on the $\Psi = \frac{\sigma_1^2 + \sigma_2^2}{\sum_j^{20} \sigma_j^2}$

(a) $M = 0.1, Re_\tau = 298$

(b) $M = 0.5, Re_\tau = 303$

(c) $M = 1.5, Re_\tau = 349$

(d) $M = 3.0, Re_\tau = 480$

Figure 4.12 The effect of Mach number at $Re_b = 5000$, $c = U(0.2\delta)$ on $\Psi = \frac{\sigma_1^2 + \sigma_2^2}{\sum_j^{20} \sigma_j^2}$

4.5 Future Directions

Future work can involve extension to more complex problems such as those exhibiting spatial development or separation. A spatially developing turbulent boundary layer and a shock-boundary layer interaction with an incoming turbulent boundary layer are ideal candidates for this study. An interesting prospect is to use these techniques to further explore the origins and mechanics of the 'low frequency unsteadiness' in turbulent shock boundary layer interactions.

CHAPTER

5

STABILITY OF HIGH-SPEED BOUNDARY LAYERS WITH THERMAL DISTORTIONS

5.1 Introduction

Transition in high-speed flows is a fairly complex physical phenomenon. The stability of high-speed boundary layers is more complicated than its low speed counterpart due to the existence of multiple unstable modes. Transition is strongly affected by a number of parameters such as: Mach number, Reynolds number, wall temperature, existence of discrete irregularities, surface roughness and environmental factors. This section focuses on the effect of thermal distortions on the stability of high speed boundary layers. In doing this a comparison is often drawn with a far more explored topic, that of the influence of surface distortions on the stability of high-speed boundary layers. Both these distortions distort the baseflow and influence instability behavior in somewhat similar ways. The stability of boundary layers is affected by the presence of small surface irregularities such as steps, gaps and roughness elements or non-uniform wall heating (arising out of the use of different materials or some active heating/cooling control mechanism). These cases are of interest because such scenarios are commonly encountered for hypersonic vehicles. The effect these distortions can have on laminar-turbulent transition needs to be fully understood because the advancing of the point of transition is extremely undesirable. This makes the stability of distorted boundary layers a problem of great engineering interest.

The effect of small surface irregularities on transition has been a subject of interest for many decades. Works ranging from as early as the 1940s have tried to ascertain the role that surface irregularities can potentially play with regard to laminar-turbulent transition. Some of the earliest

works in this direction include: Tani et al. [91], W. Liepmann & H. Fila [97] and a review of this early literature can be found in Dryden [31]. The interest stems from the importance of the transition location in the design process. The choices to be made during the design process are heavily influenced by the fact whether the flow at a certain location and at a certain time is laminar or turbulent. Laminar and turbulent flows are very different in nature in terms of drag, and surface heating among other things- and so it is critical to know if the flow is laminar or turbulent and the ways in which some distortion such as a roughness element or a thermal distortion might influence the transition from a laminar to a turbulent state.

Earlier efforts in transition research for distorted boundary layers (primarily with regard to roughness elements) were largely focused on coming up with correlations to correlate the transition location to some known parameters such as characteristics of roughness, local flow conditions and properties of the freestream flow. By contrast, the studies emphasizing on flow structure and transition mechanisms are fewer in number. With regard to transition mechanisms more work has been done for low-speed flows. In the case of hypersonic flows experimental work has been limited due to the difficulty in building quiet tunnels and in reproducing flight conditions in the wind-tunnel.

In case of numerical approaches, a temporal transient growth analysis for compressible flows was conducted by Hanifi et al. [48]. A spatial framework for examining transient growth arising due to the interaction between two or more modes of the spatial spectra was proposed by Tumin & Reshotko [93]. Reshotko & Tumin [78] used their theory based on transient growth for a spatial eigenvalue problem to show that they could use transient-growth explanation to predict transition. Their predictions showed a good match with the Reda data [77] as well as the passive nosetip technology (PANT) data [12]. Non-parallel effects were incorporated in this framework by [94]. Although most earlier studies included supersonic flows, hypersonic flows in this context were first studied by Paredes et al. [74] using a PSE based framework to compute optimal growth. Instead of using a self-similar solution they used a numerically computed mean flow obtained through a full Navier-Stokes code and this allows for incorporating the effects of shock near the leading edge [74]. Bugeat et al. [21] used the resolvent framework to compute optimal growth in compressible boundary layers. A similar use of this framework to analyze the transient growth of steady streaks was carried out by Dwivedi et al. [34] in the context of reattachment streaks in flows over compression ramps.

5.1.1 Distorted boundary layers: surface imperfections

Klebanoff & Tidstrom [56] tried to study the mechanism responsible for transition over a 2D isolated roughness, and concluded that transition was dominated by T-S unstable frequencies. Boiko et al. [17], review some of the work that has been done for 2D surface imperfections for low-speed flows. They mention two possible instability mechanisms. In the first case, external perturbations excite vortical perturbations in the separated shear layers, which are amplified spatially through modal or non-modal paths. Another possibility is the self-excitation (absolute instability) of the

separation bubble. Chang & Choudhari [23] argue that while an absolute instability mechanism may be responsible for transition in subsonic flows (oscillator), transition for supersonic and hypersonic flows would likely be dominated by a convective amplification mechanism (amplifier). Reshotko & Tumin [78] point out that an attempt to link transition with Tollmein-Schlichting mechanism has only been successful for isolated 2D disturbances. It was shown in the experimental work of Aizin & Polyakov (1979) and re-affirmed by the theoretical work of Goldstein [44] that acoustic disturbances can trigger Tollmein-Schlichting waves in the presence of two-dimensional surface discontinuities. Wilhelm et al. [99] study the effect of a forward facing step (FFS) on 2D-3D transition for an incompressible flow in a channel. They conclude that the transition from the two-dimensional regime to a third dimensional one is not an artifact of an absolute instability created due to the presence of the separation bubble but rather due to the amplification of small three-dimensional disturbances in the incoming flow. Crouch et al. [29] used the variable N-factor method to model the effect of steps. The idea is that the effect of a step can essentially be modelled as a 'jump' in the N-factor curve (or conversely as a lowering of the critical N-factor required for transition).

Discrete roughness, is generally known to aid transition and is associated with an upstream movement of the transition location. However, in some cases, roughness can have an opposite effect by delaying transition. One such scenario is the presence of a 2D roughness in hypersonic boundary layers. The effect of 2D roughness elements in hypersonic boundary layers has been investigated by Fong & Zhong [42] and Fong [41]. They found that a strategically placed 2D roughness can potentially play a stabilizing role by damping the unstable second mode. So a certain 2D roughness placed just downstream of the synchronization point for a certain frequency will damp all frequencies higher than it but amplify frequencies lower than it. Although playing a stabilizing role for the second mode the presence of roughness has a destabilizing influence on the first mode. Sawaya et al. [82] used a DNS based approach to study the effect of a range of wall deformations on the stability of a $Ma = 5.92$ boundary layer. Deformations which produced an adverse pressure gradient followed by a favorable pressure gradient (FFS, hump etc.) were shown to be far more effective in stabilizing than those inducing a favorable-pressure gradient followed by an adverse pressure gradient (eg. dip, BFS etc.). Zhao et al. [102] used an HLNS based approach to study the influence of 2D surface imperfections (humps/indentations). They found that the 2D roughness was capable of producing both - a stabilizing and a destabilizing effect. It was re-asserted, in concurrence with other works, that the synchronization frequency plays an important role in determining the way in which the roughness acts. A stabilizing effect is produced on frequencies higher than the synchronization frequency and a destabilizing effect is produced on frequencies lower than the synchronization frequency (at the location of the roughness). As the second-mode is most unstable in the 2D limit ($\beta = 0$) most of the analysis is based on 2D modes. However, for very low frequencies where the first mode dominates the effect of spanwise wavenumber has also been examined. Tang et al. [90]: performed experiments for different roughness heights ($0.125\delta - 0.375\delta$) for a $M_\infty = 6$ boundary layer and found that a taller roughness seems to provide a stronger destabilizing effect upstream of synchronization point.

5.1.2 Distorted boundary layers: thermal imperfections

Although, a large body of work is devoted to surface imperfections- much less work has been done with respect to non-uniform wall heating. Although, both of these are very different configurations, they are similar in the sense that they both impart a non-parallel character to the boundary layer and thereby modify its stability characteristics. Soudakov et al. [87] studied a $M_\infty = 6$ boundary layer with unit Reynolds' number, $Re = 2 \times 10^6$; they considered a combination of temperature jumps (between $T_w = 2T_\infty$ and $T_w = 6T_\infty$) at various streamwise locations. They did a DNS using a forcing composed of 2D planar acoustic waves at $F = \omega/Re = 1.3 \times 10^{-4}$ and found that the fate of a wave depended not only on the nature of the jump but heavily depended on the position of the jump. Zhao et al. [103] using LST and DNS investigated the effect of a heating/cooling strip of a fixed length whose position was varied along the length of the plate on the stability of a $M_\infty = 6$ boundary layer. They found out that a local heating strip downstream of the synchronization point damped disturbances, while upstream of the synchronization point amplified disturbances. A cooling strip acted in exactly the opposite manner. They proposed a stabilizing mechanism containing a cooling strip upstream or a heating strip downstream of the synchronization point. Batista & Kuehl [11] used the fact that specifically with regard to the second-mode, upstream cooling and downstream heating tends to destabilize the second mode to explore the possibility of combining upstream cooling and downstream heating to devise an optimal strategy for controlling second mode growth. Fedorov et al. [39] investigated the effect of localized heating and cooling for a Mach 6 flow over a sharp 7° half angle sharp cone at 0° angle of attack. Heating produces a region of increased pressure-followed by a region of decreased pressure (similar to a hump), whereas cooling produces a region of reduced wall pressure followed by a region of high pressure (similar to a dip). Linear stability analysis and DNS (localized periodic suction-blowing) were performed and they found that heating has a destabilizing effect whereas cooling reduces the second-mode amplitudes and delays transition. They observe that if the heating/cooling is applied upstream of the neutral point, then heating locally increases the disturbance amplitudes and so they reach the neutral point with a higher amplitude. Cooling acts in the opposite way.

A series of works ([75], [18], [19]) attempt to investigate the effects that locally heating/cooling the tip of a cone has on the stability of the flow among other things. Poplavskaya et al. [75] conducted experimental and numerical simulations for a $M_\infty = 5.95$ flow and found that the impact of local heating/cooling on the transition location seemed to depend on both- the extent (size) and the intensity of the local heating/cooling applied. Heating seemed to destabilize the flow by moving the transition location upstream. Bountin et al. [18] found that cooling seemed to advance transition and heating seemed to delay transition (this effect was weaker). These results are different from those of Fedorov et al. [39], but so is the location. Cooling leads to early transition and an increase in second mode peak height. Also, an increase in nose bluntness seems to enhance the effect of local temperature variation. Bountin et al. [19] built on the preliminary results of Bountin et al. [18] by performing experimental investigations to assess the effect of local heating/cooling on flow over cones. They notice that heating the tip of a sharp cone weakly destabilized the second mode, while

cooling stabilized disturbances over a wide range of frequencies. For a blunted cone they observe that cooling destabilizes the flow and heating stabilizes the flow (opposite trends for sharp cones). The abrupt change in the boundary layer and the significant non-parallel nature make the global frequency based tools more attractive than the traditional tools such as local LST and PSE.

5.1.3 Relevance of the global-frequency based methods

There are two main types of methods that have been employed to investigate boundary-layer stability: DNS and linearized analysis (local LST and PSE). DNS is usually performed by introducing some form of unsteady disturbances at an upstream station (for eg. periodic suction/blowing). While this yields data that fully captures the complex flow physics, it is expensive and only informs us about the evolution of that specific forcing that was used without necessarily shedding light on the larger physical trends. The linearized tools usually include LST (independent solutions at each station- modes need to be tracked through spectra obtained at each station- assumes that the boundary layer is parallel at each station) and PSE (spectra obtained at one station- and then the evolution of individual modes can be tracked through spatial marching). While these traditional stability analysis techniques have been successful in studying the stability of smooth flat-plate boundary layers and sharp cones, these methods are not extensible to more complicated configurations. This is because there are two main problems with these basic tools:

1. Spatially developing baseflows: These methods assume that either the baseflow is invariant in the streamwise direction or that it exhibits extremely slow development in the streamwise direction relative to the wavelength of the disturbances- This may no longer be a valid assumption in case of boundary layers with distortions or other factors which induce a strong non-parallel character to the boundary layer.
2. These modes are tracked/marched on a mode-by-mode basis and so it misses inter-modal interactions and these include: F & S mode synchronization, transient growth (spatial), interaction between the discrete modes and the continuous branches etc.
3. These zones of modal interactions appear in these methods as singularities and demand special treatment [37].

On the other hand, a global method of analysis such as the resolvent analysis can be used to map the growth of disturbances due to a system-level response. In the frequency domain, resolvent analysis yields a set of mode pairs (inputs and outputs): ranked by the degree to which the said input is amplified by the system while transforming it to the corresponding output. This is done by performing a singular value decomposition of the resolvent operator: $(-i\omega - \mathcal{A})^{-1}$. The potential of this method to analyze complex high-speed boundary layers was demonstrated recently by [71]. They were able to use this method to recover the classical interaction between the discrete F and S modes for a case of a flow over a sharp cone and were also able to show more complex mode

interactions made possible as a result of transient growth triggered by mode interactions between the continuous acoustic branches and the discrete F and S modes.

The global I/O framework has three significant advantages:

1. No assumptions regarding the spatial development of the baseflow
2. Organically captures multi-modal interactions (at the same frequency)
3. Accounts for transient growth (spatial)

Problems where the close-to-parallel nature of the boundary layer is significantly altered are ideal candidates to be analyzed by these methods. One such class of problems involves thermally distorted boundary layers due to non-uniformly heated walls: This can happen due to either a passive factor such as the use of different materials in the assembly of hypersonic vehicle or some active heating/cooling element applied to manipulate the flow behavior.

5.1.4 Regimes of disturbance amplification

Figure 5.1 Resolvent gain for Case B1 (Table.5.2) and 3D mode visualizations through iso-surfaces of u' ; 3D mode axes are not to scale

In Fig.5.1 the resolvent gain for the case B2 is shown. There are three distinct ‘peaks’ that can be seen:

1. For $\alpha \rightarrow 0$ a preference for a specific spanwise wavelength can be seen. These are extremely elongated streaks which get amplified via transient growth (spatial).

2. A $\beta \neq 0$ & $F \neq 0$ peaks which represents the first mode- an oblique traveling wave disturbance
3. The high frequency $\beta = 0$ peak which represents the second-mode.

Some observations about trends of instabilities in high-speed boundary layers:

- As Mach number increases second mode growth rates are generally higher than the first mode growth rates; and are typically responsible for transition for $M_\infty > 4.5$
- Both the absolute values of growth rates and the relative balance between the first and the second mode can be influenced by altering wall temperature. Wall cooling stabilizes the first mode and destabilizes the second mode; heating has the opposite effect.
- The second mode is most dangerous (exhibits highest growth) in the 2D limit ($\beta = 0$); first mode has peak growth rates when it is oblique ($\beta \neq 0$).
- A 2D roughness/heating strip placed downstream of the synchronization point (synchronization point corresponding to the most unstable second mode frequency) is able to damp those second mode frequencies- thus delaying transition- it simultaneously amplifies lower frequencies for whom this roughness/heating strip appears upstream of their synchronization point/s.
- As second mode is most unstable in the $\beta = 0$ limit, most of these analyses have involved only 2D modes; but the effect of this strategy for $\beta \neq 0$ on the mainly the first mode but also the second mode disturbances has not been extensively explored.

It is not sufficient to investigate how a distortion may affect the amplifications of one of these growth paths. For instance a strategy adopted to stabilize a dominant instability may destabilize some other instability and therefore it is imperative to look at the effect of a distortion on all the different paths available for disturbance growth lest it become an exercise that resembles a ‘game of whack-a-mode’.

5.2 Steady streaks

In classical experiments it was found that in cases where the freestream turbulence levels were high, spanwise periodic streaks were observed. The the presence of finite amplitude streaks alters the nature of the boundary layer and thus changes its stability characteristics. Depending on the amplitude of the streaks, it may admit instabilities not present in the 2D boundary layer or may in fact play a stabilizing role by suppressing T-S instabilities. Cossu & Brandt [27] discovered that in presence of these steady streaks, 2D T-S waves modify into 3D varicose modes with lower growth rates. It was observed that this experimentally observed picture was quite similar to the one predicted by the optimal growth framework and as a result it was believed that transient growth might be a possible explanation for this type of transition.

In this chapter two types of thermal distortions are considered:

- Small heating/cooling strips producing localized temperature bumps/dips along the flat plate
- A step-up or a step-down in wall temperature at some point along a flat plate

5.2.1 Setup

In this case we set the case up in a way similar to that of Bugeat et al. [21]. The length of the domain (flat plate) is L . The Re_x at the outflow is $Re_{x-out} = 1.2996 \times 10^6$. The location of the output is referred to as x_{opt} , is chosen such as $Re_{x-opt} = 1.0 \times 10^6$. The ratio $R = x_{in}/x_{opt}$ indicates the location of the input x_{in} relative to the output x_{opt} . The input and output locations are introduced through the operators B and C . B and C are spatial functions which have values of 1 in a region of width $0.04L$ and centered around x_{in} and x_{opt} respectively, and 0 everywhere else. The edges are made smooth using hyperbolic tangent profiles. Different norms are chosen for the inputs and outputs. The output norm is chosen to be the Chu norm [25] and this is a choice routinely employed for compressible flows. It has been observed in literature that the most optimal disturbances are streamwise vortices. With this in view the input norm contains the kinetic energy associated with the wall-normal and spanwise velocities. The effects of distortions created by temperature gradients along a non-uniformly heated wall are examined. Also, for a baseline case uniformly-heated walls are considered. The grids ($n_x \times n_y$) used for uniformly heated walls 401×201 . The length of the domain is $L = 1$ and the height of the domain is $H = 0.2L$. Approximately half of the points in the wall-normal direction are confined within the boundary layer (δ_{99}). The points are uniformly distributed within the boundary layer and are stretched outwards in the wall normal direction. To compute growth in a more physical context we will examine the dominant response mode for a particular set of parameters. We can then compute the total disturbance energy at each streamwise station, by integrating the Chu norm[25] in the wall normal direction. The integrated disturbance energy along the streamwise direction would indicate the growth of disturbances in space.

$\|\cdot\|_E$ and $\|\cdot\|_F$ represents two different energy norms: one for the input and the other for the output. It is possible to employ two different definitions of an energy norm for the input and the output. In any case, the absolute value of gain (G) is not necessarily physically meaningful. For instance $G > 1$ or $G < 1$ does not mean amplification or decay respectively. It must be noted that the units of $\hat{y}$ and $\hat{f}$ are not the same. $\hat{y}$ has the units of q while $\hat{f}$ has the units of q/t . As a result the units of gain (G) are the same as that of time squared. In order to relate the aforementioned maximization problem to an SVD the problem must be cast in terms of a $\|\cdot\|_2$ norm. The norms can be expressed in terms of inner products as follows:

$$G = \max_{\tilde{f}} \frac{\tilde{f}^* \mathcal{M}_F^{-1/2*} \mathcal{H}^* \mathcal{M}_E^{1/2*} \mathcal{M}_E^{1/2} \mathcal{H} \mathcal{M}_F^{-1/2} \tilde{f}}{\tilde{f}^* \tilde{f}} = \max_{\tilde{f}} \frac{\|\mathcal{M}_E^{1/2} \mathcal{H} \mathcal{M}_F^{-1/2} \tilde{f}\|_2^2}{\|\tilde{f}\|_2^2} \quad (5.1)$$

Introducing $\tilde{\mathcal{H}} = \mathcal{M}_E^{1/2} \mathcal{H} \mathcal{M}_F^{-1/2}$ the gain can be related to the singular values of the modified transfer function.

$$G = \max_{\tilde{f}} \frac{\|\mathcal{H}\tilde{f}\|_2^2}{\|\tilde{f}\|_2^2} = \sigma_1^2(\mathcal{H}) \quad (5.2)$$

$$\hat{f} = \mathcal{M}_F^{-1/2} \tilde{f} \quad \hat{y} = \mathcal{M}_E^{-1/2} \tilde{y} \quad (5.3)$$

5.2.2 Adiabatic boundary layers: effect of Mach number

In this case a Mach number study is conducted by fixing R , Re_{x-opt} and x_{opt} . The conditions for the three values of $M_\infty = 3.0, 4.5$ and 6.0 are given in Table-5.1. A grid convergence study is performed for the $M_\infty = 3.0$ case by examining the resolvent gain by fixing $R = 0.2$ for a range of spanwise wavenumbers β . Three grids ($n_x \times n_y$) are considered: 201×101 , 401×201 and 601×301 . The length of the domain is $L = 1$ and the height of the domain is $H = 0.2L$. In Fig.5.2 the resolvent gain ($G = \sigma_1^2$) is compared with the spatial transient growth results of [104] and the resolvent gain results of [21]. The results are normalized as it is not possible to directly compare between absolute values between these different methods of analyses. It can be seen that the results are fairly similar between the three grids. However, the baseflow for the coarsest grid did not fully converge- for this and in general to err on the side of safety we choose Grid 2 ($n_x \times n_y = 401 \times 201$) for all studies involving boundary layers without distortions. Another aspect of importance to us is the sensitivity of the results to baseflow numerics. The task of computing converged baseflows for complex configurations is often made easier by employing dissipative fluxes (modified Steger-Warming) and we wish to investigate if it would have any adverse impact on the results of the stability computations (which will typically be carried out using low-dissipation higher order numerics). The results can be seen in Fig.5.2(b). The absolute value of gain has little relevance and it must be read in a relative sense.

Figure 5.2 Optimal response over a range of spanwise wavenumbers β

Table 5.1 Conditions for investigating the effects of Mach number

M_∞	Re_{x-opt}	Re_{x-out}	ρ_∞	T_∞	U_∞
3.0	1.0×10^6	1.2996×10^6	0.0163478	118.928571	655.7968
4.5	1.0×10^6	1.2996×10^6	0.0078595	65.940594	732.4768
6.0	1.0×10^6	1.2996×10^6	0.0042396	40.609756	766.4281

The integrated disturbance energy along the streamwise direction would indicate the growth of disturbances in space. The results of these computations for the three Mach numbers can be seen in Fig.5.3 (a)-(c). It can be seen that the structures with smaller spanwise wavenumbers achieve maximum growth at a later stage (i.e. at a more downstream station). Also it can be seen that there is a specific wavenumber where maximum amplification is observed. This wavenumber is often referred to in literature as β_{opt} . It can also be seen that as M_∞ increases the β_{opt} shifts towards smaller wavenumbers. This effect can be more directly observed in Fig.5.4(a) which shows the resolvent gain across all spanwise wavenumbers. Not much should be read into the apparent difference in the gain values across different M_∞ as the time-scales are different in each case (both convective and acoustic). Fig.5.4(b) shows the difference in the actual amplification and it can be seen here that the difference is far less stark than indicated by G^* .

5.2.3 Isothermal boundary layers: effect of Mach number & wall temperature

To assess the effects of wall cooling we will consider isothermal cases. For each $M_\infty = 3.0, 4.5$ & 6.0 four different values of $T_w/T_{adia} = 1.0, 0.5, 0.25$ & 1.5 are considered. The details can be found in Table.5.2. The results for the effect of wall temperature on resolvent gain (G^*) can be seen in Fig.5.5. Again G^* levels drop drastically with an increase in M_∞ . For colder walls β_{opt} moves towards larger spanwise wavenumbers for all M_∞ numbers. For $M_\infty = 3.0$ the gain levels for the cold walls are roughly of the same levels as the adiabatic baseline case while the gain for the hot wall is the highest. As M_∞ increases the gain associated with the hot wall, relative to the gain for the adiabatic case at that Mach number, goes down (an opposite effect is seen in the case of cold walls).

5.2.4 Analysis of Thermal distortions

In this section the effects of a thermal distortion (intensity & location) on the transient growth of steady disturbances are investigated. The baseline for this study is the Case A1 from Table.5.2. To assess the impact of location and intensity of a thermal distortion a thermal discontinuity of width $0.04L$ was introduced along different streamwise stations ($X_{\Delta T} = 0.3R, 0.6R, 0.9R, 1.2R$) for the $M_\infty = 3.0$ nominally adiabatic (T_w/T_{adia}) case. The thermal discontinuities have four different intensities ($\Delta T = -150K, -50K, 50K, 150K$).

The grid convergence study for uniformly heated/cooled walls has been carried out in the previous sections. To ensure that these results are grid insensitive, we compare results between the uniform grid (16 cells across the thermal distortion) and the refined grid (100 cells across the

(a) $M_\infty = 3.0$

(b) $M_\infty = 4.5$

(c) $M_\infty = 6.0$

Figure 5.3 Streamwise growth of optimal response over a range of spanwise wavenumbers

Table 5.2 Conditions for investigating the effects of wall temperature

Case	M_∞	Re_{x-opt}	Re_{x-out}	ρ_∞	T_∞	T_w/T_{adia}	U_∞
A1	3.0	1.0×10^6	1.2996×10^6	0.0163478	118.928571	1	655.7968
A2	3.0	1.0×10^6	1.2996×10^6	0.0163478	118.928571	0.5	655.7968
A3	3.0	1.0×10^6	1.2996×10^6	0.0163478	118.928571	0.25	655.7968
A4	3.0	1.0×10^6	1.2996×10^6	0.0163478	118.928571	1.5	655.7968
B1	4.5	1.0×10^6	1.2996×10^6	0.0078595	65.940594	1	732.4768
B2	4.5	1.0×10^6	1.2996×10^6	0.0078595	65.940594	0.5	732.4768
B3	4.5	1.0×10^6	1.2996×10^6	0.0078595	65.940594	0.25	732.4768
B4	4.5	1.0×10^6	1.2996×10^6	0.0078595	65.940594	1.5	732.4768
C1	6.0	1.0×10^6	1.2996×10^6	0.0042396	40.609756	1	766.4281
C2	6.0	1.0×10^6	1.2996×10^6	0.0042396	40.609756	0.5	766.4281
C3	6.0	1.0×10^6	1.2996×10^6	0.0042396	40.609756	0.25	766.4281
C4	6.0	1.0×10^6	1.2996×10^6	0.0042396	40.609756	1.5	766.4281

(a) G^* v. β

(b) E_{chu} v. x

Figure 5.4 Effect of Mach number

(a) $M_\infty = 3.0$

(b) $M_\infty = 4.5$

(c) $M_\infty = 6.0$

Figure 5.5 Effects of wall temperature

thermal distortion). For the grid convergence study the cases which have the centre of the thermal distortion located at $X_{\Delta T} = 0.6R$ are selected. The results can be seen in Fig.5.6 and it is perfectly evident that the coarse grid (401×201) is sufficient for these computations even in the presence of thermal distortions.

Figure 5.6 Non-uniformly heated/cooled walls (temperature bumps): grid resolution study

The results for both the resolvent gain (G^* v. x) and the integrated E_{chu} v. x can be seen in Fig.5.7 and Fig.5.8 respectively. A thermal discontinuity located downstream of x_{opt} does not seem to affect energy amplification upstream of itself. For $X_{\Delta T} = 0.3R, 0.6R$ cases, $\Delta T = 150K$ seems to produce lowest resolvent gain, while for $X_{\Delta T} = 0.9R$ case $\Delta T = -150K$ seems to produce lowest resolvent gain. In general it can be seen in Fig.5.8 a heating strip increases streak energy locally and the cooling strip decreases streak energy locally. However as the flow relaxes downstream of the continuity it can be seen that the streak energy seems to be relaxing too but then tends to go higher than the baseline case for cooling strips and lower than the baseline case for heating strips. For $X_{\Delta T} = 0.9R$ as the output location is just downstream of the discontinuity, the local effect is still dominant and the heating strip produces the largest amplification. In the case of $X_{\Delta T} = 0.3R, 0.6R$ the streak gets sufficient space to develop downstream of the discontinuity and thus maximum amplification is seen in the case of a cooling strip.

In addition to the localized thermal distortions which may arise due to active control or difference in material of joints or rivets, another possibility is a step change in temperature due to the presence of different materials.

(a) G v. β , $X_{\Delta T} = 0.3R$

(b) G v. β , $X_{\Delta T} = 0.6R$

(c) G v. β , $X_{\Delta T} = 0.9R$

(d) G v. β , $X_{\Delta T} = 1.2R$

Figure 5.7 Effects of the location of wall temperature distortions

(a) E_{chu} v. x , $\beta = 0.25$, $X_{\Delta T} = 0.3R$

(b) E_{chu} v. x , $\beta = 0.25$, $X_{\Delta T} = 0.6R$

(c) E_{chu} v. x , $\beta = 0.25$, $X_{\Delta T} = 0.9R$

(d) E_{chu} v. x , $\beta = 0.25$, $X_{\Delta T} = 1.2R$

Figure 5.8 Effects of the location of wall temperature distortions on streak energy

5.2.5 Temperature steps

Jumps of four different intensities ($\Delta T = -150K, -50K, 50K, 150K$) are considered and the step would be placed at three different locations ($X_{\Delta T} = 0.3R, 0.6R, 0.9R$). The $Gv.\beta$ and the integrated energy for the dominant response are shown in Fig.5.9. Although significant alteration in streak energy was not seen for the hump it is clearly visible in the case of a step. A more pronounced effect is seen when the step starts further downstream. The $\Delta T = +150K, X_{\Delta T} = 0.9R$ combination produces the maximum streak energy whereas the $\Delta T = -150K, X_{\Delta T} = 0.9R$ combination produces the lowest streak energy as can be seen in Fig.5.9(b). In Fig.5.9(a) the spanwise spacing of the streaks with most potential for growth is seen. For steps which start at $x_T = 0.3R$ a clear trend can be seen that a temperature step-up favors more wider spanwise spacing and a temperature step-down favors more finer spanwise spacing. The trend for $0.6R$ or $0.9R$ are more ambiguous and the spanwise spacings are more or less equal. However in terms of the actual gain- the gain is more or less equal for steps which start at $0.3R$, but for steps which start further downstream ($0.6R, 0.9R$) the temperature step-up produces a higher gain and a temperature step-down produces a lower gain. In Fig.5.10 the streak energies for the two extreme cases ($\Delta T = \pm 150$ and $x_T = 0.9R$) is plotted alongside the streak energies for uniformly hot $T = T_{adia} + 150K$ and uniformly cold $T = T_{adia} - 150K$ walls. It can be seen that within the domain considered the temperature step-up tends to amplify the streak beyond what a uniformly hot wall could but the temperature step-down did not outperform the uniformly cold wall.

Figure 5.9 The effect of thermal distortions-steps on steady streaks

(a) E_{chu}

Figure 5.10 The effect of thermal distortions-steps on steady streaks: comparison with baseline cases (uniform wall temperature profiles)

5.3 Unsteady disturbances

5.3.1 Temperature steps

A total of 7 configurations with $\Delta T = \pm 150K$ are considered. The jumps are placed at three locations: $x=0.6R, 0.9R, 0$ (0 refers to the entire wall being held at $T = T_{adia} \pm \Delta T$). The seventh configuration is the baseline case (B1) with conditions:

Table 5.3 Conditions for investigating the effects of Mach number

M_∞	Re_{x-opt}	Re_{x-out}	ρ_∞	T_∞	U_∞
4.5	1.0×10^6	1.2996×10^6	0.0078595	65.940594	732.4768

The case is set up similar to the cases in Sec.5.2.1. A finer grid in the streamwise direction ($N_x \times N_y = 801 \times 201$) than the one used for examining steady streaks in order to capture the short wavelength unsteady disturbances. For all the seven cases a full-field resolvent analysis was performed (in the Chu norm for both the input and the output). The log of the resolvent gain for all seven configurations is shown in Fig. 5.11. The left panel contains cases with a negative ΔT and the right panel contains the cases with a positive ΔT . The central panel contains the same image in all three rows. It can be seen that the temperature step-up energizes the first mode in all three cases. The effect is stronger when the temperature step-up occurs across the maximum length of the plate. For the uniformly hot wall it almost engulfs the second mode peak. The cooling on the other hand strongly suppresses the first mode especially in the case of the uniformly cold wall. The cooling

has a more interesting effect on the second mode peak. For a cold wall the frequency associated with the peak response increases slightly. The peak gets strongly destabilized when the temperature step-down starts at $x_T = 0.9R$ or the uniformly cold wall. The peak is stabilized for the case where $x_T = 0.6R$.

In Fig.5.12 the integrated Chu norm energy ($E_{\text{chu}} \nu.x$) for the peaks associated with the first and second mode are shown. In the left pane $\log(E_{\text{chu}})$ is shown. It is not possible to compare between two lines on this plot physically. To get a fair comparison the amplification relative to some upstream location x_0 should be measured. $x_0 = 0.1$ is chosen for illustration (the choice of x_0 is somewhat arbitrary, the usual practice for a modal instability is to pick the point when the mode first becomes unstable-but in this case when considering modal and nonmodal growth at once no such obvious point exists). In the right pane these results are shown in terms of the N-factor. The uniformly cold wall produces the maximum second mode amplification (and overall as well). The first mode is shown to be stabilized by all temperature step-downs and destabilized by all temperature step-ups.

Figure 5.11 The effect of thermal distortions (steps): $\log(G)$, contour levels are the same across all cases

(a) $E_{chu} \nu.x$

(b) $N \nu.x$

(c) $E_{chu} \nu.x$

(d) $N \nu.x$

Figure 5.12 The effect of thermal distortions-steps on the first and second mode responses

CHAPTER

6

INFLOW-I/O FRAMEWORK

6.1 Introduction

Receptivity is the process whereby the instability waves acquire some initial amplitudes. The e^N method which is widely used in the hypersonics community focuses solely on the modal instability part of the problem. All other factors such as noise environment, configuration specifics, and other environmental factors need to be modeled into the critical ‘ N -factor’ and this makes it a rather arbitrary approach. The receptivity mechanisms play an important role in the transition process. The DNS approach to study the transition process is to study the evolution of specific types of forcings such as freestream waves at a certain orientations, blowing-suction through a wall, presence of roughness elements etc. This is a very expensive process and only builds a case-by-case knowledge of transition physics. In this section a formulation is developed using linearized tools which would allow an analysis of the response of flowfields to a broad range of forcings at a relatively low computational cost.

6.2 Formulation

The basic goal is to investigate the role freestream waves play in the transition process in hypersonic flows. The disturbance evolution equation for a small global disturbance vector: $q' = [\rho'_i, (\rho u)'_i, (\rho v)'_i, (\rho w)'_i, E'_i]_{i=1, \dots, n}$ is given as:

$$\frac{\partial q'}{\partial t} = \mathcal{A}(\bar{q})q' \quad (6.1)$$

This equation tracks the evolution of a small disturbance on the base flow. If the operator $\mathcal{A}$ is stable, all disturbances eventually decay. In these cases it may be useful to examine the response of the flow to continuous harmonic forcing. The forced equation is given by:

$$\frac{\partial q'}{\partial t} = \mathcal{A}(\bar{q})q' + f' \quad (6.2)$$

This forcing can exist within the flow but can also possibly originate from the boundaries of the domain. This would include cases such as wall forcing or inflow forcing. In the finite-volume context to examine the boundary forcing we also need to include the ghost cells as a part of the system. To consider this we start by extending our state vector to include the ghost cells. The global ghost cell vector is given by q_g and its size is $N_G = n_{eq} \times n_G$. Similarly the size of the original state vector (interior cells) is given by: $N = n_{eq} \times n$. The extended dynamical system can be written as:

$$\frac{\partial}{\partial t} \begin{pmatrix} q' \\ q'_g \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \widetilde{\mathcal{A}}_g \\ \widetilde{\mathcal{B}} & \widetilde{\mathcal{B}}_g \end{pmatrix} \begin{pmatrix} q' \\ q'_g \end{pmatrix}$$

For the special case of a supersonic inflow, simplifications to the above are possible. This happens because in a supersonic inflow the flow of information is strictly one-way (from the inflow to the system). Thus $\widetilde{\mathcal{B}} = 0$ and $\widetilde{\mathcal{B}}_g = -i\omega I$. The matrix $\widetilde{\mathcal{A}}_g(D, D_G)$ is built by looping over all faces, perturbing the inflow ghost cells only and calculating their influence over the interior cells. Thus it is possible to decouple the extended resolvent system for this special case.

$$\begin{aligned} \frac{\partial}{\partial t} \begin{pmatrix} q' \\ q'_g \end{pmatrix} &= \begin{pmatrix} \mathcal{A} & \widetilde{\mathcal{A}}_g \\ 0 & -i\omega I \end{pmatrix} \begin{pmatrix} q' \\ q'_g \end{pmatrix} \\ \frac{\partial q'}{\partial t} &= \mathcal{A} q' + \widetilde{\mathcal{A}}_g q'_g & \frac{\partial q'_g}{\partial t} &= -i\omega q'_g \end{aligned} \quad (6.3)$$

Thus the inflow is in a state of independent evolution and influences the dynamics of the system without being affected by it in turn. The disturbances have a modal form given by: $q' = \hat{q} e^{-i\omega t}$ and $q'_g = \hat{q}_g e^{-i\omega t}$. Substituting this modal form, the equations for a certain frequency are given by:

$$\hat{q} = (-i\omega - \mathcal{A})^{-1} \widetilde{\mathcal{A}}_g \hat{q}_g = \mathcal{H}(\omega) \hat{q}_g \quad (6.4)$$

So if a specific type of waveform is prescribed in $\hat{q}_g$ it's response in the field can be obtained by the equation $\hat{q} = \mathcal{H}(\omega) \hat{q}_g$. This type of solution is referred to as direct harmonic response (DHR) in this work. The most standard form of inflow disturbances are the characteristic free stream waves: acoustic, entropy and vorticity. The freestream wave functions are derived in the following section.

6.3 Freestream waves

Freestream waves are governed a system of linearized Euler equations. For flows with 0° angle of attack, the baseflow velocity is: $\overline{U_\infty}, 0, 0$.

$$\frac{\partial \rho'}{\partial t} + \bar{U} \frac{\partial \rho'}{\partial x} + \bar{\rho} \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right) = 0 \quad (6.5)$$

$$\bar{\rho} \frac{\partial u'}{\partial t} + \bar{\rho} \bar{U} \frac{\partial u'}{\partial x} = - \frac{\partial p'}{\partial x} \quad (6.6)$$

$$\bar{\rho} \frac{\partial v'}{\partial t} + \bar{\rho} \bar{U} \frac{\partial v'}{\partial x} = - \frac{\partial p'}{\partial y} \quad (6.7)$$

$$\bar{\rho} \frac{\partial w'}{\partial t} + \bar{\rho} \bar{U} \frac{\partial w'}{\partial x} = - \frac{\partial p'}{\partial z} \quad (6.8)$$

$$\bar{\rho} c_p \frac{\partial T'}{\partial t} + \bar{\rho} \bar{U} c_p \frac{\partial T'}{\partial x} = \frac{\partial p'}{\partial t} + \bar{U} \frac{\partial p'}{\partial x} \quad (6.9)$$

The freestream waves can be defined by an amplitude function ($\hat{a}$) and a phase given by $i(\alpha x + \epsilon y + \beta z - \omega t)$. This modal form is substituted in the linearized euler equations to form an eigenvalue problem:

$$\omega \begin{pmatrix} \hat{\rho} \\ \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{T} \end{pmatrix} = \begin{pmatrix} \alpha \bar{U} & \alpha \bar{\rho} & \epsilon \bar{\rho} & \beta \bar{\rho} & 0 \\ \alpha R \bar{T} / \bar{\rho} & \alpha \bar{U} & 0 & 0 & \alpha R \\ \epsilon R \bar{T} / \bar{\rho} & 0 & \alpha \bar{U} & 0 & \epsilon R \\ \beta R \bar{T} / \bar{\rho} & 0 & 0 & \alpha \bar{U} & \beta R \\ 0 & (\gamma - 1) \bar{T} \alpha & (\gamma - 1) \bar{T} \epsilon & (\gamma - 1) \bar{T} \beta & \alpha \bar{U} \end{pmatrix} \begin{pmatrix} \hat{\rho} \\ \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{T} \end{pmatrix}$$

The eigenvalue decomposition will provide dispersion relations (in form of the eigenvalues) and the characteristic waves in primitive variables (in form of the eigenvectors). The eigenvalues are given as (where $k^2 = \alpha^2 + \epsilon^2 + \beta^2$) :

$$\Lambda = \begin{pmatrix} \alpha \bar{U} & & & & \\ & \alpha \bar{U} & & & \\ & & \alpha \bar{U} & & \\ & & & \alpha \bar{U} - \bar{a} k & \\ & & & & \alpha \bar{U} + \bar{a} k \end{pmatrix}$$

The eigenvectors are given by:

$$\nu_1 = \begin{pmatrix} 0 \\ -\epsilon/\alpha \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \nu_2 = \begin{pmatrix} 0 \\ -\beta/\alpha \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \nu_3 = \begin{pmatrix} -\bar{\rho}/\bar{T} \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \nu_4 = \begin{pmatrix} \bar{\rho}/(\gamma-1)\bar{T} \\ -\alpha \bar{a}_0/(\gamma-1)\bar{T} k \\ -\epsilon \bar{a}_0/(\gamma-1)\bar{T} k \\ -\beta \bar{a}_0/(\gamma-1)\bar{T} k \\ 1 \end{pmatrix} \quad \nu_5 = \begin{pmatrix} \bar{\rho}/(\gamma-1)\bar{T} \\ \alpha \bar{a}_0/(\gamma-1)\bar{T} k \\ \epsilon \bar{a}_0/(\gamma-1)\bar{T} k \\ \beta \bar{a}_0/(\gamma-1)\bar{T} k \\ 1 \end{pmatrix}$$

Here ν_1 and ν_2 are vorticity waves. ν_3 is an entropy wave. ν_4 and ν_5 are the slow and fast acoustic

waves respectively.

6.3.1 2D waves

For 2D waves $\beta = 0$, $k = \sqrt{(\alpha^2 + \epsilon^2)} = \sqrt{(\alpha^2 + \alpha^2 \tan^2 \theta)} = \alpha \sec \theta = \alpha / \cos \theta$

$$\Lambda = \begin{pmatrix} \alpha \bar{U} & & & & \\ & \alpha \bar{U} & & & \\ & & \alpha \bar{U} & & \\ & & & \alpha \bar{U} - \bar{a} \frac{\alpha}{\cos \theta} & \\ & & & & \alpha \bar{U} + \bar{a} \frac{\alpha}{\cos \theta} \end{pmatrix}$$

The eigenvectors are given by:

$$\nu_1 = \begin{pmatrix} 0 \\ -\tan \theta \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \nu_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \nu_3 = \begin{pmatrix} -\bar{\rho}/\bar{T} \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \nu_4 = \begin{pmatrix} \bar{\rho}/(\gamma-1)\bar{T} \\ -\bar{a}_0 \cos \theta / (\gamma-1)\bar{T} \\ -\bar{a}_0 \sin \theta / (\gamma-1)\bar{T} \\ 0 \\ 1 \end{pmatrix} \quad \nu_5 = \begin{pmatrix} \bar{\rho}/(\gamma-1)\bar{T} \\ \bar{a}_0 \cos \theta / (\gamma-1)\bar{T} \\ \bar{a}_0 \sin \theta / (\gamma-1)\bar{T} \\ 0 \\ 1 \end{pmatrix}$$

6.4 Freestream waves: an Inflow I/O approach

6.4.1 Relation with freestream waves

In many flows it is observed that $\sigma_1 \gg \sigma_{i>1}$, that is to say that most of the information relevant to disturbance amplification is contained in the principal modes. In general it is observed that a very small number of singular modes contain most of the energy in the flow, and this is a testament to the low-rank nature of the flow operator. The fluid instability process is often preceded by a receptivity process whereby the instability wave acquires some initial amplitude. One of the mechanisms of receptivity is through the action of the freestream noise. The value of this initial amplitude is determined by the level of the noise and the nature of the noise. If a high amplitude is imparted to the wave at this stage the work required to be done by modal instability to reach critical amplitude is reduced (transition location will be correlated with smaller N factors). In our case, we hypothesize that as the set of singular values is very top-heavy most of the action of receptivity to freestream waves will be contained in only a few singular modes. Here, two things become important:

- The part of the freestream wave in direction of the singular mode (indicated by the inner product)
- The amplification experienced by the mode itself (indicated by the singular values)

This can be illustrated for a 2D case where the noise field S can be expressed as:

A is the magnitude associated with a wave type indicated by: $\hat{s}, \hat{f}, \hat{v}, \hat{e}$

$$\begin{aligned}
S = & \sum_{\omega_v} \sum_{\epsilon_v} A(\omega_v, \epsilon_v) \hat{v}(\omega_v, \epsilon_v) e^{i(\frac{\omega_v}{U} x + \epsilon_v y - \omega_v t)} \\
& + \sum_{\omega_e} \sum_{\epsilon_e} A(\omega_e, \epsilon_e) \hat{e}(\omega_e, \epsilon_e) e^{i(\frac{\omega_e}{U} x + \epsilon_e y - \omega_e t)} \\
& + \sum_{\omega_s} \sum_{\epsilon_s} A(\omega_s, \epsilon_s) \hat{s}(\omega_s, \epsilon_s) e^{i(\frac{\omega_s \cos \theta_s}{U \cos \theta_s - \bar{a}_0} x + \epsilon_s y - \omega_s t)} \\
& + \sum_{\omega_f} \sum_{\epsilon_f} A(\omega_f, \epsilon_f) \hat{f}(\omega_f, \epsilon_f) e^{i(\frac{\omega_f \cos \theta_f}{U \cos \theta_f + \bar{a}_0} x + \epsilon_f y - \omega_f t)}
\end{aligned}$$

Writing $\hat{x} e^{i(\alpha x + \epsilon y)} = \hat{x}_g$, in the frequency domain the components with a certain frequency ω are written as:

$$S_\omega = \sum_{\epsilon_v} A(\epsilon_v) \hat{v}_g(\epsilon_v) e^{-i\omega t} + \sum_{\epsilon_e} A(\epsilon_e) \hat{e}_g(\epsilon_e) e^{-i\omega t} + \sum_{\epsilon_s} A(\epsilon_s) \hat{s}_g(\epsilon_s) e^{-i\omega t} + \sum_{\epsilon_f} A(\epsilon_f) \hat{f}_g(\epsilon_f) e^{-i\omega t}$$

The harmonic response is given as: $R_\omega = \mathcal{H}(\omega) S_\omega$. Expanding the wave vectors in terms of the singular inputs of $\mathcal{H}$. $\hat{V}_i$ are the singular inputs of the transfer matrix and $x_{is}(i, s, \epsilon_s)$ for instance is the inner product of the that specific slow acoustic wave with the i^{th} singular input and N_r is the number of singular vectors selected in the expansion.

$$\begin{aligned}
R_\omega = & \sum_{\epsilon_v} A(\epsilon_v) \mathcal{H} \sum_{i=1}^{N_r} [x_{iv} \hat{V}_i] e^{-i\omega t} + \sum_{\epsilon_e} A(\epsilon_e) \mathcal{H} \sum_{i=1}^{N_r} [x_{ie} \hat{V}_i] e^{-i\omega t} \\
& + \sum_{\epsilon_s} A(\epsilon_s) \mathcal{H} \sum_{i=1}^{N_r} [x_{is} \hat{V}_i] e^{-i\omega t} + \sum_{\epsilon_f} A(\epsilon_f) \mathcal{H} \sum_{i=1}^{N_r} [x_{if} \hat{V}_i] e^{-i\omega t}
\end{aligned}$$

The resolvent operator $\mathcal{H}$ acts on a unit normal input $\hat{V}_i$ to give a unit normal output $\hat{q}_i$ with an amplification factor given by the singular value σ_i .

$$\begin{aligned}
R_\omega = & \sum_{\epsilon_v} A(\epsilon_v) \sum_{i=1}^{N_r} [\sigma_i x_{iv} \hat{q}_i] e^{-i\omega t} + \sum_{\epsilon_e} A(\epsilon_e) \sum_{i=1}^{N_r} [\sigma_i x_{ie} \hat{q}_i] e^{-i\omega t} \\
& + \sum_{\epsilon_s} A(\epsilon_s) \sum_{i=1}^{N_r} [\sigma_i x_{is} \hat{q}_i] e^{-i\omega t} + \sum_{\epsilon_f} A(\epsilon_f) \sum_{i=1}^{N_r} [\sigma_i x_{if} \hat{q}_i] e^{-i\omega t}
\end{aligned} \tag{6.10}$$

A Rank-1 approximation ($N_r = 1$) is given as:

$$R_\omega = \sum_{\epsilon_v} A(\epsilon_v)[\sigma_1 x_{1v} \hat{q}_i] e^{-i\omega t} + \sum_{\epsilon_e} A(\epsilon_e)[\sigma_1 x_{1e} \hat{q}_i] e^{-i\omega t} \\ + \sum_{\epsilon_s} A(\epsilon_s)[\sigma_1 x_{1s} \hat{q}_i] e^{-i\omega t} + \sum_{\epsilon_f} A(\epsilon_f)[\sigma_1 x_{1f} \hat{q}_i] e^{-i\omega t}$$

Here, qs and σs are readily available once the the SVD the transfer function for a certain frequency is performed. $x s$ are the inner products between the singular inputs and characteristic freestream waves which can be obtained at almost no cost. So if the amplitudes $A s$ are known a priori or can be modelled the response of the field to any kind of freestream noise environment can be reconstructed fairly quickly (the only investment in this regard would be the initial SVD computation which is not specific to any particular wave and depends only on the baseflow). Moreover, even if the amplitudes are not known but if it can be reasonably assumed that no particular freestream wave/s dominate the forcing field disproportionately then the response of the system would be dominated by the waves having high values of ' σx '.

6.4.2 Quantitative Metric

If we examine the equation for DHR $\hat{q}_g = \mathcal{H}(\omega) \hat{q}_g$, it can be seen that the transfer matrix $\mathcal{H}$ maps the inflow $\hat{q}_g$ to the response $\hat{q}$. Different types of inflow forcings would excite different levels of response in the field. The forcing shape which maximizes the energy of the response is known as the 'optimal forcing'. Thus we want to maximize the energy of the response relative to the forcing.

$$G = \frac{\|\hat{q}\|_E^2}{\|\hat{q}_g\|_E^2} = \frac{\hat{q}^* \mathcal{M}_U \hat{q}}{\hat{q}_g^* \mathcal{M}_U \hat{q}_g} = \frac{\hat{q}_g^* \mathcal{H}^* \mathcal{M}_U^{\frac{1}{2}*} \mathcal{M}_U^{\frac{1}{2}} \mathcal{H} \hat{q}_g}{\hat{q}_g^* \mathcal{M}_U^{\frac{1}{2}*} \mathcal{M}_U^{\frac{1}{2}} \hat{q}_g} \quad (6.11)$$

Here the matrix $\mathcal{M}_U$ indicates the Chu norm energy. The subscript U indicates conserved variables. For the more familiar format based on the primitive variable the matrix $\mathcal{M}$ in a cell i is given as:

$$\mathcal{M}_i = \left(\frac{\bar{c}^2}{\gamma \bar{\rho}}, \bar{\rho}, \bar{\rho}, \bar{\rho}, \frac{\bar{\rho} c_v}{T} \right)^* \mathbb{V}_i \quad (6.12)$$

The optimal forcing and response are u_1 and v_1 respectively (the columns of the unitary matrices). The energy of some global disturbance vector x is given as $E_{chu} = \int_V x^* \mathcal{M}_U x$. Naturally, this is a volume dependent quantity. As a volume independent metric we propose energy density e_{chu} .

$$e_{chu} = \frac{E_{chu}}{V} \quad (6.13)$$

In the case of DHR, the forcing is some q_g and the response to it is some q . Then,

$$\frac{E_{chu}}{E_{chu}^{(G)}} = \frac{q^* \mathcal{M}_U q}{q_g^* \mathcal{M}_U q_g} \quad (6.14)$$

this ratio indicates the amplification of the total energy contained in the domain relative to the total energy contained in the inflow plane at any instant of time. To avoid the dependence on volume in this case we look at the ratio:

$$\Phi = \frac{e_{chu}}{e_{chu}^{(G)}} = \frac{E_{chu}/\Psi}{E_{chu}^{(G)}/\Psi_G} \quad (6.15)$$

In this case if a wave travels through a freestream domain under inviscid conditions, the ratio $\frac{e_{chu}}{e_{chu}^{(G)}}$ would be unity.

6.4.3 Numerical details:

6.4.3.1 Dot product for complex numbers

For complex vectors **a** and **b**, the dot product α is given as:

$$\alpha = \langle \mathbf{a}, \mathbf{b} \rangle = \sum_i a_i \bar{b}_i \quad (6.16)$$

For general complex vectors **a**, **b**, the dot product is also complex (it is real if **a** = **b**).

6.4.3.2 Effect of phase difference:

Let us consider two waves $\mathbf{a}e^{i\omega t}$ and $\mathbf{b}e^{i\omega t}$. Their dot product is given by $\alpha = A + Bi$. And the abs. value given as: $|\alpha| = \sqrt{A^2 + B^2}$

Now if we consider the same two waves, but with a relative phase difference between the two waves: $\mathbf{a}e^{i\omega t}$ and $\mathbf{b}e^{i\omega t} e^{i\phi}$. The dot product now is $\alpha = (A + Bi)(\cos(\phi) + i \sin(\phi))$

$$\alpha = (A \cos(\phi) - B \sin(\phi)) + (A \sin(\phi) + B \cos(\phi))i \quad (6.17)$$

$$|\alpha| = \sqrt{A^2 + B^2} \quad (6.18)$$

Thus the relative phase is not a factor.

6.5 Flat plate hypersonic boundary layer (Mach 6): resolvent and inflow I/O analysis

A flat plate hypersonic boundary layer with a freestream Mach number $M_\infty = 6$ and a unit Reynolds number based on freestream conditions, $Re_\infty = 2 \times 10^6 m^{-1}$ is considered. The viscosity is based on a power law: $\mu(T) = \mu_\infty (T/T_\infty)^{0.7}$. The conditions match those used in [35]. We consider isothermal

walls for a range of wall temperatures ($T_w = 2T_\infty$ to $T_w = 6T_\infty$). The latter represents a slightly cooled wall (close to quasi-adiabatic conditions), while the former represents a highly cooled wall.

The baseflow is obtained using the sixth ordered kinetic energy consistent (KEC) fluxes of [89]. The grid used has 1778 points in the streamwise direction and 300 points in the wall normal direction. The baseflow has been shown in Fig.6.1. The region upstream of $x = 0$, is supersonic freestream and the viscous wall starts at $x = 0$. As a result, a shock emanates from this point. It is important that the shock is captured well, specially near the leading edge, as this is the region where freestream disturbances can interact with the shock and then trigger boundary layer instabilities. In view of this grid clustering is applied near the leading edge in the streamwise direction. Additionally, near wall clustering in the wall normal direction has also been used.

Figure 6.1 Baseflow: contours of Mach number

A full field resolvent analysis and an inflow-I/O analysis is conducted. The non-dimensional frequency parameter $F = \omega \mu_\infty / \rho_\infty U_\infty^2$, has been used. The results over a wide range of frequencies from $F = 1 \times 10^{-5}$ to 3×10^{-4} , for different values of wall temperatures is shown in Fig.6.2. It can be seen that the value of frequency for which the peak gain is observed shifts towards the right as the wall temperature decreases (TW'k' for instance refers to wall temperature that is 'k' times the freestream temperature). Moreover, the peak is higher for colder walls. Additionally, the results are also examined using a different frequency parameter, F_w , which is based on the wall temperature instead of the freestream temperature and can be seen in Fig.6.2 (b). In this case the peak seems to move to the left as the wall temperature decreases.

The inflow-I/O analysis for the same problem is used to calculate the the most optimal inflow shapes over a range of frequencies for different wall temperatures . The gain associated with these frequencies for different wall temperatures are shown in Fig.6.3. The absolute values of the gain between the resolvent analysis and the inflow-I/O analysis can not be compared as they have different units. Similar to the earlier results on the full resolvent analysis it can be seen that the peak shifts towards the right for F and towards the left for F_w as the wall temperature decreases. However,

unlike the previous results it can be seen that the amplitude of the peak reduces as wall temperature decreases in the inflow-I/O analysis.

Figure 6.2 Gain v. F for a full field resolvent analysis

The peak responses for both the resolvent and the inflow-I/O analysis can be seen in Fig.6.4 and Fig.6.5 respectively.

6.6 Response of a flat plate hypersonic boundary layer to freestream waves: DHR and Inflow-I/O analysis

This case involves a hypersonic boundary layer over a flatplate. The case is set up in similar fashion as Sec.5.2.1. The cases C1, C2 and C3 from Table.5.2 are considered here. The flatplate is 1m in length and the leading edge is 0.05m downstream of the inflow, in order to ensure that there is uniform supersonic inflow in the region near the inflow boundary. A symmetry boundary condition is applied at the bottom boundary in the uniform flow region. A sponge layer is applied at the top wall from $y = 0.175$ to $y = 0.2$ (the implementation details can be found in Bogey et al. [16]). The grid used is $N_x \times N_y = 831 \times 201$. The 800 cells are located on the wall and 30 cells are in the freestream region between the inflow and the leading edge.

To establish the validity of this approach, the inflow-I/O approach is compared with direct harmonic response (which is considered as a “true value”) for Case C1. The flow is subjected to freestream waves (slow/fast acoustic waves, vorticity waves and entropy waves) at different frequencies with wide range of inclination angles. A DHR computation is performed for each combination of the parametric sweep over frequency (F) and the angle of orientation (θ). For each of these computations the average energy density amplification (Φ) is recorded. This can be seen in Fig.6.6

Figure 6.3 Gain v. F for an inflow I/O analysis where the forcing is restricted to the inflow plane

Figure 6.4 Response for resolvent: contours of normalized pressure disturbance

Figure 6.5 Response for inflow-I/O: contours of normalized pressure disturbance

(f), Fig.6.7 (f), Fig.6.8 (f), and Fig.6.9 (f). The rationale behind the inflow-I/O approach is that “most” of the system response/information is dominated by a very limited (“few”) number of modes and the inflow-I/O (SVD of the transfer function $\mathcal{H} = (-i\omega - \mathcal{A})^{-1} \mathcal{A}_g$) helps to identify those modes. A reconstruction using the singular modes (and dot products) is conducted to compare with the direct harmonic response. This is done using 1 to 19 modes (with more modes being able to paint a more complete picture). In Fig.6.6 the response to the slow acoustic wave is considered. The slow acoustic wave incites the most energetic response for this case. It can be seen that just a rank-1 approximation of the inflow-I/O reproduces almost entirely the direct harmonic response. The advantage of this is that once an inflow-I/O analysis is conducted for each frequency the response over different angles (θ) can be conducted by an extremely inexpensive inner product method whereas the direct harmonic response requires an independent computation for each combination of θ and F . Unlike the slow wave, the response to the fast acoustic waves and the vorticity waves is not as fully reproduced by the rank-1 approximation, but it is almost fully reproduced by accounting for about a dozen modes as can be seen Fig.6.7 and Fig.6.8 where approximations by considering 1, 5, 10, 15 and 19 modes are shown. This number is still extremely small compared to the degrees of freedom in this system ($\approx 0.8M$). The response to the entropy wave is not as well captured, but the overall response of the flow is itself so low that the fact that inflow-I/O was not able to as fully reconstruct the response is probably inconsequential.

6.6.1 Wall temperature effects

In this section the effect of wall temperature on the response of the system to freestream waves is considered. This is done through Cases C1, C2 and C3. For each of the three cases, the rank-1 reconstruction for all the four types of waves is shown in Fig.6.10. It can be seen that the response of the system to the freestream waves is much weaker for the two colder wall cases (C2 and C3)

Figure 6.6 $\Phi(F(\times 10^{-4}), \theta)$ comparison between inflow-I/O based reconstruction (a)-(e) and direct harmonic response (f) for a slow acoustic wave

Figure 6.7 $\Phi(F(\times 10^{-4}), \theta)$ comparison between inflow-I/O based reconstruction (a)-(e) and direct harmonic response (f) for a fast acoustic wave

Figure 6.8 $\Phi(F(\times 10^{-4}), \theta)$ comparison between inflow-I/O based reconstruction (a)-(e) and direct harmonic response (f) for a vorticity wave

Figure 6.9 $\Phi(F(\times 10^{-4}), \theta)$ comparison between inflow-I/O based reconstruction (a)-(e) and direct harmonic response (f) for an entropy wave

compared to the baseline case (C1). This happens because the gain levels (σ_1^2) in this case get depressed by a cooling of the wall. This is remarkably different from the more predictable trend observed for a traditional I/O analysis where wall cooling increases the gain levels (due to the destabilization of the second mode with wall cooling). The reason here perhaps is the weakening of the disturbance by wall cooling before (in the region upstream of the point where the second mode becomes unstable) the second mode instability sets in. It can be seen that the one wave which departs from this trend is the fast wave which exhibits a more energetic response as the wall becomes cooler. This can be related to the different paths provided in Fig.1.7, where the path A1 appears to be the more dominant path (compared to A3) as the wall becomes cooler. To look at the effect of the projection alone, dot products of the canonical freestream waves with the first singular mode are visualized in Fig.6.11. The trend of the slow acoustic wave getting less effective and the fast acoustic wave getting more effective with wall cooling is also observed here.

6.7 Hypersonic flow over a sharp wedge

We investigate a hypersonic flow for a 7° half-angle sharp wedge. This case was originally investigated by Balakumar & Kegerise [6]. They performed DNS calculations for different types of freestream waves and the response of the flowfield to them. The nose radius was taken to be 0.015 mm. The flow conditions are given as:

Table 6.1 Flow Conditions

Flow Parameter	Value
M_∞	6.0
ρ_∞	0.04062 kg/m^3
T_∞	52.44° K
T_{wall}	300.0° K
U_∞	870.9 m/s
Re_∞	$10.4 * 10^6 / m$

An inflow-I/O analysis is conducted by restricting the output to only one side of the wedge. This was done purposely so as to differentiate between the response of the flowfield to waves with $+\theta$ and $-\theta$ orientation. The inflow I/O gain is shown in Fig.6.12. Two distinct peaks can be seen: the lower frequency one corresponds to the first mode whereas the higher frequency one corresponds to the second mode. The flow response at these two frequencies to the optimal forcing is shown in Fig.6.13. The left pane shows the first mode and the right pane shows the second mode. Each row shows a different region of the flow.

The inflow-I/O analysis with 2D freestream waves is conducted and the results can be seen in Fig.6.14. The slow wave by far is the most potent and it peaks for $\theta \approx 20^\circ$. This is in agreement with the results of Balakumar & Kegerise [6]. The vorticity wave is the second most effective, with

Figure 6.10 $\Phi(F(\times 10^{-4}), \theta)$ Rank-1 response: effect of wall temperature; first row: slow acoustic wave, second row: fast acoustic wave, third row: vorticity wave and fourth row: entropy wave

Figure 6.11 $\Phi(F(\times 10^{-4}), \theta)$ inner product with the first singular vector: effect of wall temperature; first row: slow acoustic wave, second row: fast acoustic wave, third row: vorticity wave and fourth row: entropy wave

Figure 6.12 Gain v. $F(\times 10^{-4})$ from Inflow I/O analysis for a flow over a 7° half-angle sharp wedge

a preference for waves orientated at $\theta \approx 45^\circ$. The response to fast acoustic and entropy waves is significantly weaker.

Figure 6.13 $M_\infty = 5.92$ flow over a sharp cone: response to the optimal inflow forcing: nose region (top row), end of the domain (middle row), full-field (bottom row)

(a) Slow wave

(b) Fast wave

(c) Vorticity wave

(d) Entropy wave

Figure 6.14 $\Phi(\theta, F \times 10^{-4})$ for a hypersonic flow over 7° half angle sharp wedge

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APPENDICES

APPENDIX

A

ARTIFICIAL CONTRIBUTIONS: VISCOUS FLUX

A standard term in the viscous flux can be shown as:

$$\frac{\partial}{\partial x_l} \beta g \frac{\partial f}{\partial x_m} \quad (\text{A.1})$$

- Type 1: x_l is homogeneous; x_m is NOT
- Type 2: x_l is NOT; x_m is homogeneous
- Type 3: x_l is homogeneous; x_m is homogeneous
- Type 4: x_l is NOT; x_m is NOT

Generalized viscous term the linearized form:

$$\begin{aligned} F^{\vec{V}} = & \left[\bar{\beta} \frac{\partial g'}{\partial x_l} + \bar{g} \frac{\partial \beta'}{\partial x_l} + \beta' \frac{\partial \bar{g}}{\partial x_l} + g' \frac{\partial \bar{\beta}}{\partial x_l} \right] \frac{\partial \bar{f}}{\partial x_m} \\ & + \left[\bar{\beta} \frac{\partial \bar{g}}{\partial x_l} + \bar{g} \frac{\partial \bar{\beta}}{\partial x_l} \right] \frac{\partial f'}{\partial x_m} \\ & + [\bar{\beta} g' + \beta' \bar{g}] \frac{\partial \bar{f}}{\partial x_l \partial x_m} \\ & + [\bar{\beta} \bar{g}] \frac{\partial f'}{\partial x_l \partial x_m} \end{aligned}$$

The case 4 is handled in the code through the usual subroutine. For the specialized cases 1, 2 and 3, the simplified forms are given by:

$$\begin{aligned} F_1^{\vec{\lambda}(V)} &= [i\alpha_l \bar{\beta} g' + i\alpha_l \bar{g} \beta'] \frac{\partial \bar{f}}{\partial x_m} + i\alpha_l [\bar{\beta} \bar{g}] \frac{\partial f'}{\partial x_m} \\ F_2^{\vec{\lambda}(V)} &= i\alpha_m \left[\bar{\beta} \frac{\partial \bar{g}}{\partial x_l} + \bar{g} \frac{\partial \bar{\beta}}{\partial x_l} \right] f' + i\alpha_m [\bar{\beta} \bar{g}] \frac{\partial f'}{\partial x_l} \\ F_3^{\vec{\lambda}(V)} &= i\alpha_l i\alpha_m [\bar{\beta} \bar{g}] f' \end{aligned}$$

A.0.1 Term 1

$$F_1^{\vec{\lambda}(V)} = \underbrace{[i\alpha_l \bar{\beta} g' + i\alpha_l \bar{g} \beta'] \frac{\partial \bar{f}}{\partial x_m}}_{\text{Diagonal}} + \underbrace{i\alpha_l [\bar{\beta} \bar{g}] \frac{\partial f'}{\partial x_m}}_{\text{Off-Diagonal}} \quad (\text{A.2})$$

The diagonal portion is handled as:

For the diagonal portion the term is: $[i\alpha_l \bar{\beta} g' + i\alpha_l \bar{g} \beta'] \frac{\partial \bar{f}}{\partial x_m}$

Finite Vol. approximation: $i\alpha_l \frac{\partial \bar{f}_i}{\partial x_m} [\bar{\beta}_i g'_i + \bar{g}_i \beta'_i]$

For conserved: $i\alpha_l \frac{\partial \bar{f}_i}{\partial x_m} [\bar{\beta}_i \frac{\partial V_i}{\partial U_i}(g, :) q'_i + \bar{g}_i \bar{\beta}_i \frac{\partial V_i}{\partial U_i}(5, :) q'_i]$

The Off-diagonal portion is handled as:

For the off-diagonal portion the term is: $i\alpha_l \bar{\beta} \bar{g} \frac{\partial f'}{\partial x_m}$

The finite volume approximation is: $i\alpha_l \bar{\beta}_i \bar{g}_i \left(\frac{\partial f'}{\partial x_m} \right)_i$

Using a Green-Gauss approximation: $i\alpha_l \bar{\beta}_i \bar{g}_i \frac{1}{V_i} \sum_k \frac{f'_i + f'_{ii}}{2} \hat{n}_{xm} S_k$

For conserved: $i\alpha_l \bar{\beta}_i \bar{g}_i \frac{1}{V_i} \sum_k \frac{1}{2} \frac{\partial V_i}{\partial U_i}(f, :) \hat{n}_{xm} S_k q'_i + i\alpha_l \bar{\beta}_i \bar{g}_i \frac{1}{V_i} \sum_k \frac{1}{2} \frac{\partial V_{ii}}{\partial U_{ii}}(f, :) \hat{n}_{xm} S_k q'_{ii}$ ii is a ghost cell:

$$i\alpha_l \bar{\beta}_i \bar{g}_i \frac{1}{V_i} \sum_k \frac{1}{2} \frac{\partial V_i}{\partial U_i}(f, :) \hat{n}_{xm} S_k q'_i + i\alpha_l \bar{\beta}_i \bar{g}_i \frac{1}{V_i} \sum_k \frac{1}{2} \underbrace{\mathbf{B}_{\text{Rot}}(f, :)}_{B_{\text{Rot}} = A_{\text{Rot}} \frac{\partial V_i}{\partial U_i}} \hat{n}_{xm} S_k q'_i$$

A.0.2 Term 2

$$F_2^{\vec{\lambda}(V)} = i\alpha_m \left[\bar{\beta} \frac{\partial \bar{g}}{\partial x_l} + \bar{g} \frac{\partial \bar{\beta}}{\partial x_l} \right] f' + i\alpha_m [\bar{\beta} \bar{g}] \frac{\partial f'}{\partial x_l} \quad (\text{A.3})$$

The diagonal portion is handled as:

For the diagonal portion the term is: $i\alpha_m \left[\bar{\beta} \frac{\partial \bar{g}}{\partial x_l} + \bar{g} \frac{\partial \bar{\beta}}{\partial x_l} \right] f'$

Finite Vol. approximation: $i\alpha_m \left[\bar{\beta}_i \frac{\partial \bar{g}_i}{\partial x_l} + \bar{g}_i \frac{\partial \bar{\beta}_i}{\partial x_l} \right] f'_i$

For conserved: $i\alpha_m \left[\bar{\beta}_i \frac{\partial \bar{g}_i}{\partial x_l} + \bar{g}_i \frac{\partial \bar{\beta}_i}{\partial \bar{T}_i} \frac{\partial \bar{T}_i}{\partial x_l} \right] \frac{\partial V_i}{\partial U_i}(f, :) q'_i$

The Off-diagonal portion is handled as:

For the off-diagonal portion the term is: $i\alpha_m [\bar{\beta} \bar{g}] \frac{\partial f'}{\partial x_l}$

The finite volume approximation is: $i\alpha_m \bar{\beta}_i \bar{g}_i \left(\frac{\partial f'}{\partial x_l} \right)_i$

Using a Green-Gauss approximation: $i\alpha_m \bar{\beta}_i \bar{g}_i \frac{1}{V_i} \sum_k \frac{f'_i + f'_{ii}}{2} \hat{n}_{xl} S_k$

For conserved: $i\alpha_m \bar{\beta}_i \bar{g}_i \frac{1}{V_i} \sum_k \frac{1}{2} \frac{\partial V_i}{\partial U_i}(f, :) \hat{n}_{xl} S_k q'_i + i\alpha_m \bar{\beta}_i \bar{g}_i \frac{1}{V_i} \sum_k \frac{1}{2} \frac{\partial V_{ii}}{\partial U_{ii}}(f, :) \hat{n}_{xl} S_k q'_{ii}$ ii is a ghost cell:

$$i\alpha_m \bar{\beta}_i \bar{g}_i \frac{1}{V_i} \sum_k \frac{1}{2} \frac{\partial V_i}{\partial U_i}(f, :) \hat{n}_{xl} S_k q'_i + i\alpha_m \bar{\beta}_i \bar{g}_i \frac{1}{V_i} \sum_k \frac{1}{2} \underbrace{\mathbf{B}_{\text{Rot}}(f, :)}_{B_{\text{Rot}} = A_{\text{Rot}} \frac{\partial V_i}{\partial U_i}} \hat{n}_{xl} S_k q'_i$$

A.0.3 Term 3

$$F_3^{\vec{\lambda}(V)} = i\alpha_l i\alpha_m [\bar{\beta} \bar{g}] f' \quad (\text{A.4})$$

The diagonal portion is handled as:

For the diagonal portion the term is: $i\alpha_l i\alpha_m [\bar{\beta} \bar{g}] f' = -\alpha_l \alpha_m \bar{\beta} \bar{g} f'$

Finite Vol. approximation: $-\alpha_l \alpha_m \bar{\beta}_i \bar{g}_i f'_i$

For conserved: $[-\alpha_l \alpha_m \bar{\beta}_i \bar{g}_i] \frac{\partial V_i}{\partial U_i}(f, :) q'_i$

APPENDIX

B

AXISYMMETRIC MODULE

B.1 Treatment of the inviscid flux

Steps to convert standard code to handle axisymmetric problems:

- Step 1. Change $\Psi_i \rightarrow \Psi'_i$, $S_k \rightarrow S'_k$
- Step 2. Make: $\hat{n}_\theta = 0$
- Step 3. Add along the diagonal: $-\frac{im}{r} A_{\theta i}$
- Step 4. Add along the diagonal: $\left(\frac{\partial \mathcal{Q}}{\partial q}\right)$

$$\frac{\partial \mathcal{Q}}{\partial q} = \frac{\partial \mathcal{Q}}{\partial v_p} \frac{\partial v_p}{\partial q} \tag{B.1}$$

$$\frac{\partial \mathcal{Q}}{\partial v_p} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{u_\theta^2}{r} & 0 & 0 & \frac{2\rho u_\theta}{r} & \frac{1}{r} \\ -\frac{u_\theta u_r}{r} & 0 & -\frac{\rho u_\theta}{r} & -\frac{\rho u_r}{r} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

B.2 Treatment of the viscous flux

The viscous flux tensor is given as:

$$\begin{pmatrix} \tau_{xx} & \tau_{xr} & \tau_{x\theta} \\ \tau_{rx} & \tau_{rr} & \tau_{r\theta} \\ \tau_{\theta x} & \tau_{\theta r} & \tau_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 2\mu \frac{\partial u}{\partial x} + \lambda \delta_0 + \lambda \frac{u_r}{r} & \mu \left(\frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x} \right) & \mu \left(\frac{\partial u_\theta}{\partial x} + \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) \\ \mu \left(\frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x} \right) & 2\mu \frac{\partial u_r}{\partial r} + \lambda \delta_0 + \lambda \frac{u_r}{r} & \mu \left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right) \\ \mu \left(\frac{\partial u_\theta}{\partial x} + \frac{1}{r} \frac{\partial u_x}{\partial \theta} \right) & \mu \left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right) & \mu \left(\frac{2}{r} \frac{\partial u_\theta}{\partial \theta} + 2 \frac{u_r}{r} \right) + \lambda \delta_0 + \lambda \frac{u_r}{r} \end{pmatrix}$$

B.2.1 Regular viscous terms

A generalized term looks like: $\frac{1}{r} \frac{1}{r^p} \frac{\partial}{\partial x_l} r \beta g \frac{1}{r^q} \frac{\partial f}{\partial x_m}$

Here: $\delta_{3l} = p$, $\delta_{3m} = q$

The case 4 is handled in the code through the usual subroutine. For the specialized cases 1, 2 and 3, the simplified forms are given by:

Type 1: x_l is homo. i.e. x_l CAN'T BE r ; x_m is NOT

$$\mathcal{F}_{visc,1} = \frac{1}{r^{p+1}} \frac{\partial}{\partial x_l} \left[\beta g \frac{1}{r^{q-1}} \frac{\partial f}{\partial x_m} \right]$$

$$\mathcal{F}_{visc,1} = \frac{1}{r^{p+1}} \left[\frac{\partial \beta}{\partial x_l} g \frac{1}{r^{q-1}} \frac{\partial f}{\partial x_m} + \beta \frac{\partial g}{\partial x_l} \frac{1}{r^{q-1}} \frac{\partial f}{\partial x_m} + \underbrace{\beta g \frac{\partial}{\partial x_l} \left(\frac{1}{r^{q-1}} \right)}_{\Psi} \frac{\partial f}{\partial x_m} + \beta g \frac{1}{r^{q-1}} \frac{\partial f}{\partial x_l \partial x_m} \right]$$

$$\begin{aligned} \mathcal{F}_{visc,1} = \frac{1}{r^{p+1}} & \left[\frac{\partial \beta'}{\partial x_l} g \frac{1}{r^{q-1}} \frac{\partial f}{\partial x_m} + \beta' \frac{\partial g}{\partial x_l} \frac{1}{r^{q-1}} \frac{\partial f}{\partial x_m} + \beta' g \Psi \frac{\partial f}{\partial x_m} + \beta' g \frac{1}{r^{q-1}} \frac{\partial f}{\partial x_l \partial x_m} \right. \\ & + \frac{\partial \beta}{\partial x_l} g' \frac{1}{r^{q-1}} \frac{\partial f}{\partial x_m} + \beta \frac{\partial g'}{\partial x_l} \frac{1}{r^{q-1}} \frac{\partial f}{\partial x_m} + \beta g' \Psi \frac{\partial f}{\partial x_m} + \beta g' \frac{1}{r^{q-1}} \frac{\partial f}{\partial x_l \partial x_m} \\ & \left. + \frac{\partial \beta}{\partial x_l} g \frac{1}{r^{q-1}} \frac{\partial f'}{\partial x_m} + \beta \frac{\partial g}{\partial x_l} \frac{1}{r^{q-1}} \frac{\partial f'}{\partial x_m} + \beta g \Psi \frac{\partial f'}{\partial x_m} + \beta g \frac{1}{r^{q-1}} \frac{\partial f'}{\partial x_l \partial x_m} \right] \end{aligned}$$

Here: $\Psi = \frac{\partial}{\partial x_l} \left(\frac{1}{r^{q-1}} \right) = \delta_{2l} \frac{\partial}{\partial r} (r^{1-q}) = \delta_{2l} (1-q) r^{-q}$. This can only be non-zero if $l = r$ & $m \neq \theta$. But for case 1, l is homogeneous and we have assumed that r needs to be an inhomogeneous direction.

$$\begin{aligned} \mathcal{F}_{visc,1} = \frac{1}{r^{p+1}} & \left[\frac{\partial \beta'}{\partial x_l} g \frac{1}{r^{q-1}} \frac{\partial f}{\partial x_m} + 0 + 0 + 0 \right. \\ & + 0 + \beta \frac{\partial g'}{\partial x_l} \frac{1}{r^{q-1}} \frac{\partial f}{\partial x_m} + 0 + 0 \\ & \left. + 0 + 0 + 0 + \beta g \frac{1}{r^{q-1}} \frac{\partial f'}{\partial x_l \partial x_m} \right] \end{aligned}$$

$$\mathcal{F}_{visc,1} = \frac{1}{r^{p+q}} \left[i\alpha_l \beta' g \frac{\partial f}{\partial x_m} + i\beta \alpha_l g' \frac{\partial f}{\partial x_m} + i\alpha_l \beta g \frac{\partial f'}{\partial x_m} \right] \quad (\text{B.2})$$

If $l = \theta$, $p + q = 1 + 0 = 1$

$$\mathcal{F}'_{visc,1} = \underbrace{\frac{i\alpha_l}{\textcolor{red}{r}} [\bar{\beta} g' + \bar{g} \beta'] \frac{\partial \bar{f}}{\partial x_m}}_{\text{Diagonal}} + \underbrace{\frac{i\alpha_l}{\textcolor{red}{r}} [\bar{\beta} \bar{g}] \frac{\partial f'}{\partial x_m}}_{\text{Off-Diagonal}} \quad (\text{B.3})$$

Type 2: x_l is NOT; x_m is homogeneous

$$\begin{aligned} \mathcal{F}_{visc,2} = & \frac{1}{r^{p+1}} [0 + 0 + 0 + 0 \\ & + 0 + 0 + 0 + 0 \\ & + \frac{\partial \beta}{\partial x_l} g \frac{1}{r^{q-1}} \frac{\partial f'}{\partial x_m} + \beta \frac{\partial g}{\partial x_l} \frac{1}{r^{q-1}} \frac{\partial f'}{\partial x_m} + \beta g \Psi \frac{\partial f'}{\partial x_m} + \beta g \frac{1}{r^{q-1}} \frac{\partial f'}{\partial x_l \partial x_m}] \end{aligned}$$

If one of the directions is θ (in this case m) then $p + q = 1$

If $l = r$ & $m \neq \theta$, $\Psi = 1$, else 0

$$\mathcal{F}'_{visc,2} = \frac{i\alpha_m}{\textcolor{red}{r}} \left[\bar{\beta} \frac{\partial \bar{g}}{\partial x_l} + \bar{g} \frac{\partial \bar{\beta}}{\partial x_l} \right] f' + \frac{i\alpha_m}{\textcolor{red}{r}} [\bar{\beta} \bar{g}] \frac{\partial f'}{\partial x_l} + \frac{i\alpha_m}{\textcolor{blue}{r}} \Psi [\bar{\beta} \bar{g}] f' \quad (\text{B.4})$$

Type 3: x_l is homogeneous; x_m is homogeneous

Ψ can only be 1 if $l = r$, but in this case l is homo.

$$\begin{aligned} \mathcal{F}_{visc,3} = & \frac{1}{r^{p+1}} [0 + 0 + 0 + 0 \\ & + 0 + 0 + 0 + 0 \\ & + 0 + 0 + 0 + \beta g \frac{1}{r^{q-1}} \frac{\partial f'}{\partial x_l \partial x_m}] \end{aligned}$$

$$\mathcal{F}'_{visc,3A} = \frac{i\alpha_l i\alpha_m}{\textcolor{red}{r}^{p+q}} [\bar{\beta} \bar{g}] f' \quad (\text{B.5})$$

B.2.2 Special terms:

The special terms are clubbed in the source terms in $\mathcal{Q}_V$

$$\begin{aligned}
\frac{1}{r_i} \mathcal{Q}_V = & \begin{pmatrix} 0 \\ 0 \\ -\frac{\tau_{\theta\theta}}{r} \\ \frac{\tau_{\theta r}}{r} \\ 0 \end{pmatrix} \\
& + \frac{\partial}{\partial x} \begin{pmatrix} 0 \\ \lambda \frac{u_r}{r} \\ 0 \\ 0 \\ u_x \lambda \frac{u_r}{r} \end{pmatrix} + \frac{1}{r_i} \frac{\partial}{\partial r} \left\{ r \begin{pmatrix} 0 \\ 0 \\ \lambda \frac{u_r}{r} \\ -\mu \frac{u_\theta}{r} \\ u_r \lambda \frac{u_r}{r} - u_\theta \mu \frac{u_\theta}{r} \end{pmatrix} \right\} \\
& + \frac{1}{r_i} \frac{\partial}{\partial \theta} \begin{pmatrix} 0 \\ 0 \\ -\mu \frac{u_\theta}{r} \\ \lambda \frac{u_r}{r} + 2\mu \frac{u_r}{r} \\ -u_r \mu \frac{u_\theta}{r} + u_\theta \lambda \frac{u_r}{r} + 2u_\theta \mu \frac{u_r}{r} \end{pmatrix}
\end{aligned}$$

Linearized:

$$\begin{aligned}
\frac{1}{r_i} \mathcal{Q}'_V = & \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{r}(\tau'_{\theta\theta}) \\ +\frac{1}{r}(\tau'_{\theta r}) \\ 0 \end{pmatrix} \\
& + \frac{\partial}{\partial x} \begin{pmatrix} 0 \\ \lambda' \frac{\bar{u}_r}{r} + \bar{\lambda} \frac{u'_r}{r} \\ 0 \\ 0 \\ \bar{u}_x \bar{\lambda} \frac{u'_r}{r} + \bar{u}_x \lambda' \frac{\bar{u}_r}{r} + u'_x \bar{\lambda} \frac{\bar{u}_r}{r} \end{pmatrix} \\
& + \frac{1}{r_i} \frac{\partial}{\partial r} \left\{ r \begin{pmatrix} 0 \\ 0 \\ \bar{\lambda} \frac{u'_r}{r} + \lambda' \frac{\bar{u}_r}{r} \\ -\bar{\mu} \frac{u'_\theta}{r} - \mu' \frac{\bar{u}_\theta}{r} \\ \bar{u}_r \bar{\lambda} \frac{u'_r}{r} + \bar{u}_r \lambda' \frac{\bar{u}_r}{r} + u'_r \bar{\lambda} \frac{\bar{u}_r}{r} - \bar{u}_\theta \bar{\mu} \frac{u'_\theta}{r} - \bar{u}_\theta \mu' \frac{\bar{u}_\theta}{r} - \bar{u}_\theta \bar{\mu} \frac{u'_\theta}{r} \end{pmatrix} \right\} \\
& + \frac{1}{r_i} \frac{\partial}{\partial \theta} \begin{pmatrix} 0 \\ 0 \\ -\bar{\mu} \frac{u'_\theta}{r} - \mu' \frac{\bar{u}_\theta}{r} \\ \bar{\lambda} \frac{u'_r}{r} + \lambda' \frac{\bar{u}_r}{r} + 2\bar{\mu} \frac{u'_r}{r} + 2\mu' \frac{\bar{u}_r}{r} \\ -\bar{u}_r \bar{\mu} \frac{u'_\theta}{r} - \bar{u}_r \mu' \frac{\bar{u}_\theta}{r} - u'_r \bar{\mu} \frac{\bar{u}_\theta}{r} + \bar{u}_\theta \bar{\lambda} \frac{u'_r}{r} + \bar{u}_\theta \lambda' \frac{\bar{u}_r}{r} + u'_\theta \bar{\lambda} \frac{\bar{u}_r}{r} + 2\bar{u}_\theta \bar{\mu} \frac{u'_r}{r} + 2\bar{u}_\theta \mu' \frac{\bar{u}_r}{r} + 2u'_\theta \bar{\mu} \frac{\bar{u}_r}{r} \end{pmatrix}
\end{aligned}$$

Expanded:

$$\begin{aligned}
& \frac{1}{r_i} \mathcal{W}'_V = \left(\begin{array}{c} 0 \\ 0 \\ -\frac{1}{r} \left[\underbrace{\frac{2\tilde{\mu}}{r} \frac{\partial u'_\theta}{\partial \theta} + \frac{2\mu'}{r} \frac{\partial \tilde{u}_\theta}{\partial \theta} + 2\tilde{\mu} \frac{u'_r}{r} + 2\mu' \frac{\tilde{u}_r}{r}}_{(4)} - \frac{2}{3} \tilde{\mu} \frac{\partial u'_x}{\partial x} - \frac{2}{3} \mu' \frac{\partial \tilde{u}_x}{\partial x} - \frac{2}{3} \tilde{\mu} \frac{\partial u'_r}{\partial r} - \frac{2}{3} \mu' \frac{\partial \tilde{u}_r}{\partial r} - \frac{2}{3} \tilde{\mu} \frac{\partial u'_\theta}{\partial \theta} - \frac{2}{3} \mu' \frac{\partial \tilde{u}_\theta}{\partial \theta} - \frac{2}{3} \tilde{\mu} \frac{u'_r}{r} - \frac{2}{3} \mu' \frac{\tilde{u}_r}{r} \right]_{(6)} \\ + \frac{1}{r} \left[\underbrace{\tilde{\mu} \frac{\partial u'_\theta}{\partial r} + \mu' \frac{\partial \tilde{u}_\theta}{\partial r}}_{(11)} + \underbrace{\tilde{\mu} \frac{\partial u'_r}{\partial \theta} + \mu' \frac{\partial \tilde{u}_r}{\partial \theta}}_{(9)} - \underbrace{\tilde{\mu} \frac{u'_\theta}{r} - \mu' \frac{\tilde{u}_\theta}{r}}_{(10)} \right] \\ 0 \end{array} \right) \\
& + \frac{\partial}{\partial x} \left(\begin{array}{c} 0 \\ -\frac{2}{3} \mu' \frac{u'_r}{r} - \frac{2}{3} \tilde{\mu} \frac{\tilde{u}'_r}{r} \\ (1) \\ 0 \\ 0 \\ -u_x \frac{2}{3} \tilde{\mu} \frac{u'_r}{r} - u_x \frac{2}{3} \mu' \frac{\tilde{u}'_r}{r} - u'_x \frac{2}{3} \tilde{\mu} \frac{\tilde{u}_r}{r} \\ (12) \end{array} \right) \\
& + \frac{1}{r_i} \frac{\partial}{\partial r} r \left(\begin{array}{c} 0 \\ 0 \\ -\frac{2}{3} \tilde{\mu} \frac{u'_r}{r} - \frac{2}{3} \mu' \frac{\tilde{u}'_r}{r} \\ (2) \\ -\tilde{\mu} \frac{u'_\theta}{r} - \mu' \frac{\tilde{u}_\theta}{r} \\ (7) \\ -u_r \frac{2}{3} \tilde{\mu} \frac{u'_r}{r} - u_r \frac{2}{3} \mu' \frac{\tilde{u}'_r}{r} - u'_r \frac{2}{3} \tilde{\mu} \frac{\tilde{u}_r}{r} - u_\theta \tilde{\mu} \frac{u'_\theta}{r} - u_\theta \mu' \frac{\tilde{u}_\theta}{r} - u_\theta \tilde{\mu} \frac{u'_r}{r} \\ (13) \quad (14) \end{array} \right) \\
& + \frac{1}{r_i} \frac{\partial}{\partial \theta} \left(\begin{array}{c} 0 \\ 0 \\ -\tilde{\mu} \frac{u'_\theta}{r} - \mu' \frac{\tilde{u}_\theta}{r} \\ (3) \\ -\frac{2}{3} \tilde{\mu} \frac{u'_r}{r} + \frac{2}{3} \mu' \frac{\tilde{u}'_r}{r} + 2\tilde{\mu} \frac{u'_r}{r} + 2\mu' \frac{\tilde{u}_r}{r} \\ (8) \\ -u_r \tilde{\mu} \frac{u'_\theta}{r} - u_r \mu' \frac{\tilde{u}_\theta}{r} - u'_r \tilde{\mu} \frac{\tilde{u}_\theta}{r} - u_\theta \frac{2}{3} \tilde{\mu} \frac{u'_r}{r} - u_\theta \frac{2}{3} \mu' \frac{\tilde{u}'_r}{r} - u'_\theta \frac{2}{3} \tilde{\mu} \frac{\tilde{u}_r}{r} + 2u_\theta \tilde{\mu} \frac{u'_r}{r} + 2u_\theta \mu' \frac{\tilde{u}_r}{r} + 2u'_\theta \tilde{\mu} \frac{\tilde{u}_r}{r} \\ (15) \quad (16) \quad (17) \end{array} \right)
\end{aligned}$$

There are two possible combinations:

B.2.2.1 Case 1: $(x, r, \theta) = (1, 0, 1)$

B.2.2.2 Case 3: $(x, r, \theta) = (0, 0, 1)$

Num	Term	Case 1: $(x, r, \theta) = (1, 0, 1)$
1	$\frac{\partial}{\partial x} \left[-\frac{2}{3}\mu' \frac{\bar{u}_r}{r} - \frac{2}{3}\bar{\mu} \frac{u'_r}{r} \right]$	$i\alpha_x \left[-\frac{2}{3}\mu' \frac{\bar{u}_r}{r} - \frac{2}{3}\bar{\mu} \frac{u'_r}{r} \right]$
2	$\frac{1}{r} \frac{\partial}{\partial r} r \left[-\frac{2}{3}\bar{\mu} \frac{u'_r}{r} - \frac{2}{3}\mu' \frac{\bar{u}_r}{r} \right]$	$\frac{-2}{3r} \left[\bar{\mu} \frac{\partial u'_r}{\partial r} + \frac{\partial \mu'}{\partial r} \bar{u}_r + \frac{\partial \bar{\mu}}{\partial r} u'_r + \mu' \frac{\partial \bar{u}_r}{\partial r} \right]$
3	$\frac{1}{r} \frac{\partial}{\partial \theta} \left[-\bar{\mu} \frac{u'_\theta}{r} - \mu' \frac{\bar{u}_\theta}{r} \right]$	$\frac{i\alpha_\theta}{r} \left[-\bar{\mu} \frac{u'_\theta}{r} - \mu' \frac{\bar{u}_\theta}{r} \right]$
4	$-\frac{1}{r} \left[\frac{2\bar{\mu}}{r} \frac{\partial u'_\theta}{\partial \theta} + \frac{2\mu'}{r} \frac{\partial \bar{u}_\theta}{\partial \theta} \right]$	$-\frac{1}{r} \left[i\alpha_\theta \frac{2\bar{\mu}}{r} u'_\theta + \frac{2\mu'}{r} \frac{\partial \bar{u}_\theta}{\partial \theta} \right]$
5	$-\frac{1}{r} \left[2\bar{\mu} \frac{u'_r}{r} + 2\mu' \frac{\bar{u}_r}{r} \right]$	$-\frac{1}{r} \left[2\bar{\mu} \frac{u'_r}{r} + 2\mu' \frac{\bar{u}_r}{r} \right]$
7	$\frac{1}{r} \frac{\partial}{\partial r} r \left[-\bar{\mu} \frac{u'_\theta}{r} - \mu' \frac{\bar{u}_\theta}{r} \right]$	$\frac{1}{r} \left[-\bar{\mu} \frac{\partial u'_\theta}{\partial r} - \frac{\partial \mu'}{\partial r} \bar{u}_\theta - \frac{\partial \bar{\mu}}{\partial r} u'_\theta - \mu' \frac{\partial \bar{u}_\theta}{\partial r} \right]$
8	$\frac{1}{r} \frac{\partial}{\partial \theta} \left[-\frac{2}{3}\bar{\mu} \frac{u'_r}{r} - \frac{2}{3}\mu' \frac{\bar{u}_r}{r} + 2\bar{\mu} \frac{u'_r}{r} + 2\mu' \frac{\bar{u}_r}{r} \right]$	$\frac{i\alpha_\theta}{r} \left[-\frac{2}{3}\bar{\mu} \frac{u'_r}{r} - \frac{2}{3}\mu' \frac{\bar{u}_r}{r} + 2\bar{\mu} \frac{u'_r}{r} + 2\mu' \frac{\bar{u}_r}{r} \right]$
9	$\frac{1}{r} \left[\frac{\bar{\mu}}{r} \frac{\partial u'_r}{\partial \theta} + \frac{\mu'}{r} \frac{\partial \bar{u}_r}{\partial \theta} \right]$	$\frac{1}{r} \left[i\alpha_\theta \frac{\bar{\mu}}{r} u'_r + \frac{\mu'}{r} \frac{\partial \bar{u}_r}{\partial \theta} \right]$
10	$\frac{1}{r} \left[-\bar{\mu} \frac{u'_\theta}{r} - \mu' \frac{\bar{u}_\theta}{r} \right]$	$\frac{1}{r} \left[-\bar{\mu} \frac{u'_\theta}{r} - \mu' \frac{\bar{u}_\theta}{r} \right]$
11	$\frac{1}{r} \left[\bar{\mu} \frac{\partial u'_\theta}{\partial r} + \mu' \frac{\partial \bar{u}_\theta}{\partial r} \right]$	$\frac{1}{r} \left[\bar{\mu} \frac{\partial u'_\theta}{\partial r} + \mu' \frac{\partial \bar{u}_\theta}{\partial r} \right]$
12	$\frac{\partial}{\partial x} \left[-\bar{u}_x \frac{2}{3}\bar{\mu} \frac{u'_r}{r} - \bar{u}_x \frac{2}{3}\mu' \frac{\bar{u}_r}{r} - u'_x \frac{2}{3}\bar{\mu} \frac{\bar{u}_r}{r} \right]$	$i\alpha_x \frac{2}{3r} \left[-\bar{u}_x \bar{\mu} u'_r - \bar{u}_x \mu' \bar{u}_r - u'_x \bar{\mu} \bar{u}_r \right]$
13	$\frac{1}{r} \frac{\partial}{\partial r} r \left[-\bar{u}_r \frac{2}{3}\bar{\mu} \frac{u'_r}{r} - \bar{u}_r \frac{2}{3}\mu' \frac{\bar{u}_r}{r} - u'_r \frac{2}{3}\bar{\mu} \frac{\bar{u}_r}{r} \right]$	$\frac{2}{3r} \left[-\frac{\partial \bar{u}_r}{\partial r} \bar{\mu} u'_r - \frac{\partial \bar{u}_r}{\partial r} \mu' \bar{u}_r - \frac{\partial u'_r}{\partial r} \bar{\mu} \bar{u}_r \right] +$ $\frac{2}{3r} \left[-\bar{u}_r \frac{\partial \bar{\mu}}{\partial r} u'_r - \bar{u}_r \frac{\partial \mu'}{\partial r} \bar{u}_r - u'_r \frac{\partial \bar{\mu}}{\partial r} \bar{u}_r \right] +$ $\frac{2}{3r} \left[-\bar{u}_r \bar{\mu} \frac{\partial u'_r}{\partial r} - \bar{u}_r \mu' \frac{\partial \bar{u}_r}{\partial r} - u'_r \bar{\mu} \frac{\partial \bar{u}_r}{\partial r} \right]$
14	$\frac{1}{r} \frac{\partial}{\partial r} \left[-\bar{u}_\theta \bar{\mu} u'_\theta - \bar{u}_\theta \mu' \bar{u}_\theta - \bar{u}_\theta \bar{\mu} u'_\theta \right]$	$\frac{1}{r} \left[-\frac{\partial \bar{u}_\theta}{\partial r} \bar{\mu} u'_\theta - \frac{\partial \bar{u}_\theta}{\partial r} \mu' \bar{u}_\theta - \frac{\partial \bar{u}_\theta}{\partial r} \bar{\mu} u'_\theta \right] +$ $\frac{1}{r} \left[-\bar{u}_\theta \frac{\partial \bar{\mu}}{\partial r} u'_\theta - \bar{u}_\theta \frac{\partial \mu'}{\partial r} \bar{u}_\theta - \bar{u}_\theta \frac{\partial \bar{\mu}}{\partial r} u'_\theta \right] +$ $\frac{1}{r} \left[-\bar{u}_\theta \bar{\mu} \frac{\partial u'_\theta}{\partial r} - \bar{u}_\theta \mu' \frac{\partial \bar{u}_\theta}{\partial r} - \bar{u}_\theta \bar{\mu} \frac{\partial u'_\theta}{\partial r} \right]$
15	$\frac{1}{r} \frac{\partial}{\partial \theta} \left[-\bar{u}_r \bar{\mu} \frac{u'_\theta}{r} - \bar{u}_r \mu' \frac{\bar{u}_\theta}{r} - u'_r \bar{\mu} \frac{\bar{u}_\theta}{r} \right]$	$\frac{i\alpha_\theta}{r} \left[-\bar{u}_r \bar{\mu} \frac{u'_\theta}{r} - \bar{u}_r \mu' \frac{\bar{u}_\theta}{r} - u'_r \bar{\mu} \frac{\bar{u}_\theta}{r} \right]$
16	$\frac{1}{r} \frac{\partial}{\partial \theta} \left[-\bar{u}_\theta \frac{2}{3}\bar{\mu} \frac{u'_r}{r} - \bar{u}_\theta \frac{2}{3}\mu' \frac{\bar{u}_r}{r} - u'_\theta \frac{2}{3}\bar{\mu} \frac{\bar{u}_r}{r} \right]$	$\frac{i\alpha_\theta}{r} \left[-\bar{u}_\theta \frac{2}{3}\bar{\mu} \frac{u'_r}{r} - \bar{u}_\theta \frac{2}{3}\mu' \frac{\bar{u}_r}{r} - u'_\theta \frac{2}{3}\bar{\mu} \frac{\bar{u}_r}{r} \right]$
17	$\frac{1}{r} \frac{\partial}{\partial \theta} \left[+2\bar{u}_\theta \bar{\mu} \frac{u'_r}{r} + 2\bar{u}_\theta \mu' \frac{\bar{u}_r}{r} + 2u'_\theta \bar{\mu} \frac{\bar{u}_r}{r} \right]$	$\frac{i\alpha_\theta}{r} \left[+2\bar{u}_\theta \bar{\mu} \frac{u'_r}{r} + 2\bar{u}_\theta \mu' \frac{\bar{u}_r}{r} + 2u'_\theta \bar{\mu} \frac{\bar{u}_r}{r} \right]$

Num	Term	Case 3: $(x, r, \theta) = (0, 0, 1)$
1	$\frac{\partial}{\partial x} \left[-\frac{2}{3}\mu' \frac{\bar{u}_r}{r} - \frac{2}{3}\bar{\mu} \frac{u'_r}{r} \right]$	$\left[-\frac{2}{3r} \frac{\partial \mu'}{\partial x} \bar{u}_r - \frac{2}{3r} \bar{\mu} \frac{\partial u'_r}{\partial x} - \frac{2}{3r} \mu' \frac{\partial \bar{u}_r}{\partial x} - \frac{2}{3r} \frac{\partial \bar{\mu}}{\partial x} u'_r \right]$
2	$\frac{1}{r} \frac{\partial}{\partial r} r \left[-\frac{2}{3}\bar{\mu} \frac{u'_r}{r} - \frac{2}{3}\mu' \frac{\bar{u}_r}{r} \right]$	$\frac{-2}{3r} \left[\bar{\mu} \frac{\partial u'_r}{\partial r} + \frac{\partial \mu'}{\partial r} \bar{u}_r + \frac{\partial \bar{\mu}}{\partial r} u'_r + \mu' \frac{\partial \bar{u}_r}{\partial r} \right]$
3	$\frac{1}{r} \frac{\partial}{\partial \theta} \left[-\bar{\mu} \frac{u'_\theta}{r} - \mu' \frac{\bar{u}_\theta}{r} \right]$	$\frac{i\alpha_\theta}{r} \left[-\bar{\mu} \frac{u'_\theta}{r} - \mu' \frac{\bar{u}_\theta}{r} \right]$
4	$-\frac{1}{r} \left[\frac{2\bar{\mu}}{r} \frac{\partial u'_\theta}{\partial \theta} + \frac{2\mu'}{r} \frac{\partial \bar{u}_\theta}{\partial \theta} \right]$	$-\frac{1}{r} \left[i\alpha_\theta \frac{2\bar{\mu}}{r} u'_\theta + \frac{2\mu'}{r} \frac{\partial \bar{u}_\theta}{\partial \theta} \right]$
5	$-\frac{1}{r} \left[2\bar{\mu} \frac{u'_r}{r} + 2\mu' \frac{\bar{u}_r}{r} \right]$	$-\frac{1}{r} \left[2\bar{\mu} \frac{u'_r}{r} + 2\mu' \frac{\bar{u}_r}{r} \right]$
7	$\frac{1}{r} \frac{\partial}{\partial r} r \left[-\bar{\mu} \frac{u'_\theta}{r} - \mu' \frac{\bar{u}_\theta}{r} \right]$	$\frac{1}{r} \left[-\bar{\mu} \frac{\partial u'_\theta}{\partial r} - \frac{\partial \mu'}{\partial r} \bar{u}_\theta - \frac{\partial \bar{\mu}}{\partial r} u'_\theta - \mu' \frac{\partial \bar{u}_\theta}{\partial r} \right]$
8	$\frac{1}{r} \frac{\partial}{\partial \theta} \left[-\frac{2}{3}\bar{\mu} \frac{u'_r}{r} - \frac{2}{3}\mu' \frac{\bar{u}_r}{r} + 2\bar{\mu} \frac{u'_r}{r} + 2\mu' \frac{\bar{u}_r}{r} \right]$	$\frac{i\alpha_\theta}{r} \left[-\frac{2}{3}\bar{\mu} \frac{u'_r}{r} - \frac{2}{3}\mu' \frac{\bar{u}_r}{r} + 2\bar{\mu} \frac{u'_r}{r} + 2\mu' \frac{\bar{u}_r}{r} \right]$
9	$\frac{1}{r} \left[\frac{\bar{\mu}}{r} \frac{\partial u'_r}{\partial \theta} + \frac{\mu'}{r} \frac{\partial \bar{u}_r}{\partial \theta} \right]$	$\frac{1}{r} \left[i\alpha_\theta \frac{\bar{\mu}}{r} u'_r + \frac{\mu'}{r} \frac{\partial \bar{u}_r}{\partial \theta} \right]$
10	$\frac{1}{r} \left[-\bar{\mu} \frac{u'_\theta}{r} - \mu' \frac{\bar{u}_\theta}{r} \right]$	$\frac{1}{r} \left[-\bar{\mu} \frac{u'_\theta}{r} - \mu' \frac{\bar{u}_\theta}{r} \right]$
11	$\frac{1}{r} \left[\bar{\mu} \frac{\partial u'_\theta}{\partial r} + \mu' \frac{\partial \bar{u}_\theta}{\partial r} \right]$	$\frac{1}{r} \left[\bar{\mu} \frac{\partial u'_\theta}{\partial r} + \mu' \frac{\partial \bar{u}_\theta}{\partial r} \right]$
12	$\frac{\partial}{\partial x} \left[-\bar{u}_x \frac{2}{3}\bar{\mu} \frac{u'_r}{r} - \bar{u}_x \frac{2}{3}\mu' \frac{\bar{u}_r}{r} - u'_x \frac{2}{3}\bar{\mu} \frac{\bar{u}_r}{r} \right]$	$\frac{2}{3r} \left[-\frac{\partial \bar{u}_x}{\partial x} \bar{\mu} u'_r - \frac{\partial \bar{u}_x}{\partial x} \mu' \bar{u}_r - \frac{\partial u'_x}{\partial x} \bar{\mu} \bar{u}_r \right] +$ $\frac{2}{3r} \left[-\bar{u}_x \frac{\partial \bar{\mu}}{\partial x} u'_r - \bar{u}_x \frac{\partial \mu'}{\partial x} \bar{u}_r - u'_x \frac{\partial \bar{\mu}}{\partial x} \bar{u}_r \right] +$ $\frac{2}{3r} \left[-\bar{u}_x \bar{\mu} \frac{\partial u'_r}{\partial x} - \bar{u}_x \mu' \frac{\partial \bar{u}_r}{\partial x} - u'_x \bar{\mu} \frac{\partial \bar{u}_r}{\partial x} \right]$
13	$\frac{1}{r} \frac{\partial}{\partial r} r \left[-\bar{u}_r \frac{2}{3}\bar{\mu} \frac{u'_r}{r} - \bar{u}_r \frac{2}{3}\mu' \frac{\bar{u}_r}{r} - u'_r \frac{2}{3}\bar{\mu} \frac{\bar{u}_r}{r} \right]$	$\frac{2}{3r} \left[-\frac{\partial \bar{u}_r}{\partial r} \bar{\mu} u'_r - \frac{\partial \bar{u}_r}{\partial r} \mu' \bar{u}_r - \frac{\partial u'_r}{\partial r} \bar{\mu} \bar{u}_r \right] +$ $\frac{2}{3r} \left[-\bar{u}_r \frac{\partial \bar{\mu}}{\partial r} u'_r - \bar{u}_r \frac{\partial \mu'}{\partial r} \bar{u}_r - u'_r \frac{\partial \bar{\mu}}{\partial r} \bar{u}_r \right] +$ $\frac{2}{3r} \left[-\bar{u}_r \bar{\mu} \frac{\partial u'_r}{\partial r} - \bar{u}_r \mu' \frac{\partial \bar{u}_r}{\partial r} - u'_r \bar{\mu} \frac{\partial \bar{u}_r}{\partial r} \right]$
14	$\frac{1}{r} \frac{\partial}{\partial r} \left[-\bar{u}_\theta \bar{\mu} u'_\theta - \bar{u}_\theta \mu' \bar{u}_\theta - \bar{u}_\theta \bar{\mu} u'_\theta \right]$	$\frac{1}{r} \left[-\frac{\partial \bar{u}_\theta}{\partial r} \bar{\mu} u'_\theta - \frac{\partial \bar{u}_\theta}{\partial r} \mu' \bar{u}_\theta - \frac{\partial \bar{u}_\theta}{\partial r} \bar{\mu} u'_\theta \right] +$ $\frac{1}{r} \left[-\bar{u}_\theta \frac{\partial \bar{\mu}}{\partial r} u'_\theta - \bar{u}_\theta \frac{\partial \mu'}{\partial r} \bar{u}_\theta - \bar{u}_\theta \frac{\partial \bar{\mu}}{\partial r} u'_\theta \right] +$ $\frac{1}{r} \left[-\bar{u}_\theta \bar{\mu} \frac{\partial u'_\theta}{\partial r} - \bar{u}_\theta \mu' \frac{\partial \bar{u}_\theta}{\partial r} - \bar{u}_\theta \bar{\mu} \frac{\partial u'_\theta}{\partial r} \right]$
15	$\frac{1}{r} \frac{\partial}{\partial \theta} \left[-\bar{u}_r \bar{\mu} \frac{u'_\theta}{r} - \bar{u}_r \mu' \frac{\bar{u}_\theta}{r} - u'_r \bar{\mu} \frac{\bar{u}_\theta}{r} \right]$	$\frac{i\alpha_\theta}{r} \left[-\bar{u}_r \bar{\mu} \frac{u'_\theta}{r} - \bar{u}_r \mu' \frac{\bar{u}_\theta}{r} - u'_r \bar{\mu} \frac{\bar{u}_\theta}{r} \right]$
16	$\frac{1}{r} \frac{\partial}{\partial \theta} \left[-\bar{u}_\theta \frac{2}{3}\bar{\mu} \frac{u'_r}{r} - \bar{u}_\theta \frac{2}{3}\mu' \frac{\bar{u}_r}{r} - u'_\theta \frac{2}{3}\bar{\mu} \frac{\bar{u}_r}{r} \right]$	$\frac{i\alpha_\theta}{r} \left[-\bar{u}_\theta \frac{2}{3}\bar{\mu} \frac{u'_r}{r} - \bar{u}_\theta \frac{2}{3}\mu' \frac{\bar{u}_r}{r} - u'_\theta \frac{2}{3}\bar{\mu} \frac{\bar{u}_r}{r} \right]$
17	$\frac{1}{r} \frac{\partial}{\partial \theta} \left[+2\bar{u}_\theta \bar{\mu} \frac{u'_r}{r} + 2\bar{u}_\theta \mu' \frac{\bar{u}_r}{r} + 2u'_\theta \bar{\mu} \frac{\bar{u}_r}{r} \right]$	$\frac{i\alpha_\theta}{r} \left[+2\bar{u}_\theta \bar{\mu} \frac{u'_r}{r} + 2\bar{u}_\theta \mu' \frac{\bar{u}_r}{r} + 2u'_\theta \bar{\mu} \frac{\bar{u}_r}{r} \right]$

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Term	$\frac{-1}{r} \left[-\frac{2}{3}\bar{\mu} \frac{\partial u'_x}{\partial x} - \frac{2}{3}\mu' \frac{\partial \bar{u}_x}{\partial x} - \frac{2}{3}\bar{\mu} \frac{\partial u'_r}{\partial r} - \frac{2}{3}\mu' \frac{\partial \bar{u}_r}{\partial r} - \frac{2}{3}\bar{\mu} \frac{\partial u'_\theta}{\partial \theta} - \frac{2}{3}\mu' \frac{\partial \bar{u}_\theta}{\partial \theta} - \frac{2}{3}\bar{\mu} \frac{u'_r}{r} - \frac{2}{3}\mu' \frac{\bar{u}_r}{r} \right]$
Case 1 (1,0,1)	$\frac{2}{3r} \left[i\alpha_x \bar{\mu} u'_x + \mu' \frac{\partial \bar{u}_x}{\partial x} + \bar{\mu} \frac{\partial u'_r}{\partial r} + \mu' \frac{\partial \bar{u}_r}{\partial r} + i\alpha_\theta \bar{\mu} u'_\theta + \mu' \frac{\partial \bar{u}_\theta}{\partial \theta} + \bar{\mu} \frac{u'_r}{r} + \mu' \frac{\bar{u}_r}{r} \right]$
Case 3 (0,0,1)	$\frac{2}{3r} \left[\bar{\mu} \frac{\partial u'_x}{\partial x} + \mu' \frac{\partial \bar{u}_x}{\partial x} + \bar{\mu} \frac{\partial u'_r}{\partial r} + \mu' \frac{\partial \bar{u}_r}{\partial r} + i\alpha_\theta \bar{\mu} u'_\theta + \mu' \frac{\partial \bar{u}_\theta}{\partial \theta} + \bar{\mu} \frac{u'_r}{r} + \mu' \frac{\bar{u}_r}{r} \right]$