Approximate Analysis of Product-Form Type Queueing Networks with Blocking and Deadlock

**by
H.G. Perros
A.A. Nilsson
and
Y.C. Liu**

**Center for Communications and Signal Processing
Department of Electrical and Computer Engineering
North Carolina State University**

December 1986

CCSP-TR-86/27

Upgrade of CCSP-TR-85/13

**Approximate Analysis of Product-Form Type
Queueing Networks with Blocking and Deadlock**

by

Harry G. Perros¹

Department of Computer Science

Center for Communications and Signal Processing

and

A.A. Nilsson

Y.C. Liu

Department of Electrical & Computer Engineering

Center for Communications and Signal Processing

North Carolina State University

Raleigh, NC 27695-7911

¹Supported in part by the National Science Foundation under grant DCR-85-02540

Abstract

A numerical procedure for analyzing exactly closed exponential queueing networks with finite queues was first developed. Due to the finiteness of these queues, blocking and deadlock may occur. Deadlocks are assumed to be detected and resolved instantaneously. The numerical procedure was then incorporated in an approximation algorithm for analyzing closed exponential queueing networks of the product-form type, in which some of the queues are finite. The approximation algorithm is based on a variant of Norton's theorem. Comparisons between the approximate results and exact numerical results were carried out and the relative error was observed to be small.

Introduction

In recent years, there has been a growing interest in the development of computational methods for the analysis of queueing networks with blocking. This is primarily due to a growing need to model actual systems which have finite capacity resources, such as computer systems, telecommunication systems, distributed systems and flexible manufacturing systems.

In this paper, we consider queueing networks of the product-form type in which some of the queues are finite. In particular, let us consider a class of closed queueing networks which is of the BCMP type, i.e., it satisfies local balance and has a product-form solution. In such queueing networks, each queue is assumed to be infinite. That is, it is large enough to accommodate all the units in the system. In this paper, we consider such queueing networks, but we assume that some of the queues have finite capacity. In view of this, the flow of units through one queue is liable to getting blocked when a destination finite queue reaches its capacity limitation. The queues are assumed to be arbitrarily interconnected. In view of this, blocking of different nodes may cause deadlocks. Deadlocks are assumed to be detected and resolved instantaneously.

Open or closed queueing networks with blocking are, in general, difficult to analyze. Closed-form solutions for the steady-state queue-length distribution, are not generally attainable. Therefore, most of the techniques employed for the analysis of such queueing networks are in the form of analytic approximations, numerical solutions, and simulation techniques. In this paper, we review the literature pertaining to closed queueing networks with blocking. For a literature review of open queueing networks with blocking see Altıok and Perros[2] and Perros[12].

Product-form solutions have been reported in the literature in connection with certain closed exponential queueing networks with blocking. In particular, if the

routing matrix is reversible then the queueing network is also reversible and it has a product-form solution (see Kelly[8] and Pittel[13]). This result holds under the following blocking mechanism. A unit at node i upon completion of its service chooses destination node j . If node j is full at that moment, the unit starts a new service at node i . Upon completion of its new service, the unit will choose a new destination node independent of the destination node chosen the previous time. This is repeated until the unit successfully enters a destination node. The same queueing networks under the above mentioned blocking mechanism, also has a product-form solution when the routing probability from nodes i to node j depends on the number of jobs present at node i and j (see Kelly[8], Yao and Buzacott[16] and Le Ny [9,10]). Other cases under which this particular queueing network has a product-form solution have also been reported in the literature (see Hordijk and Van Dijk[7] and Van Dijk[15]). Finally, Balsamo and Iazeolla[3] showed that in some cases such a queueing network with blocking is equivalent to a queueing network without blocking in terms of steady-state probabilities.

Other product-form solutions have been reported in the literature in connection with two-node closed exponential queueing networks with finite queues. (see Gordon and Newell[6], Akyildiz[1], and Suri and Diehl[14]). A survey of these results can be found in Perros[12].

A particular generalization of this two-node queueing network is the one consisting of M finite queues ($M > 2$) arranged in tandem. A unit proceeds sequentially through each of these queues, returning to the first queue after being served at the last queue. Service time at each node are exponentially distributed. Gordon and Newell[6] analyzed this queueing network under the following blocking mechanism. A unit in queue i declares its destination queue (queue j) before it begins its service. If queue j is full, the i th server becomes blocked and it cannot serve the unit. When

a departure occurs from queue j , the i th server becomes unblocked and the unit begins receiving service. They obtained a product-form solution for this system when the number of units N , is such that $N > \sum_{i=1}^M m_i - N$, where m_i is the maximum capacity of queue i , $i = 1, 2, \dots, M$. They obtained this result using the notion of "holes". Suri and Diehl[14] analyzed approximately the same tandem configuration assuming that the first queue has a buffer $m_1 \geq N$. Their analysis is based on a decomposition method.

Yao and Buzacott[16] developed an approximation algorithm for analyzing arbitrary configurations of closed queueing networks with blocking, assuming Coxian service times and reversible routing. This algorithm is applicable to a closed queueing network with blocking which has the following property. If the Coxian service distributions are substituted by exponential service distributions (with the same mean), then the resulting queueing network with blocking is reversible and has a product-form solution. Akyildiz[1] reported on an approximation algorithm for obtaining the throughput and mean queue length of closed exponential queueing networks with blocking assuming a blocking mechanism similar to the one considered in this paper. Finally, Onvural and Perros[11] obtained analytical results that give some insight into the structure of closed queueing networks with blocking.

In the following section, we present a numerical procedure for analyzing a closed queueing network with blocking and deadlock. All the queues of the queueing network are assumed to be finite. Each queue is served by a single exponential server. The blocking mechanism considered here is as follows. A unit gets blocked upon completion of its service if its destination queue is full. The unit remains blocked until it successfully enters its destination queue. In section 3, we use this numerical procedure to develop an approximation algorithm for analyzing queueing network in which some of the queues are finite. In particular, we consider a closed queueing

network where each node is represented by either a finite or infinite queue. Each queue (finite or infinite) is served by a single exponential server in a FIFO manner. One class of unit is considered. Due to the finiteness of these queues blocking and deadlock can occur. The approximation algorithm uses a variant of Norton's theorem. Comparisons between approximate and exact results are given in section 4. Finally, conclusions are given in sections 5.

2. Closed Exponential Queueing Networks with Blocking and Deadlock

Let us consider a closed queueing network consisting of M queues and N units. Each queue i , $i = 1, 2, \dots, M$, is finite in size and is served by an exponential server with mean service rate μ_i . Let m_i be the maximum capacity of queue i , including the one in service. We assume that the M queues are arbitrarily interconnected. A unit at queue i , upon completion of its service choose to join node j with probability p_{ij} , where $\sum_j p_{ij} = 1$. If queue j happens to be full at that particular instance, then the unit will get blocked. That is, the unit is forced to wait in the i th server until it succeeds in entering queue j . During this time the i th server is also blocked and it can not serve any other units that might be waiting in its queue. The i th server becomes unblocked the moment the blocked unit enters queue j . Let us assume now that there are several queues (one of which is the above mentioned queue i), all linked to queue j . Then it is possible that at any time there might be more than one blocked unit waiting to enter queue j . The maximum number of blocked units will be equal to the number of its upstream queues, i.e., queues directly linked to queue j . It is assumed that these blocked units enter queue j on a first-blocked-first-enter basis.

Due to the blocking mechanism described above, and due to the fact that these M queues are arbitrarily interconnected, it is possible that deadlocks may occur. For

instance, assume that queue i is blocked by queue j . Now, it is possible that a unit in queue j may, upon completion of its service, choose to go to queue i . If queue i is full at that time, then deadlock will occur. It is assumed that deadlocks are detected and resolved instantaneously. In this paper, it is assumed that a deadlock is resolved by instantaneously exchanging blocking units. In general, this scheme for resolving deadlocks may violate the first-blocked-first-enter priority rule described above. For instance, let us assume that queues i and k are blocked by queue j in that order. That is, if a departure occurs from queue j the blocking unit from queue i will enter queue j first. Now, let us assume that the departing unit from queue j chooses queue k as its destination, and, that queue k is full at that moment. This causes a deadlock to occur, which is resolved by simultaneously exchanging the blocking units from queues j and k . In view of this, the blocking unit from queue k enters queue j first while queue i still remains blocked. Thus, the first-blocked-first-enter priority rule has been violated.

We analyze this closed exponential queueing network with blocking and deadlock numerically. This involved the following steps: (1) generation of all the states of the system; (2) generation of the rate matrix Q ; and (3) solution of the linear system $Qx^T = 0$ in order to obtain the steady state probability distribution.

2.1 Generation of the State Space

The state of the system under investigation is given by the vector $(n_1, n_2, \dots, n_M)$, where n_i is the number of units in queue i , including the one in service. We have, $0 \leq n_i \leq m_i$. Furthermore, n_i is allowed to assume the values $m_i + 1, m_i + 2, \dots$ in order to indicate which of the "upstream" queues are blocked and in what order.

The generation of all feasible states was achieved using the following steps. (For presentation purposes, we assume a queueing network such that no more than

three queues may be blocked at any time.)

Step 1: Generate a state such that $\sum_{i=1}^M n_i = N, n_i \leq m_i$. That is a state in which no blocking occurs. We refer to such a state as a *basic* state.

Step 2: For this basic state let M' be the set of all non-empty queues. For each queue $i \in M'$, let M_i' be the set of all its downstream queues (directly linked to queue i) which are at capacity. For each queue $j \in M_i'$ generate a new state which differs from the basic state by the fact that queue i is blocked by queue j . We call such a state a *one-step blocking state*, seeing that it involves blocking of one queue only.

Step 3: Given this one-step blocking state, generate a two-step blocking state involving one more blocked queue.

Step 4: From this two-step blocking state, generate a three-step blocking state. Repeat until it is no longer feasible.

The nesting of these steps is as follows.

```
Generate a basic state until *
  begin
    Generate a one-step blocking state until *
      begin
        Generate a two-step blocking state until *
          begin
            Generate a three-step blocking state until *
          end
        end
      end
    end
  end;
```

The symbol * is used to indicate the logical condition that state generation continues until it is no longer feasible.

We note that each generated n -step blocking state ($n \geq 2$) is tested for deadlock. If the blocking of queue i by queue j is seen as a directed arc from i to j , then deadlock

occurs each time the blocked queues form a cyclic directed graph. If deadlock has occurred, then the deadlock resolution procedure described above is invoked in order to determine the final state of the system.

The number of states associated with closed queueing networks with blocking depends on the number of units in the system, the number of queues, the way these queues are interconnected and the buffer capacities of the queues. Figure 1, gives the number of states for a queueing network consisting of four fully connected queues.

Fig. 1: Number of states vs number of units in the system for four fully connected queues and $m_i=4,5,6$, $i=1,2,3,4$.

2.2 Generation of the rate matrix Q

In order to generate the rate matrix by computer, we need to be able to associate an integer with each state. This integer, referred to as the state number, is the ordinal position of the state in the space as generated using the algorithm described above. The state number for a given state is obtained via a binary "dictionary" tree. The components of the state provide the keys which describe a path to the leaf which contains the associated state number. Due to the frequency with which this routine is called, it is perhaps the most time critical routine in the matrix generation procedure.

The generation of the rate matrix, involves finding all the states which are accessible from a state, as well as their corresponding transition probabilities. A state transition occurs when a unit completes its service at a queue. Depending upon the current state, the system may move to a state in which the unit has successfully entered its destination queue, or to a state in which the unit is blocked. Blocking of this unit may or may not cause deadlock. If deadlock occurs, then the process moves with zero transition time to a state obtained by allowing all the blocked units involved in the deadlock to instantaneously move to their respective destination queues.

2.3 Numerical Solution Technique

The transition rate matrices associated with these queueing networks are typically extremely sparse. In view of this, a straight forward compact storage scheme was adopted. The non-zero elements were stored consecutively by rows in a one dimensional real array, and an integer array was used to indicate the location of these elements in their respective rows. Now, seeing that the transition rate matrix had no inherent structure, it was decided to use a simple point Gauss-Seidel iteration method. In this method, we assign an arbitrary non-zero value to the first element of the solution vector, and then we carry out a standard forward substitution. The

resulting vector has to be normalized to yield the stationary probability vector.

Experimental evidence showed that most of the CPU time was devoted to the generation of the state space and the rate matrix Q . The calculation of the stationary probability vector required only a small fraction of the total CPU time. Time complexity problems emerged as the number of queues in the queueing network increased and the interconnectivity of these queues became more complex.

As mentioned above, the non-zero elements of the rate matrix have no inherent structure. We note, that we did not search for a possible enumeration of the state space under which the rate matrix has a regular structure. The advantage of dealing with a matrix with a regular structure, is that one can avail of it in order to construct an efficient algorithm for obtaining its stationary probability vector. However, such an exercise was not necessary, seeing that the calculation of the stationary probability was not a time-critical procedure.

Finally, we note that in this paper, we assumed that deadlock is detected and resolved instantaneously. However, this may not be the case in real-life systems. In fact, it is more likely that detection and resolution may require an additional delay. Such a delay can be easily incorporated in our model in view of the above described numerical procedure.

3. The Approximate Procedure

In this section, we discuss an approximation procedure for analyzing closed queueing networks in which some of the queues are finite. A single class of customers and exponentially distributed service times is assumed. The approximation procedure is based on the well known Norton's theorem (see Chandy, Herzog, and Woo [4,5]).

Let us group all the finite queues and those infinite queues that are liable to getting

blocked into a subnetwork hereafter referred to as the *blocking subnetwork*. The remaining infinite queues are grouped into a subnetwork that will be referred to as the *non-blocking subnetwork*. For example, consider a queueing network consisting of five queues as shown in figure 2. The buffer sizes of queues 1,2 and 3 are assumed to be infinite and the buffer sizes of queues 4 and 5 are finite. Due to the routing, queue 4 may be blocked by queue 5 or vice versa. Also, queue 3 may be blocked by queue 4 or queue 5. Hence, queues 3, 4 and 5 are grouped into the blocking subnetwork, and queues 1 and 2 are grouped into the non-blocking subnetwork.

Fig. 2: A closed queueing network in which some of the queues are finite.

The approximation procedure can now be summarized as follows:

1. Analyze the non-blocking subnetwork (obtained from the original network by "shorting" the blocking subnetwork) as a product-form network assuming n fixed units, where $n=1,2,\dots,N$. For each n obtain the steady-state probabilities $p(\underline{n} | n), \underline{n} \in S_n$, where $\underline{n}$ is the state of the non-blocking subnetwork and

S_n is the set of all feasible states for given n . Based on these results calculate the throughput $T(n)$, $n=1,2,\dots,N$.

2. Construct a composite queue with a state-dependent throughput equal to $T(n)$, $n=1,2,\dots,N$. Now, in the original network substitute the non-blocking subnetwork by its equivalent composite queue. Analyze the reduced network numerically using the algorithm described in section 2, to obtain the marginal queue-length probability distribution for each queue, assuming N units in it.
3. Let $p_c(n)$, $n=1,\dots,N$ be the marginal queue-length probability that there are n units in the composite queue as calculated in step 2. Then, from step 1 we have $p(\underline{n}) = p(\underline{n} \mid n) p_c(n)$, $\underline{n} \in S_n$, and $n=1,2,\dots,N$. Using $p(\underline{n})$ the marginal queue-length probability distribution for each queue in the non-blocking subnetwork can be easily obtained.

The marginal queue-length probability distribution obtained in steps 2 and 3 constitute an approximation to the true marginal queue-length probability distribution of the original queueing network.

For the example shown in figure 2, steps 1 and 2 involve the analysis of the queueing networks shown in figures 3 and 4 respectively.

Fig. 3: Blocking subnetwork shorted

Fig. 4: Blocking subnetwork and composite queue

The above approximation algorithm also applies when the non-blocking subnetwork is represented by any closed queueing network of the BCMP type.

Finally, we note that Suri and Diehl[14] discussed briefly an approximation procedure for analyzing closed queueing networks in which some of the queues are finite. These finite queues are arranged in tandem. Units from the remaining network enter this tandem configuration through its first queue (which is assumed to have an infinite capacity) and depart through its last queue. Exponentially distributed services were assumed. Their procedure resembles steps 1 and 2 described above. Using this procedure they obtained the throughput of the system.

4. Numerical Results

The above algorithm was implemented on an IBM 3081. It was employed to analyze various configurations. The main result obtained is the queue-length distribution for each node. From this, other more commonly sought performance measures, such as throughput and mean queue length, can be easily obtained. The approximate results were compared against exact numerical values. These exact numerical values were obtained by analyzing the queueing network under study using the numerical procedure described in section 2.

The results are summarized in tables 1 to 14. Each table gives the approximate and exact results for the throughput, mean queue-length and queue-length distribution per node. The relative error (expressed as a percentage) is also given.

The approximation algorithm gives, in general, good results. The approximate results for the throughput and mean queue length have a relative error less than 1%. Also, the approximate results for the queue-length distributions have a relative error less than 5%. (Some of the approximate queue-length probabilities have a relative error as high as 60%. However, these cases are not significant seeing that the probabilities are extremely small and they have an absolute error of less than 10^{-6})

The extreme case, where the service rates in the non-blocking subnetwork are very fast compared to the service rates in the blocking subnetwork, was also considered. In this case, the customers are mostly in the blocking subnetwork, and hence, the marginal queue-length distributions of the approximate solution are almost identical to the marginal queue-length distribution of the exact solution. The same conclusion holds when the servers' rates in the blocking subnetwork are very fast compared to the service rates in the non-blocking network.

5. Conclusions

We developed a numerical procedure for analyzing closed queueing networks with blocking and deadlock. This procedure was then used to develop an approximation procedure based on Norton's theorem for analyzing queueing networks of the product-form type in which some of the queues have a finite capacity. It was shown through a number of examples that the approximation procedure is accurate and the relative error is small.

The numerical procedure was developed with a view to analyzing closed queueing networks under a particular type of blocking. It can be expanded, however, to analyze such networks under different types of blocking mechanisms. Also, it can be easily extended to allow coxian distributed service times. As was mentioned in section 2, this procedure is not very efficient when analyzing large blocking subnetworks. In this case, one can use the approximation method by Akyildiz[1], in order to obtain the throughput of the blocking subnetwork. This procedure does not give queue-length probability distributions. For the special case where the nodes are indistinguishable, one can use an efficient numerical procedure given in Onvural and Perros[11].

The approximation procedure described in section 3 can be also applied in the reverse way. That is, we can first analyze the blocking subnetwork numerically with a view to constructing an equivalent composite queue. Then, replace the blocking subnetwork in the original queueing network by its equivalent queue and analyze the resulting network as a product-form network. This algorithm gives similar results on those reported in tables 1 to 14.

Finally, in the approximation procedure, it was assumed that we can construct a single blocking subnetwork. This may not be the case in general. This algorithm can be extended to cope with multiple blocking subnetworks. In particular, each

blocking subnetwork can be replaced by an equivalent composite queue. The resulting network can then be analyzed as a product-form network. The feasibility of this extension, however, needs to be investigated.

Acknowledgement

We would like to thank an anonymous referee for suggesting an improvement on the algorithm described in section 3.

References

1. I. F. Akyildiz, "On the Exact and Approximate Analysis of Closed Queueing Networks with Blocking," Computer Science Report 86-6, Univ. of Florida.
2. T. Altioek and H.G. Perros, "Approximation Analysis of Arbitrary Configurations of Open Queueing Networks with Blocking", to appear in the Annals of Operations Research.
3. S. Balsamo and G. Iazeolla, "Some Equivalence Properties for Queueing Networks With and Without Blocking," *Performance '83*, Agrawala and Tripathi (Eds.) North-Holland (1983), 351-360.
4. K.M. Chandy, U. Herzog, and L. Woo, "Parametric Analysis of Queueing Networks," *IBM J. Res. Develop.* 19, (1975) 36-42.
5. K.M. Chandy, U. Herzog, and L. Woo, "Approximate Analysis of General Queueing Networks," *IBM J. Res. Develop.* 19, (1975) 43-49.
6. W.J. Gordon and G.F. Newell, "Cyclic Queueing Systems with Restricted Length Queues", *Operation Research* 15 (1967), 266-278.
7. A. Hordijk and N. Van Dijk, " Networks of Queues with Blocking", *Performance'81*, Kylstra (Ed.), North-Holland (1981), 51-65.
8. F.P. Kelly, *Reversibility and Stochastic Networks*, Wiley, New York, 1979.
9. L.-M. Le Ny, "Forme Produit Pour des Reseaux Multiclasses a Routage Dynamiques", *Annales Scientifiques de l'Universite de Clermont-Ferrand II*, 76 (1983) 17-34.
10. L.-M. Le Ny, "Etude Analytique de Reseaux de Files d'Attente Multiclass a Routage Variable", *RAIRO Recherche Operationnelle*, 14 (1980), 331-347.
11. R. Onvural, and H.G. Perros, "Some exact results for closed queueing networks with blocking," Computer Science report, N.C. State University, 1986.
12. H.G. Perros, "A Survey of Queueing Networks with Blocking-Part I", Computer Science Report, N.C. State University, 1986.

13. B. Pittel, "Closed Exponential Networks of Queues with Saturation: The Jackson Type Stationary Distribution and its Asymptotic Analysis," *Math. of Operation Research*, Vol. 4, (1979), 367-378.
14. R. Suri and G. W. Diehl, "A New Building Block for Performance Evaluation of Queueing Networks with Finite Buffers," *Proc. ACM SIGMETRICS on Measurement and Modeling of Computer Systems*, Cambridge, Mass., (1984) 134-142. Conference Proceedings, Cambridge, Mass.,(1984) 134-142.
15. N. Van Dijk, "Controlled Markov Processes: Time Discretization-networks of Queues", Ph.D. thesis Mathematical Centrum, Amsterdam (1983).
16. D. Yao and J. A. Buzacott, "Modelling a Class of Flexible Manufacturing Systems with Reversible Routing," Industrial Engineering Report 83-02, University of Toronto,(1983).

Table 1: The configuration analyzed is shown in Figure 2. $m_1 = m_2 = m_3 = \infty$, $m_4 = 2$, $m_5 = 3$; $\mu_1 = \mu_3 = \mu_4 = 0.5$, $\mu_2 = \mu_5 = 1$; $p_{12} = 0.7$, $p_{13} = 0.3$, $p_{21} = 0.5$, $p_{23} = 0.5$, $p_{34} = 0.5$, $p_{35} = 0.5$, $p_{41} = 0.5$, $p_{42} = 0.25$, $p_{45} = 0.25$, $p_{51} = 0.25$, $p_{52} = 0.5$, $p_{54} = 0.25$; $N = 6$.

Throughput	Exact Solution	Approximation	Error(%)
node 1	4.055e-01	4.047e-01	2.0e-01
node 2	4.596e-01	4.587e-01	2.0e-01
node 3	3.514e-01	3.516e-01	5.7e-02
node 4	2.343e-01	2.344e-01	4.3e-02
node 5	2.343e-01	2.344e-01	4.3e-02
Mean queue length	Exact Solution	Approximation	Error(%)
node 1	2.312e+00	2.310e+00	8.7e-02
node 2	7.770e-01	7.743e-01	3.5e-01
node 3	1.916e+00	1.919e+00	1.6e-01
node 4	6.689e-01	6.694e-01	7.5e-02
node 5	3.270e-01	3.274e-01	1.2e-01
Queue Length Distribution at Node 1			
Number of units	Exact Solution	Approximation	Error(%)
0	1.890e-01	1.905e-01	7.9e-01
1	1.900e-01	1.894e-01	3.2e-01
2	1.840e-01	1.831e-01	4.9e-01
3	1.673e-01	1.664e-01	5.4e-01
4	1.372e-01	1.375e-01	2.2e-01
5	9.248e-02	9.291e-02	4.6e-01
6	4.008e-02	4.003e-02	1.2e-01
Queue Length Distribution at Node 2			
Number of units	Exact Solution	Approximation	Error(%)
0	5.404e-01	5.413e-01	1.7e-01
1	2.595e-01	2.596e-01	3.9e-02
2	1.199e-01	1.196e-01	2.5e-01
3	5.199e-02	5.161e-02	7.3e-01
4	2.028e-02	2.012e-02	7.9e-01
5	6.496e-03	6.442e-03	8.3e-01
6	1.343e-03	1.325e-03	1.3e+00

Queue Length Distribution at Node 3

Number of units	Exact Solution	Approximation	Error(%)
0	2.396e-01	2.390e-01	2.5e-01
1	2.214e-01	2.213e-01	4.5e-02
2	1.962e-01	1.961e-01	5.1e-02
3	1.606e-01	1.606e-01	0.0e+00
4	1.098e-01	1.103e-01	4.6e-01
5	5.339e-02	5.354e-02	2.8e-01
6	1.894e-02	1.914e-02	1.1e+00

Queue Length Distribution at Node 4

Number of units	Exact Solution	Approximation	Error(ϵ_c)
0	5.309e-01	5.307e-01	3.8e-02
1	2.693e-01	2.693e-01	0.0e+00
2	1.998e-01	2.001e-01	1.5e-01

Queue Length Distribution at Node 5

Number of units	Exact Solution	Approximation	Error(%)
0	7.470e-01	7.468e-01	2.7e-02
1	1.923e-01	1.923e-01	0.0e+00
2	4.761e-02	4.767e-02	1.3e-01
3	1.316e-02	1.323e-02	5.3e-01

Table 2: The configuration analyzed is shown in Figure 2. $m_1 = m_2 = m_3 = \infty$, $m_4 = 2$, $m_5 = 3$; $\mu_1 = \mu_3 = \mu_4 = 0.5$, $\mu_2 = \mu_5 = 1$; $p_{12} = 0.7$, $p_{13} = 0.3$, $p_{21} = 0.5$, $p_{23} = 0.5$, $p_{34} = 0.5$, $p_{35} = 0.5$, $p_{41} = 0.5$, $p_{42} = 0.25$, $p_{45} = 0.25$, $p_{51} = 0.25$, $p_{52} = 0.5$, $p_{54} = 0.25$; $N = 15$.

Throughput	Exact Solution	Approximation	Error(%)
node 1	4.696e-01	4.698e-01	4.3e-02
node 2	5.322e-01	5.325e-01	5.6e-02
node 3	4.071e-01	4.072e-01	2.5e-02
node 4	2.714e-01	2.715e-01	3.7e-02
node 5	2.714e-01	2.715e-01	3.7e-02
Mean queue length	Exact Solution	Approximation	Error(%)
node 1	6.798e+00	6.815e+00	2.5e-01
node 2	1.113e+00	1.115e+00	1.8e-01
node 3	5.853e+00	5.833e+00	3.4e-01
node 4	8.161e-01	8.165e-01	4.9e-02
node 5	4.200e-01	4.203e-01	7.1e-02
Queue Length Distribution at Node 1			
Number of units	Exact Solution	Approximation	Error(%)
0	6.089e-02	6.035e-02	8.9e-01
1	6.440e-02	6.354e-02	1.3e+00
2	6.746e-02	6.671e-02	1.1e+00
3	6.973e-02	6.933e-02	5.7e-01
4	7.136e-02	7.132e-02	5.6e-02
5	7.260e-02	7.284e-02	3.3e-01
6	7.355e-02	7.395e-02	5.4e-01
7	7.416e-02	7.463e-02	6.3e-01
8	7.428e-02	7.474e-02	6.2e-01
9	7.362e-02	7.402e-02	5.4e-01
10	7.170e-02	7.201e-02	4.3e-01
11	6.779e-02	6.801e-02	3.2e-01
12	6.088e-02	6.099e-02	1.8e-01
13	4.967e-02	4.972e-02	1.0e-01
14	3.343e-02	3.340e-02	9.0e-02
15	1.448e-02	1.444e-02	2.8e-01

Queue Length Distribution at Node 2

Number of units	Exact Solution	Approximation	Error(%)
0	4.678e-01	4.675e-01	6.4e-02
1	2.511e-01	2.511e-01	0.0e+00
2	1.341e-01	1.340e-01	7.5e-02
3	7.102e-02	7.097e-02	7.0e-02
4	3.722e-02	3.724e-02	5.4e-02
5	1.928e-02	1.935e-02	3.6e-01
6	9.870e-03	9.939e-03	7.0e-01
7	4.981e-03	5.038e-03	1.1e+00
8	2.471e-03	2.512e-03	1.7e+00
9	1.201e-03	1.226e-03	2.1e+00
10	5.670e-04	5.813e-04	2.5e+00
11	2.574e-04	2.647e-04	2.8e+00
12	1.100e-04	1.133e-04	3.0e+00
13	4.251e-05	4.381e-05	3.1e+00
14	1.355e-05	1.398e-05	3.2e+00
15	2.797e-06	2.886e-06	3.2e+00

Queue Length Distribution at Node 3

Number of units	Exact Solution	Approximation	Error(%)
0	8.957e-02	8.908e-02	5.5e-01
1	8.594e-02	8.586e-02	9.3e-02
2	8.294e-02	8.323e-02	3.5e-01
3	8.049e-02	8.103e-02	6.7e-01
4	7.844e-02	7.909e-02	8.3e-01
5	7.661e-02	7.728e-02	8.7e-01
6	7.485e-02	7.545e-02	8.0e-01
7	7.300e-02	7.347e-02	6.4e-01
8	7.084e-02	7.114e-02	4.2e-01
9	6.806e-02	6.812e-02	8.8e-02
10	6.394e-02	6.371e-02	3.6e-01
11	5.701e-02	5.647e-02	9.5e-01
12	4.665e-02	4.589e-02	1.6e+00
13	3.149e-02	3.073e-02	2.4e+00
14	1.490e-02	1.437e-02	3.6e+00
15	5.267e-03	5.074e-03	3.7e+00

Queue Length Distribution at Node 4

Number of units	Exact Solution	Approximation	Error(%)
0	4.559e-01	4.557e-01	4.4e-02
1	2.721e-01	2.721e-01	0.0e+00
2	2.720e-01	2.722e-01	7.4e-02

Queue Length Distribution at Node 5

Number of units	Exact Solution	Approximation	Error(%)
0	6.971e-01	6.970e-01	1.4e-02
1	2.118e-01	2.118e-01	0.0e+00
2	6.503e-02	6.508e-02	7.7e-02
3	2.604e-02	2.609e-02	1.9e-01

Table 3: The configuration analyzed is shown in Figure 2. $m_1 = m_2 = m_3 = \infty$, $m_4 = 1$, $m_5 = 2$; $\mu_1 = \mu_3 = \mu_4 = 0.5$, $\mu_2 = \mu_5 = 1$; $p_{12} = 0.7$, $p_{13} = 0.3$, $p_{21} = 0.5$, $p_{23} = 0.5$, $p_{34} = 0.5$, $p_{35} = 0.5$, $p_{41} = 0.5$, $p_{42} = 0.25$, $p_{45} = 0.25$, $p_{51} = 0.25$, $p_{52} = 0.5$, $p_{54} = 0.25$; $N = 15$.

Throughput	Exact Solution	Approximation	Error(%)
node 1	4.433e-01	4.434e-01	2.3e-02
node 2	5.023e-01	5.026e-01	6.0e-02
node 3	3.839e-01	3.843e-01	1.0e-01
node 4	2.559e-01	2.562e-01	1.2e-01
node 5	2.559e-01	2.562e-01	1.2e-01
Mean queue length	Exact Solution	Approximation	Error(%)
node 1	5.102e+00	5.084e+00	3.5e-01
node 2	9.845e-01	9.880e-01	3.6e-01
node 3	7.988e+00	8.001e+00	1.6e-01
node 4	5.154e-01	5.159e-01	9.7e-02
node 5	4.100e-01	4.106e-01	1.5e-01
Queue Length Distribution at Node 1			
Number of units	Exact Solution	Approximation	Error(%)
0	1.133e-01	1.131e-01	1.8e-01
1	1.101e-01	1.093e-01	7.3e-01
2	1.039e-01	1.038e-01	9.6e-02
3	9.554e-02	9.621e-02	7.0e-01
4	8.690e-02	8.791e-02	1.2e+00
5	7.868e-02	7.967e-02	1.3e+00
6	7.107e-02	7.186e-02	1.1e+00
7	6.406e-02	6.457e-02	8.0e-01
8	5.756e-02	5.778e-02	3.8e-01
9	5.144e-02	5.137e-02	1.4e-01
10	4.550e-02	4.517e-02	7.3e-01
11	3.946e-02	3.891e-02	1.4e+00
12	3.295e-02	3.228e-02	2.0e+00
13	2.550e-02	2.483e-02	2.6e+00
14	1.676e-02	1.624e-02	3.1e+00
15	7.179e-03	6.921e-03	3.6e+00

Queue Length Distribution at Node 2

Number of units	Exact Solution	Approximation	Error(%)
0	4.977e-01	4.974e-01	6.0e-02
1	2.530e-01	2.529e-01	4.0e-02
2	1.273e-01	1.271e-01	1.6e-01
3	6.307e-02	6.305e-02	3.2e-02
4	3.081e-02	3.094e-02	4.2e-01
5	1.486e-02	1.504e-02	1.2e+00
6	7.080e-03	7.231e-03	2.1e+00
7	3.331e-03	3.436e-03	3.2e+00
8	1.543e-03	1.609e-03	4.3e+00
9	7.018e-04	7.388e-04	5.3e+00
10	3.112e-04	3.307e-04	6.3e+00
11	1.333e-04	1.427e-04	7.1e+00
12	5.410e-05	5.823e-05	7.6e+00
13	2.010e-05	2.168e-05	7.9e+00
14	6.300e-06	6.782e-06	7.7e+00
15	1.286e-06	1.383e-06	7.5e+00

Queue Length Distribution at Node 3

Number of units	Exact Solution	Approximation	Error(%)
0	3.701e-02	3.580e-02	3.3e+00
1	4.049e-02	3.955e-02	2.3e+00
2	4.429e-02	4.369e-02	1.4e+00
3	4.865e-02	4.835e-02	6.2e-01
4	5.358e-02	5.357e-02	1.9e-02
5	5.910e-02	5.936e-02	4.4e-01
6	6.519e-02	6.570e-02	7.8e-01
7	7.180e-02	7.253e-02	1.0e+00
8	7.878e-02	7.969e-02	1.2e+00
9	8.580e-02	8.677e-02	1.1e+00
10	9.207e-02	9.294e-02	9.4e-01
11	9.596e-02	9.650e-02	5.6e-01
12	9.345e-02	9.338e-02	7.5e-02
13	7.621e-02	7.555e-02	8.7e-01
14	4.579e-02	4.511e-02	1.5e+00
15	1.183e-02	1.151e-02	2.7e+00

Queue Length Distribution at Node 4

Number of units	Exact Solution	Approximation	Error(%)
0	4.846e-01	4.841e-01	1.0e-01
1	5.154e-01	5.159e-01	9.7e-02

Queue Length Distribution at Node 5

Number of units	Exact Solution	Approximation	Error(%)
0	6.850e-01	6.846e-01	5.8e-02
1	2.201e-01	2.202e-01	4.5e-02
2	9.494e-02	9.516e-02	2.3e-01

Table 4: $m_1 = m_2 = m_3 = \infty$, $m_4 = 2$; $\mu_1 = 1$, $\mu_2 = 0.5$, $\mu_3 = 1$, $\mu_4 = 0.5$; and $N = 6$.

Throughput	Exact Solution	Approximation	Error(%)
node 1	4.007e-01	4.002e-01	1.2e-01
node 2	4.007e-01	4.002e-01	1.2e-01
node 3	4.007e-01	4.003e-01	1.0e-01
node 4	4.007e-01	4.003e-01	1.0e-01
Mean queue length	Exact Solution	Approximation	Error(%)
node 1	6.205e-01	6.281e-01	1.2e+00
node 2	2.302e+00	2.295e+00	3.0e-01
node 3	1.691e+00	1.692e+00	5.9e-02
node 4	1.387e+00	1.385e+00	1.4e-01
Queue Length Distribution at Node 1			
Number of units	Exact Solution	Approximation	Error(%)
0	5.993e-01	5.998e-01	8.3e-02
1	2.505e-01	2.480e-01	1.0e+00
2	9.938e-02	9.846e-02	9.3e-01
3	3.599e-02	3.682e-02	2.3e+00
4	1.121e-02	1.255e-02	1.2e+01
5	3.005e-03	3.636e-03	2.1e+01
6	5.655e-04	7.195e-04	2.7e+01
Queue Length Distribution at Node 2			
Number of units	Exact Solution	Approximation	Error(%)
0	1.986e-01	1.996e-01	5.0e-01
1	1.912e-01	1.917e-01	2.6e-01
2	1.788e-01	1.789e-01	5.6e-02
3	1.594e-01	1.593e-01	6.3e-02
4	1.315e-01	1.312e-01	2.3e-01
5	9.395e-02	9.332e-02	6.7e-01
6	4.654e-02	4.604e-02	1.1e+00

Queue Length Distribution at Node 3

Number of units	Exact Solution	Approximation	Error(%)
0	2.650e-01	2.665e-01	5.7e-01
1	2.228e-01	2.230e-01	9.0e-02
2	2.034e-01	2.021e-01	6.4e-01
3	1.832e-01	1.794e-01	2.1e+00
4	1.179e-01	1.196e-01	1.4e+00
5	6.491e-03	7.878e-03	2.1e+01
6	1.197e-03	1.466e-03	2.2e+01

Queue Length Distribution at Node 4

Number of units	Exact Solution	Approximation	Error(%)
0	1.986e-01	1.995e-01	4.5e-01
1	2.158e-01	2.162e-01	1.9e-01
2	5.856e-01	5.843e-01	2.2e-01

Table 5: $m_1 = m_2 = m_3 = \infty$, $m_4 = 5$; $\mu_1 = 1$, $\mu_2 = 0.5$, $\mu_3 = 1$, $\mu_4 = 0.5$; and $N = 15$.

Throughput	Exact Solution	Approximation	Error(%)
node 1	4.635e-01	4.635e-01	0.0e+00
node 2	4.635e-01	4.635e-01	0.0e+00
node 3	4.635e-01	4.635e-01	0.0e+00
node 4	4.635e-01	4.635e-01	0.0e+00
Mean queue length	Exact Solution	Approximation	Error(%)
node 1	8.490e-01	8.541e-01	6.0e-01
node 2	6.563e+00	6.571e+00	1.2e-01
node 3	3.838e+00	3.840e+00	5.2e-02
node 4	3.742e+00	3.743e+00	2.7e-02
Queue Length Distribution at Node 1			
Number of units	Exact Solution	Approximation	Error(%)
0	5.365e-01	5.365e-01	0.0e+00
1	2.505e-01	2.500e-01	2.0e-01
2	1.162e-01	1.158e-01	3.4e-01
3	5.345e-02	5.336e-02	1.7e-01
4	2.427e-02	2.440e-02	5.4e-01
5	1.083e-02	1.106e-02	2.1e+00
6	4.722e-03	4.961e-03	5.1e+00
7	2.009e-03	2.200e-03	9.5e+00
8	8.360e-04	9.600e-04	1.5e+01
9	3.404e-04	4.106e-04	2.1e+01
10	1.351e-04	1.710e-04	2.7e+01
11	5.185e-05	6.858e-05	3.2e+01
12	1.892e-05	2.602e-05	3.8e+01
13	6.354e-06	9.021e-06	4.2e+01
14	1.825e-06	2.654e-06	4.5e+01
15	3.599e-07	5.315e-07	4.8e+01

Queue Length Distribution at Node 2

Number of units	Exact Solution	Approximation	Error(%)
0	7.296e-02	7.310e-02	1.9e-01
1	7.290e-02	7.302e-02	1.6e-01
2	7.279e-02	7.290e-02	1.5e-01
3	7.260e-02	7.271e-02	1.5e-01
4	7.232e-02	7.242e-02	1.4e-01
5	7.191e-02	7.201e-02	1.4e-01
6	7.136e-02	7.144e-02	1.1e-01
7	7.061e-02	7.066e-02	7.1e-02
8	6.958e-02	6.959e-02	1.4e-02
9	6.811e-02	6.807e-02	5.9e-02
10	6.586e-02	6.577e-02	1.4e-01
11	6.232e-02	6.218e-02	2.2e-01
12	5.668e-02	5.651e-02	3.0e-01
13	4.786e-02	4.769e-02	3.6e-01
14	3.477e-02	3.463e-02	4.0e-01
15	1.737e-02	1.730e-02	4.0e-01

Queue Length Distribution at Node 3

Number of units	Exact Solution	Approximation	Error(%)
0	2.003e-01	2.003e-01	0.0e+00
1	1.323e-01	1.324e-01	7.6e-02
2	1.003e-01	1.003e-01	0.0e+00
3	8.536e-02	8.544e-02	9.4e-02
4	7.858e-02	7.869e-02	1.4e-01
5	7.556e-02	7.569e-02	1.7e-01
6	7.415e-02	7.425e-02	1.3e-01
7	7.308e-02	7.299e-02	1.2e-01
8	7.102e-02	7.044e-02	8.2e-01
9	6.504e-02	6.349e-02	2.4e+00
10	4.159e-02	4.226e-02	1.6e+00
11	1.980e-03	2.730e-03	3.8e+01
12	5.347e-04	7.756e-04	4.5e+01
13	1.479e-04	2.224e-04	5.0e+01
14	3.785e-05	5.850e-05	5.5e+01
15	7.057e-06	1.111e-05	5.7e+01

Queue Length Distribution at Node 4

Number of units	Exact Solution	Approximation	Error(%)
0	7.305e-02	7.301e-02	5.5e-02
1	7.686e-02	7.683e-02	3.9e-02
2	8.448e-02	8.446e-02	2.4e-02
3	9.993e-02	9.993e-02	0.0e+00
4	1.317e-01	1.317e-01	0.0e+00
5	5.340e-01	5.340e-01	0.0e+00

Table 6: $m_1 = m_2 = m_3 = m_4 = m_5 = \infty$, $m_6 = 2$, $m_7 = 3$; $\mu_1 = \mu_5 = 1.0$, $\mu_2 = \mu_3 = \mu_4 = \mu_6 = \mu_7 = 0.5$; $p_{12} = 0.5$, $p_{13} = 0.25$, $p_{14} = 0.25$, $p_{25} = 1.0$, $p_{35} = 1.0$, $p_{45} = 1.0$, $p_{56} = 0.5$, $p_{57} = 0.5$, $p_{61} = 0.5$, $p_{67} = 0.5$, $p_{71} = 0.5$, $p_{76} = 0.5$; $N = 6$

Throughput	Exact Solution	Approximation	Error(%)
node 1	3.542e-01	3.543e-01	2.8e-02
node 2	1.770e-01	1.771e-01	5.6e-02
node 3	8.855e-02	8.857e-02	2.3e-02
node 4	8.855e-02	8.857e-02	2.3e-02
node 5	3.541e-01	3.542e-01	2.8e-02
node 6	3.542e-01	3.542e-01	0.0e+00
node 7	3.542e-01	3.542e-01	0.0e+00

Mean queue length	Exact Solution	Approximation	Error(%)
node 1	5.334e-01	5.302e-01	6.0e-01
node 2	5.301e-01	5.302e-01	1.9e-02
node 3	2.136e-01	2.136e-01	0.0e+00
node 4	2.136e-01	2.136e-01	0.0e+00
node 5	1.208e+00	1.209e+00	8.3e-02
node 6	1.359e+00	1.360e+00	7.4e-02
node 7	1.942e+00	1.943e+00	5.1e-02

Queue Length Distribution at Node 1

Number of units	Exact Solution	Approximation	Error(%)
0	6.458e-01	6.457e-01	1.5e-02
1	2.291e-01	2.312e-01	9.2e-01
2	8.466e-02	8.356e-02	1.3e-00
3	2.928e-02	2.859e-02	2.4e-00
4	8.779e-03	8.562e-03	2.5e-00
5	2.059e-03	2.012e-03	2.3e-00
6	2.926e-04	2.865e-04	2.1e-00

Queue Length Distribution at Node 2

Number of units	Exact Solution	Approximation	Error(%)
0	6.459e-01	6.457e-01	3.1e-02
1	2.310e-01	2.312e-01	8.7e-02
2	8.369e-02	8.356e-02	1.6e-01
3	2.859e-02	2.859e-02	0.0e+00
4	8.553e-03	8.561e-03	9.4e-02
5	2.011e-03	2.012e-03	5.0e-02
6	2.869e-04	2.865e-04	1.4e-01

Queue Length Distribution at Node 3

Number of units	Exact Solution	Approximation	Error(%)
0	8.229e-01	8.229e-01	0.0e+00
1	1.462e-01	1.464e-01	1.4e-01
2	2.590e-02	2.582e-02	3.1e-01
3	4.265e-03	4.253e-03	2.8e-01
4	6.088e-04	6.070e-04	3.0e-01
5	6.767e-05	6.735e-05	4.7e-01
6	4.509e-06	4.477e-06	7.1e-01

Queue Length Distribution at Node 4

Number of units	Exact Solution	Approximation	Error(%)
0	8.229e-01	8.229e-01	0.0e+00
1	1.462e-01	1.464e-01	1.4e-01
2	2.590e-02	2.582e-02	3.1e-01
3	4.265e-03	4.253e-03	2.8e-01
4	6.088e-04	6.070e-04	3.0e-01
5	6.767e-05	6.735e-05	4.7e-01
6	4.509e-06	4.477e-06	7.1e-01

Queue Length Distribution at Node 5

Number of units	Exact Solution	Approximation	Error(%)
0	2.914e-01	2.924e-01	3.4e-01
1	3.667e-01	3.651e-01	4.4e-01
2	2.151e-01	2.150e-01	4.6e-02
3	9.932e-02	9.946e-01	1.4e-01
4	2.472e-02	2.520e-02	1.9e+00
5	2.420e-03	2.575e-03	6.4e+00
6	3.367e-04	3.562e-04	5.8e+00

Queue Length Distribution at Node 6

Number of units	Exact Solution	Approximation	Error(%)
0	1.953e-01	1.950e-01	1.5e-01
1	2.502e-01	2.502e-01	0.0e+00
2	5.545e-01	5.548e-01	5.4e-02

Queue Length Distribution at Node 7

Number of units	Exact Solution	Approximation	Error(%)
0	1.442e-01	1.439e-01	2.1e-01
1	1.989e-01	1.987e-01	1.0e-01
2	2.274e-01	2.275e-01	4.4e-02
3	4.295e-01	4.300e-01	1.2e-01

Table 7: $m_1 = m_2 = m_3 = m_4 = m_5 = \infty$, $m_6 = 2$, $m_7 = 3$; $\mu_1 = \mu_5 = 1.0$, $\mu_2 = \mu_3 = \mu_4 = \mu_6 = \mu_7 = 0.5$; $p_{12} = 0.5$, $p_{13} = 0.25$, $p_{14} = 0.25$, $p_{25} = 1.0$, $p_{35} = 1.0$, $p_{45} = 1.0$, $p_{56} = 0.5$, $p_{57} = 0.5$, $p_{61} = 0.5$, $p_{67} = 0.5$, $p_{71} = 0.5$, $p_{76} = 0.5$; $N = 8$

Throughput	Exact Solution	Approximation	Error(%)
node 1	3.629e-01	3.629e-01	0.0e+00
node 2	1.814e-01	1.814e-01	0.0e+00
node 3	9.071e-02	9.072e-02	1.1e-02
node 4	9.071e-02	9.072e-02	1.1e-02
node 5	3.629e-01	3.629e-01	0.0e+00
node 6	3.629e-01	3.629e-01	0.0e+00
node 7	3.629e-01	3.629e-01	0.0e+00

Mean queue length	Exact Solution	Approximation	Error(%)
node 1	5.755e-01	5.694e-01	1.1e+00
node 2	5.698e-01	5.694e-01	7.0e-02
node 3	2.224e-01	2.223e-01	4.5e-02
node 4	2.224e-01	2.223e-01	4.5e-02
node 5	2.595e+00	2.603e+00	3.1e-01
node 6	1.540e+00	1.540e+00	0.0e+00
node 7	2.274e+00	2.274e+00	0.0e+00

Queue Length Distribution at Node 1

Number of units	Exact Solution	Approximation	Error(%)
0	6.371e-01	6.371e-01	0.0e+00
1	2.254e-01	2.289e-01	1.6e+00
2	8.694e-02	8.532e-02	1.9e-00
3	3.310e-02	3.178e-02	4.0e+00
4	1.196e-02	1.149e-02	3.9e-00
5	3.990e-03	3.888e-03	2.6e-00
6	1.166e-03	1.153e-03	1.1e-00
7	2.688e-04	2.691e-04	1.1e-01
8	3.781e-05	3.819e-05	1.0e-00

Queue Length Distribution at Node 2

Number of units	Exact Solution	Approximation	Error(%)
0	6.372e-01	6.371e-01	1.6e-02
1	2.286e-01	2.289e-01	1.3e-01
2	8.561e-02	8.532e-02	3.4e-01
3	3.182e-02	3.178e-02	1.3e-01
4	1.148e-02	1.149e-02	8.7e-02
5	3.894e-03	3.887e-03	1.8e-01
6	1.161e-03	1.153e-03	6.9e-01
7	2.728e-04	2.690e-04	1.4e+00
8	3.894e-05	3.817e-05	2.0e+00

Queue Length Distribution at Node 3

Number of units	Exact Solution	Approximation	Error(%)
0	8.186e-01	8.186e-01	0.0e+00
1	1.478e-01	1.480e-01	1.4e-01
2	2.756e-02	2.741e-02	5.4e-01
3	5.050e-03	5.025e-03	5.0e-01
4	8.885e-04	8.855e-04	3.4e-01
5	1.452e-04	1.443e-04	6.2e-01
6	2.064e-05	2.042e-05	1.1e+00
7	2.292e-06	2.252e-06	1.7e+00
8	1.527e-07	1.492e-07	2.3e+00

Queue Length Distribution at Node 4

Number of units	Exact Solution	Approximation	Error(%)
0	8.186e-01	8.186e-01	0.0e+00
1	1.478e-01	1.480e-01	1.4e-01
2	2.756e-02	2.741e-02	5.4e-01
3	5.050e-03	5.025e-03	5.0e-01
4	8.885e-04	8.855e-04	3.4e-01
5	1.452e-04	1.443e-04	6.2e-01
6	2.064e-05	2.042e-05	1.1e+00
7	2.292e-06	2.252e-06	1.7e+00
8	1.527e-07	1.492e-07	2.3e+00

Queue Length Distribution at Node 5

Number of units	Exact Solution	Approximation	Error(%)
0	6.917e-02	7.008e-02	1.3e+00
1	1.329e-01	1.341e-01	9.0e-01
2	2.467e-01	2.438e-01	1.2e+00
3	3.258e-01	3.214e-01	1.4e+00
4	1.501e-01	1.525e-01	1.6e+00
5	6.135e-02	6.317e-02	3.0e+00
6	1.298e-02	1.387e-02	6.9e+00
7	8.159e-04	9.982e-04	2.2e+01
8	1.072e-04	1.311e-04	2.2e+01

Queue Length Distribution at Node 6

Number of units	Exact Solution	Approximation	Error(%)
0	1.288e-01	1.288e-01	0.0e+00
1	2.022e-01	2.023e-01	4.9e-02
2	6.691e-01	6.689e-01	3.0e-02

Queue Length Distribution at Node 7

Number of units	Exact Solution	Approximation	Error(%)
0	8.552e-02	8.554e-02	2.3e-02
1	1.384e-01	1.385e-01	7.2e-02
2	1.926e-01	1.928e-01	1.0e-01
3	5.835e-01	5.832e-01	5.1e-02

Table 8: $m_1 = m_2 = m_3 = m_4 = \infty$, $m_5 = 3$; $\mu_1 = \mu_3 = \mu_4 = 0.5$, $\mu_2 = \mu_5 = 1.0$; $p_{12} = 0.7$, $p_{13} = 0.3$, $p_{21} = 0.5$, $p_{24} = 0.5$, $p_{31} = 0.25$, $p_{32} = 0.25$, $p_{35} = 0.5$, $p_{41} = 0.25$, $p_{42} = 0.25$, $p_{45} = 0.5$, $p_{51} = 0.5$, $p_{52} = 0.5$; $N = 6$

Throughput	Exact Solution	Approximation	Error(%)
node 1	4.649e-01	4.649e-01	0.0e+00
node 2	5.269e-01	5.269e-01	0.0e+00
node 3	1.395e-01	1.395e-01	0.0e+00
node 4	2.635e-01	2.635e-01	0.0e+00
node 5	2.015e-01	2.015e-01	0.0e+00

Mean queue length	Exact Solution	Approximation	Error(%)
node 1	3.387e+00	3.386e+00	3.0e-02
node 2	9.934e-01	9.933e-01	1.0e-02
node 3	3.777e-01	3.778e-01	2.6e-02
node 4	9.949e-01	9.951e-01	2.0e-02
node 5	2.473e-01	2.473e-01	0.0e+00

Queue Length Distribution at Node 1

Number of units	Exact Solution	Approximation	Error(%)
0	7.019e-02	7.023e-02	5.7e-02
1	1.009e-01	1.010e-01	9.9e-02
2	1.386e-01	1.386e-01	0.0e+00
3	1.777e-01	1.777e-01	0.0e+00
4	2.039e-01	2.039e-01	0.0e+00
5	1.922e-01	1.922e-01	0.0e+00
6	1.165e-01	1.165e-01	0.0e+00

Queue Length Distribution at Node 2

Number of units	Exact Solution	Approximation	Error(%)
0	4.731e-01	4.731e-01	0.0e+00
1	2.607e-01	2.607e-01	0.0e+00
2	1.406e-01	1.405e-01	7.1e-02
3	7.275e-02	7.275e-02	0.0e+00
4	3.482e-02	3.482e-02	0.0e+00
5	1.418e-02	1.418e-02	0.0e+00
6	3.856e-03	3.856e-03	0.0e+00

Queue Length Distribution at Node 3

Number of units	Exact Solution	Approximation	Error(%)
0	7.207e-01	7.206e-01	1.4e-02
1	2.046e-01	2.046e-01	0.0e+00
2	5.606e-02	5.608e-02	3.6e-02
3	1.450e-02	1.451e-02	6.9e-02
4	3.403e-03	3.405e-03	5.9e-02
5	6.653e-04	6.663e-04	1.5e-01
6	8.493e-05	8.514e-05	2.5e-01

Queue Length Distribution at Node 4

Number of units	Exact Solution	Approximation	Error(%)
0	4.724e-01	4.723e-01	2.1e-02
1	2.610e-01	2.610e-01	0.0e+00
2	1.409e-01	1.409e-01	0.0e+00
3	7.290e-02	7.291e-02	1.4e-02
4	3.485e-02	3.485e-02	0.0e+00
5	1.419e-02	1.421e-02	1.4e-01
6	3.861e-03	3.868e-03	1.8e-01

Queue Length Distribution at Node 5

Number of units	Exact Solution	Approximation	Error(%)
0	7.985e-01	7.985e-01	0.0e+00
1	1.626e-01	1.626e-01	0.0e+00
2	3.191e-02	3.192e-02	3.1e-02
3	6.969e-03	6.974e-03	7.2e-02

Table 9: $m_1 = m_2 = m_3 = m_4 = \infty$, $m_5 = 3$; $\mu_1 = \mu_3 = \mu_4 = 0.5$, $\mu_2 - \mu_5 = 1.0$; $p_{12} = 0.7$, $p_{13} = 0.3$, $p_{21} = 0.5$, $p_{24} = 0.5$, $p_{31} = 0.25$, $p_{32} = 0.25$, $p_{35} = 0.5$, $p_{41} = 0.25$, $p_{42} = 0.25$, $p_{45} = 0.5$, $p_{51} = 0.5$, $p_{52} = 0.5$; $N = 12$

Throughput	Exact Solution	Approximation	Error(%)
node 1	4.978e-01	4.978e-01	0.0e+00
node 2	5.642e-01	5.641e-01	1.8e-02
node 3	1.493e-01	1.493e+00	0.0e+00
node 4	2.821e-01	2.821e-01	0.0e+00
node 5	2.157e-01	2.157e-01	0.0e+00

Mean queue length	Exact Solution	Approximation	Error(%)
node 1	8.753e+00	8.753e+00	0.0e+00
node 2	1.271e+00	1.271e+00	0.0e+00
node 3	4.263e-01	4.263e-01	0.0e+00
node 4	1.278e+00	1.277e+00	7.8e-02
node 5	2.718e-01	2.718e-01	0.0e+00

Queue Length Distribution at Node 1

Number of units	Exact Solution	Approximation	Error(%)
0	4.447e-03	4.452e-03	1.1e-01
1	7.141e-03	7.129e-03	1.7e-01
2	1.136e-02	1.134e-02	1.8e-01
3	1.786e-02	1.784e-02	1.1e-01
4	2.767e-02	2.765e-02	7.2e-02
5	4.209e-02	4.207e-02	4.8e-02
6	6.250e-02	6.249e-02	1.6e-02
7	8.983e-02	8.984e-02	1.1e-02
8	1.233e-01	1.233e-01	0.0e+00
9	1.580e-01	1.580e-01	0.0e+00
10	1.813e-01	1.814e-01	5.5e-02
11	1.709e-01	1.709e-01	0.0e+00
12	1.036e-01	1.035e-01	9.7e-02

Queue Length Distribution at Node 2

Number of units	Exact Solution	Approximation	Error(%)
0	4.358e-01	4.359e-01	2.3e-02
1	2.468e-01	2.467e-01	4.1e-02
2	1.396e-01	1.396e-01	0.0e+00
3	7.889e-02	7.888e-02	1.3e-02
4	4.448e-02	4.448e-02	0.0e+00
5	2.498e-02	2.498e-02	0.0e+00
6	1.393e-02	1.394e-02	7.2e-02
7	7.676e-03	7.681e-03	6.5e-02
8	4.137e-03	4.140e-03	7.3e-02
9	2.141e-03	2.143e-03	9.3e-02
10	1.024e-03	1.026e-03	2.0e-01
11	4.170e-04	4.176e-04	1.4e-01
12	1.134e-04	1.136e-04	1.8e-01

Queue Length Distribution at Node 3

Number of units	Exact Solution	Approximation	Error(%)
0	7.006e-01	7.006e-01	0.0e+00
1	2.101e-01	2.101e-01	0.0e+00
2	6.282e-02	6.282e-02	0.0e+00
3	1.871e-02	1.871e-02	0.0e+00
4	5.542e-03	5.542e-03	0.0e+00
5	1.627e-03	1.627e-03	0.0e+00
6	4.712e-04	4.714e-04	4.2e-02
7	1.335e-04	1.336e-04	7.5e-02
8	3.654e-05	3.657e-05	8.2e-02
9	9.447e-06	9.459e-06	1.3e-01
10	2.216e-06	2.219e-06	1.4e-01
11	4.332e-07	4.343e-07	2.5e-01
12	5.530e-08	5.549e-08	3.4e-01

Queue Length Distribution at Node 4

Number of units	Exact Solution	Approximation	Error(%)
0	4.345e-01	4.345e-01	0.0e+00
1	2.466e-01	2.467e-01	4.1e-02
2	1.398e-01	1.399e-01	7.2e-02
3	7.919e-02	7.921e-02	2.5e-02
4	4.476e-02	4.474e-02	4.5e-02
5	2.521e-02	2.518e-02	1.2e-01
6	1.410e-02	1.407e-02	2.1e-01
7	7.786e-03	7.762e-03	3.1e-01
8	4.204e-03	4.187e-03	4.0e-01
9	2.176e-03	2.166e-03	4.6e-01
10	1.040e-03	1.036e-03	3.8e-01
11	4.237e-04	4.221e-04	3.8e-01
12	1.153e-04	1.149e-04	3.5e-01

Queue Length Distribution at Node 5

Number of units	Exact Solution	Approximation	Error(%)
0	7.843e-01	7.843e-01	0.0e+00
1	1.694e-01	1.694e-01	0.0e+00
2	3.653e-02	3.653e-02	0.0e+00
3	9.759e-03	9.764e-03	5.1e-02

Table 10: $m_1 = m_2 = m_3 = \infty$, $m_4 = m_5 = 4$, $m_6 = 5$, $\mu_1 = \mu_2 = \mu_3 = 1$, $\mu_4 = \mu_5 = \mu_6 = 0.5$; $p_{12} = 1$, $p_{23} = 1$, $p_{34} = 0.5$, $p_{35} = 0.5$, $p_{46} = 1$, $p_{56} = 1$, $p_{61} = 0.5$, $p_{64} = 0.25$, $p_{65} = 0.25$; $N = 10$.

Throughput	Exact Solution	Approximation	Error(%)
node 1	2.360e-01	2.359e-01	4.2e-02
node 2	2.360e-01	2.359e-01	4.2e-02
node 3	2.360e-01	2.359e-01	4.2e-02
node 4	2.359e-01	2.359e-01	0.0e+00
node 5	2.360e-01	2.359e-01	4.2e-02
node 6	4.719e-01	4.718e-01	2.1e-02

Mean queue length	Exact Solution	Approximation	Error(%)
node 1	3.110e-01	3.104e-01	1.9e-01
node 2	3.106e-01	3.104e-01	6.4e-02
node 3	4.697e-01	4.707e-01	2.1e-01
node 4	2.154e+00	2.153e+00	4.6e-02
node 5	2.155e+00	2.154e+00	4.6e-02
node 6	4.600e+00	4.602e+00	4.3e-02

Queue Length Distribution at Node 1			
Number of units	Exact Solution	Approximation	Error(%)
0	7.640e-01	7.641e-01	1.3e-02
1	1.788e-01	1.791e-01	1.7e-01
2	4.344e-02	4.315e-02	6.7e-01
3	1.049e-02	1.040e-02	8.6e-01
4	2.494e-03	2.478e-03	6.4e-01
5	5.749e-04	5.758e-04	1.6e-01
6	1.255e-04	1.278e-04	1.8e+00
7	2.525e-05	2.636e-05	4.4e+00
8	4.552e-06	4.881e-06	7.2e+00
9	8.802e-07	7.456e-07	1.5e+01
10	6.563e-08	7.295e-08	1.1e+01

Queue Length Distribution at Node 2

Number of units	Exact Solution	Approximation	Error(%)
0	7.640e-01	7.641e-01	1.3e-02
1	1.791e-01	1.791e-01	0.0e+00
2	4.323e-02	4.315e-02	1.9e-01
3	1.041e-02	1.040e-02	9.6e-02
4	2.479e-03	2.478e-03	4.0e-02
5	5.757e-04	5.758e-04	1.7e-02
6	1.279e-04	1.278e-04	7.8e-02
7	2.643e-05	2.636e-05	2.6e-01
8	4.916e-06	4.881e-06	7.1e-01
9	7.543e-07	7.458e-07	1.1e+00
10	7.397e-08	7.295e-08	1.4e+00

Queue Length Distribution at Node 3

Number of units	Exact Solution	Approximation	Error(%)
0	6.684e-01	6.679e-01	7.5e-02
1	2.366e-01	2.367e-01	4.2e-02
2	6.465e-02	6.489e-02	3.7e-01
3	2.092e-02	2.099e-02	3.3e-01
4	6.875e-03	6.894e-03	2.8e-01
5	2.118e-03	2.123e-03	2.4e-01
6	4.496e-04	4.577e-04	1.8e+00
7	2.994e-05	3.069e-05	2.5e+00
8	5.448e-06	5.554e-06	1.9e+00
9	8.317e-07	8.463e-07	1.8e+00
10	8.144e-08	8.285e-08	1.7e+00

Queue Length Distribution at Node 4

Number of units	Exact Solution	Approximation	Error(%)
0	1.282e-01	1.284e-01	1.6e-01
1	2.227e-01	2.232e-01	2.2e-01
2	2.290e-01	2.286e-01	1.7e-01
3	2.073e-01	2.066e-01	3.4e-01
4	2.128e-01	2.132e-01	1.9e-01

Queue Length Distribution at Node 5

Number of units	Exact Solution	Approximation	Error(%)
0	1.281e-01	1.282e-01	7.8e-02
1	2.225e-01	2.231e-01	2.7e-01
2	2.290e-01	2.286e-01	1.7e-01
3	2.074e-01	2.067e-01	3.4e-01
4	2.130e-01	2.134e-01	1.9e-01

Queue Length Distribution at Node 6

Number of units	Exact Solution	Approximation	Error(%)
0	5.158e-03	5.115e-03	8.3e-01
1	1.284e-02	1.276e-02	6.2e-01
2	2.867e-02	2.855e-02	4.2e-01
3	5.983e-02	5.969e-02	2.3e-01
4	1.169e-01	1.168e-01	8.6e-02
5	7.766e-01	7.771e-01	6.4e-02

Table 11: $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = \infty$, $m_7 = 4$, $m_8 = 5$; $\mu_1 = \mu_2 = \mu_6 = 1$,
 $\mu_3 = \mu_4 = \mu_5 = \mu_7 = \mu_8 = 0.5$; $p_{12} = 1$, $p_{23} = p_{24} = p_{25} = 0.333$, $p_{36} = p_{46} = p_{56} = 1$,
 $p_{67} = p_{68} = p_{71} = p_{78} = p_{81} = p_{87} = 0.5$; $N = 6$.

Throughput	Exact Solution	Approximation	Error(%)
node 1	3.633e-01	3.612e-01	5.8e-01
node 2	3.633e-01	3.612e-01	5.8e-01
node 3	1.211e-01	1.204e-01	5.8e-01
node 4	1.211e-01	1.204e-01	5.8e-01
node 5	1.211e-01	1.204e-01	5.8e-01
node 6	3.632e-01	3.637e-01	1.4e-01
node 7	3.633e-01	3.637e-01	1.1e-01
node 8	3.632e-01	3.637e-01	1.4e-01

Mean queue length	Exact Solution	Approximation	Error(%)
node 1	5.344e-01	5.302e-01	7.9e-01
node 2	5.343e-01	5.302e-01	7.7e-01
node 3	3.097e-01	3.076e-01	6.8e-01
node 4	3.097e-01	3.076e-01	6.8e-01
node 5	3.097e-01	3.076e-01	6.8e-01
node 6	5.808e-01	5.823e-01	2.6e-01
node 7	1.657e+00	1.662e+00	3.0e-01
node 8	1.765e+00	1.773e+00	4.5e-01

Queue Length Distribution at Node 1

Number of units	Exact Solution	Approximation	Error(%)
0	6.367e-01	6.388e-01	3.3e-01
1	2.406e-01	2.400e-01	2.5e-01
2	8.571e-02	8.475e-02	1.1e-00
3	2.756e-02	2.720e-02	1.3e-00
4	7.591e-03	7.482e-03	1.4e-00
5	1.615e-03	1.589e-03	1.6e-00
6	2.021e-04	1.987e-04	1.7e-00

Queue Length Distribution at Node 2

Number of units	Exact Solution	Approximation	Error(%)
0	6.367e-01	6.388e-01	3.3e-01
1	2.407e-01	2.400e-01	2.9e-01
2	8.569e-02	8.475e-02	1.1e+00
3	2.754e-02	2.720e-02	1.2e+00
4	7.582e-03	7.482e-03	1.3e+00
5	1.612e-03	1.589e-03	1.4e+00
6	2.016e-04	1.987e-04	1.4e+00

Queue Length Distribution at Node 3

Number of units	Exact Solution	Approximation	Error(%)
0	7.578e-01	7.592e-01	1.8e-01
1	1.878e-01	1.869e-01	4.8e-01
2	4.352e-02	4.307e-02	1.0e+00
3	9.068e-03	8.974e-03	1.0e+00
4	1.612e-03	1.596e-03	9.9e-01
5	2.201e-04	2.180e-04	9.5e-01
6	1.760e-05	1.744e-05	9.1e-01

Queue Length Distribution at Node 4

Number of units	Exact Solution	Approximation	Error(%)
0	7.578e-01	7.592e-01	1.8e-01
1	1.878e-01	1.869e-01	4.8e-01
2	4.352e-02	4.307e-02	1.0e+00
3	9.068e-03	8.974e-03	1.0e+00
4	1.612e-03	1.596e-03	9.9e-01
5	2.201e-04	2.180e-04	9.5e-01
6	1.760e-05	1.744e-05	9.1e-01

Queue Length Distribution at Node 5

Number of units	Exact Solution	Approximation	Error(%)
0	7.578e-01	7.592e-01	1.8e-01
1	1.878e-01	1.869e-01	4.8e-01
2	4.352e-02	4.307e-02	1.0e+00
3	9.068e-03	8.974e-03	1.0e+00
4	1.612e-03	1.596e-03	9.9e-01
5	2.201e-04	2.180e-04	9.5e-01
6	1.760e-05	1.744e-05	9.1e-01

Queue Length Distribution at Node 6

Number of units	Exact Solution	Approximation	Error(%)
0	5.993e-01	5.987e-01	1.0e-01
1	2.688e-01	2.688e-01	0.0e+00
2	9.502e-02	9.537e-02	3.7e-01
3	2.749e-02	2.764e-02	5.5e-01
4	7.559e-03	7.624e-03	8.6e-01
5	1.606e-03	1.629e-03	1.4e+00
6	2.007e-04	2.036e-04	1.4e+00

Queue Length Distribution at Node 7

Number of units	Exact Solution	Approximation	Error(%)
0	2.670e-01	2.667e-01	1.1e-01
1	2.459e-01	2.455e-01	1.6e-01
2	1.962e-01	1.961e-01	5.1e-02
3	1.455e-01	1.459e-01	2.7e-01
4	1.454e-01	1.466e-01	8.3e-01

Queue Length Distribution at Node 8

Number of units	Exact Solution	Approximation	Error(%)
0	2.465e-01	2.453e-01	4.9e-01
1	2.486e-01	2.481e-01	2.0e-01
2	2.075e-01	2.073e-01	9.6e-02
3	1.463e-01	1.467e-01	2.7e-01
4	9.281e-02	9.353e-02	7.8e-01
5	5.825e-02	5.921e-02	1.6e+00

Table 12: $m_1 = m_2 = m_3 = m_4 = \infty$, $m_5 = m_6 = 5$; $\mu_1 = \mu_2 = \mu_6 = 1$, $\mu_3 = \mu_4 = \mu_5 = 0.5$; $p_{12} = 1$, $p_{23} = p_{24} = 0.5$, $p_{35} = p_{46} = p_{56} = 1$, $p_{61} = p_{62} = p_{63} = p_{65} = 0.25$; $N = 10$.

Throughput	Exact Solution	Approximation	Error(%)
node 1	1.561e-01	1.561e-01	0.0e+00
node 2	3.122e-01	3.123e-01	3.2e-02
node 3	3.122e-01	3.122e-01	0.0e+00
node 4	1.561e-01	1.561e-01	0.0e+00
node 5	4.682e-01	4.682e-01	0.0e+00
node 6	6.243e-01	6.244e-01	1.6e-02

Mean queue length	Exact Solution	Approximation	Error(%)
node 1	1.862e-01	1.859e-01	1.6e-01
node 2	4.570e-01	4.570e-01	0.0e+00
node 3	3.078e+00	3.078e+00	0.0e+00
node 4	4.762e-01	4.761e-01	2.1e-02
node 5	3.946e+00	3.947e+00	2.5e-02
node 6	1.857e+00	1.856e+00	5.4e-02

Queue Length Distribution at Node 1

Number of units	Exact Solution	Approximation	Error(%)
0	8.439e-01	8.439e-01	0.0e+00
1	1.306e-01	1.310e-01	3.1e-01
2	2.147e-02	2.121e-02	1.2e+00
3	3.387e-03	3.335e-03	1.5e+00
4	5.081e-04	5.025e-04	1.1e+00
5	7.156e-05	7.159e-05	4.2e-02
6	9.354e-06	9.501e-06	1.6e+00
7	1.126e-06	1.161e-06	3.1e+00
8	1.208e-07	1.277e-07	4.1e+00
9	1.056e-08	1.104e-08	4.5e+00
10	5.807e-10	6.063e-10	4.4e+00

Queue Length Distribution at Node 2

Number of units	Exact Solution	Approximation	Error(%)
0	6.878e-01	6.877e-01	1.5e-02
1	2.116e-01	2.117e-01	4.7e-02
2	6.922e-02	6.917e-02	7.2e-02
3	2.200e-02	2.200e-02	0.0e+00
4	6.720e-03	6.722e-03	3.0e-02
5	1.945e-03	1.945e-03	0.0e+00
6	5.263e-04	5.249e-04	2.7e-01
7	1.318e-04	1.310e-04	6.1e-01
8	2.950e-05	2.921e-05	9.8e-01
9	5.407e-06	5.340e-06	1.2e+00
10	6.287e-07	6.203e-07	1.3e+00

Queue Length Distribution at Node 3

Number of units	Exact Solution	Approximation	Error(%)
0	8.722e-02	8.721e-02	1.1e-02
1	1.284e-01	1.284e-01	0.0e+00
2	1.721e-01	1.721e-01	0.0e+00
3	2.060e-01	2.059e-01	4.9e-02
4	1.991e-01	1.989e-01	1.0e-01
5	1.206e-01	1.208e-01	1.7e-01
6	4.415e-02	4.423e-02	1.8e-01
7	2.400e-02	2.404e-02	1.7e-01
8	1.202e-02	1.204e-02	1.7e-01
9	5.004e-03	5.012e-03	1.6e-01
10	1.332e-03	1.334e-03	1.5e-01

Queue Length Distribution at Node 4

Number of units	Exact Solution	Approximation	Error(%)
0	6.746e-01	6.747e-01	1.5e-02
1	2.206e-01	2.206e-01	0.0e+00
2	7.218e-02	7.212e-02	8.3e-02
3	2.284e-02	2.282e-02	8.8e-02
4	6.967e-03	6.960e-03	1.0e-01
5	2.029e-03	2.028e-03	4.9e-02
6	5.622e-04	5.619e-04	5.3e-02
7	1.462e-04	1.461e-04	6.8e-02
8	3.383e-05	3.381e-05	5.9e-02
9	6.341e-06	6.337e-06	6.3e-02
10	7.455e-07	7.449e-07	8.0e-02

Queue Length Distribution at Node 5

Number of units	Exact Solution	Approximation	Error(%)
0	3.469e-02	3.469e-02	0.0e+00
1	5.488e-02	5.488e-02	0.0e+00
2	8.357e-02	8.356e-02	1.2e-02
3	1.219e-01	1.218e-01	8.2e-02
4	1.662e-01	1.662e-01	0.0e+00
5	5.388e-01	5.389e-01	1.9e-02

Queue Length Distribution at Node 6

Number of units	Exact Solution	Approximation	Error(%)
0	2.513e-01	2.513e-01	0.0e+00
1	2.385e-01	2.386e-01	4.2e-02
2	1.888e-01	1.888e-01	0.0e+00
3	1.372e-01	1.372e-01	0.0e+00
4	9.228e-02	9.226e-02	2.2e-02
5	9.195e-02	9.193e-02	2.2e-02

Table 13: $m_1 = m_2 = m_3 = \infty$, $m_4 = m_5 = m_6 = 5$; $\mu_1 = \mu_2 = \mu_6 = 1$, $\mu_3 = \mu_4 = \mu_5 = 0.5$; $p_{13} = p_{23} = 1$, $p_{34} = p_{35} = p_{41} = p_{46} = p_{52} = p_{56} = 0.5$, $p_{61} = p_{62} = p_{63} = p_{64} = p_{65} = 0.2$; $N = 10$.

Throughput	Exact Solution	Approximation	Error(%)
node 1	2.090e-01	2.090e-01	0.0e+00
node 2	2.090e-01	2.090e-01	0.0e+00
node 3	4.778e-01	4.778e-01	0.0e+00
node 4	2.986e-01	2.986e-01	0.0e+00
node 5	2.986e-01	2.986e-01	0.0e+00
node 6	2.986e-01	2.986e-01	0.0e+00
Mean queue length	Exact Solution	Approximation	Error(%)
node 1	2.632e-01	2.631e-01	3.8e-02
node 2	2.632e-01	2.631e-01	3.8e-02
node 3	6.388e+00	6.388e+00	0.0e+00
node 4	1.323e+00	1.323e+00	0.0e+00
node 5	1.323e+00	1.323e+00	0.0e+00
node 6	4.397e-01	4.397e-01	0.0e+00
Queue Length Distribution at Node 1			
Number of units	Exact Solution	Approximation	Error(%)
0	7.910e-01	7.910e-01	0.0e+00
1	1.658e-01	1.658e-01	0.0e+00
2	3.451e-02	3.443e-02	2.3e-01
3	7.054e-03	7.029e-03	3.5e-01
4	1.400e-03	1.401e-03	7.1e-02
5	2.655e-04	2.701e-04	1.7e+00
6	4.713e-05	4.956e-05	5.2e+00
7	7.668e-06	8.457e-06	1.0e+01
8	1.105e-06	1.285e-06	1.6e+01
9	1.297e-07	1.586e-07	2.2e+01
10	9.639e-09	1.233e-08	2.8e+01

Queue Length Distribution at Node 2

Number of units	Exact Solution	Approximation	Error(%)
0	7.910e-01	7.910e-01	0.0e+00
1	1.658e-01	1.658e-01	0.0e+00
2	3.451e-02	3.443e-02	2.3e-01
3	7.054e-03	7.029e-03	3.5e-01
4	1.400e-03	1.401e-03	7.1e-02
5	2.655e-04	2.701e-04	1.7e+00
6	4.713e-05	4.956e-05	5.2e+00
7	7.668e-06	8.457e-06	1.0e+01
8	1.105e-06	1.285e-06	1.6e+01
9	1.297e-07	1.586e-07	2.2e+01
10	9.639e-09	1.233e-08	2.8e+01

Queue Length Distribution at Node 3

Number of units	Exact Solution	Approximation	Error(%)
0	6.048e-03	6.046e-03	3.3e-02
1	1.482e-02	1.481e-02	6.7e-02
2	3.112e-02	3.111e-02	3.2e-02
3	5.867e-02	5.866e-02	1.7e-02
4	9.786e-02	9.784e-02	2.0e-02
5	1.332e-01	1.332e-01	0.0e+00
6	1.430e-01	1.430e-01	0.0e+00
7	1.610e-01	1.610e-01	0.0e+00
8	1.607e-01	1.607e-01	0.0e+00
9	1.291e-01	1.291e-01	0.0e+00
10	6.456e-02	6.456e-02	0.0e+00

Queue Length Distribution at Node 4

Number of units	Exact Solution	Approximation	Error(%)
0	4.025e-01	4.025e-01	0.0e+00
1	2.438e-01	2.438e-01	0.0e+00
2	1.498e-01	1.498e-01	0.0e+00
3	9.195e-02	9.196e-02	1.1e-02
4	5.611e-02	5.610e-02	1.8e-02
5	5.583e-02	5.582e-02	1.8e-02

Queue Length Distribution at Node 5

Number of units	Exact Solution	Approximation	Error(%)
0	4.025e-01	4.025e-01	0.0e+00
1	2.438e-01	2.438e-01	0.0e+00
2	1.498e-01	1.498e-01	0.0e+00
3	9.195e-02	9.195e-02	0.0e+00
4	5.611e-02	5.610e-02	1.8e-02
5	5.586e-02	5.582e-02	7.2e-02

Queue Length Distribution at Node 6

Number of units	Exact Solution	Approximation	Error(%)
0	6.905e-01	6.905e-01	0.0e+00
1	2.163e-01	2.163e-01	0.0e+00
2	6.591e-02	6.591e-02	0.0e+00
3	1.960e-02	1.960e-02	0.0e+00
4	5.651e-03	5.651e-03	0.0e+00
5	2.040e-03	2.040e-03	0.0e+00

Table 14: $m_1 = m_2 = m_3 = \infty$, $m_4 = 1$, $m_5 = 2$, $m_6 = 3$; $\mu_1 = \mu_2 = \mu_6 = 1$, $\mu_3 = \mu_4 = \mu_5 = 0.5$; $p_{13} = p_{23} = 1$, $p_{34} = p_{35} = p_{41} = p_{46} = p_{52} = p_{56} = 0.5$, $p_{61} = p_{62} = p_{63} = p_{64} = p_{65} = 0.2$; $N = 10$.

Throughput	Exact Solution	Approximation	Error(%)
node 1	1.662e-01	1.663e-01	6.0e-02
node 2	1.663e-01	1.663e-01	0.0e+00
node 3	3.799e-01	3.799e-01	0.0e+00
node 4	2.377e-01	2.377e-01	0.0e+00
node 5	2.376e-01	2.376e-01	0.0e+00
node 6	2.378e-01	2.378e-01	0.0e+00

Mean queue length	Exact Solution	Approximation	Error(%)
node 1	1.946e-01	1.960e-01	7.2e-01
node 2	1.977e-01	1.960e-01	8.6e-01
node 3	8.030e+00	8.030e+00	0.0e+00
node 4	4.774e-01	4.774e-01	0.0e+00
node 5	7.010e-01	7.010e-01	0.0e+00
node 6	3.994e-01	3.994e-01	0.0e+00

Queue Length Distribution at Node 1

Number of units	Exact Solution	Approximation	Error(%)
0	8.338e-01	8.337e-01	1.2e-02
1	1.413e-01	1.406e-01	5.0e-01
2	2.172e-02	2.204e-02	1.5e+00
3	2.758e-03	3.125e-03	1.3e+01
4	3.254e-04	4.048e-04	2.4e+01
5	3.672e-05	4.859e-05	3.2e+01
6	4.030e-06	5.484e-06	3.6e+01
7	4.331e-07	5.872e-07	3.6e+01
8	4.477e-08	5.947e-08	3.3e+01
9	4.042e-09	5.177e-09	3.4e+01
10	2.481e-10	3.508e-10	4.1e+01

Queue Length Distribution at Node 2

Number of units	Exact Solution	Approximation	Error(%)
0	8.337e-01	8.337e-01	0.0e+00
1	1.394e-01	1.406e-01	8.6e-01
2	2.288e-02	2.204e-02	3.7e+00
3	3.436e-03	3.125e-03	9.1e+00
4	4.560e-04	4.048e-04	1.1e+01
5	5.682e-05	4.859e-05	1.4e+01
6	6.823e-06	5.484e-06	2.0e+01
7	7.996e-07	5.872e-07	2.7e+01
8	9.088e-08	5.947e-08	3.5e+01
9	9.336e-09	5.422e-09	0.0e+00
10	6.335e-10	3.508e-10	0.0e+00

Queue Length Distribution at Node 3

Number of units	Exact Solution	Approximation	Error(%)
0	8.849e-07	7.221e-07	1.8e+01
1	8.742e-06	7.648e-06	1.3e+01
2	8.148e-05	7.632e-05	6.3e+00
3	7.546e-04	7.415e-04	1.7e+00
4	6.059e-03	6.029e-03	5.0e-01
5	2.257e-02	2.255e-02	8.9e-02
6	7.817e-02	7.822e-02	6.4e-02
7	1.999e-01	1.999e-01	0.0e+00
8	3.128e-01	3.127e-01	3.2e-02
9	2.767e-01	2.767e-01	0.0e+00
10	1.030e-01	1.030e-01	0.0e+00

Queue Length Distribution at Node 4

Number of units	Exact Solution	Approximation	Error(%)
0	5.226e-01	5.226e-01	0.0e+00
1	4.774e-01	4.774e-01	0.0e+00

Queue Length Distribution at Node 5

Number of units	Exact Solution	Approximation	Error(%)
0	5.219e-01	5.219e-01	0.0e+00
1	2.552e-01	2.552e-01	0.0e+00
2	2.229e-01	2.229e-01	0.0e+00

Queue Length Distribution at Node 6

Number of units	Exact Solution	Approximation	Error(%)
0	6.987e-01	6.987e-01	0.0e+00
1	2.225e-01	2.225e-01	0.0e+00
2	5.970e-02	5.970e-02	0.0e+00
3	1.916e-02	1.916e-02	0.0e+00