**Analysis of Delta-Sigma Modulators
with Bandlimited Gaussian Inputs**

by

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Abstract

Delta-Sigma modulation is gaining widespread use as a monolithic analog to digital and digital to analog conversion technique in a wide variety of applications. The behavior of Delta-Sigma modulators depends on the statistics and nature of the input signal. Furthermore, the behavior of the circuit is input amplitude dependent due to the nonlinearity (two level quantizer). In this paper the nonlinear behavior of high order Delta-Sigma modulators with bandlimited Gaussian inputs is analyzed.

The signal to noise ratio of Delta-Sigma modulators as a function of input signal variance is computed for the first time for bandlimited Gaussian signals. The value of the input variance leading to modulator overload is calculated. Equations are derived, the solution of which yield the variance of signals within the modulator as a function of input variance. It is shown that the variance of the additive noise due to quantization is constant and independent of the input signal variance. Moreover, it is shown that if the integrators are not limited, the variance at the integrator outputs can become unbounded. Limiting will actually improve signal to noise ratio.

1.Introduction

Delta-Sigma modulators achieve high resolution analog to digital and digital to analog conversion using over-sampling techniques and by eliminating the need for precision analog components. Since most of the complexity of these circuits is in digital signal processing they are rapidly finding applications in monolithic VLSI circuits where the A/D and D/A functions are integrated on the same chip.

In this paper the performance of Delta-Sigma modulators with bandlimited Gaussian inputs is analyzed. The theoretical analysis of Delta-Sigma modulation has recently attracted attention [1,2,3]. In [3] a rigorous analysis of a first order modulator with constant input is analyzed. In [1,2], higher order modulators are treated. Previously [1,2], analytical techniques were introduced to study the performance of the modulator with DC and sinusoidal excitations. Basically the nonlinear element (two level quantizer) was replaced by two linearized gains, one for the input signal component and one for the feed back distortion components. Through this procedure, two interlocked linear systems are obtained. The input to one system consists of the input to the modulator. For the other system, the input is the random distortion components produced by the quantization process.

Since the linearized gains depend on the probability distribution functions of the signal and distortion components expressions are derived in terms of signal and noise transfer functions, and the additive noise variance. These expressions together form a system of nonlinear and integral equations the solution of which yield the linearized gains and signal variances in the modulator for a given input

variance and bandwidth.

In this paper, the following results are obtained:

- (1) The variance of the input bandlimited Gaussian signal leading to modulator overload is defined and calculated.
- (2) The variance of the signal and noise components at various nodes within the modulator are calculated as a function of input signal variance. It is shown that as the input variance increases to the overload value, the variance at the output of the second integrator in a second order system becomes unbounded.
- (3) From the result in (2) it is shown that limiting on the integrators can improve signal to noise ratio.
- (4) The signal to noise ratio is calculated as a function of input variance. As in the case of DC and sinusoidal excitation, the SNR begins to decrease as the input variance increases beyond a certain magnitude. This result is important because it is not possible to measure SNR for Gaussian inputs in simulations or actual circuits.
- (5) Finally, the effect of baseband bandwidth on the modulator performance is obtained without requiring any approximations as in [1,2].

A second order Delta-Sigma modulator is shown in Fig. 1 and an equivalent circuit is shown in Fig. 2.

2. Modeling of Nonlinearity with Multiple Gaussian Inputs

In a previous paper [1,2] it was shown that the behavior of the DSM may be analyzed by modeling the nonlinearity by two linearized gains, one for the input-signal component and one for the feed back distortion components. By this procedure, two interlocked linear systems are obtained. The input to one system consists of the input signal to the modulator. For the other system, the input is the random distortion components produced by the quantization process.

Consider the nonlinearity $N(x+y)$ with inputs x and y both random gaussian variables. See Fig. 3. Then the linearized gains which minimize the mean square error between the linearized system and the actual nonlinearity are,

$$K_x = \frac{1}{\sigma_x^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xN(x+y)p(y)q(x)dx dy \quad (2.1)$$

$$K_y = \frac{1}{\sigma_y^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} yN(x+y)p(y)q(x)dx dy \quad (2.2)$$

where,

$$p(x) = \frac{2}{\sigma_x \sqrt{\pi}} e^{\frac{-x^2}{2\sigma_x^2}} \quad (2.3)$$

and,

$$q(x) = \frac{2}{\sigma_y \sqrt{\pi}} e^{-\frac{y^2}{2\sigma_y^2}} \quad (2.4)$$

If the nonlinearity is a two level quantizer, with levels Δ and $-\Delta$, then

$$K_x = \frac{1}{\sigma_x^2} \int_{-\infty}^{\infty} 2\Delta \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma_x^2}} \sigma_y^2 p(y) dy \quad (2.5)$$

$$K_x = \frac{\Delta}{\pi} \frac{\sqrt{2}}{\sigma_x} \int_{-\infty}^{\infty} e^{-\alpha^2 y'^2} e^{-y'^2} dy' \quad (2.6)$$

Finally,

$$K_x = \frac{\Delta}{\sigma_x} \frac{\sqrt{2}}{\pi} \int_{-\infty}^{\infty} e^{-(1+\alpha^2)y'^2} dy' \quad (2.7)$$

where

$$\alpha = \frac{\sigma_y^2}{\sigma_x^2} \quad (2.8)$$

Now let,

$$A = \int_{-\infty}^{\infty} e^{-\beta x^2} dx = \frac{1}{\sqrt{\beta}} \int_{-\infty}^{\infty} e^{-x'^2} dx' = \sqrt{\frac{\pi}{\beta}} \quad (2.9)$$

where

$$\beta = (1 + \alpha^2) \quad (2.10)$$

Thus,

$$K_x = \sqrt{\frac{2}{\pi}} \frac{\Delta}{\sigma_x} \frac{1}{\sqrt{1+\alpha^2}} \quad (2.11)$$

Similarly

$$K_y = \sqrt{\frac{2}{\pi}} \frac{\Delta}{\sigma_y} \frac{\alpha}{\sqrt{1+\alpha^2}} \quad (2.12)$$

3.Application to Delta-Sigma Modulation

By replacing the quantizer by the lineared gains followed by an additive noise source, $n(k)$, the Delta-Sigma Modulator can be separated into two interlocked linear systems as illustrated in Fig. 4 and Fig. 5. In one system, the input forcing function is the bandlimited gaussian signal $x(k)$. In the other system, the forcing function is the additive noise source $n(k)$ produced by quantization. Thus, from

Fig. 4, we have,

$$H_{xe}(z) = \frac{E_x(z)}{X(z)} = \frac{H_i(z)H(z)}{1 + K_x H(z)} \quad (3.1)$$

If integrators are used in the modulator, then in the base-band region and assuming that the baseband bandwidth is small (later this restriction will be removed),

$$\frac{E_x(z)}{X(z)} \approx \frac{1}{K_x} \quad (3.2)$$

It follows that,

$$\sigma_{ex}^2 = \frac{1}{K_x^2} \sigma_x^2 \quad (3.3)$$

We are assuming that $n(k)$ is white. Furthermore, we assume that the input to the nonlinearity due to the noise source $n(k)$, $e_n(k)$, is gaussian. From Fig. 5

$$\sigma_{en}^2 = \frac{\sigma_n^2}{2\pi} \int_{-\pi}^{\pi} \frac{|H(e^{jw})|^2}{|1 + K_n H(e^{jw})|^2} dw \quad (3.4)$$

The power at the output of the Delta-Sigma Modulator is constant for a single bit quantizer. Hence,

$$E\{p^2(K)\} = \sigma_n^2 + K_x^2 \sigma_{ex}^2 + K_n^2 \sigma_{en}^2 = \Delta^2 \quad (3.5)$$

Substituting for the linearized gains and solving for the additive noise variance we obtain,

$$\sigma_n^2 = \Delta^2 \left[1 - \frac{2}{\pi} \right] \quad (3.6)$$

Substitute for K_x in (3.2), thus,

$$\sigma_{ex}^2 = \frac{\pi}{2} \frac{\sigma_{ex}^2}{\Delta^2} (1 + \alpha^2) \sigma_x^2 \quad (3.7)$$

or,

$$\alpha = \sqrt{\frac{2}{\pi} \frac{\Delta^2}{\sigma_x^2} - 1} \quad (3.8)$$

where

$$\alpha^2 = \frac{\sigma_{en}^2}{\sigma_{ex}^2} \quad (3.9)$$

Also from (2.12),

$$K_n = \sqrt{\frac{2}{\pi}} \frac{\Delta}{\sigma_{en}} \frac{\alpha}{\sqrt{1 + \alpha^2}} \quad (3.10)$$

4. Signal-to-Noise-Ratio

The noise transfer function can be written,

$$NTF(z) = \frac{P_n(z)}{N(z)} = \frac{1}{1 + K_n H(z)} \quad (4.1)$$

The in-band noise can be calculated by

$$\sigma_{nB}^2 = \frac{\sigma_n^2}{f_s} \int_{-f_B}^{f_B} P_n(e^{j2\pi \frac{f}{f_s}}) P_n(e^{-j2\pi \frac{f}{f_s}}) df \quad (4.2)$$

where we have assumed that $n(k)$ is a white random process with variance σ_n^2 .

The signal-to-noise ratio is thus defined as,

$$SNR = \frac{\sigma_x^2}{\sigma_{nB}^2} \quad (4.3)$$

To evaluate the SNR as a function of the input signal variance, we must obtain K_n by solving the coupled nonlinear and integral equation (3.4),(3.8),(3.10).

If we remove the requirement that the base-band bandwidth is small, we need to simultaneously solve the following set of equations:

$$K_x = \sqrt{\frac{2}{\pi}} \frac{\Delta}{\sigma_{ex}} \frac{1}{\sqrt{1+\alpha^2}} \quad ; \quad \alpha^2 = \frac{\sigma_{en}^2}{\sigma_{ex}^2} \quad (4.4.1)$$

$$\frac{2}{\pi} \frac{\Delta^2}{K_x^2} \frac{1}{1+\alpha^2} = \frac{\sigma_x^2}{2w_B} \int_{-w_B}^{w_B} |H_{xe}(e^{jw}, K_x)|^2 dw \quad (4.4.2)$$

$$\sigma_n^2 = \Delta^2 \left[\frac{\pi-2}{\pi} \right] \text{ constant} \quad (4.4.3)$$

$$\sigma_{en}^2 = \frac{\sigma_n^2}{2\pi} \int_{-\pi}^{\pi} \frac{|H(e^{jw})|^2}{|1+K_n H(e^{jw})|^2} \quad (4.4.4)$$

$$\sigma_{en}^2 = \frac{2}{\pi} \frac{\Delta^2}{K_n^2} \frac{\alpha^2}{1+\alpha^2} \quad (4.4.5)$$

$$K_x = K_n \quad (4.4.6)$$

Eliminating σ_{en}^2 between (4.4.4) and (4.4.5) we obtain,

$$\frac{\alpha^2}{1+\alpha^2} = \frac{\pi}{2} K_n^2 \frac{\sigma_n^2}{\Delta^2} f_1(K_n) \quad (4.5)$$

Also (2) can be written as

$$\frac{1}{1+\alpha^2} = \frac{\pi}{4} K_n^2 \frac{\sigma_x^2}{\Delta^2} f_2(K_n) \quad (4.6)$$

where

$$f_1(K_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|H(e^{jw})|^2}{|1+K_n H(e^{jw})|^2} dw \quad (4.8)$$

and

$$f_2(K_n, w_B) = \frac{1}{2w_B} \int_{-w_B}^{w_B} |H_{xe}(e^{jw}, K_n)|^2 dw \quad (4.9)$$

Eliminating α^2 between (4.5) and (4.6) ,

$$\frac{\frac{\pi}{2} K_n^2 \frac{\sigma_n^2}{\Delta^2} f_1(K_n)}{1 - \frac{\pi}{2} \frac{K_n^2 \sigma_n^2}{\Delta^2} f_1(K_n)} = \frac{1 - \frac{\pi}{2} K_n^2 \frac{\sigma_x^2}{\Delta^2} f_2(K_n, w_B)}{\frac{\pi}{2} K_n^2 \frac{\sigma_x^2}{\Delta^2} f_2(K_n, w_B)} \quad (4.10)$$

In the above expression, given the input signal variance, σ_x^2 and baseband bandwidth w_B (radians), the only unknown is K_n (note that σ_n is a constant).

5. Numerical Results

In this section numerical results are presented for the signal to noise ratio, the variance of the signal components at the quantizer input, and the linearized gains. These results are obtained by solving (4.12) for K_n as a function of input variance σ_x^2 using a computer. Once K_n is obtained, then using the equations in section 4 all other quantities of interest can be computed.

In the results to be presented a second order Delta-Sigma modulator with unity gains ($\alpha_1 = \alpha_2 = 1$) was considered. The sampling rate was 1.024 Mhz and the baseband bandwidth is 4 kHz. The quantizer step size was chosen to be one ($\Delta=1$).

Fig. 6 shows the calculated signal to noise ratio versus the input variance in dB. The zero dB reference corresponds to a unity variance. For small baseband bandwidths (3.8) shows that the maximum input variance is

$$\sigma_x^2 = \frac{2}{\pi} \Delta. \quad (5.1)$$

Beyond this value, the modulator is in overload and according to (3.8) the ratio between the variance of the signal component and noise component at the quantizer input goes to infinity. That is,

$$\rho = \frac{\sigma_{ex}^2}{\sigma_{en}^2} = \frac{1}{\sqrt{\frac{2}{\pi} \frac{\Delta^2}{\sigma_x^2} - 1}} \quad (5.2)$$

In Fig. 7, ρ and the linearized gains are plotted against the input variance. The decrease in SNR shown in Fig. 6 corresponds to the decrease in the linearized gain due to overloading.

The variances at the quantizer inputs are plotted in Fig. 8. As shown in Fig. 8 the variances of the signals at the quantizer input become very large. Due to the limiting action of the quantizer, this results in a decrease in the value of the linearized gain. In Fig. 9, a plot of the linearized gain is shown in more detail. As the linearized gain decreases, the noise spectral shaping is modified. This is shown shown in Fig. 10, where the noise transfer function is plotted for two cases corresponding to linearized gains of 0.3 and 0.6. From this figure we observe that the noise moves in-band when the gain decreases. This results in a decrease in Signal to Noise Ratio as is shown in Fig. 6.

6. Simulation Results

In order to verify the analytical results presented in this paper, simulations of a second order Delta-Sigma modulator were carried out in which the modulator was excited by bandlimited gaussian noise. In the simulations, the sampling rate was 1.024 Mhz and the gaussian noise was band limited to 4 kHz. In Fig. 11, the spectrum of the Delta-Sigma modulator is plotted for an input variance of -22 dB (with respect to unit variance). In Fig. 12 the spectrum is shown for an input variance of -12 dB. Notice that the noise has increased compared to Fig. 11. This is predicted

analytically as shown in Fig. 10 since as the variance increases the linearized gain decreases and more noise is moved in-band. Notice that in the case of bandlimited gaussian inputs, the additive noise variance is constant and independent of input variance.

Since the decrease in linearized gain is due to the very large increase in signal variances at the quantizer input, it seems that introducing limiting at the output of the second integrator should improve the modulator performance. This is because the limiting action will prevent the signal at the quantizer input from becoming too large, and therefore, the linearized gain will not decrease beyond a certain value. In Fig. 13, the spectrum of the modulator output is shown for an input variance of -9 dB. In Fig. 14, the spectrum with limiting on the integrator output is shown. Clearly, the Signal to Noise ratio is greatly improved.

In practice, the integrator output will be limited and the Signal to Noise ratio will not degrade compared to no limiting at all. With no limiting, since the signal is gaussian, the input magnitude may exceed Δ occasionally in which case the output of the integrator will increase continuously and can become unbounded as shown in Fig. 8.

7. Comparison with DC and Sinusoidal Inputs

This paper extends previously derived results for DC and sinusoidal inputs to bandlimited gaussian inputs. In Table I, a summary of expressions for overload, additive noise variance and the ratio between the variance of the signal component

and noise component , ρ , derived in [1] and this paper are provided. It is interesting to note that for bandlimited gaussian inputs, the linearized gains for the signal and noise components are equal (4.4.6). Furthermore, unlike the case for DC and sinusoidal inputs, the signal variances for bandlimited gaussian inputs can become unbounded. In this case introducing limiting on the integrator outputs can improve modulator performance.

8. Conclusions

This paper presented an analysis of the performance of higher order Delta-Sigma modulators with bandlimited gaussian inputs. It was shown that the modulator overloads for an input variance of $\frac{2}{\pi}\Delta$ where Δ is the quantizer step size. The variance of the additive noise source is constant and independent of the input variance. Furthermore, it was shown that the variance of the signal at the quantizer input can become unbounded as the input variance is increased. Therefore, the introduction of limiting on the integrator output actually improves Signal to Noise Ratio.

A system of nonlinear and integral equations were derived which yield the linearized gain value and Signal to Noise Ratio for a specified input variance and bandwidth. The Signal to Noise Ratio is shown to exhibit an overloading characteristic as the input variance increases beyond -15 dB (with respect to a variance of Δ^2) for a second order modulator.

Simulations are presented which confirm the behaviour predicted by the analysis.

9. References

- [1] Sasan Ardalan and John Paulos, "Analysis of Nonlinear Behavior in Delta-Sigma Modulators," *IEEE Trans. on Circuits and Systems*, June 1987.
- [2] Sasan Ardalan and John Paulos, " Analysis of Sinusoidally Excited Delta Sigma Modulators," *Int. Conf. on Acoustics, Speech and Sig. Proc. ICASSP 1987*, Dallas Texas.
- [3] Robert M. Gray, "Oversampled Sigma-Delta Modulation," *IEEE Trans. on Communications*, VOL. COM-35, May 1987.

HIGHER ORDER DELTA SIGMA MODULATORS

Input Excitation	Overload	Additive Noise Variance σ_n^2	Signal to Noise Component Ratio, Quantizer Input, ρ
DC level = m_x ¹	$m_x = \Delta$	$\Delta^2 \left[1 - \frac{m_x^2}{\Delta^2} - \frac{2}{\pi} \exp \left\{ -2 \left[\text{erf}^{-1} \left(\frac{m_x}{\Delta} \right) \right]^2 \right\} \right]$	$\text{erf}^{-1} \left(\frac{m_x}{\Delta} \right)$
Sinusoidal ¹ Amplitude = a_x	$a_x = \Delta$	$\Delta^2 \left[1 - \frac{a_x^2}{s \Delta^2} - \frac{2}{\pi} M^2(1/2, 1, -\rho^2) \right]$	Solution of: $\rho^2 M^2(1/2, 2, -\rho^2) = \frac{\pi}{4} \frac{a_x^2}{\Delta^2}$
Band Limited ² Gaussian, Variance σ_x^2	$\sigma_x^2 = \frac{2}{\pi} \Delta^2$	constant $\Delta^2 \left(\frac{\pi - 2}{\pi} \right)$	$\frac{1}{\sqrt{\frac{2}{\pi} \frac{\Delta^2}{\sigma_x^2} - 1}}$

Notes: $M(\alpha, \gamma, x)$ are the confluent hypergeometric functions.

¹Sasan Ardalan, John Paulos, "An Analysis of Nonlinear Behavior in Delta-Sigma Modulators," *IEEE Trans. on Circuits and Systems*, June 1987.

²Sasan Ardalan, "Analysis of Delta-Sigma Modulation with Bandlimited Gaussian Inputs," CCSP-TR-87/17, October 1987.

Figure 1. Second Order Delta-Sigma Modulator

Figure 2. Equivalent Circuit of Delta-Sigma Modulators

Figure 3. Linearized Gain Modeling of Nonlinearity

Figure 4. Block Diagram for Input Signal Component

Figure 5. Block Diagram for Noise Component

SNR, dB Bandlimited Gaussian Input

Figure 6. Calculated Signal to Noise Ratio

Figure 7. Calculated Linearized Gain and ρ

Figure 8. Calculated Variances at Quantizer Input

Variances at Quantizer Input

Figure 9. Expanded view of Linearized Gain versus Level

Figure 10. Effect of Linearized Gain on Noise Spectral Shaping

Figure 11. Spectrum With Input Variance of -22 dB

Figure 12. Spectrum With Input Variance of -12 dB

Figure 13. Spectrum With Input Variance of -9.8 dB

Figure 14. Spectrum With Input Variance of -9.8 dB and
Limiting on Second Integrator