**Corrections for Nonuniform Sampling Distortions
in Magnetic Resonance Imagery**

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Abstract

A technique for estimating and correcting nonuniform sampling distortions in magnetic resonance imagery is presented. The technique consists of first estimating the sampling grid distortion, using a parametric model, then estimating the uniformly sampled data by interpolation on the nonuniform grid. Parameterized models for two particular sampling grid distortions are presented. Maximum likelihood and maximum a posteriori estimators are developed for these distortions. Using computer simulated data, these estimators are shown to perform well.

Key Words

MRI, RESTORATION, SAMPLING, DISTORTION

1. Introduction

Magnetic resonance imaging (MRI) is a relatively new method for obtaining high resolution images of the human body [1-5]. The MRI image is a representation of different spatially varying parameters. One parameter is the relative concentration of the atomic species, typically hydrogen, being imaged. Two other parameters are the T_1 and T_2 relaxation times of the species being imaged. These two parameters reflect the local molecular environment and thus provide the possibility for distinguishing between healthy and diseased tissue.

The complex data generated during the MRI process represents samples of the Fourier transform of the plane being imaged. Both polar and Cartesian data collection strategies are used. With the polar data strategy, the data represents uniform samples taken along uniformly separated rays through the frequency domain origin. The desired image is typically obtained via a filtered backprojection algorithm. With the Cartesian data collection strategy, the data represents samples obtained using a rectangular sampling grid, i.e. the data represents samples obtained by uniformly sampling the Fourier transform of the plane being imaged in both orthogonal frequency directions. This data, obtained by using a Cartesian sampling grid, will be referred to as the Fourier data matrix. The desired image is obtained by a discrete Fourier transform of the Fourier data matrix. With no loss of generality, this paper assumes the data collection strategy to be Cartesian.

In practice there is reason to believe that the data represents samples obtained on a nonuniform sampling grid. This nonuniformity is caused by nonlinear behavior of the analog hardware. If the inverse discrete Fourier transform of the nonuniformly sampled data is taken assuming that the data is uniformly sampled data, the resulting image is distorted. Given the importance of the application, even slight

distortions may be unacceptable.

If the sampling grid distortion is known, uniformly sampled data can be estimated by interpolation using the nonuniformly sampled data. If the exact grid distortion is not known, an estimate for the distortion can be obtained using a parametric model for the distortion. This estimate can be obtained by processing the nonuniformly sampled data obtained by imaging an object with a known Fourier transform. The distortion is assumed to remain relatively constant for at least the time it takes to make several additional images.

In section 2, nonuniform sampling distortions are introduced. In section 3, the distortion estimators associated with these two distortion models are developed. The method for interpolating from nonuniform samples to uniform samples is presented in section 4. In section 5, the performance of the distortion estimators and the interpolation scheme is evaluated using computer simulated data.

2. Sampling Distortions

In the data collection strategy of interest, each row of the Fourier data matrix is obtained by repeating the somewhat simplified pulse sequence of Figure 1 with an incremented pulse amplitude, G_y . In this example, the plane being imaged is assumed to lie perpendicular to the z axis. Thus, only a gradient in the z direction is required during the RF pulse.

The Free Induction Decay (FID) signal is observed during the x gradient pulse and uniformly sampled to obtain one row of the Fourier data matrix. Let u denote the frequency axis along which the N_u columns of the Fourier data matrix lie. Let v denote the frequency axis along which the N_v rows of the Fourier data matrix lie. The pulse in the G_y direction serves to locate the frequency in the v axis of the

Fourier plane. The pulse in the G_x direction serves to move the sampling point in the u direction along the line located by the G_y pulse.

2.1 Distortion 1

In theory G_y is stepped through N_y uniformly separated values. In practice the distance between successive G_y amplitudes may decrease as the magnitude of the G_y pulse increases due to nonlinearities in the y gradient amplifier. The effect on the data is that each row of data represents samples of the Fourier transform of the plane being imaged taken at a v frequency lower in magnitude than the assumed uniform sampling frequency.

This distortion, denoted as Distortion 1, is a frequency compression distortion that occurs in the v frequency direction only and is modeled by using a sampling grid that is uniform in the u frequency direction but that has rows that are obtained by compressing the uniform sampling grid in the v frequency direction. The compression becomes more pronounced with increasing absolute frequency. A flexible model that will be used to describe this distortion is

$$v_{i \text{ (distorted)}} = \frac{v_{i \text{ (uniform)}}}{1 + \left(\frac{|v_{i \text{ (uniform)}}|}{C} \right)^q}, \quad (1)$$

where C and q are positive real numbers. For this study, it is assumed that q is known and that C is a nonrandom number to be estimated. Figure 2(a) is a plot of $v_{(distorted)}$ versus $v_{(uniform)}$ for several values of C ($q=1.0$). The more linear the curve in Figure 2(a), the higher the value of C used to generate it, and the smaller the resulting distortion. A sampling grid obtained by compressing a uniform sampling grid in the v frequency direction according to (1) is given in Figure 2(b).

The form of the model chosen to describe this distortion is a subjective decision. Another model that could be used to describe this distortion is

$$v_i \text{ (distorted)} = \frac{\gamma}{\pi} \sin \left[\frac{\pi v_i \text{ (uniform)}}{\gamma} \right] , \quad (2)$$

where γ is a positive real number and

$$\gamma \geq \max_i |v_i \text{ (uniform)}| .$$

There are many other forms for the model which are applicable. In section 5 the effect of the model choice will be discussed.

2.2 Distortion 2

In theory, G_x must be of constant amplitude during the time the FID signal is sampled in order to obtain uniformly spaced samples. In practice, square pulses are difficult to approximate. The x gradient pulse may ring or in some other way not have a constant amplitude. The pulse distortion is assumed to be the same for each row of data, that is, the pulse shape is replicated for each row. The effect on the data is that each column of data represents samples of the Fourier transform of the plane being imaged taken at a u frequency slightly displaced away from the assumed uniform sampling frequency.

This distortion, denoted as Distortion 2, occurs in the u frequency direction only and is modeled by using a sampling grid with columns that are perturbed a random distance away from their respective uniform sampling frequencies. A model that will be used to describe this distortion is

$$u_k \text{ (distorted)} = u_k \text{ (uniform)} + B_k , \quad (3)$$

where each of the perturbations, B_k is a random variable. It is convenient to

assume that each B_k is an independent, zero mean, Gaussian random variable of variance σ^2 , i.e. $N(0, \sigma)$. Figure 3(a) is a plot of u_k (*distorted*) versus u_k (*uniform*) for a particular set of values for the B_k 's. A sampling grid obtained by perturbing the columns of a uniform sampling grid in the u frequency direction according to (3) is given in Figure 3(b).

3. Distortion Estimation

Given the MRI sampling grid distortions described in the previous section, the problem of interest is to determine the extent of the grid distortion and to compensate for its effects. To apply classical estimation theory, it is necessary to have a mathematical model of the distortion which describes the distortion in terms of a few parameters. This parametric characterization of the distortion quantifies the problem and provides the variables to be estimated.

The model for Distortion 1 is given by (1) and the parameter to be estimated is C . The model for Distortion 2 is given by (3) and the parameters to be estimated are the B_k 's. Once an estimate for the distortion parameter(s) is obtained, an estimate for the uniformly sampled data can be obtained by interpolation using the nonuniformly sampled data.

There are three common estimators which are generally used: the maximum likelihood (ML) estimator, used to estimate unknown but nonrandom variables; the minimum mean square error and maximum a posteriori estimators, used for random variables with known probability distributions [6]. For the cases of interest here, the ML estimate will be derived for the parameter "C" in distortion 1; the MAP estimate will be derived for the deviations in distortion 2.

3.1 Nonuniform Sampling Distortion Estimation

To derive either a MAP or a ML estimator, the likelihood function must be specified. For this study, it is assumed that each element of $\mathbf{R}$, the data space, the $N_u \times N_v$ Fourier data matrix, is of the form

$$r_{i,k} = s(i,k,\mathbf{D}) + n_{i,k} \quad (4)$$

where $s(i,k,\mathbf{D})$ is a sample of a complex function s , $n_{i,k}$ is additive noise, $\mathbf{D}$ is an N dimensional distortion parameter vector, and i and k are data vector indices. Without any information to the contrary, the real and imaginary parts of each noise sample, $\text{Re}[n_{i,k}]$ and $\text{Im}[n_{i,k}]$, respectively, are assumed to be independent Gaussian random variables, $N(0, \sigma_n^2)$, independent of $\mathbf{D}$ and each of the other noise samples.

The likelihood function becomes

$$\begin{aligned} p_{\mathbf{r}|\mathbf{d}}(\mathbf{R}|\mathbf{D}) &= \prod_{i=1}^{N_v} \prod_{k=1}^{N_u} p_{r_{i,k}|\mathbf{d}}(R_{i,k}|\mathbf{D}) \quad (\text{since the noise samples are independent}) \\ &= \prod_{i=1}^{N_v} \prod_{k=1}^{N_u} p_{\text{Re}[r_{i,k}], \text{Im}[r_{i,k}]|\mathbf{d}}(\text{Re}[R_{i,k}], \text{Im}[R_{i,k}]|\mathbf{D}) \\ &= \prod_{i=1}^{N_v} \prod_{k=1}^{N_u} p_{\text{Re}[r_{i,k}]|\mathbf{d}}(\text{Re}[R_{i,k}]|\mathbf{D}) \cdot p_{\text{Im}[r_{i,k}]|\mathbf{d}}(\text{Im}[R_{i,k}]|\mathbf{D}) \\ &\quad (\text{since the real and imaginary parts of each noise sample are independent}) \\ &= \prod_{i=1}^{N_v} \prod_{k=1}^{N_u} \frac{1}{(2\pi)^{1/2} \sigma_n} \exp \left\{ -\frac{\{\text{Re}[R_{i,k}] - \text{Re}[s(i,k,\mathbf{D})]\}^2}{2\sigma_n^2} \right\} \cdot \frac{1}{(2\pi)^{1/2} \sigma_n} \exp \left\{ -\frac{\{\text{Im}[R_{i,k}] - \text{Im}[s(i,k,\mathbf{D})]\}^2}{2\sigma_n^2} \right\}. \end{aligned} \quad (5)$$

Deriving either the MAP or ML estimators requires determining the partial derivative of the natural logarithm of the likelihood function with respect to each element of $\mathbf{D}$. Taking the natural logarithm of (5), and taking the partial derivative

with respect to element D_j of $\mathbf{D}$, the ML equations follow:

$$\frac{1}{\sigma_n^2} \sum_{i=1}^{N_v} \sum_{k=1}^{N_u} \{ \text{Re}[R_{i,k}] - \text{Re}[s(i,k,\mathbf{D})] \} \frac{\partial \text{Re}[s(i,k,\mathbf{D})]}{\partial D_j} + \{ \text{Im}[R_{i,k}] - \text{Im}[s(i,k,\mathbf{D})] \} \frac{\partial \text{Im}[s(i,k,\mathbf{D})]}{\partial D_j} \Big|_{\mathbf{D}=\hat{\mathbf{d}}_{ml}(\mathbf{R})} = 0 \quad j=1,\dots,N. \quad (6)$$

Solving each of these N equations simultaneously for $\mathbf{D}$ yields the ML estimate for $\mathbf{D}$.

The MAP equations include the a priori density:

$$\frac{1}{\sigma_n^2} \sum_{i=1}^{N_v} \sum_{k=1}^{N_u} \left[\{ \text{Re}[R_{i,k}] - \text{Re}[s(i,k,\mathbf{D})] \} \frac{\partial \text{Re}[s(i,k,\mathbf{D})]}{\partial D_j} + \{ \text{Im}[R_{i,k}] - \text{Im}[s(i,k,\mathbf{D})] \} \frac{\partial \text{Im}[s(i,k,\mathbf{D})]}{\partial D_j} \right] + \frac{\partial}{\partial D_j} \ln[p_d(\mathbf{D})] \Big|_{\mathbf{D}=\hat{\mathbf{d}}_{map}(\mathbf{R})} = 0 \quad j=1,\dots,N. \quad (7)$$

Solving each of these N equations simultaneously for $\mathbf{D}$ yields the MAP estimate for $\mathbf{D}$.

Before inserting functional forms for $s(i,k,\mathbf{D})$, it should be noted that a closed form solution for $\hat{\mathbf{d}}_{ml}(\mathbf{R})$ or $\hat{\mathbf{d}}_{map}(\mathbf{R})$ is unlikely because neither $\text{Re}[s(i,k,\mathbf{D})]$, $\text{Im}[s(i,k,\mathbf{D})]$, nor their partial derivatives with respect to D_j are likely to be simple functions in $\mathbf{D}$. If a closed form solution for $\hat{\mathbf{d}}_{ml}(\mathbf{R})$ or $\hat{\mathbf{d}}_{map}(\mathbf{R})$ does not exist, it is possible that it can be obtained numerically.

The function s is arbitrary and can be chosen for convenience. The $s(i,k,\mathbf{D})$ used in this research is derived by nonuniformly sampling the Fourier transform of a two-dimensional rectangular pulse that has been rotated clockwise by θ and translated by α_x in the x direction and α_y in the y direction. A rectangular pulse was chosen for s because:

- it has an easily computed Fourier transform,
- the corresponding phantom (a cube) is easy to construct,
- it is spacelimited (the phantom must fit within the imaging device).

Figure 4 shows the orientation of the phantom within the imaging device.

The frame of reference with respect to the pulse, (x', y') , can be expressed in terms of the frame of reference of the imaging device, (x, y) , as:

$$f(x, y) = A \cdot \text{rect} \left[\frac{(x - \alpha_x) \cos \theta + (y - \alpha_y) \sin \theta}{\tau_x}, \frac{(-x + \alpha_x) \sin \theta + (y - \alpha_y) \cos \theta}{\tau_y} \right]. \quad (8)$$

Taking the Fourier transform of (8),

$$\begin{aligned} F(u, v) &= \text{Fourier transform}\{f(x, y)\} \\ &= A \cdot \exp[-j2\pi(u\alpha_x + v\alpha_y)] \frac{\sin[\pi\tau_x(u\cos\theta + v\sin\theta)]}{\pi(u\cos\theta + v\sin\theta)} \frac{\sin[\pi\tau_y(-u\sin\theta + v\cos\theta)]}{\pi(-u\sin\theta + v\cos\theta)}. \end{aligned} \quad (9)$$

3.2.1 ML Estimator for Distortion 1

With the parametric model for the distortion given in section 2, $\mathbf{D}$ is a one dimensional nonrandom real number C and

$$s(i, k, \mathbf{D}) = s(i, k, C) = F(u, v) \left| \begin{array}{l} u = k\Delta f_u, \quad v = \frac{i\Delta f_v}{1 + \left[\frac{|i\Delta f_v|}{C} \right]^q} \end{array} \right. \quad (10)$$

The ML equation is extremely cumbersome and is given in appendix A. As was suggested previously, an explicit solution for $\hat{c}_{ml}(\mathbf{R})$ does not exist but the ML equation can be solved numerically using a root finding routine.

The parameter q , which in this research is assumed to be known, can be included in the distortion parameter vector $\mathbf{D}$ and be estimated along with C . This complicates the estimation procedure since estimating both parameters would require that two nonlinear equations be solved simultaneously.

3.2.2. MAP Estimation for Distortion 2

With the parametric model for the distortion given in section 2, $\mathbf{D}$ is an N_u dimensional random parameter vector $\mathbf{B}$ and

$$s(i, k, \mathbf{D}) = s(i, k, \mathbf{B}) = F(u, v) \Big|_{u = k\Delta f_u + B_k, v = i\Delta f_v} \quad (11)$$

Again, the MAP equations are extremely cumbersome and are given in appendix B. As in the previous case, an explicit solution for $\hat{\mathbf{b}}_{\text{map}}(\mathbf{R})$ does not exist but the MAP equations can also be solved numerically using a root finding routine.

4. Interpolation Method

The goal of this work is to obtain an estimate for uniformly sampled data using nonuniformly sampled data. Once an estimate for the distortion vector is obtained it can be used to interpolate from the nonuniformly sampled data to uniformly sampled data. Since the object being imaged is obviously spacelimited, the continuous Fourier transform of the plane being imaged, denoted as $G(u, v)$, is uniquely determined by its uniform samples if the Nyquist condition is satisfied (i.e. if the sampling rate is high enough). The continuous Fourier transform can be reconstructed using a sinc function interpolator [7]:

$$\begin{aligned} G(u, v) &= \sum_{i=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} G(k\Delta f_u, i\Delta f_v) \frac{\sin \left[\frac{\pi}{\Delta f_u} (u - k\Delta f_u) \right]}{\frac{\pi}{\Delta f_u} (u - k\Delta f_u)} \frac{\sin \left[\frac{\pi}{\Delta f_v} (v - i\Delta f_v) \right]}{\frac{\pi}{\Delta f_v} (v - i\Delta f_v)} \\ &= K \sum_{i=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} G(k\Delta f_u, i\Delta f_v) \operatorname{sinc} \left[\frac{1}{\Delta f_u} (u - k\Delta f_u) \right] \operatorname{sinc} \left[\frac{1}{\Delta f_v} (v - i\Delta f_v) \right], \\ &\quad \text{where } K = \frac{1}{\Delta f_u} \frac{1}{\Delta f_v}. \end{aligned} \quad (12)$$

Data lengths are finite in practical situations and only a distorted version of $G(u, v)$ can be obtained with finite data. Let

$\mathbf{I} = \{\text{all integers}\},$

$\mathbf{U} = \{N_u \text{ indices corresponding to the desired uniform sampling frequencies along the } u \text{ frequency axis}\},$

$\mathbf{V} = \{N_v \text{ indices corresponding to the desired uniform sampling frequencies along the } v \text{ frequency axis}\}.$

Equation (12) can be rewritten as

$$G(u, v) = K \sum_{i \in \mathbf{V}} \sum_{k \in \mathbf{U}} G(k \Delta f_u, i \Delta f_v) \text{sinc} \left[\frac{1}{\Delta f_u} (u - k \Delta f_u) \right] \text{sinc} \left[\frac{1}{\Delta f_v} (v - i \Delta f_v) \right] \\ + K \sum_{i \in \mathbf{I} \cap \mathbf{V}} \sum_{k \in \mathbf{I} \cap \mathbf{U}} G(k \Delta f_u, i \Delta f_v) \text{sinc} \left[\frac{1}{\Delta f_u} (u - k \Delta f_u) \right] \text{sinc} \left[\frac{1}{\Delta f_v} (v - i \Delta f_v) \right]. \quad (13)$$

Let $e(u, v)$ denote the truncation error, then

$$K \sum_{i \in \mathbf{V}} \sum_{k \in \mathbf{U}} G(k \Delta f_u, i \Delta f_v) \text{sinc} \left[\frac{1}{\Delta f_u} (u - k \Delta f_u) \right] \text{sinc} \left[\frac{1}{\Delta f_v} (v - i \Delta f_v) \right] = G(u, v) - e(u, v). \quad (14)$$

Given the $N_u N_v$ nonuniform samples of $G(u, v)$, the $N_u N_v$ equations specified by equation (14) can be written in matrix notation as follows:

$$\mathbf{A} \mathbf{x} = \mathbf{b} - \mathbf{b}_\epsilon \quad (15)$$

where

$\mathbf{A}$: $(N_u N_v) \times (N_u N_v)$ sinc function interpolation matrix assumed to be invertible

$\mathbf{x}$: $(N_u N_v)$ uniform samples vector

$\mathbf{b}$: $(N_u N_v)$ nonuniform samples vector

$\mathbf{b}_\epsilon$: $(N_u N_v)$ truncation error vector.

Solving equation (15) for $\mathbf{x}$,

$$\mathbf{x} = \mathbf{A}^{-1} \mathbf{b} - \mathbf{A}^{-1} \mathbf{b}_\epsilon.$$

$\mathbf{A}^{-1} \mathbf{b}$ is a calculatable quantity but $\mathbf{A}^{-1} \mathbf{b}_\epsilon$ is not since $\mathbf{b}_\epsilon$ is not known. An approxi-

mation to $\mathbf{x}$, denoted as $\hat{\mathbf{x}}$, can be found by calculating $\mathbf{A}^{-1}\mathbf{b}$. The error in this approximation, $\mathbf{x}_\epsilon$, is equal to $\mathbf{A}^{-1}\mathbf{b}_\epsilon$.

A bound on $\mathbf{x}_\epsilon$ can be found using matrix norms,

$$\|\mathbf{x}_\epsilon\| \leq \|\mathbf{A}^{-1}\| \|\mathbf{b}_\epsilon\|. \quad (16)$$

A bound on the absolute value of the truncation error, i.e. $\|\mathbf{b}_\epsilon\|_\infty$, can be found following Papoulis [8] in terms of the total image energy and the energy in the sinc interpolating function at the uniform sampling frequencies corresponding to $\mathbf{I} \cap \mathbf{V}$ and $\mathbf{I} \cap \mathbf{U}$. The simulations in the next section show that even for large distortions the approximation is very good.

For $N_u = N_v = 512$ the dimension of $\mathbf{A}$ is 262,144 x 262,144 which makes the interpolation impractical. If the distortion occurs in the u or v frequency direction only, the interpolation becomes a one dimensional interpolation and the dimension of $\mathbf{A}$ is reduced to $N_u \times N_u$ or $N_v \times N_v$, respectively. Assuming the distortion remains constant from image to image for some period of time, $\mathbf{A}^{-1}$ need be calculated only once during that time period. $\mathbf{A}^{-1}$ can then be applied to each row/column of each Fourier transform matrix generated during the time $\mathbf{A}$ is constant.

N_u and N_v are assumed to be powers of two and the $N_u N_v$ uniform samples are assumed to be taken such that

$$\frac{1}{2}(N_u N_v) \text{ samples lie in the region } u > 0, \quad (17a)$$

$$\frac{1}{2}(N_u N_v) \text{ samples lie in the region } u \leq 0, \quad (17b)$$

$$\frac{1}{2}(N_u N_v) \text{ samples lie in the region } v > 0, \text{ and} \quad (17c)$$

$$\frac{1}{2}(N_u N_v) \text{ samples lie in the region } v \leq 0. \quad (17d)$$

4.1 The Interpolation Matrix for Distortion 1

Substituting the $N_u N_v$ nonuniform sampling frequencies specified by (10) and the assumed data range into (14), the matrix, $\mathbf{A}$ is $N_v \times N_v$ and can be shown to be

$$\mathbf{A}_{i,j} = K \operatorname{sinc} \left\{ \frac{(i - \frac{N_v}{2} + 1)}{1 + \left(\frac{|(i - \frac{N_v}{2} + 1)\Delta f_v|}{C} \right)^q} - (j - \frac{N_v}{2} + 1) \right\}. \quad (18)$$

4.2 The Interpolation Matrix for Distortion 2

Substituting the $N_u N_v$ nonuniform sampling frequencies specified by (11) and assumed data range into (14), the matrix, $\mathbf{A}$ which is $N_u \times N_u$ can be shown to be

$$\mathbf{A}_{i,j} = K \operatorname{sinc} \left\{ \frac{1}{\Delta f_u} \left[(i - \frac{N_v}{2} + 1)\Delta f_u + B_i - (j - \frac{N_u}{2} + 1)\Delta f_u \right] \right\}. \quad (19)$$

4.3 Other Interpolation Methods

As mentioned previously, when the sampling grid distortion occurs in both frequency directions, the dimension of the system of linear equations that must be solved in order to perform a sinc function interpolation is $N_u N_v \times N_u N_v$. Thus, the computational burden associated with this interpolation scheme makes it impractical for reasonable values of N_u and N_v . Any interpolation scheme that is based upon the fact that the value of $G(u,v)$ at any point (u,v) can be approximated by a combination of uniformly spaced samples of $G(u,v)$ will, in general, require that $N_u N_v$ equations be solved simultaneously in order to perform the interpolation. Schemes that do not require a system of equations to be solved simultaneously are much more attractive computationally. The sinc function interpolation scheme used in this research is a specific example of a more general interpolation procedure.

5. Computer Simulations

In this section the performance of both the distortion estimator and the interpolation scheme for both types of nonuniform sampling distortions considered in this research is evaluated using computer simulated data. The use of actual MRI data was precluded because of the proprietary nature of the data recording and preprocessing of commercial scanners. The computer simulated data is generated by nonuniformly sampling the Fourier transform of a rectangular pulse oriented as in Figure 3. The specific dimensions of the rectangular pulse are as follows: $A=1.0$, $\tau_x=0.6m$, and $\tau_y=0.6m$. The uniform sampling distance in either frequency direction is taken to be 1.0 cycle per meter, i.e. $\Delta f_u=\Delta f_v=1.0\text{ cpm}$. The number of samples in both frequency directions is the same and a power of 2. Finally, the data is generated according to (17a-d).

5.1 Distortion Estimation

The distortion estimation procedure consists of placing the phantom within the imaging device, imaging a slice of the phantom, and processing the resulting data to obtain an estimate for the distortion. The phantom parameters A , τ_x , and τ_y , are assumed to be known exactly. The orientation of the phantom, i.e. $(\theta, \alpha_x, \alpha_y)$, must be determined in order to process the data. This measurement is subject to some error, denoted as $(\theta_\epsilon, \alpha_{x,\epsilon}, \alpha_{y,\epsilon})$, where

$$\begin{aligned}\theta_\epsilon &= \theta - \theta_{assumed}, \\ \alpha_{x,\epsilon} &= \alpha_x - \alpha_x(assumed), \\ \alpha_{y,\epsilon} &= \alpha_y - \alpha_y(assumed).\end{aligned}$$

Any error in determining the orientation of the phantom reduces the accuracy of the distortion estimator. The magnitude of each of these three errors is assumed to be relatively small since if no other more accurate method exists, a relatively good

estimate for the orientation of the phantom can be obtained visually from the distorted image obtained by imaging a slice of the phantom without any restoration. The exact effect of each of these three orientation measurement errors on the accuracy of the two distortion estimators considered in this research is given in this section.

The estimate for the distortion parameter vector $\mathbf{D}$ in (6) and (7) is obtained by using a root finding routine to search a range of $\mathbf{D}$ and extract a root of these equations. The performance of this estimation procedure is therefore dependent on the size of the range of $\mathbf{D}$ that is searched as well as the behavior of the a posteriori density or likelihood equation within this range. Fortunately, for small distortions, i.e. for distortions on the order of those that would be expected to occur with an actual imaging device, the performance of this estimation procedure is excellent for both the types of distortions considered in this research.

The orientation of the phantom affects the performance of the distortion estimator. This is caused by the dependence of the distortion estimator upon the function s . In particular, the performance is highly dependent upon θ , the rotation of the phantom within the imaging device. After an exhaustive series of tests with both types of distortions, it was found that the best performance is achieved for $\theta = n45^\circ$, where n is an odd integer, and the worst performance is achieved for $\theta = n90^\circ$, where n is an integer. The very poor performance at $\theta = n90^\circ$ is believed to be due to the separability of the rectangular phantom.

Since, in general, the best performance of the estimator is achieved at $\theta = 45^\circ$, this value of θ will be used throughout the rest of this section. Since they have little effect on the performance of the estimator, α_x and α_y are, for convenience, taken to be zero throughout the rest of this section. Except where noted, the dimension of

the data (Fourier data matrices) used throughout this section is 128×128 .

5.1 Distortion Estimation for Distortion 1

With the parametric model for Distortion 1 given in (1), D is a one dimensional nonrandom real number C . In this section, the performance of the distortion estimator for Distortion 1, measured as the percentage difference between C and $\hat{c}_m(\mathbf{R})$, the ML estimate for C , is investigated using computer simulated data. Specifically, the effect of θ , θ_ϵ , $\alpha_{x,\epsilon}$, and $\alpha_{y,\epsilon}$ on the performance of the estimator is investigated.

The range of C that is searched by the root finding routine to extract a root of the ML equation is 10 to 10^9 . A value for C of 10 results in a tremendous distortion and a value of 10^9 results in virtually no distortion. The distortion curves (see (1)) corresponding to the values of C used in these tables and in section 5.2.1 are given in Figure 5. The effect of the size of the range of C searched by the root finding routine on the performance of the distortion estimator is not investigated since the performance of the estimator is excellent for all but extreme distortions using the large range of 10 to 10^9 .

The performance of the distortion estimator as a function of θ_ϵ , $\alpha_{x,\epsilon}$, and $\alpha_{y,\epsilon}$, for various values of C , is given in Table 1, Table 2, and Table 3, respectively. The effect of noisy data on the performance of the ML estimator can be investigated by processing data to which computer generated noise has been added. The performance of the distortion estimator as a function of the signal-to-noise ratio is given in Table 4 for several values of C . It is interesting that the sensitivity of the estimation procedure to any error in determining the orientation of the phantom or to noise decreases as the extent of the distortion increases.

The model used to parameterize this distortion was chosen because it was flexible and mathematically tractable. The flexibility of this estimation procedure can be investigated by processing data generated using a distortion function different from the model used to approximate the distortion. For example, equation (2), where

$$\gamma = \max_i |v_{i \text{ (uniform)}}|,$$

can be used to generate nonuniformly sampled data while equation (1) is used to obtain an estimate for the extent of the distortion. Given an appropriate value of q , the estimate for C obtained by processing data generated according to (2) corresponds to a distortion curve that very closely approximates this generating function.

Several tests were run which showed that a q could be found so that the curves for the two models were indistinguishable over the range of interest. Interpolation using this approximation to the generating function produces results that are equivalent to using the correct functional form.

5.1.2 Distortion Estimation for Distortion 2

With the parametric model for Distortion 2 given in (3), $\mathbf{D}$ is an N_u dimensional Gaussian random vector $\mathbf{B}$ ($N[0, \sigma]$). In this section, the performance of the distortion estimator for Distortion 2, measured as the mean square error between $\mathbf{B}$ and $\hat{\mathbf{b}}_{map}(\mathbf{R})$, the MAP estimate for $\mathbf{B}$, is investigated using computer simulated data. Specifically, the effect of θ , θ_ϵ , $\alpha_{x,\epsilon}$, $\alpha_{y,\epsilon}$ on the performance of the estimator is investigated. The measure of the extent of the distortion that will be used throughout the rest of this section and in section 5.2.2 is $\bar{\sigma}$, which is defined as the normalized standard deviation of $\mathbf{B}$:

$$\bar{\sigma} \equiv \frac{\sigma}{\Delta f_u}.$$

Unlike the case presented in the previous section, the range of each B_k that is searched by the root finding routine to extract roots of the MAP equations has a large effect on the performance of the estimator. A range of $-3\bar{\sigma}$ to $+3\bar{\sigma}$ produces good results for a wide range of distortions and will be used throughout the remainder of this section.

The contribution of the a priori knowledge of $\mathbf{B}$ to the MAP equations (B.1) is proportional to the ratio of the noise variance, σ_n^2 , to the assumed variance of $\mathbf{B}$, σ^2 . In the first series of experiments, the only noise added to the data is numerical roundoff noise. The ratio of σ_n^2 to σ^2 is therefore taken to be zero. This makes the MAP estimator equivalent to a ML estimator (see equations (6) and (7)) since the a priori knowledge of $\mathbf{B}$ is not used in the MAP equations. Later noise was added to test the sensitivity of the estimation procedure. The a priori knowledge of $\mathbf{B}$ is used to determine a range of $\mathbf{B}$ to search in order to obtain roots of the MAP equations.

The performance of the distortion estimator as a function of θ_ϵ , $\alpha_{x,\epsilon}$, and $\alpha_{y,\epsilon}$, for various values of $\bar{\sigma}$, is given in Table 5, Table 6, and Table 7, respectively. The effect of noisy data on the performance of the ML estimator can be investigated by processing data to which computer generated noise has been added. The performance of the distortion estimator as a function of the signal-to-noise ratio is given in Table 8 for several values of $\bar{\sigma}$. The sensitivity of the estimation procedure to any error in determining the orientation of the phantom or to noise increases as the extent of the distortion increases. The mean square error for a given amount of distortion does not necessarily increase as orientation error increases (particularly for $\bar{\sigma}=0.3$). This is because $\hat{\mathbf{b}}_{map}(\mathbf{R})$ and $\mathbf{B}$ are effectively uncorrelated when

the orientation error becomes large enough.

5.2 Interpolation

Once the distortion parameters have been estimated, it remains to use this information to restore the distorted image. In this section, the performance of the sinc function interpolation scheme is evaluated using computer simulated data. The dimension of the images used throughout this section is 128×128 . The images in this section are simulations of the images that would be obtained by imaging the phantom considered in this research.

If C and q are known, the interpolation results for Distortion 1 are excellent within the range of frequencies corresponding to the nonuniformly sampled data. The interpolation results are poor at any uniform sampling frequencies where the sample must be extrapolated. These poorly interpolated (extrapolated) rows of data result in images that are often more distorted than the images that result without any interpolation. Therefore, these rows of data are set to zero. Figure 6 shows an example of the original, the distorted and restored images for a typical value of C . Results are given in both one and two dimensions.

The interpolation results are excellent when the interpolation matrix is generated according to the same model and amount of distortion used to generate the nonuniformly sampled data. The interpolation error that results when one model is used to generate nonuniformly sampled data and another model is used to perform the interpolation, is dependent upon the extent to which the two models agree. The model used in this research for Distortion 1 is very flexible (see section 5.1.1) and the interpolation error resulting from using another distortion function to generate the nonuniformly sampled will not be investigated further.

The effect of a bad estimate for C which might occur in low SNR situations or for incorrect models was studied in one and two dimensions. These studies showed that the degradation of performance is graceful as the magnitude of the error increases.

5.2.1 Interpolation Results for Distortion 2

If $\mathbf{B}$ is known, the interpolation results for Distortion 2 are excellent. Figure 7 shows an example of both the degraded and the restored images for a typical value of $\bar{\sigma}$. Results are given in both one and two dimensions.

The effect of a bad estimate for $\mathbf{B}$ was studied by generating the nonuniformly sampled data according to one value of $\mathbf{B}$ and using another value of $\hat{\mathbf{b}}_{map}(\mathbf{R})$ to perform the interpolation. As in the case of the first distortion, the degradation in performance was graceful.

6. Summary

A technique for correcting nonuniform sampling distortions has been presented in this research. The unknown nonuniform sampling grid is determined by modeling the sampling grid distortion in terms of several distortion parameters. An estimate for these parameters can be obtained by processing samples of a known function derived using the unknown nonuniform sampling grid. Maximum likelihood and maximum a posteriori estimators were developed for the two particular distortions considered in this research. Using computer generated data, the performance of both of these estimators was shown to be excellent.

Once the sampling grid distortion is estimated, uniformly sampled data can be estimated by interpolating from the nonuniformly sampled data. A sinc function

interpolation technique was emphasized in this research and the interpolation results were shown to be very good for even large distortions. The two distortions considered in this research were distortions that occurred along only one frequency axis. For two dimensional distortions, the computational burden associated with a sinc function interpolation is tremendous and faster interpolation schemes must be used. This research is easily extended to include other sampling grid distortions.

Some other examples where the results of this research can be applied include:

- a) scan distortions (pin cushion, barrel) in television images,
- b) motion distortion in raster scanned images,
- c) positional errors in sensor arrays such as found in radio astronomy, sonar, and seismology.

7. List of References

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Appendix A.

ML Equation for Distortion 1

From (6) the ML equation used to obtain $\hat{c}_{ml}(\mathbf{R})$ is

$$\begin{aligned} \frac{1}{\sigma_n^2} \sum_{i=1}^{N_v} \sum_{k=1}^{N_u} \{ \text{Re}[R_{i,k}] - \text{Re}[s(i,k,C)] \} \frac{\partial \text{Re}[s(i,k,C)]}{\partial C} \\ + \{ \text{Im}[R_{i,k}] - \text{Im}[s(i,k,C)] \} \frac{\partial \text{Im}[s(i,k,C)]}{\partial C} \Big|_{C=\hat{c}_{ml}(\mathbf{R})} = 0. \end{aligned} \quad (\text{A.1})$$

The functional form for $s(i,k,\mathbf{D})$ from (3.14) and (3.16) is

$$\begin{aligned} s(i,k,\mathbf{D}) = s(i,k,C) = F(u,v) \Big|_{u=k\Delta f_u, v=\frac{i\Delta f_v}{1+\left(\frac{|i\Delta f_v|}{C}\right)^q}} \\ = A \exp \left\{ -j2\pi \left[\alpha_x k\Delta f_u + \frac{\alpha_y i\Delta f_v}{1+\left(\frac{|i\Delta f_v|}{C}\right)^q} \right] \right\} \frac{\sin \left[\pi \tau_x \left(k\Delta f_u \cos\theta + \frac{i\Delta f_v \sin\theta}{1+\left(\frac{|i\Delta f_v|}{C}\right)^q} \right) \right]}{\pi \left(k\Delta f_u \cos\theta + \frac{i\Delta f_v \sin\theta}{1+\left(\frac{|i\Delta f_v|}{C}\right)^q} \right)} \\ \cdot \frac{\sin \left[\pi \tau_y \left(-k\Delta f_u \sin\theta + \frac{i\Delta f_v \cos\theta}{1+\left(\frac{|i\Delta f_v|}{C}\right)^q} \right) \right]}{\pi \left(-k\Delta f_u \sin\theta + \frac{i\Delta f_v \cos\theta}{1+\left(\frac{|i\Delta f_v|}{C}\right)^q} \right)}. \end{aligned}$$

Using this functional form for $s(i,k,\mathbf{D})$, four terms must be calculated to completely specify the ML equation:

$$\text{Re}[s(i,k,C)], \text{Im}[s(i,k,C)], \frac{\partial \text{Re}[s(i,k,C)]}{\partial C}, \text{ and } \frac{\partial \text{Im}[s(i,k,C)]}{\partial C}.$$

For notational convenience let

$$v_i = \frac{i \Delta f_v}{1 + \left(\frac{|i \Delta f_v|}{C} \right)^q}$$

$$\begin{aligned} V_1 &= \sin[\pi \tau_x (k \Delta f_u \cos \theta + v_i \sin \theta)] , \\ V_2 &= \sin[\pi \tau_y (-k \Delta f_u \sin \theta + v_i \cos \theta)] , \\ V_3 &= \pi (k \Delta f_u \cos \theta + v_i \sin \theta) , \\ V_4 &= \pi (-k \Delta f_u \sin \theta + v_i \cos \theta) , \\ V_5 &= \cos[2\pi(\alpha_x k \Delta f_u + \alpha_y v_i)] , \text{ and} \\ V_6 &= -\sin[2\pi(\alpha_x k \Delta f_u + \alpha_y v_i)] . \end{aligned}$$

The real and imaginary parts of $s(i, k, C)$ are obtained by inspection,

$$\text{Re}[s(i, k, C)] = \frac{A V_1 V_2 V_5}{V_3 V_4} , \quad \text{Im}[s(i, k, C)] = \frac{A V_1 V_2 V_6}{V_3 V_4} .$$

The partial derivatives of $\text{Re}[s(i, k, C)]$ and $\text{Im}[s(i, k, C)]$ with respect to C are, respectively,

$$\begin{aligned} \frac{\partial \text{Re}[s(i, k, C)]}{\partial C} &= A \left\{ \frac{V_2 V_5}{V_3 V_4} \cos[\pi \tau_x (k \Delta f_u \cos \theta + v_i \sin \theta)] \frac{\pi \tau_x v_i \sin \theta |i \Delta f_v|^q q}{\left[1 + \left(\frac{|i \Delta f_v|}{C} \right)^q \right] C^{(q+1)}} \right. \\ &\quad + \frac{V_1 V_5}{V_3 V_4} \cos[\pi \tau_y (-k \Delta f_u \sin \theta + v_i \cos \theta)] \frac{\pi \tau_y v_i \cos \theta |i \Delta f_v|^q q}{\left[1 + \left(\frac{|i \Delta f_v|}{C} \right)^q \right] C^{(q+1)}} \\ &\quad - \frac{V_1 V_2 V_5 \pi v_i \sin \theta |i \Delta f_v|^q q}{V_4 V_3^2 \left[1 + \left(\frac{|i \Delta f_v|}{C} \right)^q \right] C^{(q+1)}} - \frac{V_1 V_2 V_5 \pi v_i \cos \theta |i \Delta f_v|^q q}{V_3 V_4^2 \left[1 + \left(\frac{|i \Delta f_v|}{C} \right)^q \right] C^{(q+1)}} \\ &\quad \left. - \frac{V_1 V_2}{V_3 V_4} \sin[2\pi(\alpha_x k \Delta f_u + \alpha_y v_i)] \frac{2\pi \alpha_y v_i |i \Delta f_v|^q q}{\left[1 + \left(\frac{|i \Delta f_v|}{C} \right)^q \right] C^{(q+1)}} \right\} , \text{ and} \end{aligned}$$

$$\begin{aligned}
\frac{\partial \text{Im}[s(i, k, C)]}{\partial C} = & A \left\{ \frac{V_2 V_6}{V_3 V_4} \cos[\pi \tau_x (k \Delta f_u \cos \theta + v_i \sin \theta)] \frac{\pi \tau_x v_i \sin \theta |i \Delta f_v|^q q}{\left[1 + \left(\frac{|i \Delta f_v|}{C}\right)^q\right] C^{(q+1)}} \right. \\
& + \frac{V_1 V_6}{V_3 V_4} \cos[\pi \tau_y (-k \Delta f_u \sin \theta + v_i \cos \theta)] \frac{\pi \tau_y v_i \cos \theta |i \Delta f_v|^q q}{\left[1 + \left(\frac{|i \Delta f_v|}{C}\right)^q\right] C^{(q+1)}} \\
& - \frac{V_1 V_2 V_6 \pi v_i \sin \theta |i \Delta f_v|^q q}{V_4 V_3^2 \left[1 + \left(\frac{|i \Delta f_v|}{C}\right)^q\right] C^{(q+1)}} - \frac{V_1 V_2 V_6 \pi v_i \cos \theta |i \Delta f_v|^q q}{V_3 V_4^2 \left[1 + \left(\frac{|i \Delta f_v|}{C}\right)^q\right] C^{(q+1)}} \\
& \left. - \frac{V_1 V_2}{V_3 V_4} \cos[2\pi(\alpha_x k \Delta f_u + \alpha_y v_i)] \frac{2\pi \alpha_y v_i |i \Delta f_v|^q q}{\left[1 + \left(\frac{|i \Delta f_v|}{C}\right)^q\right] C^{(q+1)}} \right\}.
\end{aligned}$$

Note that the limits on the summations in the ML equation must be changed to agree with the actual range of the data.

Appendix B:

MAP Equations for Distortion 2

From (7) the MAP equations used to obtain $\hat{\mathbf{b}}_{\text{map}}(\mathbf{R})$ are

$$\begin{aligned}
\frac{1}{\sigma_n^2} \sum_{i=1}^{N_v} \left[\{\text{Re}[R_{i,j}] - \text{Re}[s(i, j, \mathbf{B})]\} \frac{\partial \text{Re}[s(i, j, \mathbf{B})]}{\partial B_j} + \{\text{Im}[R_{i,j}] - \text{Im}[s(i, j, \mathbf{B})]\} \frac{\partial \text{Im}[s(i, j, \mathbf{B})]}{\partial B_j} \right] \\
- \frac{\partial}{\partial B_j} \ln[p_{\mathbf{b}}(\mathbf{B})] \Big|_{\mathbf{B}=\hat{\mathbf{b}}_{\text{map}}(\mathbf{R})} = 0 \quad j=1, \dots, N_v.
\end{aligned} \tag{B.1}$$

The functional form for $s(i, k, \mathbf{D})$ from (3.14) and (3.17) is

$$s(i, k, \mathbf{D}) = s(i, k, \mathbf{B}) = F(u, v) \Big|_{u=k \Delta f_u + B_k, v=i \Delta f_v}$$

$$= A \exp\{-j 2\pi[\alpha_x(k\Delta f_u + B_k) + \alpha_y i \Delta f_v]\}$$

$$\cdot \frac{\sin\{\pi\tau_x[(k\Delta f_u + B_k)\cos\theta + i\Delta f_v\sin\theta]\}}{\pi[(k\Delta f_u + B_k)\cos\theta + i\Delta f_v\sin\theta]} \frac{\sin\{\pi\tau_y[(-k\Delta f_u - B_k)\sin\theta + i\Delta f_v\cos\theta]\}}{\pi[(-k\Delta f_u - B_k)\sin\theta + i\Delta f_v\cos\theta]}.$$

Using this functional form for $s(i, k, \mathbf{B})$, five terms must be calculated to completely specify the MAP equations:

$$\text{Re}[s(i, k, \mathbf{B})], \text{Im}[s(i, k, \mathbf{B})], \frac{\partial \text{Re}[s(i, k, \mathbf{B})]}{\partial B_j}, \frac{\partial \text{Im}[s(i, k, \mathbf{B})]}{\partial B_j}, \text{ and } \frac{\partial}{\partial B_j} \ln[p_{\mathbf{b}}(\mathbf{B})].$$

For notational convenience let

$$\begin{aligned} V_1 &= \sin\{\pi\tau_x[(k\Delta f_u + B_k)\cos\theta + i\Delta f_v\sin\theta]\}, \\ V_2 &= \sin\{\pi\tau_y[(-k\Delta f_u - B_k)\sin\theta + i\Delta f_v\cos\theta]\}, \\ V_3 &= \pi[(k\Delta f_u + B_k)\cos\theta + i\Delta f_v\sin\theta], \\ V_4 &= \pi[(-k\Delta f_u - B_k)\sin\theta + i\Delta f_v\cos\theta], \\ V_5 &= \cos\{2\pi[\alpha_x(k\Delta f_u + B_k) + \alpha_y i \Delta f_v]\}, \text{ and} \\ V_6 &= -\sin\{2\pi[\alpha_x(k\Delta f_u + B_k) + \alpha_y i \Delta f_v]\}. \end{aligned}$$

The real and imaginary parts of $s(i, k, \mathbf{B})$ are obtained by inspection,

$$\text{Re}[s(i, k, \mathbf{B})] = \frac{A V_1 V_2 V_5}{V_3 V_4}, \quad \text{Im}[s(i, k, \mathbf{B})] = \frac{A V_1 V_2 V_6}{V_3 V_4}.$$

The partial derivatives of $\text{Re}[s(i, k, \mathbf{B})]$ and $\text{Im}[s(i, k, \mathbf{B})]$ with respect to B_j are, respectively,

$$\begin{aligned} \frac{\partial \text{Re}[s(i, k, \mathbf{B})]}{\partial B_j} &= A \left[\frac{V_2 V_5}{V_3 V_4} \cos\{\pi\tau_x[(k\Delta f_u + B_k)\cos\theta + i\Delta f_v\sin\theta]\} \pi\tau_x \cos\theta \right. \\ &\quad + \frac{V_1 V_6}{V_3 V_4} \cos\{\pi\tau_y[(-k\Delta f_u - B_k)\sin\theta + i\Delta f_v\cos\theta]\} \pi\tau_y \sin\theta \\ &\quad - \frac{V_1 V_2 V_5 \pi \cos\theta}{V_4 V_3^2} + \frac{V_1 V_2 V_6 \pi \sin\theta}{V_3 V_4^2} \\ &\quad \left. - \frac{V_1 V_2}{V_3 V_4} \sin\{2\pi[\alpha_x(k\Delta f_u + B_k) + \alpha_y i \Delta f_v]\} 2\pi\alpha_x \right] \quad k = j \end{aligned}$$

$$= 0 \quad k \neq j, \text{ and}$$

$$\begin{aligned} \frac{\partial \text{Im}[s(i, k, \mathbf{B})]}{\partial B_j} = & A \left(\frac{V_2 V_6}{V_3 V_4} \cos\{\pi \tau_x [(k \Delta f_u + B_k) \cos \theta + i \Delta f_v \sin \theta]\} \pi \tau_x \cos \theta \right. \\ & + \frac{V_1 V_6}{V_3 V_4} \cos\{\pi \tau_y [(-k \Delta f_u - B_k) \sin \theta + i \Delta f_v \cos \theta]\} \pi \tau_y \sin \theta \\ & - \frac{V_1 V_2 V_6 \pi \cos \theta}{V_4 V_3^2} + \frac{V_1 V_2 V_6 \pi \sin \theta}{V_3 V_4^2} \\ & \left. - \frac{V_1 V_2}{V_3 V_4} \cos\{2\pi [\alpha_x (k \Delta f_u + B_k) + \alpha_y i \Delta f_v]\} 2\pi \alpha_x \right) \quad k = j \\ & = 0 \quad k \neq j. \end{aligned}$$

Since it is assumed that each column perturbation is an independent, zero mean, Gaussian random variable of variance σ^2 ,

$$\begin{aligned} p_{\mathbf{b}}(\mathbf{B}) &= \prod_{m=1}^{N_u} p_{b_m}(B_m) \quad \text{since the } b_m \text{ are independent} \\ &= \prod_{m=1}^{N_u} \frac{1}{(2\pi)^{1/2} \sigma} \exp\left\{-\frac{B_m^2}{2\sigma^2}\right\}. \end{aligned} \quad (\text{B.2})$$

Taking the natural logarithm of both sides of (8.2.2),

$$\ln p_{\mathbf{b}}(\mathbf{B}) = -N_u \ln[(2\pi)^{1/2} \sigma] + \sum_{m=1}^{N_u} \frac{-B_m^2}{2\sigma^2}. \quad (\text{B.3})$$

Taking the partial derivative of both sides of (8.2.3) with respect to B_j ,

$$\frac{\partial}{\partial B_j} \ln p_{\mathbf{b}}(\mathbf{B}) = \frac{-B_j}{\sigma^2}.$$

Each of these MAP equations is thus an equation in B_j only. Therefore, to find the MAP estimate for $\mathbf{B}$ each of the equations can be solved independently. Again the

limits on the summations in the MAP equation must be changed to agree with the actual range of the data.

Figure 1: RF and gradient pulse sequence.

(a) Distorted v versus uniform v .

(b) Nonuniform sampling grid.

Figure 2: Frequency compression distortion (Distortion 1)

Figure 3: Frequency perturbation distortion (Distortion 2).

Figure 4: Phantom orientation (top view).

Figure 5: Distortion curves for $C=100$, $C=300$, and $C=1000$.

(a) Original Images

(b) Distorted Images

(b) Distorted Images

(a) Distorted Images

(b) Restored Images

$$(c) \bar{\sigma} = 0.2$$

Figure 7: Interpolation results for Distortion 2 in both one and two dimensions for $\bar{\sigma} = 0.2$

(b) Restored Images

Table 1 Estimator performance as a function of θ_ϵ .

$ \theta_\epsilon $	$\frac{\hat{c}_{ml}(\mathbf{R}) - C}{C} \times 100\%$		
	$C = 100.0$	$C = 300.0$	$C = 1000.0$
0.0°	0.00%	0.00%	0.00%
0.5°	-0.01	-0.01	0.00
1.0°	-0.02	-0.01	0.06
1.5°	0.03	1.90	26.78
2.0°	1.72	14.83	70.63
2.5°	7.51	22.99	*
3.0°	-8.58	35.41	*
4.0°	-12.76	69.35	*
5.0°	26.02	*	*

* denotes errors larger than 100% in magnitude

Table 2 Estimator performance as a function of $\alpha_{z,\epsilon}$.

$ \alpha_{z,\epsilon} (cm)$	$\frac{\hat{c}_{ml}(\mathbf{R}) - C}{C} \times 100\%$		
	$C = 100.0$	$C = 300.0$	$C = 1000.0$
0.00	0.00%	0.00%	0.00%
0.25	0.00	0.00	0.00
0.50	0.01	0.06	0.41
0.75	0.18	22.78	*
1.00	-9.91	36.78	*
1.50	-15.24	*	*
2.0	79.63	*	*

* denotes errors larger than 100% in magnitude

Table 3 Estimator performance as a function of $\alpha_{y,\epsilon}$.

$ \alpha_{y,\epsilon} \text{ (cm)}$	$\frac{\hat{c}_{ml}(\mathbf{R}) - C}{C} \times 100\%$		
	$C = 100.0$	$C = 300.0$	$C = 1000.0$
0.00	0.00%	0.00%	0.00%
0.25	-0.02	-0.05	-0.16
0.50	-0.11	-0.41	-2.28
0.75	-0.81	22.94	*
1.00	-9.91	36.81	*
1.50	-15.21	*	*
2.00	80.04	*	*

* denotes errors larger than 100% in magnitude

Table 4 Estimator performance as a function of signal to noise ratio.

SNR (dB)	$\frac{\hat{c}_{ml}(\mathbf{R}) - C}{C} \times 100\%$		
	$C = 100.0$	$C = 300.0$	$C = 1000.0$
0	0.36%	-0.67%	3.50%
20	0.04	-0.08	0.33
40	0.00	0.00	0.03

Table 5 Estimator performance as a function of θ_ϵ .

$ \theta_\epsilon $	Mean Square Error Between $\mathbf{B}$ and $\hat{\mathbf{b}}_{map}(\mathbf{R})$		
	$\bar{\sigma}=0.1$	$\bar{\sigma}=0.2$	$\bar{\sigma}=0.3$
0.0°	0.0	0.0	0.0
0.5°	0.006293	0.006887	0.013057
1.0°	0.024953	0.039455	0.063043
1.5°	0.030284	0.046613	0.134430
2.0°	0.039118	0.058090	0.138917
2.5°	0.041696	0.086527	0.204210
3.0°	0.040704	0.088770	0.602383
4.0°	0.043532	0.088299	0.205395
5.0°	0.040786	0.098333	0.178848

Table 6 Estimator performance as a function of $\alpha_{z,\epsilon}$.

$ \alpha_{z,\epsilon} (cm)$	Mean Square Error Between $\mathbf{B}$ and $\hat{\mathbf{b}}_{map}(\mathbf{R})$		
	$\bar{\sigma}=0.1$	$\bar{\sigma}=0.2$	$\bar{\sigma}=0.3$
0.00	0.0	0.0	0.0
0.25	0.001289	0.001145	0.001255
0.50	0.015355	0.030998	0.129062
0.75	0.022180	0.030069	0.105944
1.00	0.024518	0.032398	0.084698
1.50	0.019944	0.039580	0.115291
2.00	0.018315	0.042493	0.089108

Table 7 Estimator performance as a function of $\alpha_{y,\epsilon}$.

$ \alpha_{y,\epsilon} \text{ (cm)}$	Mean Square Error Between $\mathbf{B}$ and $\hat{\mathbf{b}}_{map}(\mathbf{R})$		
	$\bar{\sigma}=0.1$	$\bar{\sigma}=0.2$	$\bar{\sigma}=0.3$
0.00	0.0	0.0	0.0
0.25	0.000522	0.000513	0.000532
0.50	0.012197	0.038906	0.144700
0.75	0.019242	0.028889	0.086413
1.00	0.020938	0.036835	0.076572
1.50	0.027008	0.058439	0.108374
2.00	0.023432	0.038994	0.104144

Table 8 Estimator performance as a function of signal to noise ratio.

SNR (dB)	Mean Square Error Between $\mathbf{B}$ and $\hat{\mathbf{b}}_{map}(\mathbf{R})$		
	$\bar{\sigma}=0.1$	$\bar{\sigma}=0.2$	$\bar{\sigma}=0.3$
0	0.008352	0.032791	0.072176
20	0.006666	0.021149	0.038420
40	0.000764	0.000550	0.000343

Figure 1: RF and gradient pulse sequence.

Figure 2: Frequency compression distortion (Distortion 1)

Figure 3: Frequency perturbation distortion (Distortion 2).

Figure 4: Phantom orientation (top view).

Figure 5: Distortion curves for $C=100$, $C=300$, and $C=1000$.

Figure 6: Interpolation results for Distortion 1 in both one and two dimensions

Figure 7: Interpolation results for Distortion 2 in both one and two dimensions

Distorted Images

Restored Images