THE JACOBIANS OF CERTAIN MATRIX TRANSFORMATIONS

USEFUL IN MULTIVARIATE ANALYSIS

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FOREWORD

By Harold Hotelling

We are apparently at the beginning of a major development in the use of statistical procedures for joint treatment of a multiplicity of correlated variates. Many of the new methods depend ultimately on the distribution of the roots of certain determinantal equations. This distribution was published simultaneously in 1939 by P. L. Hsu [9] and R. A. Fisher [6] in the Annals of Eugenics and, excepting for a constant multiplier, by S. N. Roy [12] in Sankhya. Hsu's derivation, which like Roy's is otherwise complete, demonstrates the correctness of his formula for this constant multiplier only for the case of three variates. Proof that the formula is correct for the general case has turned out to be unexpectedly difficult to develop. In seeking to close this gap while lecturing on advanced multivariate analysis at Chapel Hill, Hsu developed what appears to be an extraordinarily powerful method of dealing with a wide class of transformations of the elements of one matrix into those of another induced by a functional

relation between the matrices. This technique not only leads to a solution of the original problem but appears to offer the possibility of a fresh attack on other problems of multivariate distribution theory as well. Heretofore it has been recorded only in lecture notes made by students in the spring of 1947. Since Dr. Hsu was in Peiping in poor health when last heard from, and since recent efforts to communicate with him have failed, it seems desirable that the essential ideas involved should be published. Col. Deemer and Mr. Olkin have not merely reproduced lecture notes but have re-worked the material and improved it in essential respects.

1. SUMMARY AND INTRODUCTION. In finding the distribution of multivariate statistics it is often desirable to make certain transformations. However, the calculation of the Jacobian is frequently rather difficult. We have therefore assembled in this paper the Jacobians of many of the most used transformations.

Certain general methods are applied in the calculation of the Jacobians; these methods will be found useful in the calculation of other Jacobians not given here. Three techniques of particular value for non-linear transformations are:

(a) Taking differentials to get a linear equation in the differentials; e.g., Theorem 4.1.

(b) Transformations to new variables so that the

Jacobian can be evaluated as the product of other Jacobians which are more easily calculated; e.g., Theorem 4.2.

(c) Introduction of new matrices such that the Jacobian of the original transformation is the discriminant of the bilinear form in the variables of the new matrices; e.g., Lemma 4.3 and Theorem 4.3.

2. NOTATION. Because the shape and symmetry of matrices are important in finding Jacobians, we will consistently use the following notation:

(2.1) Lower case letters will denote scalars.

(2.2) As is customary, a prime will denote the transpose of a matrix.

(2.3) Lower case letters with a superior bar will denote column vectors, e.g. $\bar{u}' = (u_1, \dots, u_p)$

(2.4) Capital letters will denote matrices. A system of superior symbols will be used as follows:

(a) $\bar{A}$ is a symmetric matrix of elements a_{ij} .

(b) $\hat{A}$ is a skew symmetric matrix of elements a_{ij} above the main diagonal and $-a_{ij}$ below the main diagonal and zeros in the main diagonal.

(c) $\tilde{A}$ is a triangular matrix with zeros above the main diagonal, e.g.

$$\begin{bmatrix} a_{11} & & & & \\ a_{12} & a_{22} & & & \\ \dots & \dots & \dots & & \\ a_{1p} & a_{2p} & \dots & a_{pp} & \end{bmatrix}$$

(d) A matrix not in classes (a), (b), (c) will be denoted by a capital letter with no superior

symbol.

(e) (dX) is the matrix of the same shape as X , whose elements are the differentials of the elements of X . This will be denoted by X^* .

(2.5) Diagonal matrices will be denoted by D with a subscript showing the elements; e.g. D_r is the diagonal matrix with elements $r_1, r_2, \dots, r_p$.

(2.6) Orthogonal matrices will be denoted by Δ and Γ .

(2.7) $A: p \times q$ means the matrix A has p rows and q columns; the same notation will be used for matrices with superior symbols.

(2.8) Since it is the absolute value of the Jacobian that is wanted, it is to be understood that all determinants are evaluated up to the sign only.

(2.9) Let X, Y be two matrices which satisfy an equation $Y = f(X)$. Let X and Y have the same number n of independent elements, and denote them by $x_1, \dots, x_n$ and $y_1, \dots, y_n$, respectively. Then the Jacobian, $\frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)}$ will be denoted by $D(X; Y)$. For purposes of calculation it is sometimes desirable to express this in terms of the elements of X and Y , for example $D(X; Y)$ might be written $D(x_{ii}, x_{ij}; y_{ii}, y_{ij})$.

More generally, if $X_1, \dots, X_k, Y_1, \dots, Y_m$ satisfy the equations $Y_i = f_i(x_1, \dots, x_k)$ ($i=1, \dots, m$) and $(X_1, \dots, X_k)$ and $(Y_1, \dots, Y_m)$ have the independent elements $x_1, \dots, x_n$ and $y_1, \dots, y_n$,

respectively, then $D(X_1, \dots, X_k; Y_1, \dots, Y_m)$ denotes $\frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)}$.

(2.10) The following abbreviations will be used: p.d. for positive definite; n.s. for non-singular; J for Jacobian; tr for trace; I_p for the identity matrix of order p.

3. THE JACOBIANS OF LINEAR TRANSFORMATIONS. We state here a number of theorems on the J of linear transformations, some without proof. When the J does not immediately follow from previous theorems we will give the proof.

THEOREM 3.1a: The J of the transformation $Y = aX$, ($Y, X: p \times q$) where a is a constant scalar $\neq 0$, is $D(X; Y) = a^{pq}$.

PROOF: $y_{ij} = ax_{ij}$ and $\frac{\partial y_{ij}}{\partial x_{ij}} = a$. Therefore, the scheme of coefficients is diagonal with pq a 's in the main diagonal.

Q.e.d.

THEOREM 3.1b: The J of the transformation $\bar{Y} = a\bar{X}$, ($\bar{Y}, \bar{X}: p \times p$), where a is a constant scalar $\neq 0$, is $D(\bar{X}; \bar{Y}) = a^{p(p+1)/2}$

THEOREM 3.2: The J of the linear transformation $\bar{y} = A\bar{x}$ ($\bar{y}, \bar{x}: p \times 1$; $A: p \times p$) is $D(\bar{x}; \bar{y}) = |A|$.

PROOF: $y_i = \sum_{k=1}^p a_{ik}x_k$ and $\frac{\partial y_i}{\partial x_j} = a_{ij}$, therefore

$J = D(\bar{x}; \bar{y}) = |A|$.

Q.e.d.

COROLLARY 3.2: The J of the transformation $\bar{x} = A^{-1}\bar{y}$ ($\bar{x}, \bar{y}: p \times 1; A: p \times p$) is $D(\bar{y}; \bar{x}) = |A|^{-1}$.

This follows from Theorem 3.2 since $|A^{-1}| = |A|^{-1}$.

THEOREM 3.3: The J of the transformation $\bar{y}' = \bar{x}' A'$ ($\bar{y}, \bar{x}: p \times 1; A: p \times p$) is $D(\bar{x}; \bar{y}) = |A|$.

THEOREM 3.4: The J of the transformation $Y = AX$ ($Y, X: p \times q; A: p \times p$) is $D(X; Y) = |A|^q$.

PROOF: This follows from Theorem 3.2 and 5B.4, since the transformation of each column of Y is independent of the others and there are q such columns of Y, the J of each column transformation being |A|.

THEOREM 3.5: The J of $Y = XA$ ($Y, X: p \times q; A: q \times q$) is $D(X; Y) = |A|^p$.

PROOF: Write $Y' = A'X'$ and the result follows directly from Theorem 3.4.

THEOREM 3.6: The J of the transformation $Y = AXB$ ($Y, X: p \times q; A: p \times p; B: q \times q$) is $D(X; Y) = |A|^q |B|^p$.

PROOF: Let $Z = AX$ and then $Y = ZB$. $D(X; Z) = |A|^q$ and $D(Z; Y) = |B|^p$. Using 5B.2 the theorem follows.

LEMMA 3.7: Let $Y = A_n A_{n-1} \dots A_1 X A_1' \dots A_{n-1}' A_n'$, then $D(X; Y) = D(X; Y_1) D(Y_1; Y_2) \dots D(Y_{n-1}; Y)$ where $Y_i = A_i Y_{i-1} A_i'$, $i = 1, \dots, n$; $Y_0 \equiv X$, $Y_n \equiv Y$.

PROOF: Follows directly from 5B.2.

THEOREM 3.7: The J of the transformation $\bar{Y} = B\bar{X}B'$ ($\bar{Y}, B, \bar{X}: p \times p$) is $D(\bar{X}; \bar{Y}) = |B|^{p+1}$.

PROOF: By 5A.5 we can write $B = F_m F_{m-1} \dots F_1$ where some of the F 's are of the form 5A.1 and the rest are of the form 5A.2. Then

$$(3.7.1) \quad \bar{Y} = F_m \dots F_1 \bar{X} F_1' \dots F_m'$$

By Lemma 3.7: $D(\bar{X}; \bar{Y}) = D(\bar{X}; \bar{Y}_1) D(\bar{Y}_1; \bar{Y}_2) \dots D(\bar{Y}_{m-1}; \bar{Y}_m)$ where $\bar{Y}_v = F_v \bar{Y}_{v-1} F_v'$, $v = 1, \dots, m$; $\bar{Y}_0 \equiv \bar{X}$, $\bar{Y}_m \equiv \bar{Y}$.

We show that $D(\bar{Y}_{v-1}; \bar{Y}_v) = |F_v|^{p+1}$ $v = 1, \dots, m$ and hence it will follow that

$$D(\bar{X}; \bar{Y}) = |F_m|^{p+1} \dots |F_1|^{p+1} = |F_m \dots F_1|^{p+1} = |B|^{p+1}$$

(1) Let G be any of the F 's which are of the form 5A.1; the transformation $\bar{Y}_v = G \bar{Y}_{v-1} G'$ implies:

$$y_{ii} = a^2 x_{ii}; y_{ij} = ax_{ij} (i \neq j); y_{jk} = x_{jk} (j, k \neq i).$$

The determinant of the partial derivatives is therefore diagonal with $p - 1$ elements a ; one element a^2 ; and the remaining elements 1. Therefore $D(\bar{Y}_{v-1}; \bar{Y}_v) = a^{p+1} = |G|^{p+1}$.

(2) Let H be any of the F 's which are of the form 5A.2; the transformation $\bar{Y}_v = H \bar{Y}_{v-1} H'$ implies:

$$y_{ii} = x_{ii} + 2ax_{ij} + a^2 x_{jj}$$

$$y_{ki} = y_{ik} = x_{ik} + ax_{jk} (k \neq i)$$

$$y_{jk} = x_{jk} (j, k \neq i).$$

The determinant of partial derivatives will have 1's in the main diagonal and 0's below the main diagonal. Hence

$$D(\bar{Y}_{v-1}; \bar{Y}_v) = 1 = |H|^{p+1}$$

Q.e.d.

COROLLARY 3.7: The J of the transformation

$$\bar{Y} = A^{-1} \bar{X} A'^{-1} \quad (\bar{Y}, \bar{X}, A: p \times p) \text{ is } D(\bar{X}; \bar{Y}) = |A|^{-(p+1)}$$

THEOREM 3.8: The J of the transformation $\tilde{Y} = \tilde{A} \tilde{X}$

$$(\tilde{Y}, \tilde{A}, \tilde{X}: p \times p) \text{ is } D(\tilde{X}; \tilde{Y}) = \prod_{i=1}^p a_{ii}^i.$$

$$\text{PROOF: } y_{ij} = \sum_{v=1}^j a_{vj} x_{iv} \quad (i \leq j).$$

The elements on one side of the main diagonal of the determinant of partial derivatives are zero. The main diagonal contains a_{11} from $\frac{\partial y_{11}}{\partial x_{11}}$; it contains a_{22} twice:

once from $\frac{\partial y_{12}}{\partial x_{12}}$, and once from $\frac{\partial y_{22}}{\partial x_{22}}$; it contains a_{ii} i

times: once from each of $\frac{\partial y_{ji}}{\partial x_{ji}}$, ($j = 1, \dots, i$).

$$\text{Therefore, the determinant is } \prod_{i=1}^p a_{ii}^i.$$

Q.e.d.

THEOREM 3.9: The J of the transformation $\bar{Y} = \tilde{A}\tilde{X}' + \tilde{X}\tilde{A}'$
 $(\bar{Y}, \tilde{A}, \tilde{X}: p \times p) \text{ is } D(\tilde{X}; \bar{Y}) = 2^p \prod_{i=1}^p a_{ii}^{p-i+1}.$

PROOF: Pre- and postmultiply by $\tilde{A}^{-1}$ and $\tilde{A}'^{-1}$ respectively:

$$(3.9.1) \quad \tilde{A}^{-1} \bar{Y} \tilde{A}'^{-1} = \tilde{X}' \tilde{A}'^{-1} + \tilde{A}^{-1} \tilde{X}$$

Let $\bar{Z} = \tilde{A}^{-1} \bar{Y} \tilde{A}'^{-1}$ and $\bar{U} = \tilde{A}^{-1} \tilde{X}$. Then

$$(3.9.2) \quad \bar{Z} = \bar{U}' + \bar{U}$$

$$(3.9.3) \quad D(\tilde{X}; \bar{Y}) = D(\tilde{X}; \bar{U}) D(\bar{U}; \bar{Z}) D(\bar{Z}; \bar{Y})$$

by 5B.2.

$$(3.9.4) \quad D(\tilde{X}; \bar{U}) = \prod_{i=1}^p a_{ii}^{-1}$$

by Theorem 3.8.

$$(3.9.5) \quad D(\bar{Z}; \bar{Y}) = |\tilde{A}|^{p+1} = \prod_{i=1}^p a_{ii}^{p+1}$$

by Corollary 3.7 and 5B.1.

$$(3.9.6) \quad z_{ii} = 2u_{ii} \quad \text{and} \quad z_{ij} = u_{ij}$$

from 3.9.2.

The determinant of partial derivatives is therefore diagonal. It contains p elements equal to 2 and the remainder 1's. Therefore,

$$(3.9.7) \quad D(\tilde{U}; \bar{Z}) = 2^p$$

Substituting in (3.9.3):

$$D(\tilde{X}; \bar{Y}) = 2^p \prod_{i=1}^p a_{ii}^{p-i+1} \quad \text{Q.e.d.}$$

THEOREM 3.10: The J of the transformation $\hat{Y} = G \hat{X} G'$ ($\hat{Y}, \hat{X}, G: p \times p$) is $D(\hat{X}; \hat{Y}) = |G|^{p-1}$

PROOF: The proof follows along the same lines as the proof of Theorem 3.7 with the necessary changes due to $\hat{X}$ and $\hat{Y}$ being skew symmetric.

4. NON LINEAR TRANSFORMATIONS. The general method for finding the J of a non-linear transformation is to take the differential of both sides and then to find the J of the resulting linear transformation in the differentials. (See 5B.3)

THEOREM 4.1: The J of the transformation $\bar{V} = \tilde{T} \tilde{T}'$ is $D(\tilde{T}; \bar{V}) = 2^p \prod_{i=1}^p t_{ii}^{p-i+1}$.

PROOF: Taking differentials of $\bar{V} = \tilde{T} \tilde{T}'$,

$$(4.1.1) \quad d\bar{V} = \tilde{T}(d\tilde{T})' + (d\tilde{T}) \tilde{T}'.$$

Let $\bar{V}^* = d\bar{V}$; $\bar{T}^* = d\bar{T}$, then

(4.1.2) $\bar{V}^* = \bar{T} \hat{T}^{*'} + \bar{T}^* \bar{T}'$. This is the transformation of Theorem 3.9, and hence the theorem follows. Q.e.d.

THEOREM 4.2¹: The J of the transformation $\bar{A} = W D_\phi W'$; $\bar{B} = W W'$ (all matrices $p \times p$) is $D(W, \phi; \bar{A}, \bar{B}) =$

$$|W|^{p+2} 2^p \prod_{i < j}^p (\phi_i - \phi_j).$$

PROOF: Taking differentials:

$$(4.2.1) \quad d\bar{A} = (dW) D_\phi W' + W(dD_\phi) W' + W D_\phi (dW)'$$

$$(4.2.2) \quad d\bar{B} = W (dW)' + (dW) W'.$$

Let $\bar{A}^* = d\bar{A}$; $\bar{B}^* = d\bar{B}$; $W^* = dW$; $D_\eta = dD_\phi$, then

$$(4.2.3) \quad \bar{A}^* = W^* D_\phi W' + W D_\eta W' + W D_\phi W^{*'}.$$

$$(4.2.4) \quad \bar{B}^* = W W^{*'} + W^{*'} W'$$

By 5B.3: $J = D(W^*, \eta; \bar{A}^*, \bar{B}^*)$. Pre- and postmultiply

(4.2.3) and (4.2.4) by W^{-1} and W'^{-1} respectively:

$$(4.2.5) \quad W^{-1} \bar{A}^* W'^{-1} = W^{-1} W^* D_\phi + D_\eta + D_\phi W^{*'} W'^{-1}$$

$$(4.2.6) \quad W^{-1} \bar{B}^* W'^{-1} = W^{*'} W'^{-1} + W^{-1} W^*$$

Let

$$(4.2.7) \quad \bar{U} = W^{-1} \bar{A}^* W'^{-1}$$

and hence $\bar{A}^* = W \bar{U} W'$

$$(4.2.8) \quad \bar{V} = W^{-1} \bar{B}^* W'^{-1}$$

and hence $\bar{B}^* = W \bar{V} W'$

$$(4.2.9) \quad R = W^{-1} W^*$$

Substituting in (4.2.5) and (4.2.6):

¹ For an application see [9] .

$$(4.2.10) \quad \bar{U} = R D_{\phi} + D_{\phi} R' + D_{\eta}$$

$$(4.2.11) \quad \bar{V} = R + R'$$

$$(4.2.12) \quad D(W^*, \eta; \bar{A}^*, \bar{B}^*) = D(W^*, \eta; R, \eta) D(R, \eta; \bar{U}, \bar{V}) D(\bar{U}, \bar{V}; \bar{A}^*, \bar{B}^*),$$

using 5B.2.

$$(4.2.13) \quad D(W^*, \eta; R, \eta) = |W|^{-p}$$

from 4.2.9 using Theorem 3.4 and 5B.4.

$$(4.2.14) \quad D(\bar{U}, \bar{V}; \bar{A}^*, \bar{B}^*) = D(\bar{U}; \bar{A}^*) D(\bar{V}; \bar{B}^*)$$

by 5B.4.

$$(4.2.15) \quad D(\bar{U}; \bar{A}^*) = D(\bar{V}; \bar{B}^*) = |W|^{p+1}$$

by Theorem 3.7. Hence

$$(4.2.16) \quad D(\bar{U}, \bar{V}; \bar{A}^*, \bar{B}^*) = |W|^{2(p+1)}.$$

To find $D(R, \eta; \bar{U}, \bar{V})$ we use (4.2.10) and (4.2.11):

$$\begin{aligned} u_{ii} &= 2r_{ii}\phi_i + \eta_i & v_{ii} &= 2r_{ii} \\ u_{ij} &= r_{ij}\phi_j + r_{ji}\phi_i & v_{ij} &= r_{ij} + r_{ji} \quad (i \neq j) \end{aligned}$$

The scheme of partial derivatives will then be as follows:

	$\eta_i (i=1, \dots, p)$	$r_{ii} (i=1, \dots, p)$	$r_{ij} (i > j)$	$r_{ij} (i < j)$
$u_{ii} (i=1, \dots, p)$	I	L	0	0
$v_{ii} (i=1, \dots, p)$	0	2I	0	0
$u_{ij} (i < j)$	0	0	D	C
$v_{ij} (i < j)$	0	0	I	I

where C is the diagonal matrix with elements: $\phi_2, \phi_3, \dots, \phi_p, \phi_3, \dots, \phi_p, \dots, \phi_{p-1}, \phi_p, \phi_p$ and D is the diagonal matrix with elements: $\phi_1, \dots, \phi_1$ (p-1 terms) $\phi_2, \dots, \phi_2$ (p-2 terms) $\dots, \phi_{p-1}$.

The matrix L does not affect the value of the determinant,

which is equal to $2^p |D - C| = 2^p \prod_{i < j}^p (\phi_i - \phi_j)$

by applying 5A.7 twice.

Hence, $J = 2^{p|W|p+2} \prod_{i < j}^p (\phi_i - \phi_j)$ q.e.d.

LEMMA 4.3: Let $X_1 = (x_{ij}^{(1)})$, $(p_1 \times q_1)$ and $X_2 = (x_{ij}^{(2)})$, $(p_2 \times q_1)$ be two matrices containing a total of N independent variables ($N \leq p_1 q_1 + p_2 q_1$) and $Y_1 = (y_{ij}^{(1)})$, $(p_1 \times q_1)$, $Y_2 = (y_{ij}^{(2)})$, $(p_2 \times q_1)$ be two other matrices containing the same number of variables.

Let $\Xi : N \times 1$ be the vector formed from all N x 's by using the elements in the first column of X_1 as the first p_1 elements of Ξ , the p_1 elements in the second column of X_1 as the next p_1 elements of Ξ , etc., through the columns of X_2 .

Let $H : N \times 1$ be the vector formed from the columns of Y_1 and Y_2 in the same way that Ξ was formed from X_1 and X_2 .

$$\begin{aligned} \text{Thus: } \Xi' &= (x_{11}^{(1)} \ x_{21}^{(1)} \ \dots \ x_{p_1 q_1}^{(1)} \ x_{11}^{(2)} \ x_{21}^{(2)} \ \dots \ x_{p_2 q_1}^{(2)}) \\ H' &= (y_{11}^{(1)} \ y_{21}^{(1)} \ \dots \ y_{p_1 q_1}^{(1)} \ y_{11}^{(2)} \ y_{21}^{(2)} \ \dots \ y_{p_2 q_1}^{(2)}) \end{aligned}$$

Let $\Xi = A H$ ($A : N \times N$), where A is a matrix of constants, be the connecting equations between the variables $x_{ij}^{(v)}$ and $y_{ij}^{(v)}$ ($v = 1, 2$).

Let $R_1 = (r_{ij}^{(1)})$, $(q_1 \times p_1)$ and $R_2 = (r_{ij}^{(2)})$, $(q_1 \times p_2)$ be two matrices with a total of N independent variables. Then $D(Y_1, Y_2; X_1, X_2)$ is equal to the discriminant of $\text{tr}(R_1 X_1 + R_2 X_2)$ where $\text{tr}(R_1 X_1 + R_2 X_2)$ is to be considered as a bilinear form in the variables of R_1, R_2 and Y_1, Y_2 .

PROOF: Let $\mu(N \times 1)$ be the vector formed from the elements R_1 and R_2 by taking the rows of R_1 in turn and then the rows of R_2 in turn:

$$\mu' = (r_{11}^{(1)} r_{12}^{(1)} \dots r_{p_1 q_1}^{(1)} r_{11}^{(2)} r_{12}^{(2)} \dots r_{p_2 q_1}^{(2)}).$$

$$\text{tr}(R_1 X_1 + R_2 X_2) = \sum_{v=1}^2 \sum_{i=1}^{q_1} \sum_{\sigma=1}^{p_v} r_{i\sigma}^{(v)} x_{\sigma i}^{(v)} = \mu' \Xi = \mu' A H$$

The discriminant of this bilinear form is $|A|$.

The $J, D(Y_1, Y_2; X_1, X_2)$ of the original transformation $\Xi = AH$ is $|A|$ (by Theorem 3.2). Q.e.d.

Introduction to Theorem 4.3.

Let $\bar{V} = \begin{bmatrix} \bar{V}_{11} & \bar{V}_{12} \\ \bar{V}_{21} & \bar{V}_{22} \end{bmatrix}$ be p.d., ($\bar{V}_{11} : p \times p$; $\bar{V}_{12} : p \times q$; $\bar{V}_{22} : q \times q$), ($p \leq q$). There exists a n.s. matrix G of the form $\begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}$

($G_1 : p \times p$; $G_2 : q \times q$) such that

$$(4.3.2) \quad G \bar{V} G' = \begin{bmatrix} I_p & D_r & 0 \\ D_r & I_q \end{bmatrix}$$

where $D_r : p \times p$ has elements $r_1, \dots, r_p$; $r_1^2 \dots r_p^2$ are the roots of the determinantal equation $(V_{12} \bar{V}_{22}^{-1} V_{21} - \theta \bar{V}_{11}) = 0$.

(See 5A.21)

$$(4.3.3) \quad \bar{V} = G^{-1} \begin{bmatrix} I_p & D_r & 0 \\ D_r & I_q \end{bmatrix} G'^{-1}$$

from 4.3.2.

Let $G^{-1} = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$, then

$$(4.3.4) \quad \bar{V} = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} I_p & D_r & 0 \\ D_r & I_q & 0 \\ 0 & 0 & I_\delta \end{bmatrix} \begin{bmatrix} P' & 0 \\ 0 & Q' \end{bmatrix}.$$

This is a transformation from the v 's to the p 's, q 's, and r 's. $\bar{V}$ contains $1/2 (p + q)(p + q + 1)$ variables and the right side of (4.3.4) contains $p^2 + q^2 + p$ variables. Let $\delta = q - p$, then $[(p^2 + q^2 + p) - 1/2 (p + q)(p + q + 1)] = 1/2 \delta(\delta - 1)$, and the number of variables on each side will be equal if $p = q$ or $p = q - 1$.

We now partition Q and rewrite (4.3.4):

$$(4.3.5) \quad \bar{V} = \begin{bmatrix} P & 0 & 0 \\ 0 & Q_{11} & Q_{12} \\ 0 & Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} I_p & D_r & 0 \\ D_r & I_p & 0 \\ 0 & 0 & I_\delta \end{bmatrix} \begin{bmatrix} P' & 0 & 0 \\ 0 & Q'_{11} & Q'_{21} \\ 0 & Q'_{12} & Q'_{22} \end{bmatrix}$$

$$(Q_{11}: p \times p; Q_{12}: p \times \delta; Q_{22}: \delta \times \delta).$$

Postmultiplying Q_{12} and Q_{22} by an orthogonal matrix Δ will not change the value of this product, i.e.

$$\bar{V} = \begin{bmatrix} P & 0 & 0 \\ 0 & Q_{11} & Q_{12}\Delta \\ 0 & Q_{21} & Q_{22}\Delta \end{bmatrix} \begin{bmatrix} I_p & D_r & 0 \\ D_r & I_p & 0 \\ 0 & 0 & I_\delta \end{bmatrix} \begin{bmatrix} P' & 0 & 0 \\ 0 & Q'_{11} & Q'_{21} \\ 0 & \Delta' Q'_{12} & \Delta' Q'_{22} \end{bmatrix}$$

as may be verified by multiplying out this expression. By 5A.10 we can choose this Δ so that $Q_{22}\Delta$ is triangular with 0's below the main diagonal. When we do this we reduce the number of variables on the right by $1/2 \delta(\delta - 1)$ and thus have the same number on each side. We now proceed to find the J of this transformation after the number of variables has been equalized.

Hsu has used this transformation in finding the distribution of the canonical correlation coefficient (see Hotelling [8]) when the population values are not

zero. Hsu's work on this has apparently not been published.

THEOREM 4.3: Let $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & \tilde{T} \end{bmatrix}$, $(Q_{11}: p \times p; Q_{12}: p \times \delta;$

$\tilde{T}: \delta \times \delta)$, $(p \leq q)$, $\delta = q - p$; $t_{ii} > 0$, $(i = 1, \dots, \delta)$.

Let $\bar{\Phi} = \begin{bmatrix} I_p & D_r & 0 \\ D_r & I_q \\ 0 & & \end{bmatrix}$, $(D_r: p \times p)$. Then the J of the

transformation

$$(4.3.6) \quad \bar{V} = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \bar{\Phi} \begin{bmatrix} P' & 0 \\ 0 & Q' \end{bmatrix}$$

is $D(P, Q, r; \bar{V}) = 2^{p+q} |P|^{q+1} |Q|^{p+1} \left(\prod_{i=1}^{\delta} t_{ii}^{i-1} \right) \left(\prod_{i=1}^p r_i^{\delta} \right) \left(\prod_{i < j}^p (r_i^2 - r_j^2) \right)$.

When $q = p$ or $q = p + 1$, the terms in t do not appear.

PROOF: Take differentials of (4.3.6) and let $\bar{V}^* = d\bar{V}$, $P^* = dP$, $Q_{11}^* = dQ_{11}$, $Q_{12}^* = dQ_{12}$, $Q_{21}^* = dQ_{21}$, $\tilde{T}^* = d\tilde{T}$, $D_{\eta} = dD_r$, and let

$$Q^* = dQ = \begin{bmatrix} Q_{11}^* & Q_{12}^* \\ Q_{21}^* & \tilde{T}^{*'} \end{bmatrix}; \text{ then}$$

$$(4.3.7) \quad \bar{V}^* = \begin{bmatrix} P^* & 0 \\ 0 & Q^* \end{bmatrix} \bar{\Phi} \begin{bmatrix} P' & 0 \\ 0 & Q' \end{bmatrix} + \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \bar{\Phi} \begin{bmatrix} P^* & 0 \\ 0 & Q^* \end{bmatrix}' +$$

$$\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} 0 & D_{\eta} & 0 \\ D_{\eta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P' & 0 \\ 0 & Q' \end{bmatrix}.$$

$J = D(P^*, Q^*, \eta; \bar{V}^*)$ by 5B.3. Premultiply (4.3.7) by

$$\begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} \text{ and postmultiply by } \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}' \text{ and introduce}$$

new variables:

$$(4.3.8) \quad \bar{X} = \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} \bar{V}^* \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}'$$

$$(4.3.9) \quad A = P^{-1} P^*$$

$$(4.3.10) \quad \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = Q^{-1} \begin{bmatrix} Q_{11}^* \\ Q_{21}^* \end{bmatrix}.$$

To simplify the notation, we write

$$(4.3.11) \quad \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} = Q^{-1} \begin{bmatrix} Q_{12}^* \\ \tilde{T}^{*'} \end{bmatrix}, \text{ but we do not consider}$$

B_{12} and B_{22} as new variables, since $Q^{-1} \tilde{T}^{*'}$ is not triangular, and hence the number of variables in B_{22} would be greater than the number of variables in $\tilde{T}^{*'}$.

We then get from (4.3.7):

$$(4.3.12) \quad \bar{X} = \begin{bmatrix} A & 0 & 0 \\ 0 & B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \end{bmatrix} \bar{\Phi} + \bar{\Phi} \begin{bmatrix} A' & 0 & 0 \\ 0 & B'_{11} & B'_{21} \\ 0 & B'_{12} & B'_{22} \end{bmatrix} + \begin{bmatrix} 0 & D_\eta & 0 \\ D_\eta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A + A' & AD_r + D_r B'_{11} + D_\eta & D_r B'_{21} \\ AD_r + D_r B'_{11} + D_\eta & B_{11} + B'_{11} & B_{12} + B'_{21} \\ D_r B'_{21} & B_{12} + B'_{21} & B_{22} + B'_{22} \end{bmatrix}$$

By 5B.2:

$$J = D(P^*, Q^*, \eta; \bar{V}^*)$$

$$(A) \quad = D(P^*, Q^*, \eta; A, B_{11}, B_{21}, Q_{12}^*, \tilde{T}^{*'}, \eta) D(A, B_{11}, B_{21}, Q_{12}^*, \tilde{T}^{*'}, \eta; \bar{X}) D(\bar{X}; \bar{V}^*) \\ = D_1 D_2 D_3.$$

$$\text{By 5B.4: } D_1 = D(P^*; A) D(Q_{11}^*, Q_{21}^*; B_{11}, B_{21})$$

$$= |P|^{-p} |Q|^{-p} \quad (\text{from (4.3.9), (4.3.10), using}$$

Theorem 3.4).

From (4.3.8), using Theorem 3.7:

$$(B) \quad D_3 = D(\bar{X}; \bar{V}^*) = |P|^{p+q+1} |Q|^{p+q+1}.$$

To compute D_2 we partition $\bar{X}$ and hence can equate the submatrices of $\bar{X}$ to the submatrices on the right of (4.3.12).

$$\bar{X} = \begin{bmatrix} \bar{X}_{11} & X_{12} & X_{13} \\ X_{12} & \bar{X}_{22} & X_{23} \\ X_{13} & X_{23} & \bar{X}_{33} \end{bmatrix}, \quad (\bar{X}_{11}, X_{12}, \bar{X}_{22}: p \times p; X_{13}, X_{23}: p \times \delta; \bar{X}_{33}: \delta \times \delta)$$

We then have:

$$(4.3.13) \quad \bar{X}_{11} = A + A'; \quad X_{12} = AD_r + D_r B'_{11} + D_r \eta; \quad \bar{X}_{22} = B_{11} + B'_{11}$$

$$(4.3.14) \quad X_{13} = D_r B'_{21}; \quad X_{23} = B_{12} + B'_{21}; \quad \bar{X}_{33} = B_{22} + B'_{22}.$$

The transformations in (4.3.13) and (4.3.14) are independent, so we can write:

$$(C) \quad D_2 = D(H, B_{11}, \eta; \bar{X}_{11}, X_{12}, \bar{X}_{22}) D(B_{21}, Q_{12}^*, \bar{T}^*; X_{13}, X_{23}, \bar{X}_{33}) \\ = D_4 D_5.$$

Computation of D_4 : For convenience we will relabel the elements of $\bar{X}$:

(4.3.15) $\bar{X}_{11} = (x_{ij}); X_{12} = (y_{ij}); \bar{X}_{22} = (z_{ij})$, all $i, j = 1, \dots, p$. and we denote the elements of the other matrices as follows:

$$(4.3.16) \quad A = (a_{ij}); B'_{11} = (b_{ij}), (i, j = 1 \dots p); \eta = (\eta_1 \dots \eta_p).$$

Using (4.3.13), (4.3.14) and (4.3.15) we get:

$$(4.3.16) \quad x_{ii} = 2a_{ii}; y_{ii} = r_i a_{ii} + r_i b_{ii} + \eta_i; z_{ii} = 2b_{ii}$$

$$(4.3.17) \quad x_{ij} = a_{ij} + a_{ji} \quad (i < j); y_{ij} = r_j a_{ij} + r_i b_{ij} \quad (i \neq j);$$

$$z_{ij} = b_{ij} + b_{ji} \quad (i < j).$$

The transformation in (4.3.16) and (4.3.17) are independent, hence

$$D_4 = D(A, B_{11}, \eta; \bar{X}_{11}, \bar{X}_{12}, \bar{X}_{22}) = D(a_{ii}, b_{ii}, \eta_i; x_{ii}, y_{ii}, z_{ii}) D(a_{ij}, b_{ij}; x_{ij}, y_{ij}, z_{ij}).$$

For $D(a_{ii}, b_{ii}, \eta_i; x_{ii}, y_{ii}, z_{ii})$ we have the scheme of coefficients:

	a_{ii}	b_{ii}	η_i	
x_{ii}	$2I$	0	0	$= 2^{2p}$, the values L and M not affecting the determinant.
z_{ii}	0	$2I$	0	
y_{ii}	L	M	I	

To compute $D(a_{ij}, b_{ij}; x_{ij}, y_{ij}, z_{ij})$ we rewrite in (4.3.17):

$$y_{ij} = r_j a_{ij} + r_i b_{ij} \quad (i < j = 1, \dots, p)$$

$$y_{ji} = r_i a_{ji} + r_j b_{ji} \quad (i < j = 1, \dots, p)$$

We then have the scheme of coefficients ($i < j$ in all cases):

	a_{ij}	b_{ji}	a_{ji}	b_{ij}	
x_{ij}	I	0	I	0	$= \left \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} - \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \right = D D - CD^{-1}C = D^2 - C^2 ,$
z_{ij}	0	I	0	I	
y_{ji}	0	C	D	0	
y_{ij}	C	0	0	D	

using 5A.6 and 5A.7 twice.

C is a diagonal matrix with elements: $r_2, r_3, \dots, r_p, r_3, \dots, r_p, \dots, r_{p-1}, r_p, r_p$; D is a diagonal matrix with elements: $r_1, \dots, r_1, (p-1 \text{ terms}) r_2, \dots, r_2, (p-2 \text{ terms}) \dots, r_{p-1}$

$$\therefore D^2 - C^2 = \prod_{i < j}^p (r_i^2 - r_j^2) \text{ and hence}$$

$$(D) \quad D_4 = D(A, B_{11}, \eta; \bar{X}_{11}, \bar{X}_{12}, \bar{X}_{22}) = 2^{2p} \prod_{i < j}^p (r_i^2 - r_j^2).$$

Computation of $D(B_{21}, Q_{12}^*, \tilde{T}^*; X_{13}, X_{23}, \bar{X}_{33}) = D_5$

The connecting equations are given in (4.3.14). B_{22} is independent of the other variables, but recall that the original variables are Q_{12}^* and Q_{22}^* and that the relation (4.3.11) does not define B_{12} and B_{22} as new variables.

The scheme of coefficients, from (4.3.14), is

$$(4.3.18) \quad \begin{array}{c} X_{13} \\ X_{23} \\ \bar{X}_{33} \end{array} \begin{array}{c|cc} B_{21} & Q_{12}^* & \tilde{T}^* \\ \hline F & 0 & 0 \\ \hline L & & G \end{array} = |F||G|,$$

so the value of L does not affect the value of the determinant. $|F|$ is the J of the transformation $X_{13} = D_r B'_{21}$

$(X_{13}, B'_{21} : p \times \delta; D_r : p \times p)$ which by Theorem 3.4 is

$|D_r|^\delta = \left(\prod_{i=1}^p r_i \right)^\delta$. Since the value of B_{21} does not affect G , we can for determining G , write: $X_{23} = B_{12}$ and $\bar{X}_{33} = B_{22} + B'_{22}$. Therefore,

$$(E) \quad D_5 = \left(\prod_{i=1}^p r_i \right)^\delta D_6; \quad D_6 = D(Q_{12}^*, \tilde{T}^*; X_{23}, \bar{X}_{33}).$$

Computation of D_6 : The connecting equations between

$B_{12}, B_{22}; Q_{12}^*, \tilde{T}^*; X_{23}, \bar{X}_{33}$ are:

$$\begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} = Q^{-1} \begin{bmatrix} Q_{12}^* \\ \tilde{T}^* \end{bmatrix}$$

$$\begin{aligned} X_{23} &= B_{12} \\ \bar{X}_{33} &= B_{22} + B'_{22} \end{aligned}$$

As previously stated, B_{12} and B_{22} cannot be considered as new variables. We therefore do not have any direct equations between the variables of X_{23} , $\bar{X}_{33}$ and $Q_{12}^*, \tilde{T}^*$.

It is clear, however, that the elements of X_{23} , $\bar{X}_{33}$ are linear functions of the elements of $Q_{12}^*, \tilde{T}^*$, and hence we can apply Lemma 4.3.

Let $W = (w_{ij})$: $(\delta \times p)$ and $\bar{S}$: $(\delta \times \delta)$, where $\bar{S} = (\epsilon_{ij} s_{ij})$, $(i, j=1 \dots \delta)$; $\epsilon_{ij} = 1$ if $i = j$; $\epsilon_{ij} = 1/2$ if $i \neq j$.

$$\begin{aligned} \text{Let } \text{tr}H &= \text{tr}(W X_{23} + \bar{S} \bar{X}_{33}) \\ &= \text{tr}(W B_{12} + \bar{S} B_{22} + \bar{S} B_{22}') = \text{tr}(W B_{12} + 2\bar{S} B_{22}) \\ (\text{by 5A.12}) &= \text{tr}(W 2\bar{S}) \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} \end{aligned}$$

$$= \text{tr}(W 2\bar{S}) K \begin{bmatrix} Q_{12}^* \\ \tilde{T}^* \end{bmatrix}, \text{ where } K = Q^{-1}.$$

Writing $Q_{12}^* = (\beta_{ij})$, $(i = 1, \dots, p; j = 1, \dots, \delta)$;

$\tilde{T}^* = (\tau_{ij})$, $(i \leq j = 1, \dots, \delta)$:

$$(4.3.19) \quad \text{tr}H = \text{tr} \begin{bmatrix} w_{11} & \dots & w_{1p} & 2s_{11} & s_{12} & \dots & s_{1\delta} \\ & \dots & & s_{12} & 2s_{22} & \dots & s_{2\delta} \\ & & & \dots & \dots & \dots & \dots \\ w_{\delta 1} & \dots & w_{\delta p} & s_{1\delta} & \dots & \dots & 2s_{\delta\delta} \end{bmatrix} K \begin{bmatrix} \beta_{11} & \dots & \beta_{1\delta} \\ \beta_{p1} & \dots & \beta_{p\delta} \\ \tau_{11} & \dots & \tau_{1\delta} \\ 0 & \dots & \tau_{\delta\delta} \end{bmatrix}.$$

H is $\delta \times \delta$ and $\text{tr}H = \sum_{j=1}^{\delta} h_{jj}$. h_{jj} is found by multiplying the j th row of $(W 2\bar{S})$ into K and then multiplying the resulting row vector into the j th column of $\begin{bmatrix} Q_{12}^* \\ \tilde{T}^* \end{bmatrix}$.

$$\therefore \text{tr}H = \sum_{j=1}^{\delta} (w_{j1} \dots w_{jp} s_{j1} \dots 2s_{jj} \dots s_{j\delta}) K \begin{bmatrix} \beta_{1j} \\ \vdots \\ \beta_{pj} \\ \tau_{1j} \\ \vdots \\ \tau_{jj} \end{bmatrix}$$

We now partition K so that the partition is a function of j :

$$(4.3.20) \quad Q^{-1} = K = \begin{bmatrix} K_{11}^{(j)} & K_{12}^{(j)} \\ K_{21}^{(j)} & K_{22}^{(j)} \end{bmatrix}, \quad (1 \leq j \leq \delta)$$

$$(K_{11}^{(j)} : (p+j) \times (p+j); K_{12}^{(j)} : (p+j) \times (\delta-j); K_{22}^{(j)} : (\delta-j) \times (\delta-j)).$$

Now we can write

$$(4.3.21) \quad \text{tr} H = \sum_{j=1}^{\delta} ((w_{j1} \dots w_{jp} s_{j1} \dots s_{jj}) K_{11}^{(j)} \begin{bmatrix} \beta_{1j} \\ \vdots \\ \beta_{pj} \\ \tau_{1j} \\ \vdots \\ \tau_{jj} \end{bmatrix} + (s_{j,j+1} \dots s_{j\delta}) K_{21}^{(j)} \begin{bmatrix} \beta_{1j} \\ \vdots \\ \beta_{pj} \\ \tau_{1j} \\ \vdots \\ \tau_{jj} \end{bmatrix}).$$

By Lemma 4.3, $D(Q_{12}^*, \tilde{T}^*; X_{23}, \bar{X}_{33})$ is the determinant of this bilinear form. Writing the terms for $j=1, 2, \dots, \delta$ in order we find that the discriminant has 0's above the main diagonal, and the product of the terms in the main diagonal is

$$(4.3.22) \quad 2|K_{11}^{(1)}| 2|K_{11}^{(2)}| \dots 2|K_{11}^{(\delta)}| = 2^{\delta} \prod_{j=1}^{\delta} |K_{11}^{(j)}|.$$

$$|K_{11}^{(j)}| = \begin{vmatrix} k_{11} & \dots & k_{1,p+j} \\ & \ddots & \\ k_{p+j,1} & \dots & k_{p+j,p+j} \end{vmatrix}, \quad \text{where the elements } k_{ih}$$

are the elements of Q^{-1} . Let A_{ih} be an element of the adjoint of Q . Then $k_{ih} = \frac{1}{|Q|} A_{hi}$.

Therefore,

$$|K_{11}^{(j)}| = \frac{1}{|Q|^{p+j}} \begin{vmatrix} A_{11} & \cdots & A_{1,p+j} \\ \vdots & \ddots & \vdots \\ A_{p+j,1} & \cdots & A_{p+j,p+j} \end{vmatrix} =$$

$$\frac{1}{|Q|} \begin{vmatrix} q_{p+j+1,p+j+1} & \cdots & q_{p+j+1,p+\delta} \\ \vdots & \ddots & \vdots \\ q_{p+\delta,p+j+1} & \cdots & q_{p+\delta,p+\delta} \end{vmatrix},$$

by 5A.11. But $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & \tilde{T}' \end{bmatrix}$, ($Q_{11} = p \times p$; $Q_{12} = p \times \delta$; $\tilde{T}' = \delta \times \delta$).

Therefore, $q_{p+k+1,p+h+1} = t_{k+1,h+1}$. Therefore

(4.3.23)

$$|K_{11}^{(j)}| = \frac{1}{|Q|} \begin{vmatrix} t_{j+1,j+1} & \cdots & t_{j+1,\delta} \\ 0 & \cdots & t_{\delta\delta} \end{vmatrix} = \frac{1}{|Q|} t_{j+1,j+1} \cdots t_{\delta\delta}.$$

On substituting this result in (4.3.22) we find

(F) $D_6 = D(Q_{12}^*, \tilde{T}^*; X_{23}, \bar{X}_{33}) = 2^\delta \prod_{j=1}^{\delta} |K_{11}^{(j)}| = 2^\delta |Q|^{-\delta} t_{22}^2 t_{33}^2 \cdots t_{\delta\delta}^{\delta-1}.$

We now consolidate the results from equations (A) through (F).

$$J = 2^{p+q} |P|^{q+1} |Q|^{p+1} \left(\prod_{i=2}^{\delta} t_{ii}^{i-1} \right) \left(\prod_{i=1}^p r_i \right)^\delta \prod_{i < j}^p (r_i^2 - r_j^2)$$

Q.e.d.

THEOREM 4.4. The J of the transformation

(4.41) $\bar{U} = |\bar{V}| \bar{V}^{-1},$

where $\bar{V}: p \times p$, is p.d., is $D(\bar{V}; \bar{U}) = (p-1) |\bar{V}|^{(p+1)(p-2)/2}$

PROOF: Take differentials of (4.4.1) and let

$\bar{U}^* = d\bar{U}$, $\bar{V}^* = d\bar{V}$; then

(4.4.2) $\bar{U}^* = |\bar{V}| \text{tr}(\bar{V}^{-1} \bar{V}^*) \bar{V}^{-1} - |\bar{V}| (\bar{V}^{-1} \bar{V}^* \bar{V}^{-1}),$

using 5A.13, 5A.14, and 5A.15.

By 5B.3, $J = D(\bar{V}^*; \bar{U}^*)$.

$\bar{V}$, and hence $\bar{V}^{-1}$, is p.d., so by 5A.16 we can factor $\bar{V}^{-1}$ into $\tilde{T} \tilde{T}'$. Substituting this for $\bar{V}^{-1}$ in (4.4.2) and pre- and postmultiplying the result by $\tilde{T}^{-1}$ and $\tilde{T}'^{-1}$ respectively:

$$(4.4.3) \quad \tilde{T}^{-1} \bar{U}^* \tilde{T}'^{-1} = |\bar{V}| \text{tr}(\tilde{T}' \bar{V}^* \tilde{T}) I - |\bar{V}| (\tilde{T}' \bar{V}^* \tilde{T}).$$

$$(4.4.4) \quad \begin{cases} \bar{Z} = \tilde{T}^{-1} \bar{U}^* \tilde{T}'^{-1} & (\bar{Z}: p \times p) \\ \bar{W} = \tilde{T}' \bar{V}^* \tilde{T} & (\bar{W}: p \times p). \end{cases}$$

By Theorem 3.7 and Corollary 3.7

$$(4.4.5) \quad \begin{cases} D(\bar{U}^*; \bar{Z}) = |\tilde{T}|^{-(p+1)} \\ D(\bar{V}^*; \bar{W}) = |\tilde{T}|^{p+1} \end{cases}$$

Putting (4.4.4) in (4.4.3):

$$(4.4.6) \quad \bar{Z} = |\bar{V}| (\text{tr} \bar{W}) I - |\bar{V}| \bar{W}.$$

$$\begin{aligned} \text{Then } D(\bar{V}^*; \bar{U}^*) &= D(\bar{V}^*; \bar{W}) D(\bar{W}; \bar{Z}) D(\bar{Z}; \bar{U}^*), \text{ (by 5B.2),} \\ &= |\tilde{T}|^{p+1} D(\bar{W}; \bar{Z}) |\tilde{T}|^{p+1} \\ &= |\bar{V}|^{-(p+1)} D(\bar{W}; \bar{Z}). \end{aligned}$$

Computation of $D(\bar{W}; \bar{Z})$.

Let

$$(4.4.7) \quad \bar{S} = |\bar{V}| \bar{W} \quad (\bar{S}: p \times p).$$

By Theorem 3.1b: $D(\bar{W}; \bar{S}) = |\bar{V}|^{p(p+1)/2}$
Substitute (4.4.7) in (4.4.6):

$$(4.4.8) \quad \bar{Z} = (\text{tr} \bar{S}) I - \bar{S}$$

whence we find

$$(4.4.9) \quad z_{ii} = \sum_{v=1}^p s_{vv} - s_{ii} = \sum_{\substack{v=1 \\ v \neq i}}^p s_{vv}.$$

Therefore $\frac{\partial z_{ii}}{\partial s_{ii}} = 0$; $\frac{\partial z_{ii}}{\partial s_{jj}} = 1, (i \neq j)$; $\frac{\partial z_{ij}}{\partial s_{ij}} = 0, (i \neq j)$.

$$z_{ij} = -s_{ij} \quad (i < j), \text{ and therefore } \frac{\partial z_{ij}}{\partial s_{ij}} = -1.$$

The scheme of coefficients is

$$(4.4.10) \quad \begin{array}{cc|cc} & & s_{ii} & s_{ij} \ (i < j) \\ z_{ii} & & H & 0 \\ z_{ij} \ (i < j) & & 0 & -I \end{array}$$

the determinant of which is

$$(4.4.11) \quad H = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ & \ddots & & & \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix} = D(\bar{S}; \bar{Z}).$$

Let $\bar{h}$ be a column vector, $p \times 1$, consisting entirely of 1's. Then $\bar{h} \bar{h}'$ is a $p \times p$ matrix consisting entirely of 1's, and $D(\bar{S}; \bar{Z}) = |\bar{h} \bar{h}' - I_p|$. Using 5A.8 we get

$$D(\bar{S}; \bar{Z}) = |1 - \bar{h}' \bar{h}| = p - 1.$$

Consolidating the results:

$$\begin{aligned} J &= D(\bar{V}^*; \bar{U}^*) = D(\bar{V}^*; \bar{W}) D(\bar{W}; \bar{S}) D(\bar{S}; \bar{Z}) D(\bar{Z}; \bar{U}^*) \\ &= |\bar{V}|^{-(p+1)} |\bar{V}|^{p(p+1)/2} (p-1) \\ &= (p-1) |\bar{V}|^{(p+1)(p-2)/2} \end{aligned}$$

Q.e.d.

COROLLARY 4.4. The J of the inverse of the transformation of Theorem 4.4 is $D(\bar{U}; \bar{V}) = \frac{1}{p-1} |\bar{U}|^{\frac{1}{p-1}(p+1)(p-2)}$.

PROOF. $|\bar{U}| = |\bar{V}|^p |\bar{V}|^{-1} = |\bar{V}|^{p-1}$, and therefore

$$|\bar{V}| = |\bar{U}|^{\frac{1}{p-1}}.$$

$$D(\tilde{U}; \tilde{V}) = \frac{1}{D(\tilde{V}; \tilde{U})} = \frac{1}{p-1} |\tilde{V}|^{-(p+1)(p-2)/2}$$

$$= \frac{1}{p-1} |\tilde{U}|^{\frac{-1}{2(p-1)}(p+1)(p-2)}$$

Q.e.d.

THEOREM 4.5. The J of the transformation
 $U = \tilde{T} [2(\hat{X} + I)^{-1} - I]$ is $D(\tilde{T}, \hat{X}; U) =$
 $2^{\frac{p(p-1)}{2}} \left(\prod_{i=1}^{p-1} t_{ii}^{p-1} \right) |\hat{X} + I|^{-(p-1)} (\hat{X}, \tilde{T}, U: p \times p; U \text{ n.s.})$

This transformation is unique if $|\tilde{T}^{-1}U + I| \neq 0$. (see 5A.18)

PROOF. Since $\hat{X}$ is skew symmetric it does not have a characteristic root -1 , so $|\hat{X} + I| \neq 0$. By 5A.17 and 5A.19 we can write $UU' = \tilde{T}\tilde{T}'$. Therefore $\tilde{T}^{-1}UU'\tilde{T}'^{-1} = I$ and hence $\Gamma = \tilde{T}^{-1}U$ is orthogonal.

Then

$$(4.5.1) \quad U = \tilde{T}\Gamma = \tilde{T} [2(\hat{X} + I)^{-1} - I] \quad (\text{by 5A.18}).$$

Take differentials of (4.5.1) and put $U^* = dU$; $\tilde{T}^* = d\tilde{T}$; $\hat{X}^* = d\hat{X}$. Then using 5A.20 we get

$$(4.5.2) \quad U^* = \tilde{T}^* \Gamma - 1/2 \tilde{T}(\Gamma + I) \hat{X}^*(\Gamma + I),$$

and by 5B.3

$$(4.5.3) \quad J = D(\tilde{T}^*, \hat{X}^*; U^*).$$

Let

$$(4.5.4) \quad \begin{aligned} W &= \tilde{T}^{-1} U^* \Gamma' \\ \tilde{R} &= \tilde{T}^{-1} \tilde{T}^* \\ \hat{Y} &= (\Gamma + I) \hat{X}^*(\Gamma' + I). \end{aligned}$$

Pre- and postmultiply (4.5.2) by $\tilde{T}^{-1}$ and Γ' respectively and substitute from (4.5.4). Then

$$(4.5.5) \quad W = \tilde{R} - 1/2 \hat{Y}.$$

By 5B.2:

$$(4.5.6) \quad D(\tilde{T}^*, \hat{X}^*; U^*) = D(\tilde{T}^*, \hat{X}^*; \tilde{R}, \hat{Y}) D(\tilde{R}, \hat{Y}; W) D(W; U^*)$$

which by 5B.4 becomes

$$= D(\tilde{T}^*; \tilde{R}) D(\hat{X}^*; \hat{Y}) D(\tilde{R}, \hat{Y}; W) D(W; U^*).$$

From (4.5.4) using Theorems 3.6, 3.8, and 3.10:

$$D(W; U^*) = |\tilde{T}|^p \quad (\text{since } |\Gamma| = 1)$$

$$D(\tilde{T}^*; \tilde{R}) = \prod_{i=1}^p t_{ii}^{-1}$$

$$D(\hat{X}^*; \hat{Y}) = |\Gamma + I|^{p-1}.$$

From (4.5.5) $w_{ii} = r_{ii}; w_{ij} = -1/2 y_{ij} \ (i < j);$

$$w_{ji} = r_{ij} + 1/2 y_{ij} \ (i < j).$$

The scheme of coefficients is

	r_{ii}	$y_{ij} \ (i < j)$	$r_{ij} \ (i < j)$
w_{ii}	I	0	0
$w_{ij} \ (i < j)$	0	$-1/2 I$	0
$w_{ji} \ (i < j)$	0	$1/2 I$	I

$$\text{Therefore } D(\tilde{R}, \hat{Y}; W) = 2^{-p(p-1)/2}$$

Hence, consolidating the results,

$$J = 2^{-p(p-1)/2} \left(\prod_{i=1}^{p-1} t_{ii}^{p-i} \right) |\Gamma + I|^{p-1}.$$

By 5A.19, $|\Gamma + I|^{p-1} = 2^{p(p-1)} |\hat{X} + I|^{-(p-1)}$, and we

have $J = 2^{p(p-1)/2} \left(\prod_{i=1}^{p-1} t_{ii}^{p-i} \right) |\hat{X} + I|^{-(p-1)}.$ Q.e.d.

DEFINITION 4.6. It is known that if $\bar{S}$ is n.s. ($p \times p$) with characteristic roots $\theta_1 > \theta_2 > \dots > \theta_p$, there exists at least one orthogonal matrix Γ such that $\bar{S} = \Gamma D_\theta \Gamma'$ (see 5A.4). Matrices Γ satisfying this equation will be called "canon matrices" of $\bar{S}$.

THEOREM 4.6. Let $\bar{S}$: $p \times p$ be n.s. with the properties:

- (i) that its characteristic roots are $\theta_1 > \theta_2 > \dots > \theta_p$,
and (ii) that its canon matrices

$$\Gamma = \begin{bmatrix} \gamma_{11} & \dots & \gamma_{1p} \\ \vdots & & \vdots \\ \gamma_{p1} & & \gamma_{pp} \end{bmatrix}$$

are such that

- a) $\gamma_{1i} \neq 0$, $i = 2, \dots, p$
b) $|\Gamma_1 + I| \neq 0$.

The J of the transformation

$$\bar{S} = [2(\hat{X} + I)^{-1} - I] D_{\theta} [2(\hat{X} + I)^{-1} - I]' \text{ is}$$

$$D(\hat{X}, \theta; \bar{S}) = 2^{p(p-1)/2} |\hat{X} + I|^{-(p-1)} \prod_{i < j}^p (\theta_i - \theta_j).$$

The requirements (i) and (ii) are necessary to make the transformation unique.

PROOF: By 5A.19 we can write

$$(4.6.1) \quad \bar{S} = \Gamma D_{\theta} \Gamma'.$$

Let

$$(4.6.2) \quad \begin{aligned} \bar{S}^* &= d\bar{S} \\ \hat{X}^* &= d\hat{X} \\ D_{\eta} &= dD_{\theta} \end{aligned}$$

Take differentials of (4.6.1) using 5A.20 and substitute from (4.6.2):

(4.6.3)

$$\bar{S}^* = -1/2 (\Gamma + I) \hat{X}^* (\Gamma + I) D_{\theta} \Gamma' + 1/2 \Gamma D_{\theta} (\Gamma' + I) \hat{X}^* (\Gamma' + I) + \Gamma D_{\eta} \Gamma'.$$

(Note that $\hat{X}' = -\hat{X}$.)

By 5B.3

$$(4.6.4) \quad J = D(\hat{X}^*, \eta; \bar{S}^*).$$

Let

$$(4.6.5) \quad \begin{aligned} \bar{V} &= \Gamma' \bar{S}^* \Gamma \\ \hat{Y} &= (\Gamma' + I) \hat{X}^* (\Gamma + I). \end{aligned}$$

Pre- and postmultiply (4.6.3) by Γ' and Γ respectively and substitute from (4.6.5):

$$(4.6.6) \quad \bar{V} = -1/2 \hat{Y} D_{\theta} + 1/2 D_{\theta} \hat{Y} + D_{\eta},$$

whence from 5B.4

$$(4.6.7) \quad D(\hat{X}^*, \eta; \bar{S}^*) = D(\hat{X}^*, \eta; \hat{Y}, \eta) D(\hat{Y}, \eta; \bar{V}) D(\bar{V}; \bar{S}^*).$$

From (4.6.5), using Theorems 3.7 and 3.10,

$$(4.6.8) \quad \begin{aligned} D(\bar{V}; \bar{S}^*) &= 1 \\ D(\hat{X}^*; \hat{Y}) &= |\Gamma + I|^{p-1}. \end{aligned}$$

$$\begin{aligned} \text{From (4.6.6): } v_{ii} &= \eta_i; v_{ij} = 1/2 \theta_i y_{ij} - 1/2 \theta_j y_{ij} = \\ &= 1/2 (\theta_i - \theta_j) y_{ij} \quad (i < j). \end{aligned}$$

The scheme of coefficients is:

$$\begin{array}{cc} & \begin{array}{cc} \eta_i & y_{ij} \end{array} \\ \begin{array}{c} v_{ii} \\ v_{ij} \end{array} & \begin{array}{|cc} \hline I & 0 \\ 0 & H \\ \hline \end{array} \end{array}, \text{ where } H \text{ is diagonal, with elements}$$

$$1/2 (\theta_1 - \theta_2), \dots, 1/2 (\theta_1 - \theta_p), 1/2 (\theta_2 - \theta_3), \dots, 1/2 (\theta_{p-1} - \theta_p),$$

a total of $p(p-1)/2$ terms. Therefore,

$$(4.6.9) \quad D(\hat{Y}, \eta; \bar{V}) = 2^{-p(p-1)/2} \prod_{i < j}^p (\theta_i - \theta_j),$$

and

$$(4.6.10) \quad J = |\Gamma + I|^{p-1} 2^{-p(p-1)/2} \prod_{i < j}^p (\theta_i - \theta_j);$$

whence, using 5A.18,

$$J = 2^{p(p-1)/2} |\hat{X} + I|^{-(p-1)} \prod_{i < j}^p (\theta_i - \theta_j). \quad \text{Q.e.d.}$$

THEOREM 4.7. The space where the conditions on $\bar{S}$

of Theorem 4.6 are not satisfied is of measure zero.

PROOF: The conditions imposed on $\bar{S}$ were

(1) That the characteristic roots of $\bar{S}$ obeyed the relation

$$\theta_1 > \theta_2 > \dots > \theta_p;$$

(2) That in the canon matrix Γ the elements $\gamma_{1i} \neq 0$,
 $i = 2 \dots p$;

(3) The submatrix Γ_1 of Γ was such that $|\Gamma_1 + I| \neq 0$.

Let Σ be the $1/2 p(p+1)$ dimensional space of $\bar{S}$.

(1) $|\bar{S} - \theta I|$ is a polynomial in θ of degree p ; let its discriminant be denoted by $f(S)$. $f(S) = 0$ is a surface F and is of measure zero. The necessary and sufficient conditions for equal roots of $|\bar{S} - \theta I| = 0$ is that $f(S) = 0$. Therefore in the space $\Sigma - F$ all the roots are distinct.

(2) $|\gamma_{1i}| = g_i(S)$ ($i = 2, \dots, p$). $g_i(S) = 0$ is a surface F_i of measure zero.

(3) $|\Gamma_1 + I| = h(S)$, and $h(S) = 0$ is a surface F' of measure zero.

Therefore the space $\Sigma - F - F_2 - \dots - F_p - F'$ satisfies the conditions of Theorem 4.6, and the measure of the sets omitted is zero. Q.e.d.

5. APPENDIX A. CERTAIN MATRIX THEOREMS

The following definitions and results from the theory of matrices and determinants are stated for reference purposes.

5A.1. Multiplication of the i th ^{row} column of a matrix by a constant $a \neq 0$ may be accomplished by ^{pre}post multiplication by a diagonal matrix D , with a in the i th position and 1's elsewhere. $|D| = a$; D^{-1} is a matrix of the same type as D . (see [7], §3.11)

5A.2. To add (a) $\begin{bmatrix} i\text{th row} \\ i\text{th column} \end{bmatrix}$ to the i th row i th column, $a \neq 0$, ^{pre}post multiply by a matrix E containing 1's in the main diagonal, a in the i th row and i th column, and 0's elsewhere. $|E| = 1$; E^{-1} is a matrix of the same type as E . (see [7], §3.11)

Any combination or succession of a finite number of operations of type 5A.1 and 5A.2 is called an elementary transformation.

5A.3. The rank of a matrix is unchanged by an elementary transformation. (see [4], §19)

5A.4. Canonical forms. A $p \times p$ matrix A of rank r can be reduced by an elementary transformation to a canonical form; i.e., to a diagonal matrix with 1's in the first r positions and 0's elsewhere. The reduction to a diagonal matrix of the characteristic roots can be effected by an orthogonal transformation. (see [4], p.280).

If the matrix A is $p \times q$, $p < q$, of rank r , there exist two orthogonal matrices $\Gamma: p \times p$ and $\Delta: q \times q$ such that $\Gamma' A \Delta = (\Lambda \ 0)$, where

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{bmatrix} \quad (p \times p)$$

and $\lambda_1^2 \dots \lambda_p^2$ are the characteristic roots of the matrix AA' . Notice that $p - r$ of the λ_i are zero.

PROOF: AA' is non-negative (see [2] , p.64), and hence its roots are non-negative and can be written $\lambda_1^2 \dots \lambda_r^2$. The rank $(A) = r$ and hence there exists an orthogonal matrix Γ such that

$$\Gamma'AA'\Gamma = \begin{bmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_r^2 \\ & & & 0 \end{bmatrix} = \Lambda^2 \text{ of the theorem. (see [5], 11.9)}$$

Since we can rearrange the position of the λ_i 's in Λ by an orthogonal transformation we can assume:

$$\Lambda^2 = \begin{bmatrix} \lambda_1^2 & & & 0 \\ & \ddots & & \\ & & \lambda_r^2 & \\ 0 & & & 0 \\ & & & \ddots & \\ & & & & 0 \end{bmatrix} = \begin{bmatrix} \Lambda_1^2 & 0 \\ 0 & 0 \end{bmatrix}$$

and $|\Lambda_1| \neq 0$.

Write $\Gamma = (\Gamma_1 \Gamma_2)$ where $\Gamma_1: p \times r$; $\Gamma_2: p \times p - r$.

Then

$$\Lambda^2 = \Gamma'AA'\Gamma = \begin{bmatrix} \Gamma_1' \\ \Gamma_2' \end{bmatrix} AA' (\Gamma_1 \Gamma_2) = \begin{bmatrix} \Gamma_1'AA'\Gamma_1 & \Gamma_1'AA'\Gamma_2 \\ \Gamma_2'AA'\Gamma_1 & \Gamma_2'AA'\Gamma_2 \end{bmatrix} = \begin{bmatrix} \Lambda_1^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(5A.4.1) \quad \Gamma_1'AA'\Gamma_1 = \Lambda_1^2$$

$$(5A.4.2) \quad \Gamma_2'AA'\Gamma_2 = 0$$

Pre- and postmultiply (5A.4.1) by Λ_1^{-1} :

$$(5A.4.3) \quad \Lambda_1^{-1} \Gamma_1' A A' \Gamma_1 \Lambda_1^{-1} = I_r$$

Let $\Delta_1 = A' \Gamma_1 \Lambda_1^{-1}$ ($\Delta_1: q \times r$) and then

$$(5A.4.4) \quad \Delta_1 \Lambda_1 = A' \Gamma_1$$

$$(5A.4.5) \quad \Delta_1' \Delta_1 = I_r \quad (\text{by } 5A.4.3).$$

From (5A.4.2): $(\Gamma_2' A)(\Gamma_2' A)' = 0$ and hence $\Gamma_2' A = 0$.

From (5A.4.5) we see that the columns of Δ_1 are orthogonal and hence there exists a matrix Δ_2 such that

$$(5A.4.6) \quad \Delta = \begin{bmatrix} \Delta_1 & \Delta_2 \end{bmatrix} \text{ is orthogonal.}$$

The matrices Γ and Δ satisfy the theorem, since on performing the multiplication we find

$$\Gamma' A = (\wedge 0) \Delta'$$

Q.e.d.

5A.5. An elementary transformation on a matrix B is equivalent to $E_1 \dots E_n B E_{n+1} \dots E_m$ where some of the E 's are of the type 5A.1, and the remainder are of type 5A.2. Hence if B is square and n.s. we can write (by 5A.4)

$$\begin{aligned} B &= E_n^{-1} \dots E_1^{-1} E_m^{-1} \dots E_{n+1}^{-1} \\ &= F_m F_{m-1} \dots F_1 \end{aligned}$$

where the F 's are of type 5A.1 or 5A.2. (see [7], § 3.12)

5A.6. Diagonal matrices are commutative. (see [7], § 1.6)

5A.7. If A and D are square, not necessarily of the same dimensions, and $|D| \neq 0$, then

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D| |A - BD^{-1}C|. \quad (\text{see [10]})$$

PROOF:
$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix} \end{vmatrix} = \begin{vmatrix} A-BD^{-1}C & B \\ 0 & D \end{vmatrix}$$

$$= |D| |A-BD^{-1}C|. \quad \text{Q.e.d.}$$

5A.8 $|I + \bar{x}\bar{y}'| = 1 + \bar{y}'\bar{x}$ (see [1], §37)

5A.9. If $\bar{x}$ is $n \times 1$, there exists an orthogonal matrix $\Gamma: n \times n$ such that $\bar{x}'\Gamma = (0, \dots, 0, a)$, $a \geq 0$, with $a = 0$ if and only if $\bar{x}' = (0, \dots, 0)$.

PROOF: If $\bar{x}' \neq (0, \dots, 0)$ choose Γ as any orthogonal matrix with elements in the last column

$$x_1 \left(\sum_{i=1}^n x_i^2 \right)^{-1/2}, (i = 1, \dots, n).$$

Then $\bar{x}'\Gamma = (0, \dots, 0, (\sum_{i=1}^n x_i^2)^{1/2})$. If $\bar{x}' = (0, \dots, 0)$, any orthogonal matrix will do, and $a = 0$. Q.e.d.

5A.10. If M is any $m \times m$ matrix, there exists an orthogonal matrix $\Delta: m \times m$, such that $M\Delta$ is triangular with 0's below the main diagonal, and with diagonal elements ≥ 0 .

PROOF: By induction. If M is 1×1 , then $\Delta = \pm 1$ will do. Assume true for a matrix of dimensions $(m-1) \times (m-1)$. Take Γ so that the product of the last row of M into Γ is $(0, \dots, 0, a)$, $a \geq 0$, (by 5A.9). Then

$$M\Gamma = \begin{bmatrix} M_1 & b \\ 0 & a \end{bmatrix}, \quad (M_1: (m-1) \times (m-1); b: (m-1) \times 1; a: 1 \times 1)$$

By the hypothesis of the induction there exists an orthogonal matrix Γ_1 such that $M_1\Gamma_1$ is triangular with 0's below the main diagonal. Let

$$\Delta = \Gamma \begin{bmatrix} \Gamma_1 & 0 \\ 0 & 1 \end{bmatrix}. \text{ Then}$$

$$M\Delta = M\Gamma \begin{bmatrix} \Gamma_1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \overline{M}_1 & b \\ 0 & a \end{bmatrix} \begin{bmatrix} \Gamma_1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \overline{M}_1\Gamma_1 & b \\ 0 & a \end{bmatrix}$$

which is of the required form. Δ is the product of two orthogonal matrices and hence is orthogonal. Q.e.d.

5A.11. If D_1 is the adjoint of any determinant D , and M and M_1 are corresponding m -rowed minors of D and D_1 respectively, then M_1 is equal to the product of D^{m-1} by the algebraic complement of M . (see [4], §11).

5A.12. If $A: m \times n$ and $B: n \times m$ then $\text{tr}(AB) = \text{tr}(BA)$.

$$\begin{aligned} \text{PROOF: } \text{tr}(AB) &= \sum_{i=1}^m \sum_{j=1}^n a_{ij} b_{ji} = \sum_{i=1}^n \sum_{j=1}^m b_{ij} a_{ji} \\ &= \text{tr}(BA) \end{aligned}$$

Q.e.d.

Definition 5A.13a: If $f(x_1, \dots, x_n)$ is a differentiable function of $x_1 \dots x_n$, then $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$ is called the differential of f .

Definition 5A.13b: If $A = (a_{ij})$, $(i=1, \dots, p; j=1, \dots, q)$ where the a_{ij} are functions of $(x_1 \dots x_r) = a_{ij}(x_1 \dots x_r)$, then $dA = (da_{ij})$, $(i=1, \dots, p; j=1, \dots, q)$.

$$5A.13. \quad d(AB) = (dA)B + A(dB).$$

$$\begin{aligned} \text{PROOF:} \quad d(AB) &= d\left(\sum_j a_{ij} b_{jk}\right) = \left(\sum_j d(a_{ij} b_{jk})\right) \\ &= \sum_j \left[(da_{ij})b_{jk} + a_{ij}(db_{jk})\right] = (dA)B + A(dB). \quad \text{Q.e.d.} \end{aligned}$$

$$5A.14. \quad \text{If } A \text{ is square, n.s., } d|A| = |A| \operatorname{tr} [A^{-1}(dA)].$$

$$\text{PROOF:} \quad \text{Let } A^{-1} = (a^{ij}).$$

$$(5A.14.1) \quad a^{ij} = (\text{cofactor } a_{ji}) |A|^{-1}$$

$$(5A.14.2) \quad d|A| = \sum_i \sum_j \frac{\partial |A|}{\partial a_{ij}} da_{ij} \quad (\text{by Definition 5A.13a}).$$

$$(5A.14.3) \quad \frac{\partial |A|}{\partial a_{ij}} = \text{cofactor } a_{ij} = |A| a^{ji} \quad (\text{by 5A.14.1}).$$

Substitute (5A.14.3) in (5A.14.2):

$$(5A.14.4) \quad d|A| = |A| \sum_i \sum_j a^{ji} da_{ij}.$$

The (i,i) term of $A^{-1}(dA)$ is $\sum_j a^{ij}(da_{ji})$, and

$\operatorname{tr} A^{-1}(dA) = \sum_i \sum_j a^{ij}(da_{ji})$, which when put in (5A.14.4) gives the result of the theorem. Q.e.d.

$$5A.15. \quad \text{If } A \text{ is square, n.s., } (dA^{-1}) = -A^{-1}(dA)A^{-1}.$$

$$\text{PROOF:} \quad d(AA^{-1}) = (dA)A^{-1} + A(dA^{-1}) = 0 \quad \text{Q.e.d.}$$

5A.16. A p.d. matrix $\bar{V}$: $p \times p$, can be factored uniquely into $\bar{V} = \bar{T}\bar{T}'$, $t_{ii} > 0, i=1, \dots, p$. (see [11]).

5A.17. If U : $p \times p$ is n.s., then UU' is p.d.

(see [3], p.246).

5A.18. If Γ is orthogonal and $|\Gamma + I| \neq 0$, there is a one to one relation between Γ and $\hat{X}$:

$$\Gamma = 2(\hat{X} + I)^{-1} - I$$

$$\hat{X} = 2(\Gamma + I)^{-1} - I$$

PROOF: Show that $\hat{X}$ is skew symmetric, i.e. $\hat{X} + \hat{X}' = 0$.

5A.19. UU' of 5A.17 can be expressed uniquely as $\tilde{T}\tilde{T}'$ where $t_{ii} > 0, i=1, \dots, p$. If $|\tilde{T}^{-1}U + I| \neq 0$, U can be expressed uniquely in the form

$$U = \tilde{T} [2(\hat{X} + I)^{-1} - I].$$

PROOF: Follows from 5A.16, 5A.17 and 5A.18.

5A.20. Given Γ as in 5A.18, then

$$d\Gamma = -1/2 (\Gamma + I) (d\hat{X}) (\Gamma + I).$$

5A.21. Let $\bar{V} = \begin{bmatrix} \bar{V}_{11} & \bar{V}_{12} \\ \bar{V}_{21} & \bar{V}_{22} \end{bmatrix}$ be p.d. ($\bar{V}_{11}: p \times p$;

$\bar{V}_{12}: p \times q$; $\bar{V}_{22}: q \times q$), $p \leq q$. Then there exists a

matrix G of the form $\begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}$ ($G_1: p \times p$; $G_2: q \times q$)

such that $G\bar{V}G' = \bar{\Phi} = \begin{bmatrix} I_p & D_r & 0 \\ D_r & I_q & 0 \\ 0 & 0 & 0 \end{bmatrix}$, where $D_r: p \times p$ has

elements $r_1, \dots, r_p$, and $r_1^2 \dots r_p^2$ are the roots of the determinantal equation $|V_{12}^* \bar{V}_{22}^{-1} V_{21} - \theta \bar{V}_{11}| = 0$. (see [8]).

PROOF: Since $\bar{V}$ is p.d., $\bar{V}_{11}$ and $\bar{V}_{22}$ are of rank p and q respectively, and hence by 5A.4 there exist two non-singular matrices F_1 and F_2 such that

$$(5A.21.1) \quad F_1 \bar{V}_{11} F_1' = I_p$$

$$(5A.21.2) \quad F_2 \bar{V}_{22} F_2' = I_q.$$

Let $A = F_1 V_{12} F_2'$; then by 5A.4 there exist two orthogonal matrices Γ_1 and Γ_2 such that

$$(5A.21.3) \quad \Gamma_1 A \Gamma_2' = \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & 0 & \\ 0 & & & \lambda_p & \\ & & & & 0 \end{bmatrix} \quad \text{where } \lambda_1^2, \dots, \lambda_p^2 \text{ are}$$

the characteristic roots of AA' ; i.e., roots of $|AA' - \lambda I| = 0$.

Pre- and postmultiply (5A.21.1) by Γ_1 and Γ_1' respectively, and pre- and postmultiply (5A.21.2) by Γ_2 and Γ_2' respectively. Then

$$(5A.21.4) \quad \Gamma_1 F_1 \bar{V}_{11} F_1' \Gamma_1' = \Gamma_1 I_p \Gamma_1' = I_p$$

$$(5A.21.5) \quad \Gamma_2 F_2 \bar{V}_{22} F_2' \Gamma_2' = \Gamma_2 I_q \Gamma_2' = I_q.$$

But $AA' = F_1 V_{12} F_2' F_2 V_{21} F_1'$, and from (5A.21.1) and 5A.21.2)

$\bar{V}_{11} = F_1^{-1} F_1'^{-1}$, and $\bar{V}_{22}^{-1} = F_2' F_2$. Hence

$$(5A.21.6) \quad \begin{aligned} AA' - \lambda I &= F_1 \left[V_{12} \bar{V}_{22}^{-1} V_{21} - \lambda F_1^{-1} F_1'^{-1} \right] F_1' \\ &= F_1 \left[V_{12} \bar{V}_{22}^{-1} V_{21} - \lambda \bar{V}_{11} \right] F_1'. \end{aligned}$$

F_1 is n.s. so $|AA' - \lambda I| = 0$ is equivalent to

$|V_{12} \bar{V}_{22}^{-1} V_{21} - \lambda \bar{V}_{11}| = 0$, and hence $\lambda_1^2, \dots, \lambda_p^2$ are the

roots of this last equation. Hence

$$(5A.21.7) \quad \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & 0 & \\ 0 & & & \lambda_p & \\ & & & & 0 \end{bmatrix} = (D_r \ 0).$$

Let $G_1 = \Gamma_1 F_1$ and $G_2 = \Gamma_2 F_2$; then G_1 and G_2 are the matrices of the theorem, since by substituting these values

in G and multiplying out $G\bar{V}G'$ we get

$$(5A.21.8) \quad G\bar{V}G' = \begin{bmatrix} \Gamma_1 F_1 \bar{V}_{11} F_1' \Gamma_1' & \Gamma_1 F_1 \bar{V}_{12} F_2' \Gamma_2' \\ \Gamma_2 F_2 \bar{V}_{21} F_1' \Gamma_1' & \Gamma_2 F_2 \bar{V}_{22} F_2' \Gamma_2' \end{bmatrix},$$

which by (5A.21.4), (5A.21.5) and (5A.21.7) is

$$\begin{bmatrix} I_p & D_r & 0 \\ D_r & & \\ 0 & & I_q \end{bmatrix}.$$

Q.e.d.

5. APPENDIX B. General Properties of Jacobians.

$$5B.1. \quad D(Y; X) = \frac{1}{D(X; Y)} \quad \text{if } D(X; Y) \neq 0.$$

5B.2. $D(X; Y) = D(X; U)D(U; V)D(V; Y)$ where U and V are any functions of X and Y such that none of the terms on the right vanish. The extension of this to more variables and to the introduction of functions of U and V is immediate.

5B.3. Given a matrix transformation, linear or not, $Y = F(X)$. Then the transformation of the differentials, $dY = dF(X)$, is linear in the differentials and the J of the original transformation is the J of the linear transformation in the differentials.

PROOF: $Y = F(X)$ implies a set of transformations in the elements of Y :

$$y_i = F_i(x_1 \dots x_m), \quad i=1 \dots m.$$

$$\text{Let } (dy)' = (dy_1 \dots dy_m); \quad (dx)' = (dx_1 \dots dx_m).$$

Then the equation $dY = dF(X)$

is of the form

$$(dy) = G(dx)$$

where

$$G = \left[\frac{\partial f_i}{\partial x_j} \right], \quad i, j = 1, \dots, m;$$

and G is by definition the J of the original transformation.

Q.e.d.

5B.4. If $X = F_1(U)$ and $Y = F_2(V)$ are a transformation from variables X and Y to new variables U and V , then

$$J = D(U, V; X, Y) = D(U; X) D(V; Y).$$

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REFERENCES

- [1] A.C.Aitken, Determinants and Matrices, Oliver and Boyd, Edinburgh, 1939.
- *[2] A.A.Albert, Introduction to Algebraic Theories, University of Chicago Press, Chicago, 1946.
- [3] G.Birkhoff and S.MacLane, A Survey of Modern Algebra, Macmillan Co., New York, 1947.
- [4] M. Bôcher, Introduction to Higher Algebra, Macmillan Co., New York, 1938.

- [5] H. Cramér, Mathematical Methods of Statistics, Princeton University Press, Princeton, 1946.
- [6] R.A.Fisher, "The Sampling Distribution of Some Statistics Obtained from Non Linear Equations", Annals of Eugenics, Vol. 9(1939), pp. 238-249.
- [7] R.G.Frazer, W.J.Duncan, A.R.Collar, Elementary Matrices and Some Applications to Dynamics and Differential Equations, Cambridge University Press, 1938.
- [8] H.Hotelling, "Relations between Two Sets of Variates", Biometrika, Vol. 28(1936), pp. 321-377.
- [9] P.L.Hsu, "On the Distribution of Roots of Certain Determinantal Equations", Annals of Eugenics, Vol. 9 (1939), pp. 250-258.
- [10] P.L.Hsu, "On the Limiting Distribution of the Canonical Correlations", Biometrika, Vol. 32 (1941-42), pp. 38-45.
- [11] P.C.Mahalanobis, R.C.Bose, S.N.Roy, "Normalisation of Statistical Variates and the Use of Rectangular Coordinates in the Theory of Sampling Distributions", Sankhyā, Vol. 3(1937), pp. 1-40.
- [12] S.N.Roy, "p-Statistics, or Some Generalisations in Analysis of Variance Appropriate to Multivariate Problems", Sankhyā, Vol. 4(1938-40), pp.381-396.