DISTRIBUTIONS OF SOME SERIAL
CORRELATION COEFFICIENTS

by

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NOTATION

The following notation and convention will be used, any departure being indicated at the proper place.

Numbers in $[]$ refer to the bibliography.

"Variate" stands for "random variable"

" $N(\mu, \sigma)$ variate" means "a variate distributed normally with mean μ and standard deviation σ ."

" $P(S)$ " stands for "the probability of the statement S ."

" E_x " stands for "the expectation of the variate x ."

" σ_x " stands for "the standard deviation of the variate x ."

$v_{r;x} = E x^r$, i.e. the r th moment of the variate x .

$\kappa_{r;x}$ denotes the r th cumulant of the variate x .

$f(x)$ denotes the probability density function of x .

$F(x)$ denotes the cumulative distribution function of x .

$\phi_x(t)$ denotes the characteristic function of x .

$\chi_x(t)$ denotes the cumulant generating function of x .

$\approx$ stands for "is approximately equal to."

$\sim$ stands for "is asymptotically equivalent to."

$O(L)$ stands for "a quantity of the order of magnitude of L ."

If A denotes a matrix then A' denotes its transpose.

If m is a positive number $[m]$ denotes the largest integer less than or equal to m .

CHAPTER I

TEST CRITERIA¹

1. Introduction and summary.

The distribution theory of serial correlation coefficients has received considerable attention lately. R. L. Anderson [2]² obtained exact distributions of serial correlation coefficients defined circularly at Hotelling's suggestion. The distributions thus obtained are complicated and are given by different analytical functions in different parts of the range. Following Koopmans' method [29] of smoothing the characteristic function of the variate, a simple approximation to the exact distribution of the circular serial correlation coefficient with known mean was found by Dixon [9] and Rubin [45] for uncorrelated normal variates. This approximation is accurate for large sample size, N , in the sense that first N moments of the serial correlation coefficient agree with the moments of the fitted distribution. Leipnik [31] extended Dixon's distribution, following a method due to Madow [32], to the case of a circular autoregressive scheme of order 1. Quenouille [43], Watson [52] and Jenkins [21, 22], all adopting the circular definition and the method of smoothing the characteristic function of the variates, have made considerable progress

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²Numbers in the square brackets refer to the Bibliography.

towards obtaining the joint distribution of several serial correlation coefficients under null and non-null hypotheses. More recently Daniels [8], by the application of the method of steepest descents, obtained the already known results and some new ones for circular and non-circular statistics.

It is difficult to give a comprehensive history of the work done in this field as the subject has been under extensive research by scores of mathematicians and statisticians. Since the present study is confined to the distribution of non-circular serial correlation coefficients under the null hypothesis, no attempt has been made to include all the names of the persons who have contributed to the problems of estimation and testing; nor is any history of the classical approach to time series analysis included. However, the bibliography includes papers and books dealing with the general problem of time series analysis. The names of Bartlett [5, 6], Quenouille [41, 42], Von Neumann [49, 50, 51], Whittle [53], T. W. Anderson [3, 4], Kendall [24, 25], Mann and Wald [33] may be specially noted in this respect.

The major part of this paper deals with the distributions of serial correlation coefficients and their functions, in uncorrelated normal variates with known means. Since these coefficients are independent of scale parameter there is no loss of generality in assuming the variance of the normal variates to be unity.

In this chapter the distribution of the sample from autore-

gressive schemes of orders 1 and 2 will be obtained. A method of inverting the covariance matrix of the sample from a general order autoregressive scheme will be suggested. Estimates, which are approximations to least-squares estimates of the parameters in the autoregressive schemes of orders 1 and 2 will be given; and statistics for testing the null hypothesis, which are approximations to the likelihood-ratio criteria, will be constructed. These are functions of first and second serial correlation coefficients.

Two alternative definitions will be proposed for the first serial correlation coefficient. r^* is the ordinary correlation coefficient between the sets $(x_1, x_2, \dots, x_{N-1})$ and $(x_2, x_3, \dots, x_N)$ except that the deviations will be taken from the population mean, which is assumed to be zero, rather than the sample mean. r_1 is the ratio of two quadratic forms obtained by replacing the denominator of r^* by $\sum_{i=1}^N x_i^2$. Bounds for $P(r^* \geq r_0)$ will be obtained in Chapter II, using a geometrical method. A geometrical approach was introduced into statistical theory by Fisher [12, 13] to obtain the distributions of ordinary, multiple and partial correlation coefficients. Other geometrical methods have been utilized by Hotelling [17, 18] in obtaining some important distributions. He also shows how to determine the order of contact of frequency curves of some statistics, with the variate axis near the ends of the range, even though the actual distribution is unknown.

In the third chapter a method will be developed to obtain dis-

tributions of serial correlation coefficients including those of partial and multiple serial correlation coefficients. Each of these distributions is represented as a product of a beta distribution and a series of Jacobi polynomials. The Hermite polynomials which are associated with the normal distribution have long been in use in distribution theory. Hotelling suggested the use of Laguerre polynomials for approximating the distribution of positive definite quadratic forms. Similarly, series of other functions such as Bessel functions have been utilized for such purposes [15].

The theory of Jacobi polynomials is well developed and many results will be quoted from Szegő [47] without proof. The same technique will be extended to obtain a bivariate distribution in Chapter IV, and could be extended to multivariate distributions as well. Since the knowledge of moments is necessary for this purpose, a considerable part of the third and fourth chapters will be devoted to calculating the cumulants and moments of the variates. A comparison with the already known distributions will be made to demonstrate the power of the proposed method.

In Chapter V, which is the last chapter, least-squares estimation of regression coefficients will be considered when the hypothesis of the independence of residuals is not satisfied.

2. The distribution of a sample in an autoregressive scheme.

Let $x_1, \dots, x_N$ be a realization of a time series at time $t = 1, \dots, N$. It is assumed that the underlying model is an auto-

regressive scheme of order k , i.e.,

$$(2.1) \quad x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \dots + \alpha_k x_{t-k} + \varepsilon_t$$

where ε_t is independent of $x_{t-1}, x_{t-2}, \dots$, and each ε is a $N(0, \sigma)$ variate, independent of all other ε 's.

In the case of $\alpha_j = 0$ for $j > 1$, it is known that (see, for instance, [20_7]) for all t

$$\begin{aligned} \mathcal{E} x_t &= 0, \\ \mathcal{E} x_t^2 &= \sigma^2 / (1 - \alpha^2), \\ \mathcal{E} x_t x_{t+k} &= \alpha_1^k \sigma_x^2, \end{aligned}$$

where $\sigma_x^2 = \mathcal{E} x_t^2$ and is independent of t . It is also known that the distribution of $x_1, \dots, x_N$ is given by

$$(2.2) \quad dF(x_1, \dots, x_N) = (2\pi\sigma^2)^{-N/2} (1-\alpha_1^2)^{1/2} \exp \left[-\frac{1}{2\sigma^2} \left\{ \sum_{i=1}^N x_i^2 + \alpha_1^2 \sum_{i=1}^{N-1} x_i^2 - 2\alpha_1 \sum_{i=1}^{N-1} x_i x_{i+1} \right\} \right] dx_1 \dots dx_N.$$

We will study a second order scheme, i.e., $\alpha_j = 0$ for $j > 2$, in more detail. Changing the notation we put $\alpha_1 = \alpha$ and $\alpha_2 = -\beta$ so that an autoregressive scheme of order 2 is

$$(2.3) \quad x_t = \alpha x_{t-1} - \beta x_{t-2} + \varepsilon_t.$$

Kendall [26, Vol. II, p. 406 ff.] has shown that the most general solution of (2.3) in our notation is

(2.4)

$$x_t = \beta^{t/2} (A \cos t\theta + B \sin t\theta) + \sum_{j=0}^{\infty} \frac{2\beta^{j/2} \sin j\theta}{(4\beta - \alpha^2)^{1/2}} \varepsilon_{t-j+1}$$

Here A and B are arbitrary constants and $2 \cos \theta = \alpha\beta^{-1/2}$, $\beta^{1/2}$ is taken with positive sign and it is assumed that $4\beta > \alpha^2$. We also assume that $0 < \beta < 1$, the first inequality being necessary for a real non-vacuous process, and the second inequality insuring that x_t will not increase without limit. Further, if we suppose the process to have been started a long time ago, then the first term on the right hand side of (2.4) may be considered to be damped out of existence. Thus

$$(2.5) \quad x_t = \sum_{j=0}^{\infty} \frac{2\beta^{j/2} \sin j\theta}{(4\beta - \alpha^2)^{1/2}} \varepsilon_{t-j+1}$$

The justification of representing x_t as a sum of infinitely many random variables is provided by the following theorem taken from Doob's Stochastic Processes [10], page 108:

"THEOREM. Let $y_1, y_2, \dots$ be mutually independent random variables, with variances $\sigma_1^2, \sigma_2^2, \dots$. Then, if $\sum_{n=1}^{\infty} \sigma_n^2 = \sigma^2 < \infty$ and if $\sum_{n=1}^{\infty} E\{y_n\}$ converges, $\sum_{n=1}^{\infty} y_n = x$ is convergent with probability 1 and also in the mean. Moreover

$$E\{x\} = \sum_{n=1}^{\infty} E\{y_n\}; \quad E\{x^2\} - E\{x\}^2 = \sigma^2$$

$$\text{and if } x_n = \sum_{i=1}^n [y_i - E\{y_i\}]$$

$$P \left\{ \text{L.U.B.}_n |x_n(\omega)| \geq \varepsilon \right\} \leq \sigma^2 / \varepsilon^2.$$

Conversely, if $\lim_{n \rightarrow \infty} \sum_{j=1}^n y_j = x$ exists, the series $\sum_{j=1}^{\infty} \sigma_j^2$ and $\sum_{j=1}^{\infty} E\{y_j\}$ converge."

We need only to point out that in the notation of the theorem above

$$(2.6) \quad y_j = \frac{2\beta^{j/2} \sin j\theta}{(4\beta - \alpha^2)^{1/2}} \varepsilon_{t-j+1},$$

$$E x_t = \sum_{j=0}^{\infty} E y_j = 0,$$

$$(2.7) \quad \sigma_x^2 = E x_t^2 = \sum_{j=0}^{\infty} E y_j^2 = \frac{(1+\beta)\sigma^2}{(1-\beta) \{(1+\beta)^2 - \alpha^2\}}.$$

Furthermore

$$(2.8) \quad E x_t x_{t+k} = \frac{2\beta^{k/2}\sigma^2}{(4\beta - \alpha^2)} \left[\frac{\cos k\theta}{1-\beta} - \frac{\cos k\theta - \beta \cos(k-2)\theta}{1 - 2\beta \cos 2\theta + \beta^2} \right].$$

If we define the autocorrelation coefficient of x_t and x_s as

$$(2.9) \quad \rho_{|t-s|} = \frac{E x_t x_s}{\sqrt{E x_t^2 E x_s^2}}$$

we have [26]

$$(2.10) \quad \rho_k = (1+\beta)^{-1} \beta^{k/2} [\sin(k+1)\theta - \beta \sin(k-1)\theta] \operatorname{cosec} \theta.$$

ρ_k satisfies the equation

$$(2.11) \quad \rho_k = \alpha \rho_{k-1} - \beta \rho_{k-2}$$

with $\rho_0 = 1$, $\rho_{-t} = \rho_t$.

From (2.11) taking $k=t$ and s in turn and solving for α and β we get

$$\alpha = \frac{\rho_t \rho_{s-2} - \rho_s \rho_{t-2}}{\rho_{t-1} \rho_{s-2} - \rho_{t-2} \rho_{s-1}}$$

$$\beta = \frac{\rho_t \rho_{s-1} - \rho_{t-1} \rho_s}{\rho_{t-1} \rho_{s-2} - \rho_{t-2} \rho_{s-1}}$$

In particular

$$(2.12) \quad \alpha = \frac{\rho_1 - \rho_1 \rho_2}{1 - \rho_1^2}$$

$$\beta = \frac{\rho_1^2 - \rho_2}{1 - \rho_1^2}$$

THEOREM 2.1. Let A be the covariance matrix of $x' = (x_1, x_2, \dots, x_N)$ where the model is (2.3), so that

$$(2.14) \quad A = \sigma_x^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 & \dots & \rho_{N-1} \\ \rho_1 & 1 & \rho_1 & \dots & \rho_{N-2} \\ \rho_2 & \rho_1 & 1 & \dots & \rho_{N-3} \\ & & & \dots & \\ \rho_{N-1} & \rho_{N-2} & \rho_{N-3} & \dots & 1 \end{bmatrix}$$

then

$$(2.15) \quad (i) \quad A^{-1} = \frac{1}{\sigma^2} \begin{bmatrix} 1 & -\alpha & \beta & 0 & \dots & 0 & 0 \\ -\alpha & 1+\alpha^2 & -\alpha-\alpha\beta & \beta & \dots & 0 & 0 \\ \beta & -\alpha-\alpha\beta & 1+\alpha^2+\beta^2 & -\alpha-\alpha\beta & \dots & 0 & 0 \\ 0 & \beta & -\alpha-\alpha\beta & 1+\alpha^2+\beta^2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1+\alpha^2 & -\alpha \\ 0 & 0 & 0 & 0 & \dots & -\alpha & 1 \end{bmatrix},$$

$$(2.16) \quad (ii) \quad |A| = \frac{\sigma^{2N}}{(1-\beta)^2 \{(1+\beta)^2 - \alpha^2\}}$$

and

(iii) if x denotes the transpose of x' , its distribution is

$$(2.17) \quad dF(x_1, \dots, x_N) = (2\pi)^{-N/2} |A|^{-1/2} \exp \left[-\frac{1}{2} x' A^{-1} x \right] dx$$

$$= (2\pi\sigma^2)^{-N/2} (1-\beta) \{(1+\beta)^2 - \alpha^2\}^{1/2} \exp \left[-\frac{1}{2\sigma^2} \left\{ \sum_{i=1}^N x_i^2 + \alpha^2 \sum_{i=2}^{N-1} x_i^2 \right. \right.$$

$$\left. \left. + \beta^2 \sum_{i=3}^{N-2} x_i^2 - 2\alpha \sum_{i=1}^{N-1} x_i x_{i+1} - 2\alpha\beta \sum_{i=2}^{N-2} x_i x_{i+1} + 2\beta \sum_{i=1}^{N-2} x_i x_{i+2} \right\} \right] dx_1 \dots dx_N.$$

PROOF. The proof of part (i) is completed by actual multiplication, taking into consideration that

$$(2.18) \quad \rho_i = \alpha \rho_{i-1} - \beta \rho_{i-2}, \quad i=2,3,\dots,N-1$$

$$\rho_1 = \alpha - \beta \rho_1$$

and

$$\frac{\sigma^2}{\sigma x^2} = \frac{(1-\beta) \{ (1+\beta)^2 - \alpha^2 \}}{1+\beta} = 1 - \alpha \rho_1 + \beta \rho_2$$

The process by which we arrive at A^{-1} is the following.

The distribution of $(\varepsilon_3, \dots, \varepsilon_N)$ is

$$dF(\varepsilon_3, \dots, \varepsilon_N) = (2\pi\sigma^2)^{-(N-2)/2} \exp \left[-\frac{1}{2\sigma^2} \sum_{t=3}^N \varepsilon_t^2 \right] d\varepsilon_3 \dots d\varepsilon_N$$

Supposing x_1 and x_2 given, make the transformation

$$\varepsilon_t = x_t - \alpha x_{t-1} + \beta x_{t-2}, \quad t=3, 4, \dots, N$$

The Jacobian of the transformation is 1, hence, given x_1 and x_2 , the distribution of $(x_3, \dots, x_N)$ is

$$(2.19) \quad dF(x_3, \dots, x_N | x_1, x_2) = (2\pi\sigma^2)^{-(N-2)/2} \exp \left[-\frac{1}{2\sigma^2} \sum_{t=3}^N (x_t - \alpha x_{t-1} + \beta x_{t-2})^2 \right] dx_3 \dots dx_N$$

Now x_1 and x_2 are distributed as bivariate normal given by

$$dF(x_1, x_2) = (2\pi\sigma_x^2)^{-1} (1 - \rho_1^2)^{-1/2} \exp \left[-\frac{1}{2\sigma_x^2(1 - \rho_1^2)} \{ x_1^2 - 2\rho_1 x_1 x_2 + x_2^2 \} \right] dx_1 dx_2$$

where $\rho_1 = \frac{\alpha}{1+\beta}$, and σ_x is given by (2.7), hence

$$(2.20) \quad dF(x_1, x_2) = (2\pi\sigma^2)^{-1} (1-\beta) \{ (1+\beta)^2 - \alpha^2 \}^{1/2} \exp \left[-\frac{1}{2\sigma^2} \{ (x_1^2 + x_2^2)(1-\beta^2) - 2\alpha(1+\beta)x_1 x_2 \} \right] dx_1 dx_2$$

Multiplying (2.19) and (2.20) we get (2.17) and also the value of $|A|$.

Thus parts (ii) and (iii) are proved.

Another method. With the help of (2.19) we write the matrix A^{-1} , with the first two rows and the first two columns blank and to be filled in, as

$$A^{-1} = \frac{1}{\sigma^2}$$

*	*	*	*		*	*	*	*
*	*	*	*		*	*	*	*
*	*	$1+\alpha^2+\beta^2$	$-\alpha-\alpha\beta$	... 0	0	0	0	0
*	*	$-\alpha-\alpha\beta$	$1+\alpha^2+\beta^2$	... 0	0	0	0	0
.	.	.	.	.	.	.	.	.
*	*	0	0		$-\alpha-\alpha\beta$	$1+\alpha^2+\beta^2$	$-\alpha-\alpha\beta$	β
*	*	0	0		β	$-\alpha-\alpha\beta$	$1+\alpha^2$	$-\alpha$
*	*	0	0		0	β	$-\alpha$	1

Now, we observe that the matrix A is persymmetric, i.e., symmetric about both diagonals, therefore A^{-1} has the same property. We fill up the first two rows and the first two columns with the help of the last two rows and the last two columns, only writing them in reverse order, and obtain (2.15). The proof is then completed by actual multiplication.

If the underlying scheme is autoregression of order k given by (2.1), A^{-1} can be obtained easily by this method if $N > 2k$.

Thus,

$$(2.21) \quad dF(x_{k+1}, \dots, x_N | x_1, \dots, x_k) =$$

$$(2\pi\sigma^2)^{-(N-k)/2} \exp \left[-\frac{1}{2\sigma^2} \sum_{i=k+1}^N (x_i - \alpha_1 x_{i-1} - \dots - \alpha_k x_{i-k})^2 \right] dx_{k+1} \dots dx_N.$$

If k is small we can find the distribution of $(x_1, \dots, x_k)$, and by multiplying it with (2.21) obtain not only the exponent but also the value of $|A|$.

In any case, however, we can write A^{-1} with the help of (2.21), filling in the first k rows and the first k columns with the help of the last k rows and the last k columns. Thus, for instance, for $k = 3$,

$$A^{-1} = \begin{bmatrix} 1 & -a_1 & -a_2 & -a_3 & 0 & \dots \\ -a_1 & 1+a_1^2 & -a_1-a_1a_2 & -a_2-a_1a_3 & -a_3 & \dots \\ -a_2 & -a_1-a_1a_2 & 1+a_1^2+a_2^2 & -a_1-a_1a_2-a_2a_3 & -a_2-a_1a_3 & \dots \\ -a_3 & -a_2-a_1a_3 & -a_1-a_1a_2-a_2a_3 & 1+a_1^2+a_2^2+a_3^2 & -a_1-a_1a_2-a_2a_3 & \dots \\ 0 & -a_3 & -a_2-a_1a_3 & -a_1-a_1a_2-a_2a_3 & 1+a_1^2+a_2^2+a_3^2 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 1+a_1^2 & -a_1 \\ \cdot & \cdot & \cdot & \cdot & -a_1 & 1 \end{bmatrix} \frac{1}{\sigma^2}$$

J . Wise [54_] has given another method of inverting A in the general case.

3. Least-squares estimates.

In the analogy with the elementary theory we find formally the least-squares estimates of the parameters in autoregressive schemes of orders 1 and 2. This will serve as a guide in constructing simpler statistics based on serial correlation coefficients.

For a Markov scheme

$$x_t = \alpha x_{t-1} + \varepsilon_t$$

least-squares estimates (which are also the maximum-likelihood estimates when x_1 is supposed to be fixed) of α and σ^2 are

$$(3.1) \quad a_1 = \frac{\sum_{i=1}^{N-1} x_i x_{i+1}}{\sum_{i=1}^{N-1} x_i^2}$$

and

$$(3.2) \quad s_1^2 = \frac{\sum_{i=1}^{N-1} (x_{i+1} - a_1 x_i)^2}{N-1}$$

respectively. The estimate of the variance of a_1 is

$$(3.3) \quad s_{a_1}^2 = \frac{\sum_{i=1}^{N-1} (x_{i+1} - a_1 x_i)^2}{(N-1) \left(\sum_{i=1}^{N-1} x_i^2 \right)}$$

L. Hurwicz [20_] has shown that a_1 is biased in small samples. Mann and Wald [33_] showed that this bias tends to zero as N tends

to infinity.

For an autoregressive scheme of order 2, that is,

$$x_t = \alpha x_{t-1} - \beta x_{t-2} + \epsilon_t$$

the least-squares estimates of α , β , and σ^2 are

$$a_2 = \frac{(\sum x_{i-1} x_i)(\sum x_{i-2}^2) - (\sum x_{i-2} x_i)(\sum x_{i-2} x_{i-1})}{(\sum x_{i-1}^2)(\sum x_{i-2}^2) - (\sum x_{i-2} x_{i-1})^2 = D}$$

$$b_2 = \frac{(\sum x_{i-2} x_{i-1})(\sum x_{i-1} x_i) - (\sum x_{i-1}^2)(\sum x_{i-2} x_i)}{D}$$

and

$$s_2^2 = \frac{\sum (x_i - a_2 x_{i-1} - b_2 x_{i-2})^2}{N-2}$$

respectively, where all the summations extend from 3 to N. The estimate of the covariance matrix of a_2 and b_2 is

$$B = \frac{s_2^2}{D} \begin{bmatrix} \sum x_{i-2}^2 & \sum x_{i-1} x_{i-2} \\ \sum x_{i-1} x_{i-2} & \sum x_{i-1}^2 \end{bmatrix}$$

Obviously these estimates are of little practical use. To simplify we propose the following estimates instead, which are suggested by and approximated to the least-squares estimates.

Writing

$$q_1 = \sum_{i=1}^{N-1} x_i x_{i+1}$$

$$q_2 = \sum_{i=1}^{N-2} x_i x_{i+2}$$

$$p = \sum_{i=1}^N x_i^2$$

let us define a first and a second serial correlation coefficient as

$$r_1 = q_1/p$$

and

$$r_2 = q_2/p$$

if $p \neq 0$.

We propose r_1 as an estimate of α in a first order autoregressive scheme,

$$s^2 = p(1-r_1^2)/(N-1)$$

as an estimate of σ^2 . Similarly an estimate of variance of r_1 is

$$s_{r_1}^2 = (1 - r_1^2)/(N-1) \quad .$$

In the analogy of elementary theory we expect

$$t^* = \frac{r_1(N-1)^{1/2}}{(1-r_1^2)^{1/2}}$$

to be distributed approximately as Student t , under the hypothesis

$\alpha = 0$.

Similarly for a scheme of order 2, given by (2.3),

$$(3.4) \quad a = \frac{r_1 - r_1 r_2}{1 - r_1^2}$$

$$(3.5) \quad b = \frac{r_1^2 - r_2^2}{1 - r_1^2}$$

and
(3.6)

$$s^2 = \frac{p(1-2r_1^2+2r_1^2r_2-r_2^2)}{(N-2)(1-r_1^2)}$$

$$= \frac{p(1-R^2)}{N-2}$$

will be our estimates of α , β and σ^2 . Here we shall define R as a multiple correlation coefficient between x_t and (x_{t-1}, x_{t-2}) . It may be noted that

$$(3.7) \quad 1 - R^2 = \frac{\begin{vmatrix} 1 & r_1 & r_2 \\ r_1 & 1 & r_1 \\ r_2 & r_1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & r_1 \\ r_1 & 1 \end{vmatrix}}.$$

Again, it is expected that when $\alpha = \beta = 0$,

$$F^*(N-2, 2) = \frac{2}{N-2} \frac{1-R^2}{R^2}$$

will be distributed, at least approximately, as F with $N-2$ and 2 degrees of freedom.

4. Likelihood-ratio criteria.

Consider the maximum of likelihood under the following hypotheses, for an autoregressive scheme of order 2 given by (2.3),

$$H_{00}: \alpha = 0, \beta = 0$$

$$H_{10}: \alpha \neq 0, \beta = 0$$

$$H_{01}: \alpha = 0, \beta \neq 0$$

$$H_{11}: \alpha \neq 0, \beta \neq 0$$

Notation: $g_{ij}(x)$ denotes the likelihood under H_{ij} ; and $\hat{\sigma}_{ij}$, $\hat{\alpha}_{ij}$, and $\hat{\beta}_{ij}$ the maximum likelihood estimates and $\hat{g}_{ij}$ the maximized likelihood.

Since we propose to study only null distributions, i.e., distributions of statistics under H_{00} , we will not consider any non-null hypothesis against another non-null hypothesis. Furthermore, for mathematical simplifications we shall assume that either x_1 , or x_1 and x_2 , are given. This results in a slight loss of information but saves many complications.

$$H_{00}: g_{00}(x_2, \dots, x_N | x_1) = (2\pi\sigma^2)^{-(N-1)/2} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=2}^N x_i^2\right]$$

$$\hat{\sigma}_{00}^2 = \left(\sum_{i=2}^N x_i^2 \right) / (N-1)$$

$$\hat{g}_{00} = (2\pi\hat{\sigma}_{00}^2 e)^{-(N-1)/2}$$

$$H_{10}: g_{10}(x_2, \dots, x_N | x_1) = (2\pi\sigma^2)^{-(N-1)/2} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=2}^N (x_i - \alpha x_{i-1})^2\right]$$

$$\hat{\sigma}_{10}^2 = \sum_{i=2}^N (x_i - \hat{\alpha} x_{i-1})^2 / (N-1)$$

$$\hat{\alpha}_{10} = \left(\sum_{i=1}^{N-1} x_i x_{i+1} \right) / \left(\sum_{i=2}^N x_i^2 \right)$$

$$\hat{g}_{10} = (2\pi\hat{\sigma}_{10}^2 e)^{-(N-1)/2}$$

Hence the likelihood-ratio criteria λ_{10} for testing H_{00} against H_{10} is

$$\begin{aligned}
 (4.1) \quad \lambda_{10}^{\frac{2}{N-1}} &= \left(\frac{\hat{g}_{10}}{\hat{g}_{00}} \right)^{\frac{2}{N-1}} = \frac{\hat{\sigma}_{00}^2}{\hat{\sigma}_{10}^2} \\
 &= \frac{\sum_{i=1}^N x_i^2}{\frac{N}{2} \left\{ 1 - 2\hat{a}_{10}^2 + \hat{a}_{10}^2 \left(\frac{\sum_{i=1}^{N-1} x_i^2}{\sum_{i=1}^N x_i^2} \right) \right\}} \\
 &\simeq \frac{1}{1 - \hat{a}_{10}^2} \simeq \frac{1}{1 - r_1^2} \quad \text{for large } N.
 \end{aligned}$$

We note that replacing $\hat{a}_{10}$ by r_1 results in some loss of power but since $\hat{a}_{10} \sim r_1$ this loss is negligible for large N . However, this or some other approximation seems necessary for further progress in the theory.

For testing H_{00} against H_{01} or H_{11} we will compare $\hat{g}_{ij}(x_3, \dots, x_N | x_1, x_2)$. $\hat{\sigma}_{00}^2$ could be considered modified accordingly.

$$H_{01}: g_{01}(x_3, \dots, x_N | x_1, x_2) = (2\pi\sigma^2)^{-(N-2)/2} \exp \left[-\frac{1}{2\sigma^2} \sum_{i=3}^N (x_i + \beta x_{i-2})^2 \right] \quad -7$$

$$\hat{\sigma}_{01}^2 = \sum_{i=3}^N (x_i + \hat{\beta}_{01} x_{i-2})^2 / (N-2)$$

$$\hat{\beta}_{01} = - \left(\sum_{i=3}^N x_{i-2} x_i \right) / \left(\sum_{i=3}^N x_i^2 \right)$$

$$\hat{g}_{01} = (2\pi\hat{\sigma}_{01}^2)^{-(N-2)/2}$$

$$H_{11}: g_{11}(x_3, \dots, x_N | x_1, x_2) = (2\pi\sigma^2)^{-(N-2)/2} \exp \left[-\frac{1}{2\sigma^2} \sum_{i=3}^N (x_i - \alpha x_{i-1} + \beta x_{i-2})^2 \right]$$

$$\hat{\sigma}_{11}^2 = \frac{1}{N-2} \sum_{i=3}^N (x_i - \hat{\alpha}_{11} x_{i-1} + \hat{\beta}_{11} x_{i-2})^2$$

$$\sum_{i=3}^N x_{i-1} x_i - \hat{\alpha}_{11} \sum_{i=3}^N x_{i-1}^2 + \hat{\beta}_{11} \sum_{i=3}^N x_{i-2} x_{i-1} = 0$$

$$\sum_{i=3}^N x_{i-2} x_i - \hat{\alpha}_{11} \sum_{i=3}^N x_{i-2} x_{i-1} + \hat{\beta}_{11} \sum_{i=3}^N x_{i-2}^2 = 0$$

Using approximations similar to those used for $\hat{\alpha}_{01}$ and $\hat{\beta}_{01}$, we find

$$(4.2) \quad \hat{\alpha}_{11} \approx \frac{r_1(1 - r_2)}{1 - r_1^2},$$

$$\hat{\beta}_{11} \approx \frac{r_1^2 - r_2}{1 - r_1^2}.$$

Thus, for testing H_{00} against H_{01} , we have

$$(4.3) \quad (\lambda_{01})^{\frac{2}{N-2}} \approx \frac{1}{1 - r_2^2},$$

and for H_{00} against H_{11}

$$(4.4) \quad (\lambda_{11})^{\frac{2}{N-2}} \approx \frac{1 - r_1^2}{1 - 2r_1^2 + 2r_1^2 r_2 - r_2^2} \\ = \frac{1}{1 - R^2}$$

where

$$1 - R^2 = \frac{\begin{vmatrix} 1 & r_1 & r_2 \\ r_1 & 1 & r_1 \\ r_2 & r_1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & r_1 \\ r_1 & 1 \end{vmatrix}} .$$

Though λ_{01} , λ_{10} and λ_{11} are not exact likelihood-ratio criteria, and their use will involve some loss of power, it seems that to develop any theory based on serial correlation coefficients some such approximations are to be used for maximum-likelihood estimates.

Other writers [2, 9, 21, 43, 52] have used circular definitions of serial correlation coefficients to circumvent some of the theoretical difficulties. We have preferred to avoid the circularity definitions and have dealt with non-circular statistics.

In the following chapters the distributions of these statistics have been obtained. Also an upper bound for $P(r^* \geq r_0)$ has been obtained where r^* is a serial correlation coefficient defined on the pattern of an ordinary correlation coefficient between two series.

CHAPTER II

DISTRIBUTION OF A FIRST SERIAL CORRELATION COEFFICIENT NEAR THE ENDS OF THE RANGE

1. Introduction

If $x_1, x_2, \dots, x_N$ are observations on the variates $X_1, X_2, \dots, X_N$, and the population means are known which may, without loss of generality, be taken to be zero, the most natural definition of the serial correlation coefficient with lag unity would be

$$r^* = \frac{\sum_{i=1}^{N-1} x_i x_{i+1}}{\sqrt{\left(\sum_{i=1}^{N-1} x_i^2\right) \left(\sum_{i=1}^{N-1} x_{i+1}^2\right)}} \quad \text{if the denominator} \neq 0,$$

which is an analogue to the definition of the ordinary correlation coefficient between two different sets of observations. However, instead of taking deviations from the sample mean, we have taken deviations from the known population means which are assumed to be zero. Due to the seemingly insurmountable mathematical difficulties involved in obtaining the distribution of r^* , several alternative definitions have been proposed as approximations to r^* . The most suitable among the alternatives seems to be

$$r_1 = \frac{\sum_{i=1}^{N-1} x_i x_{i+1}}{\sum_{i=1}^N x_i^2}.$$

Even this presents difficulties as far as exact distribution is concerned. Further, r_1 never attains the values -1 and 1, its minimum and maximum values being $-\cos \frac{\pi}{N+1}$ and $\cos \frac{\pi}{N+1}$ respectively. But it has the advantage of being (i) a good approximation to r^* for large N , (ii) a ratio of two quadratic forms, (iii) independently distributed of $\sum_{i=1}^N x_i^2$ on null hypothesis, and (iv) of not bringing any unnatural assumptions about the universe, particularly in non-null cases, as the circular definition or some other proposed definitions do.

If we consider the cumulative probability curves of r^* and r_1 , for large N , the two curves will be very close in the middle of the range but will deviate apart toward the upper end of the range. This becomes clear if we observe that

$$\sum_{i=1}^N x_i^2 \geq \frac{1}{2} x_1^2 + x_2^2 + \dots + x_{N-1}^2 + \frac{1}{2} x_N^2 \geq \sqrt{\frac{N-1}{2} \sum_{i=1}^{N-1} x_i^2 \sum_{i=1}^{N-1} x_{i+1}^2}$$

i. e. $|r_1| \leq |r^*|$, the inequality being true whenever at least two x 's are different from zero and different from each other. Hence for a given number r_0 between 0 and 1 we have $P(|r_1| \geq r_0) \leq P(|r^*| \geq r_0)$.

On the other hand we have

$$\left(\frac{r_1}{r^*}\right)^2 = \left(1 - \frac{x_N^2}{\sum x_i^2}\right) \left(1 - \frac{x_1^2}{\sum x_i^2}\right).$$

Writing $\eta_N = \left(1 - \frac{x_N^2}{\sum x_i^2}\right) \left(1 - \frac{x_1^2}{\sum x_i^2}\right)$, we observe that if x_i are independent

$N(0,1)$ variates

$$E \frac{x_1^2}{\sum x_i^2} = E \frac{x_N^2}{\sum x_i^2} = \frac{1}{N}$$

and the variances of these quantities are $O(N^{-2})$. Therefore η_N tends to 1 in probability as $N \rightarrow \infty$. Hence from section 20.6, page 254 of Cramér's Mathematical Methods of Statistics [7], we conclude that the distribution function of r_1 is asymptotically equivalent to that of r^* .

It is therefore desirable to consider the mathematical behavior of the distribution function of r^* near the ends of the range, and to use the distribution function of r_1 for the values of r_1 (or r^*) in the middle of the range. However, it may be pointed out that r_1 is easier to calculate and can be used in the whole of its range.

In this chapter the distribution of r^* near the extremities of the range is considered. The geometrical approach seemed to be particularly suitable in obtaining the order of contact of the distribution curve at $r^* = \pm 1$. It will be shown that if for a given number $r_0 < 1$ and close to 1, $P(r^* \geq r_0)$ is expanded in a series of powers of $(1 - r_0)$, the first non-zero coefficient in the expansion is that of the term $(1 - r_0)^{(N-2)/2}$. Upper and lower bounds for the coefficient of this power in the expansion will be calculated. The lower bound is positive and the upper bound gives an approximation to the upper bound of $P(r^* \geq r_0)$.

2. Geometrical representation.

Let $X_1, \dots, X_N$ be N independent $N(0,1)$ variates. Define

$$r^* = \frac{\sum_{i=1}^{N-1} X_i X_{i+1}}{\sqrt{\left(\sum_{i=1}^{N-1} X_i^2 \right) \left(\sum_{i=1}^{N-1} X_{i+1}^2 \right)}}$$

if the denominator is not equal to zero, then r^* is a variate with range $[-1, 1]$.

For every set of observations $y_1, \dots, y_N$ on these variates, we take a point S in N -dimensional Euclidean space, which may be regarded as a representation of sample space. S is determined if its co-ordinates are taken to be $(y_1, \dots, y_N)$. Denoting the origin by O , we

see that the points S are distributed with spherical symmetry about O . Furthermore, a unique value of r^* corresponds to all the points on a straight line OS , excepting the origin for which r^* is not defined. Let the straight line OS meet the $(N-1)$ -dimensional unit sphere in Q and Q' , where Q is on the same side of the origin as S is. Denoting by $(x_1, \dots, x_N)$ the Coordinates of Q , we have

$$(2.2) \quad \sum_{i=1}^N x_i^2 = 1,$$

which may also be taken as the equation of the unit sphere. The points Q and Q' may be considered to determine a unique value of r^* . Considering only the point Q , it is easily seen that the distribution of Q is uniform over the unit sphere; that is, denoting the total $(N-1)$ -dimensional surface area of (2.2) by S_{N-1} , the probability of Q falling in an area A on the sphere is A/S_{N-1} . For a given r_0 in $[-1, 1]$ there exists a set of points on the unit sphere such that for each point in this set the corresponding value of r^* lies in the interval $[r_0, 1]$, and for no other point. If this set of point covers an area A on the surface of the sphere (2.2), it follows that

$$P(r^* \geq r_0) = A/S_{N-1}.$$

We observe that $r^* = 1$ if and only if

$$x_i = \lambda x_{i-1}, \quad i = 2, 3, \dots, N \text{ and } \lambda > 0, \quad x_1 \neq 0,$$

that is,

$$x_i = \lambda^{i-1} x_1, \quad i = 2, 3, \dots, N \text{ and } \lambda > 0, \quad x_1 \neq 0.$$

Since the point $(x_1, \dots, x_N)$ lies on (2.2), we obtain for the value of x_1 ,

$$x_1 = \pm c,$$

whence

$$(2.3) \quad c = \sqrt{(1 - \lambda^2)/(1 - \lambda^{2N})}.$$

Denote the variable point $(c, \lambda c, \dots, \lambda^{N-1} c)$ by P and $(-c, -\lambda c, \dots, -\lambda^{N-1} c)$ by P' . As λ varies from zero to ∞ , each of P and P' describes a curve for every point of which — excepting the two points of each curve obtained by $\lambda = 0$ and ∞ — corresponds the value of $r^* = 1$.

Since both these curves are exactly alike, except for their position in space, we confine our attention to the curve

$$(2.4) \quad x_1 = c, x_i = \lambda^{i-1} x_1, i = 2, \dots, N, 0 < \lambda < \infty.$$

Further, from now on we reserve $(x_1, \dots, x_N)$ to denote the point on curve (2.4) which corresponds to the parameter value λ , and we use $(\xi_1, \dots, \xi_N)$ to denote any other point on the unit sphere.

To find the probability of r^* exceeding a given value r_0 which is close to 1, we consider the points within a "tube" of geodesic radius θ on the surface of the sphere (2.2) with its axial curve (2.4).

Let the length of the curve (2.4) measured from $P_0(1, 0, \dots, 0)$ to $P(x_1, \dots, x_N)$ be denoted by s , or more explicitly $s(\lambda)$, and an element of curve by ds . Denoting by primes the differential coefficient with respect to s , the direction cosines of the tangent to the curve at P are

$$x_1', x_2', \dots, x_N',$$

where

$$(2.5) \quad x_i' = \frac{d}{ds} \lambda^{i-1} c = \lambda^{i-1} \frac{d\lambda}{ds} \lambda', i=1, 2, \dots, N.$$

We note that

$$(2.6) \quad \sum_{i=1}^N x_i'^2 = 1,$$

and since

$$\sum_{i=1}^N x_i^2 = 1$$

we have

$$(2.7) \quad \sum_{i=1}^N x_i x_i' = 0$$

Let the coordinate axes be rotated, so that the new coordinates are denoted by the elements of vector η . Let $\eta = B\xi$ where

$$B = \begin{bmatrix} x_1' & x_2' & \dots & x_N' \\ x_1 & x_2 & \dots & x_N \\ b_{31} & b_{32} & \dots & b_{3N} \\ \dots & \dots & \dots & \dots \\ b_{N1} & b_{N2} & \dots & b_{NN} \end{bmatrix},$$

and

$$(2.8) \quad BB' = I$$

Here I denotes the identity matrix, B' denotes the transpose of B , ξ and η denote the column vectors $(\xi_1, \dots, \xi_N)$ and $(\eta_1, \dots, \eta_N)$ respectively.

The η_1 axis is now parallel to the tangent of the curve at P and the η_2 axis coincides with the line OP .

The $(N-3)$ -dimensional sphere given by the set of equations

$$(2.9) \quad \eta_1 = 0, \eta_2 = \cos \theta, \quad \eta_i = \gamma_i \sin \theta, \quad i = 3, 4, \dots, N,$$

with

$$\sum_{i=3}^N \gamma_i^2 = 1,$$

lies entirely on the $(N-1)$ -dimensional sphere

$$(2.10) \quad \sum_{i=1}^N \eta_i^2 = 1 = \sum_{i=1}^N \xi_i^2 .$$

The sphere (2.10) is the same as (2.2) with a change of notation. Each point on (2.9) is at a geodesic distance θ from P measured on the sphere (2.10). Further, since (2.9) lies in the plane $\eta_1 = 0$, this hypersphere is perpendicular to the tangent of curve (2.4) at P .

Changing back to the original coordinates we have $\xi = B' \eta$ or

$$\xi_i = x_{i1} \eta_1 + x_{i2} \eta_2 + b_{3i} \eta_3 + \dots + b_{Ni} \eta_N, \quad i = 1, \dots, N .$$

Equations (2.9) become

$$(2.11) \quad \xi_j = x_j \cos \theta + \sin \theta \sum_{i=3}^N b_{ij} \eta_i, \quad j=1, 2, \dots, N$$

with

$$\sum_{i=3}^N \eta_i^2 = 1 .$$

3. The value of r^* near the curve.

Let us calculate the value of r^* corresponding to a point $(\xi_1, \dots, \xi_N)$ on the hypersphere (2.11). We have

$$r^* = \left(\sum_{j=1}^{N-1} \xi_j \xi_{j+1} \right) \left\{ (1 - \xi_1^2)(1 - \xi_N^2) \right\}^{-1/2} ,$$

since

$$\sum_{j=1}^{N-1} \xi_j^2 = 1 - \xi_N^2 \quad \text{and} \quad \sum_{j=1}^{N-1} \xi_{j+1}^2 = 1 - \xi_1^2 .$$

Now

$$\begin{aligned}
 (3.1) \quad \sum_{j=1}^{N-1} \xi_j \xi_{j+1} &= \sum_{j=1}^{N-1} \left[x_j \cos \theta + \sin \theta \sum_{i=3}^N b_{ij} \gamma_i \right] \\
 &\quad \left[x_{j+1} \cos \theta + \sin \theta \sum_{i=3}^N b_{i,j+1} \gamma_i \right] \\
 &= \cos^2 \theta \sum_{j=1}^{N-1} x_j x_{j+1} \\
 &\quad + \sin \theta \cos \theta \sum_{i=3}^N \gamma_i \left(\sum_{j=1}^{N-1} b_{ij} x_{j+1} + \sum_{j=1}^{N-1} b_{i,j+1} x_j \right) \\
 &\quad + \sin^2 \theta \sum_{j=1}^{N-1} \sum_{i=3}^N \sum_{k=3}^N b_{ij} b_{k,j+1} \gamma_i \gamma_k .
 \end{aligned}$$

It is convenient to introduce here a double suffix summation convention. We will reserve the letter j appearing at two places as the suffix in a product term to represent summation over j from 1 to $N-1$. The letters i and k appearing twice as suffices with stand for summation from 3 to N . Thus, for example

$$x_j x_{j+1} \text{ means } \sum_{j=1}^{N-1} x_j x_{j+1}$$

and

$$b_{k,j+1} b_{ij} \gamma_i \gamma_k \text{ means } \sum_{j=1}^{N-1} \sum_{k=3}^N \sum_{i=3}^N b_{k,j+1} b_{ij} \gamma_i \gamma_k .$$

Now

$$1 - \xi_1^2 = 1 - (x_1 \cos \theta + \sin \theta b_{11} \gamma_1)^2$$

$$\begin{aligned}
&= 1 - x_1^2 \cos^2 \theta - 2 \sin \theta \cos \theta x_1 b_{i1} \gamma_i - \sin^2 \theta (b_{i1} \gamma_i)^2 \\
&= 1 - x_1^2 + x_1^2 \sin^2 \theta - 2 \sin \theta \cos \theta x_1 b_{i1} \gamma_i - \sin^2 \theta (b_{i1} \gamma_i)^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(3.2) \quad (1 - \xi_1^2)^{-1/2} &= (1 - x_1^2)^{-1/2} \sqrt{1 - \frac{(2 \sin \theta \cos \theta) x_1 b_{i1} \gamma_i - x_1^2 \sin^2 \theta + \sin^2 \theta (b_{i1} \gamma_i)^2}{1 - x_1^2}}^{-1/2} \\
&= (1 - x_1^2)^{-1/2} \sqrt{1 + \frac{(\sin \theta \cos \theta) x_1 b_{i1} \gamma_i}{1 - x_1^2} - \frac{\sin^2 \theta x_1^2}{2(1 - x_1^2)} + \frac{\sin^2 \theta (b_{i1} \gamma_i)^2}{2(1 - x_1^2)}} \\
&\quad + \frac{3 \sin^2 \theta \cos^2 \theta x_1^2 (b_{i1} \gamma_i)^2}{2(1 - x_1^2)^2} + O(\sin^3 \theta) \quad \text{---7.}
\end{aligned}$$

Similarly

$$\begin{aligned}
(3.3) \quad (1 - \xi_N^2)^{-1/2} &= (1 - x_N^2)^{-1/2} \sqrt{1 + \frac{\sin \theta \cos \theta x_N b_{iN} \gamma_i}{1 - x_N^2} - \frac{\sin^2 \theta x_N^2}{2(1 - x_N^2)}} \\
&\quad + \frac{\sin^2 \theta (b_{iN} \gamma_i)^2}{2(1 - x_N^2)} + \frac{3 \sin^2 \theta \cos^2 \theta x_N^2 (b_{iN} \gamma_i)^2}{2(1 - x_N^2)^2} \\
&\quad + O(\sin^3 \theta) \quad \text{---7.}
\end{aligned}$$

Remembering that the point $P(x_1, \dots, x_N)$ is on the curve (2.4) for each point of which $r^* = 1$, we have

$$(3.4) \quad 1 = (x_j x_{j+1}) (1 - x_1^2)^{-1/2} (1 - x_N^2)^{-1/2} \quad .$$

Since B is an orthogonal matrix, we obtain

$$\begin{aligned}
 (3.5) \quad b_{ij} x_{j+1} &= \lambda b_{ij} x_j = -\lambda b_{iN} x_N \\
 &= -\lambda^N b_{iN} c, \\
 b_{i,j+1} x_j &= \frac{1}{\lambda} b_{i,j+1} x_{j+1} = -\frac{1}{\lambda} b_{i1} x_1 \\
 &= -\frac{1}{\lambda} b_{i1} c.
 \end{aligned}$$

We note that N is not a dummy suffix and does not stand for summation.

Finally from (3.1) - (3.5) we obtain

$$\begin{aligned}
 (3.6) \quad r^* &= 1 - c \sin \theta \cos \theta \left[-\frac{b_{i1} \gamma_i / \lambda + \lambda^N b_{iN} \gamma_i}{(1-x_1^2)^{1/2} (1-x_N^2)^{1/2}} - \frac{b_{i1} \gamma_i}{1-x_1^2} - \frac{\lambda^{N-1} b_{iN} \gamma_i}{1-x_N^2} \right] \\
 &- \sin^2 \theta + \frac{\sin^2 \theta \cos^4 \theta x_1 x_N b_{i1} b_{kN} \gamma_i \gamma_k}{(1-x_1^2)(1-x_N^2)} \\
 &- \frac{\sin^2 \theta \cos^2 \theta}{2} \left(\frac{x_1^2}{1-x_1^2} + \frac{x_N^2}{1-x_N^2} \right) + \frac{\sin^2 \theta \cos^2 \theta}{2} \left[-\frac{(b_{i1} \gamma_i)^2}{1-x_1^2} + \frac{(b_{iN} \gamma_i)^2}{1-x_N^2} \right] \\
 &+ \frac{3}{2} \sin^2 \theta \cos^4 \theta \left[-\frac{x_1^2 (b_{i1} \gamma_i)^2}{(1-x_1^2)^2} + \frac{x_N^2 (b_{i1} \gamma_i)^2}{(1-x_N^2)^2} \right] \\
 &- \frac{\sin^2 \theta \cos^2 \theta x_1}{(1-x_1^2)^{1/2} (1-x_N^2)^{1/2}} \left[\left\{ \lambda^N b_{iN} \gamma_i + b_{i1} \gamma_i / \lambda \right\} \left\{ \frac{x_1 b_{k1} \gamma_k}{1-x_1^2} \right\} \right]
 \end{aligned}$$

$$+ \frac{x_N b_{kN} \gamma_k}{1-x_N^2} \} \int + \frac{\sin^2 \theta b_{1j} b_{k,j+1} \gamma_1 \gamma_k}{(1-x_1^2)^{1/2} (1-x_N^2)^{1/2}} + O(\sin^3 \theta) .$$

Now,

$$(3.7) \quad 1 - x_1^2 = 1 - \frac{\lambda^2 - 1}{\lambda^{2N-1}} = \frac{\lambda^2 (\lambda^{2N-2} - 1)}{\lambda^{2N-1}} ,$$

$$1 - x_N^2 = 1 - \frac{\lambda^{2N-2} (\lambda^2 - 1)}{\lambda^{2N-1}} = \frac{\lambda^{2N-2} - 1}{\lambda^{2N-1}} .$$

The coefficient of $\sin \theta \cos \theta$

$$= \frac{c(\lambda^{2N-1})}{(\lambda^{2N-2} - 1)} \int \lambda^{N-1} b_{1N} \gamma_1 + b_{11} \gamma_1 / \lambda^2 - b_{11} \gamma_1 / \lambda^2 - \lambda^{N-1} b_{1N} \gamma_1 \int$$

$$= 0 .$$

After substituting the values from (3.7) in terms of λ and replacing $\cos^2 \theta$ by $1 - \sin^2 \theta$ and simplifying, we get

$$(3.8) \quad \frac{1-r^*}{\sin^2 \theta} = 1 + \frac{(\lambda^2 - 1)(\lambda^{2N+1})}{2\lambda^2 (\lambda^{2N-2} - 1)} + \frac{(\lambda^2 - 1)(\lambda^{2N} - 1)}{\lambda^2 (\lambda^{2N-2} - 1)^2} \int \lambda^{N-1} b_{11} b_{kN} \gamma_1 \gamma_k$$

$$- \frac{\lambda (\lambda^{2N-2} - 1)}{\lambda^2 - 1} b_{1j} b_{k,j+1} \gamma_1 \gamma_k - \frac{(\lambda^{2N} - 1)}{2(\lambda^2 - 1)} \{ (b_{11} \gamma_1)^2 / \lambda^2$$

$$+ \lambda^2 (b_{1N} \gamma_1)^2 \} \int .$$

It is understood that terms of order $\sin^3 \theta$ in (3.6) have been omitted in (3.8) and in the later discussion.

4. Approximation to the value of r^* .

As an approximation, replace the terms in the square bracket in (3.8) by their expectations. Since $r_3, \dots, r_N$ are Cartesian coordinates of a point on $(N-3)$ -dimensional unit sphere

$$\sum_{i=3}^N r_i^2 = 1,$$

we have from considerations of symmetry

$$E r_i = 0$$

$$E r_i^2 = \frac{1}{N-2}, \quad i=3, 4, \dots, N,$$

$$E r_i r_k = 0 \quad \text{for } i \neq k, \quad i, k=3, 4, \dots, N.$$

Rearranging the terms of (3.8), we get

$$\begin{aligned} (4.1) \quad & \frac{\lambda^2 (\lambda^{2N-2} - 1)^2}{(\lambda^2 - 1)(\lambda^{2N} - 1)} \int \frac{1 - r^*}{\sin^2 \theta} - 1 - \frac{(\lambda^2 - 1)(\lambda^{2N} + 1)}{2\lambda^2 (\lambda^{2N-2} - 1)} - 7 \\ & \approx - \frac{(\lambda^{2N} - 1)}{2(N-2)\lambda^2 (\lambda^2 - 1)} \int \sum_{i=3}^N b_{i1}^2 + \lambda^4 \sum_{i=3}^N b_{iN}^2 - 7 \\ & + \frac{\lambda^{N-1}}{N-2} b_{i1} b_{iN} - \frac{\lambda(\lambda^{2N-2} - 1)}{(N-2)(\lambda^2 - 1)} b_{ij} b_{i,j+1}. \end{aligned}$$

Since B is orthogonal, we have

$$x_1'^2 + x_1^2 + \sum_{i=3}^N b_{i1}^2 = 1,$$

or

$$\sum_{i=3}^N b_{i1}^2 = 1 - x_1^2 - x_1'^2.$$

Similarly

$$\sum_{i=3}^N b_{iN}^2 = 1 - x_N^2 - x_N'^2 ,$$

and

$$b_{11}b_{1N} = - (x_1x_N + x_1'x_N') ,$$

and

$$b_{ij}b_{i,j+1} = - (x_jx_{j+1} + x_j'x_{j+1}') ,$$

where the double suffix summation convention is still being followed.

Hence,

$$(4.2) \quad (N-2) \left\{ \text{Right-hand side of (2.4.1)} \right\} = - \frac{(\lambda^{2N-1})}{2\lambda^2(\lambda^2-1)}$$

$$\int [1 - x_1^2 - x_1'^2 + \lambda^4(1 - x_N^2 - x_N'^2)] - \lambda^{N-1}(x_1x_N + x_1'x_N') + \frac{\lambda(\lambda^{2N-2}-1)}{(\lambda^2-1)}$$

$$- \int x_jx_{j+1} + x_j'x_{j+1}' - 7.$$

Now,

$$(4.3) \quad \frac{dx_1}{d\lambda} = x_1\lambda \int \frac{1}{\lambda^2-1} - \frac{N\lambda^{2N-2}}{\lambda^{2N}-1} - 7$$

$$= x_1\lambda \int \frac{(\lambda^{2N}-1) - N\lambda^{2N-2}(\lambda^2-1)}{(\lambda^2-1)(\lambda^{2N}-1)} - 7 ,$$

and

$$(4.4) \quad \frac{dx_i}{d\lambda} = (i-1)\lambda^{i-2}x_1 + \lambda^{i-1} \frac{dx_1}{d\lambda} , \quad i=2, \dots, N .$$

Therefore ,

$$(4.5) \quad \left(\frac{d}{d\lambda} s\right)^2 = \sum_{i=1}^N \left(\frac{dx_i}{d\lambda}\right)^2 = \frac{1}{(\lambda^2-1)^2} - \frac{N^2 \lambda^{2N-2}}{(\lambda^{2N}-1)^2}.$$

Furthermore,

$$(4.6) \quad x_j x_{j+1} = \lambda x_1^2 \sum_{j=1}^{N-1} \lambda^{2j-2} = \frac{\lambda(\lambda^{2N-2}-1)}{\lambda^{2N}-1},$$

and, since

$$x'_{j+1} = \lambda x'_j + x_j \lambda',$$

we have

$$(4.7) \quad \begin{aligned} x'_j x'_{j+1} &= \lambda \sum_{j=1}^{N-1} x_j'^2 + x_j x'_j \lambda' \\ &= \lambda(1-x_N'^2) - \lambda' x_N' x_N \end{aligned}$$

from (2.6) and (2.7). Substituting from (4.4)-(4.7) in (4.2) we get

$$(4.8) \quad \begin{aligned} &\frac{(N-2)\lambda^2(\lambda^{2N-2}-1)^2}{(\lambda^2-1)(\lambda^{2N}-1)} \int \frac{1-r^*}{\sin^2 \theta} - 1 - \frac{(\lambda^2-1)(\lambda^{2N}+1)}{2\lambda^2(\lambda^{2N-2}-1)} - 7 \\ &\approx - \frac{(\lambda^{2N}-1)}{2\lambda^2(\lambda^2-1)} \int 1 - x_1^2 - x_1'^2 + \lambda^4(1-x_N^2 - x_N'^2) - 7 \\ &\quad - \lambda^{N-1}(x_1 x_N + x_1' x_N') + \frac{\lambda(\lambda^{2N-2}-1)}{(\lambda^2-1)} \left\{ \frac{\lambda(\lambda^{2N-2}-1)}{\lambda^{2N}-1} + \lambda \right. \\ &\quad \left. - \lambda x_N'^2 - x_N x_N' \lambda' \right\}. \end{aligned}$$

Inserting the values of x_1 , x_1' , x_N , x_N' , rearranging terms and simplifying, we finally obtain

$$\frac{1-r^*}{\sin^2 \theta} \sim 1 + \frac{(\lambda^2-1)(\lambda^{2N+1})}{2\lambda^2(\lambda^{2N-2}-1)} + \frac{(\lambda^{2N-4}-1)(\lambda^{2N-1})}{(N-2)(\lambda^{2N-2}-1)^2}$$

or

$$(4.9) \quad \sin \theta \sim (1-r^*)^{1/2} \left[1 + \frac{(\lambda^2-1)(\lambda^{2N+1})}{2\lambda^2(\lambda^{2N-2}-1)} + \frac{(\lambda^{2N-4}-1)(\lambda^{2N-1})}{(N-2)(\lambda^{2N-2}-1)^2} \right]^{-1/2}.$$

5. Integral expression for $P(r^* \geq r_0)$.

To find the probability that $r^* \geq r_0$, where $r_0 < 1$ and close to 1, we proceed in the following manner. For a given λ , r_0 determines a unique positive value of $\sin \theta$; hence a unique value of θ in the interval $[0, \frac{\pi}{2}]$, say $\theta_0(\lambda)$. For a given λ , the probability that a random point $(\xi_1, \dots, \xi_N)$ falls in the elemental area of the surface of the sphere $\sum_{i=1}^N \xi_i^2 = 1$ given by the product of ds by $\int_0^{\theta_0} s'_{N-3} d\theta$ is the ratio of this area to the area of the unit

sphere; here s'_{N-3} denotes the surface area of $\sum_{i=1}^N \gamma_i^2 = \sin^2 \theta$.

This ratio equals

$$\begin{aligned} & \frac{2\pi^{\frac{1}{2}(N-2)}}{\Gamma(\frac{N-2}{2})} \frac{\Gamma(\frac{N}{2})}{2\pi^{N/2}} ds \int_0^{\theta_0(\lambda)} \sin^{N-3} \theta d\theta \\ &= \frac{N-2}{2\pi} ds \int_0^{\theta_0(\lambda)} \sin^{N-3} \theta d\theta. \end{aligned}$$

Remembering that for $r^* = 1$ there correspond two curves on the unit sphere, one traced by P and the other by P' , and noting that

changing the signs of x 's in (3.6) does not change the expression, we get

$$(5.1) \quad P(r^* \geq r_0) = \frac{N-2}{\pi} \int_0^\infty \int_0^{\Theta_0(\lambda)} \sin^{N-3} \theta d\theta ds(\lambda) .$$

It may be pointed out that for $\lambda = 0$ and ∞ , r^* is not defined, but the inclusion of these values in evaluating (5.1) does not affect the probability as the integrand remains finite near these values and is zero for $\lambda = 0$ and ∞ , as will be seen later. Now,

$$(5.2) \quad \int_0^{\Theta_0} \sin^{N-3} \theta d\theta = \int_0^{\Theta_0} \frac{\sin^{N-3} \theta \cos \theta d\theta}{\cos \theta} \\ = \frac{\sin^{N-2} \theta_0}{(N-2) \cos \theta_0} \int_0^1 \frac{1 - \frac{\sin^2 \theta_0}{N \cos^2 \theta_0} + \frac{3 \sec^2 \theta_0}{N \sin^{N-2} \theta_0} \int_0^{\Theta_0} \frac{\sin^{N+1} \theta d\theta}{\cos^4 \theta} \\ \simeq \sin^{N-2} \theta_0 / (N-2)$$

if N is large and θ_0 small.

Therefore,

$$P(r^* \geq r_0) \simeq \frac{1}{\pi} \int_0^\infty \sin^{N-2} \theta_0 ds(\lambda) .$$

From (4.5) and (4.9)

$$(5.3) \quad P(r^* \geq r_0) \simeq \frac{(1-r_0)^{(N-2)/2}}{\pi} \int_0^\infty \int_0^1 \frac{1 + \frac{(\lambda^2-1)(\lambda^{2N+1})}{2\lambda^2(\lambda^{2N-2}-1)}}{1 + \frac{(\lambda^2-1)(\lambda^{2N+1})}{2\lambda^2(\lambda^{2N-2}-1)}} d\lambda +$$

$$\frac{(\lambda^{2N-4}-1)(\lambda^{2N-1})}{(N-2)(\lambda^{2N-2}-1)^2} \int^{-(N-2)/2} \int \frac{1}{(\lambda^2-1)^2} - \frac{N^2 \lambda^{2N-2}}{(\lambda^{2N-1})^2} \lambda^{1/2} d\lambda .$$

Let us denote the integral on the right hand side by I_N and note that a change of variable $\lambda_1 = 1/\lambda$ leaves the integral unchanged; hence the integral from 0 to 1 is the same as from 1 to ∞ . Thus

$$(5.4) \quad I_N = 2 \int_0^1 \int g(\lambda) \lambda^{-(N-2)/2} h(\lambda) d\lambda$$

where

$$(5.5) \quad g(\lambda) = 1 + \frac{(\lambda^2-1)(\lambda^{2N}+1)}{2\lambda^2(\lambda^{2N-2}-1)} + \frac{(\lambda^{2N-4}-1)(\lambda^{2N-1})}{(N-2)(\lambda^{2N-2}-1)^2}$$

and

$$(5.6) \quad h(\lambda) = \int \frac{1}{(\lambda^2-1)^2} - \frac{N^2 \lambda^{2N-2}}{(\lambda^{2N-1})^2} \lambda^{1/2} .$$

We note here that E. S. keeping [23] has studied a similar but much less complicated integral.

6. Discussion about $g(\lambda)$.

Let us consider the behavior of the function $g(\lambda)$ in the range $[0, 1]$. Writing

$$(6.1) \quad g_1(y) = \frac{(1-y)(1+y^N)}{y(1-y^{N-1})} ,$$

and

$$(6.2) \quad g_2(y) = \frac{(1-y^{N-2})(1-y^N)}{(1-y^{N-1})^2} ,$$

we have

$$(6.3) \quad g(\lambda) = 1 + \frac{1}{2} g_1(\lambda^2) + \frac{1}{N-2} g_2(\lambda^2) .$$

Dividing out the factor $(1-y)$ from the numerator and the denominator of $g_1(y)$, we have

$$\begin{aligned} g_1 &= \frac{1+y^N}{y+y^2+\dots+y^{N-1}} > 0 \text{ for } y > 0 . \\ \frac{dg_1}{dy} &= \frac{(y+y^2+\dots+y^{N-1})Ny^{N-1} - (1+y^N)(1+2y+\dots+(N-1)y^{N-2})}{y^2(1+y+\dots+y^{N-2})^2} \\ &= \frac{1}{\text{Denominator}} \int y^{2N-2} + 2y^{2N-3} + \dots + (N-1)y^N - (N-1)y^{N-2} - \dots - 2y - 1 \int . \end{aligned}$$

Hence, from Descartes' rule of sign, $dg_1/dy = 0$ can have at most one positive root, as after supplying the missing term y^{N-1} we see that there is only one variation of the sign whether the coefficient of y^{N-1} be considered positive or negative. This root is given by $y = 1$. It follows that for $0 < y < 1$, dg_1/dy has a constant sign which is easily seen to be negative. Hence g_1 is a monotonically decreasing function of its argument in the range $\int 0, 1 \int$, and

$$(6.4) \quad g_1(0) = \infty, \quad g_1(1) = \frac{2}{N-1} .$$

Differentiating $g_2(y)$ with respect to y and arranging terms,

$$\frac{dg_2}{dy} = - \frac{y^{N-3}(1-y)^2}{(1-y^{N-1})^3} \int (N-2)y^{N-1} - 2y^{N-2} - 2y^{N-3} - \dots - 2y + (N-2) \int .$$

Let the expression in the square bracket be denoted by g_2^* . From Descartes' rule of sign, $g_2^* = 0$ can have at most two positive roots.

Now

$$g_2^*(1) = 0$$

and

$$\frac{dg_2^*}{dy} \Big|_{y=1} = (N-2)(N-1) - 2(N-2) - 2(N-3) - \dots - 2 \times 1 = 0.$$

Hence $y = 1$ is a repeated root of $g_2^*(y) = 0$.

Therefore,

$$\frac{dg_2}{dy} = - \frac{y^{N-3}(1-y)^4}{(1-y^{N-1})^3} \phi(y) = \frac{-y^{N-3}(1-y)}{(1+y+\dots+y^{N-2})^3} \phi(y),$$

where $\phi(y) \neq 0$ for $0 \leq y \leq 1$, and since

$$g_2^*(y) = (1-y)^2 \phi(y),$$

$\phi(y)$ has the same sign as $g_2^*(y)$ which is easily seen to be positive. Hence $dg_2/dy < 0$ for $0 < y < 1$, and for $0 \leq y \leq 1$, $g_2(y)$ is a monotonically decreasing function of y ; and

$$(6.5) \quad g_2(0) = 1, \quad g_2(1) = \frac{(N-2)N}{(N-1)^2}.$$

Since λ^2 is a monotonically increasing function of λ in $[0, 1]$, we conclude from (6.3) that $g(\lambda)$ is a monotonically decreasing function of λ in $[0, 1]$ and

$$(6.6) \quad g(0) = \infty,$$

$$\begin{aligned}
 g(1) &= 1 + \frac{1}{2} \cdot \frac{2}{N-1} + \frac{1}{N-2} \cdot \frac{(N-2)N}{(N-1)^2} \\
 &= \left(\frac{N}{N-1}\right)^2.
 \end{aligned}$$

Write

$$(6.7) \quad \psi(\lambda) = \int g(\lambda) \lambda^{-1};$$

then $\psi(\lambda)$ is a monotonically increasing function of λ in $[0, 1]$ with

$$(6.8) \quad \psi(0) = 0 \quad \text{and} \quad \psi(1) = \left(1 - \frac{1}{N}\right)^2.$$

7. Bounds on the integral I_N .

From (5.4) and (6.7)

$$I_N = 2 \int_0^1 \int \psi(\lambda) \lambda^{(N-2)/2} h(\lambda) d\lambda.$$

Now $\psi(\lambda)$ can be written as

$$(7.1) \quad \psi(\lambda) = 2\lambda^2 \left(\frac{1-\lambda^{2N-2}}{1-\lambda^{2N}}\right) \lambda^{-1} + \frac{N\lambda^2(1-\lambda^{2N-4})}{(N-2)(1-\lambda^{2N})} \lambda^{-1}.$$

Hence

$$(7.2) \quad I_N = 2^{N/2} \int_0^1 \lambda^{N-2} \lambda^{-1} + \frac{N\lambda^2(1-\lambda^{2N-4})}{(N-2)(1-\lambda^{2N})} \lambda^{-(N-2)/2} \left(\frac{1-\lambda^{2N-2}}{1-\lambda^{2N}}\right)^{N-2} d\lambda.$$

$$\int \frac{1}{(1-\lambda^2)^2} - \frac{N^2\lambda^{2N-2}}{(1-\lambda^{2N})^2} \lambda^{1/2} d\lambda.$$

Put

$$\lambda = e^{-x/N}$$

$$d\lambda = -\frac{1}{N} e^{-x/N} dx$$

then

$$(7.3) \quad I_N = \frac{2^{N/2}}{N} \int_0^{\infty} \int \frac{\sinh(1-\frac{1}{N})x}{\sinh x} \int^{N-2} \int^{1+} \frac{N \sinh(1-\frac{2}{N})x}{(N-2) \sinh x} \int^{(N-2)/2}$$

$$\int \frac{1}{4 \sinh^2 \frac{x}{N}} - \frac{N^2}{4 \sinh^2 x} \int^{1/2} dx$$

$$= \frac{2^{(N-2)/2}}{N} \int_0^{\infty} \frac{(\cosh \frac{x}{N} - \coth x \sinh \frac{x}{N})^{N-2} \int^{\cosh^2 \frac{x}{N} - N^2 \operatorname{cosech}^2 x} \int^{1/2} dx}{\int^{1+} \frac{N}{N-2} (\cosh \frac{2x}{N} - \coth x \sinh \frac{2x}{N}) \int^{(N-2)/2}}$$

Now

$$\cosh \frac{x}{N} = 1 + \frac{x^2}{2!N^2} + \frac{x^4}{4!N^4} + \dots$$

$$\sinh \frac{x}{N} = \frac{x}{N} + \frac{x^3}{3!N^3} + \dots$$

for every x , while for $|x| < \pi$,

$$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + \dots + \frac{(-1)^{n-1} 2^{2n}}{(2n)!} B_n x^{2n-1} + \dots$$

where B_n is the n th Bernoulli number. Therefore, for $|x| < \pi$,

$$\cosh \frac{x}{N} - \coth x \sinh \frac{x}{N} = 1 - \frac{1}{N} \left(1 + \frac{x^2}{3} - \frac{x^4}{45} + \frac{2x^6}{945} + \dots \right) + \frac{x^2}{2N^2} + O(N^{-3})$$

and

$$(N-2) \log(\cosh \frac{x}{N} - \coth x \sinh \frac{x}{N}) = - \left(1 + \frac{x^2}{3} - \frac{x^4}{45} + \dots \right) + \frac{1}{N} \left(\frac{3}{2} + \frac{5x^2}{6} \right)$$

$$- \frac{7}{90} x^4 + \frac{2}{189} x^6 + \dots) + O(N^{-2})$$

and finally

$$(7.4) \left(\cosh \frac{x}{N} - \coth x \sinh \frac{x}{N} \right)^{N-2} = \exp \left[- \left(1 + \frac{x^2}{3} - \frac{x^4}{45} + \dots \right) \right. \\ \left. + \frac{1}{N} \left(\frac{3}{2} + \frac{5x^2}{6} + \dots \right) + O(N^{-2}) \right] \quad 7.$$

Further

$$1 + \frac{N}{N-2} \left(\cosh \frac{2x}{N} - \coth x \sinh \frac{2x}{N} \right) = 2 \left[1 - \frac{1}{N} \left(\frac{x^2}{3} - \frac{x^4}{45} + \frac{2x^6}{945} + \dots \right) \right. \\ \left. + \frac{1}{N^2} \left(\frac{x^2}{3} + \frac{2x^4}{45} + \dots \right) + O(N^{-3}) \right] \quad 7;$$

and

$$- \frac{N-2}{2} \log \left[1 + \frac{N}{N-2} \left(\cosh \frac{2x}{N} - \coth x \sinh \frac{2x}{N} \right) \right] \quad 7 \\ = - \frac{N-2}{2} \log 2 + \frac{1}{2} \left(\frac{x^2}{3} - \frac{x^4}{45} + \frac{2x^6}{945} + \dots \right) - \frac{1}{4N} \left(2x^2 - \frac{x^4}{9} + \frac{2x^6}{135} \right. \\ \left. + \dots \right) + O(N^{-2});$$

and so

$$(7.5) \left[1 + \frac{N}{N-2} \left(\cosh \frac{2x}{N} - \coth x \sinh \frac{2x}{N} \right) \right]^{-(N-2)/2} \\ = 2^{-(N-2)/2} \exp \left[- \frac{1}{2} \left(\frac{x^2}{3} - \frac{x^4}{45} + \frac{2x^6}{945} + \dots \right) - \frac{1}{N} \left(\frac{x^2}{2} - \frac{x^4}{36} + \dots \right) + O(N^{-2}) \right] \quad 7.$$

Finally

$$(7.6) \left[\psi(e^{-x/N}) \right]^{(N-2)/2} = e^{-1} \exp \left[- \frac{x^2}{6} + \frac{x^4}{90} - \frac{x^6}{945} + \dots \right. \\ \left. + \frac{3}{2N} \left(1 + \frac{2x^2}{9} - \frac{x^4}{30} + \dots \right) + O(N^{-2}) \right] \quad 7$$

the expression being valid for $|x| < \pi$. Also

$$(7.7) \int_{-\infty}^{\infty} \operatorname{cosech}^2 \frac{x}{N} - N^2 \operatorname{cosech}^2 x \cdot e^{1/2} = \frac{N}{\sqrt{3}} \int_{-1}^1 \left(1 - \frac{x^2}{10} + \frac{137}{12600} x^4 + \dots \right) dx + o(N^{-2}).$$

We split the range of the integral I_N into the ranges $\int_{-1,1}$ and $\int_{1,\infty}$. Denoting the integral from 0 to 1 by I_{N1} , we have from (7.4) - (7.7) to order N^{-1} ,

$$\begin{aligned} (7.8) \quad I_{N1} &= \frac{e^{-1}}{\sqrt{3}} \int_0^1 \exp \left(-\frac{x^2}{6} + \frac{x^4}{90} - \frac{x^6}{945} + \dots \right) dx \\ &\quad \int_{-1}^1 \left(1 - \frac{x^2}{10} + \frac{137}{12600} x^4 + \dots \right) dx \\ &= \frac{e^{-1}}{\sqrt{3}} \int_0^1 \left(1 - \frac{x^2}{6} + \frac{x^4}{40} + \dots \right) \left(1 - \frac{x^2}{10} + \frac{137x^4}{12600} + \dots \right) dx \\ &= \frac{e^{-1}}{\sqrt{3}} \int_0^1 \left(1 - \frac{x^2}{6} - \frac{x^2}{10} + \frac{x^4}{40} + \frac{x^4}{60} + \frac{137}{12600} x^4 + \dots \right) dx \\ &= \frac{e^{-1}}{\sqrt{3}} \left(1 - \frac{1}{18} - \frac{1}{30} + \frac{1}{200} + \frac{1}{300} + \frac{137}{63000} - \dots \right) \\ &= \frac{e^{-1}}{\sqrt{3}} (.9216) = 0.196. \end{aligned}$$

Hence, to $O(N^{-1})$,

$$I_N = 0.196 + 2^{N/2} \int_0^{e^{-1/N}} \lambda^{N-2} \left(\frac{1-\lambda^{2N-2}}{1-\lambda^{2N}} \right)^{N-2} \int_{-1}^1 \frac{N \lambda^2 (1-\lambda^{2N-4})}{(N-2)(1-\lambda^{2N})} dx^{-(N-2)/2}$$

$$\int \frac{1}{(1-\lambda^2)^2} - \frac{N^2 \lambda^{2N-2}}{(1-\lambda^{2N})^2} \lambda^{1/2} d\lambda .$$

Denote the second term on the right hand side by J and substitute

$$\lambda = y^{1/2}$$

$$d\lambda = \frac{1}{2} y^{-1/2} dy ,$$

so that

$$(7.9) \quad J = 2^{(N-2)/2} \int_0^{e^{-2/N}} \frac{y^{\frac{1}{2}(N-3)}}{(1-y)^{\frac{1}{2}(N-3)}} \left(\frac{1-y^{N-1}}{1-y^N} \right)^{N-2} \int_0^{1 + \frac{Ny(1-y^{N-2})}{(N-2)(1-y^N)}} \lambda^{-(N-2)/2} d\lambda$$

$$\int_0^{1 - \frac{N^2 y^{N-1} (1-y)^2}{(1-y^N)^2}} \lambda^{1/2} dy .$$

The result (7.8) is sufficient to prove that the coefficient of $(1-r_0)^{(N-2)/2}$ in the expansion of $P(r_1^* \geq r_0)$ is non zero. We now proceed to set bounds on J . Now, for $0 \leq y < 1$,

$$(7.10) \quad (1-y^{N-1})^{N-2} (1-y^N)^{-(N-2)} = \int_0^{1 - (N-2)y^{N-1} + \binom{N-2}{2} y^{2N-2} - \binom{N-2}{3} y^{3N-3} + \dots} \lambda^{-(N-2)/2} d\lambda$$

$$\int_0^{1 + (N-2)y^N + \binom{N-1}{2} y^{2N} + \binom{N}{3} y^{3N} + \dots} \lambda^{1/2} dy$$

$$= 1 - (N-2)y^{N-1}(1-y) + \left\{ \binom{N-2}{2} y^{2N-2} - (N-2)^2 y^{2N-1} + \binom{N-1}{2} y^{2N} \right\}$$

$$- \left\{ \binom{N-2}{3} y^{3N-3} - \frac{(N-2)^2 (N-3)}{2!} y^{3N-2} + \frac{(N-2)^2 (N-1)}{2} y^{3N-1} \right.$$

$$\left. - \binom{N}{3} y^{3N} \right\} + \dots$$

and

$$(7.11) \left\{ 1 - \frac{N^2 y^{N-1} (1-y)^2}{(1-y^N)^2} \right\}^{1/2} = 1 - \frac{N^2 y^{N-1} (1-y)^2}{2(1-y^N)^2} - \frac{N^4 y^{2N-2} (1-y)^4}{8(1-y^N)^4} - \dots$$

Furthermore,

$$\begin{aligned} \log \sqrt{1 + \frac{Ny(1-y^{N-2})}{(N-2)(1-y^N)}} - \frac{1}{2}(N-2) \log(1+y) &= -\frac{1}{2}(N-2) \log(1+y) - \frac{N-2}{2} \log \sqrt{1 + \frac{2y}{(N-2)(1+y)}} \\ &\quad - \frac{Ny^{N-1}(1-y)}{(N-2)(1-y^N)} - \frac{1}{2} \\ &= -\frac{N-2}{2} \log(1+y) - \frac{y}{1+y} + \frac{Ny^{N-1}(1-y)}{2(1-y^N)} + o(N^{-1}) \end{aligned}$$

and hence

$$\frac{e^{-y/(1+y)}}{(1+y)^{(N-2)/2}} < \sqrt{1 + \frac{Ny(1-y^{N-2})}{(N-2)(1-y^N)}} - \frac{(N-2)}{2} < \frac{1}{(1+y)^{(N-2)/2}}.$$

Therefore

$$(7.12) \quad \frac{e^{-1/2}}{(1+y)^{(N-2)/2}} < \sqrt{1 + \frac{Ny(1-y^{N-2})}{(N-2)(1-y^N)}} - \frac{(N-2)}{2} < \frac{1}{(1+y)^{(N-2)/2}}.$$

Using (7.10) - (7.12) in (7.9), we obtain

$$(7.13) \quad e^{-1/2} Q < J < Q,$$

where

$$(7.14) \quad Q = 2^{\frac{1}{2}(N-2)} \int_0^{e^{-2/N}} \frac{y^{(N-3)/2} dy}{(1-y)(1+y)^{(N-2)/2}}$$

$$\begin{aligned}
& -(N-2) \int_0^{e^{-2/N}} \frac{y^{(3N-5)/2} dy}{(1+y)^{(N-2)/2}} - \frac{N^2}{2} \int_0^{e^{-2/N}} \frac{y^{(3N-5)/2} (1-y) dy}{(1-y)^N (1+y)^{(N-2)/2}} \\
& + \int_0^{e^{-2/N}} \frac{\binom{N-2}{2} y^{(5N-7)/2} - (N-2)^2 y^{(5N-5)/2} + \binom{N-1}{2} y^{(5N-3)/2} dy}{(1-y)(1+y)^{(N-2)/2}} \\
& + \frac{N^2(N-2)}{2} \int_0^{e^{-2/N}} \frac{(1-y) \{ y^{(5N-7)/2} - y^{(5N-5)/2} \} dy}{(1-y)^N (1+y)^{(N-2)/2}} \\
& - \frac{N^4}{8} \int_0^{e^{-2/N}} \frac{y^{(5N-7)/2} (1-y)^3 dy}{(1-y)^N (1+y)^{(N-2)/2}} + \dots \quad 7.
\end{aligned}$$

We observe that we have to evaluate integrals of type

$$(7.15) \quad M(p, q, e^{-2/N}) = \int_0^{e^{-2/N}} \frac{y^p}{(1+y)^q} dy$$

and

$$(7.16) \quad L(p, q, e^{-2/N}) = \int_0^{e^{-2/N}} \frac{y^p}{(1+y)^q (1-y)} dy,$$

where $q = (N-2)/2$, $p = sq+b$, $s > 0$ and $b = O(1)$. Substituting $y = e^{-2/N} z$ and expanding $(1+e^{-2/N} z)^{-q}$ in the powers of $(1-z)$ and integrating term by term, we obtain

$$\begin{aligned}
 M(p, q, e^{-2/N}) &= \frac{e^{-2(p+1)/N}}{(1+e^{-2/N})^q} \sum_{r=0}^{\infty} \frac{\Gamma(q+r)}{\Gamma(q)} \frac{\Gamma(p+1)}{\Gamma(p+r+2)} \left(\frac{1}{1+e^{2/N}} \right)^r \\
 &= \frac{e^{-2(p+1)/N}}{(p+1)(1+e^{-2/N})^q} F(1, q, p+2, \frac{1}{1+e^{2/N}});
 \end{aligned}$$

and

$$L(p, q, e^{-2/N}) = \frac{e^{-2(p+1)/N}}{(1+e^{-2/N})^q} \sum_{k=0}^{\infty} \frac{e^{-2k/N}}{p+k+1} F(1, q, p+k+2, \frac{1}{1+e^{2/N}}).$$

Now if $s > 1$ and $c, t > 0$,

$$\begin{aligned}
 F(1, t, st+c, x) &= 1 + \frac{t}{ts+c} x + \frac{t(t+1)}{(ts+c)(ts+c+1)} x^2 + \dots \\
 &> 1 + \frac{x}{s} + \frac{x^2}{s^2} + \frac{x^3}{s^3} + \dots = \frac{1}{1-x/s}
 \end{aligned}$$

and

$$F(1, t, st+c, x) < 1 + \frac{t}{ts+c} x + \frac{t(t+1)}{(ts+c)(ts+c+1)} x^2 \left\{ 1+x+x^2 + \dots \right\}.$$

If t is large, then the right hand side of the last inequality =

$$1 + \frac{x}{s} + \frac{x^2}{s^2(1-x)} + O\left(\frac{1}{st}\right). \text{ Now, for}$$

$$x = (1+e^{2/N})^{-1} = \frac{1}{2} + O(N^{-1})$$

we have

$$(7.17) \quad \frac{2s}{2s-1} < F(1, t, ts+e, \frac{1}{1+e^{2/N}}) < \frac{1+s+2s^2}{2s^2}.$$

Therefore, taking $q = \frac{N}{2} - 1$ and $p = sq + b$, to $O(N^{-1})$, we find that

$$M(p, q, e^{-2/N}) = \frac{e^{-s+1/2}}{2^{(N-2)/2} sq} F(1, q, sq+b+2, \frac{1}{1+e^{2/N}})$$

and

$$(7.18) \quad \frac{e^{-s+1/2}}{2^{(N-4)/2} q(2s-1)} < M(p, q, e^{-2/N}) < \frac{(1+s+2s^2)e^{-s+1/2}}{2^{N/2} s^3 q}.$$

Let $\lfloor q \rfloor$ denote the largest integer in q ; also, let q_1 be the largest integer in $\frac{q}{10}$, q_2 in $\frac{2q}{10}$ and so on. Then

$$(7.19) \quad \sum_{k=0}^{\infty} \frac{e^{-2k/N}}{p+k+1} F(1, q, p+k+2, \frac{1}{1+e^{2/N}})$$

$$< \sum_{k=0}^{q_1-1} \frac{e^{-2k/N}}{p+1} F(1, q, p+2, \frac{1}{1+e^{2/N}}) + \sum_{k=q_1}^{q_2-1} \frac{e^{-2k/N}}{p+q_1+1} F(1, q, p+q_1+2, \frac{1}{1+e^{2/N}})$$

$$+ \dots + \sum_{k=q_2}^{\lfloor q \rfloor - 1} \frac{e^{-2k/N}}{p+q_2+1} F(1, q, p+q_2+2, \frac{1}{1+e^{2/N}})$$

$$+ \sum_{k=\lfloor q \rfloor}^{\lfloor 2q \rfloor - 1} \frac{e^{-2k/N}}{p+\lfloor q \rfloor+1} F(1, q, p+\lfloor q \rfloor+2, \frac{1}{1+e^{2/N}})$$

$$+ \sum_{k=\lfloor 2q \rfloor}^{\lfloor 3q \rfloor - 1} \frac{e^{-2k/N}}{p+\lfloor 2q \rfloor+1} F(1, q, p+\lfloor 2q \rfloor+2, \frac{1}{1+e^{2/N}}) + \dots$$

$$< \left(\frac{1+s+2s^2}{2s^3 q} \right) \left(\frac{1-e^{-\frac{2q}{10N}}}{1-e^{-2/N}} \right) + \frac{e^{-\frac{2q}{10N}} (1-e^{-\frac{2q}{10N}})}{(s+.1)q(1-e^{-2/N})} \left(\frac{1+(s+.1)+2(s+.1)^2}{2(s+.1)^2} \right)$$

$$+ \frac{e^{-\frac{18q}{10N}} (1-e^{-\frac{2q}{10N}})}{(s+.9)q(1-e^{-2/N})} \left(\frac{1+(s+.9)+2(s+.9)^2}{2(s+.9)^2} \right)$$

$$+ \frac{e^{-2q/N} (1-e^{-2q/N})}{(s+1)q(1-e^{-2/N})} \left(\frac{1+(s+1)+2(s+1)^2}{2(s+1)^2} \right)$$

$$+ \frac{e^{-4q/N}}{(s+2)q(1-e^{-2/N})} \frac{1+(s+2)+2(s+2)^2}{2(s+2)^2} + \dots$$

$$< \frac{1-e^{-.1}}{2} \int \frac{1+s+2s^2}{s^3} + \frac{e^{-.1}}{(s+.1)^3} \{ 1+(s+.1)+2(s+.1)^2 \}$$

$$+ \frac{e^{-.2}}{(s+.2)^3} \{ 1.2+s+(s+.2)^2 \} + \dots$$

$$+ \frac{e^{-.9}}{(s+.9)^3} \{ 1.9+s+2(s+.9)^2 \} \quad]$$

$$+ \frac{1-e^{-1}}{e} \int \frac{e^{-1}}{(s+1)^3} \{ s+2+2(s+1)^2 \} + \frac{e^{-2}}{(s+2)^3} \{ s+3+2(s+2)^2 \}$$

$$+ \dots + \frac{e^{-r}}{(s+r)^3} \{ s+r+1+2(s+r)^2 \} + \dots \quad]_7.$$

Similarly

$$(7.20) \quad \sum_{k=0}^{\infty} \frac{e^{-2k/N}}{p+k+1} F(1, q, p+k+2, \frac{1}{1+e^{2/N}})$$

$$> (1-e^{-.1}) \left[\frac{2s}{2s-1} + \frac{e^{-.1}(2s+.2)}{(s+.2)(2s+.2-1)} + \frac{e^{-.2}(2s+.4)}{(s+.3)(2s+.4-1)} \right.$$

$$\left. + \dots + \frac{e^{-.9}(2s+1.8)}{(s+1)(2s+1.8-1)} \right] \quad _7$$

$$+ 2(1-e^{-1}) \left[\frac{(s+1)e^{-1}}{(s+2)(2s+1)} + \frac{e^{-2}(s+2)}{(s+3)(2s+3)} + \frac{e^{-3}(s+3)}{(s+4)(2s+5)} + \dots \right] _7.$$

For $s = 1$, $p \sim q = \frac{N}{2} - 1$, we have

$$(7.21) \quad L(p, q, e^{-2/N}) < \frac{e^{-1/2}}{2^{(N-2)/2}} \left[\frac{1-e^{-.1}}{2} \left\{ 4 + \frac{e^{-.1} \{2.1+2(1.1)^2\}}{(1.1)^3} \right. \right.$$

$$\left. + \frac{e^{-.2} \{2.2+2(1.2)^2\}}{(1.2)^3} + \dots + \frac{e^{-.9} \{2.9+2(1.9)^2\}}{(1.9)^3} \right\}$$

$$+ \frac{1-e^{-1}}{2} \left\{ e^{-1} \cdot \frac{11}{8} + e^{-2} \cdot \frac{22}{27} + e^{-3} \cdot \frac{37}{64} + e^{-4} \cdot \frac{56}{125} + \dots \right\} _7$$

$$= \frac{e^{-1/2}(1.036)}{2^{(N-2)/2}}$$

hence,

$$L(p, q, e^{-2/N}) < \frac{.629}{2^{(N-2)/2}}.$$

A similar calculation shows that for the above values of s , p and q ,

$$(7.22) \quad L(p, q, e^{-2/N}) > \frac{.542}{2^{(N-2)/2}} .$$

Denoting the integrals on the right-hand side of (7.14) by Q_1 , Q_2 etc. as they occur in order, and neglecting the sign, we see that

$$Q_1 = 2^{(N-2)/2} L\left(\frac{N-3}{2}, \frac{N-2}{2}, e^{-2/N}\right) .$$

Hence from (7.21) and (7.22)

$$(7.23) \quad 0.542 < Q_1 < 0.629 .$$

We may note here that $J < Q_1$ and hence

$$I_N = .196 + J < .196 + .629 = .825 .$$

The upper bound can be reduced further by evaluating Q_2 , Q_3 , ... and we proceed to do this. Now,

$$Q_2 = 2^{(N-2)/2} M\left(\frac{3N-5}{2}, \frac{N-2}{2}, e^{-2/N}\right) .$$

Hence, from (7.18), putting $s = 3$ and $q \sim \frac{N}{2}$, we get

$$\frac{4}{5} e^{-5/2} < Q_2 < \frac{22}{27} e^{-5/2} ,$$

or

$$0.065 < Q_2 < 0.066 .$$

Similar evaluations give the following results:

$$0.101 < Q_3 < 0.103$$

$$Q_4 = O\left(\frac{e^{-5}}{N}\right)$$

$$0.028 < Q_5 < 0.029$$

$$0.0055 < Q_6 < 0.0056 .$$

Now

$$\begin{aligned} Q &= Q_1 - Q_2 - Q_3 + Q_4 + Q_5 - Q_6 - \dots \\ &< .629 - .065 - .101 + .029 - .005 = .487 \end{aligned}$$

and

$$Q > .542 - .066 - .103 + .028 - .006 = .395 .$$

The latter terms diminish very rapidly and do not affect to the second decimal place. Now,

$$e^{-1/2} Q < J < Q ,$$

that is,

$$.239 < J < .487 .$$

Since $I_N = .196 + J$, we have

$$(7.24) \quad .435 < I_N < .683 .$$

These calculations are valid to $O(N^{-1})$ and to two decimal places.

Hence the first term, P_0 , in the expansion of $P(r^* \geq r_0)$ about $r^* = 1$ is given by

$$(7.25) \quad P_0 = \frac{I_N}{\pi} (1-r_0)^{(N-2)/2}$$

where I_N is bounded as in (7.24).

8. Bounds on $P(r^* \geq r_0)$.

From (4.9), we have

$$\sin \theta = (1-r^*)^{1/2} [\psi(\lambda)]^{1/2}.$$

Hence for a given value of r_0 , the maximum value of $\theta_0(\lambda)$, say θ'_0 , is given by

$$(8.1) \quad \begin{aligned} \sin \theta'_0 &= (1-r_0)^{1/2} \left(\frac{N-1}{N}\right) \\ &\approx (1-r_0)^{1/2} \end{aligned}$$

or

$$(8.2) \quad \cos \theta'_0 \approx r_0.$$

Furthermore, from (5.2), we have to $O(N^{-2})$,

$$\frac{\sin^{N-2} \theta_0}{N-2} < \int_0^{\theta_0} \sin^{N-2} \theta \, d\theta < \frac{\sin^{N-2} \theta_0}{(N-2) \cos \theta_0} < \frac{\sin^{N-2} \theta_0}{(N-2) \cos \theta'_0}.$$

Therefore,

$$\frac{1}{\pi} \int_0^{\infty} \sin^{N-2} \theta_0 \, ds(\lambda) < P(r^* \geq r_0) < \frac{1}{\pi \cos \theta'_0} \int_0^{\infty} \sin^{N-2} \theta_0 \, ds(\lambda)$$

or

$$\frac{I_N(1-r_0)^{(N-2)/2}}{\pi} < P(r^* \geq r_0) < \frac{I_N(1-r_0)^{(N-2)/2}}{\pi r_0}.$$

From (7.24) it follows that

$$(8.3) \quad \frac{0.435}{\pi} (1-r_0)^{(N-2)/2} < P(r^* \geq r_0) < \frac{0.683}{\pi r_0} (1-r_0)^{(N-2)/2}.$$

We remember that the results apply to such values of r_0

for which $\sin^3 \theta_0$ is negligible in comparison to 1. If, for instance, we consider 0.01 as negligible, we may take the maximum value of $\sin \theta_0 = .21$, i.e., $r_0 \geq .978$. In fact the bounds may be valid even for much lower values of r_0 , as (8.2) suggests that the neglected terms in (3.6) may be of $O(\sin^4 \theta)$ rather than $O(\sin^3 \theta)$.

A similar argument shows that $r^* = -1$ if and only if

$$x_i = \lambda x_{i-1}, \quad i=2,3,\dots,N \text{ and } \lambda < 0, \quad x_1 \neq 0.$$

If $-1 < r'_0 < 0$ and $|r'_0|$ is close to 1, it follows from (4.9) that near $r^* = -1$,

$$\sin \theta = (1+r^*)^{1/2} \int \psi(\lambda) \gamma^{1/2}$$

and if the integral corresponding to I_N be denoted by I'_N ,

$$I'_N = \int_{-\infty}^0 \int \psi(\lambda) \gamma^{(N-2)/2} h(\lambda) d\lambda.$$

A change of variable from λ to $-\lambda$ shows that $I'_N = I_N$. Therefore, the first term, P'_0 , in the expansion of $P(r_1^* \leq r'_0)$ about $r^* = -1$ is

$$(8.4) \quad P'_0 = \frac{I_N}{\pi} (1 + r'_0)^{(N-2)/2},$$

and

$$(8.5) \quad \frac{0.435}{\pi} (1+r'_0)^{(N-2)/2} < P(r^* \leq r'_0) < \frac{0.683}{\pi |r'_0|} (1+r'_0)^{(N-2)/2}.$$

It is instructive to compare (7.25) with the probability of an ordinary correlation coefficient r , based on a sample of size $(N-1)$,

exceeding a given value r_0 close to 1.

If the population means are known the frequency function of r for a sample of size $N-1$ is given by

$$\begin{aligned}
 f(r) &= \frac{\Gamma(\frac{N-1}{2})}{\sqrt{\pi} \Gamma(\frac{N-2}{2})} (1-r^2)^{(N-4)/2} \\
 &= \frac{\Gamma(\frac{N-1}{2})}{\sqrt{\pi} \Gamma(\frac{N-2}{2})} (1-r)^{(N-4)/2} \{2 - (1-r)\}^{(N-4)/2} \\
 &= \frac{2^{\frac{N-4}{2}} \Gamma(\frac{N-1}{2})}{\sqrt{\pi} \Gamma(\frac{N-2}{2})} (1-r)^{(N-4)/2} \left\{ 1 - \frac{N-4}{2} \left(\frac{1-r}{2} \right) + \dots \right\} .
 \end{aligned}$$

Therefore the first term in the expansion of $P(r \geq r_0)$ is

$$\begin{aligned}
 P(r \geq r_0) &\simeq \frac{2^{(N-4)/2} \Gamma(\frac{N-1}{2})}{\sqrt{\pi} \Gamma(\frac{N-2}{2})} \cdot \frac{2(1-r_0)^{(N-2)/2}}{N-2} \\
 &\simeq \frac{2^{(N-2)/2} N^{1/2} (1-r_0)^{(N-2)/2}}{\sqrt{2\pi} N} = \frac{2^{(N-3)/2} (1-r_0)^{(N-2)/2}}{\sqrt{\pi N}}
 \end{aligned}$$

Hence

$$(8.6) \quad \frac{P_0}{P} \simeq \frac{I_N / \sqrt{\pi N}}{\pi 2^{(N-3)/2}} = \left(\frac{N}{\pi}\right)^{1/2} \frac{I_N}{2^{(N-3)/2}} .$$

This ratio tends to zero as $N \rightarrow \infty$.

CHAPTER III

JACOBI POLYNOMIALS AND DISTRIBUTIONS OF SERIAL CORRELATION COEFFICIENTS

1. Introduction.

It was pointed out in section 1 of Chapter 2 that a statistic closely related to r^* is given by

$$(1.1) \quad r_1 = \frac{\sum_{i=1}^{N-1} x_i x_{i+1}}{\sum_{i=1}^N x_i^2},$$

which has the advantage of being a ratio of two quadratic forms. In this chapter the approximate distribution of r_1 will be obtained in terms of a beta distribution and a series of Jacobi polynomials.

Denoting by q_1 the numerator of the right hand side of (1.1), the characteristic roots of the matrix of q_1 will be utilized to calculate the cumulants and the moments of q_1 . The asymptotic normality of q_1 and of r_1 will be proved separately. Following a discussion about the properties of Jacobi polynomials a method will be developed to obtain an approximate frequency function of the ratio of two quadratic forms. Of course this method has limited applicability but suggests the possibility of using orthogonal polynomials in approximating the distributions of certain statistics whose exact distribution cannot be obtained expressly in terms of known functions and integrals. It can also be used for calculating percentile points, even when the exact distribution is available, but is tedious to handle.

2. Jacobi polynomials.

Before obtaining the distribution of r_1 it is necessary to state some properties of Jacobi polynomials. These properties are given below without proof. For the proofs a reference can be made to Orthogonal Polynomials, by G.Szegö [47].

Write

$$(2.1) \quad f(x; \alpha, \beta) = \frac{(1-x)^\alpha (1+x)^\beta}{2^{\alpha+\beta+1} B(\alpha+1, \beta+1)}, \quad \alpha, \beta > -1, -1 \leq x \leq 1.$$

The Jacobi polynomial $P_r^{(\alpha, \beta)}(x)$ is defined as the orthogonal polynomial of degree r with weight function $f(x; \alpha, \beta)$. The normalization is achieved by the restriction

$$P_r^{(\alpha, \beta)}(1) = \binom{r+\alpha}{r}.$$

Explicitly

$$(2.2) \quad P_r^{(\alpha, \beta)}(x) = \sum_{m=0}^r \binom{r+\alpha}{m} \binom{r+\beta}{r-m} \left(\frac{x-1}{2}\right)^{r-m} \left(\frac{x+1}{2}\right)^m, \quad r=0, 1, 2, \dots$$

It can be shown that

$$\int_{-1}^1 P_r^{(\alpha, \beta)}(x) P_k^{(\alpha, \beta)}(x) f(x; \alpha, \beta) dx = \begin{cases} 0 & \text{if } r \neq k \\ 1 & \text{if } r = k = 0 \\ \frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+3)} & \text{if } r = k = 1 \\ \frac{1}{r!} \frac{(\alpha+1)(\alpha+2)\dots(\alpha+r)(\beta+1)(\beta+2)\dots(\beta+r)}{(\alpha+\beta+2)(\alpha+\beta+3)\dots(\alpha+\beta+r)(\alpha+\beta+2r+1)} & \text{if } r = k > 1 \end{cases}.$$

A very useful property is

$$\begin{aligned}
 (2.3) \quad & \frac{d^k}{dx^k} \frac{(1-x)^{\alpha+k} (1+x)^{\beta+k}}{2^{\alpha+\beta+2k+1} B(\alpha+k+1, \beta+k+1)} \\
 & = (-1)^k f(x; \alpha, \beta) \frac{B(\alpha+1, \beta+1)}{B(\alpha+k+1, \beta+k+1)} \frac{k!}{2^k} P_k^{(\alpha, \beta)}(x) .
 \end{aligned}$$

$P_r^{(\alpha, \beta)}(x)$ satisfies the differential equation.

$$(2.4) \quad (1-x^2) \frac{d^2 y}{dx^2} + [\beta - \alpha - (\alpha + \beta + 2)x] \frac{dy}{dx} + r(r + \alpha + \beta + 1)y = 0$$

and is related to a hypergeometric function by the equation

$$P_r^{(\alpha, \beta)}(x) = \binom{r+\alpha}{r} F(-r, r+\alpha+\beta+1, \alpha+1, (1-x)/2) .$$

Changing the variable from x to z by

$$x = 2z - 1$$

$$dx = 2dz$$

let $f(x; \alpha, \beta)dx$ be transformed into $g(z; \alpha, \beta)dz$ and $P_r^{(\alpha, \beta)}(x)$ to $Q_r^{(\alpha, \beta)}(z)$. (Note. $Q_r^{(\alpha, \beta)}(z)$ has no connection with the second solution of (2.4), a notation used by Szegő). Then

$$(2.5) \quad g(z; \alpha, \beta) = \frac{(1-z)^\alpha z^\beta}{B(\alpha+1, \beta+1)} ; \quad 0 \leq z \leq 1, \quad \alpha, \beta > -1 ,$$

and

$$(2.6) \quad Q_r^{(\alpha, \beta)}(z) = \sum_{m=0}^r \binom{r+\alpha}{m} \binom{r+\beta}{m} (z-1)^{r-m} z^m .$$

Equation (2.3) goes into

$$(2.7) \quad \frac{d^k}{dz^k} g(z; \alpha+k, \beta+k) = (-1)^k g(z; \alpha, \beta) \frac{B(\alpha+1, \beta+1)}{B(\alpha+k+1, \beta+k+1)} k! Q_k^{(\alpha, \beta)}(z) .$$

Now, consider the characteristic function of a variate x whose frequency function is $g(x; \alpha, \beta)$. We have

$$\begin{aligned}\phi_x(t) &= \int_0^1 e^{tx} g(x; \alpha, \beta) dx \\ &= \frac{1}{B(\alpha+1, \beta+1)} \int_0^1 \sum_{r=0}^{\infty} \frac{t^r}{r!} x^{\beta+r} (1-x)^{\alpha} dx.\end{aligned}$$

The partial sums of the series under the integral sign are dominated by the function $e^{|tx|}$ in $[0, 1]$ which is integrable over this range. Therefore, we can integrate term by term and thus obtain

$$\begin{aligned}(2.8) \quad \phi_x(t) &= \sum_{r=0}^{\infty} \frac{\Gamma(\beta+1+r)\Gamma(\alpha+\beta+2)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+2+r)} \frac{t^r}{r!} \\ &= F(\beta+1, \alpha+\beta+2, t),\end{aligned}$$

where $F(\beta+1, \alpha+\beta+2, t)$ is the confluent hypergeometric function whose Taylor series expansion about $t=0$ is given by (2.8). The series (2.8) is absolutely convergent and since $\alpha, \beta > -1$,

$$\begin{aligned}1 + \frac{(\beta+1)}{(\beta+\alpha+2)} \frac{|t|}{1!} + \frac{(\beta+1)(\beta+2)}{(\beta+\alpha+2)(\beta+\alpha+3)} \frac{|t|^2}{2!} + \dots \\ < 1 + \frac{|t|}{1!} + \frac{|t|^2}{2!} + \dots = e^{|t|}.\end{aligned}$$

Hence, $F(\beta+1, \alpha+\beta+2, t)$ is an entire function and so

$$(2.9) \quad \phi_x(t) = F(\beta+1, \alpha+\beta+2, t).$$

From the inversion theorem of characteristic functions, if

$$0 < x < 1,$$

$$(2.10) \quad \frac{1}{2\pi i} \oint e^{-tx} F(\beta+1, \alpha+\beta+2, t) dt = \frac{(1-x)^\alpha x^\beta}{B(\alpha+1, \beta+1)}$$

where $\oint \dots dt$ stands for $\lim_{c \rightarrow \infty} \int_{-ic}^{ic} \dots dt$, and i is imaginary unit.

Replacing α and β by $\alpha+k$ and $\beta+k$ in (2.10) and differentiating both sides k times with respect to x we obtain

$$\begin{aligned} (2.11) \quad & \frac{1}{2\pi i} \oint t^k e^{-tx} F(\beta+k+1, \beta+\alpha+2k+2, t) dt \\ &= (-1)^k \frac{d^k}{dx^k} g(x; \alpha, \beta) \\ &= g(x; \alpha, \beta) \frac{B(\alpha+1, \beta+1)}{B(\alpha+k+1, \beta+k+1)} k! Q_k^{(\alpha, \beta)}(x) . \end{aligned}$$

Similarly, the characteristic function of a variate y with frequency function $f(y; \alpha, \beta)$ is $e^{-t} F(\beta+1, \alpha+\beta+2, 2t)$ and for $-1 < y < 1$,

$$(2.12) \quad \frac{1}{2\pi i} \oint e^{-ty} e^{-t} F(\beta+1, \alpha+\beta+2, 2t) dt = \frac{(1-y)^\alpha (1+y)^\beta}{2^{\alpha+\beta+1} B(\alpha+1, \beta+1)}$$

and

$$\begin{aligned} (2.13) \quad & \frac{1}{2\pi i} \oint t^k e^{-t(1+y)} F(\beta+k+1, \alpha+\beta+2k+2, 2t) dt \\ &= f(y; \alpha, \beta) \frac{B(\alpha+1, \beta+1)}{B(\alpha+k+1, \beta+k+1)} \frac{k!}{2^k} P_k^{(\alpha, \beta)}(y) . \end{aligned}$$

3. Distributions of certain statistics as series of Jacobi polynomials.

Let R be a statistic whose range is contained in $[0, 1]$.

Let either its characteristic function or its exact moments up to a certain order and a few dominating terms in all its moments be known.

Further, let it be possible to write its characteristic function, $\phi_R(t)$, in the form

$$\begin{aligned} (3.1) \quad \phi_R(t) &= a_0 F(\beta+1, \alpha+\beta+2, t) + a_1 t F(\beta+2, \alpha+\beta+4, t) + \dots \\ &\quad + a_k t^k F(\beta+k+1, \alpha+\beta+2k+2, t) + \dots \\ &= \sum_{k=0}^{\infty} a_k t^k F(\beta+k+1, \alpha+\beta+2k+2, t), \end{aligned}$$

where $\alpha, \beta > -1$. It is possible that $a_k = 0$ for all $k \geq r$, in which case the series will terminate. Since $|F(\beta+k+1, \alpha+\beta+2k+2, |t|)| \leq e^{|t|}$ for all $k \geq 0$, the series on the right hand side of (3.1) converges absolutely if

$$\lim_{k \rightarrow \infty} \left| \frac{a_k t}{a_{k-1}} \right| < 1;$$

and it converges absolutely for all values of t if

$$\lim_{k \rightarrow \infty} \frac{a_k}{a_{k-1}} = 0.$$

Suppose either the series (3.1) terminates or converges absolutely for all values of t . From the inversion theorem of characteristic functions the frequency function of R is given by

$$\begin{aligned}
 (3.2) \quad f(R) &= \frac{1}{2\pi i} \oint e^{-tR} \phi_R(t) dt \\
 &= \frac{1}{2\pi i} \oint e^{-tR} \sum_{k=0}^{\infty} a_k t^k F(\beta+k+1, \alpha+\beta+2k+2, t) dt.
 \end{aligned}$$

Let us formally carry out the term by term integration. From (2.11) we will then write, formally, the frequency function of R as

$$(3.3) \quad f(R) = g(R; \alpha, \beta) \left[a_0 + \sum_{k=1}^{\infty} a_k \frac{B(\alpha+1, \beta+1)}{B(\alpha+k+1, \beta+k+1)} k! Q_k^{(\alpha, \beta)}(R) \right]$$

in the range of R , and $f(R)=0$ otherwise. It will be seen that the terms of the expansions (3.1) and (3.3) correspond by means of the relation (2.11). This process is justified analytically in the case when partial sums in the integral (3.2) are dominated by an integrable function of t . However, in practical applications in most cases we are seldom interested in the convergence properties of our expansions. As Cramér states: "What we really want to know is whether a small number of terms - usually not more than two or three - suffice to give good approximation to $f(x)$ and $F(x)$ " [7, p. 224]. It will be seen later that by this method we obtain expansions for frequency functions and distribution functions of serial correlation coefficients in series which are asymptotic, and have some other desirable properties; for example, the moments of the fitted series agree to a high order of accuracy with the moments of the statistic under consideration.

Now the maximum value of $Q_k^{(\alpha, \beta)}(R)$ in the range $[0, 1]$ is

$$\max_{0 \leq R \leq 1} Q_k^{(\alpha, \beta)}(R) = \binom{k+q}{k} \text{ where } q = \max(\alpha, \beta).$$

Hence, by the ratio test, the series (3.3) converges for all values of R in $[0, 1]$ if

$$\lim_{k \rightarrow \infty} \left| \frac{4ka_k}{a_{k-1}} \right| < 1.$$

A similar procedure shows that if

$$(3.4) \quad \phi_R(t) = e^{-t} \sum_{k=0}^{\infty} a_k t^k F(\beta+k+1, \alpha+\beta+2k+2, 2t)$$

then we can formally develop the frequency function of R as

$$(3.5) \quad f(r) = \frac{(1-R)^\alpha (1+R)^\beta}{2^{\alpha+\beta+1} B(\alpha+1, \beta+1)} \sum_{k=0}^{\infty} a_k \frac{B(\alpha+1, \beta+1)}{B(\alpha+k+1, \beta+k+1)} \frac{k!}{2^k} P_k^{(\alpha, \beta)}(R).$$

This series converges if

$$\lim_{k \rightarrow \infty} \left| \frac{2ka_k}{a_{k-1}} \right| < 1.$$

The above results are summarized in the following theorems.

THEOREM 3.1. If R is a variate with range $[c_1, c_2]$ where $0 \leq c_1 \leq c_2 \leq 1$, and if the characteristic function of R , $\phi_R(t)$, can be written as (3.1) and the partial sums are dominated by a function $\xi(t)$ such that $e^{itR} \xi(t)$ is integrable over $(-\infty, \infty)$ then the frequency function of R is given by (3.3).

THEOREM 3.2. In theorem 3.1 replace

$$(i) \quad 0 \leq c_1 \leq c_2 \leq 1 \quad \text{by} \quad -1 \leq c_1 \leq c_2 \leq 1,$$

$$(ii) \quad (3.1) \text{ by } (3.4) \text{ and } (3.3) \text{ by } (3.5).$$

In the following sections we will obtain distributions of

serial correlation coefficients by the application of the method developed in this section.

4. Independence of r_1 and p .

Let $x_1, x_2, \dots, x_N$ be independent $N(0, 1)$ variates. Define

$$q_1 = \sum_{j=1}^{N-1} x_j x_{j+1} ,$$

$$p = \sum_{j=1}^N x_j^2 ,$$

and

$$r_1 = q_1/p , \text{ when } p \neq 0 .$$

THEOREM 4.1. r_1 is distributed independently of p .

PROOF. The distribution of $x_1, \dots, x_N$ is given by

$$\begin{aligned} dF(x_1, \dots, x_N) &= (2\pi)^{-N/2} e^{-\sum x_i^2/2} dx_1 \dots dx_N \\ &= (2\pi)^{-N/2} e^{-p/2} dx_1 \dots dx_N . \end{aligned}$$

Let

$$x_j = z_j p^{1/2} , \quad j = 1, 2, \dots, N, \text{ with } \sum_{j=1}^N z_j^2 = 1 .$$

The Jacobian of the transformation is [7, p. 385]

$$\left| \frac{\partial(x_1, \dots, x_N)}{\partial(p^{1/2}, z_1, \dots, z_{N-1})} \right| = \pm \frac{p^{(N-1)/2}}{(1 - z_1^2 - \dots - z_{N-1}^2)^{1/2}} .$$

Therefore the distribution of $p, z_1, \dots, z_{N-1}$ is given by

$$dF(p, z_1, \dots, z_{N-1}) =$$

$$\frac{1}{2^{N/2} \Gamma(\frac{N}{2})} e^{-p/2} p^{(N-2)/2} dp \frac{\Gamma(\frac{N}{2})}{\pi^{N/2}} \frac{dz_1 \dots dz_{N-1}}{(1-z_1^2 - \dots - z_{N-1}^2)^{1/2}}.$$

Hence the combined variable $(z_1, \dots, z_N)$ is distributed independently of p , as z_N can be expressed in terms of $z_1, \dots, z_{N-1}$ only. Now

$$r_1 = \sum_{i=1}^{N-1} \frac{x_i x_{i+1}}{p} = \sum_{i=1}^{N-1} z_i z_{i+1}$$

and is, therefore, distributed independently of p .

COROLLARY.
$$\mathcal{E} r_1^k = \frac{\mathcal{E} q_1^k}{\mathcal{E} p^k}.$$

5. Cumulants of q_1 .

Before proceeding to investigate the distribution of r_1 , we will investigate the properties of the distribution of q_1 .

As a preliminary, we will state a theorem about the cumulants of a general quadratic form in normally distributed variates.

THEOREM 5.1. Let $x' = (x_1, \dots, x_N)$ be distributed normally with mean vector zero and covariance matrix Σ , where Σ is a positive definite. Let $u = x' A x$, where x is the transpose of x' and A is a real symmetric matrix. If the characteristic roots of $A\Sigma$ are $c_1, \dots, c_N$ then

$$(5.1) \quad \kappa_{r:u} = 2^{r-1} (r-1)! \sum_{j=1}^N c_j^r$$

PROOF. The characteristic function of u , $\phi_u(t)$, is given by

$$\begin{aligned}\phi_u(t) &= (2\pi)^{-N/2} |\Sigma|^{-1/2} \int \exp\left[-\frac{1}{2}x' \Sigma^{-1}x + tx'Ax\right] dx \\ &= (2\pi)^{-N/2} |\Sigma|^{-1/2} \int \exp\left[-\frac{1}{2}x' (\Sigma^{-1} - 2tA)x\right] dx \\ &= |\Sigma|^{-1/2} |\Sigma^{-1} - 2tA|^{-1/2} \\ &= \prod_{j=1}^N (1 - 2tc_j)^{-1/2}.\end{aligned}$$

Therefore,

$$x_u(t) = -\frac{1}{2} \sum_{j=1}^N \log(1 - 2tc_j).$$

If by c_1 we denote the characteristic root of (4Σ) which is largest in magnitude, then for $|t| < \frac{1}{2|c_1|}$,

$$x_u(t) = \sum_{r=1}^{\infty} \frac{t^r}{r!} 2^{r-1}(r-1)! \sum_{j=1}^N c_j^r. \quad \text{Q.E.D.}$$

Note: If Σ is the identity matrix, $c_1, \dots, c_N$ will be the characteristic roots of A .

THEOREM 5.2. The characteristic roots of the matrix Q_1 of the form

$$Q_1 = \sum_{i=1}^{N-1} x_i x_{i+1} \quad \text{are}$$

$$(5.2) \quad \lambda_j = \cos \frac{j\pi}{N+1}, \quad j=1, 2, \dots, N.$$

PROOF. We have

$$(5.3) \quad Q_1 = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

Write

$$D_N = \begin{bmatrix} -\mu & 1 & 0 & 0 & \dots & 0 \\ 1 & -\mu & 1 & 0 & \dots & 0 \\ 0 & 1 & -\mu & 1 & \dots & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \dots & -\mu \end{bmatrix}$$

Then it can be easily shown that

$$(5.4) \quad D_N + \mu D_{N-1} + D_{N-2} = 0$$

Put

$$D_N = x^N,$$

so that

$$x^2 + \mu x + 1 = 0$$

or

$$x = \frac{-\mu \pm i(4-\mu^2)^{1/2}}{2}.$$

Let

$$\mu = 2 \cos \phi$$

so that

$$x = -e^{i\phi}, -e^{-i\phi}.$$

Now

$$D_1 = -\mu, \quad D_2 = \mu^2 - 1.$$

Introducing D_0 , which satisfies the difference equation for $N=2$, i.e.,

$$D_2 + \mu D_1 + D_0 = 0$$

we obtain

$$D_0 = 1.$$

The most general solution of (5.4) is

$$D_N = A(-e^{i\phi})^N + B(-e^{-i\phi})^N$$

where A and B are arbitrary constants. From the values of D_0 and D_1 , we obtain

$$\begin{aligned} A + B &= 1 \\ -Ae^{i\phi} - Be^{-i\phi} &= -\mu \end{aligned}$$

which give

$$B = - \frac{e^{-i\phi}}{2i \sin \phi}$$

$$A = \frac{e^{i\phi}}{2i \sin \phi}$$

and

$$\begin{aligned} D_N &= \frac{(-1)^N}{2i \sin \phi} \left[e^{i(N+1)\phi} - e^{-i(N+1)\phi} \right] \\ &= \frac{(-1)^N \sin (N+1) \phi}{\sin \phi}. \end{aligned}$$

The characteristic roots of $2Q_1$ are the roots of the equation $D_N = 0$ in μ , which is equivalent to the equation in ϕ

$$\frac{\sin(N+1)\phi}{\sin \phi} = 0.$$

This gives

$$\phi = \frac{j\pi}{N+1}, \quad j = 1, 2, \dots, N.$$

We note that $\phi = 0$ is not a solution. Since $D_N = 0$ is an N th degree equation in μ , the N roots are given by

$$\mu_j = 2 \cos \frac{j\pi}{N+1}, \quad j = 1, 2, \dots, N,$$

and hence the characteristic roots of Q_1 are

$$\lambda_j = \cos \frac{j\pi}{N+1}, \quad j = 1, 2, \dots, N.$$

COROLLARY 1. The characteristic function of q_1 is

$$\phi_{q_1}(t) = \prod_{j=1}^N (1 - 2t \cos \frac{j\pi}{N+1})^{-1/2}.$$

COROLLARY 2. The limits of the variation of r_1 are $\pm \cos \frac{\pi}{N+1}$.

Cumulants of q_1

We observe that if $N = 2m$, then

$$\lambda_{N-j+1} = -\lambda_j, \quad j = 1, 2, \dots, m;$$

and if

$$N = 2m + 1,$$

$$\lambda_{m+1} = 0, \quad \lambda_{N-j+1} = -\lambda_j, \quad j = 1, 2, \dots, m.$$

In any case

$$\sum_{j=1}^N \lambda_j^{2k+1} = 0, \quad k = 0, 1, 2, \dots$$

From theorem 5.1,

$$\kappa_{r:q_1} = 2^{r-1}(r-1)! \sum_{j=1}^N \lambda_j^r,$$

hence

$$(5.6) \quad \kappa_{2r+1:q_1} = 0, \quad r = 0, 1, 2, \dots$$

Now

$$\begin{aligned} \sum_{j=1}^N \lambda_j^{2r} &= \sum_{j=1}^N \frac{1}{2^{2r}} \left(e^{\frac{i\pi j}{N+1}} + e^{-\frac{i\pi j}{N+1}} \right)^{2r} \\ &= \frac{1}{2^{2r}} \sum_{k=0}^{2r} \binom{2r}{k} \sum_{j=1}^N e^{\frac{2\pi i j(k-r)}{N+1}}. \end{aligned}$$

If $k-r = 0, \pm(N+1), \pm 2(N+1), \dots$, then

$$\sum_{j=1}^N e^{\frac{2\pi i(k-r)j}{N+1}} = \sum_{j=1}^N 1 = N.$$

Otherwise,

$$\begin{aligned} \sum_{j=1}^N e^{\frac{2\pi i(k-r)j}{N+1}} &= \frac{e^{\frac{2\pi i(k-r)}{N+1}} - e^{\frac{2\pi i(k-r)(N+1)}{N+1}}}{1 - e^{\frac{2\pi i(k-r)}{N+1}}} \\ &= -1. \end{aligned}$$

Hence, if

$$(i) \quad r \leq N,$$

$$\begin{aligned} (5.7) \quad \sum_{i=1}^N \lambda_i^{2r} &= \frac{1}{2^{2r}} \left[\binom{2r}{r} N - \sum_{\substack{k=0 \\ k \neq r}}^{2r} \binom{2r}{k} \right] \\ &= \frac{1}{2^{2r}} \left[\binom{2r}{r} N - 2^{2r} + \binom{2r}{r} \right] \end{aligned}$$

$$= \frac{1}{2^{2r}} \left[\binom{2r}{r} (N+1) - 2^{2r} \right].$$

(ii) $r = s(N+1)+k$ where $k \leq N$ and s is a natural number

$$(5.8) \quad \sum_{j=1}^N \lambda_j^{2r} = \frac{1}{2^{2r}} \left[\binom{2r}{r} (N+1) + 2 \binom{2r}{r-(N+1)} (N+1) \right. \\ \left. + 2 \binom{2r}{r-2(N+1)} (N+1) + \dots \right. \\ \left. + 2 \binom{2r}{r-s(N+1)} (N+1) - 2^{2r} \right]$$

Finally, from theorem 5.1,

$$(5.9) \quad \kappa_{2r:q_1} = \begin{cases} \frac{(2r-1)!}{2} \left[\binom{2r}{r} (N+1) - 2^{2r} \right] & \text{if } r \leq N \\ \frac{(2r-1)!}{2} \left[\binom{2r}{r} (N+1) - 2^{2r} + 2(N+1) \sum_{j=1}^s \binom{2r}{r-j(N+1)} \right] & \text{if } s(N+1) \leq r < (s+1)(N+1) \text{ where } s = 1, 2, \dots \end{cases}$$

6. Asymptotic normality of q_1 .

THEOREM 6.1. If $x_1, x_2, \dots$ is a sequence of independent

$N(0,1)$ variates and $q_1 = \sum_{i=1}^{N-1} x_i x_{i+1}$ then q_1 , when standardized, is asymptotically $N(0,1)$.

PROOF. First we note that the Central Limit Theorem is not applicable in this case, as the successive terms in the summation are not independent of each other. Secondly, the theorem 6.1 is valid for the case when the variance of each x is the same finite quantity.

From (5.6) and (5.9) we have

$$\kappa_{1;q_1} = \kappa_{1;q_1} = 0$$

$$\kappa_{2;q_1} = \kappa_{2;q_1} = N-1.$$

Let

$$(6.1) \quad z = \frac{q_1}{\sqrt{N-1}}.$$

From (5.9) and (6.1) we have

$$(6.2) \quad x_z(t) = \frac{t^2}{2} + \sum_{r=2}^N \frac{t^{2r}}{(2r)!} \frac{(2r-1)!}{2(N-1)^r} \left\{ \binom{2r}{r}(N+1) - 2^{2r} \right\} \\ + \sum_{s=1}^{\infty} \frac{(s+1)(N+1)-1}{\sum_{r=s(N+1)}^{\infty} \frac{t^{2r}}{(2r)!} \frac{(2r-1)!}{2(N-1)^r}} \\ \left\{ \binom{2r}{r}(N+1) - 2^{2r} + 2^{(N+1)} \sum_{j=1}^s \binom{2r}{r-jN-j} \right\}.$$

As $N \rightarrow \infty$, for every r we have

$$\lim_{N \rightarrow \infty} \kappa_{2r;z} = \lim_{N \rightarrow \infty} \frac{(2r-1)!}{2(N-1)^r} \left\{ \binom{2r}{r}(N+1) - 2^{2r} \right\} = 0.$$

Furthermore, the double summation term in (6.2) recedes to infinity as $N \rightarrow \infty$; hence it follows that

$$\lim_{N \rightarrow \infty} x_z(t) = t^2/2. \quad \text{Q.E.D.}$$

The rapidity of convergence to normality could be measured by the kurtosis, as the skewness is zero.

$$\frac{\kappa_{4:q_1}}{\kappa_{2:q_1}^2} = \frac{18N-30}{(N-1)^2} = O\left(\frac{1}{N}\right).$$

The frequency function of q_1 , $f(q_1)$, in type A Edgeworth series up to $O(N^{-2})$ is given by

$$f(q_1) = \frac{1}{\sqrt{2\pi}} e^{-q_1^2/2} \left[1 + \frac{3N-5}{4(N-1)^2} H_4(q_1) + \frac{5N-11}{3(N-1)^3} H_6(q_1) \right]$$

where

$$H_r(x) = x^r - \frac{r(r-1)}{2 \cdot 1!} x^{r-2} + \frac{r(r-1)(r-2)(r-3)}{2^2 \cdot 2!} x^{r-4} - \dots$$

7. Moments of r_1 .

From theorem 4.1 the moments of r_1 are obtained by

$$v_{k:r_1} = \mathcal{E} r_1^k = \frac{\mathcal{E} q_1^k}{\mathcal{E} p^k}.$$

Utilizing the relations between moments and cumulants we obtain

$$v_{2k+1;q_1} = 0, \quad k = 0, 1, 2, \dots$$

where

$$v_{1;q_1} = \mathcal{E} q_1 \quad \text{and} \quad v_{k;q_1} = \mathcal{E} q_1^k.$$

For $N \geq 4$,

$$\begin{aligned}
 (7.1) \quad v_{2:q_1} &= N - 1, \\
 v_{4:q_1} &= 3N^2 + 12N - 27, \\
 v_{6:q_1} &= 15N^3 + 225N^2 + 525N - 2205, \\
 v_{8:q_1} &= 105(N-1)^4 + 1260(3N-5)(N-1)^2 + 1260(3N-5)^2 \\
 &\quad + 6720(5N-11)(N-1) + 1008(35N-53).
 \end{aligned}$$

In general the first two terms in the highest powers of N in $v_{2k:q_1}$ are

$$(7.2) \quad 1 \cdot 3 \cdot 5 \cdot \dots (2k-1) \left\{ N^k + k(3k-4)N^{k-1} \right\}.$$

p is distributed as χ^2 with N degrees of freedom; hence

$$E p^r = \frac{2^r \Gamma(\frac{N}{2} + r)}{\Gamma(\frac{N}{2})} = N(N+2) \dots (N+2r-2).$$

Therefore,

$$(7.3) \quad E r_1 = 0$$

$$v_{2k+1:r_1} = 0; \quad k = 1, 2, 3, \dots$$

$$v_{2:r_1} = \frac{N-1}{N(N+2)}$$

$$v_{4:r_1} = \frac{3N^2 + 12N - 27}{N(N+2)(N+4)(N+6)}$$

$$v_{6:r_1} = \frac{15N^3 + 225N^2 + 525N - 2205}{N(N+2)(N+4)(N+6)(N+8)(N+10)}$$

In general

$$\begin{aligned}
 (7.4) \quad v_{2k:r_1} &= \frac{1.3.5. \dots (2k-1) \{N^k + k(3k-4)N^{k-1}\} + O(N^{k-2})}{N(N+2) \dots (N+4k-2)} \\
 &= O(N^{-k}) .
 \end{aligned}$$

Asymptotic normality of r_1 .

Let

$$x = \frac{r_1}{\sigma_{r_1}} = \sqrt{\frac{N(N+2)}{N-1}} r_1 \simeq \sqrt{N} r_1 .$$

$\{x = 0; v_{2k+1:x} = 0, k = 1, 2, \dots, \text{ and}$

$$\begin{aligned}
 (7.5) \quad \lim_{N \rightarrow \infty} v_{2k:x} &= \lim_{N \rightarrow \infty} \frac{N^k \cdot 1.3.5 \dots (2k-1) N^k}{N^{2k}} \left[1 + O\left(\frac{1}{N}\right) \right] \\
 &= 1.3.5 \dots (2k-1) .
 \end{aligned}$$

If $\phi_x(t; N)$ denotes the characteristic function of x , then for every t

$$\begin{aligned}
 \lim_{N \rightarrow \infty} \phi_x(t; N) &= 1 + \frac{t^2}{2!} + \dots + \frac{t^{2k}}{(2k)!} 1.3.5 \dots (2k-1) + \dots \\
 &= e^{t^2/2} .
 \end{aligned}$$

It follows that the distribution function of x converges to the distribution function of a $N(0, 1)$ variate. Hence

THEOREM 7.1. r_1 , when standardized, is asymptotically $N(0, 1)$.

The rapidity of approach to normality can be measured by kurtosis as the skewness is zero. Now

$$\beta_2 = \frac{v_4}{v_2^2} = \frac{3N^2 + 12N - 27}{N(N+2)(N+4)(N+6)} \cdot \frac{N^2(N+2)^2}{(N-1)^2}$$

$$= 3\left(1 - \frac{2}{N} + \frac{10}{N^2} + O\left(\frac{1}{N^3}\right)\right)$$

$$\beta_2 - 3 = -\frac{6}{N} + O\left(\frac{1}{N^2}\right).$$

8. Approximate distribution of r_1 .

From (7.3) and (7.4) the characteristic function of $r_1^2 = z$ can be written, if $N > 6$,

$$(8.1) \quad \phi_z(t) = 1 + \frac{N-1}{N(N+2)} \frac{t}{1!} + \frac{3N^2+12N-27}{N(N+2)(N+4)(N+6)} \frac{t^2}{2!}$$

$$+ \frac{15(N^3+15N^2+35N-47)}{N(N+2)(N+4)(N+6)(N+8)(N+10)} \frac{t^3}{3!}$$

$$+ \sum_{k=4}^{\infty} v_{2k:r_1} \frac{t^k}{k!}.$$

Let us write formally,

$$(8.2) \quad \phi_z(t) = F\left(\frac{1}{2}, \frac{N+2}{2}, t\right) + \sum_{k=1}^{\infty} a_k t^k F\left(\frac{2k+1}{2}, \frac{N+4k+2}{2}, t\right).$$

Comparing (8.1) and (8.2) we determine a_k 's. Writing (8.2) in detail,

$$\phi_z(t) = 1 + \frac{1}{N+2} \frac{t}{1!} + \frac{1.3}{(N+2)(N+4)} \frac{t^2}{2!} + \dots + \frac{1.3.5 \dots (2k-1)}{(N+2) \dots (N+2k)} \frac{t^k}{k!} + \dots$$

$$+ a_1 t \left[1 + \frac{3}{N+6} \frac{t}{1!} + \frac{3.5}{(N+6)(N+8)} \frac{t^2}{2!} + \dots \right]$$

$$+ \dots$$

$$\begin{aligned}
& + a_r t^r \left[1 + \frac{2r+1}{N+4r+2} \frac{t}{1!} + \dots \right. \\
& \quad \left. + \frac{(2r+1)(2r+3)\dots(2k-1)}{(N+4r+2)\dots(N+2r+2k)} \frac{t^{k-r}}{(k-r)!} + \dots \right] \\
& + \dots \\
& + a_k t^k \left[1 + \frac{2k+1}{N+4k+2} \frac{t}{1!} + \dots \right] + \dots ;
\end{aligned}$$

or

$$(8.3) \quad \phi_z(t) = 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \left\{ a_k + \sum_{r=0}^{k-1} a_r \frac{(2r+1)(2r+3)\dots(2k-1)}{(N+4r+2)\dots(N+2r+2k)} \frac{k!}{(k-r)!} \right\}$$

where $a_0 = 1$. Therefore, we obtain

$$(8.4) \quad a_k + \sum_{r=0}^{k-1} a_r \frac{(2r+1)(2r+3)\dots(2k-1)}{(N+4r+2)\dots(N+2r+2k)} \frac{k!}{(k-r)!} = v_{2k:r_1}.$$

Solving for a_1, a_2, a_3 , we get

$$a_1 + \frac{1}{N+2} = \frac{N-1}{N(N+2)}$$

or

$$a_1 = -\frac{1}{N(N+2)} ;$$

$$\frac{3}{(N+2)(N+4)} + \frac{6a_1}{N+6} + 2a_2 = \frac{3N^2+12N-27}{N(N+2)(N+4)(N+6)} ;$$

hence,

$$a_2 = -\frac{3}{2N(N+2)(N+4)(N+6)}.$$

Similarly,

$$a_3 = -\frac{45}{3! N(N+2)(N+4)(N+6)(N+8)(N+10)}.$$

Now, all the cumulants of q_1 are known; hence, it is theoretically possible to calculate all the moments of q_1 and therefore of r_1 and z . Then we are able to determine $a_1, a_2, \dots$, to any desired number. No systematic way of determining all the coefficients could emerge because of the irregular behavior of the cumulants of q_1 for order greater than N . From corollary 2 of theorem 5.2 we see that the range of r_1 is $[-\cos \frac{\pi}{N+1}, \cos \frac{\pi}{N+1}]$ and that of z is $[0, \cos^2 \frac{\pi}{N+1}]$. Hence from (3.1) and (3.3) an approximate distribution of z to the first four terms in the expansion is

$$f(z) = \frac{(1-z)^{(N-1)/2} z^{-1/2}}{B(\frac{N+1}{2}, \frac{1}{2})} \int_0^1 \frac{B(\frac{N+1}{2}, \frac{1}{2})}{N(N+2)B(\frac{N+3}{2}, \frac{3}{2})} Q_1(z) \\ - \frac{3 B(\frac{1}{2}, \frac{N+1}{2})}{N(N+2)(N+4)(N+6)B(\frac{5}{2}, \frac{N+5}{2})} Q_2(z) \\ - \frac{45 B(\frac{1}{2}, \frac{N+1}{2})}{N(N+2)(N+4)(N+6)(N+8)(N+10)B(\frac{7}{2}, \frac{N+7}{2})} Q_3(z) \dots$$

where the superscripts for each polynomial Q are $(\frac{N-1}{2}, -\frac{1}{2})$. Or

$$(8.5) \quad f(z)dz = \frac{(1-z)^{(N-1)/2} z^{-1/2} dz}{B(\frac{N+1}{2}, \frac{1}{2})} \int_0^1 \left[1 - \frac{N+4}{2N} \left\{ z - \frac{(1-z)}{(N+1)} \right\} \right. \\ \left. - \frac{N+8}{2^2 \cdot 2! N} \left\{ z^2 - \frac{6z(1-z)}{(N+1)} + \frac{3(1-z)^2}{(N+1)(N+3)} \right\} \right. \\ \left. - \frac{3(N+12)}{2^3 \cdot 3! N} \left\{ z^3 - \frac{15z^2(1-z)}{(N+1)} + \frac{45z(1-z)^2}{(N+1)(N+3)} - \frac{15(1-z)^3}{(N+1)(N+3)(N+5)} \right\} \right] dz$$

$$\text{for } 0 \leq z \leq \cos^2 \frac{\pi}{N+1}$$

$$= 0 \quad \text{otherwise.}$$

Hence, an approximate distribution of r_1 is

$$(8.6) \quad f(r_1) dr_1 = \frac{(1-r_1^2)^{(N-1)/2} dr_1}{B(\frac{1}{2}, \frac{N+1}{2})} \left[1 - \frac{(N+4)}{2N} \left\{ r_1^2 - \frac{(1-r_1^2)}{(N+1)} \right\} \right. \\ \left. - \frac{(N+8)}{2^2 \cdot 2! N} \left\{ r_1^4 - \frac{6r_1^2(1-r_1^2)}{(N+1)} + \frac{3(1-r_1^2)^2}{(N+1)(N+3)} \right\} \right. \\ \left. - \frac{3(N+12)}{2^3 \cdot 3! N} \left\{ r_1^6 - \frac{15r_1^4(1-r_1^2)}{(N+1)} + \frac{45r_1^2(1-r_1^2)^2}{(N+1)(N+3)} - \frac{15(1-r_1^2)^3}{(N+1)(N+3)(N+5)} \right\} \right] - 7$$

$$\text{for } -\cos \frac{\pi}{N+1} \leq r_1 \leq \cos \frac{\pi}{N+1}$$

$$= 0 \quad \text{otherwise.}$$

9. Asymptotic validity of the distribution of r_1 .

In the previous section we have given an expansion of $\phi_z(t)$, first in a power series of t and then in a mixed series of powers of t and hypergeometric functions. From this latter series we have obtained the distribution of r_1 in terms of a beta function and Jacobi polynomials. Naturally questions about the behavior of series (8.5) arise. The question of convergence of (8.5) and of (8.2) depends on the knowledge of a_k for large k , which in turn depends on the specification of all the moments of r_1 . Such information is not available in exact specifications. However, first few dominant terms in all the

moments can be specified. From this it is possible to prove the asymptotic validity of the distribution of $z = r_1^2$. This proof is carried out in several steps.

Step (1). We observe that $\cos \frac{\pi}{N+1} = 1 + O(N^{-2})$. Since we are going to consider the asymptotic behavior of the series we replace the range of $z = r_1^2$ by $[-0, 1]$. This change will have negligibly small effect on the value of probabilities or of moments. For example, considering its effect on the k th moment from the first term of series (8.5),

$$\begin{aligned} & \frac{1}{B(\frac{N+1}{2}, \frac{1}{2})} \int_{\cos^2 \frac{\pi}{N+1}}^1 z^{k-1/2} (1-z)^{(N+1)/2} dz \\ & < \frac{(1 - \cos^2 \frac{\pi}{N+1})^{(n-1)/2}}{B(\frac{N+1}{2}, \frac{1}{2})} \frac{2}{2k-1} \int_{1 - \cos^{2k+1} \frac{\pi}{N+1}}^1 dz \\ & < \left(\frac{\pi}{N+1}\right)^{N+1} \frac{\Gamma(\frac{N+2}{2})}{\sqrt{\pi} \Gamma(\frac{N+1}{2})} = O\left(\frac{1}{N^{N+1/2}}\right). \end{aligned}$$

Since $E z^k = O(N^{-k})$, this change is negligible for any k less than N and as $N \rightarrow \infty$ it will be negligible for all k .

Step (2). Let us write

$$f(z) = v_0(z) + v_1(z) + v_2(z) + \dots$$

where $v_0(z), v_1(z), \dots$ are the successive terms in series (8.5); for instance,

$$v_0(z) = \frac{(1-z)^{(N+1)/2} z^{-1/2}}{B(\frac{N+1}{2}, \frac{1}{2})},$$

$$v_1(z) = - \frac{(1-z)^{(N-1)/2} z^{-1/2} \cdot Q_1^{(\frac{N-1}{2}, -\frac{1}{2})}(z)}{N(N+2) B(\frac{N+3}{2}, \frac{3}{2})}.$$

Furthermore, let us write

$$f_0(z) = v_0(z), f_1(z) = v_0(z) + v_1(z),$$

and so on, and $\xi_0(t), \xi_1(t), \dots$ for the characteristic functions and $F_0(z), F_1(z), \dots$ for the cumulative distribution functions associated with $f_0(z), f_1(z), \dots$, respectively; then

$$\xi_0(t) = F(\frac{1}{2}, \frac{N+2}{2}, t),$$

$$\xi_1(t) = F(\frac{1}{2}, \frac{N+2}{2}, t) - \frac{t}{N(N+2)} F(\frac{3}{2}, \frac{N+6}{2}, t)$$

et cetera.

Step (3). Denote the s th moment of the distributions $f_0(z), f_1(z), \dots$ by $v_{s;z}^{(0)}, v_{s;z}^{(1)}, \dots$, respectively. Then $v_{s;z}^{(0)}$ is the coefficient of $\frac{t^s}{s!}$ in the expansion of $\xi_0(t)$, $v_{s;z}^{(1)}$ is the coefficient of $\frac{t^s}{s!}$ in the expansion of $\xi_1(t)$ and so on.

Step (4). From the relations among moments and cumulants of a distribution and remembering that odd cumulants of q_1 are zero and that every even cumulant is of order N , we find

$$(9.1) \quad v_{2s;q_1} = \frac{(2s)!}{2^s s!} \kappa_2^s + \frac{(2s)!}{2^{s-2} 4! (s-2)!} \kappa_2^{s-2} \kappa_4 + \\ + \frac{(2s)!}{2^{s-4} 2! (4!)^2 (s-4)!} \kappa_2^{s-4} \kappa_4^2 + \frac{(2s)!}{2^{s-3} 6! (s-3)!} \kappa_2^{s-3} \kappa_6 + \dots$$

Therefore,

$$v_{2s;q_1} = 1.3. \dots (2s-1)(N-1)^s + 3.5 \dots (2s-1)s(s-1)(3N-5)(N-1)^{s-2} + \ell_{s-2}(N)$$

where $\ell_s(N)$ stands for an unspecified polynomial of degree s in N .

$v_{2s;q_1}$ can be arranged in the following form:

$$v_{2s;q_1} = 1.3.5 \dots (2s-1) \frac{N(N+2s+2)(N+2s+4) \dots (N+4s-2)}{N(N+2)(N+4) \dots (N+2s)} - s(N+4)(N+2s+4)(N+2s+6) \dots (N+4s-2) + \ell_{s-2}(N) \quad 7.$$

Therefore, we can write

$$v_{s;z} = \frac{1.3.5 \dots (2s-1)}{(N+2)(N+4) \dots (N+2s)} - \frac{(2s-1)(2s-3) \dots .5.3.1.s}{N(N+2)(N+6)(N+8) \dots (N+2s+2)} \\ + \frac{\ell_{s-2}(N)}{N(N+2)(N+4) \dots (N+4s-2)}.$$

We note that the first term is of $O(N^{-s})$, the second term of $O(N^{-s-1})$ and the third term of $O(N^{-s-2})$. Also that the first term is $v_{s;z}^{(0)}$ and the first two terms are equal to $v_{s;z}^{(1)}$, i.e., for $s > 1$,

$$(9.2) \quad v_{s;z} = v_{s;z}^{(0)} = \frac{\ell_{s-1}(N)}{N(N+2) \dots (N+4s-2)}$$

Furthermore, it can be easily verified that

$$\begin{aligned} v_{1:z} &= v_{1:z}^{(1)} = v_{1:z}^{(2)} = v_{1:z}^{(3)} = \dots \\ v_{2:z} &= v_{2:z}^{(2)} = v_{2:z}^{(3)} = \dots \\ v_{3:z} &= v_{3:z}^{(3)} = \dots, \end{aligned}$$

That is, by taking more of the terms of the series (8.5) more of the first moments of z become equal to the moments of the constructed distribution.

Step (5). Now,

$$\begin{aligned} \phi_z(t) &= \sum_{s=0}^{\infty} v_{s:z} \frac{t^s}{s!} \\ &= F\left(\frac{1}{2}, \frac{N+2}{2}, t\right) - \frac{t}{N(N+2)} F\left(\frac{3}{2}, \frac{N+6}{2}, t\right) \\ &\quad + \sum_{s=2}^{\infty} \frac{\ell_{s-2}(N)}{N(N+2) \dots (N+4s-2)} \frac{t^s}{s!} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{N \rightarrow \infty} \left| N \{ \phi_z(t) - \xi_0(t) \} \right| &= \lim_{N \rightarrow \infty} \left| \frac{|t|}{N+2} F\left(\frac{3}{2}, \frac{N+6}{2}, t\right) \right. \\ &\quad \left. + \sum_{s=2}^{\infty} \frac{\ell_{s-2}(N)}{(N+4) \dots (N+4s-2)} \frac{t^{s-1}}{s!} \right| = 0, \\ \lim_{N \rightarrow \infty} \left| N^2 \{ \phi_z(t) - \xi_1(t) \} \right| &= \lim_{N \rightarrow \infty} \left| \sum_{s=2}^{\infty} \frac{\ell_s(N)}{N(N+2) \dots (N+4s-2)} \frac{t^s}{s!} \right| = 0. \end{aligned}$$

Thus $\phi_z(t)$ is asymptotically equivalent to $\xi_0(t)$, $\xi_1(t)$, ...

with successively better approximations and hence $F(z)$ is asymptotically equivalent to $F_0(z)$, $F_1(z)$, ... with successively better approximations. We can improve our approximations by taking more terms of the series (8.2), stopping before the term for which a_k/a_{k-1} is no longer of $O(N^{-1})$. Thus series (8.2) is to be regarded the sum of a finite number of terms, stopping either at the term but one for which the coefficient a_k is smallest or at some earlier one when we have already achieved as much accuracy as we want. In our case, taking one, two or three terms from series (8.2), which is the same as taking the same number of terms from series (8.5), means that the moments of constructed distributions agree to the moments of z within relative orders N^{-1} , N^{-2} and N^{-3} respectively. This is clear from step (4) equations (9.2).

Step (6). It can be demonstrated that if we integrate the series (8.5) from 0 to z_0 , we actually obtain an asymptotic expansion. Writing

$$F(z_0) = \int_0^{z_0} f(z) dz$$

we have

$$\begin{aligned} F(z_0) &= \int_0^{z_0} v_0(z) dz + \int_0^{z_0} v_1(z) dz + \dots \\ &= u_0(z_0) + u_1(z_0) + u_2(z_0) + \dots, \text{ say.} \end{aligned}$$

Now,

$$u_0(z_0) = I_{z_0} \left(\frac{1}{2}, \frac{N+1}{2} \right),$$

in the notation of Karl Pearson. Furthermore,

$$|u_1(z_0)| = \frac{N+4}{2N} \left| \left\{ \frac{B(\frac{3}{2}, \frac{N+1}{2})}{B(\frac{1}{2}, \frac{N+1}{2})} I_{z_0}(\frac{3}{2}, \frac{N+1}{2}) - \frac{1}{N+1} \frac{B(\frac{1}{2}, \frac{N+3}{2})}{B(\frac{1}{2}, \frac{N+1}{2})} I_{z_0}(\frac{1}{2}, \frac{N+3}{2}) \right\} \right|$$

$$= \frac{(N+4)}{2N(N+2)} \left| I_{z_0}(\frac{3}{2}, \frac{N+1}{2}) - I_{z_0}(\frac{1}{2}, \frac{N+3}{2}) \right| ;$$

and similarly,

$$|u_2(z_0)| = \frac{3(N+8)}{8N(N+2)(N+4)} \left| I_{z_0}(\frac{5}{2}, \frac{N+1}{2}) - 2I_{z_0}(\frac{3}{2}, \frac{N+3}{2}) + I_{z_0}(\frac{1}{2}, \frac{N+5}{2}) \right| .$$

Therefore,

$$\left| \frac{u_1(z_0)}{u_0(z_0)} \right| = O(N^{-1})$$

$$\left| \frac{u_2(z_0)}{u_0(z_0)} \right| = O(N^{-2}) .$$

We may, therefore, write

$$P(r_1^2 \leq z_0) \sim I_{z_0}(\frac{1}{2}, \frac{N+1}{2}) - \frac{N+4}{2N(N+2)} \left\{ I_{z_0}(\frac{3}{2}, \frac{N+1}{2}) - I_{z_0}(\frac{1}{2}, \frac{N+3}{2}) \right\}$$

$$- \frac{3(N+8)}{8N(N+2)(N+4)} \left\{ I_{z_0}(\frac{5}{2}, \frac{N+1}{2}) - 2I_{z_0}(\frac{3}{2}, \frac{N+3}{2}) + I_{z_0}(\frac{1}{2}, \frac{N+5}{2}) \right\} .$$

10. Some other properties of the series (8.6).

(i) If the series for $f(r_1)$ given on the right hand side of (8.6) is written

$$f(r_1) = v_0(r_1) + v_1(r_1) + v_2(r_1) + \dots ,$$

where

$$v_0(r_1) = \frac{(1-r_1^2)^{(N-1)/2}}{B(\frac{1}{2}, \frac{N+1}{2})}$$

$$v_1(r_1) = - \frac{(N+4)(1-r_1^2)^{(N-1)/2}}{2N B(\frac{1}{2}, \frac{N+1}{2})} \left\{ r_1^2 - \frac{1-r_1^2}{N+1} \right\},$$

and so on, heuristic considerations show that over the effective range of r_1 ,

$$\frac{v_1(r_1)}{v_0(r_1)} = O(\frac{1}{N})$$

$$\frac{v_2(r_1)}{v_0(r_1)} = O(\frac{1}{N^2})$$

To show this we observe that the variance of r is $O(N^{-1})$, so that values of $f(r_1)$ outside some range on either side of zero, which is $O(N^{-1/2})$, are negligibly small. Thus r_1 may be regarded as $O(N^{-1/2})$ over the effective range of r_1 . Hence, over this range, i.e., for $|r_1| < A/N^{1/2}$, where A is a constant independent of N , we have

$$\frac{v_1(r_1)}{v_0(r_1)} \simeq - \frac{(N+4)}{2N} \cdot O(\frac{1}{N}) \simeq O(\frac{1}{N})$$

and

$$\frac{v_2(r_1)}{v_0(r_1)} \simeq O(\frac{1}{N^2}),$$

and so on. This statement is further strengthened by comparing the maxima of $v_0, v_1, v_2 \dots$ over the range of r_1 . It is found that

$$\max_{|x| \leq 1} |v_0(x)| = \frac{1}{B(\frac{1}{2}, \frac{N+1}{2})}$$

$$\max_{|x| \leq 1} |v_1(x)| = \frac{N+4}{2N(N+1)B(\frac{1}{2}, \frac{N+1}{2})}$$

and so

$$\frac{\max |v_1|}{\max |v_0|} = O(\frac{1}{N})$$

Similarly,

$$\frac{\max |v_2|}{\max |v_0|} = O(\frac{1}{N^2})$$

(ii) We make the transformation

$$t = \frac{r_1(N+1)^{1/2}}{(1-r_1^2)^{1/2}}$$

in (8.6). The frequency function of t is given by

$$f(t)dt = \frac{\Gamma(\frac{N+2}{2})dt}{\sqrt{(N+1)\pi} \Gamma(\frac{N+1}{2})} (1 + \frac{t^2}{N+1})^{-(N+2)/2} \int_{-1}^1 \frac{(N+4)}{2N(N+1)} (1 + \frac{t^2}{N+1})^{-1} (t^2-1)$$

$$- \frac{N+8}{8N(N+1)^2} (1 + \frac{t^2}{N+1})^{-2} (t^4 - 6t^2 + \frac{3(N+1)}{N+3}) - \dots \int, |t| \leq \sqrt{N+1} \cos \frac{\pi}{N+1}$$

= 0

otherwise .

This shows that t is approximately a Student statistic with $(N+1)$ degrees of freedom.

11. Correction for the sample mean.

In almost all practical cases we do not know the population mean of the x 's. It is, therefore, necessary to define serial correlation $\bar{r}_1$, corrected for the sample mean. Let

$$(11.1) \quad \bar{q}_1 = \sum_{i=1}^{N-1} (x_i - \bar{x})(x_{i+1} - \bar{x})$$

$$(11.2) \quad \bar{p} = \sum_{i=1}^N (x_i - \bar{x})^2$$

where

$$\bar{x} = \sum_{i=1}^N x_i / N,$$

and let

$$(11.3) \quad \bar{r}_1 = \bar{q}_1 / \bar{p}.$$

This correction complicates the mathematics considerably, as the characteristic roots of the matrix $\bar{Q}_1$ of the form $\bar{q}_1$ are not tractable. However, by straightforward expectations,

$$\begin{aligned} \mathcal{E} \bar{q}_1 &= \mathcal{E} \left\{ \sum_{j=1}^{N-1} x_j x_{j+1} - (N+1)\bar{x}^2 + (x_1 + x_N)\bar{x} \right\} \\ &= -\frac{N+1}{N} + \frac{2}{N} = -\frac{N-1}{N} \end{aligned}$$

and

$$\mathcal{E} \bar{q}_1^2 = \frac{N^3 - 2N^2 + 3}{N^2}.$$

The independence of the distribution of $\bar{r}_1$ from that of $\bar{p}$ can be proved easily. Now, $\mathcal{E} \bar{p} = N-1$, $\mathcal{E} \bar{p}^2 = (N-1)(N+1)$, we obtain

$$(11.4) \quad E \bar{r}_1 = -\frac{N-1}{N} \cdot \frac{1}{N-1} = -\frac{1}{N}$$

$$E \bar{r}_1^2 = \frac{N^3 - 2N^2 + 3}{N^2(N^2 - 1)}$$

From (3.4) and (3.5), if $N > 3$, we obtain

$$(11.5) \quad f(\bar{r}_1) d\bar{r}_1 \simeq \frac{(1-\bar{r}_1)^{(N-1)/2} (1+\bar{r}_1)^{(N-3)/2} d\bar{r}_1}{2^{N-1} B(\frac{N+1}{2}, \frac{N-1}{2})} \quad \text{if } -1 \leq \bar{r}_1 \leq 1$$

$$\frac{(2N-3)(N+1)}{64N^3(N+2)} P_2 \left(\frac{N-1}{2}, \frac{N-3}{2} \right) (\bar{r}_1) \quad \text{if } c_{N-1} \leq \bar{r}_1 \leq c_1$$

$$= 0 \quad \text{otherwise,}$$

where c_{N-1} is the least and c_1 is the highest characteristic root of $\bar{Q}_1$. One root of $\bar{Q}_1$ being zero, we always take $c_N = 0$. For $N=2$,

$$\bar{r}_1 \equiv -1/2 \quad \text{with probability } 2.$$

For $N=3$, the characteristic roots of $\bar{Q}_1$ are $0, 0, -2/3$. Hence

$$\bar{r}_1 = \frac{-2y_1^2/3}{y_1^2 + y_2^2}$$

where y_1 and y_2 are independent $N(0, 1)$. Therefore,

$$f(\bar{r}_1) d\bar{r}_1 = \frac{3}{2\pi} \left(-\frac{3}{2} \bar{r}_1 \right)^{-1/2} \left(1 + \frac{3}{2} \bar{r}_1 \right)^{-1/2} d\bar{r}_1, \quad -2/3 \leq \bar{r}_1 \leq 0.$$

For $N=4$, the characteristic roots of $\bar{Q}_1$ are

$$c_1 = (\sqrt{5} - 1)/4, c_2 = -1/4, c_3 = -(\sqrt{5} + 1)/4.$$

For $N=5$,

$$c_1 = 1/2, c_2 = (\sqrt{21} - 4)/10, c_3 = -1/2, c_4 = -(\sqrt{21} + 4)/10.$$

It is a guess that $c_1(\bar{Q}_1) = \cos \frac{2\pi}{N+1}$, and $c_{N-1} = -\cos \frac{2\pi}{N+1}$.

12. Distribution of the circular serial correlation coefficient.

Consider

$$(12.1) \quad \bar{R}_1 = \frac{\sum_{i=1}^N x_i x_{i+1} - (\sum_{i=1}^N x_i)^2/N}{\sum_{i=1}^N x_i^2 - (\sum_{i=1}^N x_i)^2/N} = \frac{c}{\bar{p}}$$

where $x_{N+1} = x_1$ and $x_1, \dots, x_N$ are independent $N(0, 1)$ variates.

The exact distribution of $\bar{R}_1$ is given by R. L. Anderson [2].

The characteristic function, $\phi_c(t)$, of c is

$$(12.2) \quad \phi_c(t) = \prod_{j=1}^{N-1} (1 - 2t \cos \frac{2\pi j}{N})^{-1/2}.$$

Let

$$c_j = \cos \frac{2\pi j}{N}, \quad j = 1, 2, \dots, N-1.$$

$$\sum_{j=1}^{N-1} c_j^k = \frac{1}{2^k} \sum_{j=1}^{N-1} (e^{2\pi i j/N} e^{-2\pi i j/N})^k$$

$$= \frac{1}{2^k} \sum_{r=1}^k \binom{k}{r} \sum_{j=1}^{N-1} e^{2\pi i j(2r-k)/N}.$$

If $k = 2m+1 < N$

$$\sum_{j=1}^{N-1} e^{2\pi i j(2r-2m-1)/N} = -1.$$

Hence

$$(12.3) \quad \sum_{j=1}^{N-1} c_j^{2m+1} = \frac{1}{2^k} \sum_{r=1}^k - \binom{k}{r} = -1.$$

If $k = 2m < N$

$$(12.4) \quad \sum_{j=1}^{N-1} c_j^{2m} = \frac{1}{2^{2m}} \left[\binom{2m}{m} N - 2^{2m} \right].$$

For $k \geq N$, we can proceed on lines similar to those used for calculating the cumulants of q_1 .

For $N > 2m$

$$\begin{aligned} \kappa_{2m-1:c} &= -2^{2m-2} (2m-2)! \\ \kappa_{2m:c} &= \frac{(2m-1)!}{2} \left[\binom{2m}{m} N - 2^{2m} \right]. \end{aligned}$$

Hence for $N > 4$,

$$\kappa_{1:c} = -1$$

$$\kappa_{2:c} = N-2$$

$$\kappa_{3:c} = -8$$

$$\kappa_{4:c} = 6(3N-8);$$

and

$$\nu_{1:c} = -1$$

$$\nu_{2:c} = N-1$$

$$\nu_{3:c} = -3(N+1)$$

$$v_{4;c} = 3(N-1)(N+5) \quad .$$

Furthermore,

$$\bar{p}^r = (N-1)(N+1)(N+3) \dots (N+2r-3) \quad .$$

Therefore the moments of $\bar{R}_1$, up to order 4, are

$$(12.4) \quad \begin{aligned} v_{1;\bar{R}_1} &= -\frac{1}{N-1} \\ v_{2;\bar{R}_1} &= \frac{N-1}{(N-1)(N+1)} = \frac{1}{N+1} \\ v_{3;\bar{R}_1} &= -\frac{3}{(N-1)(N+3)} \\ v_{4;\bar{R}_1} &= \frac{3}{(N+1)(N+3)} \quad . \end{aligned}$$

Hence

$$\phi_{\bar{R}_1}(t) = 1 - \frac{1}{N-1} \frac{t}{1!} + \frac{1}{N+1} \frac{t^2}{2!} - \frac{3}{(N-1)(N+3)} \frac{t^3}{3!} + \dots \quad ,$$

which can be written as

$$(12.5) \quad \begin{aligned} \phi_{\bar{R}_1}(t) &= e^{-t} \left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) \left(1 - \frac{1}{N-1} \frac{t}{1!} + \frac{1}{N+1} \frac{t^2}{2!} \right. \\ &\quad \left. - \frac{3}{(N-1)(N+3)} \frac{t^3}{3!} + \frac{3}{(N+1)(N+3)} \frac{t^4}{4!} + \dots \right) \\ &= e^{-t} \left(1 + \frac{N-2}{N-1} \frac{t}{1!} + \frac{N^2-N-4}{(N-1)(N+1)} \frac{t^2}{2!} \right. \\ &\quad \left. + \frac{N^3+3N^2-10N-24}{(N^2-1)(N+3)} \frac{t^3}{3!} + \frac{N^3+5N^2-14N-48}{(N^2-1)(N+3)} \frac{t^4}{4!} + \dots \right) . \end{aligned}$$

Let also

$$(12.6) \quad \phi_{\bar{R}_1}(t) = e^{-t} \left[F\left(\frac{N-2}{2}, N-1, 2t\right) + a_1 t F\left(\frac{N}{2}, N+1, 2t\right) + a_2 t^2 F\left(\frac{N+2}{2}, N+3, 2t\right) + \dots \right] .$$

Comparing (12.5) and (12.6) we determine the values of a_1, a_2, a_3 and a_4 as

$$a_1 = 0$$

$$a_2 = -\frac{1}{N^2-1}$$

$$a_3 = a_4 = 0.$$

From (3.4) and (3.5), taking $\beta = (N-4)/2$, $\alpha = (N-2)/2$, we obtain

$$f(\bar{R}_1) d\bar{R}_1 = \frac{(1-\bar{R}_1)^{(N-2)/2} (1+\bar{R}_1)^{(N-4)/2} d\bar{R}_1}{2^{N-2} B(\frac{N}{2}, \frac{N-2}{2})}$$

$$\int_{-1}^1 \frac{B(\frac{N}{2}, \frac{N-2}{2})}{2(N^2-1)B(\frac{N+4}{2}, \frac{N+2}{2})} P_2^{\frac{N-2}{2}, \frac{N-4}{2}}(\bar{R}_1) + b_5 P_5(\bar{R}_1) + \dots \quad (12.7)$$

Or,

$$(12.7) \quad f(\bar{R}_1) = \frac{(1-\bar{R}_1)^{(N-2)/2} (1+\bar{R}_1)^{(N-4)/2}}{2^{N-2} B(\frac{N}{2}, \frac{N-2}{2})} \int_{-1}^1 \frac{8}{N(N-2)} P_2(\bar{R}_1) + b_5 P_5(\bar{R}_1) + \dots \quad (12.7)$$

$$\text{for } \cos \frac{2\pi \sqrt{N/2-7}}{N} \leq \bar{R}_1 \leq \cos \frac{2\pi}{N}$$

$$= 0$$

otherwise,

where $b_5, b_6 \dots$ can be determined in the way outlined above, and where the superscripts of the polynomials are $(\frac{N-2}{2}, \frac{N-4}{2})$. The series (12.7) is exact up to 4 terms, that is, the first four moments of the fitted series are equal to the moments of $\bar{R}_1$. The remaining moments

are approximated at least to $O(N^{-2})$. Neglecting $b_5 P_5(\bar{R}_1)$ and later terms, close to 5 per cent values of the positive tail of the distribution are calculated for three values of N . It serves as a rough comparison with the exact 5 per cent values given by R.L. Anderson [2].

N	Calculations	Exact 5 per cent value
10	$P(\bar{R}_1 \geq .36) = .0496$	.360
30	$P(\bar{R}_1 \geq .26) = .0483$	.257
60	$\begin{cases} P(\bar{R}_1 \geq .18) = .0610 \\ P(\bar{R}_1 \geq .20) = .0439 \end{cases}$	.191

The closeness of the results is very satisfactory. The calculations were carried out with the help of Karl Pearson's "Tables of the Incomplete Beta-Functions", which give values of the argument only up to two decimal places. Since our object is to illustrate the theory rather than providing a table of percentage points, more refined calculations were not attempted.

A proof of the asymptotic behaviour of the distribution function of $\bar{R}_1$ obtained from $f(\bar{R}_1)$ can be developed on the lines of a proof which is given in section 9 for the distribution of r_1 .

A similar method applied to the circular serial correlation coefficient without mean correction, R_1 , where

$$R_1 = \frac{\sum_{i=1}^N x_i x_{i+1}}{\sum_{i=1}^N x_i^2}$$

gives an approximate distribution of R_1 , if N is even, as

$$(12.8) \quad f(R_1) dR_1 = \frac{(1-R_1^2)^{(N-1)/2} dR_1}{B(\frac{1}{2}, \frac{N+1}{2})} \left[1 + \frac{3\Gamma(\frac{1}{2})\Gamma(N+1)}{2^N \Gamma(N+\frac{1}{2})} Q_{\frac{N}{2}}^{(\frac{N-1}{2}, -\frac{1}{2})}(R_1^2) \right. \\ \left. - \frac{(3N+4)\Gamma(\frac{1}{2})\Gamma(N+2)}{2^{N+1} (N+1)\Gamma(N+\frac{3}{2})} Q_{\frac{N+2}{2}}^{(\frac{N-1}{2}, -\frac{1}{2})}(R_1^2) \right] \text{ if } -1 \leq R_1 \leq 1. \\ = 0 \quad \text{otherwise.}$$

The first $N+4$ moments of (12.8) are exactly equal to the moments of R_1 .

13. Concluding remarks.

We conclude this chapter with the remark that the method of developing the distribution of a serial correlation coefficient in a product of a beta distribution and a series of Jacobi polynomials produces better approximations to the true distributions as compared to the method of smoothing the characteristic function. This latter method has been widely used by several authors. For example, Dixon [9] and Jenkins [21] have obtained smoothed distributions of circular serial correlation coefficients and their functions. If we compare Dixon's distribution of the first circular serial correlation coefficient uncorrected for the mean, i.e., in our notation R_1 , with the distribution (12.8), we observe that the first term of the series (12.8) is the same as Dixon's frequency function. But whereas in the smoothing method there is no way of improving the accuracy for a fixed sample size, our technique provides a method of improving the accuracy of the fitted distribution to any desired extent. Even for small sample

sizes, by taking sufficient number of terms in the series, we can improve the accuracy to any desired level, as the successive terms decrease very rapidly. This was demonstrated in section 12 by computing the probabilities of the circular serial correlation coefficient corrected for the mean, $\bar{R}_1$, from (12.7) and comparing them with the exact values given by R.L. Anderson [27]. Even for a sample of size 10, and utilizing only first two terms of the series, we obtained a result which is correct within $O(10^{-4})$.

Finally, it may be observed that approximate distributions of partial, multiple and other ordinary serial correlation coefficients can also be obtained by the proposed method. In the case of first two serial correlation coefficients and their functions, it will be carried out in Chapter 4.

CHAPTER IV

DISTRIBUTIONS OF PARTIAL AND MULTIPLE SERIAL CORRELATION COEFFICIENTS

1. Summary.

It is indicated by section 3 of Chapter 1 that we require the distribution of a partial serial correlation coefficient, b , between x_t and x_{t-2} eliminating x_{t-1} ; and of a multiple serial correlation coefficient, R , between x_t and the combined variable (x_{t-1}, x_{t-2}) , to test different hypotheses concerning an autoregressive scheme of order 2

$$x_t = \alpha x_{t-1} - \beta x_{t-2} + e_t .$$

Since b and R are functions of r_1 and r_2 , it is necessary to investigate the joint distribution of r_1 and r_2 . A method which is due to Koopmans [28] yields the exact distribution of r_2 , when the sample number, N , is even. It is also applicable to the derivation of exact distributions of serial correlations with even lags when N is even. These distributions will be developed in the following two sections.

The distribution of r_2 for N even is defined by different parts of the range; furthermore, its exact distribution for N odd is not available. Therefore, an approximate distribution of r_2 , N even or odd, will be obtained in terms of a beta distribution and Jacobi polynomials. In later sections approximate distributions of b and R will be obtained.

2. Cumulants of q_s .

Let $\phi_s(t;N)$ and $\chi_s(t;N)$ denote the characteristic and cumulant generating functions, respectively, of the form

$$(2.1) \quad q_s = \sum_{i=1}^{N-s} x(i)x(i+s),$$

where s is a natural number $< N/2$ and $x(i)$, $i=1,2,\dots,N$ are independent $N(0,1)$ variates. We will use the notation $x(i)$ and x_i interchangeably. If C denotes the $(N \times N)$ matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

then the matrix Q_s of the quadratic form q_s is

$$(2.2) \quad Q_s = (C^s + C'^s)/2$$

where C' is the transpose of C .

Let ms be the greatest multiple of s in N and j the remainder, so that $N = ms + j$.

q_s can be written as the sum of the following s quadratic forms:

$$(2.3) \quad \begin{aligned} q_{sr} &= \sum_{k=0}^{m-1} x(r+ks)x(r+k+1s); \quad r = 1, 2, \dots, j, \\ q_{sr} &= \sum_{k=0}^{m-2} x(r+ks)x(r+k+1s); \quad r=j+1, \dots, s. \end{aligned}$$

If $s = 1$ then $j = 0$, and we have found the cumulants of q_1 . If $s > 1$, then $q_{s1}, q_{s2}, \dots, q_{ss}$ are all forms like q_1 , and no x which appears in one of them appears in any other of them; hence they are mutually independent. Each of the first j forms involves $(m+1)$ of the x 's and each of the last $(s-j)$ forms involves m of the x 's. Therefore,

$$(2.4) \quad \phi_s(t; N) = \prod_{k=1}^{m+1} \phi_1(t; m+1)^j \prod_{k=1}^m \phi_1(t; m)^{s-j}.$$

Or

$$(2.5) \quad \phi_s(t; N) = \prod_{k=1}^{m+1} (1 - 2t \cos \frac{k\pi}{m+2})^{-1/2j} \prod_{k=1}^m (1 - 2t \cos \frac{k\pi}{m+1})^{-1/2(s-j)}$$

which, incidentally, proves

THEOREM 2.1. The characteristic roots of the matrix Q_s are

$$\cos \frac{k\pi}{m+2}, \quad k=1, 2, \dots, m+1, \text{ each repeated } j \text{ times;}$$

$$\cos \frac{k\pi}{m+1}, \quad k=1, 2, \dots, m, \text{ each repeated } (s-j) \text{ times.}$$

We observe that if N is a multiple of s , i.e., $j = 0$, there are only m distinct roots of Q_s given by

$$\cos \frac{k\pi}{m+1}, \quad k=1, 2, \dots, m, \text{ each repeated } s \text{ times.}$$

Since $\cos \theta$ is a decreasing function of θ in the range $[0, \pi]$, we have

$$\cos \frac{k\pi}{m+2} > \cos \frac{k\pi}{m+1} > \cos \frac{(k+1)\pi}{m+2}; \quad k=1, \dots, m+1.$$

If $j \neq 0$ and the distinct roots of Q_s are denoted by $c_1, c_2, \dots$,

c_{2m+1} in decreasing order of magnitude, i.e.,

$$(2.6) \quad c_1 > c_2 > \dots > c_{2m+1},$$

then

$$(2.7) \quad c_{2k-1} = \cos \frac{k\pi}{m+2}, \quad k=1, 2, \dots, m+1;$$

$$c_{2k} = \cos \frac{k\pi}{m+1}, \quad k=1, 2, \dots, m.$$

Taking the logarithm on both sides of (2.4)

$$\chi_s(t; N) = j \chi_1(t; m+1) + (s-j) \chi_1(t; m).$$

If the r th cumulant of q_s is denoted by $\kappa_r(q_s; N)$, then it follows from (5.6) of Chapter 3 that

$$(2.8) \quad \kappa_{2r+1}(q_s; N) = 0; \quad r = 0, 1, \dots$$

Furthermore, if $r \leq m$, then from (5.9) of Chapter 3,

$$(2.9) \quad \begin{aligned} \kappa_{2r}(q_s; N) &= \frac{(2r-1)!}{2} \int j \binom{2r}{r} (m+2)^{-2r} j^{(s-j)} \binom{2r}{r} (m+1)^{-2r} (s-j)^{-r} \\ &= \frac{(2r-1)!}{2} \int \binom{2r}{r} (N+s)^{-2r} s^{-r}. \end{aligned}$$

For $r > m$, we can similarly write down expressions for $\kappa_{2r}(q_s; N)$

with the help of equation (5.9) of Chapter 3. Hence, all the cumulants of q_s are known.

THEOREM 2.2. q_s , when standardized, is an asymptotically $N(0, 1)$ variate.

The proof is similar to that given for the asymptotic normality of q_1 .

3. The exact distribution of r_{2s} when N is even.

Let

$$p = \sum_{i=1}^N x_i^2$$

and

$$r_{2s} = q_{2s}/p \quad \text{when } p \neq 0.$$

THEOREM 3.1. If $N = 2sm + 2j$, where $2sm$ is the greatest multiple of $2s$ in N , so that $j < s$, then the frequency function $f(y)$ of $r_{2s} = y$, when $j \neq 0$, is given by

$$(3.1) \quad f(y) = \frac{(N-2)/2}{(j-1)!} \sum_{i=1}^r \int \frac{\partial^{j-1}}{\partial w^{j-1}} \frac{(w-y)^{(N-4)/2}}{\prod_{k=1}^{m+1} (w-c_{2k-1})^j \prod_{k=1}^m (w-c_{2k})^{s-j}} \Big|_{w=c_{2i-1}} \\ + \frac{(N-2)/2}{(s-j-1)} \sum_{i=1}^{r-1} \int \frac{\partial^{s-j-1}}{\partial w^{s-j-1}} \frac{(w-y)^{(N-4)/2}}{\prod_{k=1}^m (w-c_{2k-1})^j \prod_{k=1}^m (w-c_{2k})^{s-j}} \Big|_{w=c_{2i}}$$

$$\text{if } c_{2r} < y \leq c_{2r-1}, r=1, 2, \dots, m,$$

$$= 0$$

otherwise ,

where $\prod'$ indicates that from the product that factor is omitted which involves the characteristic root with respect to which differentiation is being performed.

If $j = 0$, the first summation is to be disregarded.

PROOF. We only consider the case $j \neq 0$, $j = 0$ being a special case of it.

The joint characteristic function of p and q_{2s} denoted by $\phi(t, \theta)$, where t corresponds to p and θ to q_{2s} , is given by

$$\begin{aligned}\phi(t, \theta) &= \prod_{k=1}^{m+1} (1 - 2t - 2\theta c_{2k-1})^{-j} \prod_{k=1}^m (1 - 2t - 2\theta c_{2k})^{-(s-j)} \\ &= (1-2t)^{-N/2} \prod_{k=1}^{m+1} \left(1 - \frac{2\theta c_{2k-1}}{1-2t}\right)^{-j} \prod_{k=1}^m \left(1 - \frac{2\theta c_{2k}}{1-2t}\right)^{-(s-j)}.\end{aligned}$$

From here we can proceed on the lines of the argument given by Koopmans [28] in his discussion of the distribution of a ratio of two quadratic forms. Without reproducing his arguments, we arrive at the following integral expression for the distribution of $y = r_{2s}$,

$$f(y) = \frac{(N-2)/2}{2\pi i} \int_{\gamma_y} \frac{(z-y)^{(N-4)/2} dz}{\prod_{r=1}^{m+1} (z - c_{2r-1})^j \prod_{r=1}^m (z - c_{2r})^{s-j}},$$

where γ_y is any curve proceeding from $z = y$ into the lower half-plane, crossing the real axis at a point $z > c_1$, and returning to $z = y$ through the upper half-plane.

Now, if $g(z)$ is a function analytic within and on a closed curve γ , we have

$$\begin{aligned}\frac{1}{2\pi i} \int_{\gamma} \frac{g(z) dz}{(z-a)^r} &= \frac{1}{(r-1)!} \frac{\partial^{r-1}}{\partial a^{r-1}} g(a), \text{ if } a \text{ is within or on } \gamma, \\ &= 0, \text{ if } a \text{ is outside of } \gamma.\end{aligned}$$

Hence the theorem follows.

4. The distribution of r_2 .

If $N = 2m$, the distribution of r_2 is obtained from theorem 3.1 by taking $s = 1$ and $j = 0$.

$$f(r_2) = (m-1) \sum_{i=1}^u \frac{c_{2i}^{-r_2}}{\prod_{k=1}^m (c_{2i} - c_{2k})}, \quad \text{if } c_{2u+2} < r_2 \leq c_{2u},$$

$$u = 1, 2, \dots, m-1$$

$$= 0 \quad \text{otherwise,}$$

where

$$c_{2u} = \cos \frac{u\pi}{m+1}, \quad u = 1, 2, \dots, m.$$

The frequency function (4.1) is exact but of little practical use if N is even moderately large, say 10 or greater, due to the involved calculations for percentile points.

For N odd, the exact distribution can be put only in the integral form.

We therefore use the theorem 3.1 of Chapter 3 and obtain the distribution of r_2 in terms of a beta distribution and Jacobi polynomials.

Following the method which gives cumulants and moments of q_1 , we obtain the moments of q_2 , if $N \geq 6$, up to order 6 as

$$\xi_{q_2} = 0$$

$$v_{2k+1;q_2} = 0 \quad k = 1, 2, \dots,$$

$$v_{2;q_2} = N-2$$

$$v_{4:q_2} = 3N^2 + 6N - 48$$

$$v_{6:q_2} = 15N^3 + 180N^2 - 60N - 3600.$$

In general, the term of highest power in N of $v_{2k:q_2}$ is $1.3.5 \dots (2k-1) N^k$.

Hence, the moments of r_2 are

$$\xi_{r_2} = 0$$

$$v_{2k+1:r_2} = 0, \quad k = 1, 2, \dots,$$

$$v_{2:r_2} = \frac{N-2}{N(N+2)}$$

$$v_{4:r_2} = \frac{3N^2 + 6N - 48}{N(N+2)(N+4)(N+6)}$$

$$v_{6:r_2} = \frac{15N^3 + 180N^2 - 60N - 3600}{N(N+2)(N+4)(N+6)(N+8)(N+10)}$$

and in general

$$v_{2k:r_2} = \frac{1.3.5 \dots (2k-1) N^k \mu_{k-1}(N)}{N(N+2)(N+4) \dots (N+4k-2)}.$$

On the lines of the proof given for the approach to normality of r_1 , it can be proved that r_2 , when standardized, tends to a $N(0,1)$ variate as $N \rightarrow \infty$.

Writing $r_2^2 = z$, $\phi_z(t)$ can be written

$$(4.2) \quad \phi_z(t) = 1 + \frac{N-2}{N(N+2)} \frac{t}{1!} + \frac{3N^2 + 6N - 48}{N(N+2)(N+4)(N+6)} \frac{t^2}{2!} + \dots$$

Let us also write

$$(4.3) \quad \phi_z(t) = \sum_{k=0}^{\infty} a_k F\left(\frac{2k+1}{2}, \frac{N+4k+2}{2}, t\right).$$

Comparing (4.2) and (4.3), we determine up to a_3 ,

$$a_0=1, a_1 = -\frac{2}{N(N+2)}, a_2=0, a_3=0.$$

Hence from section 3 of Chapter 3, the expansion of the frequency function of z , up to third degree polynomial, is

$$(4.4) \quad f(z) dz = \frac{z^{-1/2}(1-z)^{(N-1)/2} dz}{B\left(\frac{1}{2}, \frac{N+1}{2}\right)} \int_0^1 \frac{2}{N(N+2)} \cdot \frac{B\left(\frac{1}{2}, \frac{N+1}{2}\right)}{B\left(\frac{3}{2}, \frac{N+3}{2}\right)} Q_1\left(\frac{N-1}{2}, -\frac{1}{2}\right)(z) dz$$

$$\text{if } 0 \leq z \leq \cos^2 \frac{\pi}{\sqrt{\frac{N+1}{2}}+1},$$

$$= 0 \quad \text{otherwise}.$$

Now, $Q_1\left(\frac{N-1}{2}, -\frac{1}{2}\right)(r_2^2)$ can be expressed in terms of $P_2\left(\frac{N-1}{2}, \frac{N-1}{2}\right)(r_2)$, and

thus we obtain

$$(4.5) \quad f(r_2) dr_2 = \frac{(1-r_2^2)^{(N-1)/2} dr_2}{B\left(\frac{1}{2}, \frac{N+1}{2}\right)} \int_0^1 \frac{8(N+4)}{N(N+1)(N+3)} P_2\left(\frac{N-1}{2}, \frac{N-1}{2}\right)(r_2) dz$$

$$= \frac{(1-r_2^2)^{(N-1)/2} dr_2}{B\left(\frac{1}{2}, \frac{N+1}{2}\right)} \int_0^1 \frac{(N+2)(N+4)}{N(N+1)} \left(r_2^2 - \frac{1}{N+2}\right) dz$$

$$\text{if } -\cos \frac{\pi}{\sqrt{\frac{N+1}{2}}+1} \leq r_2 \leq \cos \frac{\pi}{\sqrt{\frac{N+1}{2}}+1}$$

$$= 0 \quad \text{otherwise}.$$

Now, if $z_0 = r_0^2$, then from (4.4),

$$P(r_2^2 \leq r_0^2) = P(z \leq z_0) = F(z_0), \text{ say.}$$

$$\begin{aligned} F(z_0) &= I_{z_0} \left(\frac{1}{2}, \frac{N+1}{2} \right) - \frac{(N+2)(N+4)}{N(N+1)} \int \frac{B\left(\frac{3}{2}, \frac{N+1}{2}\right) I_{z_0} \left(\frac{3}{2}, \frac{N+1}{2} \right)}{B\left(\frac{1}{2}, \frac{N+1}{2}\right)} - \frac{I_{z_0} \left(\frac{1}{2}, \frac{N+1}{2} \right)}{N+2} dz \\ &= I_{z_0} \left(\frac{1}{2}, \frac{N+1}{2} \right) - \frac{N+4}{N(N+1)} \int I_{z_0} \left(\frac{3}{2}, \frac{N+1}{2} \right) - I_{z_0} \left(\frac{1}{2}, \frac{N+1}{2} \right) dz. \end{aligned}$$

Hence, apart from the fact that the difference

$$I_{z_0} \left(\frac{3}{2}, \frac{N+1}{2} \right) - I_{z_0} \left(\frac{1}{2}, \frac{N+1}{2} \right)$$

will be considerably smaller than $I_{z_0} \left(\frac{1}{2}, \frac{N+1}{2} \right)$, we observe that the second term is $O(N^{-1})$ as compared to the first term; and therefore the series is an asymptotic series. We can write

$$(4.6) P(r_2^2 \leq r_0^2) = I_{z_0} \left(\frac{1}{2}, \frac{N+1}{2} \right) - \frac{(N+4)}{N(N+1)} \int I_{z_0} \left(\frac{3}{2}, \frac{N+1}{2} \right) - I_{z_0} \left(\frac{1}{2}, \frac{N+1}{2} \right) dz.$$

Since $a_2 = a_3 = 0$, the error in (4.6) will be $O(N^{-3})$ or less.

5. Bivariate moments of r_1 and r_2 .

To find the bivariate moments of r_1 and r_2 , it is not possible to use the joint characteristic function of q_1 and q_2 , as the matrices of these forms are not commutative and cannot be reduced to their diagonal forms simultaneously. Hence we proceed with the straightforward method of calculating the expectations of products.

We first note that

$$\mathcal{E} x_i^{2r+1} = 0, \quad r = 0, 1, 2, \dots,$$

and, therefore, in taking the expectations we disregard all the terms which contain odd powers of x 's as a factor. For example:

$$\begin{aligned} \mathcal{E} q_1^2 q_2^2 &= \mathcal{E} \left(\sum_{i=1}^{N-1} x_i - x_{i+1} \right)^2 \left(\sum_{j=1}^{N-2} x_j x_{j+2} \right)^2 \\ &= \mathcal{E} \left[\sum_{i \geq j+3} \sum x_j^2 x_{j+2}^2 x_i^2 x_{i+1}^2 + \sum_{j=1}^{N-3} x_j^2 x_{j+2}^4 x_{j+3}^2 \right. \\ &\quad + \sum_{j=1}^{N-2} x_j^2 x_{j+1}^2 x_{j+2}^4 + \sum_{j=1}^{N-2} x_j^4 x_{j+1}^2 x_{j+2}^2 \\ &\quad + \sum_{j=1}^{N-3} x_j^2 x_{j+1}^4 x_{j+3}^2 + \sum_{i \leq j-2} \sum x_i^2 x_{i+1}^2 x_j^2 x_{j+2}^2 \\ &\quad \left. + 4 \sum_{i=1}^{N-3} x_i^2 x_{i+1}^2 x_{i+2}^2 x_{i+3}^2 + \text{Terms containing odd} \right] \end{aligned}$$

powers of x 's as a factor, 7,

the double summation extends over values of i and j such that no subscript of x is less than 1 or greater than N . Thus

$$\begin{aligned} \mathcal{E} q_1^2 q_2^2 &= \frac{(N-4)(N-3)}{2} + 3(N-3) + 3(N-2) + 3(N-2) + 3(N-3) \\ &\quad + \frac{(N-4)(N-3)}{2} + 4(N-3), \end{aligned}$$

or

$$\mathcal{E} q_1^2 q_2^2 = N^2 + 9N - 30.$$

We have used the fact that $\mathcal{E} x^2 = 1$, $\mathcal{E} x^4 = 3$.

The results of the calculations are the following:

$$\mathcal{E} q_1 q_2 = 0 ,$$

$$\mathcal{E} q_1^2 q_2 = 2(N-2), \quad \mathcal{E} q_1 q_2^2 = 0 ,$$

$$\mathcal{E} q_1^3 q_2 = 0, \quad \mathcal{E} q_1^2 q_2^2 = N^2 + 9N - 30, \quad \mathcal{E} q_1 q_2^3 = 0 ,$$

$$\mathcal{E} q_1^4 q_2 = 12N^2 + 60N - 216, \quad \mathcal{E} q_1^3 q_2^2 = 0 ,$$

$$\mathcal{E} q_1^2 q_2^3 = 6N^2 + 48N - 216, \quad \mathcal{E} q_1 q_2^4 = 0 ,$$

$$\mathcal{E} q_1^5 q_2 = 0, \quad \mathcal{E} q_1^4 q_2^2 = 3N^3 + 108N^2 + 159N - 1434, \quad \mathcal{E} q_1^3 q_2^3 = 0,$$

$$\mathcal{E} q_1^2 q_2^4 = 3N^3 + 66N^2 + 417N - 2526, \quad \mathcal{E} q_1 q_2^5 = 0 .$$

Writing $v_{ks} = \mathcal{E} r_1^k r_2^s$, we can summarize the moments, including the univariate moments, up to order six as follows:

Order 1: $v_{10} = v_{01} = 0 :$

Order 2: $v_{20} = \frac{N-1}{N(N+2)}, \quad v_{11} = 0, \quad v_{02} = \frac{N-2}{N(N+2)} ;$

Order 3: $v_{30} = 0, \quad v_{21} = \frac{2(N-2)}{N(N+2)(N+4)}, \quad v_{12} = v_{03} = 0;$

Order 4: $v_{40} = \frac{3N^2 + 12N - 27}{N(N+2)(N+4)(N+6)}, \quad v_{31} = 0,$

$$v_{22} = \frac{N^2 + 9N - 30}{N(N+2)(N+4)(N+6)}, \quad v_{13} = 0, \quad v_{04} = \frac{3N^2 + 6N - 48}{N(N+2)(N+4)(N+6)} ;$$

Order 5: $v_{50} = 0, \quad v_{41} = \frac{12N^2 + 60N - 216}{N(N+2)(N+4)(N+6)(N+8)}, \quad v_{32} = 0$

$$v_{23} = \frac{6N^2 + 48N - 216}{N(N+2)(N+4)(N+6)(N+8)}, \quad v_{14} = v_{05} = 0$$

Order 6: $v_{60} = \frac{15N^3 + 225N^2 + 525N - 2205}{N(N+2)(N+4)(N+6)(N+8)(N+10)}, v_{51} = 0,$

$$v_{42} = \frac{3N^3 + 108N^2 + 159N - 1434}{N(N+2) \dots (N+10)}, v_{33} = 0;$$

(5.1) $v_{24} = \frac{3N^3 + 66N^2 + 417N - 2526}{N(N+2) \dots (N+10)}, v_{15} = 0,$

$$v_{06} = \frac{15N^3 + 180N^2 - 60N - 3600}{N(N+2) \dots (N+10)}.$$

6. The distribution of partial serial correlation coefficient.

In an autoregressive scheme of order 2,

$$x_t = \alpha_{t-1} - \beta x_{t-2} + \epsilon_t.$$

we have seen that

(6.1)
$$\beta = \frac{\rho_1^2 - \rho_2}{1 - \rho_1^2}$$

where ρ_1 and ρ_2 are population autocorrelation coefficients with lag 1 and 2 respectively. It is easy to see that $-\beta$ is the partial autocorrelation between x_t and x_{t-2} when x_{t-1} is eliminated. Substituting the estimates of ρ_1 and ρ_2 we get a consistent estimate of β as

(6.2)
$$b = \frac{r_1^2 - r_2}{1 - r_1^2}.$$

Further, define the multiple autocorrelation between x_t and (x_{t-1}, x_{t-2}) as

(6.3)
$$\rho = (1 - \beta^2)(1 - \rho_1^2) = \frac{1 - 2\rho_1^2 + 2\rho_1^2\rho_2 - \rho_2^2}{1 - \rho_1^2}.$$

A sample multiple serial correlation R can be defined by

$$(6.4) \quad 1 - R^2 = (1 - b^2)(1 - r_1^2) = \frac{1 - 2r_1^2 + 2r_1^2 r_2^2 - r_2^2}{1 - r_1^2}.$$

Since $r_1^2 < 1$, the condition $R \geq 0$ gives

$$r_2 \geq 2r_1^2 - 1.$$

We already know that if $m = \left\lfloor \frac{N+1}{2} \right\rfloor$,

$$r_2 \leq \cos \frac{\pi}{m+1},$$

from which we conclude that the limits of the variation of R are $\left[0, 1\right]$ and those of b are $\left[-\cos \frac{\pi}{m+1}, 1\right]$.

We determine approximate moments of b and R by the following method.

From (6.2)

$$\begin{aligned} (6.5) \quad 1 + b &= \frac{1 - r_2}{1 - r_1^2} = (1 - r_2)(1 - r_1^2)^{-1} \\ &= \left\{ 1 - kr_2 + \binom{k}{2} r_2^2 - \binom{k}{3} r_2^3 \dots \right\} \left\{ 1 + kr_1^2 + \binom{k+1}{2} r_1^4 \right. \\ &\quad \left. + \binom{k+2}{3} r_1^6 + \dots \right\} \\ &= 1 - kr_2 + kr_1^2 + \binom{k}{2} r_2^2 - k^2 r_1^2 r_2^2 - \binom{k}{3} r_2^3 \\ &\quad + \frac{k^2(k-1)}{2} r_1^2 r_2^2 + \dots \end{aligned}$$

Taking the expectations on both sides of (6.5) with the help of (5.1), and rearranging the terms, we obtain

$$(6.6) \quad \begin{aligned} \mathcal{E}(1+b)^k = & 1 + \frac{k(k+1)}{2(N+2)} + \frac{k(k^3-2k^2-5k+6)}{8N(N+2)} \\ & + \frac{k(k^4-16k^3+51k^2-76k+16)}{8N(N+2)(N+4)} + O\left(\frac{1}{N(N+2)(N+4)(N+6)}\right). \end{aligned}$$

This approximation for $\mathcal{E}(1+b)^k$ is valid for k small in comparison to N , i.e., for $k < \sqrt{N}$. For higher values of k we will need more terms. Omitting the terms of $O(N^{-4})$ we have

$$\mathcal{E}(1+b) \simeq \frac{N+3}{N+2} - \frac{3}{N(N+2)(N+4)}$$

and

$$\mathcal{E}(1+b)^2 \simeq \frac{N+5}{N+2} - \frac{1}{N(N+2)} - \frac{11}{2N(N+2)(N+4)}.$$

Unfortunately it is not possible to determine exactly the moments of $(1+b)$. However, working with the approximations to the order N^{-4} for these moments, let us write the characteristic function of $(1+b)$ as

$$\begin{aligned} \phi_{1+b}(t) = & F\left(\frac{N+3}{2}, N+2, 2t\right) + a_1 t F\left(\frac{N+5}{2}, N+4, 2t\right) \\ & + a_2 t^2 F\left(\frac{N+7}{2}, N+6, 2t\right), \end{aligned}$$

terminating the series at a_2 . Proceeding in the manner of section 3 of Chapter 3 we obtain

$$\frac{N+3}{N+2} + a_1 = \frac{N+3}{N+2} - \frac{3}{N(N+2)(N+4)},$$

or

$$a_1 = -\frac{3}{N(N+2)(N+4)} ;$$

$$2a_2 + 2a_1 \frac{N+5}{N+4} + \frac{N+5}{N+2} = \frac{N+5}{N+2} - \frac{1}{N(N+2)} - \frac{11}{N(N+2)(N+4)} ,$$

or

$$a_2 = -\frac{N+9}{2N(N+2)(N+4)} .$$

Hence ,

$$(6.7) \quad f(b)db \sim \frac{(1-b)^{(N-1)/2}(1+b)^{(N+1)/2}db}{2^{N+1} B(\frac{N+1}{2}, \frac{N+3}{2})} \int_0^1 \frac{6}{N(N+1)(N+4)} P_1^{(\frac{N-1}{2}, \frac{N+1}{2})}(b) \\ - \frac{4(N+9)}{N(N+1)(N+3)} P_2^{(\frac{N-1}{2}, \frac{N+1}{2})}(b) \int_0^1 \frac{\pi}{\sqrt{1-b^2}} \leq b \leq 1, \\ = 0 \quad \text{otherwise.}$$

Now ,

$$P_1^{(\frac{N-1}{2}, \frac{N+1}{2})}(b) = \frac{N+2}{2} (b - \frac{1}{N+2})$$

and

$$P_2^{(\frac{N-1}{2}, \frac{N+1}{2})}(b) = \frac{(N+1)(N+3)}{32} \int_0^1 (1+b)^2 - \frac{2(N+5)}{N+1} (1-b^2) \\ + \frac{N+5}{N+1} (1-b)^2 \int_0^1 .$$

Furthermore ,

$$\mathcal{E} b = O(N^{-1})$$

and

$$\sigma_b^2 = O(N^{-1}) .$$

Hence, for large N , the effective range of the variation of b is

$O(N^{-1/2})$. Therefore, in this range, the second and third terms of the series in the square bracket are $O(N^{-5/2})$ and $O(N^{-1})$ respectively. Ordinarily, we would expect the second term to be $O(N^{-1/2})$; but in this particular case a_1 is $O(N^{-3})$ instead of $O(N^{-1})$. However it can be proved, as in the case of r_1 , that the distribution function of b obtained from (6.7) will be asymptotically valid representation of the true distribution.

If we consider $\cos \frac{\pi}{\int (N+1)/2+1} \sim 1$ and denote by

$\mathcal{E}_f(1+b)^k$ the expectation of $(1+b)^k$ with respect to the function (6.7), we find after some reduction

$$\mathcal{E}_f(1+b)^k = 1 + \frac{k(k+1)}{2(N+2)} + \frac{k(k^3-2k^2-5k+6)}{8N(N+2)} + O\left(\frac{1}{N^3}\right).$$

A comparison with (6.6) shows that $\mathcal{E}_f(1+b)^k$ agrees with $\mathcal{E}(1+b)^k$ to $O(N^{-3})$.

7. The distribution of multiple serial correlation coefficient.

In section 6 a definition of a multiple serial correlation, R , between x_t and (x_{t-1}, x_{t-2}) was given by the equation

$$1-R^2 = (1-2r_1^2+2r_1^2r_2-r_2^2)(1-r_1^2)^{-1}.$$

Hence,

$$\mathcal{E}(1-R^2)^k = \mathcal{E}(1-2r_1^2+2r_1^2r_2-r_2^2)^k (1-r_1^2)^{-k}.$$

Expanding the factors, we obtain after rearranging

$$(7.1) \quad \mathcal{E}(1-R^2)^k = \mathcal{E} \left[1 - k(r_1^2 + r_2^2) + 2kr_1^2r_2 - \frac{k^2-3k}{2}r_1^4 + (k^2-2k)r_1^2r_2^2 + \frac{k(k-1)}{2}r_2^4 + \dots \right]$$

$$= 1 - \frac{2k}{N+2} + \frac{k(4k-1)}{N(N+2)} + O\left(\frac{1}{N^3}\right),$$

the approximation being good for k small as compared to N .

Writing $T = 1 - R^2$ and the characteristic function of T , $\phi_T(t)$, in the form

$$\phi_T(t) = F\left(\frac{N}{2}, \frac{N+2}{2}, t\right) + a_1 t F\left(\frac{N+2}{2}, \frac{N+6}{2}, t\right) + \dots,$$

we can only determine a_1 effectively as $a_2, a_3, \dots$ require more exact knowledge of the moments of T . We therefore consider the series for $\phi_T(t)$ to be terminated after two terms. Omitting the terms $O(N^{-3})$ and less, we have

$$a_1 = \frac{3}{N(N+2)}.$$

Thus the distribution of T may be written up to the first degree Jacobi polynomial as

$$(7.2) \quad f(T)dT = \frac{N}{2} T^{(N-2)/2} dT \left[1 + \frac{3(N+4)}{2N^2} \left(\frac{N+2}{2} T - \frac{N}{2} \right) \right] \text{ for } 0 \leq T \leq 1,$$

$$= 0 \quad \text{otherwise.}$$

Or, the distribution of R^2 is, to the same degree of approximation,

$$(7.3) \quad f(R^2)dR^2 = \frac{N(1-R^2)^{(N-2)/2}}{2} dR^2 \left[1 + \frac{3(N+4)(N+2)}{4N^2} \left(1-R^2 - \frac{N}{N+2} \right) \right]$$

$$\text{for } 0 \leq R^2 \leq 1,$$

$$= 0 \quad \text{otherwise.}$$

First we observe that the moments of the distribution (7.3) agree to the order N^{-2} with the actual moments of R^2 . Secondly, since $\mathcal{E} R^2 = O(N^{-1})$ and the variance of R^2 is $O(N^{-2})$, in the effective

range of R the second term in (7.3) is $O(N^{-1})$ as compared to the first term. Thirdly, the cumulative distribution function of R^2 is

$$F(R_0^2) = P(R^2 \leq R_0^2) = 1 - (1 - R_0^2)^{N/2} + \frac{3(N+4)}{4N} R_0^2 (1 - R_0^2)^{N/2},$$

or

$$1 - F(R_0^2) = (1 - R_0^2)^{N/2} \left[1 + \frac{3(N+4)}{4N} R_0^2 \right].$$

If R_0^2 is $O(N^{-1})$, the second term is $O(N^{-1})$ relative to the first term.

Another method

The k th moment of the distribution

$$p(x) dx = \frac{N}{2} x^{(N-2)/2} dx, \quad 0 \leq x \leq 1,$$

is

$$x^k = \frac{N}{N+2k}.$$

If $2k < N$, this can be expanded as $(1 - \frac{2k}{N} + \frac{4k^2}{N^2} + \dots)$. If

k is small compared to N , these moments agree with the moments of T to order N^{-1} . Let us, then, write an approximate distribution of T as

$$f^*(T) dT = \frac{N}{2} T^{(N-2)/2} dT \left[1 + a_1 \left(\frac{N+2}{2} T - \frac{N}{2} \right) \right],$$

where $(\frac{N+2}{2} T - \frac{N}{2})$ is first degree orthogonal polynomial with weight function $T^{(N-2)/2}$ and range $[0, 1]$. From this, we have

$$\int_0^1 T f^*(T) dT = \frac{N}{N+2} + \frac{N}{2} a_1 \left(\frac{N+2}{N+4} - \frac{N}{N+2} \right) .$$

Equating this to the first moment of T , we obtain

$$a_1 = \frac{3(N+4)}{2N^2} + O\left(\frac{1}{N^2}\right) .$$

Therefore, neglecting terms $O(N^{-2})$ in a_1 ,

$$f^*(T) dT = \frac{N}{2} T^{(N-2)/2} \left[1 + \frac{3(N+4)}{2N^2} \left(\frac{N+2}{2} T - \frac{N}{2} \right) \right] ,$$

which agrees with (7.2). This series may be extended by taking additional terms and equating corresponding moments.

In (7.3), make the transformation

$$F = \frac{(1-R^2)/N}{R^2/2} .$$

Then, the distribution of F is given by

$$\begin{aligned} f(F) dF &= \left(\frac{N}{2}\right)^{(N+2)/2} \frac{F^{(N-2)/2} dF}{\left(1 + \frac{NF}{2}\right)^{(N+2)/2}} \left[1 + \frac{3(N+4)(N+2)}{4N^2} \left(\frac{NF}{2+NF} - \frac{N}{N+2} \right) \right] \\ &= \left(\frac{N}{2}\right)^{\frac{N+2}{2}} \frac{F^{(N-2)/2}}{\left(1 + \frac{NF}{2}\right)^{(N+2)/2}} \left[1 + \frac{3(N+4)}{4N} \frac{F-1}{\left(1 + \frac{NF}{2}\right)} \right] , \end{aligned}$$

$$0 \leq F \leq \infty ,$$

which shows that F is approximately distributed as Fisher's e^{2z} , with N and 2 degrees of freedom.

8. Extension to multivariate case.

We shall state a theorem suggested by theorems 3.1 and 3.2 of Chapter 3. The proof is similar in nature.

THEOREM 8.1. Let $\phi(t_1, \dots, t_u, \theta_1, \dots, \theta_v)$ be the joint characteristic function of variates $z_1, \dots, z_u, y_1, \dots, y_v$, where $0 \leq b_i \leq z_i \leq c_i \leq 1$, $i = 1, 2, \dots, u$; and $-1 \leq d_j \leq y_j \leq e_j \leq 1$, $j = 1, 2, \dots, v$. Let it be possible to write

$$(8.1) \quad \phi(t_1, \dots, t_u, \theta_1, \dots, \theta_v) = \sum_{i_1, \dots, i_u, j_1, \dots, j_v=0}^{\infty} a_{i_1 \dots i_u j_1 \dots j_v} e^{-(\theta_1 + \dots + \theta_v)} \prod_{s=1}^u t_s^{i_s} F(\beta_s + i_s + 1, \alpha_s + \beta_s + 2i_s + 2, t_s) \prod_{r=1}^v \theta_r^{j_r} F(\delta_r + j_r + 1, \gamma_r + \delta_r + 2j_r + 2, 2\theta_r),$$

all $\alpha, \beta, \gamma, \delta > -1$,

the series (8.1) being absolutely convergent for all values of $t_1, \dots, t_u, \theta_1, \dots, \theta_v$. If the partial sums of the series are dominated by a function ξ of these variables such that $\xi \exp |t_1 z_1 + \dots + t_u z_u + \theta_1 y_1 + \dots + \theta_v y_v|$ is integrable, then the joint frequency function of the variates is

$$(8.2) \quad f(z_1, \dots, z_u, y_1, \dots, y_v) = \prod_{s=1}^u \frac{(1-z_s)^{\beta_s} z_s^{\alpha_s}}{B(\alpha_s+1, \beta_s+1)} \prod_{r=1}^v \frac{(1-y_r)^{\gamma_r} (1+y_r)^{\delta_r}}{2^{\delta_r+\gamma_r+1} B(\gamma_r+1, \delta_r+1)}$$

$$\int \sum_{i_1, \dots, j_1, \dots=0}^{\infty} a_{i_1 \dots j_1 \dots j_v} \prod_{s=1}^u \frac{B(\alpha_s+1, \beta_s+1) i_s!}{B(\alpha_s+i_s+1, \beta_s+i_s+1)}$$

$$Q_{i_s}^{(\alpha_s, \beta_s)}(z_s) = \prod_{r=1}^v \frac{B(\gamma_r+1, \delta_r+1)}{B(\gamma_r+j_r+1, \delta_r+j_r+1)} \frac{j_r!}{2^{j_r}} P_{j_r}^{(\gamma_r, \delta_r)}(y_r)$$

within the range of the variates and zero otherwise.

9. Bivariate distribution of r_1 and r_2 .

For a given r_1 the condition $R \geq 0$ gives the range of r_2 as $[-2r_1^2 - 1, 1]$. But since r_2 in absolute value cannot exceed $\cos \frac{\pi}{m+1}$ where $m = \lfloor \frac{N+1}{2} \rfloor$, we have

$$\max(-\cos \frac{\pi}{m+1}, 2r_1^2 - 1) \leq r_2 \leq \cos \frac{\pi}{m+1}.$$

Similarly, for a given r_2 ,

$$-\sqrt{(1+r_2)/2} \leq r_1 \leq \sqrt{(1+r_2)/2}.$$

For a given r_1 , the range of the variation of b is

$$\left(-1 + \frac{1 - \cos \frac{\pi}{m+1}}{1 - r_1^2}, \cos \frac{\pi}{m+1}\right).$$

As $N \rightarrow \infty$ so does m and this range approaches to $[-1, 1]$. Furthermore, $E r_1 b - E r_1 E b \approx 0$. This indicates that r_1 and b , though not independent, are asymptotically independently distributed. Hence, we shall consider the joint distribution of r_1 and b as an intermediate step to finding the joint distribution of r_1 and r_2 .

Write $z = r_1^2$ and let $\phi(t, \theta)$ be the joint characteristic

function of z and b . From the knowledge of the first few moments of z and b let us write

$$(9.1) \quad \phi(t, \theta) = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} a_{ks} t^k \theta^s e^{-\theta} F\left(\frac{N+2s+3}{2}, N+2s+2, 2\theta\right) F\left(\frac{2k+1}{2}, \frac{N+4k+2}{2}, t\right)$$

where $a_{00}=1$. But

$$(9.2) \quad \phi(t, \theta) = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\theta^s}{s!} \mathcal{E} e^{-\theta} z^k (1+b)^s.$$

Therefore, we obtain the equations

$$\mathcal{E} z = a_{10} + \frac{1}{N+2}$$

$$\mathcal{E} (1+b) = a_{01} + \frac{N+3}{N+2}$$

$$\mathcal{E} z^2 = 2a_{20} + \frac{6a_{10}}{N+6} + \frac{3}{(N+2)(N+4)}$$

$$\mathcal{E} z(1+b) = a_{11} + \frac{a_{01}}{N+2} + \frac{(N+3)a_{10}}{N+2} + \frac{N+3}{(N+2)^2}$$

$$\mathcal{E} (1+b)^2 = 2a_{02} + \frac{2(N+5)a_{01}}{N+4} + \frac{N+5}{N+4},$$

and so on. Since the moments of $(1+b)$ have been calculated to order N^{-3} we do not attempt to determine coefficients beyond those listed above. We find to order N^{-3} ,

$$(9.3) \quad a_{01} = -\frac{3}{N(N+2)(N+4)}, \quad a_{10} = -\frac{1}{N(N+2)},$$

$$a_{02} = -\frac{N+9}{2N(N+2)(N+4)}, \quad a_{20} = -\frac{3}{2N(N+2)(N+4)(N+6)}.$$

Now,

$$\xi_{z(1+b)} = \frac{1}{N+2} - \frac{3}{N(N+2)(N+4)} + O\left(\frac{1}{N^4}\right).$$

The value of a_{11} turns out to be $O(N^{-4})$, which we shall neglect. Hence to the second order polynomials the joint distribution of z and b is

$$(9.4) \quad f(z, b) dz db = \frac{(1-z)^{(N-1)/2} z^{-1/2} (1-b)^{(N-1)/2} (1+b)^{(N+1)/2} dz db}{2^{N+1} B\left(\frac{N+1}{2}, \frac{1}{2}\right) B\left(\frac{N+1}{2}, \frac{N+3}{2}\right)}$$

$$\int_0^1 \int_0^1 \left[1 - \frac{N+4}{2N} \left(z - \frac{1-z}{N+1} - \frac{3}{2N(N+4)} \left\{ (1+b) - \frac{N+3}{N+1} (1-b) \right\} - \frac{N+8}{8N} \left\{ z^2 - \frac{6z(1-z)}{N+1} + \frac{3(1-z)^2}{(N+1)(N+3)} \right\} - \frac{N+9}{8N} \left\{ (1+b)^2 - \frac{2(N+5)(1-b)^2}{N+1} + \frac{(N+5)(1-b)^2}{N+1} \right\} \right] dz db,$$

in the range of z and b and zero otherwise. Making the transformation

$$z = r_1^2$$

$$1+b = \frac{1-r_2^2}{1-r_1^2},$$

we finally obtain the joint distribution of r_1 and r_2 as

$$(9.5) \quad f(r_1, r_2) dr_1 dr_2 = \left\{ \frac{(1-2r_1^2+r_2^2)(1-r_2^2)}{1-r_1^2} \right\}^{(N-1)/2} \frac{(1-r_2^2)}{(1-r_1^2)^2} dr_1 dr_2$$

$$\int_0^1 \int_0^1 \left[1 - \frac{N+4}{2N} \left(r_1^2 - \frac{1-r_1^2}{N+1} - \frac{3}{2N(N+4)} \left\{ \frac{1-r_2^2}{1-r_1^2} - \frac{N+3}{N+1} \right\} \right) \right] dr_1 dr_2,$$

$$\begin{aligned}
& \left(\frac{1-2r_1^2+r_2}{1-r_1^2} \right) \left\{ -\frac{N+8}{8N} \left\{ r_1^4 - \frac{6}{N+1} r_1^2 (1-r_1^2) + \frac{3(1-r_1^2)^2}{(N+1)(N+3)} \right\} \right. \\
& - \frac{N+9}{8N} \left\{ \left(\frac{1-r_2}{1-r_1^2} \right)^2 - \frac{2(N+5)}{N+1} \frac{(1-r_2)(1-2r_1^2+r_2)}{(1-r_1^2)^2} \right. \\
& \left. \left. + \frac{N+5}{N+1} \left(\frac{1-2r_1^2+r_2}{1-r_1^2} \right)^2 \right\} \right\} - 7
\end{aligned}$$

within the range of variation of r_1 and r_2 , and zero otherwise.

The asymptotic behavior of the series (9.1) for the characteristic function of r_1 and b can be established in the same manner as was given in section 9 of Chapter 3 for the series representation of the characteristic function of r_1 . Also, in the effective range of r_1 and r_2 , it can be shown that the terms in (9.5) decrease rapidly and the first term may serve a good approximation for even moderately large N .

10. Concluding remarks.

Starting from the joint distribution of r_1 and r_2 given by (9.2), we can obtain an approximate distribution of any function of these variates.

It may be noted that the distributions obtained in third and fourth chapters are quite good even for small samples as the theory permits development of the series to any desired order of approximation.

CHAPTER V

LEAST-SQUARES ESTIMATION WHEN RESIDUALS ARE CORRELATED

1. Summary.

In this chapter we will study the effect on the least-squares estimates of regression coefficients and of the variance of residuals when the condition of independence of residuals is violated. The maxima and minima of a bilinear and a quadratic form will be obtained. These will be utilized to set up bounds on the variances and covariances of the least-squares estimates of regression coefficients. The cases of a linear trend and of trigonometric functions will be considered in more detail. The material in section 2 up to equation (2.11) is not new except for its presentation, and is adapted from Hotelling's lectures on the theory of least squares and regression analysis. The rest of the chapter is original.

2. Least-squares estimates and their covariances.

Let $y_1, \dots, y_N$ be observations on a variate and let

$$(2.1) \quad y_\gamma = \sum_{i=1}^p \beta_i x_{i\gamma} + \Delta_\gamma, \quad \gamma = 1, 2, \dots, N.$$

The observations on $x_1, \dots, x_p$ are considered fixed from sample to sample. Let us write $\Delta' = (\Delta_1, \dots, \Delta_N)$ and Δ for its transpose. In regression theory it is usually assumed that $N > p$ and that

$$(i) \quad \sum \Delta = 0$$

$$(ii) \quad \sum \Delta_\gamma^2 = \sigma^2 \text{ for } \gamma = 1, 2, \dots, N,$$

$$(iii) \quad \hat{C}_{\gamma\delta} \Delta_{\gamma} \Delta_{\delta} = 0 \quad \text{for } \gamma \neq \delta ,$$

(iv) Δ is distributed normally.

If $N(\mu, \Sigma)$ stands for a normal distribution with mean vector μ and covariance matrix Σ , the above conditions may be summarized as

(1) Δ is distributed as $N(0, I_N)$, where I_N is the $N \times N$ identity matrix and 0 stands for the $N \times 1$ zero vector.

Write

$$(2.2) \quad y' = (y_1, \dots, y_N)$$

$$\beta' = (\beta_1, \dots, \beta_p)$$

$$x(p \times N) = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1N} \\ x_{21} & x_{22} & \dots & x_{2N} \\ \dots & \dots & \dots & \dots \\ x_{p1} & x_{p2} & \dots & x_{pN} \end{bmatrix}$$

$$= (x_{i\gamma}); \quad i = 1, 2, \dots, p, \quad \gamma = 1, 2, \dots, N ,$$

$$a(p \times p) = xx'$$

$$c = a^{-1}$$

$$g = xy$$

and

$$b' = (b_1, \dots, b_p).$$

We note that

$$a' = a \quad \text{and} \quad c' = c .$$

We will use a prime on a letter to denote the transpose of a matrix or a vector represented by that letter.

We suppose that $N > p$ and that the rank of x is p . Then the least-squares estimates, b of β and s^2 of σ^2 , are given by

$$(2.3) \quad b = cg$$

$$(2.4) \quad s^2 = \frac{y'y - b'g}{N-p} \\ = \frac{y'y - b'g}{n},$$

where $n = N - p$. Furthermore, if we write $\delta q = q - \mathcal{E}q$, where q is a variate, then

$$(2.5) \quad B(p \times p) = \mathcal{E}\delta b\delta b' = \sigma^2 c.$$

Now,

$$(2.6) \quad \mathcal{E}b = \mathcal{E}cg = \mathcal{E}(xx')^{-1}xy \\ = (xx')^{-1}xx'\beta \\ = \beta$$

as $\mathcal{E}y = \mathcal{E}(x'\beta + \Delta) = x'\beta$. It is also known that under the condition (1)

$$(2.7) \quad \mathcal{E}s^2 = \sigma^2.$$

We propose to consider these estimates when condition (iii) is replaced by

$$(iii)^* \quad \mathcal{E}\Delta_{\gamma}\Delta_{\delta}' = \sigma^2 \rho_{|\gamma-s|}, \quad \rho_0 = 1.$$

Which means that condition (1) is replaced by

$$(2) \quad \Delta \text{ is distributed } N(0, \sigma^2 P), \text{ where}$$

$$\begin{aligned}
 (2.8) \quad P(N \times N) &= \begin{bmatrix} 1 & \rho_1 & \rho_2 & \dots & \rho_{N-1} \\ \rho_1 & 1 & \rho_1 & \dots & \rho_{N-2} \\ \rho_2 & \rho_1 & 1 & \dots & \rho_{N-3} \\ \dots & \dots & \dots & \dots & \dots \\ \rho_{N-1} & \rho_{N-2} & \rho_{N-3} & \dots & 1 \end{bmatrix} \\
 &= I_N + \sum_{i=1}^{N-1} \rho_i (C^i + C'^i),
 \end{aligned}$$

and where

$$C = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

We still find $E b = \beta$.

Let

$$(2.9) \quad v = y - x'b.$$

Then

$$\begin{aligned}
 v'v &= (y' - b'x)(y - x'b) \\
 &= y'y - b'g
 \end{aligned}$$

and

$$s^2 = \frac{v'v}{n}.$$

Also

$$\begin{aligned}
 v &= x'\beta + \Delta - x'b \\
 &= x'(\beta - b) + \Delta
 \end{aligned}$$

$$= (I_N - x' c x) \Delta$$

as

$$b = \beta + c x \Delta.$$

Write $m = x' c x$; then $m' = m$ and $m^2 = m$. It follows that if λ is a root of m then $\lambda = \lambda^2$, so that, $\lambda = 0$ or 1 . Writing tr for trace of a matrix, we have

$$\begin{aligned} \text{tr } m &= \text{tr } x' c x = \text{tr } x x' c = \text{tr } I_p \\ &= p. \end{aligned}$$

It follows that p roots of m are 1 and $N-p = n$ of them are zero. This can also be seen by observing that

$$\begin{aligned} x(I_N - m) &= x - x x' c x \\ &= 0, \end{aligned}$$

i.e., there are p relations among the columns of $I_N - m$. Now,

$$\begin{aligned} \xi v' v &= \xi \text{tr } v' v \\ &= \xi \text{tr } \Delta' (I_N - m)' (I_N - m) \Delta \\ &= \xi \text{tr } \Delta' (I_N - m) \Delta \\ &= \xi \text{tr } \Delta \Delta' (I_N - m) \\ &= \sigma^2 \text{tr } P - \sigma^2 \text{tr } P m \\ &= N \sigma^2 - \sigma^2 \text{tr } P m. \end{aligned}$$

If $P = I_N$ then $\text{tr } P m = \text{tr } m = p$, $\xi v' v = n \sigma^2$, and $\xi s^2 = \sigma^2$ as stated in (2.7).

If $L = (\ell_{ij})$, $i, j = 1, 2, \dots, N$, is a matrix, we observe that

$$\text{tr } L(C^k + C'^k) = 2 \sum_{i=1}^{N-k} \ell_{k, k+i}$$

and

$$\text{tr } L I_N = \sum_{i=1}^N \ell_{ii}.$$

Therefore,

$$\begin{aligned} \text{tr } P m &= \text{tr } m P = \text{tr} \left[m I_N + \sum_{k=1}^{N-1} \rho_k \text{tr } m (C^k + C'^k) \right] \\ &= p + 2 \sum_{k=1}^{N-1} \rho_k \sum_{\gamma=1}^{N-k} m_{\gamma, k+\gamma} \end{aligned}$$

and

$$(2.10) \quad \xi s^2 = \xi \frac{v' v}{n} = \sigma^2 \left[1 - \frac{2}{n} \sum_{k=1}^{N-1} \sum_{\gamma=1}^{N-k} \rho_k m_{\gamma, k+\gamma} \right].$$

Now,

$$\begin{aligned} \xi \delta g \delta g' &= x' \xi \delta y \delta y' x \\ &= \sigma^2 x' P x \end{aligned}$$

and

$$(2.11) \quad B = \xi \delta b \delta b' = \sigma^2 c' x' P x c$$

Write

$$d(p \times N) = (d_{j\gamma}) = cx, \quad j = 1, 2, \dots, p, \quad \gamma = 1, 2, \dots, N,$$

and let $B_{ij} = B_{ji}$ be the element of B in i th row and j th column.

We obtain from

$$\begin{aligned} B &= \sigma^2 d P d' \\ &= \sigma^2 d \left[I_N + \sum_{k=1}^{N-1} \rho_k (C^k + C'^k) \right] d' \end{aligned}$$

and

$$\left[d (C^k + C'^k) d' \right]_{ij} = \sum_{\gamma=1}^{N-k} (d_{j\gamma} d_{i, k+\gamma} + d_{i\gamma} d_{j, k+\gamma})$$

that

$$(2.12) \quad B_{ij} = \text{cov}(b_i, b_j) \\ = \sigma^2 \int \sum_{\gamma=1}^N d_{i\gamma} d_{j\gamma} + \sum_{k=1}^{N-1} \sum_{\gamma=1}^{N-k} \rho_k (d_{j\gamma} d_{i, k+\gamma} + d_{i\gamma} d_{j, k+\gamma}) \quad \text{--} \int.$$

Since each b_i is a linear function of Δ 's, the vector b is $N(\beta, B)$.

If by a proper choice of x 's we make

$$xx' = I_p,$$

so that $a = c = I_p$, we have

$$(2.13) \quad B_{ij} = \sigma^2 \int \delta_{ij} + \sum_{k=1}^{N-1} \sum_{\gamma=1}^{N-k} \rho_k (x_{i\gamma} x_{j, k+\gamma} + x_{j\gamma} x_{i, k+\gamma}) \quad \text{--} \int,$$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$.

3. Bounds on a quadratic and a bilinear form.

We notice that ξs^2 and the B_{ij} 's contain bilinear and quadratic forms in the x 's. We wish to set some uniform bounds on these quantities for different sets of non-stochastic variables.

THEOREM 3.1. Let $A = (a_{ij})$ be an $N \times N$ real symmetric matrix. Let $\alpha_1, \alpha_2, \dots, \alpha_N$ be the characteristic roots of A where

$$\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{N-1} \leq \alpha_N.$$

If $x' = (x_1, \dots, x_N)$, $y' = (y_1, \dots, y_N)$ are row vectors in real quantities with x and y their transposed column vectors, then, under the conditions

(i) $x'x=1$, (ii) $y'y=1$, and (iii) $x'y=0$, $\lambda = x'Ay$ has a maximum $(\alpha_N - \alpha_1)/2$ and a minimum $-(\alpha_N - \alpha_1)/2$.

PROOF. Since the problem is invariant under any simultaneous orthogonal transformation of x and y , we assume that A is a diagonal matrix. Hence

$$(3.2) \quad \lambda = \sum_{i=1}^N \alpha_i x_i y_i .$$

Using Laplace's method of multipliers, we write

$$(3.3) \quad T = \lambda - \frac{1}{2}k(\sum x_i^2 - 1) - \frac{1}{2}\ell(\sum y_i^2 - 1) - m \sum x_i y_i .$$

The maximum and minimum values of λ and T are the same. For these values

$$\frac{\partial T}{\partial x_i} = \alpha_i y_i - m y_i - k x_i = 0, \quad i = 1, 2, \dots, N,$$

$$\frac{\partial T}{\partial y_j} = \alpha_j x_j - m x_j - \ell y_j = 0, \quad j = 1, 2, \dots, N .$$

Therefore,

$$(3.4) \quad \sum x_i \frac{\partial T}{\partial x_i} = \lambda - k = 0$$

$$\sum y_i \frac{\partial T}{\partial x_i} = \sum \alpha_i y_i^2 - m = 0$$

$$\sum x_j \frac{\partial T}{\partial y_j} = \sum \alpha_j x_j^2 - m = 0$$

$$\sum y_j \frac{\partial T}{\partial y_j} = \lambda - \ell = 0 ,$$

which give

$$\ell = k = \lambda$$

$$m = \sum \alpha_i y_i^2 = \sum \alpha_i x_i^2 = \nu, \text{ say.}$$

The equations (3.4) become

$$-\lambda x_i + (\alpha_i - \nu)y_i = 0, \quad i = 1, 2, \dots, N,$$

$$(\alpha_i - \nu)x_i - \lambda y_i = 0, \quad i = 1, 2, \dots, N.$$

Let $z' = (x' \ y')$. Then for a non-null solution in z of the above equations it is necessary that

$$\begin{vmatrix} -\lambda I & A - \nu I \\ A - \nu I & -\lambda I \end{vmatrix} = 0$$

or

$$\prod_{i=1}^N [\alpha_i - (\nu + \lambda)] [\alpha_i - (\nu - \lambda)] = 0.$$

This means that either

$$\nu + \lambda = \alpha_i \quad \text{for } i = 1 \text{ or } 2 \text{ or } \dots \text{ or } N$$

or

$$\nu - \lambda = \alpha_i \quad \text{for } i = 1 \text{ or } 2 \text{ or } \dots \text{ or } N.$$

Hence the stationary values of λ are obtained from the above given values of $\nu + \lambda$ and $\nu - \lambda$ provided they are consistent with conditions (i) - (iii). Now,

$$(3.5) \quad \frac{\partial^2 T}{\partial x_i \partial x_j} = -\lambda \delta_{ij}$$

$$\frac{\partial^2 T}{\partial y_j \partial x_i} = (\alpha_i - \nu) \delta_{ij}$$

$$\frac{\partial^2 T}{\partial y_i \partial y_j} = -\lambda \delta_{ij},$$

$$i, j=1, 2, \dots, N \text{ and } \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} .$$

The Hessian, i.e., the matrix of the second partial derivatives, H , is given by

$$(3.6) \quad H(2N \times 2N) = \begin{bmatrix} -\lambda I & A - \nu I \\ A - \nu I & -\lambda I \end{bmatrix} .$$

The characteristic roots of H are given by the equation in k

$$|(k + \lambda I)^2 - (A - \nu I)^2| = 0 ,$$

i.e., $k = (\nu - \lambda) - \alpha_i, \alpha_i - (\nu + \lambda), i=1, 2, \dots, N$. All the $2N$ roots will be non-negative if and only if

$$\nu + \lambda = \alpha_1$$

$$\nu - \lambda = \alpha_N$$

i.e., if

$$\lambda = -(\alpha_N - \alpha_1)/2$$

$$\nu = (\alpha_1 + \alpha_N)/2 .$$

Hence the minimum value of λ is $-(\alpha_N - \alpha_1)/2$. One vector for which λ attains its minimum is

$$z^i = (1/\sqrt{2}, 0, \dots, 0, 1/\sqrt{2}, 1/\sqrt{2}, 0, \dots, 0, -1/\sqrt{2}) ,$$

i.e.,

$$x^i = (1/\sqrt{2}, 0, \dots, 0, 1/\sqrt{2})$$

$$y^i = (1/\sqrt{2}, 0, \dots, 0, -1/\sqrt{2}) .$$

Similarly, all the $2N$ roots of H are non-positive if

$$\nu + \lambda = \alpha_N$$

$$\nu - \lambda = \alpha_1$$

and the maximum value of λ is $(\alpha_N - \alpha_1)/2$. Thus

$$(3.7) \quad -(\alpha_N - \alpha_1)/2 = \lambda_{\min} \leq \lambda \leq \lambda_{\max} = (\alpha_N - \alpha_1)/2. \quad \text{Q.E.D.}$$

THEOREM 3.2. In the notation of theorem 3.1 and under the same conditions

$$(\alpha_1 + \alpha_2)/2 = v_{\min} \leq v \leq v_{\max} = (\alpha_N + \alpha_{N-1})/2, \quad ,$$

where

$$v = x'Ax = y'Ay.$$

PROOF.

$$v = \sum_{i=1}^N \alpha_i x_i^2 \leq \alpha_N \sum_{i=1}^N x_i^2 = \alpha_N.$$

If $\alpha_{N-1} = \alpha_N$, the vectors

$$x^* = (0, 0, \dots, 0, 1/\sqrt{2}, 1/\sqrt{2})$$

$$y^* = (0, 0, \dots, 0, -1/\sqrt{2}, 1/\sqrt{2})$$

make $v = \alpha_N$ and the maximum of v , in this case, is α_N .

If $\alpha_{N-1} \neq \alpha_N$, $v \neq \alpha_N$, as this implies $x_N^2 = y_N^2 = 1$, $x_1 = x_2 = \dots = x_{N-1} = y_1 = y_2 = \dots = y_{N-1} = 0$, which is contrary to the condition $x'y = 0$.

For the vectors x^* and y^* the value of v is

$(\alpha_{N-1} + \alpha_N)/2$. Hence

$$(\alpha_{N-1} + \alpha_N)/2 \leq v_{\max} < \alpha_N$$

and for $v_{\max}$ $x_N \neq 1$, $y_N \neq 1$. Now,

$$v \leq \alpha_N x_N^2 + \alpha_{N-1} \sum_{i=1}^{N-1} x_i^2 = \alpha_{N-1} + (\alpha_N - \alpha_{N-1})x_N^2.$$

If $x_N^2 < 1/2$, $v < (\alpha_N + \alpha_{N-1})/2$; therefore, for $v_{\max}$ $x_N^2 \geq 1/2$.

Similarly, $y_N^2 \geq 1/2$, i.e.,

$$(3.8) \quad (x_N y_N)^2 \geq 1/4 .$$

But

$$(3.9) \quad (x_N y_N)^2 = (x_1 y_1 + \dots + x_{N-1} y_{N-1})^2 \\ \leq \left(\sum_{i=1}^{N-1} x_i^2 \right) \left(\sum_{i=1}^{N-1} y_i^2 \right) \\ = (1 - x_N^2)(1 - y_N^2) \leq 1/4 \text{ for } x_N, y_N \geq 1/2 .$$

It follows from (3.8) and (3.9) that for $v_{\max}$, $x_N^2 = 1/2$, $y_N^2 = 1/2$.

For these values of x_N^2 and y_N^2

$$v = \alpha_N/2 + (\alpha_1 x_1^2 + \dots + \alpha_{N-1} x_{N-1}^2)$$

where $\sum_{i=1}^{N-1} x_i^2 = 1/2$. The maximum value of $\sum_{i=1}^{N-1} \alpha_i x_i^2$ under the condi-

tion $\sum_{i=1}^{N-1} x_i^2 = 1/2$ is $\alpha_{N-1}/2$ which is achieved for $x_{N-1}^2 = 1/2$,

$x_1 = \dots = x_{N-2} = 0$. Similarly, $y_{N-1}^2 = 1/2$, $y_1 = \dots = y_{N-2} = 0$.

Thus the maximum value of v ,

$$v_{\max} = (\alpha_N + \alpha_{N-1})/2 ,$$

which is attained for $z = (x^* y^*)$. A similar proof shows that

$$v_{\min} = (\alpha_1 + \alpha_2)/2 . \quad \text{Q.E.D.}$$

4. Bounds on covariances.

In this section we revert to the notation of section 2 and

define x, y, a, β etc., as in (2.2).

If x_i denotes the i th row of x and if by proper choice of the x 's we make

$$\sum_{\gamma=1}^N x_{i\gamma} x_{j\gamma} = \delta_{ij},$$

then $a = c = I_p$ and we get (2.13), i.e.,

$$B_{ij} = \sigma^2 \delta_{ij} + \sum_{k=1}^{N-1} \rho_k x_i (c^k + c^{k'}) x_j' \quad (4.1)$$

From theorem 2.1 of Chapter 4 the maximum characteristic root of $c^k + c^{k'}$ is $2 \cos \frac{\pi}{\lfloor \frac{N+k-1}{k} \rfloor + 1}$ which is less than 2. Therefore from theorem 3.1,

$$(4.1) \quad |B_{ij} - \sigma^2 \delta_{ij}| < 2\sigma^2 \left| \sum_{k=1}^{N-1} \rho_k \right|.$$

For first order autoregressive scheme

$$\Delta_t = a \Delta_{t-1} + \varepsilon_t,$$

where ε_t satisfies the conditions given in section 2 of Chapter 1, we have

$$\rho_k = a^k.$$

Hence, if $a > 0$,

$$(4.2) \quad \sigma^2 \left[1 - \frac{2(a-a^N)}{1-a} \right] < B_{ii} < \sigma^2 \left[1 + \frac{2(a-a^N)}{1-a} \right]$$

and

$$(4.3) \quad -\frac{2\sigma^2(a-a^N)}{1-a} < B_{ij} < \frac{2\sigma^2(a-a^N)}{1-a}.$$

Similarly with the help of (2.10) we can set bounds on B_{ij} for a second order scheme. We note that (4.1) gives a uniform bound for all schemes and for all x 's such that $xx' = I_p$. In special cases we can improve these bounds to a considerable extent.

5. Linear trend.

Let us suppose that $N = 2s+1$ and consider the linear trend in the form

$$(5.1) \quad y_\gamma = \beta_1(2s+1)^{-1/2} + \beta_2(\gamma-s-1)/u + \Delta_\gamma, \quad \gamma=1, 2, \dots, N,$$

where

$$(5.2) \quad u^2 = s(s+1)(2s+1)/3.$$

In the notation of section 2

$$(5.3) \quad \begin{aligned} x_{1\gamma} &= (2s+1)^{-1/2} \\ x_{2\gamma} &= (\gamma-s-1)/u, \quad \gamma=1, 2, \dots, N, \\ b_1 &= (2s+1)^{1/2} \bar{y} = N^{1/2} \bar{y} \\ b_2 &= \frac{\sum_\gamma y_\gamma - (s+1)\sum_\gamma y_\gamma}{u}, \end{aligned}$$

where the summation over γ is from 1 to N . Also

$$(5.4) \quad \begin{aligned} a &= c = I_p \\ b &= g \\ m_{ij} &= (x'x)_{ij} = \frac{1}{N} + \frac{3(2i-N-1)(2j-N-1)}{N(N^2-1)} \end{aligned}$$

$$p = 2$$

$$n = N-2$$

$$s^2 = \frac{\mathbf{v}'\mathbf{v}}{n} = \frac{\sum_{\gamma} y_{\gamma}^2 - b_1^2 - b_2^2}{n}$$

and

$$B_{ij} = \sigma^2 \sum_{\gamma=1}^N x_{i\gamma} x_{j\gamma} + \sum_{k=1}^{N-1} \sum_{\gamma=1}^{N-k} \rho_k (x_{i\gamma} x_{j, k+\gamma} + x_{j\gamma} x_{i, k+\gamma}) \quad , i, j=1, 2 .$$

After substituting the values of $x_{1\gamma}$ and $x_{2\gamma}$ we obtain

$$(5.5) \quad B_{11} = \sigma^2 \sum_{\gamma=1}^{N-1} \frac{(N-k)}{N} \rho_k$$

$$B_{12} = 0$$

$$B_{22} = \sigma^2 \sum_{k=1}^{N-1} \rho_k - \frac{2}{N} \left(3 + \frac{2}{N^2-1} \right) \sum_{k=1}^{N-1} k \rho_k$$

$$+ \frac{4}{N(N^2-1)} \sum_{k=1}^{N-1} k^3 \rho_k$$

and

$$\ell s^2 = \sigma^2 \sum_{k=1}^{N-1} \rho_k + \frac{2}{nN} \left(4 + \frac{2}{N^2-1} \right) \sum_{k=1}^{N-1} k \rho_k - \frac{4}{nN(N^2-1)} \sum_{k=1}^{N-1} k^3 \rho_k .$$

For the first order autoregressive scheme, $\rho_k = a^k$. Now, for any number $z \neq 1$, we have

$$(5.6) \quad \sum_{k=1}^q z^k = \frac{z - z^{q+1}}{1-z}$$

$$\sum_{k=1}^q k z^k = \frac{z - (q+1)z^{q+1} + qz^{q+2}}{(1-z)^2}$$

$$\sum_{k=1}^q k^2 z^k = \frac{1}{(1-z)^3} \left[z + z^2 - (q+1)^2 z^{q+1} + (2q^2 + 2q - 1)z^{q+2} - q^2 z^{q+3} \right]$$

$$\sum_{k=1}^q k^3 z^k = \frac{1}{(1-z)^4} \left[z + 4z^2 + z^3 - (q+1)^3 z^{q+1} + (3q^3 + 6q^2 - 4)z^{q+2} \right.$$

$$\left. - (3q^3 + 3q^2 - 3q + 1)z^{q+3} + q^3 z^{q+4} \right].$$

Hence for the first order scheme

$$\mathcal{E}_{\sigma}^2 = 1 - \frac{4}{n} \frac{\alpha - \alpha^N}{1 - \alpha} + \frac{8}{nN} \frac{\alpha - N\alpha^N + \alpha^{N+1}}{(1 - \alpha)^2} + \frac{4}{nN(N^2 - 1)} \frac{\alpha - N\alpha^N + \alpha^{N+1}}{(1 - \alpha)^2} - \frac{4}{nN(N^2 - 1)} \frac{1}{1} \sum_{k=1}^{N-1} k^3 \alpha^k$$

$$\approx 1 - \frac{4\alpha}{n(1 - \alpha)},$$

(5.7)

$$B_{11} = \sigma^2 \left[1 - \frac{2}{N} \frac{\alpha - N\alpha^N + (N-1)\alpha^{N+1}}{(1 - \alpha)^2} + 2 \frac{\alpha - \alpha^N}{1 - \alpha} \right]$$

$$\approx \sigma^2 \left[\frac{1 + \alpha}{1 - \alpha} - \frac{2\alpha}{N(1 - \alpha)^2} \right]$$

$$B_{12} = 0$$

$$B_{22} \approx \sigma^2 \left[\frac{1 + \alpha}{1 - \alpha} - \frac{6\alpha}{N(1 - \alpha)^2} \right].$$

If we put

$$b_1' = b_1 / \sqrt{N} = \bar{y}$$

$$b_2' = \frac{\sqrt{12} b_2}{N(N^2-1)}$$

we have

$$y_\gamma = b_1' + b_2' (\gamma - s - 1) + \Delta_\gamma$$

and under the first order scheme for Δ_γ ,

$$(5.8) \quad \sigma_{\bar{y}}^2 \approx \frac{\sigma^2}{N} \left(\frac{1+\alpha}{1-\alpha} \right)$$

$$\sigma_{b_2'}^2 \approx \frac{12\sigma^2}{N(N^2-1)} \left(\frac{1+\alpha}{1-\alpha} \right)$$

$$\text{cov}(\bar{y}, b_2') = 0.$$

Since b_1 and b_2 are linear functions of A 's and since their covariance is zero, we conclude that b_i are independently distributed as $N(\beta_i, B_{ii})$, $i = 1, 2$. A consistent estimate of α may be taken

$$r_1 = \frac{\sum_{\gamma=1}^{N-1} v_\gamma v_{\gamma+1}}{\sum_{\gamma=1}^N v_\gamma^2}.$$

6. Trigonometric functions.

Consider

$$(6.1) \quad y_\gamma = \frac{\beta_1}{\sqrt{N}} + \sqrt{\frac{2}{N}} \sum_{i=1}^p \beta_{2i} \cos \lambda_i \gamma + \sqrt{\frac{2}{N}} \sum_{i=1}^p \beta_{2i+1} \sin \lambda_i \gamma + \Delta_\gamma, \gamma=1, 2, \dots, N,$$

where $\lambda_i = 2\pi w_i/N$ and w_i is an integer, $i=1, 2, \dots, p$; and Δ_γ

satisfies condition (2) of section 2.

Now

$$\sum_{\gamma} \cos \lambda_i \gamma = \sum_{\gamma} \sin \lambda_i \gamma = 0$$

$$\frac{2}{N} \sum_{\gamma} \cos^2 \lambda_i \gamma = \frac{2}{N} \sum_{\gamma} \sin^2 \lambda_i \gamma = 1, \quad i = 1, \dots, p.$$

In our notation

$$(6.2) \quad x_{1\gamma} = 1/\sqrt{N}$$

$$x_{2i,\gamma} = \cos \lambda_i \gamma$$

$$x_{2i+1,\gamma} = \sin \lambda_i \gamma, \quad i=1,2,\dots,p; \quad \gamma=1,2,\dots,N.$$

x is a $(2p+1) \times N$ matrix such that $xx' = I_{2p+1}$. Therefore

$$a = c = I_{2p+1},$$

and we find

$$(6.3) \quad b_1 = \sqrt{N} \bar{y}$$

$$b_{2i} = \sqrt{\frac{2}{N}} \sum_{\gamma} y_{\gamma} \cos \lambda_i \gamma$$

$$b_{2i+1} = \sqrt{\frac{2}{N}} \sum_{\gamma} y_{\gamma} \sin \lambda_i \gamma, \quad i=1,2,\dots,p.$$

Furthermore,

$$(6.4) \quad m_{\gamma\delta} = \frac{1}{N} + \frac{2}{N} \sum_{i=1}^p (\cos \lambda_i \gamma \cos \lambda_i \delta + \sin \lambda_i \gamma \sin \lambda_i \delta) \\ = \frac{1}{N} + \frac{2}{N} \sum_{i=1}^p \cos (\gamma - \delta) \lambda_i$$

$$n = N - 2p - 1$$

$$s^2 = \left(\sum_{\gamma=1}^N y_{\gamma}^2 - \sum_{i=1}^{2p+1} b_i^2 \right) / n$$

and

$$\hat{s}^2 = \sigma^2 \left[1 - \frac{2}{N} \sum_{k=1}^{N-1} \rho_k + \frac{2}{nN} \sum_{k=1}^{N-1} k \rho_k \right]$$

$$- \frac{4}{nN} \sum_{k=1}^{N-1} \sum_{i=1}^p (N-k) \rho_k \cos k\lambda_i \quad] .$$

For the covariances of b_i and b_j we obtain from (2.13)

$$B_{11} = \sigma^2 \left[1 + 2 \sum_{k=1}^{N-1} \frac{N-k}{N} \rho_k \right]$$

$$B_{1,2i} = \sigma^2 \left[\frac{2}{N} \sum_{k=1}^{N-1} \sum_{\gamma=1}^{N-k} \rho_k \left\{ \cos(k+\gamma)\lambda_i + \cos \gamma\lambda_i \right\} \right]$$

$$B_{2i,2i} = \sigma^2 \left[1 + \frac{4}{N} \sum_{k=1}^{N-1} \sum_{\gamma=1}^{N-k} \rho_k \cos(k+\gamma)\lambda_i \cos \gamma\lambda_i \right]$$

and $B_{1,2i+1}$ and $B_{2i+1,2i+1}$ are obtainable from the expressions

for $B_{1,2i}$ and $B_{2i,2i}$ by replacing cosine by sine. Finally

$$B_{2i,2j+1} = \sigma^2 \left[\sum_{k=1}^{N-1} \sum_{\gamma=1}^{N-k} \frac{2}{N} \left\{ \cos \gamma\lambda_i \sin(k+\gamma)\lambda_i \right. \right. \\ \left. \left. + \cos(k+\gamma)\lambda_i \sin \gamma\lambda_i \right\} \right] .$$

If we consider $\rho_2, \rho_3, \dots$ to be negligible, we obtain to order N^{-1}

$$\xi^2 \simeq \sigma^2 \left[1 - \frac{2\rho_1}{N} \right]$$

$$B_{11} \simeq \sigma^2 \left[1 + 2\rho_1 - \frac{2\rho_1}{N} \right]$$

$$B_{1,2i} \simeq -\frac{2}{N} \sigma^2 \rho_1 (1 + \cos \lambda_i)$$

$$B_{1,2i+1} \simeq -\frac{2}{N} \sigma^2 \rho_1 (1 + \sin \lambda_i)$$

$$B_{2i,2i} \simeq \sigma^2 (1 + 2\rho_1 \cos \lambda_i)$$

$$B_{2i+1,2i+1} \simeq \sigma^2 (1 + 2\rho_1 \cos \lambda_i)$$

$$B_{2i,2j+1} \simeq \frac{\sigma^2 \rho_1}{N} \sum_{\gamma=1}^{N-1} \left[\sin \{ k\lambda_j + \gamma(\lambda_j + \lambda_i) \} + \sin \{ k\lambda_j + \gamma(\lambda_j - \lambda_i) \} \right. \\ \left. + \sin \{ \gamma(\lambda_j + \lambda_i) + k\lambda_i \} + \sin \{ \gamma(\lambda_j - \lambda_i) - k\lambda_i \} \right]$$

We can get more precise expressions if the underlying scheme is known. For example, if the underlying scheme is first order autoregressive scheme, we have

$$\xi^2 \simeq 1 - \frac{2\alpha}{n(1-\alpha)} - \frac{4\alpha}{N} \sum_{i=1}^p \left(\frac{\cos \lambda_i - \alpha}{1 - 2\alpha \cos \lambda_i + \alpha^2} \right).$$

We observe that the vector b is distributed as $N(\beta, B)$.

7. Concluding remarks.

We conclude this chapter with the remarks that in most practical cases the underlying scheme for Δ 's will not be known. Even if it is known, we will be required to estimate the parameters involved

which will further reduce the degrees of freedom of s^2 . Similarly, B_{ij} 's will have to be estimated from the sample. It may not be very difficult to estimate σ^2 , but the question of estimating ρ_1 , ρ_2 , ... remains open. One suggestion obviously is to estimate these quantities by sample serial correlations; or, in case when higher order autocorrelations are functions of first k autocorrelations, to estimate the first k autocorrelations by corresponding sample serial correlations. In any case, the distribution of resulting estimates will be extremely difficult to obtain. However, these estimates will be a definite improvement over the estimates generally used whenever there is evidence of significant serial correlations among the residuals.

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