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STRONGLY REGULAR GRAPHS, PARTIAL GEOMETRIES AND PARTIALLY BALANCED DESIGNS

by

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March, 1963

Grant No. AF AFOSR-60-21

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This research was supported in part by the Mathematics Division of the Air Force Office of Scientific Research.

Institute of Statistics
Mimeo Series No. 358

Strongly Regular Graphs, Partial Geometries and Partially Balanced Designs[1)]

by

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0. Summary.

This paper introduces the concept of a partial geometry, which serves to unify and generalize certain theorems on embedding of nets, and uniqueness of association schemes of partially balanced designs, by Bruck, Connor, Shrikhande and others. Certain lemmas and theorems are direct generalizations of those proved by Bruck [5], for the case of nets, which are a special class of partial geometries.

1) This research was supported in part by the United States Air Force under grant no AF AFOSR-60-21, monitered by the Office of Scientific Research.

The author also wishes to acknowledge, that some of the ideas on which the present paper is based originated during discussions at the Symposium on Combinatorial Mathematics held at the Rand Corporation, Santa Monica, California, in the summer of 1961, in which the author participated.

2) Now visiting professor to the University of Geneva, on leave from the University of North Carolina.

1. Introduction.

We use graph theoretic methods for the study of association schemes of partially balanced incomplete block (PBIB) designs. For this purpose it is convenient to switch from graph theoretic language to the language of designs and vice versa as necessary.

As we shall be concerned with finite graphs only, we shall use the word graph in the sense of finite graphs.

A graph G is said to be regular if each vertex is joined to n_1 other vertices, and unjoined to n_2 other vertices. Clearly

$$(1.1) \qquad v - 1 = n_1 + n_2.$$

If further any two joined vertices of G, are both joined to exactly p^1_{11} other vertices, and any two unjoined vertices are both joined to exactly p^2_{11} other vertices, then the graph G is defined to be strongly regular, with parameters

$$(1.2) \qquad n_1, \quad n_2, \quad p^1_{11}, \quad p^2_{11}.$$

The concept of strongly regular graphs is isomorphic with the concept of association schemes of PBIB designs (with two associate classes), which was first introduced by Bose and Shimamoto [4]. Such a scheme they defined as a scheme of relations between v treatements such that (i) any two objects are either first associates or second associates (ii) each treatement has n_i i-th associates (i = 1,2) (iii) If two treatments are i-th associates, then the number of treatments common to the j-th associates of the first and k-th associates of the second is p^i_{jk} and is independent of the pair of treatments with which we start. Also $p^i_{jk} = p^i_{kj}$.

Bose and Clatworthy [1] showed that it is unnecessary to assume the constancy of all the p^i_{jk}'s. If we assume that n_1, n_2, p^1_{11} and p^2_{11} are constant, then the constancy of the p^1_{12}, p^1_{21}, p^1_{22}, p^2_{12}, p^2_{21}, p^2_{22} follows and $p^1_{12} = p^1_{21}$, $p^2_{12} = p^2_{21}$.

If we now identify the v treatments of the association scheme with the v vertices of a graph G, and consider two vertices as joined or unjoined according as the corresponding treatments are first or second associates, it is clear that a strongly regular graph G with the parameters (1.2) is isomorphic with an association scheme with the same parameters.

We have introduced for the first time in this paper the concept of a partial geometry.

A partial geometry (r,k,t) is a system of undefined points and lines, and an undefined relation incidence satisfying the axioms A1 - A4. To avoid cumbersome expression we may use standard geometric language. Thus a point incident with a line may be said to lie on it and the line may be said to pass through the point. If two lines are incident with the same point, we say that they intersect.

A1. Any two points are incident with not more than one line.

A2. Each point is incident with r lines.

A3. Each line is incident with k points.

A4. If the point P is not incident with the line l, there pass through P exactly t lines $(t \geqq 1)$ intersecting l.

We show that the number of points v and the number of lines b in the partial geometry (r,k,t) are given by

(1.3) $$v = k\left[(r-1)(k-1)+t\right]/\,t,$$

(1.4) $$b = r\left[(r-1)(k-1)+t\right]/\,t.$$

The graph G of a partial geometry is defined as a graph whose vertices correspond to the points of the geometry, and in which two vertices are joined or unjoined according as the corresponding points are incident or non-incident with a common line.

We then prove :

Theorem. The graph G of a partial geometry (r,k,t) is strongly regular with parameters

(1.5) $$n_1 = r\,(k-1), \quad n_2 = (r-1)(k-1)(k-t)\,/\,t,$$

(1.6) $$p^1_{11} = (t-1)(r-1)+k-2, \quad p^2_{11} = rt.$$

where

(1.7) $1 \leqq t \leqq r, \quad 1 \leqq t \leqq k.$

Using certain theorems of Bose and Mesner [3] relating to association schemes, we derive

Theorem. A necessary condition for the existence of a partial geometry (r,k,t) is that the number

$$(1.8) \qquad \alpha = \frac{rk\,(r-1)(k-1)}{t\,(k+r-t-1)} ,$$

is integral.

A strongly regular graph with parameters (1.5), (1.6) and for which (1.7) is satisfied will be defined to be a <u>pseudo-geometric</u> graph with characteristics (r,k,t). Such a graph may or may not be the graph of a partial geometry (r,k,t). It is therefore of interest to study the conditions under which a strongly regular graph, and in particular a pseudo-geometric graph with characteristics (r,k,t) is the graph of a partial geometry (r,k,t).

A subset of vertices of a graph G, any two of which are joined is called a <u>clique</u> of G. When G is the graph of a partial geometry (r,k,t) there will exist in G a set Σ of distinct cliques, $K_1, K_2, \ldots, K_b$ corresponding to the lines of the geometry satisfying the following axioms

A*1. Any two joined vertices of G are contained in one and only one clique of Σ.

A*2. Each vertex of G is contained in r cliques of Σ.

A*3. Each clique of Σ contains k vertices of G.

A*4. If P is a vertex of G not contained in a clique K_i of Σ there are exactly t vertices in K_i, which are joined to $P (i = 1, 2, \ldots, b)$.

Hence any graph G in which there exists a set Σ of cliques $K_1, K_2, \ldots, K_b$, satisfying axioms A*1 to A*4, is the graph of a partial geometry (r,k,t). In fact G together with the cliques of Σ is isomorphic to a partial geometry (r,k,t) the vertices of G corresponding to the points and the cliques of Σ corresponding to the lines of the geometry. Such a graph will be called <u>geometrisable</u> (r,k,t).

One may consider graphs in which there exists a set of cliques $K_1, K_2, \ldots, K_b$ satisfying one or more but not all of the axioms A_1^*, A_2^*, A_3^*, A_4^*. Thus a previous result due to Bose and Clatworthy [1], is equivalent to the following :

Theorem. If in a strongly regular graph G, there exists a set Σ of cliques $K_1, K_2, \ldots, K_b$, satisfying the axioms A_1^*, A_2^*, A_3^*, and if $k > r$, then the graph is geometrisable (r,k,t), the vertices of G, and the cliques of Σ being the points and lines of the corresponding geometry.

We further prove

Theorem. If in a pseudo-geometric graph with characteristics (r,k,t), there exists a set Σ of cliques $K_1, K_2, \ldots, K_b$, satisfying axioms A_1^* and A_2^*, and if $k > r$, then G is geometrisable (r,k,t), the vertices of G, and the cliques of Σ being the points and lines of the corresponding geometry.

There are many interesting examples of partial geometries some of which are given in section 7. In particular the partial geometry (r,k,t) becomes a net of degree r and order k when $t = r-1$. A pseudo-geometric graph with characteristics $(r,k,r-1)$ may be defined to be a <u>pseudo-net graph</u>, of degree r and order k. Bruck [5] has proved a series of Lemmas for pseudo-net graphs and in particular has shown that a pseudo-net graph of degree r and order k is geometrisable $(r,k,r-1)$ if

$$k > \frac{1}{2}(r-1)(r^3-r^2+r+2). \tag{1.9}$$

The special case $r = 2$, was proved by Shrikhande [19]. In this paper we give the following generalization of Bruck's result.

Theorem. A pseudo-geometric graph with characteristics (r,k,t) is geometrisable (r,k,t) if

$$k > \frac{1}{2}\left[r(r-1) + t(r+1)(r^2-2r+2)\right]. \tag{1.10}$$

We have proved a series of Lemmas which are direct generalizations of the Lemmas used by Bruck for his proof. In fact it is surprising how smoothly the technique devised by Bruck for the special case of nets, works in the general case.

In particular the concept of grand cliques introduced by Bruck for the case of nets is capable of easy generalization. If G is a pseudo-geometric graph with characteristics (r,k,t), then a major clique of G is defined as a clique which contains at least $k - (r-1)^2 (t-1)$ points. A major clique which is also maximal is defined as a grand clique.

Given a pseudo-geometric graph G with characteristics (r,k,t) the set of grand cliques is unambiguously defined. We may take this set to be the set Σ and enquire under what circumstances, the axioms A*1 - A*4 will be satisfied. We then show that (1.10) is a sufficient condition for this.

A pseudo-geometric graph with characteristics (r,k,t) has the same parameters as the triangular association scheme if we take $r = t = 2$, $k = n-1$. Substituting these values in (1,10) we get $n > 8$. Thus for these special values of r,k,t our result is equivalent to the uniqueness of the triangular scheme for $n > 8$, a result first proved by Connor [9]. In fact our result may be interpreted as a generalized uniqueness theorem as explained in section 12.

A net of degree r and order k is defined to have deficiency $d = r + 1 - k$. Bruck [5] showed that a net of order k and deficiency d can be completed to an affine plane by the addition of new lines, if

$$k > \frac{1}{2}(d-1)(d^3 - d^2 + d + 2).$$

We generalize Bruck's result to the following :

Theorem. Given a partially balanced incomplete block (PBIB) design $(r, k, \lambda_1, \lambda_2)$, $\lambda_1 > \lambda_2$, based on an association scheme with parameters

$$n_1 = (d-1)(k-1)(k-t) / t, \qquad n_2 = d(k-1),$$

$$p_{11}^1 = \left[(d-1)(k-1)(k-t) - d(k-t-1) - t\right] / t,$$

$$p_{11}^2 = (d-1)(k-t)(k-t-1) / t,$$

we can extend the design by adding new blocks, containing the same treatments, in such a way that the extended design is a balanced incomplete block (BIB) design, with $r_0 = r + d(\lambda_1 - \lambda_2)$ replications, block size k and in which every pair of treatments occur together in λ_1 blocks.

2. Strongly regular graphs.

A finite graph G consists of a finite set of v vertices, and a relation adjacency such that any two distinct vertices of G may be either adjacent or non-adjacent. Adjacent vertices may be said to be joined and non-adjacent vertices to be unjoined. We shall only be concerned with finite graphs only, and use the word graph in the sense of finite graphs.

The graph G is said to be regular (of degree n_1) if each vertex of G is joined to exactly n_1 other vertices. In this case each vertex will be unjoined to exactly n_2 other vertices, where

$$n_1 + n_2 = v - 1. \tag{2.1}$$

A regular graph G will be said to be strongly regular if (i) any two vertices which are joined in G, are both simultaneously joined to exactly p^1_{11} other vertices (ii) any two pairs of vertices which are unjoined in G, are both simultaneously joined to exactly p^2_{11} vertices.

A strongly regular graph G thus depends on four parameters n_1, n_2, p^1_{11} and p^2_{11}, the number of vertices being given by (2.1).

Let two vertices of a strongly regular graph G be called first associates if they are joined, and second associates if they are unjoined. If the vertices θ and ϕ of G are i-th associates, we shall denote by $p^i_{jk}(\theta, \phi)$ the number of vertices which are j-th associates of θ and k-th associates of ϕ. From definition the number $p^i_{11}(\theta, \phi)$ is independent of the pair θ, ϕ so long as they are i-th associates. Thus

$$p^i_{11}(\theta, \phi) = p^i_{11} \qquad i = 1, 2.$$

We shall show that a similar situation prevails with respect to all the numbers $p^i_{jk}(\theta,\phi)$, so that we can write

$$p^i_{jk}(\theta,\phi) = p^i_{jk} \qquad i, j, k = 1,2$$

and that

$$p^i_{jk} = p^i_{kj}. \tag{2.2}$$

This follows from the following theorems proved by Bose and Clatworthy 1 , in connection with partially balanced incomplete block (PBIB) designs (if we identify treatments with vertices).

Theorem 2.1. Let there exist a relationship of association between every pair among v treatments satisfying the conditions :

(a) Any two treatments are either first associates or second associates (b) Each treatment has n_1 first and n_2 second associates (c) For any pair of treatments which are first associates the number p^1_{11} of treatments common to the first associates of the first and the first associates of the second is independent of the pair of treatments with which we start.

Then, for every pair of first associates among the v treatments the numbers p^1_{12}, p^1_{21} and p^1_{22} are constants, and $p^1_{12} = p^1_{21}$.

Theorem 2.2. Let there exist a relationship of association between every pair among v treatments satisfying the conditions :

(a) Any two treatments are either first associates (b) Each treatment has n_1 first associates and n_2 second associates (c) For any pair of treatments which are second associates, the number p^2_{11} of treatments common to the first associates of the first and the first associates of the second is independent of the pair of treatments with which we start.

Then, for every pair of second associates among the v treatments the numbers p^2_{12}, p^2_{21} and p^2_{22} are constants, and $p^2_{12} = p^2_{21}$.

It appears from the proof of Bose and Clatworthy that

$$(2.3)\qquad p^1_{12} = n_1 - p^1_{11} - 1 = p^1_{21},\quad p^1_{22} = n_2 - n_1 + p^1_{11} + 1,$$

$$(2.4)\qquad p^2_{12} = n_1 - p^2_{11} = p^2_{21},\qquad p^2_{22} = n_2 - n_1 + p^2_{11} - 1.$$

Actually the relations (2.3), (2.4) were obtained much earlier by Bose and Nair [3], but in their formulation they started with the constancy of all the numbers p^i_{jk} $(i,j,k = 1,2)$. It is also easy to prove as shown by Bose and Nair [3], that

$$(2.5)\qquad n_1\ p^1_{12} = n_2\ p^2_{11},\qquad n_1\ p^1_{22} = n_2\ p^2_{12}.$$

The concept of a strongly regular graph is isomorphic with the concept of an association scheme with two associate classes, as introduced by Bose and Shimamoto [4], in connection with the theory of partially balanced designs, if treatments are identified with vertices, a pair of first associates with a pair of joined vertices, and a pair of second associates with a pair of unjoined vertices. Thus strongly regular graphs first arose in connection with the theory of partially balanced designs.

The numbers v, n_1, n_2, p^i_{jk} are called the parameters of the regular graph G. They are connected by the relations (2.1) - (2.5), and only four are independent. These may be conveniently chosen as n_1, n_2, p^1_{11} and p^2_{11}.

3. Partial geometries.

Consider two undefined classes of objects called points and lines, together with a relation incidence, such that a point and a line, may or may not be incident. Then the points and lines are said to form a partial geometry (r,k,t) provided that the following axioms are satisfied :

A1. A pair of distinct points is not incident with more than one line.

A2. Each point is incident with exactly r lines.

A3. Each line is incident with exactly k points.

A4. Given a point P not incident with a line l, there are exactly t lines $(t \geqq 1)$ which are incident with P, and also incident with some point incident with l.

If there were two distinct lines l and m each incident with two distinct points P_1 and P_2, then A1 would be contradicted. Hence we have,

A'1. A pair of distinct lines is not incident with more than one point.

For convenience we will use the ordinary geometric language. Thus if a point is incident with a line, we shall say that the point lies on the line or is contained in the line, and that the line passes through the point. If two points are incident with the line, then we speak of the line as joining the two points. By A1 there cannot be more than one line joining two points. Thus either two points are unjoined or joined by a unique line. If two lines are incident with a common point, they are said to intersect, and the common point is said to be their point of intersection. By A'1 two lines cannot intersect in more than one point. Hence two lines are either non-intersecting or intersect in a unique point.

Theorem 3.1. Given a partial geometry (r,k,t) there exist a dual partial geometry $PG(k,r,t)$ obtained by calling the points of the first, the lines of the second ; and the lines of the first the points of the second.

This follows by noting the duality of A1 and A'1, the duality A2 and A3 and the self dual nature of A4. In fact A4 can be rephrased as

A'4. Given a line l not incident with a point P there are exactly t points which are incident with l and also incident with some line incident with P.

In terms of the alternative geometric language introduced we may write A4 and A'4 as

A4. Through any point P not lying on a line l there pass exactly t lines intersecting l.

A'4. On any line l not passing through a point P, there lie exactly t points, joined to P.

The equivalence of A4 and A'4 is now obvious.

4. Graph of a partial geometry.

The graph G of a partial geometry (r,k,t) is defined as follows : The vertices of G are the points of the partial geometry. Two vertices of G are joined (adjacent) if the corresponding points of the geometry are joined (incident with the same line). Two vertices of G are unjoined (non-adjacent) if the corresponding points of the partial geometry are unjoined (i.e. there exists no line incident with both the points).

Theorem 4.1. The graph G of partial geometry (r,k,t) is strongly regular with parameters

$$n_1 = r(k-1), \qquad n_2 = (r-1)(k-1)(k-t)\ /\ t, \tag{4.1}$$

$$p_{11}^1 = (t-1)(r-1) + k-2, \qquad p_{11}^2 = rt, \tag{4.2}$$

$$1 \leqq t \leqq r, \qquad 1 \leqq t \leqq k. \tag{4.3}$$

Let there be v points and b lines in the partial geometry. Since the points of the geometry have been identified with the vertices of the graph G, we can call two points of the geometry first associates if they are joined by a line, and second associates if they are not joined by a line. Now through any point P of the geometry there pass r lines, each of which contains $k-1$ other points besides P. Hence P has exactly $r(k-1)$ first associates. Hence

$$n_1 = r(k-1). \tag{4.4}$$

This shows that G is a regular graph. Consider the $b-r$ lines not passing through P. From A4 each of these lines contains exactly t first associates of P. Any particular first associate Q of P, lies on $r-1$ such lines, since one of the r lines passing through Q joins it to P. Hence the number of first associates is

$$n_1 = t(b-r)\ /\ (r-1). \tag{4.5}$$

Comparing (4.4) and (4.5) we have

$$b = r\left[(r-1)(k-1) + t\right] / t. \tag{4.6}$$

Again each of the $b-r$ lines not passing through P contains exactly $k-t$ second associates of P. Any particular second associate R of P lies on r such lines. Hence the number of second associates of P is

$$n_2 = (k-t)(b-r) / r.$$

Substituting for b from (4.6) we have

$$n_2 = (r-1)(k-1)(k-t) / t. \tag{4.7}$$

Consider any two points P and Q which are first associates. They are joined by a line l. We shall count the number of points which are first associates to each of P and Q. The $k-2$ points on l other than P and Q are first associates of both P and Q. Now there pass $(r-1)$ lines through P, other than l. By A4 each of these contains $t-1$ first associates of Q other than P. Thus these $(t-1)(r-1)$ points are first associates of both P and Q. It is easy to see that there are no other first associates of both P and Q. Hence

$$p^1_{11} = (t-1)(r-1) + k-2. \tag{4.8}$$

Consider two points P and R which are second associates. There is no line joining them. Each of the r lines passing through P contains t first associates of R. Hence

$$p^2_{11} = rt. \tag{4.9}$$

We have now verified the formulae (4.1) and (4.2), showing that G is a strongly regular graph. The values of the other parameters p^i_{jk} follow from the identities (2.3) and (2.4). The inequality (4.3) clearly follows from axiom A4.

<u>Corollary</u>. The number of points v and the number of lines b in a partial geometry (r,k,t) is given by

$$v = k\left[(r-1)(k-1)+t\right]/t, \tag{4.10}$$

$$b = r\left[(r-1)(k-1)+t\right]/t. \tag{4.11}$$

In view of the isomorphism of association schemes with two associate classes, and strongly regular graphs, the definition of partially balanced incomplete block (PBIB) designs given by Bose and Shimamoto [4], may now be rephrased as follows :

Given a strongly regular graph G with parameters n_1, n_2, p^1_{11}, p^2_{11}, we may identify its v vertices with v treatments. Then a PBIB design is an arrangement of the v treatments into b sets (called blocks) such that.

(α) Each treatment is contained in exactly r blocks

(β) Each block contains k distinct treatments

(γ) Any two tretments which are first associates (joined in G) occur together in exactly λ_1 blocks. Any two treatments which are second associates (unjoined in G) occur together in λ_2 blocks

The design may be called a PBIB design $(r,k,\lambda_1,\lambda_2)$ based on the strongly regular graph G.

Given a partial geometry (r,k,t), with graph G, it is clearly a PBIB design $(r,k,1,0)$ based on G, and this PBIB design is a connected design. This follows because two first associates always occur together in a block, and if two treatments θ_0 and θ_2 are second associates, we can find a treatment θ_1 in $p^2_{11} = rt > 0$ ways, such that θ_0 and θ_1 are first associates, and θ_1 and θ_2 are first associates. The <u>incidence matrix</u> of a partial geometry may be defined as the matrix $N = (n_{ij})$ where $n_{ij} = 1$ if the i-th point is incident with the j-th line and 0 otherwise. Then N is also the incidence matrix of the corresponding PBIB design. Now Connor and Clatworthy [11] and Bose and Mesner [2] , have shown that for a connected PBIB design, with two associate classes, NN' has only three distinct characteristic roots, whose multiplicities are 1, α and β where

$$(4.12) \qquad \alpha, \beta = \frac{n_1 + n_2}{2} \mp \frac{(n_1 - n_2) + \gamma(n_1 + n_2)}{2\sqrt{\Delta}},$$

and

$$\gamma = p_{12}^2 - p_{12}^1, \quad \Delta = \gamma^2 + 2(p_{12}^1 + p_{12}^2) + 1.$$

Now these multiplcities are necessarily integral. Using the formulae (4.1), (4.2), (2.3) and (2.4) we find that

$$(4.13) \qquad \alpha = \frac{rk}{t}\frac{(r-1)(k-1)}{(k+r-t-1)}.$$

Hence we have the theorem

Theorem (4.2). A necessary condition for the existence of a partial geometry (r,k,t) is that the number

$$\alpha = \frac{rk}{t}\frac{(r-1)(k-1)}{(k+r-t-1)}$$

is a positive integer.

5. <u>Partially balanced designs, which are partial geometrées.</u>

We have already shown that a partial geometry (r,k,t) is isomorphic to a PBIB design (r,k,1,0) based on the graph (association scheme) of the geometry. However a PBIB design based on a strongly regular graph need not necessarily be a partial geometry. It would therefore be of interest to find sufficient conditions under which a PBIB design (r,k,1,0) based on a strongly regular graph is isomorphic to a partial geometry.

Now Bose and Clatworthy [1] have shown that if there exists a PBIB design (r,k,1,0) based on a strongly regular graph G for which $r < k$, then the parameters of G are given by the formulae (4.1), (4.2) i.e. are the same as the parameters of the graph of some partial geometry (r,k,t). We shall show that the design is indeed a partial geometry and thus establish the following theorem :

Theorem(5.1). If there exists a PBIB design $(r,k,1,0)$ based on a strongly regular graph G, then if $r<k$, the design must be a partial geometry (r,k,t) for some $t \leqq r$. The parameters of G are given by the formulae (4.1), (4.2).

The parameters of G are given by (4.1), (4.2) in view of the result of Bose and Clatworthy already cited. Also the axioms A1, A2, A3 for a partial geometry are implicit in the definition of a PBIB design $(r,k,1,0)$. It therefore only remains to prove axiom A4 which amounts to saying that each block of the design contains exactly t treatments which are first associates of a given treatment not contained in the block.

Let K be the set of k treatments contained in a particular block, and let $\bar{K}$ be the set of the remaining $v-k$ treatments. Let $g(x)$ denote the number of treatments in $\bar{K}$ which have exactly x first associates in K. Then

$$(5.1) \qquad \sum_{x=0}^{k} g(x) = v - k = k\,(k-1)(r-1)\,/\,t.$$

Let us count the number of pairs (P,Q), where P is a treatment in K, Q is a treatment in $\bar{K}$, and P and Q are first associates. Now each treatment in K has $k-1$ first associates in K, and consequently n_1-k+1 first associates in $\bar{K}$. Hence the required number is $k\,(n_1-k+1)$. Again there are $g(x)$ treatments in $\bar{K}$, which have exactly x treatments in K. These treatments contribute $xg(x)$ to our count. Hence

$$(5.2) \qquad \sum_{x=0}^{k} xg(x) = k\,(n_1-k+1) = k\,(r-1)(k-1).$$

Again let us count the number of triplets (P_1P_2,Q) where P_1P_2 is an ordered pair of distinct treatments in K, and Q is a treatment in $\bar{K}$ which is a first associate of both P_1 and P_2. Since P_1 and P_2 have $k-2$ common first associates in K, they have $p^1_{11} - k + 2$ common first associates in $\bar{K}$. Hence the required number of triplets like (P_1P_2,Q) is $k\,(k-1)(p^1_{11}-k+2)$. Now each of the $g(x)$ treatments in $\bar{K}$, which have x first associates in K contribute $x\,(x-1)\,g\,(x)$ to our count. Hence we have the equation

$$(5.3) \qquad \sum_{x=0}^{k} x(x-1)\ g(x) = k(k-1)(p_{11}^{1}-k+2) = k(k-1)(t-1)(r-1).$$

Using (5.1), (5.2) and (5.3) a simple calculation shows that

$$\bar{x} = \sum xg(x) \ / \ \sum g(x) = t,$$

i.e. the average value of x (the number of first associates in K of any treatment of $\bar{K}$ is t. Also

$$\sum_{x=0}^{k} g(x)(x-t)^2 = 0,$$

which shows that x must always have the value t. This proves our theorem.

6. Geometrisable and pseudo-geometric graphs.

A strongly regular graph G which has parameters (4.1) and (4.2) and for which the inequality (4.3) is satisfied, is defined to be a pseudo-geoemtric graph with characteristics (r,k,t). Thus a pseudo-geometric graph with characteristics (r,k,t) has the same parameters as the graph of a partial geometry (r,k,t). However a graph may be pseudo-geometric without being the graph of a partial geometry.

A subset of vertices of a graph G, any two of which are joined is called a clique of G. When G is the graph of a partial geometry there will exist in G a set Σ of distinguished cliques $K_1, K_2, \ldots, K_b$, corresponding to the lines of the geometry satisfying the following axioms :

A*1. Any two joined vertices of G are contained in one and only one clique of Σ .

A*2. Each vertex of G is contained in r cliques of Σ .

A*3. Each clique of Σ contains k vertices of G.

A*4. If P is a vertex of G not contained in a clique K_i of Σ , there are exactly t vertices in K_i which are joined to P $(i = 1, 2, \ldots, b)$.

Hence any graph G in which there exists a set $\sum$ of cliques $K_1, K_2, \ldots, K_b$, satisfying axioms A*1 - A*4 is the graph of a partial geometry (r,k,t). In fact G together with the cliques of $\sum$ is isomorphic to a partial geometry (r,k,t), the vertices of G corresponding to the points and cliques of $\sum$ to the lines of the geometry. Such a graph will be said to be <u>geometrisable</u> (r,k,t).

One may consider graphs in which there exist a set $\sum$ of cliques $K_1, K_2, \ldots, K_b$ satisfying one or more but not all of the axioms A*1, A*2, A*3, A*4, and investigate under what additional conditions the graph will be geometrisable. Thus theorem (5.1) may be rephrased as

Theorem (6.1). If there exists a set $\sum$ of cliques $K_1, K_2, \ldots, K_b$ in a strongly regular graph G, satisfying axioms A*1, A*2, A*3 and if $k > r$, then G is geometrisable (r,k,t).

Theorem (6.2). Let G be a pseudo-geometric graph with characteristics (r,k,t). If it is possible to find in G a set $\sum$ of cliques $K_1, K_2, \ldots, K_b$, satisfying axioms A*1 and A*2, and if $k > r$, then G is geometrisable (r,k,t).

We shall prove that each of the cliques $K_1, K_2, \ldots, K_b$ contains exactly k vertices, and that if Q is any vertex not contained in any clique K_i $(1 \leqq i \leqq b)$, then there are exactly t vertices in K_i which are first associates of (joined to) Q. This will show that if the vertices of G are taken as points, and the cliques $K_1, K_2, \ldots, K_b$ as lines, then we have a partial geometry (r,k,t).

Let P be any vertex. Without loss of generality we can take $K_1, K_2, \ldots, K_r$ to contain P. Now the sets $K_1-P, K_2-P, \ldots, K_r-P$ are disjoint and must contain between them all the r(k-1) first associates of P. From this it follows that the average number of vertices in the r cliques of the set $K_1, K_2, \ldots, K_r$ is k. Hence there exists a clique containing at least k vertices. Let K_j be such a clique. Let us take a subset K of K_j, such that K contains exactly k vertices. Let $\bar{K}$ be the set of vertices not contained in K. Let g(x) be the number of vertices in $\bar{K}$ which have exactly x first associates in K. Then it follows exactly as in the proof of theorem (5.1) that

$$\sum_{x=0}^{k} g(x) = k(k-1)(r-1) \;/\; t,$$

and

$$\sum_{x=0}^{k} x\, g(x) = k(r-1)(k-1).$$

Hence the average value of x is t. Also as before

$$\sum_{x=0}^{k} (x-t)^2\, g(x) = 0,$$

which is only possible if x is constant and equal to t. Hence every point of $\bar{K}$ has exactly t first associates in K.

If K_j contains any vertex Q other than those already contained in K, then Q belongs to $\bar{K}$, and therefore has exactly t first associates in K. But each point of K is a first associate of Q hence $t=k$, which contradicts $1 \leqq t \leqq r < k$. This shows that none of the cliques K_j containing P, contains more than k vertices. Thus each contains exactly k vertices. Since each clique contains at least one vertex each of the clique $K_1, K_2, \ldots, K_b$ contains exactly k vertices. Also if Q is a point not contained in K_i $(1 \leqq i \leqq b)$, then there are exactly t vertices in K_i which are first associates of Q_i. This completes the proof of the theorem.

N.B. Compare theorems (6.1) and (6.2).

7. Examples of partial geometries.

(a) A net (r,k) of degree r and order k is a system of undefined points and lines together with an incidence relation subject to the following axioms (i) There is at least one point (ii) The lines of the net can be partitioned into r disjoint, non-empty, "parallel classes " such that each point of the net is incident with exactly one line of each class and given two lines belonging to distinct classes there is exactly one point of the net which is incident with both lines.

For convenience we can use phrases such as "point is on a line" istead of speaking of incidence. Then it can be readily proved (see for example Bruck [5]) that

(1) Each line of the net contains exactly k distinct points where $k \geqq 1$.

(2) Each point of the net lies on exactly r distinct lines where $r \geqq 1$.

(3) The net has exactly rk distinct lines. These lines fall into r parallel classes of k lines each. Distinct lines of the same parallel class have no common points. Two lines of different classes have one common point.

(4) The net has exactly k^2 distinct points.

We shall show that <u>a net (r,k) of degree r and order k is a partial geometry $(r,k,r-1)$</u>. The properties (1) and (2) above show that axioms A3 and A2 of a partial geometry hold.

Two lines cannot intersect in more than one point, for they either belong to the same parallel class and have no common point, or different parallel classes in which they have one common point. From this follows the fact that two distinct points cannot be incident with more than one line. Hence axiom A1 for a partial geometry holds.

Again given a point P not incident with the line l, There are exactly r lines through P, one belonging to each parallel class. One of these is parallel to l (i.e. belongs to the same parallel class as l), and does not intersect l. The other $r-1$ lines through P each intersects l in one point, these points being all distinct. Hence axiom A4 for a partial geometry holds with $t = r-1$. This completes the proof of our statement.

It follows from theorem 4.1, that the parameters of the graph G_N of a net (r,k) are given by

$$(7.1) \qquad n_1 = r\,(k-1), \qquad n_2 = (k-1)(k-r+1),$$

$$(7.2) \qquad p^1_{11} = (r-2)(r-1)+(k-2), \qquad p^2_{11} = r\,(r-1).$$

If a strongly regular graph has parameters (7.1), (7.2) we shall call it a pseudo-net graph with characteristics (r,k). A pseudo-net graph with characteristics (r,k) is pseudo-geometric with characteristics $(r,k,r-1)$.

Bruck [5], defines the deficiency d of a net (r,k) by

$$(7.3) \qquad d = k - r + 1.$$

The interpretation of the deficiency d is that if it were possible to add d more parallel classes, each consisting of k lines, so that the extended net now has $k+1$ classes of parallels, the net would become an affine plane, in which any two points are joined by a unique line.

If we take the k^2 points of the net as treatments and the rk lines as blocks, we obtain what is known as the <u>lattice</u> design. This is a PBIB design $(r,k,1,0)$ based on the strongly regular graph G_N with parameters given by (7.1), (7.2). Lattice designs were introduced by Yates [24]. The association scheme corresponding to G_N is the L_r scheme defined by Bose and Shimamoto [4].

It is well known that a lattice design with r replications and blocksize k is equivalent to a system of $r-2$ mutually orthogonal Latin squares of order k.

If $r-2$ mutually orthogonal Latin squares of order k are given, we can superpose them. Then each cell contains $r-2$ symbols belonging in order to the different Latin squares. The k^2 cells are now identified with k^2 treatments. Treatments belonging to the same row give one set of k blocks. Treatments belonging to the same column give another set of k blocks. Treatments (cells) which contain the same symbol of the i-th Latin square give a set of blocks for each value of i $(i = 1, 2, \ldots, r-2)$. We thus get r sets of blocks. The treatments and blocks so obtained constitute a lattice design.

Conversely given a Lattice design with r replications and block size k, we can construct a set of $r-2$ mutually orthogonal Latin squares of order k.

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Conversely given a Lattice design with r replications and ~~blocks~~ block size k, we can construct a set of $r-2$ mutually orthogonal Latin squares of order k.

(b) Take an $n \times n$ squares and write down the numbers $1, 2, \ldots, n(n-1)/2$ in the cells above the main deagonal. Fill up the cells below the main diagonal symmetrically. The case $n = 5$ is exemplified below.

*	1	2	3	4
1	*	5	6	7
2	5	*	8	9
3	6	8	*	10
4	7	9	10	*

Fig. 1

The cells containing the same number are identifies with the same point. Thus there are two different cells representing the same point, there being $v = n(n-1)/2$ points all together. Let the n rows constitute lines. Thus there are n lines. It is clear that axioms A1, A2, A3 of a partial geometry are satisfied with $r = 2$, $k = n-1$. It is easy to see that any two lines intersect in one point. Thus $t = 2$, and we have a partial geometry $(2, n-1, 2)$.

If two points which lie on a line are called first associates, and two points which do not lie on any line are called second associates, we have the triangular association scheme first defined by Bose and Shimamoto [4], and extensively studied by Connor [9], Shrikhande [21], Hoffman [13,14] and Chang [6,7]. The parameters of the association scheme or the corresponding strongly regular graph are

$$(7.4)\qquad n_1 = 2(n-2), \qquad\qquad n_2 = (n-2)(n-3)/2,$$
$$(7.5)\qquad p^1_{11} = n-2, \qquad\qquad p^2_{11} = 4.$$

If a strongly regular graph has the parameters (7.4), (7.5) we shall call it a pseudo-triangular graph with characteristic n. A pseudo-triangular graph with characteristic n is pseudo-geometric with characteristics $(2, n-1, 2)$.

(c) A balanced incomplete block design BIB is an arrangement of a set of v_o objects or treatments in b_o sets or blocks, such that (i) each block contains k_o distinct treatments (ii) each treatment is contained in r_o blocks (iii) each pair of distinct treatments is contained in λ_o blocks. This design has sometimes been called a (v_o, k_o, λ_o) configuration. The dual of a design is defined as a new design whose treatments and blocks are in (1,1) correspondence with the blocks and treatments of the original design, and incidence is preserved (where a block and a treatment are incident if the treatment is contained in the block, and non-incident otherwise), Shrikhande [23] has shown that the dual of a BIB design with $\lambda_o = 1$ is a PBIB design. Now a BIB design with $\lambda_o = 1$ is clearly a partial geometry (r_o, k_o, k_o). Hence the dual design is the dual partial geometry PG (k_o, r_o, k_o). If we set $k_o = r$ and $r_o = k$, so as to make r the replication number and k the block size in the dual design, then the dual design is the partial geometry (r,k,r). Any two blocks (lines) of this design intersect in a unique treatment (point). Hence the association scheme of this design has been called the SLB (singly linked block) scheme by Bose and Shimamoto [4]. The parameters of the corresponding strongly regular graph can be written down directly from theorem (4.1). We have

$$(7.6) \qquad n_1 = r(k-1), \qquad n_2 = (k-r)(r-1)(k-1)/r,$$

$$(7.7) \qquad p_{11}^1 = (r-1)^2 + k - 2, \qquad p_{11}^2 = r^2.$$

If a strongly regular graph has the parameters (7.6), (7.7) we shall call it a pseudo-SLB graph with characteristics (r,k). A pseudo-SLB graph with characteristics (r,k) is a pseudo-geometric graph with characteristics (r,k,r).

(d) To conclude we shall give a rather less obvious example of a partial geometry.

Consider an elliptic non-degenerate quadric Q_5 in the finite projective space $PG(5,p^n)$. This quadric is ruled by straight lines, called generators, but contains no plane. As shown by Primrose [16] and Ray-Chaudhuri [17], there are $(s^3+1)(s+1)$ points and $(s^3+1)(s^2+1)$ generators in Q_5, each generator contains $s+1$ points, and through each point there pass s^2+1 generators, where $s = p^n$.

If P is a point on Q_5 not contained in a generator l, then the polar 4-space of P intersects l in a single point P^*, and PP^* is a generator of Q_5. It can be readily verified by using theorems proved by Ray-Chaudhuri that PP^* is the only generator through P, which intersects l. This shows that if we consider the points and generators of Q_5 as points and lines, they constitute a partial geometry $(s^2+1,\ s+1,\ 1)$. The parameters of the graph of this partial geometry can be easily written down using theorem (4.1). They are

$$n_1 = s(s^2+1), \qquad n_2 = s^4+1,$$
$$p_{11}^1 = s-1, \qquad p_{11}^2 = s^2+1.$$

The partial geometry $(s^2+1,\ s+1,\ 1)$ is of course a PBIB design. This was obtained by Ray-Chaudhuri [18], the special case $s = 2$ was given earlier by Bose and Clatworthy [1]. An interesting point in the present formulation is that to verify that the configuration of points and generators on Q_5 is a PBIB design we have only to check the constancy of r, k and t instead of the constancy of

r, k, n_1, n_2, p_{11}^1, p_{11}^2 as was done by Bose and Clatworthy [1] and by Ray-Chaudhuri [18]. The dual partial geometry $(s+1,\ s^2+1,\ 1)$ is also of interest.

In the same way one can show that the configuration of points and generators on a non-degenerate quadric Q_4 in $PG(4,p^n)$ is a partial geometry $(s+1,\ s+1,\ 1)$, where $s = p^n$. The corresponding design was first obtained by Clatworthy [8].

8. Lemmas on claws in pseudo-geometric graphs.

We have shown in theorem (5.1), that if we can base a PBIB design $(r,k,1,0)$ on a strongly regular graph G, then G is pseudo-geometric and the design is a partial geometry. We can ask the converse question : If the graph G is pseudo-geometric (r,k,t) can we base a PBIB design $(r,k,1,0)$ on it ? In graph theoretic language this question may be put as : If the graph G is pseudo-geometric (r,k,t) can we find a set of cliques K_1, K_2, ..., K_b satisfying the axioms A*1 - A*4 of section 6.

In the rest of the paper we shall frequently use the following functions :

(8.1) $\qquad \gamma(r,t) = 1 + (r-1)^2\ (t-1),$

(8.2) $\qquad q(r,t) = 1 + (r-1)(2r-1)(t-1),$

(8.3) $\qquad \rho(r,t) = rt + (r-1)(t-1)(2r-1),$

(8.4) $\qquad p(r,t) = \frac{1}{2}\left[r\ (r-1) + t\ (r+1)(r^2-2r+2)\right].$

We note that in view of the inequality $1 \leqq t \leqq r$

(8.5) $\qquad p(r,t) \geqq \rho(r,t) \geqq q(r,t) \geqq \gamma(r,t)$

The concept of a <u>claw</u> was suggested by Alan Hoffman in conversation with R.H. Bruck and the author. By a claw $[P,S]$ of a pseudo-geometric graph G is meant an ordered pair consisting of a vertex P, the <u>vertex</u> of the claw, and a non-empty set S of vertices distinct from P such that every vertex in S is joined to P in G but no two vertices in S are joined in G.

The number of elements in a finite set S will be denoted by $|S|$. The order of the claw $[P,S]$ is defined as $s = |S|$.

In Lemmas 8.1 - 8.3, G denotes a pseudo-geometric graph (r,k,t).

Lemma 8.1. If $k > \gamma(r,t) = 1 + (t-1)(r-1)^2$ then for any s, $1 \leq s \leq r$, each vertex P of G is the vertex of a claw $[P,S]$ of order s. We can choose S to include any vertex A joined to P in G.

Let P be a vertex of G. Suppose there exists a claw $[P,S]$ of order s. Let T be the set of all vertices other than those belonging to $[P,S]$ and which are joined to P (are first associates of P). Let $f(x)$ be the number of vertices Q in T, such that Q is joined to exactly x vertices in S ($f(x)$ is the number of vertices in T each of which has $f(x)$ first associates in S). Then we have

$$(8.6) \qquad \sum_{x=0}^{s} f(x) = n_1 - s = r(k-1) - s,$$

since the left hand side of (8.6) counts all first associates of P, which are not in S. Now let us count pairs (A,Q) where A is in S and Q is in T and is a first associate of both A and P. Since A and P have exactly p_{11}^1 common first associates (none of which can belong to S by the definition of a claw) we have sp_{11}^1 pairs like (A,Q).

Again there are $f(x)$ vertices in T each of which has exactly x first associates in S. They contribute $xf(x)$ to our count, Hence

$$(8.7) \qquad \sum_{x=0}^{s} x\, f(x) = sp_{11}^1 = s\left\{(t-1)(r-1)+(k-2)\right\}.$$

Hence

$$(8.8) \qquad f(0) - \sum_{x=1}^{s} (x-1)\, f(x) = (r-s)(k-1) - s(t-1)(r-1).$$

Hence if $s < r$ and $k > \rho(r,t) = 1 + (r-1)^2 (t-1)$, then $f(0)$ is positive, i.e. there is at least one vertex A_{s+1} in T which is not joined to $A_1, A_2, \ldots, A_s$. We can therefore add A_{s+1} to S and get a claw of order $s+1$. In this way we can go on extending a claw till we get a claw of any required order not exceeding r. We can start this process with any claw $[P,A]$ of order 1. This proves the Lemma.

Lemma (8.2). If $k > \rho(r,t) = 1 + (r-1)^2 (t-1)$, and if $[P,S]$ is a claw of order $r-1$, then there exist at least $k - (r,t)$ distinct vertices Q of G such that $[P, S \cup Q]$ is a claw of order r.

If $[P,S]$ is a claw of order $r-1$, then from (8.3)

$$f(0) \geqq k - \rho(r,t).$$

Hence there exist at least $k - \rho(r,t)$ vertices Q, each of which taken together with S give a claw $[P,S^*]$ of order r where $S^* = S \cup Q$.

Lemma (8.3). If $k > p(r,t) = \frac{1}{2}\left[r(r-1) + t(r+1)(r^2-2r+2)\right]$ then there exists in G no claw of order $r+1$.

Let $[P,S]$ be a claw of order s in G. Let the set T be as in Lemma (8.1). We shall count the number of triplets $(A_1 A_2, Q)$ where A_1, A_2 is an ordered pair of distinct vertices in S, and Q is a vertex in T which is a first associate of both A_1 and A_2. Since A_1 and A_2 are second associates they have exactly $p_{11}^2 - 1$ common first associates other than P. Some of these may not lie in T. Hence an upper bound for the required number of triplets is $s(s-1)\, p_{11}^2$. However the $f(x)$ vertices in T, each of which has exactly x first associates in S, contribute $x(x-1)\, f(x)$ to our count. Hence

$$(8.9) \qquad \sum_{x=0}^{s} x\,(x-1)\,f(x) \leqq s\,(s-1)(p_{11}^2-1) = s\,(s-1)(rt-1).$$

If possible let $s = r+1$. Then adding (8.8) to (8.9) multiplied by a half, and noting that on the left hand side of (8.9) the term $x = 0$ contributes nothing we get

$$f(0) + \frac{1}{2}\sum_{x=1}^{s}(x-1)(x-2)\ f(x) \leqq -\ k+\frac{1}{2}\left[r(r-1)+t(r+1)(r^2-2r+2)\right]$$

Hence if $k > p(r,t)$, there cannot exist a claw of order $r+1$, as the left hand side is essentially positive.

9. Lemmas on cliques in pseudo-geometric graphs.

In the Lemmas (9.1) to (9.6), G denotes a pseudo-geometric graph (r,k,t). The definition of a major clique generalizes Bruck's definition, and the concept of a grand clique is taken over from Bruck [5].

A major clique K of G is a clique such that

$$|K| \geqq k+1 - \rho(r,t) = k - (r-1)^2\ (t-1).$$

A grand clique is a major clique which is also a maximal clique.

If K and L are distinct maximal cliques, then $K \cup L$ cannot be a clique. Since K and L are distinct, there must be a vertex P in one of them (say K), not belonging to the other (L). Now if $K \cup L$ is a clique, then P is joined to every vertex of L. Thus $P \cup L$ is a clique which contradicts the fact that L is maximal. Since grand cliques are maximal the union of two grand cliques cannot be a clique.

If we take the set of grand cliques as the set Σ of section 6, we may enquire under what conditions the axioms A*1, A*2, A*3 and A*4 are satisfied. The Lemmas which follow are directed to this purpose.

Lemma (9.1). If $k > (r,t)$ and if G has no claw of order $r+1$, then for every pair of distinct joined vertices P and Q in G, there exists at least one major clique containing both P and Q.

From Lemma (8.1) we can find a claw $[P,S]$ of order r such that $Q \in S$. Let $A_1, A_2, \ldots, A_{r-1}$ be other vertices of S. Let Ω be the set of vertices R which when adjoined to S, give a claw P,S^* of order r, $S^* = S \cup R$. From Lemma (8.2), the number of points in $\Omega = |\Omega| \geqq k - \rho(r,t)$.

The vertices in Ω are all joined to one another. If any two were not joined they could be added to $A_1, A_2, \ldots, A_{r-1}$ to give a claw of order $r+1$. Thus P and the vertices in Ω are all mutually joined. Hence $P \cup \Omega = K$ is a clique of order $\geqq k+1 - \rho(r,t)$, i.e. a major clique.

Cor 1. In Lemma (9.1) the hypothesis may be replaced by $k > \not p\,(r,t)$.

This follows from Lemma (8.3) by noting that $p\,(\not t,t) \geqq \rho(r,t)$ for $1 \leqq t \leqq r$.

Cor 2. When the conditions of Lemma (9.1) or Cor. 1 are satisfied, P and Q are contained in at least one grand clique.

We can extend the major clique K by adding new vertices till it becomes maximal, and therefore a grand clique.

Lemma (9.2). If K and L are cliques of G and $K \cup L$ is not a clique then $|K \cap L| \leqq rt$.

Since $K \cup L$ is not a clique, there must exist in $K \cup L$ a pair of vertices P_1, P_2 not joined to one another such that $P_1 \in K$, $P_2 \in L$. Any vertex belonging to $|K \cap L|$ must be joined to both P_1 and P_2. Hence the number of vertices in $|K \cap L|$ cannot exceed p^2_{11}. Thus

$$|K \cap L| \leqq rt.$$

Lemma (9.3). If K and L are cliques of G and $K \cap L$ contains at least two vertices A and B, then

$$|K \cup L| \leqq k + (r-1)(t-1).$$

Every vertex in $K \cup L$, other than A and B, is a first associate of both A and B. Hence

$$|K \cup L| \leqq p^1_{11} + 2 = k + (r-1)(t-1).$$

Lemma (9.4). If K and L are cliques of G such that (i) $K \cup L$ is not a clique (ii) $K \cap L$ contains at least two vertices, then

$$|K| + |L| \leqq k + rt + (r-1)(t-1).$$

Since conditions of Lemmas (9.2) and (9.3) are satisfied

$$|K| + |L| = |K \cap L| + |K \cup L|$$

$$\leqq rt + k + (r-1)(t-1).$$

Lemma (9.5). If $k > \rho(r,t) = rt + (r-1)(t-1)(2r-1)$ and if G has no claw of order $r+1$, then two distinct vertices of G which are joined, are contained in one and only one grand clique.

Suppose there are two distinct grand cliques K and L both containing P and Q. Then $K \cup L$ cannot be a clique. Also $K \cap L$ has at least two vertices P and Q. Hence from Lemma (9.4),

$$|K| + |L| \leqq rt + k + (r-1)(t-1).$$

Since K and L are grand cliques, they are both major. Hence

$$2\left[k - (r-1)^2 (t-1)\right] \leqq |K| + |L| \leqq rt + k + (r-1)(t-1).$$

$$k \leqq rt + (r-1)(t-1)(2r-1) = \rho(t).$$

which is a contradiction.

Cor. In Lemma (9.5) the hypothesis may be replaced by $k > p(r,t)$.

Lemma (9.6) If (i) $k > q(r,t) = 1 + (r-1)(2r-1)(t-1)$, (ii) two distinct vertices of G are contained in utmost one grand clique of G, (iii) there exists no claw of order $r+1$ in G, then each point of G is contained in exactly r grand cliques.

From Lemma (9.1), Cor. 2 any two vertices are contained in at least one grand clique. It follows from assumption (ii), that any two vertices of G are contained in one and only one grand clique. Again from Lemma (8.1), there exists a claw $[P,S]$ of order r, where $S = \{A_1, A_2, \ldots, A_r\}$. As in Lemma (8.1), let T be the set of first associates of P, other than $A_1, A_2, \ldots, A_r$. We define H_j as the set consisting of P, A_j and all $Q \in T$, such that Q is a first associate of A_j but not of A_i when $i \neq j$. Let $f(x)$ be as in

Lemma (8.1). Now f(0), since there are no claws of order $r+1$. Hence from (8.5) and (8.6) we have

$$\sum_{x=1}^{r} f(x) = n_1 - r = r(k-2),$$

$$\sum_{x=1}^{r} xf(x) - (x-1)f(x) = r(k-2) + r(r-1)(t-1).$$

$$\sum_{x=2}^{r} (x-1)\ f(x) = r(r-1)(t-1)$$

Now

$$\sum_{2}^{r} f(x) \leqq \sum_{2}^{r} (x-1)\ f(x) \leq (r-1) \sum_{2}^{r} f(x).$$

$$r(t-1) \leq \sum_{2}^{r} f(x) \leq r\ (r-1)(t-1).$$

$$r\ (t-1) \leq r(k-2) - f(1) \leq r(r-1)(t-1).$$

Hence we have

$$(9.1) \qquad r(k-t-1) \geq f(1) \geq r\ \{(k-2) - (r-1)(t-1)\}.$$

Any two vertices in H_j are joined together otherwise there would be a claw of order $r+1$. Thus H_j is a clique.

If we put $H_j^* = H_j - (A_j \cup P)$, then H_j^* consists of exactly those vertices of T which are joined to A_j but to no other vertex of S. Hence $H_1^*, H_2^*, \ldots, H_r^*$ are disjoint sets and the total number of vertices in these sets is $f(1)$, which satisfies (9.1).

Now there is a unique grand clique K_j containing A_j and P $(j = 1, 2, \ldots, r)$. The number of vertices in K_j cannot be less than the number of vertices in H_j. If possible let $|K_j| < |H_j|$. Since K_j is a grand clique it follows that H_j is a major clique and is contained in some grand clique K_j'. Since A_j and P are contained in K_j and K_j', they must coincide. Hence K_j contains H_j which contradicts $|K_j| < |H_j|$.

Now consider the r grand cliques $K_1, K_2, \ldots, K_r$. Then $K_1-P, K_2-P, \ldots, K_r-P$ are disjoint. For if K_i-P and K_j-P $(i \neq j)$ have a common point Q, then K_i and K_j would coincide and would contain both A_i and A_j which is impossible since A_i is not joined to A_j. Now

$$|K_1-P| + |K_2-P|+\ldots+|K_t-P| \geqq |H_1-P|+|H_2-P|+\ldots+|H_r-P|$$

$$= r + |H_1^*|+|H_2^*|+\ldots+|H_r^*|$$

$$= r + f(1)$$

$$\geq r\left\{(k-1) - (r-1)(t-1)\right\}.$$

If possible suppose there is another grand clique K_r+1 containing P. The vertices in $K_{r+1}-P$ must be disjoint from the vertices in $K_1-P, \ldots, K_r-P$. Also $K_{r+1}-P$ must have at least $(k-1) - (r-1)^2 (t-1)$ vertices. If we remember that the number of first associates of P is $r(k-1)$ we have

$$r\left\{(k-1) - (r-1)(t-1)\right\} + (k-1) - (r-1)^2 (t-1) \leqq r(k-1).$$

$$k \leqq 1 + (r-1)(2r-1)(t-1).$$

Hence $k \leqq q(r,t)$, which gives a contradiction. This finally proves our Lemma.

Cor. 1. The hypothesis of the Lemma may be replaced by (i) $k > \rho(r,t)$ and (ii) there exists no claw of order $r+1$ in G.

This follows from Lemma (9.5), and the inequality $\rho(r,t) \geqq q(r,t)$.

Cor. 2. The hypothesis of the Lemma may be replaced by $k > p(r,t)$.

This follows from Cor. 1, Lemma (8.3), and the inequality $p(r,t) \geqq \rho(r,t)$.

Theorem (9.1). Let G be a pseudo-geometric graph (r,k,t). If (i) $k > q(r,t)$, (ii) two distinct vertices of G are contained in utmost one grand clique, (iii) there exists no claw of order $r+1$ in G; then G is geometrisable (r,k,t).

Theorem (9.2). Let G be a pseudo-geometric graph (r,k,t). If (i) $k > \rho(r,t)$, (ii) there exists no claw of order $r+1$ in G; then G is geometrisable (r,k,t).

Theorem (9.3). Let G be a pseudo-geometric graph (r,k,t). If $k > p(r,t)$, then G is geometrisable (r,k,t).

If we take the set of grand cliques of G, as the set of section 6, then Lemmas (9.5) and (9.6), together with their corolla-ries show that the axioms A*1 and A*2 are satisfied, under the conditions of any of the Theorems (9.1), (9.2), (9.3). The result now follows from Theorem (6.2).

10. The uniqueness of the triangular association scheme for $n > 8$.

Consider a pseudo-triangular graph with characteristic n, which is a pseudo-geometric graph with characteristics $(2,n-1,2)$ and with parameters (7.4), (7.5). In this case $r = 2$, $t = 2$ and the function $p(r,t)$ given by (8.4) is equal to 7. Hence from Theorem (9.3) the graph is geometrisable $(2,n-1,2)$ if $n-1 > 7$, i.e. $n > 8$.

Now $v = n(n-1)/2$ and $r = 2$. Thus each point occurs in exactly two lines. Given any point P not contained in a line b, the two lines m_1 and m_2 containing P, must both intersect b, since $r = t = 2$. Hence any two lines intersect in a unique point. If we designate the lines by the numbers $1, 2, \ldots, n$; then we may make a (1,1) correspondence between ~~the~~ points and the unordered number of pairs (i,j), $i \neq j$, $i,j = 1, 2, \ldots, n$, where the point corresponding to (i,j) is the intersection of the lines i and j. If we now take an $n \times n$ square and write down in the cells (i,j) and (j,i) the treatment corresponding to the unordered pair (i,j), then clearly the points occurring in the same row (or same column) are those occurring in the same line (see Fig. 1 for the case $n = 5$). Thus the association relations between the vertices of the graph will be exhibited in the form known as the triangular scheme.

for $n > 8$. This result was first obtained by Connor [9]. Of course when we use design of experiments language the vertices of the graph or points are treatments.

Shrikhande [21] has proved the uniqueness of the triangular scheme for $n = 5, 6$ and Hoffman [13] and Chang [6] have proved the same for $n = 7$. Both Hoffman [14] and Chang [7] have shown that for $n = 8$, the parameters (7.4), (7.5) do not completely determine the scheme. There are three other possible schemes with the same parameters besides the triangular. This may be expressed as follows : <u>There are four non-isomorphic strongly regular graphs with parameters</u>

$$n_1 = 12, \quad n_2 = 15, \quad p_{11}^1 = 6, \quad p_{11}^2 = 4$$

<u>only one of which is geometrisable</u> (2,7,2).

Consider a BIB design

$$v^* = \frac{1}{2}(n-1)(n-2), \quad b^* = \frac{1}{2}n(n-1), \quad r^* = n, \quad k^* = n-2, \quad \lambda^* = 2 \tag{9.1}$$

Hall and Connor [12] have shown that if this design exists then it can be embedded in a symmetric BIB design

$$v_o = b_o = \frac{1}{2}n(n-1) + 1, \quad r_o = k_o = n, \quad \lambda_o = 2 \tag{9.2}$$

Their proof does not cover the case $n = 8$, for which Connor [10] seperately showed that the design (9.1) does not exist.

Shrikhande [22], has proved the Hall-Connor theorem for the case $n \neq 8$ by using the uniqueness of the triangular scheme for $n \neq 8$. It is interesting to observe that $n = 8$, the case not covered in Hall and Connor's entirely different proof, is exactly the case when the parameters (7.4), (7.5) do not uniquely characterise the scheme as triangular.

11. Theorems of Shrikhande, Bruck and Mesner on the uniqueness of the L_r scheme.

Consider G_N as a pseudo-net graph with characteristics (r,k) or the corresponding association scheme with parameters (7.1), (7.2). Since $t = r-1$ in this case, G_N is geometrisable $(r,k,r-1)$ if

$$(11.1) \qquad k > p(r,r-1) = \frac{1}{2}(r-1)(r^3 - r^2 + r + 2).$$

In particular if $r = 2$, this reduces to $k > 4$.

In the case $r = 2$, the geometry consists of two sets of parallel lines. Each parallel class contains k lines, and each line contains k points. Lines of the same class do not intersect. Lines of different classes intersect in a point. Thus each point is uniquely determined as the intersection of one line from each class. We can number the lines of each class $1, 2, \ldots, k$; and we can number the points or vertices of the graph $1, 2, \ldots, k^2$. We now take a $k \times k$ square and identify the i-th line of the first class with the i-th row, the j-th line of the second class with the j-th column, and the cell (i,j) with the point which is intersection of the line i of the first class, with the line j of the second class, then the association relations between the vertices of the graph are exhibited as an L_2 scheme. This proves the uniqueness of the L_2 scheme for $n > 4$ a result obtained by Shrikhande [19] and by Mesner [15].

In the general case the geometry consists of a set of k^2 points (the vertices of G_N) and a r classes of parallel lines, each class containing k lines. Lines of the same class do not intersect. Lines of different classes intersect in one point. Let the parallel classes be designated by (R), (C), (U_1), ..., (U_{r-2}). To the k lines within each class we assign the symbols $1, 2, \ldots, k$. Each point is uniquely given by the intersection of a line of (R), with a line of (C). Hence as in the case $r = 2$, the lines of (R) may be identified with the rows, and the lines of (C) of the columns of a $k \times k$ square. Then the intersection of the line of (R) and the line of (C) is identified with the cell (i,j). If in each cell (i,j) we put the number of the line of (U_α) which passes through the point corresponding to the cell, we get a Latin square L_α $(\alpha = 1, 2, \ldots, r-2)$

and the Latin squares $L_1, L_2, \ldots, L_{r-2}$ are mutually orthogonal. Two points (cells) are first associates if they lie in the same row, same column or correspond to the same letter of the same Latin square. Thus the association relations between the points or vertices of G_N can be exhibited by the L_r scheme defined by Bose and Shimamoto [4]. Thus the L_r scheme with parameters (7.1) - (7.2) is unique (up to type) if (11.1) holds. It is necessary to add the words up to type, since there may be many non-isomorphic sets of $r-2$ mutually orthogonal Latin squares. This result is implicit in Bruck's paper [5]. A slightly weaker result was proved by Mesner [15].

12. The SLB scheme and the general uniqueness theorem.

Let consider the SLB scheme or the pseudo-SLB graph with characteristics (r,k) for which the parameters are given by (7.6), (7.7). Then theorem (9.3) states that the graph is geometrisable (r,k,r) if

$$k > \frac{1}{2} r(r^3 - r^2 + r + 1). \tag{12.1}$$

In the language of designs this would mean that if there is an association scheme with parameters (7.6), (7.7) then if (12.1) holds the association relations can be exhibited by the dual of a BIB design (with $r_o = k$ and $k_o = r$, $\lambda_o = 1$) so that the first associates are exactly those which occur together in a block of this dual and the second associates are those which do not occur together in a block of this dual. Thus (7.6), (7.7) determine the structure of the association scheme up to type. It is necessary to add the words up to type since there will exist in general non-isomorphic BIB designs with ($r_o = k$, $k_o = r$, $\lambda_o = 1$) and their duals will automatically be non-isomorphic. When (12.1) does not hold we cannot say that their will exist a dual of a BIB design i.e. a partial geometry (r,k,r), whose structure will exhibit the association relations.

In general then we can say that if we have a pseudo-geometric association scheme with parameters (4.1), (4.2), then if

$$(12.2)\qquad k > p(r,t) = \frac{1}{2}\left[r(r-1) + t(r+1)(r^2 - 2r + 2)\right],$$

the association structure can be exhibited by means of a partial geometry (r,k,t), the first associates being those treatments which correspond to points on a line of the geometry. Such schemes may called geometric schemes. Thus when (12.2) is true, the association scheme will be determined up to type, for there will exist non-isomorphic partial geometries with the same parameters r,k,t. This may be regarded as a generalized uniqueness theorem. When (12.2) is not true very little is known except for schemes which have the same parameters as the triangular scheme or the L_2 scheme ($r = 2$, k, $t = 1$) These two cases have been fully investigated.

13. A general embedding theorem.

Theorem (13.1). Given a PBIB design $(r,k,\lambda_1,\lambda_2)$, $\lambda_1 > \lambda_2$ based on a strongly regular graph G (association scheme) with parameters

$$(13.1)\qquad n_1 = (d-1)(k-1)(k-t)/t, \qquad n_2 = d(k-1),$$

$$(13.2)\qquad p_{11}^1 = \left[(d-1)(k-1)(k-t) - d(k-t-1) - t\right]/t\ ,$$

$$p_{11}^2 = (d-1)(k-t)(k-t-1)/t.$$

we can extend the design by adding new blocks, containing the same treatments, in such a way that the extended design is a balanced incomplete block (BIB) design with $r_0 = r + d(\lambda_1 - \lambda_2)$ replications, block size k and in which every pair of treatments occur together in λ_1 blocks, provided that

$$(13.3)\qquad k > p(d,t) = \frac{1}{2}\left[d(d-1) + t(d+1)(d^2 - 2d + 2)\right].$$

Let G^* be the complementary of G, i.e. G^* is the graph with the same vertices as G, but with the relation of adjacency reversed, i.e. just those vertices in G^* are joined which were unjoined in G. This means that first associates become second associates and vice versa. The parameters of G^* are obtained from G by interchanging the subscripts and superscripts 1 and 2. Hence for G^*

$$(13.3)\quad n_1^* = d(k-1), \qquad n_2^* = (d-1)(k-1)(k-t)/t,$$

$$(13.4)\quad p_{22}^{2*} = \left[(d-1)(k-1)(k-t)-d(k-t-1)-t\right]/t,$$

$$p_{22}^{1*} = (d-1)(k-t)(k-t-1)/t.$$

Using the identities (2.3), (2.4), we find that

$$p_{11}^{1*} = (t-1)(d-1) + k-2, \qquad p_{11}^{2*} = dt.$$

Hence G^* is pseudo-geometric with characteristics (d,k,t).

In view of (13.3) it follows from theorem (9.3) that G^* is geometrisable (d,k,t). From (4.1) the geometry has $d\left[(d-1)(k-1) + t\right]/t$ blocks, and every pair of treatments which were second associates in the original PBIB design occur once in the new blocks. If we add these new blocks repeated $\lambda_1 - \lambda_2$ times to the original blocks then each pair occurs λ_1 times and each treatment occurs $r + d(\lambda_1 - \lambda_2)$ times. This proves our theorem.

We shall now drive from this the embedding theorem on orthogonal Latin squares due to Shrikhande and Bruck.

In theorem (13.1) take $t = d-1$, then the G^* is a pseudo-geometric graph $(d,k,d-1)$ i.e. a pseudo-net graph. If

$$k > p(d,d-1) = \frac{1}{2}(d-1)(d^3 - d^2 + d + 2),$$

it is geometrisable.

Also let us take $r = k + 1 - d$, $\lambda_1 = 1$, $\lambda_2 = 0$. Then the PBIB design becomes the design $(k + 1 - d, k, 1, 0)$ based on the strongly regular graph with parameters (7.1), (7.2). This is easy to check by substituting $d = k + 1 - r$ in (13.1), (13.2) and noting that they reduce to (7.1), (7.2). Hence the PBIB design is a net of degree $k + 1 - d$, or a lattice design with $r = k + 1 - d$ and block size k. Hence the extended design is a BIB design with $r + d$ i.e. $k + 1$ replications in which every pair of treatments occurs in one block. This is an affine plane of order k. Hence we have

Theorem (13.2A). A lattice design (or a net) with $r = k + 1 - d$ and block size k can be completed to an affine plane

$$v_o = k^2, \quad b_o = k(k+1), \quad r_o = k+1, \quad k_o = k, \quad \lambda_o = 1,$$

by adding kd new blocks, if $k > \frac{1}{2}(d-1)(d^3 - d^2 + d + 2)$.

Now we have already notised the equivalence of a lattice design with r replications and block size k, with a set of $r-2$ mutually orthogonal Latin squares of order k. Since an affine plane of order k can be regarded as a lattice with $k+1$ replications, theorem (13.2A) may alternatively be stated as

Theorem (13.2B). If there exist $k-1-d$ mutually orthogonal Latin squares of order k, it is possible to get a complete set of $k-1$ mutually orthogonal Latin squares, by adding d new suitably chosen squares, provided that $k > \frac{1}{2}(d-1)(d^3 - d^2 + d + 2)$.

The case $d = 2$, was first obtained by Shrikhande [20] and the general case was obtained by Bruck [5].

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