FOURIER METHODS IN THE STUDY OF VARIANCE FLUCTUATIONS
IN TIME SERIES ANALYSIS

by

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1. INTRODUCTION

Almost all the literature on time series analysis is devoted to stationary processes; that is, in the wide sense definition, the class of stochastic processes where the variance is finite and the covariance between any two observations depends only on the difference in their times of observation. One way to violate this stationarity property is to assume that a process $[X_t]$ possesses a probability distribution with a certain mean and variance depending on t . There are many practical situations where the assumption of the constant variability of a process may not have justifiable ground: the crop yield of a certain area is a function of the amount of daily rainfall which may have different variability from one day to the other; a random noise x_t in an electronic device may possess a probability distribution with zero mean and variance σ_t^2 . In climatological studies Bliss [1958], who studies data on the monthly mean temperature, showed that the variability of temperatures is more in winter than in summer. Perhaps in mentioning an analogous situation in the field of design of experiments, where the variance of the error term of the linear model is not assumed constant, we would give a clear picture of the variety of situations in which the present problem may be encountered.

This thesis is concerned with time series models where the observations are normally distributed with means zero and unequal but finite variances which are expressed as a Fourier series. The purpose of the thesis is: (1) to obtain consistent estimators for the coefficients in the Fourier expansion of the variance, and (2) to test the

hypothesis of the constancy of the variance in a linear model. The study of these problems involves the study of two separate models. The first is the random process model presented by Herbst [1963a]. The model for this process X_t is such that

$$X_t = \sigma_t \eta_t \quad (t = 0, \pm 1, \pm 2, \dots),$$

where η_t are uncorrelated normally distributed random variables with mean zero and variance 1, NID (0,1), and the σ_t are finite real constants, not necessarily equal. Fourier methods are employed in this thesis to give an approximate expression for the set of σ_t . If X_t ($t = 1, \dots, N$) is a realization of the above process, then the Fourier expansion associated with σ_t is given by

$$\sigma_t^2 = \sum_{s=-m}^m \gamma_s Z^{-st/N} \quad (t = 1, 2, \dots, N), \quad Z^\lambda = e^{2\pi i \lambda}, \quad i = \sqrt{-1},$$

and $m = \lfloor \frac{N}{2} \rfloor$.

Thus

$$\gamma_s = N^{-1} \sum_{t=1}^N \sigma_t^2 Z^{ts/N}, \quad (|s| = 0, 1, 2, \dots, m-1, \text{ } N \text{ even}) \text{ or} \\ (|s| = 0, 1, 2, \dots, m, \text{ } N \text{ odd}).$$

We assume that few of the Fourier coefficients are non-zero. Following Schuster's periodogram analysis of the scheme of hidden periodicities,

the periodogram of X_t^2 , $N^{-2} \left| \sum_{t=1}^N x_t^2 Z^{t\lambda} \right|^2$, which differs from the one

proposed by Herbst (1963a) by the factor N^{-2} , is used to detect the

dominant frequencies in the Fourier expansion of σ_t^2 . It will be shown

that the above periodogram is, in a sense which will be clarified in the

sequel, a consistent estimate of $|\gamma_p|^2$, where $\gamma_p = \gamma_{p,N}$. $|\gamma_p|^2$ provides a measure of coincidence of σ_t^2 with $\cos 2\pi t \frac{p}{N}$ and $\sin 2\pi t \frac{p}{N}$.

From the above model Herbst [1965] has generated a circular complex stationary process $[J_\alpha]$. It will be shown that the sampling properties of the covariance functions associated with this process are somewhat similar to those of the covariance functions in the real process.

The second model is the "moving average model", studied by Herbst [1963a], for a process such that

$$X_t = \sum_{j=0}^h \alpha_j \sigma_{t-j} \eta_{t-j} \quad (t = 0, \pm 1, \pm 2, \dots),$$

where the α_j 's are real constants, h is a positive integer, σ_t and η_t are as described for the first model. The "modified" first order autoregressive model, for example, is a special case of this model where the constants, α_j 's, are powers of the parameter. It is shown that the periodogram of X_t^2 is, in a sense which will be clarified in the sequel,

a consistent estimate of $|\gamma_p|^2 \left| \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \right|^2$. Chapter 6 of this thesis

is devoted to finding a consistent estimator of $|\gamma_p|^2$ separate from the

factor $\left| \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \right|^2$.

In Chapter 7, we construct a large-sample test that σ_t is constant in the moving average model, the alternative hypothesis being that σ_t is not constant. The test statistic is based on the periodogram of X_t^2 , and

its asymptotic distribution, which is independent of $\left| \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \right|^2$, is

found to be that of the Kolmogorov-Smirnov statistic. Intuitively, it

seems to us that the test statistic based on the periodogram of X_t^2 provides a better test than that proposed by Herbst [1963b] based on the periodogram of X_t .

The theory proved in this thesis is supported by artificial examples in which the variance σ_t^2 takes different Fourier expansions. For the estimation problem, uncorrelated and correlated series are constructed where only one or two Fourier coefficients are non-zero in the variance expansion. For the testing problem, series are constructed in such a way that the variance will not be constant; the Fourier expansions will include a few non-zero coefficients.

2. REVIEW OF LITERATURE

As it was clear from the context of Chapter 1, except for the work done by Herbst, there is no known study done on the specific models studied in this thesis. In connection with the model $X_t = \sigma_t \eta_t$ ($t = 0, \pm 1, \pm 2, \dots$) and for a sample of size N , Herbst [1963a] proposed the use of the periodogram of X_t^2 :

$$I_x^{(N)}(\lambda) = N^{-1} |J_x^2(\lambda)|^2 = N^{-1} \left| \sum_{t=1}^N X_t^2 e^{it\lambda} \right|^2,$$

to detect the periodicities associated with the Fourier expansion of σ_t^2 . The argument given is as follows. Let $Y_t = X_t^2$ and $Z_t = Y_t - \sigma_t^2$, then $Y_t = \sigma_t^2 + Z_t$, where $Z_t = \sigma_t^2(\eta_t^2 - 1)$. Thus, Z_t are independent variables with zero means and variances $2\sigma_t^4$ ($1 \leq t \leq N$). Therefore, Y_t resembles the classical scheme of hidden periodicities and, by showing that $J_y(\frac{p}{N})$ ($p = 1, 2, \dots, m-1$) is a linear function of γ_p , he concludes that the above periodogram would be a suitable estimate of the periodicities in σ_t^2 and its high peaks would correspond to "active" frequencies $\frac{p}{N}$ with $\gamma_p \neq 0$.

In the same paper Herbst [1963a] extends the above model to the "moving average" model. He proved the following lemma which corresponds to a lemma of Hannan [1960] in the stationary processes. Let

$$D_t = \sum_{j=0}^h a_j X_{t-j}, \text{ where } X_t \text{ is as defined above. Then}$$

$$2^{\frac{1}{2}} J_D(\frac{p}{N}) = \sum_{j=0}^h a_j Z^{j\lambda} \cdot 2^{\frac{1}{2}} J_X(\frac{p}{N}) + N^{-\frac{1}{2}} \theta(\frac{p}{N}),$$

where, provided that $\sigma_t^2 < \infty$ ($t = 0, \pm 1, \pm 2, \dots$)

$$E|\theta(\frac{p}{N})|^2 < \infty.$$

Hannan [1960] proved the following lemma. Let $X_t = \sum_{j=0}^{\infty} \alpha_j \epsilon_{t-j}$, where the ϵ_t are independent random variables with means zero, variance σ^2 and fourth cumulant K_4 . Then for a sample of size N

$$I_N(\lambda, x) = f(\lambda) \cdot I_N(\lambda, \epsilon) + O(N^{-\frac{1}{2}}),$$

$$(i = \sqrt{-1}), \text{ where } I_N(\lambda, x) = N^{-1} \left| \sum_{t=1}^N X_t Z^{t\lambda} \right|^2 \quad \text{and} \quad f(\lambda) = \left| \sum_{j=0}^{\infty} \alpha_j e^{2\pi i j \lambda} \right|^2.$$

The term $O(N^{-\frac{1}{2}})$ indicates that the neglected quantities will have mean square which is of order N^{-1} , if $\sum_0^{\infty} |\alpha_j| j^{\frac{1}{2}} < \infty$. Earlier Bartlett [1954] proved this lemma, but in the following form:

$$I_N(\lambda, x) = f(\lambda) I_N(\lambda, \epsilon) [1 + O(N^{-\frac{1}{2}})].$$

Herbst [1964] proved that for the moving average model $X_t = \sum_{j=0}^p a_j \sigma_{t-j} \eta_{t-j}$ and using the weight function

$$D_N(\lambda, \mu) = \begin{cases} \frac{1}{2} m_N & \text{if } \lambda - \frac{1}{m_N} < \mu < \lambda + \frac{1}{m_N} \\ 0 & \text{otherwise,} \end{cases}$$

the function $f_N(\lambda) = N^{-1} \sum_{j=0}^m D_N(\lambda, \frac{j}{N}) I_{x,N}(j/N)$ is a consistent estimator

$$\text{of } \gamma_0 \left| \sum_{j=0}^p a_j Z^{j\lambda} \right|^2, \text{ where } \gamma_0 = \lim_{N \rightarrow \infty} N^{-1} \sum_{t=1}^N \sigma_t^2 \text{ and } I_{x,N}(\frac{j}{N}) = N^{-1} \left| \sum_{t=1}^N X_t Z^{jt/N} \right|^2.$$

Herbst [1965] generated the following circular complex stationary process

$$J_{\alpha} = N^{-\frac{1}{2}} \sum_{t=1}^N X_t Z^{-t\alpha/N} \quad (X_t = \sigma_t \eta_t),$$

and showed that $E J_{\alpha} J_{\alpha+s}^* = \gamma_s$ and $J_{p+N} = J_p$. The main purpose of that paper was to construct test statistics for the variance σ_t^2 and in this connection he proved that

$$E \left[N^{-1} \sum_{t=1}^N X_t^2 \cos\left(\frac{2\pi ts}{N}\right) \right] = \text{Re}(\gamma_s)$$

and

$$E \left[N^{-1} \sum_{t=1}^N X_t^2 \sin\left(\frac{2\pi ts}{N}\right) \right] = \text{Im}(\gamma_s)$$

Herbst [1963b] proposed large sample test statistics, depending on the periodogram of X_t and distributed asymptotically as the Kolmogorov-Smirnov statistic, for testing the homogeneity of variance in a moving average model. He introduced two test statistics and showed that, using an argument based on the extreme conditions, the power of the tests depends on the correlogram of the periodogram ordinates.

Priestley [1962a, 1962b] studied normal stationary process X_t given by $X_t = Y_t + Z_t$, where Y_t is a stationary process with an absolutely continuous spectral density and Z_t is a stationary process with a discrete spectrum. Although the structure of this problem is entirely different from the one we are studying at present, the testing problem involved has some similarity to what we are dealing with. One of the testing procedure approaches Priestley proposed in his papers is the grouping of the periodogram, a procedure which gives better results than others depending on the periodogram.

3. PERIODOGRAM OF SQUARES OF GAUSSIAN DATA

3.1. Introduction

The periodogram, in the classical time series analysis, is used to detect the active frequencies in the mean of an observed series. Herbst [1963a] studied the use of the periodogram of squares of Gaussian data with zero means in the detection of the frequencies $\frac{p}{N}$ with which the non-zero Fourier coefficients γ_p in a Fourier series expansion of the variances are associated. In this chapter we will study the properties and use of the periodogram, which is defined in a slightly different way from the usual definition, of squares of this kind of data. In this case harmonic analysis will have different interpretations from the classical one, as we deal with non-stationary stochastic processes.

Consider the set $[X_t: t = 1, 2, \dots, N]$ of N random variables, which is a finite length record of a time series, such that

$$X_t = \sigma_t \eta_t \quad (3.1)$$

where η_t are NID $(0, 1)$ and σ_t are non-negative real constants which vary with t . Let

$$Y_t = X_t^2 = \sigma_t^2 \eta_t^2. \quad (3.2)$$

Every real valued function, which takes a finite number of values, can be approximated by a Fourier series [see for example Franklin, 1958]. Let us assume the Fourier series expansion for σ_t^2 of the form:

$$\sigma_t^2 = \sum_{s=-m}^m \gamma_s Z^{\frac{-ts}{N}} \quad (3.3)$$

where

$$Z^\lambda = e^{2\pi i \lambda}, \quad i = \sqrt{-1} \quad \text{and} \quad m = \left[\frac{N}{2} \right].$$

Our definition of the periodogram of the random variables $Y_1, Y_2, \dots, Y_N$ is given by:

$$I_Y(\lambda) = |J_Y(\lambda)|^2 \quad (0 \leq \lambda \leq 1) \quad (3.4)$$

where

$$J_Y(\lambda) = N^{-1} \sum_{t=1}^N Y_t Z^{t\lambda}.$$

$I_Y(\lambda)$ is a continuous random function of λ and is usually evaluated at a finite set of frequencies λ , where $\lambda = \frac{p}{N}$, ($p = 0, 1, \dots, m$). This provides us with a relative measure of coincidence of σ_t^2 with $\cos(2\pi \frac{tp}{N})$ and $\sin(2\pi \frac{tp}{N})$.

The main concern of the present chapter is in finding an exact expansion for $\text{cov} [I_Y^{(N)}(\frac{p}{N}), I_Y^{(N)}(\frac{p'}{N})]$, $1 \leq p, p' \leq m$, and studying the asymptotic properties of this result together with some examples. These examples are constructed by using random normal deviates multiplied by variances which follow certain Fourier expansion with specific frequencies. It is found that these series give clear support to the use of this type of periodogram in detecting the frequencies in the Fourier form of the variances.

The above covariance can be found directly by applying the definition of the periodogram given in (3.4), but the use of the results of section 3 is a more direct way of arriving at this covariance.

3.2. Harmonic Analysis

The periodogram given above may be written as

$$\begin{aligned} I_y^{(N)}\left(\frac{p}{N}\right) &= N^{-2} \sum_{t=1}^N \sum_{t'=1}^N Y_t Y_{t'} Z^{(t-t')\frac{p}{N}} \\ &= N^{-2} \sum_{t=1}^N \sum_{t'=1}^N \sigma_t^2 \sigma_{t'}^2 \eta_t^2 \eta_{t'}^2 Z^{(t-t')\frac{p}{N}}. \end{aligned}$$

Therefore, taking expectations of both sides, we get:

$$E[I_y^{(N)}\left(\frac{p}{N}\right)] = 3N^{-2} \sum_{t=1}^N \sigma_t^4 + N^{-2} \sum_{\substack{t, t'=1 \\ t \neq t'}}^N \sigma_t^2 \sigma_{t'}^2 Z^{(t-t')\frac{p}{N}}, \quad (3.5)$$

since $E(\eta_t^4) = 3$, $E(\eta_t^2) = 1$.

From the expression of σ_t^2 we notice that

$$\sigma_t^4 = \sigma_t^2 (\sigma_t^2)^* = \sum_{\alpha, \alpha'=-m}^m \gamma_\alpha \gamma_{\alpha'}^* Z^{\frac{t}{N}(\alpha' - \alpha)},$$

where the asterisks indicate complex conjugates.

Therefore, summing up both sides, from 1 to N, we get:

$$\sum_{t=1}^N \sigma_t^4 = \sum_{\alpha, \alpha'=-m}^m \gamma_\alpha \gamma_{\alpha'}^* \sum_{t=1}^N Z^{\frac{t}{N}(\alpha' - \alpha)},$$

and using the fact that

$$\sum_{t=1}^N Z^{\frac{tk}{N}} = \begin{cases} N & \text{for } k = 0, \pm N, \pm 2N, \dots \\ 0 & \text{otherwise} \end{cases}, \quad (3.6)$$

we see that

$$N^{-2} \sum_{t=1}^N \sigma_t^4 = N^{-1} \sum_{\alpha=-m}^m |\gamma_\alpha|^2. \quad (3.7)$$

The second term in (3.5) may be written as

$$N^{-2} \left(\sum_{t=1}^N \sigma_t^2 \frac{tp}{N} \right) \left(\sum_{t=1}^N \sigma_t^2 \frac{-tp}{N} \right) - N^{-2} \sum_{t=1}^N \sigma_t^4,$$

or

$$N^{-2} \left| \sum_{t=1}^N \sigma_t^2 \frac{tp}{N} \right|^2 - N^{-2} \sum_{t=1}^N \sigma_t^4 = |\gamma_p|^2 - N^{-1} \sum_{\alpha=-m}^m |\gamma_\alpha|^2.$$

Therefore

$$E[I_y^{(N)}(\frac{p}{N})] = |\gamma_p|^2 + 2N^{-1} \sum_{\alpha=-m}^m |\gamma_\alpha|^2. \quad (3.8)$$

Let us assume that $\sum_{\alpha=-m}^m |\gamma_\alpha|^2$ tends to some limit as $N \rightarrow \infty$,

$$\lim_{N \rightarrow \infty} \sum_{\alpha=-m}^m |\gamma_\alpha|^2 = C_0, \text{ say.} \quad (3.9)$$

Then we can conclude from (3.8) that

$$\left| E I_y^{(N)}(\frac{p}{N}) - |\gamma_p|^2 \right| \rightarrow 0 \text{ as } N \rightarrow \infty. \quad (3.10)$$

The form of $|\gamma_p|^2$ is of a spectral density type which measures the relative coincidence of σ_t^2 with $\cos(2\pi t \frac{p}{N})$ and $\sin(2\pi t \frac{p}{N})$. For a large value of N , $I_y^{(N)}(\frac{p}{N})$ is an unbiased estimate of this function. Thus, we would expect that a large value of $I_y^{(N)}(\frac{p}{N})$ at a certain frequency $\frac{p}{N}$ will reveal a high value of $|\gamma_p|^2$, thereby indicating the existence of this frequency in the variances. Therefore, dominant frequencies in the variances will show up in the high peaks of the graph of $I_y^{(N)}(\frac{p}{N})$, against p , at those frequencies.

3.3. Covariance between Two Sets of Variates

In this section we will prove a theorem and corollaries concerning the covariance between two sets of variates which constitute four random variables. The results in this section are motivated by the need for a formula, applicable to the random variables described in lemma 3.1, which would have the characteristic of the well-known lemma of Isserlis:

$$\text{cov}(xy, uw) = E(xw) \cdot E(yu) + E(xu) \cdot E(yw), \quad (3.11)$$

where s , y , u , and w are normally distributed random variables; cov is used for covariance.

The interesting point in the following results is that the random variables considered are not necessarily normally distributed.

Lemma 3.1

Let the random variables X , Y , U , W be identically distributed random variables with mean μ , such that each pair of these random variables consists either of statistically independent random variables or random variables which are identical with probability 1 (e.g., either $X=Y$ with probability 1 or X and Y are statistically independent). Consider x , y , u , and w as being these random variables when measured around their means. Then

$$\text{cov}(xy, uw) = \begin{cases} E(x^4), & \text{if } x, y, u, \text{ and } w \text{ are the same} \\ E(xu) \cdot E(yw) + E(xw) \cdot E(yu) & \text{otherwise.} \end{cases} \quad (3.12)$$

Proof

The proof is obvious when the above random variables are the same.

To prove the second part we notice that we have only the following cases to be considered:

- (1) All of the four random variables are independent (i.e., all different). Then $\text{cov}(xy, uw) = 0$.
- (2) Two of the variables are independent and the other two are the same, but independent of the first two. There are $\binom{4}{2} = 6$ cases under this category. Then it is clear that
- $$\begin{aligned} \text{cov}(xy, uw) &= E(xyuw) - E(xy) \cdot E(uw) \\ &= 0 - 0 = 0 \end{aligned}$$
- for each case.
- (3) The four random variables form two sets of identical variables which are independent. There are $\binom{4}{2} \div 2 = 3$ cases under this condition, which are listed as follows:
- (a) x, y and u, w are the two sets. Then
- $$\begin{aligned} \text{cov}(xy, uw) &= E(xyuw) - E(xy) \cdot E(uw) \\ &= E(x^2) \cdot E(u^2) - E(x^2) \cdot E(u^2) \\ &= 0. \end{aligned}$$
- (b) x, u and y, w are the two sets. Then
- $$\begin{aligned} \text{cov}(xy, uw) &= E(xyuw) - E(xy) \cdot E(uw) \\ &= E(xu) \cdot E(yw) - 0 \\ &= E(xu) \cdot E(yw). \end{aligned}$$
- (c) x, w and y, u are the two sets. Then
- $$\begin{aligned} \text{cov}(xy, uw) &= E(xyuw) - E(xy) \cdot E(uw) \\ &= E(xw) \cdot E(yu). \end{aligned}$$
- (4) Three of the random variables are the same and the fourth one is different. Then it is obvious that $\text{cov}(xy, uw) = 0$.

Now, if $x, y, u,$ and w satisfy $\exists a$, then

$$E(xu) \cdot E(yw) = E(xw) \cdot E(yu) = 0,$$

so that $\text{cov}(xy, uw) = 0$. If they satisfy $\exists b$, then

$$E(xw) \cdot E(yu) = 0.$$

Similarly, if they satisfy $\exists c$, then

$$E(xu) \cdot E(yw) = 0.$$

Also both the results of $\exists b$ and $\exists c$ are zero whenever any of the other cases are satisfied.

Therefore,

$$\text{cov}(xy, uw) = E(xu) \cdot E(yw) + E(xw) \cdot E(yu)$$

is satisfied by all of the above cases. Hence, the lemma.

The following corollary is an obvious result of the lemma.

Corollary 3.1: $\text{cov}(x^2, uw) = 2E(xu) \cdot E(xw)$.

Theorem 3.1

If $X, Y, U,$ and W satisfy the conditions of Lemma 3.1, then

$$\text{cov}(XY, UW) = \begin{cases} \text{Var}(X^2) & \text{if all the variables are the same} \\ E(xu) \cdot E(yw) + E(xw) \cdot E(yu) + \mu[E(xyu) + E(xyw) \\ + E(xuw) + E(yuw)] + \mu(2-\mu)[E(xu) + E(xw) \\ + E(yu) + E(yw)], & \text{otherwise.} \end{cases}$$

Proof

The first part of the theorem is trivial.

To prove the other part, we consider:

$$\begin{aligned}\text{cov}(xy, uw) &= \text{cov}[(X-\mu)(Y-\mu), (U-\mu)(W-\mu)] \\ &= \text{cov}[XY - \mu X - \mu Y + \mu^2, UW - \mu W - \mu U + \mu^2]\end{aligned}$$

or

$$\begin{aligned}\text{cov}(xy, uw) &= \text{cov}(XY, UW) - \mu \text{cov}(XY, W) - \mu \text{cov}(XY, U) - \mu \text{cov}(X, UW) \\ &\quad + \mu^2 \text{cov}(X, W) + \mu^2 \text{cov}(X, U) - \mu \text{cov}(Y, UW) + \mu^2 \text{cov}(Y, W) \\ &\quad + \mu^2 \text{cov}(Y, U).\end{aligned}$$

Now, to find $\text{cov}(XY, U)$ and other similar terms, consider

$$\begin{aligned}\text{cov}(xy, u) &= \text{cov}[(X-\mu)(Y-\mu), (U-\mu)] \\ &= \text{cov}(XY, U) - \text{cov}(X, U) - \text{cov}(Y, U).\end{aligned}$$

But

$$\text{cov}(X, U) = \text{cov}(x, u) = E(xu)$$

and

$$\text{cov}(Y, U) = E(yu).$$

Therefore,

$$\text{cov}(XY, U) = \text{cov}(xy, u) + E(xu) + E(yu).$$

Also

$$\begin{aligned}\text{cov}(xy, u) &= E(xyu) - E(xy) - E(u) \\ &= E(xyu).\end{aligned}$$

Hence

$$\text{cov}(XY, U) = E(xyu) + E(xu) + E(yu),$$

and a similar result for the other terms.

Therefore,

$$\begin{aligned}
 \text{cov}(XY, UW) = & \mu[E(xyu) + E(xu) + E(yu)] \\
 & + \mu[E(xyu) + E(xw) + E(yw)] \\
 & + \mu[E(xuw) + E(xu) + E(xw)] \\
 & + \mu[E(yuw) + E(yu) + E(yw)] \\
 & - \mu^2[E(xu) + E(xw) + E(yu) + E(yw)] \\
 & + \text{cov}(xy, uw).
 \end{aligned}$$

Substituting for $\text{cov}(xy, uw)$ from lemma 3.1 and arranging terms, we get

$$\begin{aligned}
 \text{cov}(XW, UW) = & E(xu) \cdot E(yw) + E(xw) \cdot E(yu) \\
 & + \mu[E(xyu) + E(xyw) + E(xuw) + E(yuw)] \\
 & + \mu(2-\mu)[E(xu) + E(xw) + E(yu) + E(yw)].
 \end{aligned}$$

Note: The above holds true for normally distributed random variables.

The following corollaries are obvious results of theorem 3.1.

Corollary 3.2

If $\mu = 1$ in theorem 3.1, then

$$\text{cov}(XY, UW) = \begin{cases} \text{Var}(X^2) & \text{if all variables are the same} \\ E(xu) \cdot E(yw) + E(xw) \cdot E(yu) + E(xyu) + E(xyw) \\ + E(xuw) + E(yuw) + E(xu) + E(xw) + E(yu) + E(yw) & (3.13) \\ \text{otherwise.} \end{cases}$$

Corollary 3.3

$$\text{cov}(X^2, UW) = \begin{cases} \text{Var}(X^2) & \text{if all variables are the same} \\ 2 E(xu) \cdot E(xw) + E(xu) + E(xw) + E(xuw) \\ + E(x^2u) + E(x^2w), & \text{otherwise.} \end{cases} \quad (3.14)$$

3.4. Using Corollary 3.2 in Finding $\text{cov}[I_y^{(N)}(\frac{p}{N}), I_y^{(N)}(\frac{p'}{N})]$

$$(p \neq p', p, p' = 1, 2, \dots, m)$$

From the definition of $I_y^{(N)}(\frac{p}{N})$ we write

$$\text{cov}[I_y^{(N)}(\frac{p}{N}), I_y^{(N)}(\frac{p'}{N})] = \text{cov}[J_y(\frac{p}{N}) \cdot J_y^*(\frac{p}{N}), J_y(\frac{p'}{N}) \cdot J_y^*(\frac{p'}{N})] \quad (3.15)$$

where, as before, the asterisk indicates complex conjugates. Or

$$\text{cov}[I_y^{(N)}(\frac{p}{N}), I_y^{(N)}(\frac{p'}{N})] = N^{-4} \text{cov}[\sum_{t=1}^N \sum_{t'=1}^N Y_t Y_{t'} Z_N^{\frac{p}{N}(t-t')}, \sum_{\alpha=1}^N \sum_{\alpha'=1}^N Y_\alpha Y_{\alpha'} Z_N^{\frac{p}{N}(\alpha-\alpha')}]$$

by substituting for J_y , or

$$= N^{-4} V,$$

where

$$V = \text{cov}[\sum_{t=1}^N \sum_{t'=1}^N Y_t Y_{t'} Z_N^{\frac{p}{N}(t-t')}, \sum_{\alpha=1}^N \sum_{\alpha'=1}^N Y_\alpha Y_{\alpha'} Z_N^{\frac{p}{N}(\alpha-\alpha')}] \quad (3.16)$$

First we shall find V , which can be written as

$$\begin{aligned} V &= \sum_{t=1}^N \sum_{t'=1}^N \sum_{\alpha=1}^N \sum_{\alpha'=1}^N \text{cov}(Y_t Y_{t'}, Y_\alpha Y_{\alpha'}) Z_N^{\frac{p}{N}(t-t') + \frac{p'}{N}(\alpha-\alpha')} \\ &= \sum_{t, t', \alpha, \alpha'=1}^N \sigma_t^2 \sigma_{t'}^2 \sigma_\alpha^2 \sigma_{\alpha'}^2 \text{cov}(\eta_t^2 \eta_{t'}^2, \eta_\alpha^2 \eta_{\alpha'}^2) Z_N^{\frac{p}{N}(t-t') + \frac{p'}{N}(\alpha-\alpha')} \quad (3.17) \end{aligned}$$

Applying the above corollary, setting

$$\lambda_{tt'\alpha\alpha'} = \sigma_t^2 \sigma_{t'}^2 \cdot \sigma_\alpha^2 \sigma_{\alpha'}^2 Z_N^{\frac{p}{N}(t-t') + \frac{p'}{N}(\alpha-\alpha')}, \quad (3.18)$$

and

$$\xi = (\eta^2 - 1)$$

we get:

$$\begin{aligned}
 V = & \sum_{t,t',\alpha,\alpha'=1}^N \lambda_{tt'\alpha\alpha'} E(\xi_t \xi_{\alpha'}) \cdot E(\xi_t, \xi_{\alpha'}) \\
 & + \sum_{t,t',\alpha,\alpha'=1}^N \lambda_{tt'\alpha\alpha'} E(\xi_t \xi_{\alpha'}) \cdot E(\xi_t, \xi_{\alpha'}) \\
 & + \sum_{t,t',\alpha,\alpha'=1}^N \lambda_{tt'\alpha\alpha'} E(\xi_t \xi_{t'} \xi_{\alpha'}) + \sum_{t,t',\alpha,\alpha'=1}^N \lambda_{tt'\alpha\alpha'} E(\xi_t \xi_{t'} \xi_{\alpha'}) \\
 & + \sum_{t,t',\alpha,\alpha'=1}^N \lambda_{tt'\alpha\alpha'} E(\xi_{\alpha} \xi_{\alpha'} \xi_t) + \sum_{t,t',\alpha,\alpha'=1}^N \lambda_{tt'\alpha\alpha'} E(\xi_{\alpha} \xi_{\alpha'} \xi_{t'}) \\
 & + \sum_{t,t',\alpha,\alpha'=1}^N \lambda_{tt'\alpha\alpha'} E(\xi_t \xi_{\alpha'}) + \sum_{t,t',\alpha,\alpha'=1}^N \lambda_{tt'\alpha\alpha'} E(\xi_t \xi_{\alpha'}) \\
 & + \sum_{t,t',\alpha,\alpha'=1}^N \lambda_{tt'\alpha\alpha'} E(\xi_t \xi_{\alpha'}) + \sum_{t,t',\alpha,\alpha'=1}^N \lambda_{tt'\alpha\alpha'} E(\xi_{t'} \xi_{\alpha'}) \\
 & + \sum_{t=1}^N \sigma_t^8 \text{Var}(\eta_t^4) , \tag{3.19}
 \end{aligned}$$

where the subscripts t, t', α, α' cannot all be equal under the summation signs.

From the definition of ξ , we see that

$$E(\xi_i \cdot \xi_j) = \begin{cases} 0 & \text{for } i \neq j \\ 2 & \text{for } i = j \end{cases} \tag{3.20}$$

and

$$E(\xi_i \xi_j \xi_k) = \begin{cases} 8 & \text{for } i = j = k \\ 0 & \text{otherwise.} \end{cases}$$

Also from the definition of η we see that

$$\begin{aligned}\text{Var}(\eta_t^4) &= E(\eta_t^8) - [E(\eta_t^4)]^2 \\ &= 105 - 9 = 96.\end{aligned}\tag{3.21}$$

Therefore

$$\begin{aligned}V &= 4 \sum_{t,t'=1}^N \lambda_{tt'tt'} + 4 \sum_{t,t'=1}^N \lambda_{tt't't} \\ &\quad + 8 \sum_{t,\alpha'=1}^N \lambda_{ttt\alpha'} + 8 \sum_{t,\alpha=1}^N \lambda_{tt\alpha t} \\ &\quad + 8 \sum_{\alpha,t'=1}^N \lambda_{\alpha t' \alpha \alpha} + 8 \sum_{\alpha,t=1}^N \lambda_{t \alpha \alpha \alpha} \\ &\quad + 2 \sum_{t,t',\alpha'=1}^N \lambda_{tt't\alpha'} + 2 \sum_{t,t',\alpha=1}^N \lambda_{tt'\alpha t} \\ &\quad + 2 \sum_{t,t',\alpha'=1}^N \lambda_{tt't'\alpha'} + 2 \sum_{t,t',\alpha=1}^N \lambda_{tt'\alpha t'} \\ &\quad + 96 \sum_{t=1}^N \lambda_{tttt},\end{aligned}\tag{3.22}$$

where, as before, the subscripts t, t', α, α' cannot all be equal under the summation signs.

Now, allowing for the subscripts t, t', α, α' to be equal in the summation, it is easily seen that V will reduce to the form:

$$\begin{aligned}
V = & 4 \sum_{t,t'=1}^N \lambda_{tt'tt'} + 4 \sum_{t,t'=1}^N \lambda_{tt't't} \\
& + 8 \sum_{t,\alpha'=1}^N \lambda_{ttt\alpha'} + 8 \sum_{t,\alpha=1}^N \lambda_{ttt\alpha} \\
& + 8 \sum_{\alpha,t'=1}^N \lambda_{\alpha t'\alpha\alpha} + 8 \sum_{\alpha,t=1}^N \lambda_{t\alpha\alpha\alpha} \\
& + 2 \sum_{t,t',\alpha'=1}^N \lambda_{tt't\alpha'} + 2 \sum_{t,t',\alpha=1}^N \lambda_{tt'\alpha t} \\
& + 2 \sum_{t,t',\alpha'=1}^N \lambda_{tt't'\alpha'} + 2 \sum_{t,t',\alpha=1}^N \lambda_{tt'\alpha t'} \\
& + 48 \sum_{t=1}^N \lambda_{tttt} .
\end{aligned} \tag{3.23}$$

Substituting from λ 's from (3.18) in (3.23) we get:

$$\begin{aligned}
V = & 4 \sum_{t,t'=1}^N \sigma_t^4 \sigma_{t'}^4 \frac{t}{N}(p+p') - \frac{t'}{N}(p+p') \\
& + 4 \sum_{t,t'=1}^N \sigma_t^4 \sigma_{t'}^4 \frac{t}{N}(p-p') - \frac{t'}{N}(p-p') \\
& + 8 \sum_{t,\alpha=1}^N \sigma_t^6 \sigma_{\alpha'}^2 \frac{tp'}{N} - \frac{\alpha'}{N} p' \\
& + 8 \sum_{t,\alpha=1}^N \sigma_t^6 \sigma_{\alpha}^2 \frac{tp'}{N} + \frac{\alpha}{N} p' \\
& + 8 \sum_{t',\alpha=1}^N \sigma_{t'}^6 \sigma_{\alpha}^2 \frac{t'p}{N} + \frac{\alpha p}{N}
\end{aligned}$$

$$\begin{aligned}
& + 8 \sum_{t, \alpha=1}^N \sigma_t^6 \sigma_\alpha^2 \frac{tp}{N} - \frac{\alpha p}{N} \\
& + 2 \sum_{t, t', \alpha'=1}^N \sigma_t^4 \sigma_{t'}^2 \sigma_{\alpha'}^2 \frac{t}{N} (p+p') - \frac{t' p}{N} - \frac{\alpha' p'}{N} \\
& + 2 \sum_{t, t', \alpha=1}^N \sigma_t^4 \sigma_{t'}^2 \sigma_\alpha^2 \frac{t}{N} (p-p') - \frac{t' p}{N} + \frac{\alpha p'}{N} \\
& + 2 \sum_{t, t', \alpha'=1}^N \sigma_t^4 \sigma_{t'}^2 \sigma_{\alpha'}^2 \frac{t}{N} (p-p') + \frac{t' p'}{N} \\
& + 2 \sum_{t, t', \alpha=1}^N \sigma_t^4 \sigma_{t'}^2 \sigma_\alpha^2 \frac{t}{N} (p+p') + \frac{t' p}{N} + \frac{\alpha p'}{N} \\
& + 48 \sum_{t=1}^N \sigma_t^8. \tag{3.24}
\end{aligned}$$

From (3.24) we notice that the first term can be written as:

$$4 \left(\sum_{t=1}^N \sigma_t^4 \frac{t}{N} (p+p') \right) \left(\sum_{t=1}^N \sigma_t^4 \frac{t}{N} (p+p') \right)^*$$

which is

$$= 4 \left| \sum_{t=1}^N \sigma_t^4 \frac{t}{N} (p+p') \right|^2,$$

and a similar result for the second term.

Also, we notice that the 4th term is a complex conjugate of the 3rd; the 5th term is a complex conjugate of the 6th; the 10th term is a complex conjugate of the 7th; and finally the 9th term is a complex conjugate of the 8th.

For any complex number Z we have $Z + Z^* = 2\text{Re}(Z)$, where Re indicates the real part of Z . Thus, we see that V reduces to the following form:

$$\begin{aligned}
 V = & 4 \left| \sum_{t=1}^N \sigma_t^4 Z^{\frac{t}{N}} (p+p') \right|^2 + 4 \left| \sum_{t=1}^N \sigma_t^4 Z^{\frac{t}{N}} (p-p') \right|^2 \\
 & + 16 \text{Re} \left[\sum_{t,t'=1}^N \sigma_t^6 \sigma_{t'}^2 Z^{\frac{tp'}{N} - \frac{t'p}{N}} \right] \\
 & + 16 \text{Re} \left[\sum_{t,t'=1}^N \sigma_t^6 \sigma_{t'}^2 Z^{\frac{tp}{N} - \frac{t'p'}{N}} \right] \\
 & + 4 \text{Re} \left[\sum_{t,t',\alpha=1}^N \sigma_t^4 \sigma_{t'}^2 \sigma_{\alpha}^2 Z^{\frac{t}{N}(p+p') - \frac{t'p}{N} - \frac{\alpha p'}{N}} \right] \\
 & + 4 \text{Re} \left[\sum_{t,t',\alpha=1}^N \sigma_t^4 \sigma_{t'}^2 \sigma_{\alpha}^2 Z^{\frac{t}{N}(p-p') - \frac{t'p}{N} + \frac{\alpha p'}{N}} \right] \\
 & + 48 \sum_{t=1}^N \sigma_t^8. \tag{3.25}
 \end{aligned}$$

Now we need to express V in terms of the Fourier coefficients (γ_s) of the expansion of σ_t^2 given in (3.3). We have σ_t^4 , σ_t^6 and σ_t^8 for which we need to find the Fourier coefficients in terms of γ_s . We proceed as follows:

(1) Suppose

$$\sigma_t^4 = \sum_{s=-m}^m \delta_s Z^{\frac{-ts}{N}},$$

where

$$\delta_s = N^{-1} \sum_{t=1}^N \sigma_t^4 Z^{\frac{ts}{N}}.$$

Then

$$\begin{aligned}\delta_s &= N^{-1} \sum_{t=1}^N \sigma_t Z^{\frac{ts}{N}} = N^{-1} \sum_{t=1}^N \left(\sum_{\alpha, \alpha'=-m}^m \gamma_\alpha \gamma_{\alpha'}^* Z^{\frac{-\alpha t}{N} + \frac{\alpha' t}{N}} \right) Z^{\frac{ts}{N}}, \\ &= N^{-1} \sum_{\alpha, \alpha'=-m}^m \gamma_\alpha \gamma_{\alpha'}^* \left(\sum_{t=1}^N Z^{\frac{(-\alpha + \alpha' + s)t}{N}} \right).\end{aligned}$$

But $\sum_{t=1}^N Z^{\frac{tp}{N}} = N$ for $p = 0, \pm N, \dots$; 0 otherwise.

Therefore

$$\sum_{t=1}^N Z^{\frac{(\alpha' - \alpha + s)t}{N}} = N \text{ when } \alpha' - \alpha + s = 0, \text{ or when } \alpha' = \alpha - s.$$

Therefore

$$\delta_s = \sum_{\alpha=-m}^m \gamma_\alpha \gamma_{\alpha-s}^* \quad (3.26)$$

$$(2) \text{ Let } \sigma_t^6 = \sum_{s=-m}^m \lambda_s Z^{\frac{-ts}{N}} \text{ where } \lambda_s = N^{-1} \sum_{t=1}^N \sigma_t^6 Z^{\frac{ts}{N}}.$$

Then by the same argument as in (1) we see that $\lambda_s = \sum_{j=-m}^m \delta_j \gamma_{j-s}^*$.

Therefore

$$\lambda_s = \sum_{j=-m}^m \left(\sum_{i=-m}^m \gamma_i \gamma_{i-j}^* \right) \gamma_{j-s}^* \text{ or } \lambda_s = \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-s}^*. \quad (3.27)$$

$$(3) \text{ Let } \sigma_t^8 = \sum_{s=-m}^m \Delta_s Z^{\frac{ts}{N}}, \text{ where } \Delta_s = N^{-1} \sum_{t=1}^N \sigma_t^8 Z^{\frac{ts}{N}}.$$

By the same procedure as above we see that

$$\Delta_s = \sum_{k=-m}^m \delta_k \delta_{k-s}^* = \sum_{k=-m}^m \left(\sum_{i=-m}^m \gamma_i \gamma_{i-k}^* \right) \left(\sum_{j=-m}^m \gamma_j \gamma_{j-k+s}^* \right)^*$$

or

$$\Delta_s = \sum_{i,j,k=-m}^m \gamma_i \gamma_{i-k}^* \gamma_j^* \gamma_{j-k+s} \quad (3.28)$$

Now, substituting (3.26), (3.27) and (3.28) into (3.25) and arranging the sums so that the Z factors are under summations for t, t' and α , we get:

$$\begin{aligned} V = & 4 \left| \sum_{s=-m}^m \sum_{i=-m}^m \gamma_i \gamma_{i-s}^* \left(\sum_{t=1}^N Z^{\frac{t}{N}(p+p') - \frac{ts}{N}} \right) \right|^2 \\ & + 4 \left| \sum_{s=-m}^m \sum_{i=-m}^m \gamma_i \gamma_{i-s}^* \left(\sum_{t=1}^N Z^{\frac{t}{N}(p-p') - \frac{ts}{N}} \right) \right|^2 \\ & + 16 \operatorname{Re} \left[\sum_{s,s'=-m}^m \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-s}^* \gamma_{s'} \left(\sum_{t,t'=1}^N Z^{\frac{tp'}{N} - \frac{ts}{N} - \frac{t'p'}{N} - \frac{t's'}{N}} \right) \right] \\ & + 16 \operatorname{Re} \left[\sum_{s,s'=-m}^m \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-s}^* \gamma_{s'} \left(\sum_{t,t'=1}^N Z^{\frac{tp}{N} - \frac{ts}{N} - \frac{t'p}{N} - \frac{t's'}{N}} \right) \right] \\ & + 4 \operatorname{Re} \left[\sum_{s,s',s''=-m}^m \sum_{i=-m}^m \gamma_i \gamma_{i-s}^* \gamma_{s'} \gamma_{s''} \left(\sum_{t,t',\alpha=1}^N Z^{\frac{t}{N}(p+p') - \frac{ts}{N} - \frac{t'p}{N} - \frac{t's'}{N} - \frac{\alpha p'}{N} - \frac{\alpha s''}{N}} \right) \right] \\ & + 4 \operatorname{Re} \left[\sum_{s,s',s''=-m}^m \sum_{i=-m}^m \gamma_i \gamma_{i-s}^* \gamma_{s'} \gamma_{s''} \left(\sum_{t,t',\alpha=1}^N Z^{\frac{t}{N}(p-p') - \frac{ts}{N} - \frac{t'p}{N} - \frac{t's'}{N} + \frac{\alpha p'}{N} - \frac{\alpha s''}{N}} \right) \right] \\ & + 48N \sum_{i,j,k=-m}^m \gamma_i \gamma_{i-k}^* \gamma_j^* \gamma_{j-k} \quad (3.29) \end{aligned}$$

All the sums over t , t' and α in (3.29) can be factorized into a sum over t , a sum over t' and a sum over α . Therefore, using the identity (3.6) for each sum and remembering that

$$\gamma_s = \gamma_{N-s}^* \quad \text{and} \quad \gamma_{-s} = \gamma_s^*, \quad (3.30)$$

we see that V reduces to:

$$\begin{aligned} V = & 4N^2 \left| \sum_{i=-m}^m \gamma_i \gamma_{i-(p+p')}^* \right|^2 + 4N^2 \left| \sum_{i=-m}^m \gamma_i \gamma_{i-(p-p')}^* \right|^2 \\ & + 16N^2 \operatorname{Re} \left[\gamma_{p'}^* \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{i-p'}^* \right] + 16N^2 \operatorname{Re} \left[\gamma_p^* \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-p}^* \right] \\ & + 4N^3 \operatorname{Re} \left[\gamma_{p'}^* \gamma_{p'}^* \sum_{i=-m}^m \gamma_i \gamma_{i-(p+p')}^* \right] + 4N^3 \operatorname{Re} \left[\gamma_p^* \gamma_p^* \sum_{i=-m}^m \gamma_i \gamma_{i-(p-p')}^* \right] \\ & + 48N \sum_{i,j,k=-m}^m \gamma_i \gamma_{i-k}^* \gamma_j^* \gamma_{j-k}. \end{aligned}$$

Therefore, referring to (3.16), we get the exact expression required:

$$\begin{aligned} \operatorname{cov}[I_y^{(N)}(\frac{p}{N}), I_y^{(N)}(\frac{p'}{N})] = & \frac{4}{N^2} \left| \sum_{i=-m}^m \gamma_i \gamma_{i-(p+p')}^* \right|^2 \\ & + \frac{4}{N^2} \left| \sum_{i=-m}^m \gamma_i \gamma_{i-(p-p')}^* \right|^2 + \frac{16}{N^2} \operatorname{Re} \left[\gamma_{p'}^* \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-p'}^* \right] \\ & + \frac{16}{N^2} \operatorname{Re} \left[\gamma_p^* \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-p}^* \right] + \frac{4}{N} \operatorname{Re} \left[\gamma_{p'}^* \gamma_{p'}^* \sum_{i=-m}^m \gamma_i \gamma_{i-(p+p')}^* \right] \\ & + \frac{4}{N} \operatorname{Re} \left[\gamma_p^* \gamma_p^* \sum_{i=-m}^m \gamma_i \gamma_{i-(p-p')}^* \right] + \frac{48}{N^3} \sum_{i,j,k=-m}^m \gamma_i \gamma_{i-k}^* \gamma_j^* \gamma_{j-k}. \quad (3.32) \end{aligned}$$

From this we immediately obtain:

$$\begin{aligned}
 \text{Var}[I_x^{(N)}(\frac{p}{N})] &= \frac{4}{N^2} \left| \sum_{i=-m}^m \gamma_i \gamma_{i-2p}^* \right|^2 + \frac{4}{N^2} \left| \sum_{i=-m}^m |\gamma_i|^2 \right|^2 \\
 &+ \frac{32}{N^2} \text{Re}[\gamma_p^* \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-p}^*] + \frac{4}{N} \text{Re}[(\gamma_p^*)^2 \sum_{i=-m}^m \gamma_i \gamma_{i-2p}^*] \\
 &+ \frac{4}{N} |\gamma_p|^2 \sum_{i=-m}^m |\gamma_i|^2 + \frac{48}{N^3} \sum_{i,j,k=-m}^m \gamma_i \gamma_{i-k}^* \gamma_j^* \gamma_{j-k} . \quad (3.32a)
 \end{aligned}$$

The two expressions in (3.31) and (3.32a) are important results for the description of behavior of the periodograms $I_y^{(N)}(\lambda)$. These results, due to their complex appearance, may not give us a clear idea of how they can be useful in understanding the sampling properties and application of the periodogram. We will present in the next two sections the asymptotic behavior of $I_y^{(N)}(\lambda)$ for large N and some special examples of the Fourier expansion for σ_t^2 . The purpose of the first is to give simpler form to the above covariance by considering only prominent terms in the above expressions.

3.5. Asymptotic Properties of $\text{cov}[I_y^{(N)}(\frac{p}{N}), I_y^{(N)}(\frac{p'}{N})]$

Consider γ_i ($i = 0, 1, \dots, m$), where by definition

$$\gamma_i = N^{-1} \sum_{t=1}^N \sigma_t^2 Z^{\frac{ti}{N}} .$$

Therefore,

$$|\gamma_i| \leq N^{-1} \sum_{t=1}^N \sigma_t^2 \left| Z^{\frac{ti}{N}} \right| = N^{-1} \sum_{t=1}^N \sigma_t^2$$

or $|\gamma_i| \leq \gamma_{0,N}$, where

$$\gamma_{0,N} = N^{-1} \sum_{t=1}^N \sigma_t^2. \quad (3.33)$$

Assume that

$$\lim_{N \rightarrow \infty} \gamma_{0,N} = \gamma_0, \quad (3.34)$$

where γ_0 is a constant. Also, we assumed at the beginning of this chapter that $\lim_{N \rightarrow \infty} \sum_{s=-m}^m |\gamma_s|^2 = C_0$. Then in considering the asymptotic behavior of our covariance, we proceed as follows. First, consider the first term in (3.32), which is

$$\frac{4}{N^2} \left| \sum_{i=-m}^m \gamma_i \gamma_{i-(p+p')}^* \right|^2.$$

It is clear that

$$\left| \sum_{i=-m}^m \gamma_i \gamma_{i-(p+p')}^* \right| \leq \sum_{i=-m}^m |\gamma_i|^2 = C_{0,N}, \text{ say.}$$

Therefore,

$$\frac{4}{N^2} \left| \sum_{i=-m}^m \gamma_i \gamma_{i-(p+p')}^* \right|^2 \leq \frac{4}{N^2} (C_{0,N})^2. \quad (3.35)$$

A similar result holds for the second term in (3.32), i.e.,

$$\frac{4}{N^2} \left| \sum_{i=-m}^m \gamma_i \gamma_{i-(p-p')}^* \right|^2 \leq \frac{4}{N^2} (C_{0,N})^2. \quad (3.36)$$

Second, consider the third term in (3.32)

$$\frac{16}{N^2} \operatorname{Re} \left[\sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-p'}^* \right] \leq \frac{16}{N^2} \left| \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-p'}^* \right|.$$

Since

$$\lambda_s = \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-s}^*,$$

we get that

$$\left| \gamma_{p'}^* \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-p'}^* \right| \leq |\gamma_{p'}^*| \cdot |\lambda_{p'}|.$$

But

$$|\lambda_s| \leq \lambda_{0,N} = N^{-1} \sum_{t=1}^N \sigma_t^6 \leq \sigma_M^2 C_{0,N},$$

where

$$\sigma_M^2 = \max_t (\sigma_t^2).$$

Thus

$$\left| \gamma_{p'}^* \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-p'}^* \right| \leq |\gamma_{p'}^*| \cdot \sigma_M^2 C_{0,N}.$$

Therefore

$$\frac{16}{N^2} \operatorname{Re} \left[\gamma_{p'}^* \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-p'}^* \right] \leq \frac{16}{N^2} |\gamma_{p'}^*| \sigma_M^2 C_{0,N}, \quad (3.37)$$

and a similar result for the fourth term of (3.32), i.e.,

$$\frac{16}{N^2} \operatorname{Re} \left[\gamma_p^* \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-p}^* \right] \leq \frac{16}{N^2} |\gamma_p^*| \sigma_M^2 C_{0,N}. \quad (3.38)$$

Again, by the same argument as above, it is easily seen that for the fifth term of (3.32):

$$\frac{4}{N} \operatorname{Re} [\gamma_p^* \gamma_{p'}^* \sum_{i=-m}^m \gamma_i \gamma_{i-(p+p')}^*] \leq \frac{4}{N} |\gamma_p^*| \cdot |\gamma_{p'}^*| C_{0,N}, \quad (3.39)$$

and for the sixth term:

$$\frac{4}{N} \operatorname{Re}[\gamma_p^* \gamma_{p'} \sum_{i=-m}^m \gamma_i \gamma_{i-(p-p')}^*] \leq \frac{4}{N} |\gamma_p^*| \cdot |\gamma_{p'}| C_{0,N} . \quad (3.40)$$

Finally, for the last term of (3.32), we see that

$$\frac{48}{N^3} \sum_{i,j,k=-m}^m \gamma_i \gamma_{i-k}^* \gamma_j^* \gamma_{j-k+s} \leq \frac{48}{N^3} \sigma_M^4 C_{0,N} . \quad (3.41)$$

Therefore, combining results in (3.35) through (3.41) and using the fact that $|\gamma_s| = |\gamma_s^*|$, we see that:

$$\begin{aligned} \operatorname{cov}[I_y^{(N)}(\frac{p}{N}), I_y^{(N)}(\frac{p'}{N})] &\leq \frac{8}{N^2} (C_{0,N})^2 + \frac{16}{N^2} \sigma_M^2 (|\gamma_{p'}| + |\gamma_{p'}|) C_{0,N} \\ &+ \frac{8}{N} |\gamma_p| \cdot |\gamma_{p'}| \cdot (C_{0,N}) + \frac{48}{N^3} (\sigma_M^4) (C_{0,N}) . \end{aligned} \quad (3.42)$$

This can be written as

$$\operatorname{cov}[I_y^{(N)}(\frac{p}{N}), I_y^{(N)}(\frac{p'}{N})] \leq c_1/N^2 + c_2/N^2 + c_3/N + c_4/N^3, \quad (3.43)$$

where,

$$c_1 = 8(C_{0,N})^2 ,$$

$$c_2 = 16(\sigma_M^2) (|\gamma_{p'}^*| + |\gamma_{p'}^*|) (C_{0,N})$$

$$c_3 = 8|\gamma_p| \cdot |\gamma_{p'}| \cdot C_{0,N}$$

$$c_4 = 48 \sigma_M^2 C_{0,N} .$$

Now, from this result, it is clear that

$$\text{cov}[I_y^{(N)}(\frac{p}{N}), I_y^{(N)}(\frac{p'}{N})] \rightarrow 0 \text{ as } N \rightarrow \infty, \quad (3.44)$$

provided σ_t^2 is bounded uniformly in t . Also, we see immediately that

$$\text{Var}[I_y^{(N)}(\frac{p}{N})] \rightarrow 0 \text{ as } N \rightarrow \infty.$$

From (3.10) and the above equation, $E\left|I_y^{(N)}(\frac{p}{N}) - |\gamma_p|^2\right|^2 \rightarrow 0$ as $N \rightarrow \infty$, and in this sense, $I_y^{(N)}(\frac{p}{N})$ is a consistent estimator of $|\gamma_p|^2$.

Neglecting all terms but $O(N^{-1})$ in (3.32) we may conclude that for large N , $\text{cov}[I_y^{(N)}(\frac{p}{N}), I_y^{(N)}(\frac{p'}{N})]$ can be approximated by the following:

$$\begin{aligned} \text{cov}[I_y^{(N)}(\frac{p}{N}), I_y^{(N)}(\frac{p'}{N})] &\sim \frac{4}{N} \text{Re}[\gamma_p^* \gamma_{p'}^*, \sum_{i=-m}^m \gamma_i \gamma_{i-(p+p')}^*] \\ &+ \frac{4}{N} \text{Re}[\gamma_p^* \gamma_{p'}, \sum_{i=-m}^m \gamma_i \gamma_{i-(p-p')}^*]. \end{aligned} \quad (3.45)$$

From this it immediately follows that

$$\text{var}[I_y^{(N)}(\frac{p}{N})] \sim \frac{4}{N} \left\{ \text{Re}[(\gamma_p^*)^2 \sum_{i=-m}^m \gamma_i \gamma_{i-2p}^*] + |\gamma_p|^2 \sum_{i=-m}^m |\gamma_i|^2 \right\}. \quad (3.46)$$

3.6. Special Cases for the Fourier Form of σ_t^2

To understand the significance and applications of the above results we are presenting in this section some special cases of the Fourier expansion for σ_t^2 . It is known that the Fourier form of σ_t^2 ,

$$\sum_{s=-m}^m \gamma_s Z^{\frac{-ts}{N}}, \text{ can be written in the form:}$$

$$\begin{aligned}\sigma_t^2 = & \gamma_s + 2A_1 \cos\left(\frac{2\pi t}{N}\right) + 2A_2 \cos\left(\frac{4\pi t}{N}\right) + \dots + 2A_m \cos\left(\frac{m\pi t}{N}\right) \\ & + 2B_1 \sin\left(\frac{2\pi t}{N}\right) + 2B_2 \sin\left(\frac{4\pi t}{N}\right) + \dots + 2B_m \sin\left(\frac{m\pi t}{N}\right),\end{aligned}\quad (3.47)$$

where

$$A_j = \operatorname{Re}(\gamma_j), \quad B_j = \operatorname{Im}(\gamma_j), \quad j = \pm 1, \pm 2, \dots, \pm m.$$

The special cases follow.

(I) Let σ_t^2 be such that $\gamma_s = 0$ for all $s \neq f$, $|f| \leq m$, i.e. σ_t^2 has only one frequency f/N . Then σ_t^2 has an expansion given by:

$$\sigma_t^2 = \gamma_s + 2A_f \cos\left(\frac{2\pi t}{N}\right) + 2B_f \sin\left(\frac{2\pi t}{N}\right).$$

Considering the exact expressions for the covariance and variance in (3.32) and (3.32a) and since $\gamma_p = 0$ for all $p \neq f$, we see that:

$$(a) \quad \sum_{i=-m}^m \gamma_i \gamma_{i-(p+p')}^* = 2\gamma_0 \gamma_f \quad \text{if } p=f, p'=0. \quad \text{For there are}$$

only two terms, $\gamma_0 \gamma_f^*$ and $\gamma_f \gamma_0^*$, which are non-zero, and $\gamma_0 \gamma_{-f}^* = \gamma_0 \gamma_f$. For any other values of p and p' the above sum is zero, since $\gamma_s = 0$ for $s \neq f$. Obviously we will get the same result if we replace p by p' . Therefore

$$\sum_{i=-m}^m \gamma_i \gamma_{i-(p+p')}^* = \begin{cases} 2\gamma_0 \gamma_f & \text{for } p=f, p'=0, \text{ or } p=0, p'=f \\ 0 & \text{otherwise.} \end{cases}$$

$$(b) \quad \sum_{i=-m}^m \gamma_i \gamma_{i-(p-p')}^* = \gamma_0^2 + 2|\gamma_f|^2 \quad \text{for } p=p'. \quad \text{This is obvious}$$

since the only possible values for i which yield non-zero terms in the above sum are $-f$, 0 and f . If $p=f$ and $p'=0$, the only non-zero terms for this sum are $\gamma_0 \gamma_{-f}^*$ and $\gamma_f \gamma_0^*$, which are the same, corresponding to $i=0$

and f respectively. By the same argument as above we see that by setting $p=0$, $p'=f$, the sums will be $2\gamma_0\gamma_f^*$. No other values of p , p' will yield a non-zero sum. Thus

$$\sum_{i=-m}^m \gamma_i \gamma_{i-(p-p')}^* = \begin{cases} \gamma_0^2 + 2|\gamma_f|^2 & \text{if } p=p' \\ 2\gamma_0\gamma_f & \text{if } p=f, p'=0 \\ 2\gamma_0\gamma_f^* & \text{if } p=0, p'=f \\ 0 & \text{otherwise.} \end{cases}$$

(c) $\gamma_{p,i,j}^* \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-p}^* = 3\gamma_0^2 \cdot |\gamma_f|^2 + \gamma_0^4$ for $p=0$. For the non-zero terms are obtained from four possible combinations for i and j : $i=j=0$; $i=f$; $j=0$; $i=-f$, $j=0$, and $i=j=f$. These values of i and j will yield the terms γ_0^4 and three of the term $\gamma_0^2 |\gamma_f|^2$ respectively. If $p=f$, then the above four combinations of i and j will yield the terms $\gamma_0^2 |\gamma_f|^2$, $|\gamma_f|^4$, and two of the term $\gamma_0^2 |\gamma_f|^2$ respectively. Therefore

$$\gamma_{p,i,j}^* \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_{j-p}^* = \begin{cases} 3\gamma_0^2 \cdot |\gamma_f|^2 + \gamma_0^4 & \text{for } p=0 \\ 3\gamma_0^2 \cdot |\gamma_f|^2 + |\gamma_f|^4 & \text{for } p=f \\ 0 & \text{otherwise.} \end{cases}$$

The same result will be obtained by replacing p with p' .

(d) For the sum $\sum_{i,j,k=-m}^m \gamma_i \gamma_{i-k}^* \gamma_j^* \gamma_{j-k}$, the term γ_0^4 corresponds to the case $i=j=k=0$. The subscripts in this sum can take the values 0 or f . All the combinations of these values, except in the case $i=j=k=0$, will yield the same term, $\gamma_0^2 \cdot |\gamma_f|^2$. Therefore, there are $(2^3-1)=7$ terms $\gamma_0^2 \cdot |\gamma_f|^2$. Also, there are seven terms of the form $\gamma_0^2 \cdot |\gamma_f|^2$,

if f is replaced by $-f$. Thus

$$\sum_{i,j,k=-m}^m \gamma_i \gamma_{i-k}^* \gamma_j^* \gamma_{j-k} = \gamma_0^4 + 14|\gamma_f|^2 \gamma_0^2.$$

Now applying the results (a)-(d) to the covariance and variance expressions in (3.32) and (3.32a), we get:

$$\begin{aligned} \text{cov}[I_y^{(N)}(\frac{f}{N}), I_y^{(N)}(\frac{p}{N})] &= 16N^{-2} \text{Re}[3\gamma_0^2 |\gamma_f|^2 + |\gamma_f|^4] \\ &\quad + 48N^{-3} [\gamma_0^4 + 14\gamma_0^2 |\gamma_f|^2], \end{aligned}$$

or

$$\begin{aligned} \text{cov}[I_y^{(N)}(\frac{f}{N}), I_y^{(N)}(\frac{p}{N})] &= 16N^{-2} [3\gamma_0^2 |\gamma_f|^2 + |\gamma_f|^4 + 4N^{-1} (\gamma_0^4 + 14\gamma_0^2 |\gamma_f|^2)] , \\ p \neq f, p &= 1, 2, \dots, m. \end{aligned} \quad (3.48)$$

and

$$\text{cov}[I_y^{(N)}(\frac{p}{N}), I_y^{(N)}(\frac{p'}{N})] = 48N^{-3} [\gamma_0^4 + 14\gamma_0^2 |\gamma_f|^2] \quad (3.49)$$

for all values of $p, p' = 1, 2, \dots, m$, not equal to f .

The variances will then follow in the same way as above:

$$\begin{aligned} \text{var}[I_y^{(N)}(\frac{f}{N})] &= 4N^{-2} (\gamma_0^2 + 2|\gamma_f|^2)^2 + 16N^{-2} [2(3\gamma_0^2 |\gamma_f|^2 + |\gamma_f|^4)] \\ &\quad + 4N^{-1} [|\gamma_f|^2 (\gamma_0^2 + 2|\gamma_f|^2)] + 48N^{-3} [\gamma_0^4 + 14\gamma_0^2 |\gamma_f|^2] \end{aligned} \quad (3.50)$$

and

$$\text{var}[I_y^{(N)}(\frac{p}{N})] = 4N^{-2} (\gamma_0^2 + 2|\gamma_f|^2)^2 + 48N^{-3} (\gamma_0^4 + 14\gamma_0^2 |\gamma_f|^2) \quad (3.51)$$

for $p \neq f, p = 1, 2, \dots, m$.

Here it is to be noted that $|\gamma_f|^2 = A_f^2 + B_f^2$ and that all of the above expressions do not depend on the value of p , so that if $p = f+s$, then $\text{cov}[I_y^{(N)}(\frac{f}{N}); I_y^{(N)}(\frac{f+s}{N})]$ is still given by (3.48). Therefore, if $|\gamma_s|^2 \rightarrow 0$ as $s \rightarrow \infty$, this covariance will not change, i.e., it does not decrease as $|\gamma_s|^2$ decreases with the increase of lag s .

On the other hand, considering expressions (3.48) through (3.51), one easily notices that, as $N \rightarrow \infty$, the covariances and variances approach zero rapidly.

II. Let σ_t^2 be such that $\gamma_s = 0$ for all $s \neq f_\mu, |f_\mu| \leq m, \mu = 1, 2$, i.e. σ_t^2 contains only two frequencies $\frac{f_1}{N}$ and $\frac{f_2}{N}$. Then, σ_t^2 may be written in the form

$$\sigma_t^2 = \gamma_0 + 2A_{f_1} \cos(\frac{2\pi t}{N}) + 2A_{f_2} \cos(\frac{4\pi t}{N}) + 2B_{f_1} \sin(\frac{2\pi t}{N}) + 2B_{f_2} \sin(\frac{4\pi t}{N}).$$

Then, in a similar way of proof, obtained in the first example, we arrive at the following results:

$$(a) \quad \sum_{i=-m}^m \gamma_i \gamma_{i-(p+p')}^* = \begin{cases} 2\gamma_0 \gamma_{f_j} & \text{if } p=f_j, p'=0 \text{ or } p'=f_j, p=0, j=1,2 \\ 0 & \text{otherwise.} \end{cases}$$

$$(b) \quad \sum_{i=-m}^m \gamma_i \gamma_{i-(p-p')}^* = \begin{cases} \gamma_0^2 + 2|\gamma_{f_1}|^2 + 2|\gamma_{f_2}|^2 & \text{if } p=p' \\ 2\gamma_0 \gamma_{f_j} & \text{if } p=f_j, p'=0 \text{ or } p'=f_j, p=0 \\ 2\gamma_0 \gamma_{f_j}^* & \text{if } p=0, p'=f_j \text{ or } p=f_j, p'=0 (j=1,2) \\ 0 & \text{otherwise} \end{cases}$$

$$(c) \gamma_p \sum_{i,j=-m}^m \gamma_i \gamma_{i-j}^* \gamma_j^* \gamma_{j-p} = \begin{cases} 3\gamma_0^2(|\gamma_{f_1}|^2 + |\gamma_{f_2}|^2) + \gamma_0^4 & \text{for } p=0 \\ 3\gamma_0^2|\gamma_{f_j}|^2 + |\gamma_{f_j}|^4 & \text{for } p=f_j, (j=1,2) \\ 0 & \text{otherwise.} \end{cases}$$

$$(d) \sum_{i,j,k=-m}^m \gamma_i \gamma_{i-k}^* \gamma_j^* \gamma_{j-k} = \gamma_0^4 + 14\gamma_0^2(|\gamma_{f_1}|^2 + |\gamma_{f_2}|^2).$$

The covariances will then be as follows:

$$\begin{aligned} \text{cov}[I_y^{(N)}(\frac{f_j}{N}); I_y^{(N)}(\frac{p'}{N})] &= 16N^{-2}[3\gamma_0^2|\gamma_{f_j}|^2 + |\gamma_{f_j}|^4] \\ &+ 48N^{-3}[\gamma_0^4 + 14\gamma_0^2(|\gamma_{f_1}|^2 + |\gamma_{f_2}|^2)], \\ &\quad (j = 1, 2) \end{aligned} \quad (3.52)$$

$$\begin{aligned} \text{cov}[I_y^{(N)}(\frac{f_1}{N}); I_y^{(N)}(\frac{f_2}{N})] &= 16N^{-2}[3\gamma_0^2(|\gamma_{f_1}|^2 + |\gamma_{f_2}|^2) + |\gamma_{f_1}|^4 + |\gamma_{f_2}|^4] \\ &+ 48N^{-3}[\gamma_0^4 + 14\gamma_0^2(|\gamma_{f_1}|^2 + |\gamma_{f_2}|^2)] \end{aligned} \quad (3.53)$$

and

$$\text{cov}[I_y^{(N)}(\frac{p}{N}); I_y^{(N)}(\frac{p'}{N})] = 48N^{-3}[\gamma_0^4 + 14\gamma_0^2(|\gamma_{f_1}|^2 + |\gamma_{f_2}|^2)] \quad (3.54)$$

for p and $p' \neq f_1$ or f_2 , $p, p' = 1, 2, \dots, m$.

And the variances will be:

$$\begin{aligned} \text{var}[I_y^{(N)}(\frac{f_j}{N})] &= 4N^{-2}[\gamma_0^2 + 2(|\gamma_{f_1}|^2 + |\gamma_{f_2}|^2)]^2 + 16N^{-2}[2(3\gamma_0^2|\gamma_{f_j}|^2 + |\gamma_{f_j}|^4)] \\ &\quad + 4N^{-1}|\gamma_{f_j}|^2(\gamma_0^2 + 2|\gamma_{f_j}|^2) + 48N^{-3}[\gamma_0^4 + 14\gamma_0^2(|\gamma_{f_1}|^2 + |\gamma_{f_2}|^2)] \end{aligned} \quad (3.55)$$

and

$$\text{var}[I_y^{(N)}(\frac{p}{N})] = 4N^{-2}[\gamma_0^2 + 2(|\gamma_{f_1}|^2 + |\gamma_{f_2}|^2)]^2 + 48N^{-3}[\gamma_0^4 + 14\gamma_0^2(|\gamma_{f_1}|^2 + |\gamma_{f_2}|^2)]. \quad (3.56)$$

Similar remarks as those for example (I) are applied to this example.

3.7. Artificial Examples of Time Series

3.7.1. Description of Series

In this section we present three artificial series, using 200 random normal deviates: η_t , $t = 1, 2, \dots, 200$. Each series is constructed by multiplying these random deviates by the square root of the variance σ_t^2 , which follows a specific Fourier expansion. Therefore, they are uncorrelated observations which may be written as $Y_t = \sigma_t \eta_t$, $t = 1, 2, \dots, 200$.

The Fourier expansions of σ_t^2 are chosen in such a way that one or a few of the coefficients γ_s is not zero while all the remaining γ 's are zero. The sum of the absolute values of the real and imaginary parts of these coefficients must not exceed $\frac{\gamma_0}{2}$.

It is found by experience gathered from constructing these series that some "unplanned frequencies" will show up in the relatively high peaks of $I_y^{(N)}(\frac{p}{N})$. But after adjusting the coefficients, γ 's, it is found that the high values of the periodogram at those unplanned frequencies will disappear to a certain extent. The difficulty of adjusting these coefficients is bigger whenever two or more frequencies for the set of σ_t^2 are desired. The generated series are listed below.

Series I: The variances follow the expansion

$$\sigma_t^2 = 20.6 - 7.4 \cos\left(\frac{7\pi t}{100}\right) + 10.4 \sin\left(\frac{7\pi t}{100}\right), \quad t = 1, 2, \dots, 200.$$

This means that $\gamma_0 = 20.6$, $2\text{Re}(\gamma_7) = -7.4$, $2\text{Im}(\gamma_7) = 10.4$ and the remaining coefficients are zero. The periodogram of squares of the series together with the true values of $|\gamma_p|^2 + N^{-1}C_{0,N}$ are listed at different values of p , $p = 0, 1, 2, \dots, 100$, in Table (3.1). A high peak for this periodogram is recorded at $p = 7$, which indicates the frequency $\frac{7}{200}$ associated with the variances. One notices the irregular behavior in the irregular fluctuations of the periodogram around the true value of $|\gamma_p|^2 + N^{-1}C_{0,N}$ as one moves farther away from $p = 7$. The highest value of the true periodogram is at $p = 7$ and close to the value of the estimator at this place. As N becomes larger, the sampling fluctuation will decrease in magnitude but not in irregularity for large values of p .

Series II: There are two frequencies, $\frac{3}{200}$ and $\frac{7}{200}$, associated with the Fourier expansion of σ_t^2 which is given by

Table 3.1. Values of the periodogram of Series I and the true values of $|\gamma_p|^2$ and $N^{-1}C_{0,N}$ at $p=0,1,2,\dots,100$

p	G(p)	p	G(p)	p	G(p)	p	G(p)
1	1.072	26	6.651	51	8.163	76	4.378
2	.211	27	6.812	52	.893	77	3.304
3	6.621	28	13.148	53	.071	78	9.158
4	1.858	29	3.136	54	13.226	79	3.339
5	1.707	30	3.530	55	7.570	80	5.906
6	.326	31	2.015	56	1.758	81	2.718
7	40.864	32	1.771	57	4.623	82	.170
8	2.954	33	.614	58	5.658	83	1.454
9	.396	34	5.408	59	4.632	84	5.742
10	8.162	35	8.485	60	2.839	85	7.366
11	7.141	36	15.039	61	1.995	86	1.656
12	2.641	37	5.824	62	7.582	87	.945
13	5.807	38	6.219	63	6.109	88	1.360
14	.419	39	1.325	64	2.257	89	5.581
15	2.223	40	.704	65	.191	90	.764
16	.815	41	1.392	66	1.437	91	6.881
17	.806	42	8.392	67	.297	92	2.393
18	5.126	43	1.935	68	2.512	93	.679
19	7.377	44	.938	69	1.980	94	.168
20	1.136	45	4.777	70	8.305	95	.207
21	2.978	46	2.362	71	.041	96	10.102
22	8.524	47	2.309	72	1.582	97	1.961
23	9.541	48	1.893	73	20.941	98	.027
24	.010	49	9.598	74	.537	99	2.868
25	1.292	50	1.808	75	.428	100	2.771

G(0)	$ \gamma_7 ^2$	$ \gamma_p ^2, p \neq 7, p=1,2,\dots,100$	γ_0^2	$N^{-1}C_{0,N}$
350.197	40.730	0.000	404.360	2.552

$$\sigma_t^2 = 20.2 - 7.2 \cos\left(\frac{3\pi t}{100}\right) - 12.2 \cos\left(\frac{7\pi t}{100}\right), \quad t = 1, 2, \dots, 200, \quad \gamma_0 = 20.2,$$

$2\text{Re}(\gamma_3) = -7.2$, $2\text{Re}(\gamma_7) = -12.2$, $\text{Im}(\gamma_3) = \text{Im}(\gamma_7) = 0$, and all the other coefficients are zero. The periodogram of squares of this series and the true values of $|\gamma_p|^2 + N^{-1}C_{0,N}$ are listed in Table (3.2).

Series III: The Fourier expansion of σ_t^2 in this series contains the two frequencies $\frac{2}{200}$ and $\frac{5}{200}$:

$$\sigma_t^2 = 20.4 - 10.9 \cos\left(\frac{2\pi t}{100}\right) - 8.8 \cos\left(\frac{5\pi t}{100}\right),$$

where $\gamma_0 = 20.4$, $2\text{Re}(\gamma_2) = -10.9$, $2\text{Re}(\gamma_5) = 8.8$, $\text{Im}(\gamma_2) = \text{Im}(\gamma_5) = 0$ and all the remaining γ 's are zero. Values of periodogram and true values of $|\gamma_p|^2 + N^{-1}C_{0,N}$ are listed in Table (3.3). High peaks at $p = 2$ and $p = 5$ for the periodogram estimate are seen with other relative high peaks at other values of p due to its irregular fluctuations around the true values of $|\gamma_p|^2 + N^{-1}C_{0,N}$.

3.7.2. Calculating Procedure

The above series with the corresponding periodograms were constructed using an IBM 1410. A FORTRAN language program was written following two stages of calculations:

- (1) The 200 random normal deviates (η_t) and the necessary coefficients for σ_t^2 are fed into the computer. X_t is calculated from $X_t = \sigma_t \eta_t$ ($t = 1, 2, \dots, 200$).

Table 3.2. Values of the periodogram of Series II and the true values of $|\gamma_p|^2$ and $N^{-1}C_{0,N}$ for $p=0,1,2,\dots,100$

p	G(p)	p	G(p)	p	G(p)	p	G(p)
1	.125	26	4.341	51	6.912	76	5.778
2	.682	27	3.735	52	1.554	77	3.537
3	24.876	28	9.182	53	.474	78	7.026
4	.357	29	1.852	54	13.979	79	2.585
5	2.862	30	2.859	55	5.198	80	4.804
6	.210	31	2.959	56	5.011	81	1.389
7	24.798	32	1.880	57	4.588	82	.315
8	2.075	33	.162	58	5.276	83	.856
9	.467	34	3.261	59	4.563	84	2.783
10	9.194	35	2.313	60	6.185	85	4.899
11	3.786	36	11.963	61	3.638	86	1.773
12	4.542	37	4.009	62	2.636	87	.851
13	6.020	38	4.877	63	7.604	88	4.226
14	.432	39	3.150	64	4.000	89	4.670
15	6.158	40	.677	65	.788	90	2.801
16	.520	41	1.456	66	2.705	91	4.384
17	1.583	42	5.573	67	.194	92	1.785
18	3.662	43	1.180	68	2.783	93	1.334
19	10.714	44	1.759	69	.006	94	.164
20	.275	45	4.640	70	8.268	95	.844
21	1.446	46	1.933	71	1.747	96	7.935
22	11.064	47	1.564	72	.463	97	.815
23	7.958	48	1.542	73	22.477	98	.453
24	.663	49	6.588	74	.170	99	3.175
25	1.682	50	.585	75	1.318	100	1.802

G(0)	$ \gamma_3 ^2$	$ \gamma_7 ^2$	$ \gamma_p ^2, p \neq 3, 7$	γ_0^2	$N^{-1}C_{0,N}$
p=1,2,...,100					
314.943	12.960	37.210	0.000	408.400	2.542

Table 3.3. Values of the periodogram of Series III and the true values of $|\gamma_p|^2$ and $N^{-1}C_{0,N}$ for $p=0,1,2,\dots,100$

p	G(p)	p	G(p)	p	G(p)	p	G(p)
1	2.537	26	3.418	51	5.086	76	3.180
2	20.542	27	10.805	52	2.784	77	6.664
3	3.794	28	10.514	53	2.057	78	7.409
4	2.216	29	.587	54	16.345	79	9.902
5	24.692	30	5.749	55	11.058	80	4.153
6	1.518	31	3.524	56	.470	81	3.844
7	4.927	32	2.497	57	16.435	82	4.551
8	1.044	33	.579	58	3.327	83	.516
9	.487	34	.362	59	11.379	84	12.084
10	4.473	35	6.529	60	7.231	85	7.032
11	5.702	36	14.250	61	.434	86	.640
12	.703	37	5.990	62	3.211	87	.633
13	3.574	38	8.706	63	2.940	88	1.494
14	.977	39	.240	64	3.109	89	10.431
15	1.132	40	.194	65	.497	90	3.394
16	1.002	41	.677	66	.152	91	8.216
17	.979	42	4.249	67	1.118	92	2.679
18	4.464	43	2.185	68	4.150	93	1.248
19	3.288	44	.074	69	.442	94	.651
20	5.199	45	2.195	70	2.993	95	.395
21	.321	46	2.994	71	.070	96	11.350
22	14.771	47	.278	72	4.060	97	.150
23	4.104	48	3.693	73	24.391	98	.759
24	1.667	49	4.843	74	3.469	99	1.312
25	4.865	50	4.212	75	2.244	100	2.431

G(0)	$ \gamma_2 ^2$	$ \gamma_5 ^2$	$ \gamma_p ^2, p \neq 2, 5$ $p=1, 2, \dots, 100$	γ_0^2	$N^{-1}C_{0,N}$
338.754	29.703	19.360	0.000	416.159	2.570

(2) Taking the square of X_t , the periodogram is computed by applying the following expansion:

$$I_{x^2}^{(N)}\left(\frac{p}{200}\right) = \frac{1}{(200)^2} \left[\left[\sum_{t=1}^{200} X_t^2 \cos\left(\frac{\pi tp}{100}\right) \right]^2 + \left[\sum_{t=1}^{200} X_t^2 \sin\left(\frac{\pi tp}{100}\right) \right]^2 \right]$$

$p = 0, 1, 2, \dots, 100.$

4. CIRCULAR STATIONARY PROCESS

4.1. Introduction

The series Y_t which is obtained by squaring the series $X_t = \sigma_t \eta_t$ discussed in Chapter 3 was shown not to be a second order stationary process. The definition of a second order stationary process [Hannan, 1960] is that $\text{cov}(Y_t, Y_{t+s}) = \tau_s$ which depends only on s and $\text{var}(Y_t) = \tau_0 < \infty$. From the series X_t it is possible to obtain a circular stationary process [Herbst, 1965] $[J_\alpha]$, ($\alpha = 0, \pm 1, \pm 2, \dots$) which is defined as follows:

$$J_\alpha = N^{-\frac{1}{2}} \sum_{t=1}^N X_t Z^{\frac{-t\alpha}{N}} \quad (t = 0, \pm 1, \pm 2, \dots) \quad (4.1)$$

where J_α is a complex-valued random variable with zero mean and variance:

$$E(J_\alpha J_\alpha^*) = \gamma_0. \quad (4.2)$$

The fact that $J_\alpha = J_{\alpha+N}$ and $E(J_\alpha J_{\alpha+s}^*) = \gamma_s$, depending only on s , (4.3) implied that $[J_\alpha]$ is a circular second order stationary process.

Herbst [1965] presented test procedures, using the estimates of the real and imaginary parts of γ_s , for the variance σ_t^2 . In this chapter we will study the sampling properties of the covariance functions obtained for this process. It is found that the process $[J_\alpha]$ has properties similar to the real second order stationary process. The approximate expressions for the covariance functions have a form similar to that in the classical situation.

The covariances of the function

$$C_s = \frac{1}{N-s} \sum_{\alpha=1}^{N-s} J_{\alpha} J_{\alpha+s}^* \quad (s \geq 0) \quad (4.4)$$

and that of $\bar{C}_s$, which has a form similar to that of C_s except for the replacement of $N-s$ for N , are derived in Sections 2 and 3. In Section 4, real and imaginary parts of $\bar{C}_s$ are considered, which are the estimates of real and imaginary parts of γ_s .

4.2. The Function C_s

In this section we will derive the exact expression for the covariance of C_s and $C_{s'}$ ($s, s' \geq 0$), and from this we will obtain an approximate expression for $\text{cov}(C_s, C_{s'})$.

C_s , defined by (4.4), is a complex-valued function, considered as a bivariate function, whose components are the real and imaginary parts of C_s . C_s is an unbiased estimate of γ_s , i.e. the real and imaginary parts of C_s are unbiased estimates of the real and imaginary parts of γ_s . It will be shown in this section that C_s has the same properties as the classical sample covariance R_s [Bartlett, 1956]. The results of this section will be used in the next section when we consider the modified covariance function $\bar{C}_s$.

We start by recalling the generalized definition of covariance applicable to complex random variables. For any complex-valued function $Z(t)$ the covariance is given by:

$$\text{cov}[Z(t), Z(t+s)] = E[Z(t) \cdot Z^*(t+s)] - E[Z(t)] \cdot E[Z^*(t+s)]. \quad (4.5)$$

Thus, considering the covariance function C_s , we have

$$\begin{aligned} \text{cov}(C_s, C_{s+t}) &= \frac{1}{(N-s)(N-s-t)} \text{cov}\left(\sum_{\alpha=1}^{N-s} J_{\alpha} J_{\alpha+s}^*, \sum_{\alpha=1}^{N-s-t} J_{\alpha} J_{\alpha+s+t}^*\right) \\ &= \frac{1}{(N-s)(N-s-t)} \sum_{\alpha=1}^{N-s} \sum_{\alpha'=1}^{N-s-t} \text{cov}(J_{\alpha} J_{\alpha+s}^*, J_{\alpha'} J_{\alpha'+s+t}^*), \end{aligned}$$

or, using (4.5),

$$\begin{aligned} \text{cov}(C_s, C_{s+t}) &= \frac{1}{(N-s)(N-s-t)} \sum_{\alpha=1}^{N-s} \sum_{\alpha'=1}^{N-s-t} [E(J_{\alpha} J_{\alpha+s}^* J_{\alpha'} J_{\alpha'+s+t}^*) \\ &\quad - E(J_{\alpha} J_{\alpha+s}^*) \cdot E(J_{\alpha'} J_{\alpha'+s+t}^*)] . \end{aligned} \quad (4.6)$$

But from (4.1) and (4.3) one can see that

$$\begin{aligned} E(J_{\alpha}^* J_{\alpha'+s+t}) &= E(J_{\alpha'} J_{\alpha'+s+t}^*)^* \\ &= [E(J_{\alpha'} J_{\alpha'+s+t}^*)]^* \end{aligned}$$

$$\text{or} \quad E(J_{\alpha}^* J_{\alpha'+s+t}) = \gamma_{s+t}^* . \quad (4.7)$$

Therefore, using (4.6) in (4.5) we find that

$$\text{cov}(C_s, C_{s+t}) = \frac{1}{(N-s)(N-s-t)} \sum_{\alpha=1}^{N-s} \sum_{\alpha'=1}^{N-s-t} E(J_{\alpha} J_{\alpha+s}^* J_{\alpha'} J_{\alpha'+s+t}^*) - \gamma_s \gamma_{s+t}^* . \quad (4.8)$$

Now we need to find first, $E(J_{\alpha} J_{\alpha+s}^* J_{\alpha'} J_{\alpha'+s+t}^*)$. By using (4.1)

we have:

$$\begin{aligned} E(J_{\alpha} J_{\alpha+s}^* J_{\alpha'} J_{\alpha'+s+t}^*) &= N^{-2} E \sum_{i,j,k,\ell=1}^N X_i X_j X_k X_{\ell} Z^{\frac{-\alpha i}{N} + (\alpha+s) \frac{j}{N} + \frac{\alpha' k}{N} + (\alpha'+s+t) \frac{\ell}{N}} \\ &= N^{-2} \sum_{i,j,k,\ell=1}^N E(X_i X_j X_k X_{\ell}) Z^{\frac{-\alpha i}{N} + (\alpha+s) \frac{j}{N} + \frac{\alpha' k}{N} + (\alpha'+s+t) \frac{\ell}{N}} . \end{aligned} \quad (4.9)$$

It is easily seen that

$$E(X_i X_j X_k X_\ell) = \begin{cases} 3\sigma_i^4 & i=j=k=\ell \\ \sigma_i^2 \sigma_k^2 & i=j, k=\ell, j \neq k \\ \sigma_i^2 \sigma_k^2 & i=\ell, k=j, i \neq k \\ \sigma_i^2 \sigma_j^2 & i=k, j=\ell, i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

Or, if we allow for equality of subscripts in $\sigma_j^2 \sigma_k^2$, $\sigma_i^2 \sigma_k^2$ and $\sigma_i^2 \sigma_j^2$ in (4.9) we get

$$E(X_i X_j X_k X_\ell) = \begin{cases} \sigma_i^2 \sigma_k^2 & i=j, k=\ell \\ \sigma_i^2 \sigma_k^2 & i=\ell, k=j \\ \sigma_i^2 \sigma_j^2 & i=k, j=\ell \\ 0 & \text{otherwise.} \end{cases} \quad (4.10)$$

Substituting in (4.9) we get

$$\begin{aligned} E(J_\alpha J_{\alpha+s}^* J_{\alpha'}^* J_{\alpha'+s+t}) &= N^{-2} \left[\sum_{i,k=1}^N \sigma_i^2 \sigma_k^2 Z^{\frac{si}{N} - (s+t) \frac{k}{N}} \right. \\ &\quad + \sum_{i,k=1}^N \sigma_i^2 \sigma_k^2 Z^{-(\alpha'+\alpha+t+s) \frac{i}{N} + (\alpha'+\alpha+s) \frac{k}{N}} \\ &\quad \left. + \sum_{i,j=1}^N \sigma_i^2 \sigma_j^2 Z^{(\alpha'-\alpha) \frac{i}{N} - (\alpha'-\alpha+t) \frac{j}{N}} \right], \end{aligned}$$

or,

$$\begin{aligned}
 E(J_{\alpha} J_{\alpha+s}^* J_{\alpha'}^* J_{\alpha'+s+t}) &= N^{-2} \left(\sum_{i=1}^N \sigma_i^2 Z^{\frac{si}{N}} \right) \left(\sum_{k=1}^N \sigma_k^2 Z^{\frac{(s+t)k}{N}} \right)^* \\
 &\quad + N^{-2} \left(\sum_{i=1}^N \sigma_i^2 Z^{\frac{(\alpha'+\alpha+t+s)i}{N}} \right)^* \left(\sum_{k=1}^N \sigma_k^2 Z^{\frac{(\alpha'+\alpha+s)k}{N}} \right) \\
 &\quad + N^{-2} \left(\sum_{i=1}^N \sigma_i^2 Z^{\frac{(\alpha'-\alpha)i}{N}} \right) \left(\sum_{j=1}^N \sigma_j^2 Z^{\frac{(\alpha'-\alpha+t)j}{N}} \right)^* .
 \end{aligned} \tag{4.11}$$

Remembering that

$$\sigma_t^2 = \sum_{p=-m}^m \gamma_p Z^{\frac{-tp}{N}} \quad \text{and} \quad \gamma_p = N^{-1} \sum_{t=1}^N \sigma_t^2 Z^{\frac{tp}{N}} , \tag{4.12}$$

we see immediately that

$$E(J_{\alpha} J_{\alpha+s}^* J_{\alpha'}^* J_{\alpha'+s+t}) = \gamma_s \gamma_{s+t}^* + \gamma_{\alpha'+\alpha+s} \gamma_{\alpha'+\alpha+s+t}^* + \gamma_{(\alpha'-\alpha)} \gamma_{(\alpha'-\alpha)+t}^* . \tag{4.13}$$

Substituting this into (4.8), we get:

$$\text{cov}(C_s, C_{s+t}) = \frac{1}{(N-s)(N-s-t)} \sum_{\alpha=1}^{N-s} \sum_{\alpha'=1}^{N-s-t} (\gamma_{\alpha'+\alpha+s} \gamma_{\alpha'+\alpha+s+t}^* + \gamma_{(\alpha'-\alpha)} \gamma_{(\alpha'-\alpha)+t}^*) . \tag{4.14}$$

Not, consider $\sum_{\alpha=1}^{N-s} \sum_{\alpha'=1}^{N-s-t} \gamma_{\alpha'+\alpha+s} \gamma_{\alpha'+\alpha+s+t}^* .$

Expanding this term, we get:

$$\begin{aligned}
& \gamma_{2+s} \gamma_{2+s+t}^* + \gamma_{3+s} \gamma_{3+s+t}^* + \dots + \gamma_{(N-s-t+1)+s} \gamma_{(N-s-t+1)+s+t}^* \\
& \gamma_{3+s} \gamma_{3+s+t}^* + \gamma_{4+s} \gamma_{4+s+t}^* + \dots + \gamma_{(N-s-t+2)+s} \gamma_{(N-s-t+2)+s+t}^* \\
& \quad \cdot \quad \cdot \quad \dots \quad \cdot \quad \cdot \\
& \quad \cdot \quad \cdot \quad \dots \quad \cdot \quad \cdot \\
& \quad \cdot \quad \cdot \quad \dots \quad \cdot \quad \cdot \\
& \gamma_{(N-s-t+1)+s} \gamma_{(N-s-t+1)+s+t}^* + \dots + \gamma_{(2N-2s-2t)+s} \gamma_{(2N-2s-2t)+s+t}^* \\
& \gamma_{(N-s-t+2)+s} \gamma_{(N-s-t+2)+s+t}^* + \dots + \gamma_{(2N-2s-2t+1)+s} \gamma_{(2N-2s-2t+1)+s+t}^* \\
& \quad \cdot \quad \cdot \quad \dots \quad \cdot \quad \cdot \\
& \quad \cdot \quad \cdot \quad \dots \quad \cdot \quad \cdot \\
& \gamma_{(N-s)+s} \gamma_{(N-s)+s+t}^* + \dots + \gamma_{(2N-2s-t-1)+s} \gamma_{(2N-2s-t-1)+s+t}^* \\
& \gamma_{(N-s+1)+s} \gamma_{(N-s+1)+s+t}^* + \dots + \gamma_{(2N-2s-t)+s} \gamma_{(2N-2s-t)+s+t}^*
\end{aligned} \tag{4.15}$$

The terms in (4.15) can be represented by an $(N-s) \times (N-s-t)$ matrix P whose elements, A_{ij} , are the variable parts of the subscripts in a one-one correspondence with the above terms, e.g. $\beta \longleftrightarrow \gamma_{\beta+s} \gamma_{\beta+s+t}^*$. Recalling that $\gamma_{\beta} = \gamma_{-N+\beta} = \gamma_{+N+\beta}$, we can replace the matrix P by the following matrix:

$$Q = \begin{bmatrix} -(N-2), -(N-3), & \dots, -(s+t-1) \\ -(N-3), -(N-4), & \dots, -(s+t-2) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & -(s-1) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot \\ -(s+t-1), -(s+t-2), \dots, (N-2s-2t) \\ -(s+t-2), -(s+t-3), \dots, (N-2s-2t+1) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ -s, -(s-1), \dots, (N-2s-t-1) \\ -(s-1), -(s-2), \dots, (N-2s-t) \end{bmatrix} \quad (4.16)$$

From the above representation of (4.15) we notice that

$(N-2s-t)$ occurs once,

$(N-2s-t-1)$ occurs twice,

$\cdot$

$\cdot$

$\cdot$

0 occurs $(N-2s-t+1)$ times,

$\cdot$

$\cdot$

$\cdot$

$-(s-1)$ occurs $(N-s-t)$ times.

All terms from $-s$ through $-(s+t-1)$ occur $(N-s-t)$ times each. Also, terms from $-(s+t)$ through $-(N-2)$ occur $(N-s-t-1)$ times, $(N-s-t-2)$ times, ..., and one time respectively. Therefore, we can write:

$$\sum_{\alpha=1}^{N-s} \sum_{\alpha'=1}^{N-s-t} \gamma_{\alpha'+\alpha+s} \gamma_{\alpha'+\alpha+s+t}^* = \sum_{v=-(N-2)}^{N-2s-t} [(N-s-t)-\eta(v)] \gamma_{v+s} \gamma_{v+s+t}^* \quad (4.17)$$

where

$$\eta(v) = \begin{cases} v+s-1 & v \geq 0, -(s-1) \leq v \leq -1 \\ 0 & -(s+t-1) \leq v \leq -(s-1) \\ -(v+s+t)+1 & -(N-2) \leq v \leq -(s+t) \end{cases}$$

Now, let $v+s=u$. We see that

$$\sum_{\alpha=1}^{N-s} \sum_{\alpha'=1}^{N-s-t} \gamma_{\alpha'+\alpha+s} \gamma_{\alpha'+\alpha+s+t}^* = \sum_{u=-(N-s)+2}^{N-s-t} [(N-s-t)-\eta(u)] \gamma_u \gamma_{u+t}^* \quad (4.18)$$

where

$$\eta(u) = \begin{cases} u-1 & u \geq 1 \\ 0 & 1-t \leq u \leq 1 \\ -(u+t-1) & -(N-s-2) \leq u \leq -t \end{cases}$$

The second term of (4.14) can easily be proved to have the following form:

$$\sum_{\alpha=1}^{N-s} \sum_{\alpha'=1}^{N-s-t} \gamma_{(\alpha'-\alpha)+s} \gamma_{(\alpha'-\alpha)+s+t}^* = \sum_{u=-(N-s)+1}^{N-s-t-1} [(N-s-t) - \eta_1(u)] \gamma_u \gamma_{u+t}^* \quad (4.19)$$

where

$$\eta_1(u) = \begin{cases} u & u \geq 0 \\ 0 & -t \leq u \leq 0 \\ -(u+t) & -(N-s-1) \leq u \leq -t \end{cases} \quad (4.20)$$

Therefore, (4.14) becomes

$$\begin{aligned} \text{cov}(C_s, C_{s+t}) &= \frac{1}{(N-s)} \sum_{u=-(N-s)+2}^{N-s-t} \left[1 - \frac{\eta_1(u)}{(N-s-t)} \right] \gamma_u \gamma_{u+t}^* \\ &+ \sum_{u=-(N-s)+1}^{N-s-t-1} \left[1 - \frac{\eta_1(u)}{(N-s-t)} \right] \gamma_u \gamma_{u+t}^* \quad (4.21) \end{aligned}$$

For large values of N this covariance takes the following approximate form:

$$\text{cov}(C_s, C_{s+t}) \sim \frac{1}{(N-s)} \sum_{u=-\infty}^{\infty} \gamma_u \gamma_{u+t}^* + \sum_{u=-\infty}^{\infty} \gamma_u \gamma_{u+t}^* = \frac{2}{(N-s)} \sum_{u=-\infty}^{\infty} \gamma_u \gamma_{u+t}^* \quad (4.22)$$

and from this we get

$$\text{var}(C_s) \sim \frac{2}{(N-s)} \sum_{u=-\infty}^{\infty} |\gamma_u|^2.$$

These are in exactly the same form as that for the covariance and variance functions of a real stationary process. One will notice that, if $\gamma_s \rightarrow 0$ as $s \rightarrow \infty$, then the above expressions in (4.22) will not be changed, since they do not depend on γ_s . This suggests that the sampling fluctuations in the covariance function C_s depend on the unknown γ_s even if γ_s decreases with the increase of s , i.e. the true

function γ_s damps down with the increase of s . Thus C_s provides little information, and hence is an unreliable estimate of γ_s as s becomes larger.

4.3. Modification on C_s

4.3.1. The Function $\bar{C}_s$

The modified function of C_s , $\bar{C}_s$, is defined by:

$$\bar{C}_s = \frac{1}{N} \sum_{\alpha=1}^N J_{\alpha} J_{\alpha+s}^* , \quad (4.23)$$

where $\bar{C}_s$ is again a complex-valued function. Since clearly $E(\bar{C}_s) = \gamma_s$, $\bar{C}_s$ is an unbiased estimate of γ_s and the real and imaginary parts of $\bar{C}_s$ are the unbiased estimates of the real and imaginary parts of γ_s . In this instance $\bar{C}_s$ has simple real and imaginary parts [Herbst, 1965] which are given by

$$\text{Re}(\bar{C}_s) = N^{-1} \sum_{t=1}^N X_t^2 \cos\left(\frac{2\pi ts}{N}\right) \text{ and } \text{Im}(\bar{C}_s) = N^{-1} \sum_{t=1}^N X_t^2 \sin\left(\frac{2\pi ts}{N}\right) .$$

In this section we shall employ some of the results of the last section in finding a simple expression for the covariance of $\bar{C}_s$ and $\bar{C}_{s'}$, ($s \neq s'$). It will be found that $\bar{C}_s$ behaves in the same manner as C_s (discussed in the last section) for large N and also for the case in which $\gamma_s \rightarrow 0$ as $s \rightarrow \infty$.

Using (4.23) and the definition of covariance we get:

$$\text{cov}(\bar{C}_s, \bar{C}_{s+t}) = N^{-2} \sum_{\alpha, \alpha'=1}^N E(J_{\alpha} J_{\alpha+s}^* J_{\alpha'} J_{\alpha'+s+t}^*) - \gamma_s \gamma_{s+t}^* . \quad (4.24)$$

In Section 3 of this chapter it was shown that

$$E(J_{\alpha} J_{\alpha+s}^* J_{\alpha'} J_{\alpha'+s+t}^*) = \gamma_s \gamma_{s+t}^* + \gamma_{\alpha'+\alpha+s} \gamma_{\alpha'+\alpha+s+t}^* + \gamma_{\alpha'-\alpha} \gamma_{\alpha'-\alpha+t}^* \quad (4.25)$$

Therefore, substituting (4.25) into (4.24), we get

$$\text{cov}(\bar{C}_s, \bar{C}_{s+t}) = N^{-2} \sum_{\alpha, \alpha'=1}^N [\gamma_{\alpha'+\alpha+s} \gamma_{\alpha'+\alpha+s+t}^* + \gamma_{\alpha'-\alpha} \gamma_{\alpha'-\alpha+t}^*] \quad (4.26)$$

Let us consider the term $\sum_{\alpha, \alpha'=1}^N \gamma_{\alpha'+\alpha+s} \gamma_{\alpha'+\alpha+s+t}^* = T(s, t)$, say, and the matrix Q presented in the last section. Since α, α' vary from 1 to N , the first row of Q , for example, will be $-(N-2), -(N-3), \dots, 0, 1$ and the second row will be $-(N-3), -(N-4), \dots, 1, 2$. Thus, one can easily see that Q will take the form Q_1 such that

$$Q_1 = \begin{bmatrix} -(N-2), & -(N-3), & \dots, & 0, & 1 \\ -(N-3), & -(N-4), & \dots, & 1, & 2 \\ \cdot & \cdot & \dots, & \cdot & \cdot \\ \cdot & \cdot & \dots, & \cdot & \cdot \\ \cdot & \cdot & \dots, & \cdot & \cdot \\ 0, & 1, & \dots, & N-2, & N-1 \\ 1, & 2, & \dots, & N-1, & N \end{bmatrix} \quad (4.27)$$

Thus, it is easily seen that $T(s, t)$ will take the form:

$$T(s, t) = \sum_{u=-(N-2)}^N [N-\eta(u)] \gamma_{u+s} \gamma_{u+s+t}^*, \quad (4.28)$$

where

$$\eta(u) = \begin{cases} u-1 & \text{if } u \geq 0 \\ -(u-1) & \text{if } u < 0 \end{cases}.$$

Or $T(s,t)$ can be written as:

$$T(s,t) = \sum_{v=-(N-1)}^{N-1} [N-|v|] \gamma_{v+s+1} \gamma_{v+s+t+1}^* \quad (4.29)$$

Also, the second term of (4.26) can be shown to take the form:

$$N^{-2} \sum_{v=-(N-1)}^{N-1} [N-|v|] \gamma_v \gamma_{v+t}^* \quad (4.30)$$

Therefore, the final form for (4.24) is

$$\text{cov}(\bar{C}_s, \bar{C}_{s+t}) = N^{-1} \sum_{v=-(N-1)}^{N-1} \left[1 - \frac{|v|}{N} \right] [\gamma_{v+s+1} \gamma_{v+s+t+1}^* + \gamma_v \gamma_{v+t}^*] \quad (4.31)$$

Now, for large values of N we may approximate (4.31) by the following form:

$$\text{cov}(\bar{C}_s, \bar{C}_{s+t}) \sim N^{-1} \sum_{v=-\infty}^{\infty} [\gamma_{v+s+1} \gamma_{v+s+1}^* + \gamma_v \gamma_{v+t}^*] \quad (4.32)$$

and if $\gamma_s \rightarrow 0$ whenever $s \rightarrow \infty$, the above expression for $\text{cov}(\bar{C}_s, \bar{C}_{s+t})$ will tend to have the form:

$$\text{cov}(\bar{C}_s, \bar{C}_{s+t}) \sim N^{-1} \sum_{v=-\infty}^{\infty} \gamma_v \gamma_{v+t}^* \quad (4.33)$$

From the above one notices that the covariance for the new definition of C_s has a similar form to that for the classical covariance function for a stationary process [Bartlett, 1956], and behaves in the same manner for large N and also for the asymptotic case in which $\gamma_s \rightarrow 0$ as $s \rightarrow \infty$ for a fixed value of N . Thus $\bar{C}_s$ provides us with little information about γ_s for large values of s .

4.3.2. Real and Imaginary Parts of $\bar{C}_s$

In this section we will study the sampling fluctuations of the real and imaginary parts of $\bar{C}_s$ by deriving the exact and approximate covariance and variance functions.

Let

$$\bar{C}_s = A_s + iB_s, \quad (4.34)$$

where A_s, B_s are the real and imaginary parts of $\bar{C}_s$. Then, since

$$N^{-1} \sum_{\alpha=1}^N J_{\alpha} J_{\alpha+s}^* = N^{-1} \sum_{t=1}^N X_t^2 Z^{\frac{st}{N}}, \quad (4.35)$$

we see that

$$A_s = N^{-1} \sum_{t=1}^N X_t^2 \cos \frac{2\pi st}{N} \quad (4.36)$$

and

$$B_s = N^{-1} \sum_{t=1}^N X_t^2 \sin \frac{2\pi st}{N}.$$

Herbst [1965] showed that $E(A_s) = \text{Re}(\gamma_s)$ and $E(B_s) = \text{Im}(\gamma_s)$. Now, to find $\text{cov}(A_s, A_{s'})$ and $\text{cov}(B_s, B_{s'})$, using (4.36) we proceed as follows:

$$\begin{aligned} N^2 \text{cov}(A_s, A_{s'}) &= E \left[\sum_{t, t'=1}^N X_t^2 X_{t'}^2 \cos \frac{2\pi st}{N} \cos \frac{2\pi s't'}{N} \right] \\ &\quad - E \left(\sum_{t=1}^N X_t^2 \cos \frac{2\pi st}{N} \right) E \left(\sum_{t=1}^N X_t^2 \cos \frac{2\pi s't}{N} \right) \\ &= 3 \sum_{t=1}^N \sigma_t^4 \cos \frac{2\pi st}{N} + \sum_{\substack{t, t'=1 \\ t \neq t'}}^N \sigma_t^2 \sigma_{t'}^2 \cos \frac{2\pi st}{N} \cos \frac{2\pi s't'}{N} \\ &\quad - \left(\sum_{t=1}^N \sigma_t^2 \cos \frac{2\pi st}{N} \right) \left(\sum_{t=1}^N \sigma_t^2 \cos \frac{2\pi s't}{N} \right). \end{aligned}$$

In the last expression by taking one sum of the first term and adding it to the second term, we get a quantity which would be cancelled out by the negative term.

$$N^2 \text{cov}(A_s, A_{s'}) = 2 \sum_{t=1}^N \sigma_t^2 \cos \frac{2\pi s t}{N} \cos \frac{2\pi s' t}{N}. \quad (4.37)$$

Let $\sigma_t^4 = \sum_{i=-m}^m \delta_i Z^{\frac{it}{N}}$, and noting that $\cos 2\pi x = \frac{1}{2}(Z^x + Z^{-x})$, we get:

$$\begin{aligned} N^2 \text{cov}(A_s, A_{s'}) &= \frac{1}{2} \sum_{t=1}^N \left(\sum_{i=-m}^m \delta_i Z^{\frac{-it}{N}} \right) \left(Z^{\frac{st}{N}} + Z^{\frac{-st}{N}} \right) \left(Z^{\frac{s't}{N}} + Z^{\frac{-s't}{N}} \right) \\ &= \frac{1}{2} \sum_{i=-m}^m \delta_i \sum_{t=1}^N Z^{\frac{-it}{N}} \left[Z^{\frac{(s+s')t}{N}} + Z^{\frac{(s'-s)t}{N}} + Z^{\frac{-(s'-s)t}{N}} \right. \\ &\quad \left. + Z^{\frac{-(s+s')t}{N}} \right], \end{aligned}$$

which reduced to

$$N^2 \text{cov}(A_s, A_{s'}) = \frac{N}{2} [\delta_{s+s'} + \delta_{s'-s} + \delta_{s'-s}^* + \delta_{s'+s}^*],$$

or

$$\text{cov}(A_s, A_{s'}) = \frac{1}{N} \text{Re}[\delta_{s'+s} + \delta_{s'-s}]. \quad (4.38)$$

But, as was proved in Chapter 3,

$$\delta_k = \sum_{i=-m}^m \gamma_i \gamma_{i-k}^*. \quad (4.39)$$

Therefore, using (4.39) in (4.38), we get:

$$\text{cov}(A_s, A_{s'}) = \frac{1}{N} \text{Re} \left[\sum_{i=-m}^m \gamma_i \gamma_{i-(s'+s)}^* + \sum_{i=-m}^m \gamma_i \gamma_{i-(s'-s)}^* \right]. \quad (4.40)$$

Remembering that $\sin 2\pi x = \frac{1}{2i}(Z^x - Z^{-x})$, in exactly the same procedure as above we can easily see that:

$$N^2 \text{cov}(B_s, B_{s'}) = -\frac{1}{2} \sum_{j=-m}^m \delta_j \sum_{t=1}^N Z^{\frac{-jt}{N}} [Z^{(s+s')\frac{t}{N}} Z^{(s'-s)\frac{t}{N}} - Z^{-(s'-s)\frac{t}{N}} + Z^{-(s'+s)\frac{t}{N}}]$$

or

$$\text{cov}(B_s, B_{s'}) = -\frac{1}{2N} [\delta_{s+s'} - \delta_{s'-s} - \delta_{s'-s}^* + \delta_{s'+s}^*].$$

Therefore, $\text{cov}(B_s, B_{s'}) = \frac{1}{N} \text{Re}[\delta_{s'-s} - \delta_{s'+s}]$, which can be written as

$$\text{cov}(B_s, B_{s'}) = \frac{1}{N} \text{Re} \left[\sum_{i=-m}^m \gamma_i \gamma_{i-(s'-s)}^* - \sum_{i=-m}^m \gamma_i \gamma_{i-(s'+s)}^* \right]. \quad (4.41)$$

Now, for large values of N we can approximate (4.40) and (4.41) by the following:

$$\text{cov}(A_s, A_{s'}) \sim N^{-1} \text{Re} \left[\sum_{i=-\infty}^{\infty} \gamma_i \gamma_{i-(s'+s)}^* + \gamma_i \gamma_{i-(s'-s)}^* \right] \quad (4.42)$$

and

$$\text{cov}(B_s, B_{s'}) \sim N^{-1} \text{Re} \left[\sum_{i=-\infty}^{\infty} \gamma_i \gamma_{i-(s'-s)}^* - \gamma_i \gamma_{i-(s'+s)}^* \right].$$

Again, these formulae for covariances are of the same type of covariance of a sample covariance function for a stationary process that is well known in time series literature, and behave in the same fashion for large N as well as in the case in which $\gamma_s \rightarrow 0$ as $s \rightarrow \infty$. In the latter case it is clear that $\gamma_{i-(s'+s)}^* \rightarrow 0$ as $s \rightarrow \infty$ and that, if $s'-s = t$, formulae (4.42) reduce to the following form:

$$\text{cov}(A_s, A_{s+t}) \sim N^{-1} \text{Re} \left(\sum_{i=-\infty}^{\infty} \gamma_i \gamma_{i-t}^* \right) \text{ and } \text{cov}(B_s, B_{s+t}) \sim N^{-1} \text{Re} \left(\sum_{i=-\infty}^{\infty} \gamma_i \gamma_{i-t}^* \right). \quad (4.43)$$

Thus, if whenever $s \rightarrow \infty$, $\gamma_s \rightarrow 0$, we get

$$\text{cov}(A_s, A_{s+t}) = \text{cov}(B_s, B_{s+t}) \quad (4.44)$$

and

$$\text{var}(A_s) = \text{var}(B_s) = N^{-1} \sum_{i=-\infty}^{\infty} |\gamma_i|^2 = N^{-1} C_0. \quad (4.45)$$

(4.45) proves that A_s and B_s are consistent estimators of $\text{Re}(\gamma_s)$ and $\text{Im}(\gamma_s)$ respectively, since $\text{var}(A_s) = \text{var}(B_s) \rightarrow 0$ as $N \rightarrow \infty$. Thus, since $\text{Re}(\gamma_s) = N^{-1} \sum_{t=1}^N \sigma_t^2 \cos(\frac{2\pi ts}{N})$ and $\text{Im}(\gamma_s) = N^{-1} \sum_{t=1}^N \sigma_t^2 \sin(\frac{2\pi ts}{N})$, A_s and B_s provide consistent measures of coincidence of σ_t^2 with $\cos(\frac{2\pi ts}{N})$ and $\sin(\frac{2\pi ts}{N})$ respectively.

4.3.3. The Relation between $\bar{C}_s$ and $I_{x^2}^{(N)}(\frac{s}{N})$

As defined in Chapter 3, $I_{x^2}^{(N)}(\frac{s}{N}) = N^{-2} \left| \sum_{t=1}^N X_t^2 Z \frac{ts}{N} \right|^2$. Hence

$$I_{x^2}^{(N)}(\frac{s}{N}) = N^{-2} \left| \sum_{\alpha=1}^N J_{\alpha} J_{\alpha+s}^* \right|^2. \quad (4.46)$$

Therefore, using the previous definition of $\bar{C}_s$, we see that

$$I_{x^2}^{(N)}(\frac{s}{N}) = N^{-2} |\bar{C}_s|^2. \quad (4.47)$$

From a proof in Chapter 3 for large values of N , we have:

$$\text{cov}[I_{x^2}^{(N)}(\frac{s}{N}), I_{x^2}^{(N)}(\frac{s'}{N})] \sim \frac{4}{N} \text{Re}[\gamma_s^* \gamma_{s'} \sum_{i=-\infty}^{\infty} \gamma_i \gamma_{i-(s+s')}^* + \gamma_s^* \gamma_{s'} \sum_{i=-\infty}^{\infty} \gamma_i \gamma_{i-(s-s')}^*]. \quad (4.48)$$

This means that for $s' = s+t$,

$$\text{cov}[|\bar{C}_s|^2, |\bar{C}_{s+t}|^2] \sim \frac{4}{N} \text{Re}[\gamma_s^* \gamma_{s+t} \sum_{i=-\infty}^{\infty} \gamma_i \gamma_{i-(2s+t)}^* + \gamma_s^* \gamma_{s+t} \sum_{i=-\infty}^{\infty} \gamma_t \gamma_{i+t}^*] \quad (4.49)$$

and that

$$\text{var}[|\bar{C}_s|^2] \sim \frac{4}{N} \text{Re}[(\gamma_s^*)^2 \sum_{i=-\infty}^{\infty} \gamma_i \gamma_{i-2s}^* + \frac{4}{N} |\gamma_s|^2 \sum_{i=-\infty}^{\infty} |\gamma_i|^2]. \quad (4.50)$$

It appears from (4.49) and (4.50) that if $\gamma_s \rightarrow 0$ whenever $s \rightarrow \infty$ and with N fixed, then:

$$\text{cov}[|\bar{C}_s|^2, |\bar{C}_{s+t}|^2] \rightarrow 0$$

and

$$\text{var}[|\bar{C}_s|^2] \rightarrow 0 \text{ as } s \rightarrow \infty. \quad (4.51)$$

This means that if we ignore the terms of order N^{-2} in the expressions for the covariance and variance in (3.32) and (3.32a) of Chapter 3, then the sampling fluctuations in $|\bar{C}_s|^2$ will damp down as s increases, if $\gamma_s \rightarrow 0$ as s increases. On the other hand, if we include terms of $O(N^{-2})$, this is not the case.

5. PERIODOGRAM OF SQUARES OF A MOVING AVERAGE

5.1. Introduction

A more applicable model in practice than the random model introduced in the third chapter is the linear process model. This model is a linear combination of independent random variables with zero means and variances which are not necessarily equal, e.g. $X_t = \sum_{j=0}^h b_j Y_{t-j}$. In time series terminology this is the model of "a moving average of innovations." This type of model may be applied to a physical phenomenon where an observation at a certain time depends on several other observations on another factor made at a sequence of time previous to t . We would also arrive at this form of model by considering a k -order autoregressive scheme, KOARS, where $k \geq 1$ and whose characteristic equation has roots $\mu_1, \mu_2, \dots, \mu_k$ with $|\mu_j| < 1$ ($j = 1, 2, \dots, k$).

Herbst [1963a] studied the following moving average model

$$X_t = \sum_{j=0}^h \alpha_j \sigma_{t-j} \eta_{t-j}, \quad (5.1)$$

where α_j 's are real constants, σ_t and η_t are as defined in Chapter 3. In this chapter we will present the periodogram of squares of X_t and study its sampling properties with some artificial examples at the end of the chapter. In Chapter 3 it was proved that the periodogram of squares of a random process is a consistent estimator of the function $|\gamma_p|^2$. In the model considered at the present one would expect that the corresponding periodogram is an estimate of two factors, that which involves the α_j 's and the other a function of $|\gamma_p|^2$. In the third section of this chapter a relation between the periodogram of X_t^2 and

that of $Y_t^2 = \sigma_t^2 \eta_t^2$ will be proved, having the same character as that for the stationary moving average model [Hannan, 1960] and where the variances are assumed to be constants.

5.2. The Origin of a Moving Average Model

Many physical and natural phenomena can be described by the model discussed in the last section. Slutsky [1927] presented an original discussion of fitting moving average models, with constant variances, to economical and other data. A numerous number of examples can be found among these phenomena where the constancy, with respect to time, of variances cannot be assumed. Our moving average model may arise, for example, in the crop yields during one season in which the amount of rainfall daily is the "random cause", i.e. the crop yield X_t at time t is some constant times the rainfall at time $t-1$ plus another constant times the rainfall at time $t-2$, ..., etc., where the amount of rainfall is assumed to be a random variable with zero mean and variance σ_t^2 . A moving average model may be considered as a limiting form of an autoregressive scheme. Consider the k -order autoregressive model

$$X_t = b_1 X_{t-1} + b_2 X_{t-2} + \dots + b_k X_{t-k} + \sigma_t \eta_t \quad (5.2)$$

$$(t = 0, \pm 1, \pm 2, \dots),$$

where σ_t are finite constants, η_t are uncorrelated random variables with zero means and variance 1, and k is an integer $k \geq 1$. To solve this equation (in terms of $\sigma_t \eta_t$) we notice that it can be written as $(E^t - b_1 E^{t-1} - b_2 E^{t-2} - \dots - b_k E^{t-k})X_t = \sigma_t \eta_t$, where the operator E is such that $E^j(X_t) = X_{t-j}$, $j = 0, 1, 2, \dots, k$. The characteristic equation is

then

$$E^t - b_1 E^{t-1} - b_2 E^{t-2} - \dots - b_k E^{t-k} = 0, \quad (5.3)$$

which will have the roots, real and complex, $\mu_1, \mu_2, \dots, \mu_k$. The solution of (5.2) can be written as

$$X_t = A_1 \mu_1^{t-u} + A_2 \mu_2^{t-u} + \dots + A_k \mu_k^{t-u} + \sum_{j=0}^{t-u-k} \alpha_j \sigma_{t-j} \eta_{t-j},$$

where the A_k 's and α_j 's are constants and $X_t = X_u$ at the initial time u .

Now, if $u \rightarrow -\infty$, X_t will take the form $X_t = \sum_{j=0}^{\infty} \alpha_j \sigma_{t-j} \eta_{t-j}$,

which is our model described in (5.1) ($h=\infty$) and we further assume

$$\text{var } X_t = \sum_{j=0}^{\infty} \alpha_j^2 \sigma_{t-j}^2 < \infty \quad (\text{all } t).$$

As an example, consider the modified model of the first order autoregressive scheme:

$$X_t = \rho X_{t-1} + \sigma_t \eta_t \quad (t = 0, \pm 1, \pm 2, \dots). \quad (5.4)$$

Then, solving (5.4) by iteration, we obtain

$$X_t = \rho^{r-u} X_u + \sum_{s=0}^{r-u-1} \rho^s \sigma_{t-s} \eta_{t-s},$$

where $X_t = X_u$ at initial time u . If $|\rho| < 1$, then as $u \rightarrow -\infty$ this will give the solution

$$X_t = \sum_{s=0}^{\infty} \rho^s \sigma_{t-s} \eta_{t-s},$$

which is the moving average model presented here with $h=\infty$ and

$$\alpha_j = \rho^j \quad (j = 0, 1, \dots) \text{ and we assume } \sum_{s=0}^{\infty} \rho^{2s} \sigma_{t-s}^2 < \infty \quad (\text{all } t).$$

5.3. Periodogram of X_t^2

The periodogram of X_t^2 is defined by

$$I_{X^2}^{(N)}\left(\frac{p}{N}\right) = |J_{X^2}\left(\frac{p}{N}\right)|^2, \quad 1 \leq p \leq m, \text{ where}$$

$$J_{X^2}\left(\frac{p}{N}\right) = N^{-1} \sum_{t=1}^N X_t^2 Z^{\frac{tp}{N}}. \quad (5.5)$$

Using (5.1) in (5.5), we get:

$$J_{X^2}\left(\frac{p}{N}\right) = N^{-1} \sum_{t=1}^N \left(\sum_{j,j'=0}^h \alpha_j \alpha_{j'} \sigma_{t-j} \sigma_{t-j'} \eta_{t-j} \eta_{t-j'} \right) Z^{\frac{tp}{N}}, \text{ or}$$

$$J_{X^2}\left(\frac{p}{N}\right) = N^{-1} \sum_{j=0}^h \sum_{t=1}^N \alpha_j^2 \sigma_{t-j}^2 \eta_{t-j}^2 Z^{\frac{tp}{N}} + N^{-1} \sum_{j \neq j'=0}^h \sum_{t=1}^N \alpha_j \alpha_{j'} \sigma_{t-j} \sigma_{t-j'} \eta_{t-j} \eta_{t-j'} Z^{\frac{tp}{N}}$$

$$= B_1 + N^{-\frac{1}{2}} B_2, \text{ say, where}$$

$$B_1 = N^{-1} \sum_{j=0}^h \sum_{t=1}^N \alpha_j^2 \sigma_{t-j}^2 \eta_{t-j}^2 Z^{\frac{tp}{N}}, \text{ and}$$

$$B_2 = N^{-\frac{1}{2}} \sum_{j \neq j'=0}^h \sum_{t=1}^N \alpha_j \alpha_{j'} \sigma_{t-j} \sigma_{t-j'} \eta_{t-j} \eta_{t-j'} Z^{\frac{tp}{N}}. \quad (5.7)$$

Consider B_1 , and suppose first that $h \leq N$. Then, B_1 can be written

as:

$$\begin{aligned} B_1 &= N^{-1} \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \sum_{t=1}^N \sigma_{t-j}^2 \eta_{t-j}^2 Z^{\frac{(t-j)p}{N}} \\ &= N^{-1} \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \sum_{t'=1-j}^{N-j} \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{t'p}{N}}, \end{aligned} \quad (5.8)$$

or

$$B_1 = N^{-1} \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \left(\sum_{t'=1-j}^{N-j} \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{t'p}{N}} - \sum_{t'=1-j}^0 \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{t'p}{N}} \right. \\ \left. + \sum_{t'=1-j}^0 \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{t'p}{N}} + \sum_{t'=N-j+1}^N \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{t'p}{N}} - \sum_{t'=N-j+1}^N \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{t'p}{N}} \right),$$

which is obtained by adding and subtracting terms to get a sum over t' from 1 to N . Therefore,

$$B_1 = N^{-1} \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \left(\sum_{t'=1}^N \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{t'p}{N}} \right) + N^{-1} \sum_{j=0}^h \sum_{t'=1-j}^0 \alpha_j^2 \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{p}{N}(j+t')} \\ - N^{-1} \sum_{j=0}^h \sum_{t'=N-j+1}^N \alpha_j^2 \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{p}{N}(j+t')},$$

which is equal to

$$B_1 = \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} J(Y, \frac{p}{N}) + N^{-1} \sum_{j=0}^h \left(\sum_{k=0}^{j-1} \alpha_j^2 \sigma_{j-k}^2 \eta_{j-k}^2 Z^{\frac{p}{N}(j-k)} - \sum_{k=0}^{j-1} \alpha_j^2 \sigma_{N-k}^2 \eta_{N-k}^2 Z^{\frac{p}{N}(N+j-k)} \right), \quad (5.9)$$

where

$$J(Y, \frac{p}{N}) = N^{-1} \sum_{t=1}^N Y_t Z^{\frac{tp}{N}} = N^{-1} \sum_{t=1}^N \sigma_t^2 \eta_t^2 Z^{\frac{tp}{N}} = J(\sigma^2 \eta^2, \frac{p}{N}). \quad (5.10)$$

For a reason which will be seen later, we may write B_1 in the following form:

$$B_1 = \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} J(Y, \frac{p}{N}) + N^{-1} \sum_{j=0}^h \left(\sum_{k=\max(0, j-N)}^{j-1} \alpha_j^2 \sigma_{j-k}^2 \eta_{j-k}^2 Z^{\frac{p}{N}(j-k)} - \sum_{k=0}^{\min(j-1, N-1)} \alpha_j^2 \sigma_{N-k}^2 \eta_{N-k}^2 Z^{\frac{p}{N}(N+j-k)} \right). \quad (5.11)$$

Second, suppose that $h > N$; then, B_1 in (5.8) would take the form

A such that:

$$A = N^{-1} \sum_{j=0}^N \alpha_j^2 Z^{\frac{jp}{N}} \sum_{t'=1-j}^{N-j} \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{t'p}{N}} + N^{-1} \sum_{j=N+1}^h \alpha_j^2 Z^{\frac{jp}{N}} \sum_{t'=1-j}^{N-j} \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{t'p}{N}}. \quad (5.12)$$

Working similarly to the above procedure, we see that

$$A = \sum_{j=0}^N \alpha_j^2 Z^{\frac{jp}{N}} J(Y, \frac{p}{N}) + N^{-1} \sum_{j=0}^N \left(\sum_{k=0}^{j-1} \sigma_{-k}^2 \eta_{-k}^2 Z^{\frac{p}{N}(j-k)} - \sum_{k=0}^{j-1} \sigma_{N-k}^2 \eta_{N-k}^2 Z^{\frac{p}{N}(N+j-k)} \right) \\ + N^{-1} \sum_{j=N+1}^h \alpha_j^2 Z^{\frac{jp}{N}} \left(\sum_{t'=1-j}^{N-j} \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{t'p}{N}} \right). \quad (5.13)$$

The last term of (5.13) can be written as:

$$N^{-1} \sum_{j=N+1}^h \alpha_j^2 Z^{\frac{jp}{N}} \left(\sum_{t'=1-j}^{N-j} \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{t'p}{N}} + \sum_{t'=1}^N \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{t'p}{N}} - \sum_{t'=1}^N \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{t'p}{N}} \right),$$

or as:

$$\sum_{j=N+1}^h \alpha_j^2 Z^{\frac{jp}{N}} J(Y, \frac{p}{N}) + N^{-1} \sum_{j=N+1}^h \alpha_j^2 Z^{\frac{jp}{N}} \left(\sum_{t'=1-j}^{N-j} \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{t'p}{N}} - \sum_{t'=1}^N \sigma_{t'}^2 \eta_{t'}^2 Z^{\frac{t'p}{N}} \right)$$

or as:

$$\sum_{j=N+1}^h \alpha_j^2 Z^{\frac{jp}{N}} J(Y, \frac{p}{N}) + N^{-1} \sum_{j=N+1}^h \alpha_j^2 \left(\sum_{k=j-N}^{j-1} \sigma_{-k}^2 \eta_{-k}^2 Z^{\frac{(j-k)p}{N}} - \sum_{k=0}^{N-1} \sigma_{N-k}^2 \eta_{N-k}^2 Z^{\frac{p}{N}(N+j-k)} \right). \quad (5.14)$$

Therefore, A has the form:

$$\begin{aligned}
A = & \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} J(Y, \frac{p}{N}) + N^{-1} \sum_{j=0}^N \alpha_j^2 \left(\sum_{k=0}^{j-1} \sigma_{-k}^2 \eta_{-k}^2 Z^{\frac{p}{N}(j-k)} - \sum_{k=0}^{j-1} \sigma_{N-k}^2 \eta_{N-k}^2 Z^{\frac{p}{N}(N+j-k)} \right) \\
& + N^{-1} \sum_{j=N+1}^h \alpha_j^2 \left(\sum_{k=j-N}^{j-1} \sigma_{-k}^2 \eta_{-k}^2 Z^{\frac{p}{N}(j-k)} - \sum_{k=0}^{N-1} \sigma_{N-k}^2 \eta_{N-k}^2 Z^{\frac{p}{N}(N+j-k)} \right). \quad (5.15)
\end{aligned}$$

Hence, we can write (5.15) in the form:

$$\begin{aligned}
A = & \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \cdot J(Y, \frac{p}{N}) + N^{-1} \sum_{j=0}^h \alpha_j^2 \left(\sum_{k=\max(0, j-N)}^{j-1} \sigma_{-k}^2 \eta_{-k}^2 Z^{\frac{p}{N}(j-k)} \right. \\
& \left. - \sum_{k=0}^{\min(j-1, N-1)} \sigma_{N-k}^2 \eta_{N-k}^2 Z^{\frac{p}{N}(N+k-j)} \right). \quad (5.16)
\end{aligned}$$

Thus, A has a form exactly similar to that of B_1 (5.11), where we assumed that $h \leq N$, and in the meantime we arrived at a form similar to that of a lemma proved for a stationary process [Hannan, 1960] and valid for any value of h .

$$\text{Thus } A = A_1 + N^{-\frac{1}{2}} A_2,$$

where

$$A_1 = \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \cdot J(Y, \frac{p}{N}), \quad (5.17)$$

$$A_2 = N^{-\frac{1}{2}} \sum_{j=0}^h \alpha_j^2 \left(\sum_{k=\max(0, j-N)}^{j-1} \sigma_{-k}^2 \eta_{-k}^2 Z^{\frac{p}{N}(j-k)} - \sum_{k=0}^{\min(j-1, N-1)} \sigma_{N-k}^2 \eta_{N-k}^2 Z^{\frac{p}{N}(N+k-j)} \right).$$

Going back to (5.6), we see that for any value of h

$$J_x^2(\frac{p}{N}) = \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \cdot J(Y, \frac{p}{N}) + N^{-\frac{1}{2}} (A_2 + B_2), \quad (5.18)$$

where A_2 is defined in (5.17) and B_2 in (5.7).

Now, applying the Minkowski inequality to the last two terms, we have:

$$[E|A_2 + B_2|^2]^{\frac{1}{2}} \leq [E|A_2|^2]^{\frac{1}{2}} + [E|B_2|^2]^{\frac{1}{2}}. \quad (5.19)$$

From (5.17) we see that:

$$[E|A_2|^2]^{\frac{1}{2}} \leq N^{-\frac{1}{2}} \sum_{j=0}^h |\alpha_j|^2 [E \sum_{\max(0, j-N)}^{j-1} \sigma_{-k}^2 \eta_{-k}^2 Z_N^{p(j-k)} \sum_{k=0}^{\min(j-1, N-1)} \sigma_{N-k}^2 \eta_{N-k}^2 Z_N^{p(N+j-k)}]^{\frac{1}{2}},$$

by applying the same inequality. Or

$$[E|A_2|^2]^{\frac{1}{2}} \leq N^{-\frac{1}{2}} \sum_{j=0}^h |\alpha_j|^2 [\sum_{\max(0, j-N)}^{j-1} |\sigma_{-k}^2|^2 E|\eta_{-k}^2|^2 + \sum_{k=0}^{\min(j-1, N-1)} |\sigma_{N-k}^2|^2 E|\eta_{N-k}^2|^2]^{\frac{1}{2}}.$$

But $E|\eta^2|^2 = E\eta^4 = 3$. Therefore,

$$[E|A_2|^2]^{\frac{1}{2}} \leq \sqrt{3/N} \sum_{j=0}^h |\alpha_j|^2 [\sum_{\max(0, j-N)}^{j-1} |\sigma_{-k}^2|^2 + \sum_0^{\min(j-1, N-1)} |\sigma_{N-k}^2|^2]^{\frac{1}{2}} \quad (5.20)$$

Now,

$$\sigma_t^2 \leq \sigma_M^2 = \max_t \sigma_t^2. \quad (5.21)$$

Hence, (5.20) becomes

$$[E|A_2|^2]^{\frac{1}{2}} \leq \sqrt{3/N} \sigma_M^2 \sum_{j=0}^h |\alpha_j|^2 [\sum_{\max(0, j-N)}^{j-1} 1 + \sum_0^{\min(j-1, N-1)} 1]^{\frac{1}{2}}. \quad (5.22)$$

Now, if $h \leq N$, then

$$\begin{aligned}
 P &= \sum_{j=0}^h |\alpha_j|^2 \left[\sum_{\max(0, j-N)}^{j-1} 1 + \sum_0^{\min(j-1, N-1)} 1 \right]^{\frac{1}{2}} = \sum_{j=0}^h |\alpha_j|^2 \left[\sum_0^{j-1} 1 + \sum_0^{j-1} 1 \right]^{\frac{1}{2}} \\
 &= \sum_{j=0}^h |\alpha_j|^2 \sqrt{2} j^{\frac{1}{2}}. \quad (5.23)
 \end{aligned}$$

If $h > N$, then

$$\begin{aligned}
 P &= \sum_{j=0}^N |\alpha_j|^2 \left[\sum_0^{j-1} 1 + \sum_0^{j-1} 1 \right]^{\frac{1}{2}} + \sum_{j=N+1}^h |\alpha_j|^2 \left[\sum_{j-N}^{j-1} 1 + \sum_0^{N-1} 1 \right]^{\frac{1}{2}} \\
 &= \sum_{j=0}^N |\alpha_j|^2 \sqrt{2} j^{\frac{1}{2}} + \sum_{j=N+1}^h |\alpha_j|^2 \sqrt{2} N^{\frac{1}{2}}. \quad (5.24)
 \end{aligned}$$

But in the second term of (5.24) we have $N < j$. Therefore,

$$P < \sum_{j=0}^N |\alpha_j|^2 \sqrt{2} j^{\frac{1}{2}} + \sum_{j=N+1}^h |\alpha_j|^2 \sqrt{2} j^{\frac{1}{2}},$$

or

$$P < \sqrt{2} \sum_{j=0}^h |\alpha_j|^2 j^{\frac{1}{2}}. \quad (5.25)$$

Hence, for any value of h we have that:

$$[E|A_2|^2]^{\frac{1}{2}} \leq \sqrt{6} N^{-\frac{1}{2}} \sigma_M^2 \sum_{j=0}^h |\alpha_j|^2 j^{\frac{1}{2}}. \quad (5.26)$$

Then $E|A_2|^2$ is bounded if $\sum_{j=0}^h |\alpha_j|^2 j^{\frac{1}{2}}$ is convergent and σ_t^2 is bounded uniformly in t . If $\alpha_j = o(j^{-3/2})$, then $\sum_{j=0}^h |\alpha_j|^2 j^{\frac{1}{2}}$ is certainly convergent.

We now consider the term B_2 which is given by

$$B_2 = N^{-\frac{1}{2}} \sum_{j \neq j'=0}^h \sum_{t=1}^N \alpha_j \alpha_{j'} \sigma_{t-j} \sigma_{t-j'} \eta_{t-j} \eta_{t-j'} Z^{\frac{tp}{N}}.$$

This can be written in the form

$$B_2 = N^{-\frac{1}{2}} \sum_{j \neq j'=0}^h \alpha_j \alpha_{j'} u_{jj'},$$

where

$$u_{jj'} = \sum_{t=1}^N \sigma_{t-j} \sigma_{t-j'} \eta_{t-j} \eta_{t-j'} Z^{\frac{tp}{N}}. \quad (5.27)$$

The variance of B_2 is then given by

$$\text{var}(B_2) = N^{-1} \sum_{j \neq j', k \neq k'=0}^h \alpha_j \alpha_{j'} \alpha_k \alpha_{k'} E(u_{jj'} u_{kk'}) ,$$

$$[E(u_{jj'}) = 0] .$$

Now

$$E[u_{jj'} u_{kk'}] = \sum_{t, t'=1}^N \sigma_{t-j} \sigma_{t-j'} \sigma_{t-k} \sigma_{t-k'} E[\eta_{t-j} \eta_{t-j'} \eta_{t-k} \eta_{t-k'}] Z^{\frac{tp}{N} - \frac{t'p}{N}}.$$

But

$$E[\eta_{t-j} \eta_{t-j'} \eta_{t-k} \eta_{t-k'}] = E[\eta_{t-j}^2 \eta_{t-j'}^2] = 1, \quad \text{if}$$

$$t' = t+k-j \quad \text{and} \quad t' = t+k-j' \quad \text{or if}$$

$$t' = t+k'-j' \quad \text{and} \quad t' = t+k'-j ,$$

and equal to zero otherwise.

Thus

$$E[u_{jj'}u_{kk'}] = \sum_{t=1}^N \sigma_{t-j}^2 \sigma_{t-j'}^2 Z^{(j-k)\frac{p}{N}} + \sum_{t=1}^N \sigma_{t-j}^2 \sigma_{t-j'}^2 Z^{(j'-k)\frac{p}{N}} \quad (j'-j=k'-k)$$

Therefore, it is easily seen that the variance of B_2 would take the following form:

$$\begin{aligned} \text{var}(B_2) = N^{-1} [& \sum_{\substack{j \neq j', k \neq k' \\ j'-j=k'-k}}^h \alpha_j \alpha_{j'} \alpha_k \alpha_{k'} Z^{(j-k+j'-k')\frac{p}{2N}} \sum_{t=1}^N \sigma_{t-j}^2 \sigma_{t-j'}^2 \\ & + \sum_{\substack{j \neq j', k \neq k' \\ (j'-j=k-k')}}^h \alpha_j \alpha_{j'} \alpha_k \alpha_{k'} Z^{(j'-k+j-k')\frac{p}{2N}} \sum_{t=1}^N \sigma_{t-j}^2 \sigma_{t-j'}^2] \end{aligned} \quad (5.28)$$

Let us assume that $\sigma_t^2 < \text{constant}$, uniformly in t . Then

$$N^{-1} \sum_{t=1}^N \sigma_{t-j}^2 = \gamma_{0,N}^{(j)} \longrightarrow \gamma_0 \quad \text{as } N \longrightarrow \infty, \text{ uniformly in } j.$$

Allowing for the subscripts j, j', k and k' to be equal we can see that the sum for the α 's will be less or equal to $\left| \sum_{j=0}^h \alpha_j Z^{\frac{jp}{2N}} \right|^4$. Therefore, from (5.28) we get that

$$\text{var}(B_2) \leq 2\sigma_M^2 \gamma_{0,N}^{(j)} \left| \sum_{j=0}^h \alpha_j Z^{\frac{jp}{2N}} \right|^4$$

or

$$\text{var}(B_2) \leq 2\sigma_M^2 \gamma_{0,N}^{(j)} \left(\sum_{j=0}^h |\alpha_j| \right)^4 < \infty \quad (5.29)$$

Due to the fact that $E(B_2) = 0$, we see that

$$E|B_2|^2 \leq 2\sigma_M^2 \gamma_{0,N}^{(j)} \left(\sum_{j=0}^h |\alpha_j| \right)^4 < \infty \quad (5.30)$$

In exactly the same way as above we prove that $E|B|^4 < \infty$. Therefore $E|B_2|^2$ is bounded if $\sum_{j=0}^h |\alpha_j|$ is convergent. (It is certainly convergent for $\alpha_j = o(j^{3/2})$.)

Therefore, combining (5.26) and (5.30) into (5.19), we get:

$$\lim_{N \rightarrow \infty} [E|\sqrt{N} A_2 + B_2|^2]^{\frac{1}{2}} \leq \sqrt{6} \sigma_M^2 \sum_{j=0}^h |\alpha_j|^2 j^{\frac{1}{2}} + 2\sigma_M \sqrt{\gamma_0} \left(\sum_{j=0}^h |\alpha_j| \right)^2. \quad (5.31)$$

Hence, $E|\sqrt{N} A_2 + B_2|^2$ is bounded whenever $\sum_{j=0}^h |\alpha_j|^2 j^{\frac{1}{2}}$ and $\sum_{j=0}^h |\alpha_j|$ are convergent.

By a similar procedure, as above, we can show that $E|\sqrt{N} A_2 + B_2|^4$ is bounded under the same conditions.

Therefore, we can write (5.10) in the form

$$J_{x^2} \left(\frac{p}{N} \right) = \sum_{j=0}^h \alpha_j^2 \cdot J(Y, \frac{p}{N}) + N^{-\frac{1}{2}} T \left(\frac{p}{N} \right), \quad (5.32)$$

where $E|T|^2 < +\infty$ and $E|T|^4 < +\infty$.

Therefore, from (5.32) we get:

$$I_{x^2}^{(N)} \left(\frac{p}{N} \right) = |J_{x^2} \left(\frac{p}{N} \right)|^2 = \left| \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \right|^2 \cdot |J(Y, \frac{p}{N})|^2 + N^{-\frac{1}{2}} T^* \left(\frac{p}{N} \right) \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \\ J(Y, \frac{p}{N}) + N^{-\frac{1}{2}} T \left(\frac{p}{N} \right) \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} J^* \left(Y, \frac{p}{N} \right) + N^{-1} |T \left(\frac{p}{N} \right)|^2. \quad (5.33)$$

For the second term in (5.33) we have:

$$|T^* \left(\frac{p}{N} \right) \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \cdot J(Y, \frac{p}{N})|^2 = |T \left(\frac{p}{N} \right)|^2 \cdot \left| \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \right|^2 \cdot I_Y^{(N)} \left(\frac{p}{N} \right).$$

Applying the Schwarz inequality, we get:

$$\begin{aligned} E|T^*(\frac{p}{N}) \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} J(Y, \frac{p}{N})|^2 &= \left| \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \right|^2 E(|T(\frac{p}{N})|^2 \cdot I_y^{(N)}(\frac{p}{N})) \\ &\leq \left| \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} \right|^2 E^{\frac{1}{2}}(|T(\frac{p}{N})|^4) \cdot E^{\frac{1}{2}}(I_y^{(N)}(\frac{p}{N}))^2. \end{aligned} \quad (5.34)$$

Now, from previous results we find that $E^{\frac{1}{2}}|I_y^{(N)}(\frac{p}{N})|^2$ and $E^{\frac{1}{2}}|T(\frac{p}{N})|^4$ are bounded. Therefore,

$$E|T^*(\frac{p}{N}) \sum_{j=0}^h \alpha_j^2 Z^{\frac{jp}{N}} J(Y, \frac{p}{N})|^2$$

is bounded, and a similar result holds for the third term.

Also, it is obvious that the last term of (5.33) has a bounded second absolute moment. Therefore, we establish the following theorem:

Theorem (5.1)

Let X_t be defined as in (5.1) such that $\sum_{j=0}^h |\alpha_j| < \infty$ and $\sum_{j=0}^h |\alpha_j|^2 j^{\frac{1}{2}} < \infty$. Then,

$$I_p^i = g(\frac{p}{N}) I_p + O(N^{-\frac{1}{2}}), \quad (5.35)$$

where $I_p^i = I_x^{(N)}(\frac{p}{N})$, $g(\frac{p}{N}) = \left| \sum_{j=0}^h \alpha_j^2 Z^{jp/N} \right|^2$, $I_p = I_y^{(N)}(\frac{p}{N})$ and the O term indicates terms which have absolute second moments of order N^{-1} .

Ignoring the term $O(N^{-\frac{1}{2}})$ and taking expectations of both sides of (5.35) we obtain:

$$EI'_p \sim g\left(\frac{p}{N}\right) \cdot (|\gamma_p|^2 + 2N^{-1}c_{0,N}), \quad (5.36)$$

$$= s(p), \text{ say,}$$

$$\text{where, as before, } c_{0,N} = \sum_{j=-m}^m |\gamma_j|^2.$$

Now, applying the ordinary definition of covariance, we get:

$$\text{cov}(I'_p, I'_{p'}) \sim g\left(\frac{p}{N}\right) \cdot g\left(\frac{p'}{N}\right) \cdot \text{cov}(I_p, I_{p'}). \quad (5.37)$$

This indicates that, assuming boundedness of $g\left(\frac{p}{N}\right)$, I'_p behaves in a somewhat similar manner as the periodogram of squares of independent random processes discussed in Chapter 3.

Theorem(5.2)

I'_p is a consistent estimator of $g\left(\frac{p}{N}\right)|\gamma_p|^2$ in the sense of equation (5.39).

Proof:

From (5.37) it is obvious that

$$\text{var}(I'_p) \sim (g\left(\frac{p}{N}\right))^2 \text{var}(I_p)$$

which tends to zero as $N \rightarrow \infty$, and from (5.36) we have

$$E\left|I'_p - g\left(\frac{p}{N}\right) \cdot |\gamma_p|^2\right|^2 \rightarrow 0 \text{ as } N \rightarrow \infty. \quad (5.39)$$

Therefore, I'_p is a consistent estimate of $g\left(\frac{p}{N}\right) \cdot |\gamma_p|^2$, the product of two factors, the first being a relative measure of coincidence of α_j^2 with $\cos(2\pi j \frac{p}{N})$ and $\sin(2\pi j \frac{p}{N})$ and the other being the function defined in Chapter 3 which measures the relative coincidence of σ_t^2 with $\cos(\frac{2\pi tp}{N})$ and $\sin(\frac{2\pi tp}{N})$.

For future use, we shall prove the following lemma.

Lemma (5.1)

All absolute moments of I_p' of finite order are bounded.

Proof:

From (5.3.5), and ignoring the term $O(N^{-\frac{1}{2}})$, we have that

$$I_p' = g\left(\frac{p}{N}\right) \left| N^{-1} \sum_{t=1}^N Y_t Z^{tp/N} \right|^2. \quad (5.40)$$

Since $g\left(\frac{p}{N}\right) = \left| \sum_{j=0}^h \alpha_j^2 Z^{jp/N} \right|^2$, it is obvious that

$$g\left(\frac{p}{N}\right) \leq \left(\sum_{j=0}^h |\alpha_j| \right)^4 = G < \infty.$$

Therefore, by the Minkowski inequality, we get from (5.40):

$$[E |I_p'|^2]^{\frac{1}{2}} \leq N^{-1} G \sum_{t=1}^N E^{\frac{1}{2}} |Y_t|^2. \quad (5.41)$$

But

$$E |Y_t|^2 = \sigma_t^4 E(\eta_t^4) = 3\sigma_t^4.$$

Therefore (5.41) becomes

$$[E |I_p'|^2]^{\frac{1}{2}} \leq \sqrt{3} N^{-1} \sum_{t=1}^N \sigma_t^2 = \sqrt{3} \gamma_{0,N}.$$

Thus, $E |I_p'|^2$ is bounded since $\lim_{N \rightarrow \infty} \gamma_{0,N} = \gamma_0 < \infty$. Similarly,

we can prove the same thing for higher moments. Hence the lemma.

5.4. Artificial Series

5.4.1. Description of the Series

In this section we will present three examples of artificial series generated by using a number of random normal deviates (η_t) . The normal

deviates are multiplied by the different values of the square root of the variance which follows a certain Fourier expansion containing two or more terms. Different series are constructed by choosing suitable constants, α_j 's, to form a moving average model with two or more terms. Periodograms of the squares of these series are computed together with the true values of $S(p)$. The series are given as follows.

Series IV. The frequency associated with the Fourier expansion of σ_t^2 is $2/200$ such that

$$\sigma_t^2 = 22.5 - 6.0 \cos\left(\frac{2\pi t}{100}\right) + 15.4 \sin\left(\frac{2\pi t}{100}\right), \quad (t = 1, 2, \dots, N = 200),$$

i.e., $\gamma_0 = 22.5$, $2\text{Re}(\gamma_2) = -6.0$, $2\text{Im}(\gamma_2) = 15.4$ and all the other γ 's are zero.

The model for the series is $X_t = \sigma_t \eta_t + 0.5 \sigma_{t-1} \eta_{t-1}$, where $h=1$, $\alpha_0=1.0$, $\alpha_1=0.5$ and 201 values of the random normal deviates η_t are used.

Periodogram values and the true values of $S(p)$ at $p=0, 1, 2, \dots, 100$, are listed in Table 5.1 and plotted in Figure 5.1. A peak at $p=2$ is observed for the periodogram and the true value of $S(p)$, but there are irregular fluctuations of the periodogram as we move farther away from $p=2$. Again, this type of behavior is due to the fact that the periodogram does not damp as $\gamma_s \rightarrow 0$ with the increase of lag s . For large values of N these fluctuations will relatively decrease in magnitude and the periodogram will then be an estimate of $\left| \sum_{j=0}^h \alpha_j^2 Z^{jp/N} \right|^2 |\gamma_p|^2$.

Series V. This series is a three-term moving average given by

$$X_t = \sigma_t \eta_t + 0.65 \sigma_{t-1} \eta_{t-1} - 0.50 \sigma_{t-2} \eta_{t-2},$$

Table 5.1. Values of the periodogram of Series IV; the true value of $S(p)$;
values of $|\gamma_p|^2$ and $N^{-1}C_{0,N}$ at $p = 0, 1, 2, \dots, 100$

p	$G(p)$	$S(p)$	p	$G(p)$	$S(p)$	p	$G(p)$	$S(p)$	p	$G(p)$	$S(p)$
1	5.435	5.021	26	.455	4.515	51	7.553	3.365	76	5.527	2.244
2	114.908	111.655	27	24.247	4.478	52	.381	3.314	77	29.102	2.210
3	8.401	5.015	28	50.634	4.439	53	6.756	3.264	78	9.327	2.177
4	12.220	5.009	29	20.760	4.400	54	5.870	3.214	79	5.162	2.145
5	13.305	5.002	30	14.175	4.360	55	26.624	3.164	80	7.957	2.115
6	1.338	4.994	31	5.796	4.318	56	4.870	3.114	81	1.575	2.086
7	16.827	4.983	32	9.175	4.276	57	36.903	3.064	82	3.594	2.058
8	3.089	4.972	33	2.869	4.233	58	14.728	3.015	83	11.537	2.032
9	6.817	4.958	34	4.999	4.189	59	4.332	2.967	84	6.295	2.007
10	6.281	4.943	35	9.505	4.145	60	6.435	2.918	85	3.661	1.983
11	26.707	4.927	36	48.558	4.099	61	.542	2.871	86	3.202	1.961
12	10.365	4.909	37	10.051	4.053	62	10.974	2.823	87	3.898	1.940
13	26.365	4.890	38	13.427	4.007	63	13.345	2.777	88	1.658	1.921
14	1.240	4.869	39	6.303	3.959	64	1.619	2.731	89	1.732	1.903
15	8.100	4.847	40	5.785	3.912	65	4.836	2.685	90	.129	1.887
16	12.313	4.823	41	1.261	3.863	66	.936	2.641	91	10.983	1.872
17	.340	4.798	42	10.033	3.815	67	4.946	2.597	92	5.403	1.858
18	23.156	4.772	43	.382	3.766	68	15.726	2.554	93	2.277	1.847
19	8.329	4.744	44	10.498	3.716	69	6.138	2.512	94	13.266	1.836
20	14.069	4.715	45	7.621	3.666	70	5.190	2.470	95	3.941	1.828
21	3.956	4.685	46	8.884	3.616	71	3.351	2.430	96	13.038	1.821
22	24.599	4.653	47	6.521	3.566	72	4.874	2.391	97	1.508	1.815
23	29.697	4.621	48	11.695	3.516	73	27.880	2.352	98	8.148	1.811
24	.667	4.587	49	6.690	3.466	74	.219	2.315	99	2.822	1.809
25	22.908	4.551	50	19.469	3.415	75	16.835	2.279	100	19.617	1.808
$G(0)$ $ \gamma_2 ^2$ $ \gamma_p ^2$ $p \neq 2, p=1, 2, \dots, 100$ γ_0^2 $N^{-1}C_{0,N}$											
740.948		68.29	0.000		506.250		3.214

Figure 5.1. Periodogram values and the true values of $S(p)$ for Series IV

where

$$\sigma_t^2 = 20.6 - 7.4 \cos\left(\frac{3\pi t}{100}\right) + 10.4 \sin\left(\frac{3\pi t}{100}\right) \quad (t = 1, 2, \dots, 200),$$

i.e., the Fourier expansion for σ_t^2 contains only one frequency $3/200$. Here we have $h=2$, $\alpha_0=1.0$, $\alpha_1=0.65$ and $\alpha_2=-0.50$. The coefficients in the σ_t^2 expansion are $\gamma_0=20.6$, $2\text{Re}(\gamma_3)=-7.4$, $2\text{Im}(\gamma_3)=10.4$ and all the remaining coefficients are zero. For this series 202 random normal deviates (η_t) were used, as required by the model. Periodogram values and the true values of $S(p)$ are listed in Table 5.2 and plotted in Figure 5.2, where a high peak for the periodogram and the true value of $S(p)$ is noticed at $p=3$. This peak corresponds to the frequency $3/200$ associated with the variance σ_t^2 .

Series VI. The Fourier expansion for the variance in this series contains the two frequencies $2/200$ and $5/200$ such that

$$\sigma_t^2 = 20.4 - 10.9 \cos\left(\frac{2\pi t}{100}\right) - 8.8 \cos\left(\frac{5\pi t}{100}\right),$$

so that $\gamma_0=20.4$, $2\text{Re}(\gamma_2)=-10.9$, $2\text{Re}(\gamma_5)=-8.8$, $\text{Im}(\gamma_2)=\text{Im}(\gamma_5)=0$ and all the other coefficients are zero. The model of the moving average is $X_t = \sigma_t \eta_t + 0.65 \sigma_{t-1} \eta_{t-1}$, which means that $h=1$, $\alpha_0=1.0$ and $\alpha_1=0.65$.

For this model, 201 random normal deviates were used to furnish the 200 values of the series. Periodogram values and the true values of $S(p)$ ($p=0, 1, 2, \dots, 100$) are listed in Table 5.3 and plotted in Figure 5.3. Two peaks for the periodogram and the true values of $S(p)$ at $p=2$ and $p=5$ are observed, which correspond to the two frequencies $2/200$ and $5/200$ in the variance model respectively. There are other peaks for the periodogram, but relatively smaller in magnitude than the first ones, due to the periodogram's irregular fluctuations.

Table 5.2. Values of the periodogram of Series V; the true value of $S(p)$; values of $|\gamma_p|^2$ and $N^{-1}C_{O,N}$ at $p = 0, 1, 2, \dots, 100$

p	$G(p)$	$S(p)$	p	$G(p)$	$S(p)$	p	$G(p)$	$S(p)$	p	$G(p)$	$S(p)$
1	5.159	7.071	26	2.921	4.888	51	8.488	1.793	76	7.504	1.271
2	6.096	7.059	27	15.217	4.747	52	3.009	1.716	77	20.248	1.293
3	103.115	120.421	28	78.109	4.604	53	2.758	1.645	78	8.139	1.317
4	12.415	7.014	29	3.950	4.461	54	.761	1.579	79	3.081	1.342
5	10.492	6.980	30	6.773	4.318	55	15.787	1.518	80	21.902	1.368
6	1.825	6.938	31	19.644	4.175	56	.705	1.462	81	2.576	1.395
7	19.674	6.890	32	5.558	4.032	57	30.827	1.412	82	2.006	1.422
8	6.195	6.834	33	3.470	3.889	58	16.469	1.366	83	16.213	1.448
9	1.827	6.772	34	1.485	3.748	59	8.107	1.326	84	1.212	1.475
10	13.695	6.702	35	28.935	3.608	60	9.406	1.291	85	1.616	1.502
11	9.844	3.470	36	40.764	3.470	61	.548	1.259	86	1.766	1.528
12	17.830	6.544	37	14.706	3.334	62	22.197	1.233	87	1.178	1.553
13	23.398	6.456	38	19.611	3.200	63	13.081	1.212	88	3.661	1.577
14	1.851	6.362	39	1.474	3.069	64	1.969	1.195	89	2.828	1.600
15	10.497	6.262	40	2.126	2.941	65	1.707	1.183	90	1.250	1.621
16	10.080	6.157	41	1.227	2.816	66	2.370	1.174	91	20.356	1.641
17	1.241	6.047	42	13.818	2.695	67	2.134	1.170	92	3.279	1.659
18	14.039	5.933	43	.180	2.577	68	13.626	1.169	93	.314	1.676
19	8.491	5.814	44	2.783	2.463	69	5.858	1.172	94	14.797	1.690
20	11.291	5.691	45	10.885	2.354	70	6.479	1.178	95	7.354	1.703
21	1.077	5.564	46	4.445	2.249	71	1.310	1.187	96	5.252	1.713
22	31.886	5.434	47	6.121	2.148	72	14.049	1.199	97	9.312	1.721
23	27.816	5.301	48	5.022	2.052	73	27.803	1.214	98	1.827	1.727
24	.461	5.165	49	5.903	1.960	74	.119	1.231	99	2.496	1.731
25	22.531	5.028	50	20.697	1.874	75	12.061	1.250	100	14.107	1.732
$G(0)$ $ \gamma_3 ^2$ $ \gamma_p ^2$ $p \neq 3, p=1, 2, \dots, 100$ γ_0^2 $N^{-1}C_{O,N}$											
842.573 40.73 0.000 424.360 2.529

Figure 5.2. Periodogram values and the true values of $S(p)$ of Series V

Table 5.3. Values of the periodogram of Series VI; the true value of $S(p)$; values of $|\gamma_p|^2$ and $N^{-1}C_{0,N}$ at $p = 0, 1, 2, \dots, 100$

p	G(p)	S(p)	p	G(p)	S(p)	p	G(p)	S(p)	G(p)	S(p)
1	6.578	5.202	26	4.743	4.518	51	2.763	2.962	3.875	1.446
2	85.620	65.253	27	28.960	4.467	52	3.362	2.894	22.420	1.401
3	9.558	5.194	28	59.987	4.415	53	4.043	2.826	7.316	1.356
4	9.772	5.186	29	7.137	4.362	54	11.477	2.758	2.833	1.314
5	54.277	44.150	30	30.312	4.308	55	29.409	2.691	10.184	1.273
6	3.912	5.165	31	15.494	4.252	56	3.369	2.623	4.685	1.233
7	20.040	5.151	32	4.235	4.195	57	41.736	2.556	4.544	1.196
8	5.470	5.135	33	.618	4.137	58	11.341	2.490	10.518	1.160
9	3.663	5.117	34	.427	4.077	59	12.582	2.424	5.361	1.126
10	4.853	5.097	35	23.825	3.017	60	3.561	2.359	3.113	1.094
11	17.052	5.075	36	61.407	3.956	61	1.295	2.294	1.927	1.064
12	5.909	5.051	37	22.586	3.893	62	10.603	2.231	4.413	1.036
13	8.999	5.025	38	33.377	3.830	63	9.259	2.167	1.698	1.010
14	1.511	4.996	39	.872	3.766	64	.012	2.105	2.549	.986
15	5.607	4.996	40	.029	3.702	65	2.061	2.044	.099	.964
16	16.380	4.935	41	5.002	3.637	66	1.715	1.984	15.701	.944
17	.385	4.901	42	14.343	3.571	67	6.878	1.924	1.500	.926
18	20.695	4.865	43	4.720	3.504	68	17.204	1.866	2.006	.910
19	9.944	4.828	44	2.686	3.438	69	3.015	1.809	4.522	.896
20	18.562	4.788	45	7.066	3.370	70	.933	1.753	3.739	.884
21	2.769	4.747	46	9.595	3.302	71	.867	1.699	6.051	.875
22	35.606	4.705	47	3.175	3.235	72	13.340	1.645	4.267	.867
23	13.414	4.660	48	11.115	3.167	73	44.722	1.594	1.460	.862
24	1.259	4.614	49	3.289	3.099	74	1.649	1.543	1.960	.859
25	16.365	4.567	50	17.015	3.030	75	21.404	1.494	7.560	.858
<hr/>										
G(0)	$ \gamma_2 ^2$	$ \gamma_5 ^2$	$ \gamma_p ^2$	$p \neq 2, 5, p=1, 2, \dots, 100$	γ_0^2	$N^{-1}C_{0,N}$				
813.490	29.703	19.360		0.000	416.159	2.51

Figure 5.3. Periodogram values and the true values of $S(p)$ for Series VI

5.4.2. Computation Procedure

FORTTRAN language programs were written for an IBM 1410 to generate and compute the above three series. The computing procedure is as follows.

(1) 201 or more random normal deviates, specific values for the coefficients of σ_t^2 expansion and suitable values of α_j 's are fed into the computer. The 200 values of σ_t are computed, applying the corresponding model for each series. Then the 200 values of

$X_t = \sum_{j=0}^h \alpha_j \sigma_{t-j} \eta_{t-j}$ are computed for each specific series.

(2) The periodogram of X_t^2 is computed by using the same expansion as that given in 3.7.2 in Chapter 3.

6. ESTIMATION PROCEDURES FOR $|\gamma_p|^2$ ($p=1,2,\dots,m$) IN A MOVING AVERAGE MODEL

6.1. Introduction

In Chapter 5 the periodogram of squares of the moving average scheme, X_t ($t=0, \pm 1, \pm 2, \dots$), proved to be a consistent estimator of the product of two functions: $|\gamma_p|^2$ and $g(\frac{p}{N})$ at the frequency of $\frac{p}{N}$. The interest of the present work is concentrated on the study of the existence of estimators and in particular consistent estimators of a function of $|\gamma_p|^2$, where the function $|\gamma_p|^2$ is our tool in discovering the frequencies associated with the variances. It may happen that α_j 's have an oscillatory behavior, so that the periodogram of squares of X_t will not only estimate the dominant periodicities due to the variances, but also those periodicities of the coefficients α_j . Therefore, an attempt to derive an estimator of $|\gamma_p|^2$ separately is necessary. Unfortunately, there is no direct way to find such an estimator, for there is no way to separate $|\gamma_p|^2$ from $g(\frac{p}{N})$; they are two "confounded" factors. The only way to approach this problem is to find an estimate of a constant times $g(\frac{p}{N})$, a constant in the sense that it has no periodicities, or, if there are any, they could be neglected. Then this will lead to an estimation procedure where an estimator of $|\gamma_p|^2$ / constant is a ratio of two other estimators. But due to the difficulty in obtaining the exact distributions of these estimators, we shall attempt to find approximate methods for solution.

In this chapter we shall present this type of estimation procedure, by deriving an estimator which is a ratio of two other estimators. A

consistent estimator was derived on the assumption that

$g(\lambda) = \left| \sum_{j=0}^h \alpha_j^2 Z^{j\lambda} \right|^2$ is differentiable in the interval $0 \leq \lambda \leq 1/2$ and

$g'(\lambda) < a < \infty$. A similar assumption was made by Herbst [1964] in

deriving a consistent estimator of $\gamma_0 \left| \sum_{j=0}^h \alpha_j Z^{j\lambda} \right|^2$, where

$\gamma_0 = \lim_{N \rightarrow \infty} (N^{-1} \sum_{t=1}^N \sigma_t^2)$. Numerical computations, applied to Series IV-

VI, are given in support of the theoretical discoveries.

6.2. A Consistent Estimator of $\frac{|\gamma_p|^2}{c_0}$

6.2.1. Derivation of the Estimator

In this section we will derive an estimator of $\frac{|\gamma_p|^2}{\text{constant}}$ on the assumption of regular behavior of $g(\lambda)$ in the interval $0 \leq \lambda \leq 1/2$. It is interesting to observe that the present problem of estimation encountered in this chapter has some similarity with the problem of estimation of noise spectrum studied by physicists and engineers. Consider, for instance, the following example given by Grenander and Rosenblatt [1957]. A resistance R is connected in series with an inductance L and a capacitance C . There is a random current y_t produced by the random voltage x_t due to the random motion (noise) of the electrons. The current y_t is considered as a result of a filter operating on x_t with frequency response $\phi(\lambda)$, where $\phi(\lambda)$ is a function of R, L, C and the frequency λ of the noise. If the spectral densities corresponding to x_t, y_t and $f_x(\lambda)$ and $f_y(\lambda)$ respectively, then

$$f_y(\lambda) = f_x(\lambda) |\phi(\lambda)|^2.$$

Due to the physical properties of this experiment, the filter allows only frequencies in the neighborhood of the resonant frequency λ_0 . Thus, for the purpose of spectrum estimation, $f_x(\lambda)$ can be regarded as constant for all these frequencies near λ_0 .

We shall prove the following theorem.

Theorem 6.1.

Consider the function

$$R_n\left(\frac{p}{N}\right) = \sum_{j=-n}^n I'_{p+j} \quad (6.1)$$

when n is a sequence of integers such that $n \rightarrow \infty$ and $\frac{n}{\sqrt{N}} \rightarrow 0$ as $N \rightarrow \infty$. Then, on the assumption that

$$I'_j = g\left(\frac{j}{N}\right) I_j + O(N^{-\frac{1}{2}}) \quad (6.2)$$

($O(N^{-\frac{1}{2}})$ indicates terms which have absolute second moment of order N^{-1})

$$\left| R_n\left(\frac{p}{N}\right) - \sum_{j=-n}^n g\left(\frac{p+j}{N}\right) I_{p+j} \right| \rightarrow 0$$

in probability.

Proof:

Consider the quantity

$$\left| \sum_{j=-n}^n I'_{p+j} - \sum_{j=-n}^n g\left(\frac{p+j}{N}\right) I_{p+j} \right| = \left| \sum_{j=-n}^n O(N^{-\frac{1}{2}}) \right|.$$

Applying the Minkowski inequality we get:

$$E \left| \sum_{-n}^n I'_{p+j} - \sum_{-n}^n g\left(\frac{p+j}{N}\right) I_{p+j} \right| = E \left| \sum_{-n}^n O(N^{-\frac{1}{2}}) \right| \leq \sum_{-n}^n E |O(N^{-\frac{1}{2}})| \leq n O_a(N^{-\frac{1}{2}}),$$

where $O_a(N^{-\frac{1}{2}})$ indicates terms which are of order $N^{-\frac{1}{2}}$.

Applying the Markov inequality, for $\epsilon > 0$, we get that

$$P \left[\left| \sum_{-n}^n (I'_{p+j} - g\left(\frac{p+j}{N}\right) I_{p+j}) \right| > \epsilon \right] \leq \frac{n O_a(N^{-\frac{1}{2}})}{\epsilon}.$$

But

$$n O_a(N^{-\frac{1}{2}}) \rightarrow 0 \text{ as } N \rightarrow \infty,$$

since $\frac{n}{\sqrt{N}} \rightarrow 0$ as $N \rightarrow \infty$. Hence the theorem.

Thus, we can replace $\sum_{-n}^n I'_{p+j}$ by $\sum_{-n}^n g\left(\frac{p+j}{N}\right) I_{p+j}$, since from theorem

(6.1) the other terms will tend to zero in probability. Therefore,

(6.1) becomes

$$R_n\left(\frac{p}{N}\right) \sim \sum_{-n}^n g\left(\frac{p+j}{N}\right) I_{(p+j)}. \quad (6.3)$$

Letting $g\left(\frac{p+j}{N}\right) = g\left(\frac{p}{N}\right) + \frac{j}{N} g'(\theta)$, $\left(\frac{p}{N} < \theta < \frac{p+j}{N}\right)$, and remembering

that $E I_s = |\gamma_s|^2 + N^{-1} \sum_{-m}^m |\gamma_j|^2$, we see that

$$E R_n\left(\frac{p}{N}\right) \sim \sum_{-n}^n \left[g\left(\frac{p}{N}\right) + \frac{j}{N} g'(\theta) \right] \left[|\gamma_{p+j}|^2 + N^{-1} \sum_{-m}^m |\gamma_j|^2 \right]. \quad (6.3a)$$

Now consider the term $\sum_{-n}^n \frac{j}{N} g'(\theta) |\gamma_{p+j}|^2$, resulting from the pro-

duct of quantities under the sum in (6.3a). It is clear that

$$\sum_{-n}^n \left(\frac{j}{N} g'(\theta) \right) |\gamma_{p+j}|^2 \leq a\left(\frac{n}{N}\right) \sum_{-n}^n |\gamma_{p+j}|^2 \leq a\left(\frac{n}{N}\right) \sum_{-m}^m |\gamma_j|^2.$$

But

$$\lim_{N \rightarrow \infty} \sum_{-m}^m |\gamma_1|^2 = C_0 < \infty.$$

Therefore, as $N \rightarrow \infty$ $\sum_{-n}^n \frac{j}{N} g'(\theta) |\gamma_{p+j}|^2 \rightarrow 0$. It is clear that,

except for $\sum_{-n}^n g(\frac{p}{N}) |\gamma_{p+j}|^2$, all the other terms vanish with the increase of N . Therefore, for large N

$$E[R_n(\frac{p}{N})] \sim g(\frac{p}{N}) C_0. \quad (6.4)$$

6.2.2. Variance of $R_n(\frac{p}{N})$, $p = 1, 2, \dots, m$

By definition we have

$$\text{var}[R_n(\frac{p}{N})] = \sum_{-n}^n \text{var}[I'_{p+j}] + \sum_{j \neq j'} \text{cov}[I'_{p+j}, I'_{p+j'}]. \quad (6.5)$$

$$\begin{aligned} \text{var}(I'_{p+j}) &= [g(\frac{p+j}{N})]^2 \text{var } I_{p+j} \\ &= [g(\frac{p+j}{N})]^2 [\Gamma(p+j) + O_1(N^{-2})], \end{aligned}$$

where

$$\Gamma(p+j) = 4N^{-1} [\text{Re}[(\gamma_{p+j}^*)^2 \sum_{i=-m}^m \gamma_i \gamma_{i-2(p+j)}^*] + |\gamma_{p+j}|^2 \sum_{i=-m}^m |\gamma_i|^2] \quad (6.6)$$

and $O_1(N^{-2})$ indicates terms of order N^{-2} .

$$\begin{aligned} \text{cov}(I'_{p+j}, I'_{p+j'}) &= g(\frac{p+j}{N}) \cdot g(\frac{p+j'}{N}) \text{cov}[I_{p+j}, I_{p+j'}] \\ &= g(\frac{p+j}{N}) g(\frac{p+j'}{N}) [\Gamma(p+j, p+j') + O_1(N^{-2})], \end{aligned}$$

where

$$\begin{aligned} \Gamma(p+j, p+j') &= 4N^{-1} \operatorname{Re} [\gamma_{p+j}^* \gamma_{p+j'}^* \sum_{i=-m}^m \gamma_i \gamma_{i-(2p+j+j')}^* \\ &\quad + \gamma_{p+j} \gamma_{p+j'}^* \sum_{i=-m}^m \gamma_i \gamma_{i-(j-j')}^*] . \end{aligned} \quad (6.7)$$

Since $\frac{n}{\sqrt{N}} \rightarrow 0$ as $N \rightarrow \infty$, the above sums in (6.5) over terms of order N^{-2} will tend to zero as $N \rightarrow \infty$. Thus, considering large values of N we can write (6.5) as follows:

$$\operatorname{var}[R_n(\frac{p}{N})] = \sum_{j=-n}^n [g(\frac{p+j}{N})]^2 \Gamma(p+j) + \sum_{j \neq j'=-n}^n g(\frac{p+j}{N}) g(\frac{p+j'}{N}) \Gamma(p+j, p+j') . \quad (6.8)$$

The vanishing of the other terms (as $N \rightarrow \infty$) could easily be seen by remembering that $g(\frac{p}{N}) \leq (\sum_{j=0}^h |\alpha_j|)^4 = G < \infty$, ($p = 1, 2, \dots, m$), since we assumed in Chapter 5 that $\sum_{j=0}^h |\alpha_j|$ is convergent. Thus, we can write

$$\begin{aligned} \operatorname{var}[R_n(\frac{p}{N})] &\leq G^2 \sum_{j=-n}^n \Gamma(p+j) + G^2 \sum_{j \neq j'=-n}^n \Gamma(p+j, p+j') \\ &= G^2 (I_1 + I_2) , \end{aligned} \quad (6.9)$$

where

$$I_1 = \sum_{j=-n}^n \Gamma(p+j) \quad \text{and} \quad I_2 = \sum_{j \neq j'=-n}^n \Gamma(p+j, p+j') .$$

Let us consider I_1 .

$$I_1 = 4N^{-1} \left[\sum_{j=-n}^n \operatorname{Re} [(\gamma_{p+j}^*)^2 \sum_{i=-m}^m \gamma_i \gamma_{i-2(p+j)}^*] + |\gamma_{p+j}|^2 \sum_{i=-m}^m |\gamma_i|^2 \right] . \quad (6.10)$$

Now, for any complex function Z , $\operatorname{Re}(Z) \leq |Z|$. Therefore,

$$\begin{aligned} \operatorname{Re}[(\gamma_{p+j}^*)^2 \sum_{i=-m}^m \gamma_i \gamma_{i-2(p+j)}^*] &\leq |(\gamma_{p+j}^*)^2 \sum_{i=-m}^m \gamma_i \gamma_{i-2(p+j)}^*| \\ &= |\gamma_{p+j}^*|^2 \left| \sum_{i=-m}^m \gamma_i \gamma_{i-2(p+j)}^* \right|. \end{aligned}$$

But from the relation $\gamma_s = N^{-1} \sum_{t=1}^N \sigma_t^2 Z^{ts/N}$ we see that

$$|\gamma_s| \leq N^{-1} \sum_{t=1}^N \sigma_t^2 = \gamma_{0,N} \text{ which } \rightarrow \gamma_0 < \infty \text{ as } N \rightarrow \infty. \text{ Also, since}$$

$$\sum_{i=-m}^m \gamma_i \gamma_{i-s}^* = \delta_s = N^{-1} \sum_{t=1}^N \sigma_t^4 Z^{ts/N}, \text{ we see that } \left| \sum_{i=-m}^m \gamma_i \gamma_{i-s}^* \right| = |\delta_s| \leq$$

$$N^{-1} \sum_{t=1}^N \sigma_t^4 = \sum_{i=-m}^m |\gamma_i|^2 \text{ which tends to } C_0 < \infty \text{ as } N \rightarrow \infty. \text{ Therefore,}$$

$$\operatorname{Re}[(\gamma_{p+j}^*)^2 \sum_{i=-m}^m \gamma_i \gamma_{i-2(p+j)}^*] \leq (\gamma_{0,N})^2 \sum_{i=-m}^m |\gamma_i|^2.$$

For the second term of (6.10) we see that

$$\sum_{j=-n}^n |\gamma_{p+j}|^2 \sum_{i=-m}^m |\gamma_i|^2 \leq \left(\sum_{j=-n}^n (\gamma_{0,N})^2 \right) \left(\sum_{i=-m}^m |\gamma_i|^2 \right).$$

Therefore

$$\begin{aligned} I_1 &\leq 4N^{-1} \sum_{j=-n}^n [(\gamma_{0,N})^2 \sum_{i=-m}^m |\gamma_i|^2 + (\gamma_{0,N})^2 \sum_{i=-m}^m |\gamma_i|^2] \\ &\leq 4N^{-1} \sum_{j=-n}^n 2\gamma_{0,N}^2 \sum_{i=-m}^m |\gamma_i|^2 = \frac{8(2n+1)}{N} \gamma_{0,N}^2 \sum_{i=-m}^m |\gamma_i|^2 \\ &\rightarrow 0 \text{ as } N \rightarrow \infty \end{aligned}$$

Now, we consider the function I_2 .

$$I_2 = 4N^{-1} \sum_{j \neq j' = -n}^n \operatorname{Re} \left[\gamma_{p+j}^* \gamma_{p+j'}^* \sum_{i=-m}^m \gamma_i \gamma_{i-(2p+j+j')}^* + \gamma_{p+j} \gamma_{p+j'}^* \sum_{i=-m}^m \gamma_i \gamma_{i-(j-j')}^* \right]$$

$$= 4N^{-1} (A_1 + A_2),$$

where

$$A_1 = \sum_{j \neq j' = -n}^n \operatorname{Re} \left[\gamma_{p+j}^* \sum_{i=-m}^m \gamma_i \gamma_{i-(2p+j+j')}^* \right]$$

and

$$A_2 = \sum_{j \neq j' = -n}^n \operatorname{Re} \left[\gamma_{p+j} \gamma_{p+j'}^* \sum_{i=-m}^m \gamma_i \gamma_{i-(j-j')}^* \right].$$

(6.11)

Considering A_1 , we see that

$$A_1 \leq \sum_{j \neq j' = -n}^n |\gamma_{p+j}^* \gamma_{p+j'}^*| \cdot \left| \sum_{i=-m}^m \gamma_i \gamma_{i-(2p+j+j')}^* \right|$$

$$\leq (2n+1)^2 (\gamma_{0,N})^2 \sum_{i=-m}^m |\gamma_i|^2.$$

A similar inequality could easily be found for A_2 . Thus,

$$I_2 \leq 8N^{-1} (2n+1)^2 (\gamma_{0,N})^2 \sum_{i=-m}^m |\gamma_i|^2.$$

Since $\frac{n}{\sqrt{N}} \rightarrow 0$ as $N \rightarrow \infty$, it is clear that $I_2 \rightarrow 0$ as $N \rightarrow \infty$.

From (6.9) we have that $\operatorname{var} R_n(\frac{p}{N}) \leq G^2(I_1 + I_2)$, where now I_1 and I_2 were proved to approach zero as $N \rightarrow \infty$. Thus,

$$\operatorname{var} R_n(\frac{p}{N}) \rightarrow 0 \text{ as } N \rightarrow \infty.$$

This means that $R_n(\frac{p}{N})$ is a consistent estimator of $C_0 g(\frac{p}{N})$ in the sense that $E|R_n(\frac{p}{N}) - C_0 g(\frac{p}{N})|^2 \rightarrow 0$ ($N \rightarrow \infty$). At least heuristically the following theorem would seem to be true:

Theorem (6.2)

$$A(p) = \frac{I_2^{(N)}(\frac{p}{N})}{R_n(\frac{p}{N})} \text{ is a consistent estimator of } \frac{|\gamma_p|^2}{C_0}, p = 1, 2, \dots, m,$$

in the sense that $E\left|A(p) - \frac{|\gamma_p|^2}{C_0}\right|^2 \rightarrow 0$ ($N \rightarrow \infty$).

6.3. Numerical Applications

In this section numerical applications of the theory derived in the last section are given, applied to the Series IV-VI introduced in Chapter 5. The computations were carried out using an IBM 1410 with FORTRAN language for the programs. For each series $R_n(\frac{p}{N})$ was found for $n=10$ then the estimator $A(p)$ ($p=1, 2, \dots, 100$) was calculated. The values of $A(p)$ (each multiplied by 20) are listed in Tables 6.1 through 6.3 and plotted in Figures 6.1 through 6.3. As we move farther away from the values of p where high peaks are expected, there are still the irregular fluctuations in the values of $A(p)$. There are two reasons for this. The first is possibly due to the influence of terms of $O(N^{-1})$ in the variance of $I_2^{(N)}(\frac{p}{N})$, since such terms will not decrease as $s \rightarrow \infty$. The second explanation could be attributed to the fact that the estimator $A(p)$ is a ratio of two estimators, each of which is a function of the periodogram and consequently will occasionally yield high values for this ratio at values of p where the corresponding Fourier coefficients are zero. This irregular behavior will be clear when we compare Figures 6.1 through 6.3 with the corresponding ones in Chapter 5. We

Table 6.1. Values of $A(p)$, the estimator of $\frac{|\gamma_p|^2}{c_0}$
for Series IV

p	A(p)	p	A(p)	p	A(p)	p	A(p)
1	.278	26	.029	51	.801	76	.671
2	6.110	27	1.548	52	.038	77	3.472
3	.433	28	3.109	53	.672	78	1.134
4	.675	29	1.341	54	.581	79	.683
5	.763	30	.923	55	2.706	80	1.094
6	.078	31	.393	56	.511	81	.209
7	1.044	32	.610	57	3.953	82	.470
8	.187	33	.207	58	1.507	83	1.534
9	.424	34	.385	59	.456	84	.922
10	.393	35	.714	60	.682	85	.523
11	1.733	36	3.842	61	.063	86	.469
12	.635	37	.777	62	1.355	87	.588
13	2.150	38	1.089	63	1.355	88	.294
14	.104	39	.616	64	.170	89	.325
15	.653	40	.568	65	.481	90	.021
16	1.044	41	.128	66	.103	91	1.857
17	.026	42	1.045	67	.484	92	.880
18	1.598	43	.040	68	1.767	93	.362
19	.543	44	1.090	69	.727	94	2.220
20	.897	45	.715	70	.602	95	.662
21	.253	46	.851	71	.400	96	2.135
22	1.659	47	.660	72	.571	97	.239
23	2.052	48	1.157	73	3.258	98	1.253
24	.050	49	.692	74	.027	99	.409
25	1.655	50	2.011	75	2.024	100	2.660

Table 6.2. Values of $A(p)$, the estimator of $\frac{|\gamma_p|^2}{c_0}$
for Series V

p	A(p)	p	A(p)	p	A(p)	p	A(p)
1	.298	26	.183	51	1.061	76	.916
2	.351	27	.939	52	.335	77	2.488
3	5.789	28	4.572	53	.308	78	.991
4	.745	29	.240	54	.084	79	.400
5	.645	30	.419	55	1.755	80	2.930
6	.115	31	1.251	56	.082	81	.317
7	1.310	32	.341	57	3.636	82	.244
8	.414	33	.234	58	1.864	83	2.140
9	.125	34	.109	59	.913	84	.174
10	.948	35	2.052	60	1.056	85	.221
11	.712	36	3.078	61	.069	86	.253
12	1.182	37	1.098	62	2.692	87	.167
13	1.451	38	1.519	63	1.388	88	.592
14	.165	39	.156	64	.212	89	.478
15	.896	40	.207	65	.173	90	.194
16	.888	41	.119	66	.251	91	3.590
17	.104	42	1.446	67	.206	92	.562
18	.950	43	.019	68	1.466	93	.052
19	.579	44	.299	69	.677	94	2.716
20	.758	45	1.091	70	.696	95	1.287
21	.071	46	.515	71	.146	96	.877
22	2.128	47	.750	72	1.551	97	1.481
23	1.945	48	.609	73	3.169	98	.274
24	.035	49	.767	74	.015	99	.360
25	1.549	50	2.564	75	1.472	100	1.931

Table 6.3. Values of $A(p)$, the estimator of $\frac{|\gamma_p|^2}{c_0}$
for Series VI

p	A(p)	p	A(p)	p	A(p)	p	A(p)
1	.328	26	.267	51	.286	76	.432
2	4.532	27	1.606	52	.339	77	2.462
3	.527	28	3.060	53	.417	78	.826
4	.577	29	.382	54	1.213	79	.347
5	3.523	30	1.666	55	3.117	80	1.270
6	.241	31	.883	56	.367	81	.537
7	1.312	32	.234	57	4.611	82	.519
8	.351	33	.037	58	1.167	83	1.280
9	.240	34	.026	59	1.348	84	.851
10	.314	35	1.454	60	.386	85	.486
11	1.149	36	3.822	61	.153	86	.340
12	.364	37	1.412	62	1.184	87	.776
13	.704	38	2.204	63	.847	88	.362
14	.122	39	.070	64	.001	89	.575
15	.442	40	.002	65	.183	90	.021
16	1.584	41	.431	66	.170	91	3.582
17	.033	42	1.301	67	.627	92	.343
18	1.540	43	.429	68	1.843	93	.460
19	.736	44	.233	69	.338	94	1.132
20	1.256	45	.547	70	.106	95	.956
21	.181	46	.804	71	.098	96	1.527
22	2.426	47	.289	72	1.478	97	1.038
23	.930	48	1.062	73	4.958	98	.357
24	.090	49	.347	74	.187	99	.457
25	1.085	50	1.772	75	2.383	100	1.702

Figure 6.1. Values of $A(p)$ for Series IV

Figure 6.2. Values of $A(b)$ for Series V

notice that for the present estimator there are relatively higher peaks than for the corresponding ones in the periodogram at the values of p which correspond to zero Fourier coefficients. On the other hand, except for Series VI, the highest peaks for $A(p)$ correspond to dominant frequencies in the expansion for σ_t^2 .

Figure 6.3. Values of $A(p)$ for Series VI

7. A TESTING PROCEDURE FOR THE VARIANCE σ_t^2

7.1. Introduction

The last chapters were concerned with developing a theory for using models with variances depending on the time of observation. The question which arises, before using such models to describe the real world, is how to make the decision about the non-constancy of the variance within the observations. Thus, an important problem which needs to be investigated is that of testing the homogeneity of the variance for the model proposed.

In this chapter we shall study the following testing hypothesis problem.

$$H_0: \sigma_t^2 = \text{constant} \quad (\text{or } |\gamma_p| = 0 \text{ for } p = 1, 2, \dots, m) \\ \text{vs. } H: |\gamma_p| \neq 0$$

using the moving average model $X_t = \sum_{j=1}^h \alpha_j \sigma_{t-j} \eta_{t-j}$. Herbst [1963b] studied the same problem and developed two test statistics depending on the periodogram of X_t . In this chapter we shall present a large sample test statistic which depends on the periodogram of X_t^2 . It seems that the testing procedure presented in this chapter is better, as far as detecting the departure from the null hypothesis, than the ones proposed by Herbst.

Priestly [1962a] and [1962b] studied a problem similar in character, but quite different in structure to the present one, as it deals with stationary processes which contain harmonic components superimposed on a non-uniform continuous spectral density function. The idea of grouping the periodogram ordinates which is used in this chapter is found in

one approach Priestley used in his papers to test for the existence of harmonic components.

An argument for the power of the test shows that it is an increasing function, though may not be monotonic, with the departure from the null hypothesis. Examples are presented in support of this test criterion at the end of the chapter.

7.2. The Test Statistic

7.2.1. Statistics Based on the Periodogram

Let us subdivide the set of the m periodogram estimates into $(\frac{m}{k})$ subsets as follows:

$$I'_1, I'_2, \dots, I'_k; I'_{k+1}, \dots, I'_{2k}; I'_{2k+1}, \dots, I'_{3k}, \dots, (I'_j = I_x^{(N)}(\frac{j}{N})),$$

where k is some integer such that $k = O(N^{1/3})$; thus $\frac{k}{\sqrt{N}} \rightarrow 0$ as $N \rightarrow \infty$. Then we form the following statistics:

$$B_\ell(p) = \frac{\sum_{j=(\ell-1)k+1}^{(\ell-1)k+p} I'_j}{\sum_{j=(\ell-1)k+1}^{\ell k} I'_j} \quad (7.1)$$

($p = 1, 2, \dots, k$), where $\ell = 1, 2, \dots, \frac{m}{k}$; i.e., we get $\frac{m}{k}$ sets of statistics $B_\ell(p)$, each of which takes k values. Letting $C = (\ell-1)k$, it is easy to see that $B_\ell(p)$ takes the form:

$$B_\ell(p) = \frac{\sum_{j=1}^p I'_{j+C}}{\sum_{j=1}^k I'_{j+C}} \quad (7.2)$$

Recalling that $g'(\lambda) < a$, $0 \leq \lambda \leq 1/2$, and using the result of Theorem (5.1) we see that

$$I'_{j+c} = I_{j+c} \left[g\left(\frac{c}{N}\right) + \frac{j}{N} g'(\theta) \right] + O(N^{-\frac{1}{2}}) \quad \left(\frac{j+c}{N} > \theta > \frac{c}{N} \right).$$

By lemma (5.1) all moments of I_j are bounded and since $j \leq k = O(N^{1/3})$, we can easily see that

$$I'_{j+c} = I_{j+c} g\left(\frac{c}{N}\right) + O(N^{-\frac{1}{2}}) \quad (7.3)$$

where the O term indicates terms which have absolute second moment of order N^{-1} .

Now, we shall prove a theorem which asserts that the distribution of the statistic $B_\ell(p)$ is asymptotically the same as the one which involves the set of I_j . First we shall prove the following preliminary theorem.

Theorem (7.1)

The random variable $|K_{j+c} - I_{j+c}|$ converges to zero in probability as $N \rightarrow \infty$, $(K_{j+c} = \frac{I'_{j+c}}{g(\frac{c}{N})})$.

Proof:

Consider the quantity $[K_{j+c} - I_{j+c}]^2$. It is clear from (7.3) that it has mean less than $C_1 N^{-1}$ and variance $C_2 N^{-2}$, where C_1 and C_2 are some constants, not depending on j or N . Then by the Markov inequality we get that:

$$P[|K_{j+c} - I_{j+c}|^2 > \lambda_1] \leq \frac{C_1 N^{-1}}{\lambda_1},$$

where λ_1 is some positive number. Letting $\lambda_1 = N^{-\frac{1}{2}}$, we get

$$P[|K_{j+c} - I_{j+c}| > N^{-1/4}] \leq \frac{C_1 N^{-1}}{N^{-1/2}} = C_1 N^{-1/2}.$$

Thus, the above probability tends to zero as $N \rightarrow \infty$. Hence the theorem.

Theorem (7.2)

The random variable $|B_\ell(p) - \bar{B}_\ell(p)|$ converges to zero in probability as $N \rightarrow \infty$, where

$$\bar{B}_\ell(p) = \frac{\sum_{j=1}^p I_{j+c}}{\sum_{j=1}^k I_{j+c}} \quad [c = (\ell-1)k].$$

Proof:

It is clear that we can write $B_\ell(p)$ as follows:

$$B_\ell(p) = \frac{\sum_{j=1}^p K_{j+c}}{\sum_{j=1}^k K_{j+c}}.$$

Consider the following identity (similar to the one suggested by Hannon, 1960]:

$$\frac{\sum_{j=1}^p K_{j+c}}{\sum_{j=1}^k K_{j+c}} = \left\{ \frac{\sum_{j=1}^p I_{j+c}}{\sum_{j=1}^k I_{j+c}} + \frac{\sum_{j=1}^p (K_{j+c} - I_{j+c})}{\sum_{j=1}^k I_{j+c}} \right\} \left\{ 1 + \frac{\sum_{j=1}^k (K_{j+c} - I_{j+c})}{\sum_{j=1}^k I_{j+c}} \right\}^{-1} \quad (7.4)$$

Now

$$E \sum_{j=1}^k I_{j+c} = \sum_{j=1}^k (|\gamma_{j+c}|^2 + N^{-1} \sum_{i=-m}^m |\gamma_i|^2) \leq \sum_{j=-m}^m |\gamma_j|^2 (1+N^{-1}).$$

Thus, $E \sum_{j=1}^k I_{j+c} \leq C_0$ as $N \rightarrow \infty$. Thus $\sum_{j=1}^k I_{j+c}$ converges in probability to a non-zero constant.

By theorem (7.1) $|K_{j+c} - I_{j+c}| \xrightarrow{p} 0$. Therefore the left-hand side of (7.4) minus the right-hand side of (7.4) converges in probability to zero, i.e.,

$$|B_{\ell}(p) - \bar{B}_{\ell}(p)| \xrightarrow{p} 0.$$

7.2.2. The Distribution of $\bar{B}_{\ell}(p)$ ($p=1,2,\dots,k$) under the Null

Hypothesis

In this section we shall derive an asymptotic distribution for $\bar{B}_{\ell}(p)$ under the null hypothesis.

Theorem (7.3)

Under the null hypothesis that $\sigma_t^2 = \text{constant}$, $\bar{B}_{\ell}(p)$ is asymptotically distributed as $\beta_{p,k}$ (Beta distribution with parameters p and k).

Proof:

Let $\sigma_1^2 = \sigma_2^2 = \dots = \sigma_N^2 = \sigma^2$. Then, by definition:

$$\begin{aligned} I_p &= N^2 \left| \sum_{t=1}^N \sigma^2 \eta_t^2 Z^{tp/N} \right|^2 \\ &= \sigma^4 N^{-2} \left| \sum_{t=1}^N \eta_t^2 Z^{tp/N} \right|^2. \end{aligned}$$

Or

$$I_p = \sigma^4 N^{-2} \left[\sum_{t=1}^N \eta_t^2 \cos\left(\frac{2\pi tp}{N}\right) \right]^2 + \left[\sum_{t=1}^N \eta_t^2 \sin\left(\frac{2\pi tp}{N}\right) \right]^2. \quad (7.5)$$

(7.5) can be written as:

$$I_p = \sigma^4 N^{-2} \left[\sum_{t=1}^N U_t(p) \right]^2 + \left[\sum_{t=1}^N V_t(p) \right]^2,$$

where $U_t(p) = \eta_t^2 \cos(\frac{2\pi tp}{N})$ and $V_t(p) = \eta_t^2 \sin(\frac{2\pi tp}{N})$, or $I_p = \sigma^4 N^{-2} [U_p^2 + V_p^2]$,

$$U_p = \sum_{t=1}^N U_t(p), \quad V_p = \sum_{t=1}^N V_t(p).$$

To find the asymptotic distribution of $I_y^{(N)}(\frac{p}{N})$ under the null hypothesis we consider first U_p and V_p , ($p = 1, 2, \dots, m-1$).

$U_t(p)$ ($t = 1, 2, \dots, N$) is a set of independent random variables with means $E U_t(p) = \cos(\frac{2\pi tp}{N})$ and variances equal to

$$E[U_t(p)]^2 - [\cos(\frac{2\pi tp}{N})]^2 = 3 \cos^2(\frac{2\pi tp}{N}) - \cos^2(\frac{2\pi tp}{N}) = 2 \cos^2(\frac{2\pi tp}{N}) = \delta_t^2$$

($p = 1, 2, \dots, m-1$).

Therefore

$$E U_p = \sum_{t=1}^N \cos(\frac{2\pi tp}{N}) = 0, \text{ since } p \neq 0,$$

and

$$\text{var } U_p = 2 \sum_{t=1}^N \cos^2(\frac{2\pi tp}{N}) = \sum_{t=1}^N [1 + \cos(\frac{4\pi tp}{N})]$$

$$= N + 0 = N.$$

Similarly, we prove that $E V_p = 0$ and $\text{var } V_p = N$.

Let us find the absolute third moment, ρ_t^3 , of $U_t(p)$. It is defined as:

$$\rho_t^3 = E |U_t(p) - \cos(\frac{2\pi tp}{N})|^3.$$

Using the well-known inequality

$$|a+b|^r \leq 2^{r-1}(|a|^r + |b|^r) \quad (r \geq 1),$$

we get that, after taking the expectations:

$$E|U_t(p) - \cos(\frac{2\pi tp}{N})|^3 \leq 4[E|U_b(p)|^3 + |\cos(\frac{2\pi tp}{N})|^3], \quad \text{since } r=3.$$

$$\begin{aligned} E|U_t(p)|^3 &= E|\eta_t^2 \cos(\frac{2\pi tp}{N})|^3 \\ &= |\cos(\frac{2\pi tp}{N})|^3 E|\eta_t^2|^3 = 15|\cos(\frac{2\pi tp}{N})|^3, \end{aligned}$$

$$\text{since } E|\eta_t^2|^3 = E\eta_t^6 = 15.$$

Therefore,

$$\begin{aligned} \rho^3 &= \rho_1^3 + \rho_2^3 + \dots + \rho_N^3 \\ &\leq 4[15 \sum_{t=1}^N |\cos(\frac{2\pi tp}{N})|^3 + \sum_{t=1}^N |\cos(\frac{2\pi tp}{N})|^3]. \end{aligned}$$

Hence

$$\rho^3 \leq 64 \sum_{t=1}^N |\cos(\frac{2\pi tp}{N})|^3 \leq 64N, \quad \text{since } |\cos(\frac{2\pi tp}{N})|^3 \leq 1.$$

$$\text{Also, } \sigma^2 = \delta_1^2 + \delta_2^2 + \dots + \delta_N^2 = N \text{ as proved earlier.}$$

$$\text{Therefore } \frac{\rho}{\sigma} \leq \frac{(64)^{1/3} N^{1/3}}{N^{1/2}}, \text{ and } \lim_{N \rightarrow \infty} \frac{\rho}{\sigma} = 0.$$

This means that U_p is asymptotically normally distributed with zero mean and variance N since the conditions for Liapounoff's theorem are satisfied. Therefore, $\frac{U_p^2}{N}$ is asymptotically distributed as χ^2 with one degree of freedom.

In a similar way to the above we can prove that $\frac{V_p^2}{N}$ is distributed as χ^2 with one degree of freedom.

Now

$$\begin{aligned} \text{cov}[U_p, V_p] &= E\left[\sum_{t=1}^N U_t(p) \cdot \sum_{t=1}^N V_t(p)\right] \\ &= \sum_{t=1}^N E[U_t(p)V_t(p)] + \sum_{t=1}^N \sum_{t'=1}^N E[U_t(p)V_{t'}(p)]. \quad (7.6) \end{aligned}$$

The second term of (7.6) is obviously zero and the first term is given by

$$\begin{aligned} \sum_{t=1}^N E \eta_t^4 \sin\left(\frac{2\pi tp}{N}\right) \cos\left(\frac{2\pi tp}{N}\right) &= 3 \sum_{t=1}^N \sin\left(\frac{2\pi tp}{N}\right) \cos\left(\frac{2\pi tp}{N}\right) \\ &= 3/2 \sum_{t=1}^N \sin\left(\frac{4\pi tp}{N}\right) = 0 \quad (p = 1, 2, \dots, m). \end{aligned}$$

Thus, U_p and V_p are uncorrelated, each one of which is asymptotically normally distributed with zero mean and variance N . Thus, they are asymptotically independent. Therefore, $\frac{U_p^2}{N} + \frac{V_p^2}{N}$ is distributed as χ_2^2 with two degrees of freedom and this means that

$$I_p = \sigma^4 N^{-1} \chi_2^2.$$

We have proved in Chapter 3 that the periodogram ordinates I_p ($p = 1, 2, \dots, m$) are asymptotically uncorrelated. Therefore, the periodogram ordinates under H_0 form a set of independently distributed random variables each of which is distributed as χ_2^2 with two degrees of freedom. Therefore, the statistic

$$\bar{B}_\ell(p) = \frac{\sum_{j=1}^p I_{j+c}}{k} \quad (p = 1, 2, \dots, k; c = (\ell-1)k+1),$$

is distributed as $\beta_{p,k}$. Hence, the theorem.

7.2.3. The Development of the Test

Having grouped the periodograms ordinates into m/k subsets, the null hypothesis may then be presented in the following form:

$$H_0 = \bigcap_{\ell=1}^{m/k} H_0^{(\ell)},$$

where $H_0^{(\ell)}$ is the null hypothesis corresponding to the subset $I_{(\ell-1)k+1}^1, \dots, I_{\ell k}^1$ of the periodogram given as follows:

$$H_0^{(\ell)} : |\gamma_p| = 0 \text{ for all } p = (\ell-1)k+1, \dots, \ell k \quad (\ell = 1, 2, \dots, m/k).$$

Thus, we accept H_0 if we accept all the subhypotheses $H_0^{(\ell)}$ ($\ell=1, 2, \dots, m/k$).

Now, we need to construct a test statistic for the subhypotheses, then develop a criterion for the general hypothesis H_0 .

It is clear that $\bar{B}_\ell(1) \leq \bar{B}_\ell(2) \leq \dots \leq \bar{B}_\ell(k)$. Bartlett [1956] has shown that the joint distribution of $\bar{B}(p)$ is $k!$ and it is easy to prove this assertion. But this is the joint distribution of the order statistics of k independent rectangular, distributed random variables on the interval $(0,1)$. $\bar{B}_\ell(p)$ here is the p^{th} order statistic of such variables since $\bar{B}_\ell(1) \leq \bar{B}_\ell(2) \leq \dots \leq \bar{B}_\ell(k)$. From the distribution theory of order statistics $E[X_{(r)}] = \frac{r}{n+1}$ where $X_{(r)}$ is the r^{th} order statistic and n is the number of the uniformly distributed variables over $(0,1)$. Thus, $E[\bar{B}_\ell(p)] \sim \frac{p}{k+1}$ and consequently $E[B_\ell(p)] \sim \frac{p}{k+1}$.

Therefore, following Bartlett, $\bar{B}_\ell(p)$ may be tested against its expectation by using the Kolmogorov-Smirnov statistic. Considering the statistic $B_\ell(p)$ in place of $\bar{B}_\ell(p)$, we would consider the following statistic

$$D_\ell = \max_p \sqrt{(k+1)} \left| B_\ell(p) - \frac{p}{k+1} \right| ,$$

as a test statistic for the null hypothesis $H_0^{(\ell)}$, where

$$P[D_\ell \leq x_0] \longrightarrow \sum_{j=-\infty}^{\infty} (-1)^j e^{-2x_0^2 j^2} .$$

If the hypothesis is not true, then this probability is high (for a suitable value of x_0), i.e., the situation will be in favor of the alternative hypothesis.

If α is the significance level of the general hypothesis, then the appropriate significance level, α_1 , of a test based on D_ℓ will be given by:

$$\alpha = 1 - [P[D_\ell \leq d_{\alpha_1} | H_0]]^{m/k} = 1 - (1 - \alpha_1)^{m/k} .$$

Hence, $\alpha_1 \sim k/m \alpha$, i.e., $\alpha_1 = P[D_\ell > d_{\alpha_1} | H_0]$.

If $m/k = 5$, then $\alpha = .05$ and $.01$ for $\alpha_1 = .01$ and $.002$ respectively. From Smirnov [1948] the corresponding critical values, d_{α_1} , are 1.63 and 1.93 respectively. Now, we form the following testing procedure:

accept H_0 if $D_L \leq d_{\alpha_1}$

and reject H_0 if $D_L > d_{\alpha_1}$, where $D_L = \max_\ell (D_\ell)$.

7.2.4. The Mean and Variance of $\bar{B}_\ell(p)$

For the first order of approximation, we may write

$$E \bar{B}_\ell(p) = \frac{\sum_{j=1}^p E(I_{j+c})}{\sum_{j=1}^k E(I_{j+c})} = \frac{\sum_{j=1}^p |\gamma_{j+c}|^2}{\sum_{j=1}^k |\gamma_{j+c}|^2}, \quad [(\ell-1)k = c]. \quad (7.7)$$

It is easy to see that $\sum_{j=1}^k I_{j+c} = \sum_{j=-n}^n I_{j+c}$, where $a = \ell k - \frac{(k-1)}{2}$ and $n = \frac{(k-1)}{2}$. Therefore

$$E \sum_{j=1}^k I_{j+c} = E \sum_{j=-n}^n I_{j+c} = \sum_{j=-n}^n |\gamma_{j+a}|^2.$$

Thus, for large n , $E(\sum_{j=1}^k I_{j+c}) \sim \sum_{-\infty}^{\infty} |\gamma_{j+a}|^2 = \sum_{-\infty}^{\infty} |\gamma_j|^2 = C_0$, and we can write

$$E[\bar{B}_\ell(p)] \sim \frac{\Delta_p}{C_0}, \quad (7.8)$$

where $\Delta_p = \sum_{j=1}^p |\gamma_{j+c}|^2$.

To find the variance of $\bar{B}_\ell(p)$, we note that the approximate expression for the variance of a ratio of two functions, say $\frac{U_1}{U_2}$, is given by

$$\text{var}\left(\frac{U_1}{U_2}\right) = \frac{\text{var}(U_1)}{\mu_2^2} - 2 \frac{\mu_1}{\mu_2^3} \text{cov}(U_1, U_2) + \frac{\mu_1^2}{\mu_2^4} \text{var}(U_2), \quad (7.9)$$

when $\mu_j = E(U_j)$, $j = 1, 2$.

Let $U_1 = \sum_{j=1}^p I_{j+c}$, $U_2 = \sum_{j=1}^k I_{j+c}$. Then $\mu_1 = \Delta_p$ and $\mu_2 = C_0$.

$$\text{var}(U_1) = \sum_{j=1}^p \text{var}(I_{j+c}) = \sum_{j=1}^p \Gamma(j+c), \text{ where } \Gamma(s) \text{ is defined in (6.6)}$$

$$\text{cov}(U_1, U_2) = \sum_{j=1}^p \sum_{j'=1}^k \text{cov}(I_{j+c}, I_{j'+c}) = \sum_{j=1}^p \sum_{j'=1}^k \Gamma(j+c, j'+c),$$

where $\Gamma(s, s')$ is defined in (6.7)

$$\text{var}(U_2) = \sum_{j=1}^k \Gamma(j+c).$$

Therefore, substituting in (7.9), we get:

$$\text{var}[\bar{B}_\ell(p)] = \frac{\sum_{j=1}^p \Gamma(j+c)}{C_0^2} - 2 \frac{\Delta_p}{C_0^3} \sum_{j=1}^p \sum_{j'=1}^k \Gamma(j+c, j'+c) + \frac{\Delta_p^2}{C_0^4} \sum_{j=1}^p \Gamma(j+c). \quad (7.10)$$

To understand the behavior of this variance we may consider $\bar{B}_1(p)$, i.e., $C=0$, and let us assume that $\gamma_p=0$ for all $p \neq 0, 1$. This means that

$$\sigma_t^2 = \sum_{j=-1}^1 \gamma_j z^{-jt/N}.$$

Then it is clear that

$$\sum_{i=-m}^m \gamma_i \gamma_{i-(s+s')}^* = 0 \text{ for all values of } s \text{ and } s' \quad (s, s' \neq 0),$$

and

$$\sum_{i=-m}^m \gamma_i \gamma_{i-(s-s')}^* = \begin{cases} \gamma_0^2 + 2|\gamma_1|^2 & \text{if } s=s' \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\sum_{j=1}^p \Gamma(j) = 4N^{-1} |\gamma_1|^2 (\gamma_0^2 + 2|\gamma_1|^2) ,$$

$$\sum_{j=1}^p \sum_{j'=1}^k \Gamma(j, j') = \sum_{j=1}^p \Gamma(j) ,$$

since all the terms inside the sum will be zero for $j \neq j'$. Hence,

$$\sum_{j=1}^p \sum_{j=1}^k \Gamma(j, j') = 4N^{-1} |\gamma_1|^2 (\gamma_0^2 + 2|\gamma_1|^2) .$$

Also, it is clear that $\Delta p = |\gamma_1|^2$ and $C_0 = (\gamma_0^2 + 2|\gamma_1|^2)$. Therefore (7.10) becomes

$$\begin{aligned} \text{var } \bar{B}_1(p) &= \frac{4N^{-1} |\gamma_1|^2 (\gamma_0^2 + 2|\gamma_1|^2)}{C_0^2} - \frac{8N^{-1} |\gamma_1|^2 |\gamma_1|^2 (\gamma_0^2 + |\gamma_1|^2)}{C_0^3} \\ &\quad + \frac{4N^{-1} |\gamma_1|^4 |\gamma_1|^2 (\gamma_0^2 + 2|\gamma_1|^2)}{C_0^4} = \frac{4N^{-1} |\gamma_1|^2}{(\gamma_0^2 + 2|\gamma_1|^2)} \\ &\quad \left[1 - \frac{2|\gamma_1|^2}{(\gamma_0^2 + 2|\gamma_1|^2)} + \frac{|\gamma_1|^4}{(\gamma_0^2 + 2|\gamma_1|^2)^2} \right] \\ &= \frac{4N^{-1} |\gamma_1|^2}{(\gamma_0^2 + 2|\gamma_1|^2)} \left[1 - \frac{|\gamma_1|^2}{(\gamma_0^2 + 2|\gamma_1|^2)} \right]^2 \end{aligned} \quad (7.11)$$

Or, approximating further, we see that

$$\text{var } \bar{B}_1(p) = \frac{4N^{-1} |\gamma_1|^2}{(\gamma_0^2 + 2|\gamma_1|^2)} . \quad (7.12)$$

7.3. An Argument for the Power of the Test

It is extremely difficult to find an explicit expression for the power function of the test statistic discussed in this chapter, but we shall present the following heuristic argument in this connection.

Massey [1950] proved the following theorem thus.

Let $D_n = \max_x \sqrt{n} |F(x) - S_n(x)|$ be the Kolmogorov-Smirnov statistic. Suppose that we are testing the null hypothesis $H_0: F(x) = F_0(x)$ vs. $H: F(x) = F_1(x)$ ($F_0(x) \neq F_1(x)$). Then the power P is given by

$$P > 1 - \frac{1}{\sqrt{2\pi}} \int_{\lambda_1}^{\lambda_2} e^{-t^2/2} dt \quad (7.13)$$

where

$$\lambda_1 = (-d_\alpha \pm \Delta\sqrt{n}), \quad \lambda_2 = (d_\alpha \pm \Delta\sqrt{n}), \quad \Delta = \max_x |F_1(x) - F_0(x)|,$$

and d_α is such that

$$P[D_n > d_\alpha | H_0] = \alpha,$$

that is d_α is the critical value of the test.

Now suppose that $\lambda_1 = (-d_\alpha + \Delta\sqrt{n})$, $\lambda_2 = (d_\alpha + \Delta\sqrt{n})$ (the sign before Δ depends on whether $F_1(x) < \text{or} > F_0(x)$). Then it is clear that

$$I = \frac{1}{\sqrt{2\pi}} \int_{\lambda_1}^{\lambda_2} e^{-t^2/2} dt$$

is a monotone decreasing function of Δ , since d_α is fixed. In exactly

the same way we see that this is true if a minus sign is in place of the plus sign before Δ . Therefore, the right-hand side of (7.13) is a monotone increasing function of Δ and the power is always greater than this monotone function.

Let us examine the value of Δ by considering, for simplicity, the test statistic $B_1(p)$. Then for the present testing hypothesis problem we have

$$F_0(x) = \bar{B}_1(p) = \frac{\sum_{j=1}^p I_j}{\sum_{j=1}^k I_j} \quad (\text{for } \ell=1)$$

under the assumption that $\sigma_t^2 = \text{constant}$. Let the alternative be that

$$\sigma_t^2 = \sum_{-1}^1 \gamma_j Z^{-jt/N}.$$

Then the alternative $F_1(x)$ will be

$$F_1(x) = \tilde{B}_1(p) = \frac{\sum_{j=1}^p \tilde{I}_j}{\sum_{j=1}^k \tilde{I}_j},$$

where $\sim$ indicates the periodogram estimates under the assumption of non-constant variance. Since

$$E \tilde{I}_j = |\gamma_j|^2 + N^{-1} \sum_{i=-m}^m |\gamma_i|^2,$$

$$E \tilde{I}_1 = |\gamma_1|^2 + N^{-1} \sum_{i=-m}^m |\gamma_i|^2 \quad \text{and} \quad E \tilde{I}_j = N^{-1} \sum_{i=-m}^m |\gamma_i|^2$$

for all $j \neq 1$. Thus, we may consider the I_j 's ($j \neq 1$) as not affected by the alternative hypothesis assumption and hence

$$\tilde{B}_1(p) = \frac{\tilde{I}_1 + I_2 + \dots + I_p}{\tilde{I}_1 + I_2 + \dots + I_k}.$$

Therefore

$$\begin{aligned} |\tilde{B}_1(p) - \bar{B}_1(p)| &= \left| \frac{\tilde{I}_1 + I_2 + \dots + I_p}{\tilde{I}_1 + I_2 + \dots + I_k} - \frac{I_1 + I_2 + \dots + I_p}{I_1 + I_2 + \dots + I_k} \right| \\ &= \left| \frac{\tilde{I}_1(I_{p+1} + \dots + I_k) + I_1(I_2 + \dots + I_p) - I_1(I_2 + \dots + I_k)}{(\tilde{I}_1 + \dots + I_k)(I_1 + \dots + I_k)} \right| \\ &= \frac{|\tilde{I}_1(I_{p+1} + \dots + I_k) - I_1(I_{p+1} + \dots + I_k)|}{(\tilde{I}_1 + \dots + I_k)(I_1 + \dots + I_k)} \\ &= \frac{(I_{p+1} + \dots + I_k) |\tilde{I}_1 - I_1|}{(\tilde{I}_1 + \dots + I_k)(I_1 + \dots + I_k)}. \end{aligned}$$

Thus

$$\begin{aligned} \Delta &= \max_p |\tilde{B}_1(p) - \bar{B}_1(p)| \\ &= \frac{(I_2 + \dots + I_k) |\tilde{I}_1 - I_1|}{(\tilde{I}_1 + \dots + I_k)(I_1 + \dots + I_k)} \sim \frac{|\tilde{I}_1 - I_1|}{(\tilde{I}_1 + I_2 + \dots + I_k)}. \quad (7.14) \end{aligned}$$

Now $E \sum_{j=2}^k I_j = \frac{(k-1)\gamma_0^2}{N}$, since $\sigma_t^2 = \text{constant} = \gamma_0$. Therefore

$$\begin{aligned}
E[\tilde{I}_1 + \sum_{j=2}^k I_j] &= |\gamma_1|^2 + \frac{2}{N} |\gamma_1|^2 + \frac{\gamma_0^2}{N} + \left(\frac{k-1}{N}\right) \gamma_0^2 \\
&= \frac{(N+2)}{N} |\gamma_1|^2 + \frac{k\gamma_0^2}{N},
\end{aligned}$$

and

$$\begin{aligned}
E(\tilde{I}_1) - E(I_1) &= (|\gamma_1|^2 + \frac{2}{N} |\gamma_1|^2 + \frac{\gamma_0^2}{N}) - \frac{\gamma_0^2}{N} \\
&= |\gamma_1|^2 \left(\frac{N+2}{N}\right).
\end{aligned}$$

Let us look at these periodogram estimates in terms of their expectations. We get

$$\tilde{\Delta} = \frac{\frac{(N+2)}{N} |\gamma_1|^2}{\left(\frac{N+2}{N}\right) |\gamma_1|^2 + \frac{k\gamma_0^2}{N}} = \frac{1}{1 + \frac{k\gamma_0^2}{(N+2) |\gamma_1|^2}}. \quad (7.15)$$

For fixed values of N , $\tilde{\Delta}$ is a monotone increasing function; it varies from 0 to 1 as $|\gamma_1|^2$ increases from 0 to ∞ .

Thus, we can conclude that Δ on the average is a monotone increasing function of the departure from the null hypothesis and hence that

$$\text{Power} > \left(\begin{array}{l} \text{a monotone increasing function of the} \\ \text{departure from the null hypothesis} \end{array} \right).$$

This means that the present test will have an increasing probability of detecting the departure from the null hypothesis $H_0: \sigma_t^2 = \text{constant}$.

The above argument does not prove the monotonicity of the power of the test, but at least it shows that the test is sensitive to the departure from the null hypothesis.

7.4. Numerical Examples

In this section we are presenting three artificial series constructed by using a number of random deviates. The method of constructing these series is similar to those described in the previous chapters. FORTRAN language was used for the program to obtain the periodogram values and the values of the Kolmogorov-Smirnov statistic. There are 100 periodogram values for each series subdivided into 5 sets: thus each set contains 20 periodogram ordinates. Values of the periodograms with the corresponding values of the statistic D_ℓ ($\ell=1,2,\dots,5$) are listed in Tables 7.1 through 7.3.

The series are listed as follows.

(1) Series VII.

The expression for σ_t^2 is given by

$$\sigma_t^2 = [1 + \sum_{s=1}^5 (0.36)^s \cos(\frac{2\pi s t}{200}) + \sum_{s=1}^5 (0.81)^s \sin(\frac{2\pi s t}{200})]^2$$

and

$$X_t = \sigma_t \eta_t + 0.72 \sigma_{t-1} \eta_{t-1}.$$

From Table 7.1 it can be seen that the value of $D_L = \max(D_1, \dots, D_5) = D_1 = 2.107$, which is highly significant at the 0.01 significance level. Thus, the test is in agreement with the structure of the series.

(2) Series VIII.

The Fourier expression for σ_t^2 is given by

$$\sigma_t^2 = [1 + \sum_{s=1}^4 (.48)^s \cos(\frac{2\pi s t}{200}) + \sum_{s=1}^4 (.68)^s \sin(\frac{2\pi s t}{200})]^2.$$

The model is given by

$$X_t = \sigma_t \eta_t + .56 \sigma_{t-1} \eta_{t-1}.$$

Table 7.1. Periodogram values of Series VII and the corresponding values of D_l ($l=1,2,\dots,5$)

p	G(p)	p	G(p)	p	G(p)	p	G(p)
1	3.529	26	.083	51	.107	76	.387
2	1.864	27	.345	52	.087	77	.342
3	.904	28	.625	53	.072	78	.189
4	.354	29	.441	54	.084	79	.204
5	.247	30	.185	55	.121	80	.153
6	.088	31	.061	56	.157	81	.113
7	.010	32	.016	57	.162	82	.129
8	.007	33	.013	58	.039	83	.097
9	.050	34	.100	59	.015	84	.031
10	.224	35	.179	60	.044	85	.005
11	.429	36	.205	61	.101	86	.003
12	.577	37	.059	62	.160	87	.039
13	.521	38	.003	63	.202	88	.043
14	.342	39	.035	64	.195	89	.058
15	.378	40	.049	65	.149	90	.101
16	.176	41	.050	66	.088	91	.220
17	.064	42	.072	67	.025	92	.163
18	.061	43	.048	68	.015	93	.164
19	.273	44	.020	69	.013	94	.207
20	.349	45	.009	70	.003	95	.132
21	.326	46	.060	71	.021	96	.292
22	.218	47	.171	72	.037	97	.253
23	.112	48	.273	73	.067	98	.249
24	.064	49	.350	74	.200	99	.214
25	.051	50	.249	75	.322	100	.267
D_1		D_2		D_3		D_4	
2.107		1.360		.766		1.121	
						1.161	

Table 7.2. Periodogram values of Series VIII and the corresponding values of D_ℓ ($\ell=1,2,\dots,5$)

p	G(p)	p	G(p)	p	G(p)	p	G(p)
1	2.071	26	.024	51	.137	76	.117
2	.956	27	.207	52	.097	77	.125
3	.303	28	.505	53	.047	78	.099
4	.023	29	.388	54	.011	79	.114
5	.032	30	.126	55	.062	80	.078
6	.077	31	.011	56	.164	81	.041
7	.058	32	.027	57	.203	82	.045
8	.010	33	.071	58	.074	83	.082
9	.023	34	.166	59	.000	84	.072
10	.161	35	.194	60	.011	85	.050
11	.325	36	.176	61	.071	86	.018
12	.387	37	.029	62	.165	87	.024
13	.287	38	.003	63	.171	88	.015
14	.123	39	.054	64	.128	89	.018
15	.078	40	.048	65	.058	90	.038
16	.024	41	.028	66	.015	91	.084
17	.005	42	.013	67	.004	92	.052
18	.008	43	.010	68	.024	93	.056
19	.071	44	.036	69	.003	94	.062
20	.154	45	.045	70	.007	95	.038
21	.231	46	.030	71	.010	96	.083
22	.225	47	.052	72	.015	97	.089
23	.193	48	.113	73	.040	98	.098
24	.088	49	.205	74	.064	99	.118
25	.022	50	.216	75	.091	100	.167
D_1		D_2		D_3		D_4	
2.293		1.130		.899		.879	
						.723	

Table 7.3. Periodogram values of Series IX and the corresponding values of D_ℓ ($\ell=1,2,\dots,5$)

p	G(p)	p	G(p)	p	G(p)	p	G(p)
1	2.136	26	.039	51	.163	76	.118
2	.978	27	.253	52	.112	77	.141
3	.308	28	.569	53	.051	78	.108
4	.045	29	.458	54	.012	79	.139
5	.093	30	.161	55	.077	80	.098
6	.152	31	.026	56	.191	81	.039
7	.113	32	.033	57	.245	82	.057
8	.013	33	.112	58	.113	83	.088
9	.025	34	.209	59	.007	84	.082
10	.174	35	.226	60	.013	85	.053
11	.366	36	.161	61	.097	86	.027
12	.429	37	.023	62	.191	87	.011
13	.300	38	.005	63	.179	88	.009
14	.126	39	.072	64	.122	89	.013
15	.067	40	.066	65	.052	90	.031
16	.041	41	.037	66	.004	91	.073
17	.021	42	.024	67	.010	92	.040
18	.023	43	.025	68	.029	93	.057
19	.050	44	.062	69	.011	94	.056
20	.144	45	.075	70	.010	95	.027
21	.229	46	.059	71	.019	96	.070
22	.262	47	.052	72	.029	97	.078
23	.234	48	.118	73	.063	98	.085
24	.137	49	.215	74	.074	99	.111
25	.042	50	.225	75	.100	100	.157
<hr/>							
D_1		D_2		D_3		D_4	
D_5							
2.143		1.112		.709		.821	
						.662	

From Table 7.2 it is clear that $D_1 = 2.293$ is the greatest value of the statistics D_ℓ . This is highly significant at the .01 significance level. Thus, we reject the hypothesis of the σ_t^2 being constant.

(3) Series IX.

For this series we get the following expression for the variance σ_t^2 :

$$\sigma_t^2 = \left[1 + \sum_{s=1}^3 (.48)^s \cos\left(\frac{2\pi st}{200}\right) + \sum_{s=1}^3 (.68)^s \sin\left(\frac{2\pi st}{200}\right) \right]^2.$$

The model is again as in Series VIII. The maximum value of D_ℓ is $D_1 = 2.143$, and this is again significant at the .01 significance level.

8. SUMMARY AND CONCLUSIONS

8.1. The Problem

Two problems are considered in this thesis:

(1) To find consistent estimators for the coefficients in the Fourier expansion of the variance of data with zero means and unequal variances. Two models were studied in this connection. The first one is the random Gaussian data model with zero means and variances, which are not necessarily equal. The sampling properties of a stationary process generated from this model were studied. The second model is the moving average model which is a linear function of the process in the first model.

(2) To propose a testing procedure for the null hypothesis that the variances in the moving average model are the same, against the general alternative that the variances are not equal. The variances were expressed in a short Fourier series expansion.

8.2. The Random Process

Let X_t ($t = 1, 2, \dots, N$) be a realization of a random process such that $X_t = \sigma_t \eta_t$, where η_t are NID with zero means and variance one and σ_t possesses a Fourier expansion

$$\sigma_t^2 = \sum_{s=-m}^m \gamma_s Z^{-st/N} \quad m = \left[\frac{N}{2} \right].$$

Then it was shown that the periodogram of X_t ,

$$I_p = I_{x^2}^{(N)}\left(\frac{p}{N}\right) = N^{-2} \left| \sum_{t=1}^N X_t^2 Z^{tp/N} \right|^2,$$

is a consistent estimator of $|\gamma_p|^2$ ($p = 1, 2, \dots, m$). An exact expression for the covariance associated with this periodogram, using a lemma proved in this thesis, was derived. From this result an approximate expression, ignoring terms of order N^{-2} , was shown to take the form

$$\text{cov}[I'_p, I'_{p'}] \sim 4N^{-1} \text{Re}[\gamma_p^* \gamma_{p'}^*, \sum_{i=-m}^m \gamma_i \gamma_{i-(p+p')}^* + \gamma_p^* \gamma_{p'} \sum_{i=-m}^m \gamma_i \gamma_{i-(p-p')}^*]$$

Some special examples were given to clarify the meaning and applications of the above expressions. Three artificial series were constructed in support of the use of the periodogram of X_t^2 . Irregular fluctuations were noticeable in all of these examples.

8.3. Circular Complex Stationary Process

The complex-valued circular stationary process $J_\alpha = N^{-\frac{1}{2}} \sum_{t=1}^N X_t Z^{-t\alpha/N}$ was shown to behave in the same way as the real stationary process. The covariance of the function $C_s = \frac{1}{(N-s)} \sum_{\alpha=1}^{N-s} J_\alpha J_{\alpha+s}^*$ was shown to take the approximate form

$$\text{cov}(C_s, C_{s'}) \sim \frac{2}{(N-s)} \sum_{u=-\infty}^{\infty} \gamma_u \gamma_{u+t}^*$$

When the function $\bar{C}_s = N^{-1} \sum_{\alpha=1}^N J_\alpha J_{\alpha+s}^*$ was considered, the covariance function took the form

$$\text{cov}(\bar{C}_s, \bar{C}_{s'}) \sim N^{-1} \sum_{v=-\infty}^{\infty} \gamma_v \gamma_{v+t}^*$$

Thus, it was indicated that C_s and $\bar{C}_s$ behave in the same manner as the classical covariance function R_s in that the sampling fluctuations

tend to zero with the increase of N , but they never do so whenever $\gamma_s \rightarrow 0$ as $s \rightarrow \infty$.

The covariances for the real and imaginary parts of $\bar{C}_s$, A_s and B_s respectively, was proved to take the forms

$$\text{cov}(A_s, A_{s'}) \sim N^{-1} \text{Re} \sum_{i=-\infty}^{\infty} [\gamma_i \gamma_{i-(s'+s)}^* + \gamma_i \gamma_{i-(s'-s)}^*]$$

and

$$\text{cov}(B_s, B_{s'}) \sim N^{-1} \text{Re} \sum_{i=-\infty}^{\infty} [\gamma_i \gamma_{i-(s'-s)}^* - \gamma_i \gamma_{i-(s'+s)}^*].$$

Again A_s and B_s were shown to have the same properties as those of C_s and $\bar{C}_s$.

8.4. The Moving Average Model

Let $X_t = \sum_{j=0}^h \alpha_j \sigma_{t-j} \eta_{t-j}$, where the α_j 's are real constants; σ_t and η_t are as defined above in Section 2. Then it was shown that

$$I'_p = g\left(\frac{p}{N}\right) I_p + O(N^{-\frac{1}{2}}),$$

where $g\left(\frac{p}{N}\right) = \left| \sum_{j=0}^h \alpha_j^2 Z^{jp/N} \right|^2$, $I_p = I_y^{(N)}\left(\frac{p}{N}\right)$, $(Y_t = \sigma_t^2 \eta_t^2)$ and the $O(N^{-\frac{1}{2}})$ indicates that the neglected terms have mean square of order N^{-1} . Thus, it was shown that I'_p is a consistent estimator of $g\left(\frac{p}{N}\right) |\gamma_p|^2$.

Three artificial series were constructed using a number of random normal deviates. The different series were generated by choosing specific Fourier expansion for σ_t^2 and suitable coefficients α_j 's. The graph of the periodogram vs. p showed high peaks corresponding to the dominant frequencies in the expression for σ_t^2 . This periodogram

possesses properties similar to that shown in the first problem in that the sampling fluctuations decrease as $N \rightarrow \infty$, but they do not do so whenever $|\gamma_s| \rightarrow 0$ as $s \rightarrow \infty$.

8.5. A Consistent Estimator for $|\gamma_p|^2/c_0$

Using the moving average model of 8.4 the function $R_n(\frac{p}{N}) = \sum_{-n}^n I'_{p+j}$ was shown to be a consistent estimator of $C_0 g(\frac{p}{N})$. Thus, using this function and the periodogram of X_t^2 , it was shown that $A(p) = \frac{I'_p}{R_n(\frac{p}{N})}$ is a consistent estimator of $\frac{|\gamma_p|^2}{c_0}$.

Numerical applications using the series generated in 8.4 were considered. They revealed that high peaks of $A(p)$ correspond to the dominant frequency in σ_t^2 . Graphs were drawn for these estimates. The present estimation procedure, being an approximate one and in a ratio form, was another cause (in addition to the one explained in the last section for a different estimator) for the irregular behavior of the estimate $A(p)$ at values of p which do not correspond to the dominant frequencies of σ_t^2 .

8.6. Testing Procedure

The null hypothesis $H_0: \sigma_t^2 = \text{constant}$ vs. the alternative $H: \sigma_t^2 \neq \text{constant}$ was considered. A large sample testing procedure was proposed as follows. Subdivide the set of the periodogram ordinates as $I'_1, \dots, I'_k; I'_{k+1}, \dots, I'_{2k}, \dots$ where k is an integer such that $k \rightarrow \infty$ and $\frac{k}{\sqrt{N}} \rightarrow 0$ as $N \rightarrow \infty$. Then the set of Kolmogorov-Smirnov statistics were formed as follows:

$$D_\ell = \max_p \sqrt{k+1} \left| B_\ell(p) - \frac{p}{k+1} \right| ,$$

where

$$B_\ell(p) = \frac{\sum_{j=(\ell-1)k+1}^{(\ell-1)k+p} I'_j}{\sum_{j=(\ell-1)k+1}^{\ell k} I'_j} \quad (\ell=1, 2, \dots, \frac{m}{k}) .$$

The test statistic was then proposed as D_L , where $D_L = \max_\ell D_\ell$, and the null hypothesis would be rejected if $D_L > d_\alpha$, otherwise it would be accepted; where d_α is such that

$$P[D_L \leq d_\alpha \mid H_0] = 1 - \alpha ,$$

α is the level of significance for each D_ℓ .

An argument was given to show that this test has a power which is increasing with the departure from the null hypothesis. Artificial examples were given in support of this test. These series were constructed in a way so that they yield non-constant variances (following different Fourier expansions). All the tests were in agreement with the construction of the examples, i.e., the tests were in favor of the alternative hypothesis.

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