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THREE-STAGE NESTED DESIGNS FOR ESTIMATING VARIANCE COMPONENTS

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ABSTRACT

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Based on a completely random three-stage nested model, five fundamental sampling structures were identified. These five fundamental structures were formed by allowing samples of size one or two to be selected from the second or third stages. From the five fundamental structures, ten classes of designs were enumerated such that each class contained no more than three fundamental structures and each design contained a multiple of twelve third-stage samples. The designs were also restricted to those that would permit the ANOVA estimation of all three variance components. This enumeration technique generated sixty-one designs, of which eleven have been discussed previously by other workers in the area. Each of the ten tables not only contains the number of replications of each fundamental structure, but also a design code for each design, which can be used to identify the degrees of freedom structure for the design. Additionally, each design has a set of four coefficient numbers which simplify the computation of the expected mean square coefficients in the ANOVA table.

General formulas for the variances and covariances of the ANOVA estimators of the variance components are given. For each of the sixty-one designs, the general formulas were used to compute the variances and covariances of the variance component estimators conditional on the sample size, $12r$, $r = 1(1)10$, and the specific sizes of the three variance components. Forty-nine different configurations of the variance components were studied. In all cases the third stage component was set to unity, while the second and first stage components took on values 2^k , $k = -3(1)3$, and 2^m , $m = -3(1)3$, respectively.

The designs were compared using the three criteria: the trace, the determinant and the adjusted trace of the covariance matrix of the variance component estimators. The adjusted trace criterion uses the covariance matrix when each element is scaled by the sizes of the variance components involved in the computation of the element. The smallest value of the criterion was used to indicate the optimum design. Optimum and worst designs as well as their criterion ratio were determined for all three criteria over five classifications of the designs.

Of the fifty new designs found in this study, only six designs were never chosen to be optimum by at least one of the three criteria over the five classifications. The trace criterion does the most reasonable job of discriminating amongst the designs. Sample size is critical when it is small. Most optimum designs contain only two fundamental structures.

BIOGRAPHY

The author was born August 27, 1939, in Flin Flon, Manitoba. He was raised in Flin Flon and was graduated from the Hapnot Collegiate Institute in 1957.

He received the general Bachelor of Science degree from the University of Manitoba in 1961. In 1963, he received the Master of Science degree in statistics from the University of Manitoba. During 1963-65, he was a lecturer in the Department of Mathematics, Carleton University, Ottawa, Ontario. From 1965 through 1969, the author was a doctoral student in Experimental Statistics at North Carolina State University, as well as a research associate in the Department of Biostatistics of the University of North Carolina. The requirements for the Ph.D. in Experimental Statistics were completed in 1969. He then accepted a joint appointment as Assistant Professor of applied mathematics and biostatistics at McMaster University in Hamilton, Ontario.

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CHAPTER 1. INTRODUCTION AND REVIEW OF LITERATURE

1.1 Introduction

It is quite common to use a completely balanced arrangement in a three-stage nested experimental situation for the simultaneous estimation of the variance components. However, this arrangement choice may lead to large disparities in the precision with which the variance components are estimated, as well as adversely affecting the economy of conducting the experiment. On the other hand, unbalanced arrangements involve more complicated experimental plans and likewise more difficult analyses. However, the unbalanced arrangements may be optimum in the sense that they minimize some criteria based on the variances and covariances of the variance component estimators. How an experimenter may judge this optimality is the main objective of this study.

1.2 Review of Literature

In recent years, much attention has been given to methods and designs for estimating variance components using the completely random model. Crump [1954] proposed some designs for estimating functions of variance components from the two-stage nested classification. Gaylor [1960] and Mostafa [1967] proposed designs for estimating variance component functions from the two-way crossed classification. Anderson and Bancroft [1952], Anderson [1960], Calvin and Miller [1961], Prairie [1962], Bainbridge [1963] and Anderson [1966] proposed a few designs

for estimating functions of variance components from nested classifications with three or more stages. Except for Prairie's designs, a more detailed discussion of the designs proposed by these authors will be postponed until the fundamental structures have been defined in Chapter 2.

Prairie's [1962] designs are formed by the following schemes. Suppose we let A, B and E identify the classes at the first, second and third stages respectively. What Prairie calls D2 designs are constructed as follows:

- i) Choose n , the total number of observations, and a , the number of A-classes.
- ii) Write

$$n = aq_1 + r_1, \quad 0 \leq r_1 < a$$

and assign $q_1 + 1$ experimental units to each of r_1 of the A-classes and q_1 units to each of the remaining $a - r_1$ A-classes.

- iii) Choose $b > a$, the number of B-classes and write

$$b = aq_2 + r_2, \quad 0 \leq r_2 < a.$$

Assign $q_2 + 1$ B-classes to each of r_2 of the A-classes and q_2 B-classes to the remaining $a - r_2$ A-classes.

- iv) Within each A-class, assign the E-classes or units to the B-classes as equally as possible.

Prairie's D1 designs are constructed by dividing the A-classes into groups and then assigning the B-classes and E-classes such that balance is maintained within each group and such that the design constructed does not constitute a D2 design. Based on total sample sizes

of 48 observations, Prairie considered 8 different degrees of freedom assignments and then constructed a D1 and D2 design within each assignment except the one that generates the completely balanced design.

Much of the previous work has not attempted to systematically enumerate designs for estimating all three variance components, and when it has, the work has been restricted to designs of a specific sample size.

The only known attempt to compare any of these designs quantitatively was conducted by Prairie [1962]. The technique used there was to compare the variance of the estimator for each variance component from the competitive design to the completely balanced design. This method gives rise to three separate comparisons which in some instances gave conflicting answers as to which is a better design.

1.3 Objectives of the Present Study

The objectives of the present study will be to (i) systematically enumerate and classify some three stage nested designs for estimating all three variance components; (ii) propose some criteria to evaluate the designs which are based on functions of the covariance matrix of the variance component estimators; and (iii) for various variance component ratios and different sample sizes, compare the designs by three criteria in five ways that may be of interest to the experimenter who wants to use these kinds of designs.

A general computer program for the computation of the variances and covariances of the variance component estimators as well as the three judgment criteria for each design is available from the author.

CHAPTER 2. THE MODEL AND ENUMERATION OF DESIGNS

2.1 Introduction

In this chapter the assumed model is defined. Five fundamental structures are identified and are used to enumerate ten classes of designs for estimating all three variance components. When a design has been discussed by some previous author, this fact is noted, and an example of how the design tables may be employed by an experimenter is discussed.

2.2 The Model

The model that will be used throughout this study is a completely random nested three-stage classification.

$$y_{ijk} = \mu + \alpha_i + \beta_{(i)j} + \varepsilon_{(ij)k}, \quad (2.2.1)$$

$$\begin{aligned} i &= 1, 2, \dots, a, & b &= \sum_{i=1}^a b_i, & n_i &= \sum_{j=1}^{b_i} n_{ij}, \\ j &= 1, 2, \dots, b_i, & n &= \sum_{i=1}^a n_i, \\ k &= 1, 2, \dots, n_{ij}, \end{aligned}$$

where μ is a constant, α_i is the effect of the i -th class sampled, $\beta_{(i)j}$ is the effect of the j -th sub-class sampled in the i -th class sampled, and $\varepsilon_{(ij)k}$ is the effect of the k -th sub-sub-class sampled in the ij -th sub-class sampled. For the purposes of this study, the following distributional assumptions will be made:

- i) each of the members α_i , $\beta_{(i)j}$, and $\varepsilon_{(ij)k}$ are mutually independent and normally distributed with zero means

$$\begin{aligned} \text{ii) } E(\alpha_i \alpha_{i'}) &= \sigma_a^2, \quad \text{if } i = i' \\ &0, \quad \text{otherwise} \end{aligned} \quad (2.2.2)$$

$$\begin{aligned} \text{iii) } E(\beta_{(i)j} \beta_{(i')j'}) &= \sigma_b^2, \quad \text{if } i = i', j = j' \\ &0, \quad \text{otherwise} \end{aligned} \quad (2.2.3)$$

$$\begin{aligned} \text{iv) } E(\varepsilon_{(ij)k} \varepsilon_{(i'j')k'}) &= \sigma_e^2, \quad \text{if } i = i', j = j', k = k' \\ &0, \quad \text{otherwise.} \end{aligned} \quad (2.2.4)$$

The basic implication of the above assumptions is that

$$\begin{aligned} \text{cov}(y_{ijk}, y_{i'j'k'}) &= 0, \quad i \neq i', j \neq j', k \neq k' \\ &\sigma_a^2, \quad i = i', j \neq j', k \neq k' \\ &\sigma_b^2 + \sigma_a^2, \quad i = i', j = j', k \neq k' \\ &\sigma_e^2 + \sigma_b^2 + \sigma_a^2, \quad i = i', j = j', k = k' \end{aligned} \quad (2.2.5)$$

This result will be used in the computation of the variances and covariances of the variance component estimators.

2.3 Fundamental Structures

The designs to be considered in this study are made up of five fundamental structures, each based on the model (2.2.1). The five fundamental structures are formed by taking all possible combinations of structures formed when splits of either one or two are allowed when passing from one stage to the next lower stage in the design. The five fundamental structures so formed are pictorially represented by the stick diagrams shown in table 2.3.1.

Table 2.3.1. Fundamental Structures

	Structure Number				
Stage	1	2	3	4	5
1
2
3
b_i	2	2	2	1	1
n_{ij}	2	2,1	1	2	1
n_i	4	3	2	2	1

The first structure is used when two second-stage samples are taken from each first-stage sample and in turn, two third-stage samples are taken from each second-stage sample. Similarly, the second structure is used when two second-stage samples are taken from each first-stage sample and in one of the second-stage samples, two third-stage samples are selected, while in the other second-stage sample, only one third-stage sample is selected.... Likewise, the fifth structure is used in a design when only one second-stage sample is selected from each first-stage sample, and in turn only one third-stage sample is selected from each second-stage sample.

2.4 Identification of Previously Enumerated Designs

A balanced design is made up entirely of replications of structure 1 by itself. Anderson and Bancroft [1952], Anderson [1960] and Anderson [1966] suggested a "staggered" design, called an Anderson

design throughout this study, which is made up of an equal number of replications of the two structures 1 and 3. Additionally, a new design is studied, called a modified Anderson design, which is made up of four replications of structure 3 for each replication of structure 1. A Calvin-Miller [1961] design, is made up entirely of replications of structure 2 by itself. A Bainbridge [1963] design is made up of equal replications of each of the structures 2, 3 and 5. Prairie's [1962] designs are denoted by a code e.g., Prairie (8:D1)--this indicates that the design is a Prairie D1 design, as described in section 1.2, based on the degrees of freedom assignment number eight. The exception to this coding rule is the design denoted as Prairie (1), which is the balanced design.

2.5 Design Enumeration

The designs were restricted to contain $12r$ observations, $r = 1(1)10$, and as many as three fundamental structures found in table 2.3.1 such that each of the three variance components could be estimated by means of the standard analysis of variance estimators. The number 12 was chosen as a multiplier of r since it is the lowest common multiple of the n_1 found in table 2.3.1. Originally it was thought that letting r go as high as 20 would be necessary, however, preliminary investigations showed that there was very little change in the results between $r = 11$ and $r = 20$. By varying r from 1 to 10, samples sizes 12 through 120 are encountered and these cover many common sizes of experiments.

These restrictions enabled the enumeration of 61 different designs that are naturally classified into ten not necessarily mutually

exclusive classes. Of these 61 designs, only 11 have been suggested previously by other authors.

In each of the ten following tables, the designs are systematically enumerated, numbered, identified by a design code as well as having a set of coefficient numbers which will be useful for computing the expectations of the mean squares in the standard analysis of variance table. Tables 2.5.2 to 2.5.9 have an additional column to indicate that a particular design has been previously enumerated in another class, and thus identify it.

Table 2.5.1. Class I Designs

Design	Structure			Code	Coefficient Numbers			
	1	3	5		c	d	e	g
1.1	1	0	8	(9,1,2)	60	8	12	2
1.2	1	1	6	(8,2,2)	54	8	13	3/2
1.3	1	2	4	(7,3,2)	48	8	14	4/3
1.4	1	3	2	(6,4,2)	42	8	15	5/4
1.5	1	4	0	(5,5,2)	36	8	16	2
1.6	2	0	4	(6,2,4)	48	10	18	2
1.7	2	1	2	(5,3,4)	42	10	19	5/3
1.8	2	2	0	(4,4,4)	36	10	20	3/2
1.9	3	0	0	(3,3,6)	36	12	24	2

Table 2.5.2. Class II Designs

Design	Structure			Code	Coefficient Numbers			
	2	3	5		c	d	e	g
2.1	1	0	9	(10,1,1)	64	7	9	4/3
2.2	1	1	7	(9,2,1)	58	7	10	7/6
2.3	1	2	5	(8,3,1)	52	7	11	10/9
2.4	1	3	3	(7,4,1)	46	7	12	13/12
2.5	1	4	1	(6,5,1)	40	7	13	16/15
2.6	2	0	6	(8,2,2)	56	8	12	4/3
2.7	2	1	4	(7,3,2)	50	8	13	11/9
2.8	2	2	2	(6,4,2)	44	8	14	7/6
2.9	2	3	0	(5,5,2)	38	8	15	17/15
2.10	3	0	3	(6,3,3)	48	9	15	4/3
2.11	3	1	1	(5,4,3)	42	9	16	5/4
2.12	4	0	0	(4,4,4)	40	10	18	4/3

Table 2.5.3. Class III Designs

Design	Structure			Code	Coefficient Numbers				Previous Number
	1	4	5		c	d	e	g	
3.1	1	0	8	(9,1,2)	60	8	12	2	(1.1)
3.2	1	1	6	(8,1,3)	60	9	13	2	
3.3	1	2	4	(7,1,4)	60	10	14	2	
3.4	1	3	2	(6,1,5)	60	11	15	2	
3.5	1	4	0	(5,1,6)	60	12	16	2	
3.6	2	0	4	(6,2,4)	48	10	18	2	(1.6)
3.7	2	1	2	(5,2,5)	48	11	19	2	
3.8	2	2	0	(4,2,6)	48	12	20	2	
3.9	3	0	0	(3,3,6)	36	12	24	2	(1.9)

Table 2.5.4. Class IV Designs

Design	Structure			Code	Coefficient Numbers				Previous Number
	2	4	5		c	d	e	g	
4.1	1	0	9	(10,1,1)	64	7	9	4/3	(2.1)
4.2	1	1	7	(9,1,2)	64	8	10	4/3	
4.3	1	2	5	(8,1,3)	64	9	11	4/3	
4.4	1	3	3	(7,1,4)	64	10	12	4/3	
4.5	1	4	1	(6,1,5)	64	11	13	4/3	
4.6	2	0	6	(8,2,2)	56	8	12	4/3	(2.6)
4.7	2	1	4	(7,2,3)	56	9	13	4/3	
4.8	2	2	2	(6,2,4)	56	10	14	4/3	
4.9	2	3	0	(5,2,5)	56	11	15	4/3	
4.10	3	0	3	(6,3,3)	48	9	15	4/3	(2.10)
4.11	3	1	1	(5,3,4)	48	10	16	4/3	
4.12	4	0	0	(4,4,4)	40	10	18	4/3	(2.12)

Table 2.5.5. Class V Designs

Design	Structure			Code	Coefficient Numbers				Previous Number
	1	2	5		c	d	e	g	
5.1	1	0	8	(9,1,2)	60	8	12	2	(1.1)
5.2	1	1	5	(7,2,3)	52	9	15	5/3	
5.3	1	2	2	(5,3,4)	44	10	18	14/9	
5.4	2	0	4	(6,2,4)	48	10	18	2	(1.6)
5.5	2	1	1	(4,3,5)	40	11	21	16/9	
5.6	3	0	0	(3,3,6)	36	12	24	2	(1.9)

Table 2.5.6. Class VI Designs

Design	Structure			Code	Coefficient Numbers				Previous Number
	1	2	3		c	d	e	g	
6.1	1	0	4	(5,5,2)	36	8	16	2	(1.5)
6.2	1	2	1	(4,4,4)	38	10	19	17/12	
6.3	2	0	2	(4,4,4)	36	10	20	3/2	(1.8)
6.4	3	0	0	(3,3,6)	36	12	24	2	(1.9)

Table 2.5.7. Class VII Designs

Design	Structure			Code	Coefficient Numbers				Previous Number
	1	2	4		c	d	e	g	
7.1	1	0	4	(5,1,6)	60	12	16	2	(3.5)
7.2	1	2	1	(4,3,5)	44	11	19	14/9	
7.3	2	0	2	(4,2,6)	48	12	20	2	(3.8)
7.4	3	0	0	(3,3,6)	36	12	24	2	(1.9)

Table 2.5.8. Class VIII Designs

Design	Structure			Code	Coefficient Numbers				Previous Number
	1	3	4		c	d	e	g	
8.1	1	0	4	(5,1,6)	60	12	16	2	(3.5)
8.2	1	1	3	(5,2,5)	54	11	16	3/2	
8.3	1	2	2	(5,3,4)	48	10	16	4/3	
8.4	1	3	1	(5,4,3)	42	9	16	5/4	
8.5	1	4	0	(5,5,2)	36	8	16	2	(1.5)
8.6	2	0	2	(4,2,6)	48	12	20	2	(3.8)
8.7	2	1	1	(4,3,5)	42	11	20	5/3	
8.8	2	2	0	(4,4,4)	36	10	20	3/2	(1.8)
8.9	3	0	0	(3,3,6)	36	12	24	2	(1.9)

Table 2.5.9. Class IX Designs

Design	Structure			Code	Coefficient Numbers				Previous Number
	2	3	4		c	d	e	g	
9.1	2	0	3	(5,2,5)	56	11	15	4/3	(4.9)
9.2	2	1	2	(5,3,4)	50	10	15	11/9	
9.3	2	2	1	(5,4,3)	44	9	15	7/6	
9.4	2	3	0	(5,5,2)	38	8	15	17/15	(2.9)
9.5	4	0	0	(4,4,4)	40	10	18	4/3	(2.12)

Table 2.5.10. Class X Designs

Design	Structure			Code	Coefficient Numbers			
	3	4	5		c	d	e	g
10.1	1	1	8	(10,1,1)	66	7	8	1
10.2	1	2	6	(9,1,2)	66	8	9	1
10.3	1	3	4	(8,1,3)	66	9	10	1
10.4	1	4	2	(7,1,4)	66	10	11	1
10.5	1	5	0	(6,1,5)	66	11	12	1
10.6	2	1	6	(9,2,1)	60	7	9	1
10.7	2	2	4	(8,2,2)	60	8	10	1
10.8	2	3	2	(7,2,3)	60	9	11	1
10.9	2	4	0	(6,2,4)	60	10	12	1
10.10	3	1	4	(8,3,1)	54	7	10	1
10.11	3	2	2	(7,3,2)	54	8	11	1
10.12	3	3	0	(6,3,3)	54	9	12	1
10.13	4	1	2	(7,4,1)	48	7	11	1
10.14	4	2	0	(6,4,2)	48	8	12	1
10.15	5	1	0	(6,5,1)	42	7	12	1

Some of the designs enumerated have been suggested by other authors. This correspondence is indicated in Table 2.5.11.

Table 2.5.11. Correspondence between enumerated designs and those suggested by other workers

Design in Present Study	Presented in Table	Previous Name(s)
1.2	2.5.1	Prairie (8:D1)
1.4	2.5.1	Prairie (7:D1)
1.5	2.5.1	Modified Anderson
1.6	2.5.1	Prairie (4:D1)
1.8	2.5.1	Anderson; Prairie (3:D1)
1.9	2.5.1	Balanced; Prairie (1)
2.8	2.5.2	Bainbridge
2.12	2.5.2	Calvin-Miller; Prairie (3:D2)
10.7	2.5.10	Prairie (8:D2)
10.9	2.5.10	Prairie (4:D2)
10.14	2.5.10	Prairie (7:D2)

Classes 2, 5 and 6 in Prairie's enumeration are not considered by this scheme, however, they would be enumerated if splits of size greater than two were allowed when forming the fundamental structures.

For the tables 2.5.1 to 2.5.10, the designs are systematically numbered, the fundamental structures that make up the design are identified, as well as a coefficient, which when multiplied by r determines how many of that type of structure appears in the design. The code that accompanies each design is a 3-tuple (u,v,w) which serves as a design code as well as a way of identifying the degrees of freedom breakdown in the analysis of variance table of a three-stage nested design. The degrees of freedom for a design can be determined by means of the formulae in the following table.

Table 2.5.12. Correspondence between the Degrees of Freedom and the Design Codes

Source	Formula
A	$a - 1 = ur - 1$
B in A	$b - a = vr$
Error	$n - b = wr$

The coefficient numbers c, d, e and g are used to determine the coefficients for the expected mean squares in the analysis of variance table of a three stage nested design.

$$k_1 = \frac{cr-d}{6(a-1)}, \quad k_2 = \frac{72r-e}{6(a-1)}, \quad k_3 = g \quad (2.5.1)$$

The coefficients k_1 , k_2 and k_3 will be identified in Chapter 3.

2.6 An Example

Suppose an experimenter decided to run an experiment with design 2.8. In table 2.5.2 it is seen that design 2.8 is the eighth design in the second class of designs that were systematically enumerated from the fundamental structures 2, 3 and 5. The integers 2 2 2 beneath the pictorial representations of the three structures indicate that the experimenter would select 2r replicates of each of the structures 2, 3 and 5. Pictorially, the experimental plan for $r = 1$ would be the appropriate randomization of the following sampling pattern.

The design code (6,4,2) indicates that there are $6r - 1$ degrees of freedom for the A-classes in the analysis of variance table, $4r$ d.f. for the B in A classes source and $2r$ d.f. for the error source. From the coefficient numbers, the expected mean square coefficients are:

$$k_1 = \frac{44r-8}{6(6r-1)} , \quad k_2 = \frac{72r-14}{6(6r-1)} , \quad k_3 = \frac{7}{6} .$$

Some of these coefficients would quite obviously simplify; however, this form permits the same usage for each of the 61 designs.

CHAPTER 3. VARIANCE COMPONENT ESTIMATION AND PRECISION

3.1 Introduction

In this chapter, the ANOVA estimators, their variances and covariances will be derived for the three variance components. An example is presented. The implication of negative variance component estimates will be cited in the summary but not explicitly studied here.

3.2 Variance Component Estimators

In this study, only the analysis of variance estimators of the variance components will be considered. It will be helpful to have the notation solidified in the standard analysis of variance table.

Table 3.2.1. Analysis of Variance for a General Three-Stage Nested Design

Source	d.f.	M.S.	E(MS)
A-classes	a-1	MSA	$\sigma_e^2 + k_1\sigma_b^2 + k_2\sigma_a^2$
B in A classes	b-a	MSB	$\sigma_e^2 + k_3\sigma_b^2$
Error	n-b	MSE	σ_e^2
Total	n-1		

$$\frac{1}{k_1} = \sum_{i=1}^a \sum_{j=1}^{b_i} n_{ij}^2 f_i, \quad k_2 = \sum_{i=1}^a n_i^2 f_i, \quad k_3 = \sum_{i=1}^a \sum_{j=1}^{b_i} n_{ij}^2 f_{ij}$$

¹Since the summation over i will always range from 1 through a and the summation over j will always range from 1 through b_i, in future statements, the ranges will be omitted. Only the appropriate subscript will be used to indicate the type of summation.

where $f_i = \frac{n-n_i}{nn_1(a-1)}$ and $f_{ij} = \frac{n_i-n_{ij}}{n_1n_{ij}(b-a)}$.

The analysis of variance estimators are formed by setting the appropriate mean squares equal to their expectations so that

$$\hat{\sigma}_e^2 = \text{MSE} \quad (3.2.1)$$

$$\hat{\sigma}_b^2 = \frac{\text{MSB}-\text{MSE}}{k_3} \quad (3.2.2)$$

$$\hat{\sigma}_a^2 = \frac{k_3\text{MSA} - k_1\text{MSB} - (k_3-k_1)\text{MSE}}{k_2k_3} \quad (3.2.3)$$

Their variances are:

$$\text{var}(\hat{\sigma}_e^2) = \text{var}(\text{MSE}) \quad (3.2.4)$$

$$\text{var}(\hat{\sigma}_b^2) = \frac{1}{k_3^2} \{ \text{var}(\text{MSB}) + \text{var}(\text{MSE}) - 2 \text{cov}(\text{MSB}, \text{MSE}) \} \quad (3.2.5)$$

$$\begin{aligned} \text{var}(\hat{\sigma}_a^2) = & \frac{1}{k_2^2k_3^2} \{ k_3^2 \text{var}(\text{MSA}) + k_1^2 \text{var}(\text{MSB}) + (k_3-k_1)^2 \text{var}(\text{MSE}) \\ & - 2k_1k_3 \text{cov}(\text{MSA}, \text{MSB}) - 2k_3(k_3-k_1) \text{cov}(\text{MSA}, \text{MSE}) \\ & + 2k_1(k_3-k_1) \text{cov}(\text{MSB}, \text{MSE}) \} \end{aligned} \quad (3.2.6)$$

and the covariances are:

$$\text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_b^2) = \frac{-1}{k_3} \text{var}(\text{MSE}) \quad (3.2.7)$$

$$\text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_a^2) = \frac{-(k_3-k_1)}{k_2k_3} \text{var}(\text{MSE}) \quad (3.2.8)$$

$$\begin{aligned} \text{cov}(\hat{\sigma}_b^2, \hat{\sigma}_a^2) = & \frac{1}{k_2k_3^2} \{ -k_1 \text{var}(\text{MSB}) + k_3 \text{cov}(\text{MSA}, \text{MSB}) \\ & + (k_3-k_1) \text{var}(\text{MSE}) \} \end{aligned} \quad (3.2.9)$$

In order to obtain a computational form for these variances and

covariances, it will help to have the variances and covariances of quadratic forms in normally distributed variables.

3.3 Expectations, Variances and Covariances of Mean Squares

It is quite well known, Whittle [1953], that if $y_1, \dots, y_n$ is multivariate normal $N(\underline{0}, \underline{V})$, and, $Q = \underline{y}' \underline{M} \underline{y}$ and $P = \underline{y}' \underline{N} \underline{y}$ are two quadratic forms in $\underline{y}$, then

$$E\{Q\} = \text{tr} \{ \underline{V} \underline{M} \} \quad (3.3.1)$$

$$\text{var}\{Q\} = 2 \text{tr} \{ \underline{V} \underline{M} \underline{V} \underline{M} \} \quad (3.3.2)$$

$$\text{cov}\{Q, P\} = 2 \text{tr} \{ \underline{V} \underline{M} \underline{V} \underline{N} \} \quad (3.3.3)$$

where $\underline{M}$ and $\underline{N}$ are symmetric matrices of rank q and p respectively. Two general properties of the trace operator,

$$\text{tr}(\underline{A} \underline{B}) = \text{tr}(\underline{B} \underline{A}), \quad \text{tr}(\underline{A} \underline{B} + \underline{C} \underline{D}) = \text{tr}(\underline{A} \underline{B}) + \text{tr}(\underline{C} \underline{D}) \quad (3.3.4)$$

provided all of the multiplication and the addition operations are conformable, will be used quite extensively in arriving at expressions for the variances and covariances of the variance component estimators.

Suppose $\underline{0}_{ij}$ is a column vector with n_{ij} elements all zeros, $(n_{ij} \times 1)$ and that $\underline{1}_{ij}$ is a column vector with n_{ij} elements all unity, then define $(n_{ij} \times 1)$

$$\underline{W}'_{ij} = [\underline{0}'_{i1} \mid \underline{0}'_{i2} \mid \cdots \mid \underline{0}'_{i,j-1} \mid \underline{1}'_{ij} \mid \underline{0}'_{i,j+1} \mid \cdots \mid \underline{0}'_{ib_i}] ,$$

$(1 \times n_i)$

a row vector that contains a unit row vector in the j -th position. It

will also be expedient to define $\underline{1}_i$ as a column vector with n_i elements all unity, $(n_i \times 1)$

elements all unity, $\underline{1}$ a column vector with n elements all unity, $(n \times 1)$

$\tilde{J}$ a square matrix with all elements unity where the order will be n_{ij} if the subscript ij is used, n_i if the subscript i is used and n if no subscript is used. Likewise, let $\tilde{I}$ denote an identity matrix where the order is n_{ij} if the subscript is ij , n_i if the subscript is i and n if there is no subscript.

The following properties will be used extensively:

$$\begin{matrix} \tilde{W}'_{ij} & \tilde{W}_{im} \\ (1 \times n_i) & (n_i \times 1) \end{matrix} = \begin{matrix} n_{ij} & , \text{ if } j = m \\ 0 & , \text{ otherwise} \end{matrix}$$

$$\begin{matrix} \tilde{W}'_{ij} & \tilde{1}_i \\ (1 \times n_i) & (n_i \times 1) \end{matrix} = \begin{matrix} \tilde{1}'_i & \tilde{W}_{ij} \\ (1 \times n_i) & (n_i \times 1) \end{matrix} = n_{ij}$$

$$\begin{matrix} \tilde{W}_{ij} & \tilde{W}'_{ij} & \tilde{1}_i & \tilde{1}'_i \\ (n_i \times 1) & (1 \times n_i) & (n_i \times 1) & (1 \times n_i) \end{matrix} = n_{ij} \begin{matrix} \tilde{W}_{ij} & \tilde{1}'_i \\ (n_i \times 1) & (1 \times n_i) \end{matrix}$$

$$\begin{matrix} \tilde{1}'_{ij} & \tilde{1}_{ij} \\ (1 \times n_{ij}) & (n_{ij} \times 1) \end{matrix} = n_{ij}$$

$$\begin{matrix} \tilde{1}'_i & \tilde{1}_i \\ (1 \times n_i) & (n_i \times 1) \end{matrix} = n_i$$

$$\begin{matrix} \tilde{1}' & \tilde{1} \\ (1 \times n) & (n \times 1) \end{matrix} = n$$

$$\begin{matrix} \tilde{1}_{ij} & \tilde{1}'_{ij} \\ (n_{ij} \times 1) & (1 \times n_{ij}) \end{matrix} = \tilde{J}_{ij}$$

$$\begin{matrix} \tilde{1}_i & \tilde{1}'_i \\ (n_i \times 1) & (1 \times n_i) \end{matrix} = \tilde{J}_i$$

$$\begin{matrix} \tilde{1} & \tilde{1}' \\ (n \times 1) & (1 \times n) \end{matrix} = \tilde{J}$$

$$\begin{matrix} \mathbf{1}_{\sim i} & \mathbf{1}'_{\sim i} & \mathbf{1}_{\sim i} & \mathbf{1}'_{\sim i} \\ (n_i \times 1) & (1 \times n_i) & (n_i \times 1) & (1 \times n_i) \end{matrix} = \mathbf{J}_{\sim i}^2 = n_i \mathbf{J}_{\sim i}$$

$$\begin{matrix} \mathbf{1}_{\sim ij} & \mathbf{1}'_{\sim ij} & \mathbf{1}_{\sim ij} & \mathbf{1}'_{\sim ij} \\ (n_{ij} \times 1) & (1 \times n_{ij}) & (n_{ij} \times 1) & (1 \times n_{ij}) \end{matrix} = \mathbf{J}_{\sim ij}^2 = n_{ij} \mathbf{J}_{\sim ij}$$

$$\text{tr} \left(\begin{matrix} \mathbf{W}_{\sim ij} & \mathbf{1}'_{\sim i} \\ (n_i \times 1) & (1 \times n_i) \end{matrix} \right) = n_{ij}$$

$$\text{tr}(\mathbf{I}_{\sim}) = \text{tr}(\mathbf{J}_{\sim}) = n$$

$$\text{tr}(\mathbf{I}_{\sim i}) = \text{tr}(\mathbf{J}_{\sim i}) = n_i$$

$$\text{tr}(\mathbf{I}_{\sim ij}) = \text{tr}(\mathbf{J}_{\sim ij}) = n_{ij}.$$

The general form of $\mathbf{V}_{\sim}$, the covariance matrix of $\mathbf{y}_{\sim}$, is a block diagonal matrix

$$\mathbf{V}_{\sim} = \begin{bmatrix} \mathbf{V}_{\sim 1} & & & & \\ & \mathbf{V}_{\sim 2} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \mathbf{V}_{\sim a} \end{bmatrix} \quad (3.3.5)$$

where

$$\mathbf{V}_{\sim i} = \sigma_e^2 \mathbf{I}_{\sim i} + \sigma_b^2 \sum_j \mathbf{W}_{\sim ij} \mathbf{W}'_{\sim ij} + \sigma_a^2 \mathbf{1}_{\sim i} \mathbf{1}'_{\sim i}.$$

In order to make use of these properties for computing the expectations, variances and covariances of the mean square, it will be necessary to know the structure of the matrix of the quadratic form used in computing the mean square. The following relationships give this information.

$$MSA = \frac{1}{a-1} \mathbf{y}'_{\sim} \mathbf{A} \mathbf{y}_{\sim} = \frac{P}{a-1}$$

where $\mathbf{A} = \text{diag} \left(\frac{1}{n_i} \mathbf{1}_{\sim i} \mathbf{1}'_{\sim i} \right) - \frac{1}{n} \mathbf{1}_{\sim} \mathbf{1}'_{\sim}$. (3.3.6)

$$MSB = \frac{1}{b-a} \mathbf{y}'_{\sim} \mathbf{B} \mathbf{y}_{\sim} = \frac{Q}{b-a}$$

where $\mathbf{B} = \text{diag} (\mathbf{B}_i)$ and (3.3.7)

$$\mathbf{B}_i = \sum_j \frac{1}{n_{ij}} \mathbf{w}_{ij} \mathbf{w}'_{ij} - \frac{1}{n_i} \mathbf{1}_{\sim i} \mathbf{1}'_{\sim i} .$$

$$MSE = \frac{1}{n-b} \mathbf{y}'_{\sim} \mathbf{E} \mathbf{y}_{\sim} = \frac{R}{n-b} \quad (3.3.8)$$

where $\mathbf{E} = \text{diag} (\mathbf{E}_i)$ and

$$\mathbf{E}_i = \mathbf{I}_{\sim i} - \sum_j \frac{1}{n_{ij}} \mathbf{w}_{ij} \mathbf{w}'_{ij} .$$

Now, by computing the expectations, variances and covariances of the quadratic forms P, Q, and R, it will be a simple matter to deduce the expectations, variances and covariances of MSA, MSB and MSE.

Consider first of all, the expectations and variances of the quadratic forms P, Q and R.

$$P = \mathbf{y}'_{\sim} \mathbf{A} \mathbf{y}_{\sim}$$

Let $r_i = \frac{1}{n_i} - \frac{1}{n}$, then from (3.3.5) and (3.3.6), we have

$$\mathbf{V} \mathbf{A} = \begin{bmatrix} r_1 \mathbf{V}_{\sim 1 \sim 1} \mathbf{1}_{\sim 1} \mathbf{1}'_{\sim 1} & -\frac{1}{n} \mathbf{V}_{\sim 1 \sim 1 \sim 2} \mathbf{1}_{\sim 1} \mathbf{1}'_{\sim 2} & \cdots & -\frac{1}{n} \mathbf{V}_{\sim 1 \sim 1 \sim a} \mathbf{1}_{\sim 1} \mathbf{1}'_{\sim a} \\ -\frac{1}{n} \mathbf{V}_{\sim 2 \sim 2 \sim 1} \mathbf{1}_{\sim 2} \mathbf{1}'_{\sim 1} & r_2 \mathbf{V}_{\sim 2 \sim 2 \sim 2} \mathbf{1}_{\sim 2} \mathbf{1}'_{\sim 2} & \cdots & -\frac{1}{n} \mathbf{V}_{\sim 2 \sim 2 \sim a} \mathbf{1}_{\sim 2} \mathbf{1}'_{\sim a} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} \mathbf{V}_{\sim a \sim a \sim 1} \mathbf{1}_{\sim a} \mathbf{1}'_{\sim 1} & \cdot & \cdot & r_a \mathbf{V}_{\sim a \sim a \sim a} \mathbf{1}_{\sim a} \mathbf{1}'_{\sim a} \end{bmatrix}$$

Hence

$$\begin{aligned}
 E(P) &= \text{tr}(\underline{VA}) = \sum_i \text{tr}(r_i \underline{V}_{i\sim i} \underline{1} \underline{1}') \\
 &= \sum_i r_i \text{tr} \{ [\sigma_e^2 \underline{I}_{i\sim i} + \sigma_b^2 \sum_j \underline{W}_{ij} \underline{W}_{ij}' + \sigma_a^2 \underline{1}_{i\sim i} \underline{1}_{i\sim i}'] \} \\
 &= \sum_i r_i \{ \sigma_e^2 \text{tr}[\underline{1}_{i\sim i} \underline{1}_{i\sim i}'] + \sigma_b^2 \sum_j \text{tr}[\underline{W}_{ij} \underline{W}_{ij}' \underline{1}_{i\sim i} \underline{1}_{i\sim i}'] + \sigma_a^2 \text{tr}[\underline{1}_{i\sim i} \underline{1}_{i\sim i} \underline{1}_{i\sim i} \underline{1}_{i\sim i}'] \} \\
 &= \sigma_e^2 \sum_i r_i n_i + \sigma_b^2 \sum_i \sum_j r_i n_{ij}^2 + \sigma_a^2 \sum_i r_i n_i^2 \\
 &= \sigma_e^2 (a-1) + \sigma_b^2 \sum_i \sum_j r_i n_{ij}^2 + \sigma_a^2 \sum_i r_i n_i^2 . \tag{3.3.9}
 \end{aligned}$$

Combining (3.3.6) and (3.3.9), after some algebra,

$$\begin{aligned}
 E(\text{MSA}) &= \sigma_e^2 + \sigma_b^2 \sum_i \sum_j n_{ij}^2 f_i + \sigma_a^2 \sum_i n_i^2 f_i \\
 &= \sigma_e^2 + k_1 \sigma_b^2 + k_2 \sigma_a^2 .
 \end{aligned}$$

In order to compute $\text{var}(P)$, $\text{tr}(\underline{VAVA})$ is needed. The diagonal submatrix of $\underline{VAVA}$ is

$$(r_i \underline{V}_{i\sim i} \underline{1}_{i\sim i} \underline{1}_{i\sim i}')^2 + \frac{1}{n} \sum_{\substack{\ell \\ \ell \neq i}} (\underline{V}_{i\sim i} \underline{1}_{i\sim i} \underline{1}_{i\sim i}') (\underline{V}_{\ell\sim \ell} \underline{1}_{\ell\sim \ell} \underline{1}_{\ell\sim \ell}')$$

and

$$\begin{aligned}
 \text{var}(P) &= 2 \text{tr}(\underline{VAVA}) = 2 \left\{ \sum_i \text{tr}(r_i \underline{V}_{i\sim i} \underline{1}_{i\sim i} \underline{1}_{i\sim i}')^2 \right. \\
 &\quad \left. + \frac{1}{n} \sum_i \sum_{\substack{\ell \\ i \neq \ell}} \text{tr}[(\underline{V}_{i\sim i} \underline{1}_{i\sim i} \underline{1}_{i\sim i}') (\underline{V}_{\ell\sim \ell} \underline{1}_{\ell\sim \ell} \underline{1}_{\ell\sim \ell}')] \right\} \tag{3.3.10}
 \end{aligned}$$

Consider, first of all, the first term in the brackets on the right hand side of (3.3.10)

$$\sum_i \text{tr}(\underline{V}_{i\sim i} \underline{1}_{i\sim i} \underline{1}_{i\sim i}')^2 = \sum_i r_i^2 \text{tr}(\underline{V}_{i\sim i} \underline{1}_{i\sim i} \underline{1}_{i\sim i}')^2 \tag{3.3.11}$$

In the following, we will only consider the trace part of (3.3.11); replacing the summation and r_i^2 when the value for the trace is known.

Using (3.3.5),

$$\begin{aligned}
 \text{tr}(V_{\sim i \sim i \sim i} 1 1')^2 &= \text{tr}(\sigma_e^2 1_{\sim i \sim i} 1' + \sigma_b^2 \sum_j W_{ij} W'_{ij} 1_{\sim i \sim i} 1' + \sigma_a^2 1_{\sim i \sim i \sim i \sim i} 1 1')^2 \\
 &= \text{tr}(\sigma_e^4 1_{\sim i \sim i \sim i \sim i} 1 1' + \sigma_b^4 [\sum_j n_{ij} W_{ij} 1'] [\sum_m n_{im} W_{im} 1'] \\
 &\quad + \sigma_a^4 n_i^2 1_{\sim i \sim i \sim i \sim i} 1 1' + 2\sigma_e^2 \sigma_b^2 \sum_j n_{ij} 1_{\sim i \sim i} W_{ij} 1' \\
 &\quad + 2\sigma_e^2 \sigma_a^2 n_i 1_{\sim i \sim i \sim i \sim i} 1 1' + 2\sigma_b^2 \sigma_a^2 n_i \sum_j n_{ij} W_{ij} 1' 1 1') \\
 &\quad (3.3.12)
 \end{aligned}$$

Now, if (3.3.4) is invoked, along with some of the earlier defined properties, and noticing that

$$\begin{aligned}
 \text{tr}(\sum_j n_{ij} W_{ij} 1') (\sum_m n_{im} W_{im} 1') &= \text{tr}(\sum_j n_{ij}^3 W_{ij} 1') \\
 + \text{tr}(\sum_{j \neq m} \sum_m n_{ij} n_{im}^2 W_{ij} 1') &= \sum_j n_{ij}^4 + \sum_{j \neq m} \sum_m n_{ij}^2 n_{im}^2 = (\sum_j n_{ij}^2)^2
 \end{aligned}$$

then

$$\begin{aligned}
 \text{tr}(V_{\sim i \sim i \sim i} 1 1')^2 &= \sigma_e^4 n_i^2 + \sigma_b^4 (\sum_j n_{ij}^2)^2 + \sigma_a^4 n_i^4 \\
 + 2\sigma_e^2 \sigma_b^2 n_i \sum_j n_{ij}^2 &+ 2\sigma_e^2 \sigma_a^2 n_i^3 + 2\sigma_b^2 \sigma_a^2 n_i \sum_j n_{ij}^2 \quad (3.3.13)
 \end{aligned}$$

In a similar fashion, consider the second term in the brackets on the right hand side of (3.3.10)

$$\frac{1}{n^2} \sum_{i \neq l} \sum_l \text{tr}(V_{\sim i \sim i \sim l} 1 1') (V_{\sim l \sim l \sim i} 1 1') \quad (3.3.14)$$

Considering just the trace part of (3.3.14), then

$$\begin{aligned}
 \text{tr}(V_{i\sim i\sim l}^{1,1'}) (V_{l\sim l\sim i}^{1,1'}) &= \text{tr}\{(\sigma_e^2 \mathbf{1}_{i\sim l}^{1,1'} + \sigma_b^2 \sum_j n_{ij\sim ij\sim l}^{W_{1,1'}} \\
 &+ \sigma_a^2 n_{i\sim i\sim l}^{1,1'}) (\sigma_e^2 \mathbf{1}_{l\sim i}^{1,1'} + \sigma_b^2 \sum_m n_{lm\sim lm\sim i}^{W_{1,1'}} + \sigma_a^2 n_{l\sim l\sim i}^{1,1'})\} \\
 &= \text{tr}\{\sigma_e^4 \mathbf{1}_{i\sim l\sim l\sim i}^{1,1'} + \sigma_b^4 (\sum_j n_{ij\sim ij\sim l}^{W_{1,1'}}) (\sum_m n_{lm\sim lm\sim i}^{W_{1,1'}}) \\
 &+ \sigma_a^4 n_{i\sim l\sim i\sim l\sim i}^{1,1'} + \sigma_e^2 \sigma_b^2 (\sum_m n_{lm\sim i\sim l\sim lm\sim i}^{1,1'} W_{1,1'} + \sum_j n_{ij\sim ij\sim l\sim l\sim i}^{1,1'} W_{1,1'}) \\
 &+ \sigma_e^2 \sigma_a^2 (n_{l\sim i\sim l\sim l\sim i}^{1,1'} + n_{i\sim i\sim l\sim l\sim i}^{1,1'}) \\
 &+ \sigma_b^2 \sigma_a^2 (n_l \sum_j n_{ij\sim ij\sim l\sim l\sim i}^{1,1'} + n_i \sum_m n_{lm\sim i\sim l\sim lm\sim i}^{1,1'})\} \quad (3.3.15)
 \end{aligned}$$

Again, if (3.3.4) is invoked, along with some of the earlier defined properties, then

$$\begin{aligned}
 \text{tr}(V_{i\sim i\sim l}^{1,1'}) (V_{l\sim l\sim i}^{1,1'}) &= \sigma_e^4 n_i n_l + \sigma_b^4 \sum_j \sum_m n_{ij}^2 n_{lm}^2 \\
 &+ \sigma_a^2 n_i^2 n_l^2 + \sigma_e^2 \sigma_b^2 (n_i \sum_m n_{lm}^2 + n_l \sum_j n_{ij}^2) \\
 &+ \sigma_e^2 \sigma_a^2 (n_i^2 n_l^2 + n_i^2 n_l) + \sigma_b^2 \sigma_a^2 (n_i^2 \sum_m n_{lm}^2 + n_l^2 \sum_j n_{ij}^2) \quad (3.3.16)
 \end{aligned}$$

If we now substitute (3.3.16) into (3.3.14) and (3.3.13) into (3.3.11), and substitute (3.3.14) and (3.3.11) into (3.3.10), then

$$\begin{aligned}
 \text{var}(P) &= 2\{\sigma_e^4 (\sum_i r_i^2 n_i^2 + \frac{1}{n^2} \sum_i \sum_{l \neq i} n_i n_l) \\
 &+ \sigma_b^4 (\sum_i r_i^2 [\sum_j n_{ij}^2]^2 + \frac{1}{n^2} \sum_i \sum_{l \neq i} \sum_j \sum_m n_{ij}^2 n_{lm}^2)
 \end{aligned}$$

$$\begin{aligned}
& + \sigma_a^4 \left(\sum_i r_i^2 n_i^4 + \frac{1}{n^2} \sum_{\substack{i \neq \ell}} \sum_{\ell} n_i^2 n_{\ell}^2 \right) \\
& + \sigma_e^2 \sigma_b^2 \left(2 \sum_i r_i^2 n_i^2 \sum_j n_{ij}^2 + \frac{1}{n^2} \sum_{\substack{i \neq \ell}} \sum_{\ell} [n_i \sum_m n_{\ell m}^2 + n_{\ell} \sum_j n_{ij}^2] \right) \\
& + \sigma_e^2 \sigma_a^2 \left(2 \sum_i r_i^2 n_i^3 + \frac{1}{n^2} \sum_{\substack{i \neq \ell}} \sum_{\ell} [n_i n_{\ell}^2 + n_i^2 n_{\ell}] \right) \\
& + \sigma_b^2 \sigma_a^2 \left(2 \sum_i r_i^2 n_i^2 \sum_j n_{ij}^2 + \frac{1}{n^2} \sum_{\substack{i \neq \ell}} \sum_{\ell} [n_i^2 \sum_m n_{\ell m}^2 + n_{\ell}^2 \sum_j n_{ij}^2] \right) \}.
\end{aligned} \tag{3.3.17}$$

Equation (3.3.17) reduces to the computationally easier and more appealing form

$$\begin{aligned}
\text{var}(P) = & 2 \{ \sigma_e^4 (a-1) \\
& + \sigma_b^4 \left(\frac{1}{n^2} \left[n^2 \sum_i \frac{1}{n_i} \left(\sum_j n_{ij}^2 \right)^2 - 2n \sum_i \frac{1}{n_i} \left(\sum_j n_{ij}^2 \right)^2 + \left(\sum_i \sum_j n_{ij}^2 \right)^2 \right] \right) \\
& + \sigma_a^4 \left(\frac{1}{n^2} \left[n^2 \sum_i n_i^2 - 2n \sum_i n_i^3 + \left(\sum_i n_i^2 \right)^2 \right] \right) \\
& + \sigma_e^2 \sigma_b^2 \left(\frac{2}{n} \left[n \sum_i \frac{1}{n_i} \sum_j n_{ij}^2 - \sum_i \sum_j n_{ij}^2 \right] \right) + \sigma_e^2 \sigma_a^2 \left(\frac{2}{n} \left[n^2 - \sum_i n_i^2 \right] \right) \\
& + \sigma_b^2 \sigma_a^2 \left(\frac{2}{n^2} \left[n^2 \sum_i \sum_j n_{ij}^2 - 2n \sum_i n_i \sum_j n_{ij}^2 + \sum_i n_i^2 \sum_{\ell} \sum_j n_{\ell j}^2 \right] \right) \}.
\end{aligned} \tag{3.3.18}$$

Now, using (3.3.6), and since $a-1$ is fixed for a particular design

$$\text{var(MSA)} = \frac{1}{(a-1)^2} \text{var}(P) \tag{3.3.19}$$

where $\text{var}(P)$ is detailed in (3.3.18).

Consider next the quadratic form

$$Q = \underline{y}' \underline{B} \underline{y}.$$

Then from (3.3.5) and (3.3.7),

$$\underline{V}\underline{B} = \text{diag} (\underline{V}_i \underline{B}_i).$$

Hence

$$E(Q) = \text{tr}(\underline{V}\underline{B}) = \sum_i \text{tr}(\underline{V}_i \underline{B}_i) \quad (3.3.20)$$

Considering just the trace portion of (3.3.20),

$$\begin{aligned} \text{tr}(\underline{V}_i \underline{B}_i) &= \text{tr}\{(\sigma_e^2 \underline{I}_i + \sigma_b^2 \sum_j \underline{W}_{ij} \underline{W}'_{ij} + \sigma_a^2 \underline{1}_i \underline{1}'_i) \\ &\quad (\sum_m \frac{1}{n_{im}} \underline{W}_{im} \underline{W}'_{im} - \frac{1}{n_i} \underline{1}_i \underline{1}'_i)\} \\ &= \sigma_e^2 [\sum_j \frac{1}{n_{ij}} \text{tr}(\underline{W}_{ij} \underline{W}'_{ij}) - \frac{1}{n_i} \text{tr}(\underline{1}_i \underline{1}'_i)] \\ &\quad + \sigma_b^2 [\sum_j \sum_m \frac{1}{n_{im}} \text{tr}(\underline{W}_{ij} \underline{W}'_{ij} \underline{W}_{im} \underline{W}'_{im}) - \frac{1}{n_i} \sum_j \text{tr}(\underline{W}_{ij} \underline{W}'_{ij} \underline{1}_i \underline{1}'_i)] \\ &\quad + \sigma_a^2 [\sum_m \frac{1}{n_{im}} \text{tr}(\underline{1}_i \underline{1}'_i \underline{W}_{im} \underline{W}'_{im}) - \frac{1}{n_i} \text{tr}(\underline{1}_i \underline{1}'_i \underline{1}_i \underline{1}'_i)] \end{aligned}$$

which, after some simplification reduces to

$$\text{tr}(\underline{V}_i \underline{B}_i) = \sigma_e^2 (b_i - 1) + \sigma_b^2 (n_i - \frac{1}{n_i} \sum_j n_{ij}^2). \quad (3.3.21)$$

Substituting (3.3.21) into (3.3.20) and then (3.3.20) into (3.3.7),

gives

$$E(\text{MSB}) = \sigma_e^2 + \sigma_b^2 \sum_i \sum_j n_{ij}^2 f_{ij} = \sigma_e^2 + k_3 \sigma_b^2.$$

Now, consider $\text{var}(Q)$, which will be needed in the computation of $\text{var}(\text{MSB})$.

$$\text{var}(Q) = 2 \text{tr}(\underline{V}\underline{B})^2 = 2 \text{tr} \text{diag}(\underline{V}_i \underline{B}_i)^2 = 2 \sum_i \text{tr}(\underline{V}_i \underline{B}_i)^2. \quad (3.3.22)$$

Considering just the trace portion of (3.3.22),

$$\begin{aligned}
 \text{tr}(\mathbf{V}_{i\sim i} \mathbf{B}_i)^2 &= \text{tr} \left\{ \sigma_e^2 \left(\sum_j \frac{1}{n_{ij}} \mathbf{W}_{ij} \mathbf{W}'_{ij} - \frac{1}{n_i} \mathbf{1}_i \mathbf{1}'_i \right) \right. \\
 &\quad \left. + \sigma_b^2 \left(\sum_j \mathbf{W}_{ij} \mathbf{W}'_{ij} - \frac{1}{n_i} \sum_j \mathbf{W}_{ij} \mathbf{W}'_{ij} \mathbf{1}_i \mathbf{1}'_i \right) \right\}^2 \\
 &= \text{tr} \left\{ \sigma_e^4 \left(\sum_j \sum_m \frac{1}{n_{ij}} \frac{1}{n_{im}} \mathbf{W}_{ij} \mathbf{W}'_{ij} \mathbf{W}_{im} \mathbf{W}'_{im} \right. \right. \\
 &\quad \left. - \frac{2}{n_i} \sum_j \frac{1}{n_{ij}} \mathbf{W}_{ij} \mathbf{W}'_{ij} \mathbf{1}_i \mathbf{1}'_i + \frac{1}{n_i^2} \mathbf{1}_i \mathbf{1}'_i \mathbf{1}_i \mathbf{1}'_i \right) \\
 &\quad + \sigma_b^4 \left(\sum_j \mathbf{W}_{ij} \mathbf{W}'_{ij} \mathbf{W}_{ij} \mathbf{W}'_{ij} - \frac{2}{n_i} \sum_j \mathbf{W}_{ij} \mathbf{W}'_{ij} \mathbf{W}_{ij} \mathbf{W}'_{ij} \mathbf{1}_i \mathbf{1}'_i \right. \\
 &\quad \left. + \frac{1}{n_i^2} \left[\sum_j \mathbf{W}_{ij} \mathbf{W}'_{ij} \mathbf{1}_i \mathbf{1}'_i \mathbf{W}_{ij} \mathbf{W}'_{ij} \mathbf{1}_i \mathbf{1}'_i + \sum_{\substack{j, m \\ j \neq m}} \mathbf{W}_{ij} \mathbf{W}'_{ij} \mathbf{1}_i \mathbf{1}'_i \mathbf{W}_{im} \mathbf{W}'_{im} \mathbf{1}_i \mathbf{1}'_i \right] \right) \\
 &\quad + 2\sigma_e^2 \sigma_b^2 \left(\sum_j \frac{1}{n_{ij}} \mathbf{W}_{ij} \mathbf{W}'_{ij} \mathbf{W}_{ij} \mathbf{W}'_{ij} - \frac{1}{n_i} \sum_j \mathbf{1}_i \mathbf{1}'_i \mathbf{W}_{ij} \mathbf{W}'_{ij} \right. \\
 &\quad \left. + \frac{1}{n_i^2} \sum_j \mathbf{1}_i \mathbf{1}'_i \mathbf{W}_{ij} \mathbf{W}'_{ij} \mathbf{1}_i \mathbf{1}'_i - \frac{1}{n_i} \sum_j \frac{1}{n_{ij}} \mathbf{W}_{ij} \mathbf{W}'_{ij} \mathbf{W}_{ij} \mathbf{W}'_{ij} \mathbf{1}_i \mathbf{1}'_i \right) \left. \right\}
 \end{aligned}$$

Now, if (3.3.4) is invoked, along with some of the earlier defined properties, then

$$\begin{aligned}
 \text{tr}(\mathbf{V}_{i\sim i} \mathbf{B}_i)^2 &= \sigma_e^4 (b_i - 1) + 2\sigma_e^2 \sigma_b^2 \left(n_i - \frac{1}{n_i} \sum_j n_{ij}^2 \right) \\
 &\quad + \sigma_b^4 \left(\sum_j n_{ij}^2 - \frac{2}{n_i} \sum_j n_{ij}^3 + \frac{1}{n_i^2} \left[\sum_j n_{ij}^2 \right]^2 \right). \quad (3.3.23)
 \end{aligned}$$

Substituting (3.3.23) into (3.3.22) gives

$$\begin{aligned}
 \text{var}(Q) &= 2 \left\{ \sigma_e^4 (b-a) + 2\sigma_e^2 \sigma_b^2 \sum_i \left(n_i - \frac{1}{n_i} \sum_j n_{ij}^2 \right) \right. \\
 &\quad \left. + \sigma_b^4 \left(\sum_i \sum_j n_{ij}^2 - 2 \sum_i \frac{1}{n_i} \sum_j n_{ij}^3 + \sum_i \frac{1}{n_i^2} \left[\sum_j n_{ij}^2 \right]^2 \right) \right\}. \quad (3.3.24)
 \end{aligned}$$

Now, using (3.3.7), and since $b-a$ is fixed for a particular design

$$\text{var}(\text{MSB}) = \frac{1}{(b-a)^2} \text{var}(Q) \quad (3.3.25)$$

where $\text{var}(Q)$ is detailed in (3.3.24).

Suppose we now consider the quadratic form

$$R = \mathbf{y}' \mathbf{E} \mathbf{y}.$$

Then from (3.3.5) and (3.3.8)

$$\mathbf{V} \mathbf{E} = \text{diag}(\mathbf{V}_i \mathbf{E}_i).$$

Hence

$$E(R) = \text{tr}(\mathbf{V} \mathbf{E}) = \sum_i \text{tr}(\mathbf{V}_i \mathbf{E}_i). \quad (3.3.26)$$

Considering just the trace portion of (3.3.26),

$$\begin{aligned} \text{tr}(\mathbf{V}_i \mathbf{E}_i) &= \text{tr}\left\{(\sigma_e^2 \mathbf{I}_i + \sigma_b^2 \sum_j \mathbf{W}_{ij} \mathbf{W}'_{ij} + \sigma_a^2 \mathbf{1}_i \mathbf{1}'_i)(\mathbf{I}_i - \sum_j \frac{1}{n_{ij}} \mathbf{W}_{ij} \mathbf{W}'_{ij})\right\} \\ &= \text{tr}\left\{\sigma_e^2 (\mathbf{I}_i - \sum_j \frac{1}{n_{ij}} \mathbf{W}_{ij} \mathbf{W}'_{ij})\right. \\ &\quad + \sigma_b^2 (\sum_j \mathbf{W}_{ij} \mathbf{W}'_{ij} - \sum_j \frac{1}{n_{ij}} \mathbf{W}_{ij} \mathbf{W}'_{ij} \mathbf{W}_{ij} \mathbf{W}'_{ij}) \\ &\quad \left. + \sigma_a^2 (\mathbf{1}_i \mathbf{1}'_i - \sum_j \frac{1}{n_{ij}} \mathbf{1}_i \mathbf{1}'_i \mathbf{W}_{ij} \mathbf{W}'_{ij})\right\} \end{aligned}$$

which, after some simplification and noting ^{1/} that the coefficients of σ_b^2 and σ_a^2 reduce to zero, yield

$$\text{tr}(\mathbf{V}_i \mathbf{E}_i) = \sigma_e^2 (n_i - b_i). \quad (3.3.27)$$

Substituting (3.3.27) into (3.3.26), and then (3.3.26) into (3.3.8),

gives

$$E(\text{MSE}) = \sigma_e^2. \quad (3.3.28)$$

^{1/} It is instructive to note that the coefficient of σ_b^2 reduces to zero before the trace operator is applied.

Now, consider $\text{var}(R)$, which will be needed in the computation of $\text{var}(\text{MSE})$.

$$\text{var}(R) = 2 \text{tr}(\tilde{V}\tilde{E}) = 2 \text{tr} \text{diag}(\tilde{V}_i \tilde{E}_i)^2 = 2 \sum_i \text{tr}(\tilde{V}_i \tilde{E}_i)^2 \quad (3.3.29)$$

Considering just the trace portion of (3.3.29),

$$\begin{aligned} \text{tr}(\tilde{V}_i \tilde{E}_i)^2 &= \sigma_e^4 \text{tr}(\tilde{I}_i - \sum_j \frac{1}{n_{ij}} \tilde{W}_{ij} \tilde{W}'_{ij})^2 \\ &= \sigma_e^4 \text{tr}(\tilde{I}_i - \sum_j \frac{1}{n_{ij}} \tilde{W}_{ij} \tilde{W}'_{ij}), \end{aligned}$$

since $\tilde{I}_i - \sum_j \frac{1}{n_{ij}} \tilde{W}_{ij} \tilde{W}'_{ij}$ is idempotent, hence

$$= \sigma_e^4 (n_i - b_i). \quad (3.3.30)$$

Substituting (3.3.30) into (3.3.29) yields

$$\text{var}(R) = 2\sigma_e^4 \sum_i (n_i - b_i) = 2\sigma_e^4 (n-b). \quad (3.3.31)$$

Now, using (3.3.8), and since $n-b$ is fixed for a particular design,

$$\text{var}(\text{MSE}) = \frac{1}{(n-b)^2} \text{var}(R) = \frac{2\sigma_e^4}{n-b}. \quad (3.3.32)$$

Now, consider the covariances between the quadratic forms so that the covariances between the mean squares can be computed.

Firstly, consider the covariance between the quadratic forms P and Q .

$$\text{cov}(P, Q) = 2 \text{tr}(\tilde{V} \tilde{A} \tilde{V} \tilde{B}) = 2 \sum_i r_i \text{tr}(\tilde{V}_i \tilde{A}_i \tilde{V}_i \tilde{B}_i). \quad (3.3.33)$$

Considering just the trace portion of (3.3.33),

$$\begin{aligned}
\text{tr}(V_{\sim i \sim i \sim i \sim i}^{1,1'} V_{B_i}) &= \text{tr}\{(\sigma_e^2 \frac{1}{n_i} \frac{1}{n_{ij}} W_{ij} W'_{ij} - \frac{1}{n_i} \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij} + \sigma_a^2 \frac{1}{n_i} \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij}) \\
&\cdot (\sigma_e^2 [\sum_j \frac{1}{n_{ij}} W_{ij} W'_{ij} - \frac{1}{n_i} \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij}] + \sigma_b^2 [\sum_j W_{ij} W'_{ij} - \frac{1}{n_i} \sum_j n_{ij} W_{ij} W'_{ij}] \\
&+ \sigma_a^2 [\sum_j \frac{1}{n_{ij}} W_{ij} W'_{ij} - \frac{1}{n_i} \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij}])\} \\
&= \text{tr}\{\sigma_e^4 (\sum_j \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij} - \frac{1}{n_i} \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij}) \\
&+ \sigma_b^4 (\sum_j \sum_m n_{ij} W_{ij} W'_{ij} - \frac{1}{n_i} \sum_j \sum_m n_{ij} n_{im} W_{ij} W'_{im}) \\
&+ \sigma_a^4 (n_i \sum_j \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij} - n_i \frac{1}{n_{ij}} \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij}) \\
&+ \sigma_e^2 \sigma_b^2 (\sum_j \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij} - \frac{1}{n_i} \sum_j n_{ij} \frac{1}{n_{ij}} W_{ij} W'_{ij}) \\
&+ \sum_j \sum_m \frac{n_{ij}}{n_{im}} W_{ij} W'_{im} - \frac{1}{n_i} \sum_j n_{ij} W_{ij} W'_{ij}) \\
&+ \sigma_e^2 \sigma_a^2 (\sum_j \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij} - \frac{1}{n_i} \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij} + n_i \sum_j \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij} - \frac{1}{n_i} \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij}) \\
&+ \sigma_b^2 \sigma_a^2 (\sum_j \sum_m n_{ij} W_{ij} W'_{im} - \sum_j n_{ij} W_{ij} W'_{ij}) \\
&+ n_i \sum_j n_{ij} W_{ij} W'_{ij} - \sum_j n_{ij}^2 W_{ij} W'_{ij})\}.
\end{aligned}$$

Now if (3.3.4) is invoked, along with some of the previously stated simplifications of the trace operator on the matrices involved, all of the coefficients reduce to zero except for the coefficient of σ_b^4 , and the simplified result is

$$\text{tr}(V_{\sim i \sim i \sim i \sim i}^{1,1'} V_{B_i}) = \sigma_b^4 (\sum_j n_{ij}^3 - \frac{1}{n_i} [\sum_j n_{ij}^2]^2). \quad (3.3.34)$$

Substituting (3.3.34) into (3.3.33), it is easily seen that

$$\text{cov}(P, Q) = 2 \sum_i r_i \left(\sum_j n_{ij}^3 - \frac{1}{n_i} \left[\sum_j n_{ij}^2 \right]^2 \right) \sigma_b^4. \quad (3.3.35)$$

However,

$$\text{cov}(\text{MSA}, \text{MSB}) = \frac{1}{(a-1)(b-a)} \text{cov}(P, Q). \quad (3.3.36)$$

Hence, if (3.3.35) is substituted into (3.3.36)

$$\text{cov}(\text{MSA}, \text{MSB}) = \frac{1}{(a-1)(b-a)} 2 \sum_i r_i \left(\sum_j n_{ij}^3 - \frac{1}{n_i} \left[\sum_j n_{ij}^2 \right]^2 \right) \sigma_b^4. \quad (3.3.37)$$

Theorem 3.3.1

If n_{ij} is a constant for all j , given in particular i , then there is no covariance contribution from that fundamental structure.

Proof:

Since n_{ij} is a constant for all j , given i , then $n_i = \sum_j n_{ij} = b_i n_{ij}$, and consequently

$$\begin{aligned} \sum_j n_{ij}^3 - \frac{1}{n_i} \left[\sum_j n_{ij}^2 \right]^2 &= b_i n_{ij}^3 - \frac{1}{b_i n_{ij}} [b_i n_{ij}^2]^2 \\ &= b_i n_{ij}^3 - b_i n_{ij}^3 = 0. \quad \text{Q.E.D.} \end{aligned}$$

The major implication of theorem 3.3.1 is that if the design is constructed only from structures 1, 3, 4 or 5, as identified in table 2.3.1, then $\text{cov}(\text{MSA}, \text{MSB}) = 0$; and likewise, if structure 2 is in the design then $\text{cov}(\text{MSA}, \text{MSB}) \neq 0$.

Secondly, consider the covariance between the quadratic forms P and R .

$$\text{cov}(P, R) = 2 \text{tr}(\text{VAVE}) = 2 \sum_i r_i \text{tr}(\underset{\sim}{V}_i \underset{\sim}{1}_i \underset{\sim}{1}'_i \underset{\sim}{V}_i \underset{\sim}{E}_i). \quad (3.3.38)$$

Considering only the trace portion of (3.3.38),

$$\begin{aligned}
 \text{tr}(V_{\sim i \sim i \sim i \sim i} B_{\sim i \sim i} V_{\sim i \sim i} E_{\sim i}) &= \text{tr}\left\{(\sigma_e^2 \frac{1}{n_{\sim i \sim i}} + \sigma_b^2 \sum_j n_{ij} W_{ij} W'_{ij}) \right. \\
 &\quad \left. + \sigma_a^2 n_{\sim i \sim i} (\sigma_e^2 [I_{\sim i} - \sum_j \frac{1}{n_{ij}} W_{ij} W'_{ij}] + \sigma_a^2 [\frac{1}{n_{\sim i \sim i}} - \sum_j \frac{1}{n_{ij}} W'_{ij}]) \right\} \\
 &= \text{tr}\left\{\sigma_e^4 (\frac{1}{n_{\sim i \sim i}} - \sum_j \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij}) \right. \\
 &\quad \left. + \sigma_a^4 (n_{\sim i \sim i} \frac{1}{n_{\sim i \sim i}} - n_{\sim i} \sum_j \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij}) \right. \\
 &\quad \left. + \sigma_e^2 \sigma_b^2 (\sum_j n_{ij} W_{ij} W'_{ij} - \sum_j \sum_m \frac{n_{ij}}{n_{im}} W_{ij} W'_{ij} W_{im} W'_{im}) \right. \\
 &\quad \left. + \sigma_e^2 \sigma_a^2 (\frac{1}{n_{\sim i \sim i}} - \sum_j \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij} + n_{\sim i} \frac{1}{n_{\sim i \sim i}} - n_{\sim i} \sum_j \frac{1}{n_{ij}} \frac{1}{n_{ij}} W_{ij} W'_{ij}) \right\}.
 \end{aligned}$$

Now if (3.3.4) is invoked, along with some of the previously stated matrix and trace simplifications, all of the variance component coefficients reduce to zero. Hence, $\text{cov}(P, R) = 0$ and likewise, $\text{cov}(MSA, MSE) = 0$.

Finally, consider the covariance between the quadratic forms Q and R.

$$\text{cov}(Q, R) = 2 \text{tr}(VBVE) = 2 \sum_i \text{tr}(V_{\sim i \sim i \sim i \sim i} B_{\sim i \sim i} V_{\sim i \sim i} E_{\sim i}). \quad (3.3.39)$$

Considering only the trace portion of (3.3.39),

$$\begin{aligned}
 \text{tr}(V_{\sim i \sim i \sim i \sim i} B_{\sim i \sim i} V_{\sim i \sim i} E_{\sim i}) &= \text{tr}\left\{(\sigma_e^2 [\sum_j \frac{1}{n_{ij}} W_{ij} W'_{ij} - \frac{1}{n_{\sim i}} \frac{1}{n_{\sim i}}]) \right. \\
 &\quad \left. + \sigma_b^2 [\sum_j W_{ij} W'_{ij} - \frac{1}{n_{\sim i}} \sum_j n_{ij} W_{ij} W'_{ij}] + \sigma_a^2 [\sum_j \frac{1}{n_{ij}} W'_{ij} - \frac{1}{n_{\sim i}} \frac{1}{n_{\sim i}}]) \right. \\
 &\quad \left. \cdot (\sigma_e^2 [I_{\sim i} - \sum_j \frac{1}{n_{ij}} W_{ij} W'_{ij}] + \sigma_a^2 [\frac{1}{n_{\sim i}} - \sum_j \frac{1}{n_{ij}} W'_{ij}]) \right\} \\
 &= \text{tr}\left\{\sigma_e^4 (\sum_j \frac{1}{n_{ij}} W_{ij} W'_{ij} - \frac{1}{n_{\sim i}} \frac{1}{n_{\sim i}} - \sum_j \sum_m \frac{1}{n_{ij}} \frac{1}{n_{im}} W_{ij} W'_{ij} W_{im} W'_{im}) \right.
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{n_i} \sum_j \frac{1}{n_{ij}} 1_{i\sim i} 1'_{W_{ij} W'_{ij}} + \sigma_a^4 \left(\sum_j \frac{1}{n_{ij}} 1_{W_{ij} W'_{ij}} 1_{1_{i\sim i}} \right. \\
& - \frac{1}{n_i} 1_{1_{i\sim i}} 1_{1_{i\sim i}} - \sum_j \sum_m \frac{1}{n_{im}} 1_{W_{ij} W'_{ij}} 1_{W'_{im}} + \sum_j \frac{1}{n_{ij}} 1_{1_{i\sim i}} 1_{W'_{ij}} \left. \right) \\
& + \sigma_e^2 \sigma_b^2 \left(\sum_j \frac{1}{n_{ij}} W_{ij} W'_{ij} - \sum_j \sum_m \frac{1}{n_{im}} W_{ij} W'_{ij} W_{im} W'_{im} \right. \\
& - \frac{1}{n_i} \sum_j n_{ij} W_{ij} 1'_{i\sim i} + \frac{1}{n_i} \sum_j \sum_m \frac{n_{ij}}{n_{im}} W_{ij} 1'_{W_{im} W'_{im}} \left. \right) \\
& + \sigma_e^2 \sigma_a^2 \left(\sum_j \frac{1}{n_{ij}} W_{ij} W'_{ij} 1_{1_{i\sim i}} - \frac{1}{n_i} 1_{1_{i\sim i}} 1_{1_{i\sim i}} - \sum_j \sum_m \frac{1}{n_{ij}} W_{ij} W'_{ij} 1_{W'_{im}} \right. \\
& + \frac{1}{n_i} \sum_j \frac{1}{n_{ij}} 1_{1_{i\sim i}} 1_{W'_{ij}} + \sum_j \frac{1}{n_{ij}} 1_{W'_{ij}} - \frac{1}{n_i} 1_{1_{i\sim i}} - \sum_j \sum_m \frac{1}{n_{im}} \frac{1}{n_{ij}} W_{ij} W'_{im} W_{im} W'_{ij} \\
& + \sum_j \frac{1}{n_{ij}} \frac{1}{n_{ij}} 1_{1_{i\sim i}} W_{ij} W'_{ij} \left. \right) + \sigma_b^2 \sigma_a^2 \left(\sum_j \frac{1}{n_{ij}} W_{ij} W'_{ij} 1_{1_{i\sim i}} - \frac{1}{n_i} \sum_j n_{ij} W_{ij} 1'_{1_{i\sim i}} \right. \\
& \left. - \sum_j \sum_m W_{ij} W'_{ij} 1_{W'_{im}} + \frac{1}{n_i} \sum_j \sum_m n_{ij} W_{ij} 1'_{1_{i\sim i}} W'_{im} \right) \}.
\end{aligned}$$

Now, if (3.3.4) is invoked along with some of the previously stated matrix and trace simplifications, all of the variance component coefficients reduce to zero. Hence, $\text{cov}(Q, R) = 0$ and likewise, $\text{cov}(\text{MSB}, \text{MSE}) = 0$. These results have been previously obtained by Prairie (1962), but the notation and derivation have been somewhat simplified here.

3.4 Variances and Covariances of the Variance Component Estimators

By considering the variance and covariance forms (3.2.4), (3.2.5), (3.2.6), (3.2.7), (3.2.8) and (3.2.9), as well as (3.3.19), (3.3.25), (3.3.32), and (3.3.37), the variances and covariances of the variance component estimators can be written down in a closed form.

From (3.2.4) and (3.3.32),

$$\text{var}(\hat{\sigma}_e^2) = \frac{2\sigma_e^4}{n-b} \quad (3.4.1)$$

Since it has been shown in section 3.3 that $\text{cov}(\text{MSB}, \text{MSE}) = 0$, and by substituting (3.3.32) and (3.3.25) into (3.2.5), then

$$\begin{aligned} \text{var}(\hat{\sigma}_b^2) = & \frac{1}{k_3^2} \left\{ \frac{2}{(b-a)^2} (\sigma_e^4 [b-a] + 2\sigma_e^2 \sigma_b^2 \sum_i [n_i - \frac{1}{n_i} \sum_j n_{ij}^2] \right. \\ & \left. + \sigma_b^4 [\sum_i \sum_j n_{ij}^2 - 2 \sum_i \frac{1}{n_i} \sum_j n_{ij}^3 + \sum_i \frac{1}{n_i^2} \{\sum_j n_{ij}^2\}^2]) + \frac{2\sigma_e^4}{n-b} \right\}. \end{aligned} \quad (3.4.2)$$

Now, if $\rho_1 = \sigma_b^2/\sigma_e^2$, then after collecting terms and simplifying the expressions, (3.4.2) reduces to:

$$\begin{aligned} \text{var}(\hat{\sigma}_b^2) = & \frac{2\sigma_e^4}{k_3^2(b-a)^2} \left\{ \frac{n-a}{n-b} (b-a) + 2(n - \sum_i \sum_j \frac{n_{ij}^2}{n_i}) \rho_1 \right. \\ & \left. + (\sum_i \sum_j n_{ij}^2 - 2 \sum_i \sum_j \frac{n_{ij}^3}{n_i} + \sum_i \frac{1}{n_i^2} [\sum_j n_{ij}^2]^2) \rho_1^2 \right\} \end{aligned} \quad (3.4.3)$$

In addition, it has also been shown in section 3.3 that

$\text{cov}(\text{MSA}, \text{MSE}) = 0$; hence, by substituting (3.3.32), (3.3.25), (3.3.19)

and (3.3.37) into (3.2.6), then

$$\begin{aligned} \text{var}(\hat{\sigma}_a^2) = & \frac{1}{k_2^2 k_3^2} \left\{ \frac{2k_3^2}{(a-1)^2} (\sigma_e^4 [a-1] + \sigma_b^4 [\frac{1}{n^2} \{n^2 \sum_i \frac{1}{n_i^2} (\sum_j n_{ij}^2)^2 \right. \\ & \left. - 2n \sum_i \frac{1}{n_i} (\sum_j n_{ij}^2)^2 + (\sum_i \sum_j n_{ij}^2)^2\}] \right. \\ & \left. + \sigma_a^4 [\frac{1}{n^2} \{n^2 \sum_i n_i^2 - 2n \sum_i n_i^3 + (\sum_i n_i^2)^2\}] \right. \\ & \left. + \sigma_e^2 \sigma_b^2 [\frac{2}{n} \{n \sum_i \frac{1}{n_i} \sum_j n_{ij}^2 - \sum_i \sum_j n_{ij}^2\}] + \sigma_e^2 \sigma_a^2 [\frac{2}{n} \{n^2 - \sum_i n_i^2\}] \right\} \end{aligned}$$

$$\begin{aligned}
& + \sigma_b^2 \sigma_a^2 \left[\frac{2}{n^2} \left\{ n^2 \sum_i \sum_j n_{ij}^2 - 2n \sum_i n_i \sum_j n_{ij}^2 + \sum_i n_i^2 \sum_j \sum_l n_{lj}^2 \right\} \right] \\
& + \frac{2k_1^2}{(b-a)^2} \left(\sigma_e^4 [b-a] + 2\sigma_e^2 \sigma_b^2 \sum_i \left[n_i - \frac{1}{n_i} \sum_j n_{ij}^2 \right] \right. \\
& \left. + \sigma_b^4 \left[\sum_i \sum_j n_{ij}^2 - 2 \sum_i \frac{1}{n_i} \sum_j n_{ij}^3 + \sum_i \frac{1}{n_i^2} \left\{ \sum_j n_{ij}^2 \right\}^2 \right] \right) \\
& + \frac{2(k_3 - k_1)^2}{n-b} \sigma_e^4 - \frac{4k_3 k_1}{(a-1)(b-a)} \sum_i r_i \left(\sum_j n_{ij}^3 - \frac{1}{n_i} \left[\sum_j n_{ij}^2 \right]^2 \right) \sigma_b^4 \}
\end{aligned}
\tag{3.4.4}$$

Now, if $\rho_2 = \sigma_a^2 / \sigma_e^2$, after some simplification, (3.4.4) becomes

$$\begin{aligned}
\text{var}(\hat{\sigma}_a^2) &= \frac{2\sigma_e^4}{k_2^2 k_3^2} \left\{ \frac{k_3^2}{(a-1)^3} (a-1 + \rho_1^2 \left[\frac{1}{n^2} \left\{ n^2 \sum_i \frac{1}{n_i} \left(\sum_j n_{ij}^2 \right)^2 \right. \right. \right. \right. \\
& \quad \left. \left. \left. - 2n \sum_i \frac{1}{n_i} \left(\sum_j n_{ij}^2 \right)^2 + \left(\sum_i \sum_j n_{ij}^2 \right)^2 \right\} \right] \right. \right. \\
& \quad \left. + \rho_2^2 \left[\frac{1}{n^2} \left\{ n^2 \sum_i n_i^2 - 2n \sum_i n_i^3 + \left(\sum_i n_i^2 \right)^2 \right\} \right] \right. \\
& \quad \left. + \rho_1 \left[\frac{2}{n} \left\{ n \sum_i \frac{1}{n_i} \sum_j n_{ij}^2 - \sum_i \sum_j n_{ij}^2 \right\} \right] + \rho_2 \left[\frac{2}{n} \left\{ n^2 - \sum_i n_i^2 \right\} \right] \right. \\
& \quad \left. + \rho_1 \rho_2 \left[\frac{2}{n^2} \left\{ n^2 \sum_i \sum_j n_{ij}^2 - 2n \sum_i n_i \sum_j n_{ij}^2 + \sum_i n_i^2 \sum_j \sum_l n_{lj}^2 \right\} \right] \right) \\
& \quad + \frac{k_1^2}{(b-a)^2} (b-a + 2\rho_1 \sum_i \left[n_i - \frac{1}{n_i} \sum_j n_{ij}^2 \right] \\
& \quad + \rho_1^2 \left[\sum_i \sum_j n_{ij}^2 - 2 \sum_i \frac{1}{n_i} \sum_j n_{ij}^3 + \sum_i \frac{1}{n_i^2} \left\{ \sum_j n_{ij}^2 \right\}^2 \right]) \\
& \quad + \frac{(k_3 - k_1)^2}{n-b} - \frac{2k_3 k_1}{(a-1)(b-a)} \sum_i r_i \left(\sum_j n_{ij}^3 - \frac{1}{n_i} \left[\sum_j n_{ij}^2 \right]^2 \right) \rho_1^2 \}.
\end{aligned}
\tag{3.4.5}$$

From (3.2.7) and (3.3.32),

$$\text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_b^2) = -\frac{2\sigma_e^4}{k_3(n-b)} \quad (3.4.6)$$

Likewise, from (3.2.8) and (3.3.32),

$$\text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_a^2) = -\frac{(k_3-k_1)2\sigma_e^4}{k_2k_3(n-b)} = \frac{(k_3-k_1)}{k_2} \text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_b^2). \quad (3.4.7)$$

If (3.3.25), (3.3.32) and (3.3.37) are substituted into (3.2.9), then

$$\begin{aligned} \text{cov}(\hat{\sigma}_b^2, \hat{\sigma}_a^2) &= \frac{2\sigma_e^4}{k_2k_3^2} \left\{ -\frac{k_1}{(b-a)^2} (b-a + 2\rho_1 \sum_i [n_i - \frac{1}{n_i} \sum_j n_{ij}^2] \right. \\ &\quad + \rho_1^2 [\sum_{ij} n_{ij}^2 - 2 \sum_i \frac{1}{n_i} \sum_j n_{ij}^3 + \sum_i \frac{1}{n_i^2} \{ \sum_j n_{ij}^2 \}^2]) \\ &\quad \left. + \frac{k_3\rho_1^2}{(a-1)(b-a)} \sum_i r_i (\sum_j n_{ij}^3 - \frac{1}{n_i} [\sum_j n_{ij}^2]^2) + \frac{k_3-k_1}{n-b} \right\}. \end{aligned} \quad (3.4.8)$$

Now, from equations (3.4.1), (3.4.3), (3.4.5), (3.4.6), (3.4.7) and (3.4.8), it is possible to compute the variances and covariances of the analysis of variance estimators of the variance components if σ_e , ρ_1 and ρ_2 are known along with the structure of the design. It would certainly be possible to combine and simplify some of the terms in these variance and covariance expressions; however, it proved to be computationally easier to employ the expressions in the forms already presented.

3.5 An Example Linked to the Computer Program

Just as in section 2.6, the example chosen to demonstrate the linkage between the variance and covariance formulas and the computer program to evaluate them is design 2.8 found in table 2.5.2. The

designs are identified in the computer program by the same identification numbers used in the text here.

In the discussion that follows, σ_e^2 was set to one, and the following relationships will serve to link together terms in the expressions of the text with those found in the computer program.

$$R(I) = \rho_1$$

$$P(J) = \rho_2$$

$$T = \rho_2 / \rho_1$$

$$B = \rho_1^2$$

$$C = \rho_1 \rho_2$$

$$D = \rho_2^2$$

$$A = r$$

$$A2 = r^2$$

$$V3 = \text{var}(\hat{\sigma}_e^2)$$

$$V2 = \text{var}(\hat{\sigma}_b^2)$$

$$V1 = \text{var}(\hat{\sigma}_a^2)$$

$$CVEB = \text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_b^2)$$

$$CVEA = \text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_a^2)$$

$$CVBA = \text{cov}(\hat{\sigma}_b^2, \hat{\sigma}_a^2)$$

$$E = (72r - e)^2$$

The terms F, G, H and W are identified respectively with the terms $\text{var}(\text{MSA})$, $\text{var}(\text{MSB})$, $\text{var}(\text{MSE})$ and $\text{cov}(\text{MSA}, \text{MSB})$ in equations (3.2.6) and (3.4.5) as

$$\begin{aligned} F = & 72 \left\{ a-1 + \rho_1^2 \left(\frac{1}{n^2} \left[n^2 \sum_i \frac{1}{n_i^2} \left\{ \sum_j n_{ij}^2 \right\}^2 - 2n \sum_i \frac{1}{n_i} \left\{ \sum_j n_{ij}^2 \right\}^2 \right. \right. \right. \\ & \left. \left. \left. + \left\{ \sum_i \sum_j n_{ij}^2 \right\}^2 \right] \right) + \rho_2^2 \left(\frac{1}{n^2} \left[n^2 \sum_i n_i^2 - 2n \sum_i n_i^3 + \left\{ \sum_i n_i^2 \right\}^2 \right] \right) \right\} \end{aligned}$$

$$\begin{aligned}
& + \rho_1 \left(\frac{2}{n} \left[n \sum_i \frac{1}{n_i} \sum_j n_{ij}^2 - \sum_i \sum_j n_{ij}^2 \right] \right) + \rho_2 \left(\frac{2}{n} \left[n^2 - \sum_i n_i^2 \right] \right) \\
& + \rho_1 \rho_2 \left(\frac{2}{n^2} \left[n^2 \sum_i \sum_j n_{ij}^2 - 2n \sum_i n_i \sum_j n_{ij}^2 + \sum_i n_i^2 \sum_j \sum_l n_{lj}^2 \right] \right) \}. \quad (3.5.1)
\end{aligned}$$

In terms of design 2.8, equation (3.5.1) was found to be

$$\begin{aligned}
F &= 432.* \frac{1}{A} - 72. + (688.*A - 144.)*B + (1056.*A - 192.)*R(I) \\
&+ (2016.*A - 472.)*D + (1728.*A - 336.)*P(J) \\
&+ (2304.*A - 512.)*C. \\
G &= \frac{2(cr-d)^2}{k_3^2(b-a)^2} \left\{ b-a + 2\rho_1 \sum_i (n_i - \frac{1}{n_i} \sum_j n_{ij}^2) \right. \\
&\left. + \rho_1^2 \left(\sum_i \sum_j n_{ij}^2 - 2 \sum_i \frac{1}{n_i} \sum_j n_{ij}^3 + \sum_i \frac{1}{n_i^2} \left[\sum_j n_{ij}^2 \right]^2 \right) \right\}. \quad (3.5.2)
\end{aligned}$$

In terms of design 2.8, equation (3.5.2) was found to be

$$\begin{aligned}
G &= ((44.*A - 8.)*\frac{2}{2.})*(18. + 42.*R(I) + 25.*B)/(49.*A) \\
H &= \frac{72(a-1)^2(k_3-k_1)^2}{k_3^2(n-b)} \quad (3.5.3)
\end{aligned}$$

In terms of design 2.8, equation (3.5.3) was found to be

$$\begin{aligned}
H &= ((12.*A - 6.)*\frac{2}{2.})*(49.*A) \\
W &= \frac{144(a-1)k_1}{k_3^2(b-a)} \sum_i r_i \left(\sum_j n_{ij}^3 - \frac{1}{n_i} \left[\sum_j n_{ij}^2 \right]^2 \right) \rho_1^2. \quad (3.5.4)
\end{aligned}$$

¹The symbol * indicates multiplication in FORTRAN.

²The symbol ** indicates exponentiation in FORTRAN.

In the case of design 2.8, equation (3.5.4) was found to be

$$W = (44.*A - 8.)*(16.*A - 4.)*B/(7.*A).$$

Using the definitions (3.5.1), (3.5.2), (3.5.3) and (3.5.4), then

$$V1 = (F + G + H - W)/E$$

Additionally, for design 2.8,

$$V3 = 1./A$$

$$V2 = (54. + 42.*R(I) + 25.*B)/(49.*A)$$

$$E = (72.*A - 14.)*2.$$

$$CVEB = -6./(7.*A)$$

$$CVEA = (1. - 2.*A)*CVEB/(72.*A - 14.)$$

$$CVBA = ((4. - 22.*A)*(18. + 42.*R(I) + 25.*B)$$

$$+ (28.*A - 7.)*B - 36.*A + 18.)/(1764.*A^2 - 343.*A).$$

By employing this example, a user of the designs could search the computer program, find the appropriate design heading, and hence, easily find the computational forms for the variances and covariances of the variance component estimators for that design.

CHAPTER 4. DESIGN EVALUATION

4.1 Introduction

In this chapter, three criteria based on functions of the variance-covariance matrix of the variance component estimators are defined. The criteria are evaluated for 49 variance component configurations and 10 different sample sizes. A configuration is defined as the 3-tuple $(\sigma_e^2, \sigma_b^2, \sigma_a^2)$ which can be reduced to $(1, \rho_1, \rho_2)$. Based on these criteria, the optimum and worst designs are cited for five different classifications of the designs. Specific recommendations are made for using the designs.

4.2 Criteria for Design Evaluation

Since all of the designs enumerated in this study were constructed such that they would permit the estimation of all three of the variance components, this is a trivariate problem. Because the problem is in this sense multivariate, it was felt that criteria that are useful in Multivariate Analysis would be a logical starting point to find criteria that would be useful in evaluating optimality of designs for estimating variance components.

To this end, it was decided to study some functions of the variance-covariance matrix of the variance component estimators, in particular, the trace, the determinant and what is called the adjusted trace.

Let $\tilde{Z}$ be the variance-covariance matrix of the variance component estimators.

$$\tilde{Z} = \begin{bmatrix} \text{var}(\hat{\sigma}_e^2) & \text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_b^2) & \text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_a^2) \\ & \text{var}(\hat{\sigma}_b^2) & \text{cov}(\hat{\sigma}_b^2, \hat{\sigma}_a^2) \\ \text{sym.} & & \text{var}(\hat{\sigma}_a^2) \end{bmatrix}. \quad (4.2.1)$$

Similarly, define the adjusted variance-covariance matrix $\tilde{Z}_a$ as

$$\tilde{Z}_a = \begin{bmatrix} \text{var}(\hat{\sigma}_e^2) & \frac{\text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_b^2)}{\rho_1} & \frac{\text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_a^2)}{\rho_2} \\ & \frac{\text{var}(\hat{\sigma}_b^2)}{\rho_1^2} & \frac{\text{cov}(\hat{\sigma}_b^2, \hat{\sigma}_a^2)}{\rho_1 \rho_2} \\ \text{sym.} & & \frac{\text{var}(\hat{\sigma}_a^2)}{\rho_2^2} \end{bmatrix}. \quad (4.2.2)$$

It should be noticed that $\tilde{Z}_a$ is simply $\tilde{Z}$ with each of its elements scaled by the size of the variance components involved in the computation of the element.

On the basis of these two matrices, three criteria were suggested. The first criterion is called the trace criterion,

$$\text{tr}(\tilde{Z}) = \text{var}(\hat{\sigma}_e^2) + \text{var}(\hat{\sigma}_b^2) + \text{var}(\hat{\sigma}_a^2). \quad (4.2.3)$$

The second criterion is called the determinant criterion,

$$\begin{aligned}
\det(\tilde{Z}) = & \text{var}(\hat{\sigma}_e^2) [\text{var}(\hat{\sigma}_b^2) \text{var}(\hat{\sigma}_a^2) - \text{cov}^2(\hat{\sigma}_b^2, \hat{\sigma}_a^2)] \\
& + \text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_b^2) [\text{cov}(\hat{\sigma}_b^2, \hat{\sigma}_a^2) \text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_a^2) - \text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_b^2) \text{var}(\hat{\sigma}_a^2)] \\
& + \text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_a^2) [\text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_b^2) \text{cov}(\hat{\sigma}_b^2, \hat{\sigma}_a^2) - \text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_a^2) \text{var}(\hat{\sigma}_b^2)]. \quad (4.2.4)
\end{aligned}$$

Finally, the third criterion is called the adjusted trace, since it is computed from the adjusted variance-covariance matrix as

$$\text{tr}(\tilde{Z}_a) = \text{var}(\hat{\sigma}_e^2) + \frac{\text{var}(\hat{\sigma}_b^2)}{\rho_1^2} + \frac{\text{var}(\hat{\sigma}_a^2)}{\rho_2^2}. \quad (4.2.5)$$

A fourth criterion, which was called the adjusted determinant, was also thought to be useful, however, it is easily seen that

$$\det(\tilde{Z}_a) = \frac{1}{\rho_1^2 \rho_2^2} \det(Z). \quad (4.2.6)$$

Since closeness to optimality is judged by computing the ratio of the criterion from the design in consideration to the criterion for the optimal design, then

$$\frac{\det(\tilde{Z}_a)}{\det(\tilde{Z}_a)_{\text{opt.}}} = \frac{\det(\tilde{Z})}{\rho_1^2 \rho_2^2} \cdot \frac{\rho_1^2 \rho_2^2}{\det(\tilde{Z})_{\text{opt.}}} = \frac{\det(Z)}{\det(\tilde{Z})_{\text{opt.}}}, \quad (4.2.7)$$

i.e., the adjusted determinant ratio is equivalent to the determinant ratio. Because of this fact, only the first three criteria will be considered in the following discussion.

An optimum design in a class is that design which has the smallest value of the particular criterion for a given sample size and variance component configuration. Similarly, the worst design in a class is that design which has the largest value of the particular criterion for a given sample size and variance component configuration.

In order to obtain numerical values for the criteria, it was necessary to specify a sample size for the experiment, in this case 12r observations where $r = 1(1)10$, and the variance component configuration since the variances and covariances of the variance component estimators require the size of the variance component to be known in order to obtain a numerical value. Forty-nine different variance component configurations were studied, since it was thought that these configurations would cover many of the common situations. These configurations were formed by setting $\rho_1 = 2^k$ and $\rho_2 = 2^m$ where $k = -3(1)3$, $m = -3(1)3$. The designs were evaluated in various ways that were considered to be of interest to the researcher.

4.3 Comparison of All Designs Enumerated

Since the enumeration technique discussed in Chapter 2 evolved sixty-one designs for the estimation of the variance components, comparing all sixty-one designs together was determined to be a useful comparison. The results of such a comparison are found in table 4.3.1.

In table 4.3.1 as well as all of the tables found in the Appendix, Chapter 7, ρ_1 and ρ_2 identify the variance component configuration, r indicates for which sample size the design is optimum, and ratio indicates the ratio of the worst design's criterion to the optimum designs criterion.

A standard format was employed in order to construct table 4.3.1 as well as all of the tables in Chapter 7. Consider the first line in the body of table 4.3.1. Here $\rho_1 = 0.125$, $\rho_2 = 0.125$ and $r = -$. The dash for r is to be interpreted as for all $r = 1(1)10$, i.e., the sample size does not change the results here. In this instance, using

Table 4.3.1. Comparison of all enumerated designs

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	1.9	10.1	10.19	1.9	10.1	103.32	1.9	10.1	12.71
	1/4	-	1.9	10.1	9.34	1.9	10.1	78.21	1.9	10.1	12.32
	1/2	-	1.9	10.1	7.60	1.9	10.1	55.24	1.9	10.1	12.38
	1	1	8.7	10.1	3.96	1.9	2.5	31.06	1.9	10.1	12.23
		2	1.9	10.1	4.49	1.9	2.5	35.20	1.9	10.1	12.41
		others	1.9	10.1	4.90	1.9	10.1	38.90	1.9	10.1	12.50
	2	1	5.2	10.1	2.28	1.9	2.5	27.05	1.9	10.1	12.37
		2-5	1.6	10.1	2.49	1.9	2.5	32.69	1.9	10.1	12.57
		others	3.7	10.1	2.53	1.9	2.5	33.29	1.9	10.1	12.59
	4	1-3	1.1	1.9	1.49	1.9	2.5	29.28	1.9	10.1	12.60
		4-9	1.1	2.5	1.46	1.9	2.5	30.56	1.9	10.1	12.64
		10	3.2	2.5	1.46	1.9	2.5	30.62	1.9	10.1	12.64
	8	-	2.1	1.9	1.92	1.9	2.5	29.20	1.9	10.1	12.67
1/4	1/8	-	1.9	10.1	9.50	1.9	10.1	83.29	1.9	10.1	10.99
	1/4	-	1.9	10.1	8.76	1.9	10.1	65.35	1.9	10.1	9.79
	1/2	-	1.9	10.1	7.27	1.9	10.1	47.62	1.9	10.1	9.51
	1	1	5.5	10.1	3.94	1.9	2.5	26.86	1.9	10.1	9.08
		others	1.9	10.1	4.87	1.9	10.1	34.08	1.9	10.1	9.66
	2	1-3	1.6	10.1	2.49	1.9	2.5	27.73	1.9	10.1	9.74
		4-5	5.3	10.1	2.53	1.9	2.5	28.45	1.9	10.1	9.83
		others	2.12	10.1	2.57	1.9	2.5	28.97	1.9	10.1	9.83
	4	1-2	1.1	1.9	1.51	1.9	2.5	24.63	1.9	10.1	9.79
		3	1.1	2.5	1.45	1.9	2.5	25.52	1.9	10.1	9.86
		4	10.12	2.5	1.45	1.9	2.5	25.95	1.9	10.1	9.89
		others	10.12	10.1	1.47	1.9	2.5	25.69	1.9	10.1	9.95
	8	1-6	2.1	1.9	1.89	1.9	2.5	25.14	1.9	10.1	9.99
		others	10.7	1.9	1.86	1.9	2.5	25.45	1.9	10.1	10.01
1/2	1/8	1	1.9	10.1	7.80	1.9	10.1	47.54	1.8	10.1	7.98
		others	1.9	10.1	8.37	1.9	10.1	61.78	1.9	10.1	9.52
	1/4	1	1.9	10.1	7.11	1.9	10.1	39.14	1.8	10.1	6.63
		others	1.9	10.1	7.81	1.9	10.1	50.96	1.9	10.1	7.91
	1/2	-	1.9	10.1	6.68	1.9	10.1	38.96	1.9	10.1	6.78
	1	1	1.8	10.1	4.01	1.9	2.5	22.10	1.9	10.1	5.75
		others	1.9	10.1	4.78	1.9	10.1	28.66	1.9	10.1	6.51
	2	1	8.4	10.1	2.43	1.9	2.5	19.58	1.9	10.1	5.99
		others	6.2	10.1	2.76	1.9	2.5	24.10	1.9	10.1	6.63
	4	1	1.2	1.9	1.72	1.9	2.5	18.01	1.9	10.1	6.22
		others	10.14	10.1	1.59	1.9	2.5	22.26	1.9	10.1	6.79
	8	1	2.1	1.9	2.41	1.9	2.5	17.13	1.9	10.1	6.36
		others	10.6	1.9	1.79	1.9	2.5	21.22	1.9	10.1	6.89

Table 4.3.1. Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1	1/8	-	1.9	10.1	6.90	1.9	10.1	44.05	1.5	10.5	9.05
	1/4	-	1.9	10.1	6.56	1.9	10.1	38.44	1.5	10.1	7.42
	1/2	1	1.8	10.1	5.27	1.9	10.1	23.89	1.8	10.1	5.07
		others	1.9	10.1	5.88	1.9	10.1	31.27	1.8	10.1	5.77
	1	-	1.8	10.1	4.64	1.9	10.1	24.03	1.8	10.1	4.64
	2	1-2	1.5	10.1	2.93	1.9	2.5	18.30	1.8	10.1	4.14
		others	1.8	10.1	3.11	1.9	2.5	19.80	1.9	10.1	4.35
	4	1	10.14	10.5	1.75	1.9	2.5	14.87	1.8	10.1	3.99
		2	10.14	10.5	1.79	1.9	2.5	16.95	1.9	10.1	4.20
		3-4	10.14	10.1	1.83	1.9	2.5	17.86	1.9	10.1	4.34
		others	2.9	10.1	1.86	1.9	2.5	18.37	1.9	10.1	4.42
	8	1	2.2	1.9	2.23	1.9	2.5	14.14	1.8	10.1	4.09
		others	10.10	1.9	1.69	1.9	2.5	17.51	1.9	10.1	4.52
2	1/8	-	1.8	10.1	5.87	1.9	10.1	32.44	1.5	10.5	9.11
	1/4	-	1.8	10.1	5.71	1.9	10.1	29.78	1.5	10.5	8.18
	1/2	-	1.8	10.1	5.38	1.9	10.1	25.84	1.5	10.5	6.51
	1	1-5	1.5	10.1	4.69	1.9	10.1	20.53	1.5	10.1	4.57
		others	1.8	10.1	4.73	1.9	10.1	21.04	1.5	10.1	4.60
	2	-	1.5	10.1	3.77	1.9	2.5	16.76	1.8	10.1	3.63
	4	1	2.4	10.5	2.38	2.12	2.5	13.00	2.12	10.1	3.06
		others	2.9	10.5	2.55	1.9	2.5	15.67	2.12	10.1	3.38
	8	1	2.4	1.9	2.05	2.12	2.5	12.75	2.12	2.5	3.14
		2	2.4	1.9	1.77	1.9	2.5	13.78	2.12	2.5	3.29
		others	2.4	10.5	1.75	1.9	2.5	14.94	2.12	2.5	3.40
4	1/8	-	2.9	10.1	5.95	1.9	10.1	25.91	1.5	10.5	9.03
	1/4	1	2.9	10.1	5.69	1.8	10.1	18.77	1.5	10.5	7.91
		others	2.9	10.1	5.86	1.9	10.1	24.67	1.5	10.5	8.55
	1/2	1	2.9	10.1	5.49	1.8	10.1	17.27	1.5	10.5	7.00
		others	2.9	10.1	5.69	1.9	10.1	22.60	1.5	10.5	7.57
	1	1	2.9	10.5	5.10	1.8	10.1	15.08	1.5	10.5	5.46
		others	2.9	10.1	5.33	1.9	10.1	19.60	1.5	10.5	5.87
	2	1	2.9	10.5	4.42	2.12	10.1	12.69	1.5	10.1	3.76
		2	2.9	10.5	4.57	1.9	10.1	14.42	1.5	10.1	3.92
		others	2.9	10.5	4.68	1.9	10.1	16.07	2.9	10.1	4.05
	4	1	2.9	10.5	3.34	2.12	2.5	12.45	2.9	10.1	2.82
		2	2.9	10.5	3.49	2.12	2.5	13.24	2.12	10.1	3.02
		others	2.9	10.5	3.61	1.9	2.5	14.12	2.12	10.1	3.17
	8	1	2.4	10.5	2.38	2.12	2.5	12.29	2.12	10.1	2.67
		others	10.15	10.5	2.50	2.12	2.5	13.64	2.12	10.1	2.98

Table 4.3.1. Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
8	1/8	1	2.9	10.5	6.05	1.8	10.1	17.88	1.5	10.5	8.35
		others	2.9	10.5	6.21	1.9	10.1	22.50	1.5	10.5	8.95
	1/4	1	2.9	10.5	6.00	1.8	10.1	17.43	1.5	10.5	8.12
		others	2.9	10.5	6.17	1.9	10.1	21.91	1.5	10.5	8.71
	1/2	1	2.9	10.5	5.89	1.8	10.1	16.60	1.5	10.5	7.64
		others	2.9	10.5	6.07	1.9	10.1	20.85	1.5	10.5	8.20
	1	1	2.9	10.5	5.67	1.8	10.1	15.24	1.5	10.5	6.69
		others	2.9	10.5	5.87	1.9	10.1	19.09	1.5	10.5	7.19
	2	1-2	2.9	10.5	5.37	2.12	10.1	14.91	1.5	10.5	5.34
		3	2.9	10.5	5.41	1.9	10.1	15.61	1.5	10.5	5.40
		others	2.9	10.5	5.47	1.9	10.1	16.58	2.9	10.5	5.50
	4	1	2.9	10.5	4.42	2.12	2.5	12.06	1.5	10.1	3.47
		2	2.9	10.5	4.58	2.12	2.5	12.83	2.9	10.1	3.63
		3-8	2.9	10.5	4.70	2.12	10.1	13.59	2.9	10.1	3.75
		others	2.9	10.5	4.70	1.9	10.1	13.66	2.9	10.1	3.76
	8	1	10.15	10.5	3.40	2.12	2.5	11.97	2.11	10.1	2.63
		others	10.15	10.5	3.60	2.12	2.5	13.27	2.12	10.1	2.95

the trace criterion, design 1.9 is optimum, design 10.1 is the worst and just how bad it is can be seen from the ratio which has a value of 10.19. This ratio means that the trace criterion from design 10.1 is 10.19 times as large as the trace criterion from design 1.9. When the determinant is used as the criterion, design 1.9 is optimum, design 10.1 is worst and with a ratio 103.32. Likewise, when the adjusted trace criterion is used, design 1.9 is optimal, design 10.1 is worst, however the ratio is 12.71 now. For this particular configuration all of the criteria arrive at the same optimum design, the same worst design, but a different ratio for each criterion.

Consider another configuration in table 4.3.1; in this case $\rho_1 = 0.25$, $\rho_2 = 2$. Here the optimum design depends on the sample size for the trace criterion but not for the determinant or the adjusted trace. This time there are three entries in the r column. When $r = 1-3$, i.e., for samples of size 12, 24 or 36, the trace criterion determined that design 1.6 was optimum, the worst design is 10.1 and the ratio is 2.49. The value 2.49 corresponds to the case $r = 3$, and this technique was employed since the ratios converge as r increases.

If $r = 4-5$, i.e., for samples of size 48 or 60, design 5.3 was optimum and 10.1 was worst by means of the trace criterion. When $r =$ others, i.e. for samples of 72(12)120, the trace determined design 2.12 to be optimum and design 10.1 to be worst where the ratio is 2.57. On the other hand, the determinant and the adjusted trace criteria are not affected by the sample sizes studied for this configuration. The determinant criterion found design 1.9 to be optimum and design 2.5 to be worst, a ratio of 28.97 for others. Likewise, the adjusted trace

criterion found design 1.9 to be optimum, however design 10.1 was found to be worst with a ratio of 9.83 for others. The ratio values 2.57, 28.97 and 9.83 are the values for the case $r = 10$. The procedure was employed since the ratios converge as the sample sizes increase. For this configuration the three criteria do not always agree as to which design is optimum and which design is worst.

By means of the trace criterion, twenty-five designs were found to be optimum and four designs appeared as the worst choice for the variance component configurations and sample sizes considered. The optimum designs were: 1.1, 1.2, 1.5, 1.6, 1.8, 1.9, 2.1, 2.2, 2.4, 2.9, 2.12, 3.2, 3.7, 5.2, 5.3, 5.5, 6.2, 8.4, 8.7, 10.6, 10.7, 10.10, 10.12, 10.14 and 10.15. Of these twenty-five designs, only eight designs: 1.2, 1.5, 1.6, 1.8, 1.9, 2.12, 10.7 and 10.14 have been cited by other workers in this area. The other seventeen optimum designs were enumerated in this study. The four worst designs by the criterion were: 1.9, 2.5, 10.1 and 10.5. The interesting point to notice here is that design 1.9 can be either the optimum or the worst choice depending upon the variance component configuration. The particular choice of a design would depend on the variance component configuration and the sample size to be used.

The trace criterion seems to do a credible job since it determined design 1.9 as optimum when both ρ_1 and ρ_2 were less than or equal to 0.5. The design code of 1.9 is (3,3,6) which means that design 1.9 concentrates one-half of its degrees of freedom on estimating σ_e^2 , while only one-quarter on each of σ_b^2 and σ_a^2 . If $\rho_1 \leq 1$ and $0.5 < \rho_2 \leq 1$ or $0.5 < \rho_1 \leq 1$ and $\rho_2 \leq 1$, then either design 1.8 or 1.9 was found to

be optimum. Design 1.8 has a code (4,4,4) which gives equal weight to each component when they are approximately equal in size. Design 8.7 with a code (4,3,5) was optimum when $r=1$, $\rho_1 = 0.125$ and $\rho_2 = 1$, while design 5.5 with a code (4,3,5) was optimum when $r=1$, $\rho_1 = 0.25$ and $\rho_2 = 1$. These are reasonable since their codes are intermediate to that for design 1.8 and that for design 1.9. If $1 < \rho_1 \leq 2$ and $\rho_2 \leq 0.5$, design 1.8 was found to be optimum. If $0.5 < \rho_1 \leq 1$ and $1 < \rho_2 \leq 2$ or $1 < \rho_1 \leq 2$ and $0.5 < \rho_2 \leq 1$, then designs 1.5 and 1.8 were found to be optimum. Design 1.5 has a code (5,5,2) which is now starting to concentrate more of the d.f. in the first two stages of the design as it intuitively should. If $1 < \rho_1 \leq 2$ and $1 < \rho_2 \leq 2$ then design 1.5 was optimum. When $\rho_1 \leq 0.5$ and $1 < \rho_2 \leq 2$, designs 1.6, 2.12, 3.7, 5.2, 5.3, 6.2 and 8.4 with design codes (6,2,4), (4,4,4), (7,2,3), (5,3,4), (4,4,4), (5,4,3) respectively, were found to be optimum. These seem to be reasonable choices since they tend to concentrate more of their d.f. in the first stage as they should. When $2 < \rho_1 < 8$ and $\rho_2 \leq 4$, then design 2.9 with a code (5,5,2) was found to be optimum. Since 5 is the maximum number for the second position of the design code, this seems to be a reasonable choice in this case. If $4 < \rho_2 \leq 8$ and $2 < \rho_1 \leq 8$, then design 10.15 with a code (6,5,1) was chosen as optimum. Again, this seems reasonable since the sampling is concentrated in the first two stages when their components are large. When $\rho_1 \leq 2$ and $2 < \rho_2 \leq 8$, designs 1.1, 1.2, 2.1, 2.2, 2.4, 2.9, 3.2, 10.6, 10.7, 10.10, 10.12, 10.14 with design codes (9,1,2), (8,2,2), (10,1,1), (9,2,1), (7,4,1), (5,5,2), (8,1,3), (9,2,1), (8,2,2), (8,3,1), (6,3,3), and (6,4,2) respectively, were

found to be optimum. These are reasonable choices since they concentrate their d.f. in the first stage.

The trace criterion also does an admirable job with selecting the worst designs. When $\rho_1 \leq 4$ and $\rho_2 \leq 1$, $\rho_1 \leq 2$ and $1 < \rho_2 \leq 2$, or $0.25 \leq \rho_1 \leq 1$ and $2 < \rho_2 \leq 4$, design 10.1 with a code (10,1,1) was the worst choice. This seems reasonable since it concentrates too many d.f. in the first stage when the first stage component is not that large. Likewise, when $\rho_1 \leq 0.125$ and $4 \leq \rho_2 \leq 8$ or $0.25 \leq \rho_1 \leq 1$ and $\rho_2 = 8$, design 1.9 with a code (3,3,6) was found to be worst. This design concentrates its d.f. in the third stage when it is the first stage that needs them. When $\rho_1 = 8$, $\rho_1 = 4$ and $2 \leq \rho_2 \leq 8$, or $\rho_1 = 2$ and $4 \leq \rho_2 \leq 8$, design 10.5 with a code (6,1,5) was found to be worst. This design concentrates too few d.f. in the second stage when it is a relatively large component to be estimated.

In terms of the determinant criterion, there are three different designs determined to be optimum and two designs that are worst. The optimum designs are: 1.8, 1.9 and 2.12, all designs that have been cited by other workers. The worst designs are 2.5 and 10.1. For this criterion, none of the designs found to be optimum ever appeared as the worst choice.

When $\rho_1 \leq 1$, $\rho_2 \leq 2$ and $\rho_1 = 2$ or $\rho_1 = 4$ and $\rho_2 = 0.125$, design 1.9 with a code (3,3,6) was found to be optimum by the determinant. When $\rho_2 \leq 1$ and $4 \leq \rho_2 \leq 8$, design 1.8 with a code (4,4,4) and design 1.9 were found to be optimum. In this case, design 1.8 was optimum only when $r=1$. When $\rho_2 = 2$ and $4 \leq \rho_1 \leq 8$, $\rho_2 = 4$ and $2 \leq \rho_1 \leq 4$, or $\rho_1 = 2$ and $\rho_2 = 8$, designs 2.12 with a code (4,4,4) and 1.9 with a

code (3,3,6) were optimum. Again, design 2.12 was only optimum for small sample sizes when $r=1$ or 2. If $\rho_1 \geq 4$ and $\rho_2 = 8$ or $\rho_1 = 8$ and $\rho_2 \geq 4$, then design 2.12 with a code (4,4,4) was found to be optimum. From these optimum choices, it appears that the determinant criterion is relatively insensitive to configuration changes.

Also, when the worst designs were detected by the determinant criterion, there was very little change when the configuration was varied. When $\rho_2 \leq 1$, $\rho_2 = 2$ and $4 \leq \rho_1 \leq 8$ or $\rho_2 = 4$ and $\rho_1 = 8$, design 10.1 with a code (10,1,1) was found to be worst. The other cases studied found design 2.5 with a code (6,5,1) as the worst choice. There was some small sample mixing of designs 2.5 and 10.1 at the interface. The design code does not seem to correspond with either the optimum or worst choice as would be anticipated.

When the adjusted trace criterion was employed to discriminate amongst the designs, six designs, 1.5, 1.8, 1.9, 2.9, 2.11 and 2.12 were found to be optimum, while the worst choices were: 1.1, 1.2, 1.3, 2.5, 10.1 and 10.5. Two of the optimum designs, 2.9 and 2.11 were enumerated in this study.

When $\rho_1 \leq 0.5$ or $\rho_1 = 1$ and $2 \leq \rho_2 \leq 8$, design 1.9 with a code (3,3,6) was found to be optimum while design 1.8 with a (4,4,4) code was the optimum choice for the case $r=1$ where $\rho_1 = 0.5$ and $0.125 \leq \rho_2 \leq 0.25$ and also $\rho_1 = 1$ and $2 \leq \rho_2 \leq 8$. Design 1.5 with a (5,5,2) code was found to be optimum for the configurations $1 \leq \rho_1 \leq 8$, $0.125 \leq \rho_2 \leq 0.25$ and $2 \leq \rho_1 \leq 8$, $0.5 \leq \rho_2 \leq 1$. Design 1.8 with a code (4,4,4) was found to be optimum when $\rho_1 = 1$, $0.5 \leq \rho_2 \leq 1$ and $\rho_1 = 2$, $\rho_2 = 2$. When $4 \leq \rho_1 \leq 8$, $\rho_2 = 2$ and

$\rho_1 = 8, \rho_2 = 4$, design 2.9 with a (5,5,2) code was found to be optimum. However, when $2 \leq \rho_1 \leq 4, \rho_2 = 4$ and $2 \leq \rho_1 \leq 8, \rho_2 = 8$, design 2.12 with a (4,4,4) code was found as optimum. There again is a small amount of mixing between the optimum cases of designs 1.5, 2.9 and 2.12. The interesting point about the adjusted trace criterion is that when an optimum design was selected, the first two positions of the design code are the same. It appears that this criterion does not distinguish between the sizes of the first two stage's components, just their joint size relative to the third-stage component.

When $1 \leq \rho_1 \leq 8$ and $\rho_2 = 0.125$, $2 \leq \rho_1 \leq 8$ and $0.25 \leq \rho_2 \leq 0.5$, $4 \leq \rho_1 \leq 8$ and $\rho_2 = 1$, or $\rho_1 = 8$ and $\rho_2 = 2$, design 10.5 with a (6,1,5) code was found to be worst. Except for the configuration $\rho_1 = 2, \rho_2 = 8$ all other configurations caused the adjusted trace criterion to find design 10.1 with a (10,1,1) code as worst. The case $\rho_1 = 2, \rho_2 = 8$ exhibited design 2.5 with a code (6,5,1) as the worst choice. Except for this last configuration, the adjusted trace criterion determined as worst designs that had the minimum d.f. in the second stage, without regard to the sample size or ratios of the other components with respect to the second-stage component.

4.4 Comparisons of Other Design Classifications

In this section, two major and two minor design classifications that could be of use to the researchers wanting to employ the designs are discussed. The two major classifications are the designs in the classes as they were enumerated and the designs grouped by the number

of first-stage samples. The two minor classifications are the designs grouped by the number of second-stage degrees of freedom and the designs grouped by the number of third-stage degrees of freedom.

4.4.1 Comparisons Within Design Classes as Enumerated

From each of the ten classes of designs, the optimum, worst and their ratio were determined for the forty-nine variance component configurations, ten sample sizes and the three criteria. The tables associated with this classification are found in the appendix, Chapter 7.

When the optimum designs were determined for these ten classes of designs, each design was checked to see if the following inequalities were satisfied for each sample size and variance component configuration.

$$\text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_b^2)^2 \leq \text{var}(\hat{\sigma}_e^2) \text{var}(\hat{\sigma}_b^2) \quad (4.4.1.1)$$

$$\text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_a^2)^2 \leq \text{var}(\hat{\sigma}_e^2) \text{var}(\hat{\sigma}_a^2) \quad (4.4.1.2)$$

$$\text{cov}(\hat{\sigma}_b^2, \hat{\sigma}_a^2)^2 \leq \text{var}(\hat{\sigma}_b^2) \text{var}(\hat{\sigma}_a^2) \quad (4.4.1.3)$$

Within each computer program, a counter was set to zero. If (4.4.1.1) was not satisfied, 1 was added to the counter; if (4.4.1.2) was not satisfied, 2 was added to the counter and similarly, if (4.4.1.3) was not satisfied, then 4 was added to the counter. The final counter value then indicated which of the six variances and covariances should be checked for algebraic accuracy. The correspondence can be found in table 4.4.1.1.

Table 4.4.1.1. Correspondence between the counter value and the variance or covariance to be checked for algebraic accuracy

Counter Value	Value to be Checked
0	none
1	$\text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_b^2)$
2	$\text{cov}(\hat{\sigma}_e^2, \hat{\sigma}_a^2)$
3	$\text{var}(\hat{\sigma}_e^2)$
4	$\text{cov}(\hat{\sigma}_b^2, \hat{\sigma}_a^2)$
5	$\text{var}(\hat{\sigma}_b^2)$
6	$\text{var}(\hat{\sigma}_a^2)$
7	all

Although this technique does not guarantee correct formulas in the computation of the variances and covariances of the variance component estimators, it did enable gross errors to be detected, and to this extent, the equations used in the computer programs are correct.

Consider first of all the designs in class I. From the results in table 7.1, all of the designs in the class were found to be optimum for some variance component configuration and sample size when the trace criterion was used as the judgment criterion. In this case, designs 1.1, 1.5 and 1.9 were also found to be worst designs. Just as with the comparison of all the designs together, designs 1.1, 1.5, and 1.9 were either an optimum or worst choice depending on the variance component configuration and sample size.

When the determinant was used to distinguish amongst the designs, two designs, 1.8 and 1.9 were determined to be optimum while design 1.1 was always the worst choice.

On the basis of the adjusted trace criterion, design 1.5, 1.8 and 1.9 were deemed optimum, while designs 1.1, 1.2 and 1.3 were deemed worst designs.

The results of determining optimum designs from class II are found in table 7.2. On the basis of the trace criterion, all the designs in the class except design 2.5 were found to be optimum, while designs 2.1, 2.5 and 2.12 were found to be worst designs. Here designs 2.1 and 2.12 appear as both optimum and worst designs.

When the determinant criterion was used to compare the designs in class II, only design 2.12 was found to be optimum, while designs 2.1 and 2.5 were found to be worst.

On the other hand, when the adjusted trace criterion was used to discriminate amongst the designs in class II, designs 2.9, 2.11 and 2.12 were found to be optimum, while designs 2.1 and 2.5 were worst.

As with the class I designs, the choice of a design for a given experimental situation can be made directly from table 7.2 if the variance component configuration and sample size were known.

Table 7.3 presents the results of the comparisons conducted on the designs in class III. For this class of designs, the following design equivalences should be noted:

$$3.1 \equiv 1.1, \quad 3.6 \equiv 1.6, \quad 3.9 \equiv 1.9.$$

By means of the trace criterion, six of the nine designs were determined to be optimum; they were: 3.1, 3.2, 3.6, 3.7, 3.8 and 3.9, while designs 3.1, 3.5 and 3.9 were determined to be worst designs.

However, design 3.9 was always optimum by both the determinant and adjusted trace criteria, with design 3.1 always being worst when the determinant was used to discriminate amongst the designs and designs 3.1 and 3.5 being worst by the adjusted trace criterion.

The class IV designs are evaluated in table 7.4. For this class of designs, the following equivalences should be noted:

$$4.1 \equiv 2.1, \quad 4.6 \equiv 2.6, \quad 4.10 \equiv 2.10, \quad 4.12 \equiv 2.12.$$

When the trace criterion was used as the discriminating device, six designs: 4.1, 4.6, 4.7, 4.10, 4.11 and 4.12 were determined to be optimum while three designs: 4.1, 4.5 and 4.12 were determined to be worst. By means of the determinant criterion, design 4.12 was the only optimum design, while design 4.1 was always the worst design. Similarly, the adjusted trace criterion found design 4.12 as the only optimum design, with designs 4.1 and 4.5 being worst.

The class V designs are evaluated in table 7.5. For this class of designs, the following design equivalences should be noted:

$$5.1 \equiv 1.1, \quad 5.4 \equiv 1.6, \quad 5.6 \equiv 1.9.$$

All six of the designs in this class appeared as optimum when the trace criterion was employed. At the same time, designs 5.1 and 5.6 were found to be worst. When the determinant criterion was used, design 5.6 was always optimum and design 5.1 was always worst. Similarly, when the adjusted trace criterion was used, designs 5.3, 5.5 and 5.6 were found to be optimum, while design 5.1 was always worst.

The evaluation of the class VI designs is found in table 7.6. The following design equivalences should be noted for this class:

$$6.1 \equiv 1.5, \quad 6.3 \equiv 1.8, \quad 6.4 \equiv 1.9.$$

Both the trace and the adjusted trace found all of the designs in this class to be optimum. The trace determined designs 6.1 and 6.4 as worst, while the adjusted trace criterion found designs 6.1, 6.2 and 6.4 as worst. When the determinant was employed to discriminate amongst the designs, designs 6.2, 6.3 and 6.4 were found to be optimum, while designs 6.1 and 6.2 were found to be worst.

The class VII designs evaluation is found in table 7.7. The following design equivalences should be noted:

$$7.1 \equiv 3.5, \quad 7.3 \equiv 3.8, \quad 7.4 \equiv 1.9.$$

By means of the trace criterion, all four of the designs in this class were determined to be optimum, while designs 7.1 and 7.4 were determined to be worst. On the other hand, the determinant criterion always found design 7.4 optimum and design 7.1 worst. Similarly, designs 7.4 and 7.2 were found to be optimum by means of the adjusted trace criterion, while design 7.1 was always worst.

The evaluation of the class VIII designs can be found in table 7.8. The following design equivalences exist for this class:

$$8.1 \equiv 3.5, \quad 8.5 \equiv 1.5, \quad 8.6 \equiv 3.8, \quad 8.8 \equiv 1.8, \quad 8.9 \equiv 1.9.$$

When the trace criterion was employed as the discriminating device, seven designs: 8.3, 8.4, 8.5, 8.6, 8.7, 8.8 and 8.9 were determined to be optimum, while three designs: 8.1, 8.5 and 8.9 were found to be the worst choices. By means of the determinant criterion, designs 8.8 and 8.9 were found to be optimum while designs 8.1, 8.4 and 8.5 were found to be worst choices. On the other hand, when the adjusted trace criterion was employed, designs 8.4, 8.5, 8.8 and 8.9 were found to be optimum while designs 8.1 and 8.5 were found to be worst.

The class IX evaluation can be found in table 7.9. For this class of designs, the following design equivalences should be noted:

$$9.1 \equiv 4.9, \quad 9.4 \equiv 2.9, \quad 9.5 \equiv 2.12.$$

Designs 9.2, 9.3, 9.4 and 9.5 were found to be optimum by the trace criterion, while designs 9.1, 9.4 and 9.5 were found to be worst. In terms of the determinant criterion, design 9.5 was always found to be optimum, however, designs 9.1, 9.3 and 9.4 were found to be worst. By means of the adjusted trace criterion, designs 9.4 and 9.5 were found to be optimum, while designs 9.1 and 9.4 were found to be worst choices.

Table 7.10 contains the results of the evaluation conducted on the class X designs. Eight designs, 10.6, 10.7, 10.10, 10.11, 10.12, 10.13, 10.14 and 10.15 were judged to be optimum by the trace criterion, while designs 10.1 and 10.5 were found to be worst. On the other hand, when the determinant criterion was employed, designs 10.12 and 10.14 were found to be optimum with design 10.1 always being the worst choice. Likewise, by means of the adjusted trace criterion, designs 10.12, 10.14 and 10.15 were found to be optimum and the designs 10.1 and 10.5 were found to be worst.

Generally, to find the optimum design in a class, it is necessary to specify the sample size as well as the variance component configuration and the optimality criterion before the choice can be made from the appropriate table in the appendix.

4.4.2 Comparisons within Design Classes with an Equal Number of First-Stage Samples

The sixty-one designs in this study were classified into classes with equal numbers of first-stage samples. Such a classification can

be found in table 4.4.2.1. This table was generated by grouping together all of those designs that had the same number in the first position of the design codes found in the tables of section 2.5.

Tables 7.11 through 7.17 contain the results of comparing these designs by means of the three criteria: trace, determinant and adjusted trace.

Table 4.4.2.1. Designs grouped by the number of first-stage samples

First-Stage Samples	Design Numbers
3r	1.9
4r	1.8;2.12;3.8;5.5;6.2;7.2;8.7
5r	1.5;1.7;2.9;2.11;3.5;3.7;4.9;4.11;5.3;8.2;8.3;8.4;9.2;9.3
6r	1.4;1.6;2.5;2.8;2.10;3.4;4.5;4.8;10.5;10.9;10.12;10.14;10.15
7r	1.3;2.4;2.7;3.3;4.4;4.7;5.2;10.4;10.8;10.11;10.13
8r	1.2;2.3;2.6;3.2;4.3;10.3;10.7;10.10
9r	1.1;2.2;4.2;10.2;10.6
10r	2.1;10.1

If the experimenter decides to use 3r first-stage samples, then there is only one choice to make, and that happens to be design 1.9, the balanced design.

However, if the desire were to select 4r first-stage samples, there are seven designs to choose from. By means of the trace criterion, an optimum design would be one of the designs: 1.8, 2.12, 5.5, 6.2, 7.2 or 8.7 and the worst choice would be one of the designs 1.8, 2.12, 3.8 or 5.5, depending on the sample size and variance component configuration.

These results can be found in table 7.11. If the determinant criterion was used to decide on an optimum design, then one of the designs: 1.8, 2.12, 3.8, 5.5 or 8.7 would be chosen, while designs 2.12, 3.8 and 6.2 were worst choices. Similarly, when the adjusted trace criterion was employed, designs 1.8, 2.12 and 5.5 were found to be optimum choices, while designs 3.8 and 2.12 were worst.

When the experimenter decides to employ 5r first-stage samples, there are fourteen different designs to choose from. The results of comparing these designs are found in table 7.12. The trace criterion found designs 1.5, 1.7, 2.9, 3.7, 8.3, 8.4, 9.2 and 9.3 as optimum, while designs 1.7, 2.9, 3.5 and 3.7 were found to be worst. When the determinant criterion was used to discriminate among the designs, designs 1.5, 1.7, 2.9, 2.11, 3.7, 4.11, 5.3 and 9.3 were found to be optimum, and designs 2.9, 3.5, 8.4 and 9.3 were found to be worst. The adjusted trace criterion, on the other hand, found designs 1.5, 1.7, 2.9, 2.11, 3.7 and 9.3 as optimum designs, while designs 2.9, 3.5 and 4.9 were found to be worst.

If the experimenter decides to use 6r first-stage samples, there are thirteen designs to choose from. The results of comparing these designs can be found in table 7.13. Using the trace criterion to discriminate among the designs, designs 1.4, 1.6, 2.8, 2.10, 10.12, 10.14 and 10.15 were determined to be optimum, while designs 1.6, 2.5 and 10.5 were found to be worst. When the determinant was used, designs 1.4, 1.6, 2.8 and 2.10 were found to be optimum, whereas design 2.5 was always worst. Similarly, when the adjusted trace criterion was employed, designs 1.4, 1.6, 2.8, 2.10 and 10.15 turned up as optimum, while designs 2.5 and 10.5 were found to be worst.

Whenever the experimenter wants to use 7r first-stage samples in his design, there are eleven designs to choose from. The results of comparing these designs can be found in table 7.14. By means of the trace criterion, designs 1.3, 2.4, 2.7, 3.3, 4.7, 5.2 and 10.11 were found to be optimum, while designs 5.2 and 10.4 were found to be worst. Similarly, when the determinant was used, designs 1.3, 2.4, 2.7, 4.7, 5.2 and 10.11 were found to be optimum and design 10.4 was always worst. On the other hand, using the adjusted trace criterion, designs 1.3, 2.4, 2.7 and 5.2 were found to be optimum with design 10.4 again always being worst.

If there are 8r first-stage samples to be taken, there are eight designs to choose from. Table 7.15 contains the results of comparing these eight designs. When the trace criterion was used to discriminate among the designs, designs 1.2, 2.6, 3.2, 10.7 and 10.10 were found to be optimum and designs 1.2, 3.2 and 10.3 were found to be worst. On the other hand, when the determinant was employed, designs 1.2, 2.6 and 3.2 were optimum with design 10.3 always being worst. Similarly, with the adjusted trace criterion, designs 1.2, 2.3, 2.6 and 3.2 were found to be optimum, while design 10.3 was found to be worst again.

When it is desired to select 9r first-stage samples in the experiment, there are five designs to choose from. The results of comparing these designs can be found in table 7.16. The trace criterion found designs 1.1, 2.2, 4.2 and 10.6 to be optimum, while designs 1.1 and 10.2 were found to be worst. Likewise, with the determinant criterion, designs 1.1, 2.2, 4.2 and 10.6 were found to be optimum, but design 10.2 was always worst. On the other hand, the adjusted trace

criterion found designs 1.1 and 2.2 to be optimum while 10.2 was always worst.

If the desire is to use 10r first-stage samples in the design, then there are two designs to choose from. The results of comparing these two designs are found in table 7.17. Regardless of which of the three criteria was employed, design 2.1 was always optimum and design 10.1 was always worst.

Again, once the choice has been made as to the number of first-stage samples to be taken, an optimum design can be found by whatever of the criteria seem best for the sample size and variance component configuration present. This choice can be made by referring to the appropriate table in chapter 7, in this case, tables 7.11 through 7.17.

4.4.3 Comparisons within Design Classes with Equal Number of Second-Stage Degrees of Freedom

Although this classification of the designs was not considered to be of as much interest to a researcher as the first two classifications discussed here, it was considered that this classification could be a basis for selecting a design.

Table 4.4.3.1 was formed by grouping together all those designs that have the same number in the second position of the design codes found in the tables of section 2.5.

The detailed tables of the optimum designs for this classification were not entered into the appendix; however, a short discussion is given here in order that the optimum designs for this classification will be noted.

Table 4.4.3.1. Designs grouped by the number of second-stage degrees of freedom

Second-Stage Degrees of Freedom	Design Numbers
r	1.1;2.1;3.2;3.3;3.4;3.5;4.2;4.3;4.4;4.5;10.1;10.2;10.3;10.4;10.5
2r	1.2;1.6;2.2;2.6;3.7;3.8;4.7;4.8;4.9;5.2;8.2;10.6;10.7;10.8;10.9
3r	1.3;1.7;1.9;2.3;2.7;2.10;4.11;5.3;5.5;7.2;8.3;8.7;9.2;10.10;10.11;10.12
4r	1.4;1.8;2.4;2.8;2.11;2.12;6.2;8.4;9.3;10.13;10.14
5r	1.5;2.5;2.9;10.15

Suppose it were desired to have r d.f. in the second-stage, then there are fifteen designs to choose from. By means of the trace criterion, designs 1.1, 2.1, 3.2, 3.3, 3.4 and 3.5 were found to be optimum, while designs 3.5, 10.1 and 10.5 were found to be worst. When the determinant criterion was used to distinguish among the designs, only design 3.5 was found to be optimum and design 10.1 was always worst. On the other hand, the adjusted trace criterion found designs 1.1, 3.2, 3.3, 3.4 and 3.5 as optimum, while designs 10.1 and 10.5 were found to be worst.

There are also fifteen designs to choose from when it is desired to have $2r$ d.f. in the second-stage. The trace criterion found designs 1.2, 1.6, 2.2, 2.6, 3.7, 3.8, 4.7, 5.2, 10.6 and 10.7 as optimum designs, while designs 3.8, 8.2, 10.6 and 10.9 were found to be worst. However, when the determinant criterion was used, design 3.8 was always optimum with designs 2.2 and 10.6 being worst. When the adjusted trace

criterion was employed, the optimum designs were 1.6, 3.7 and 3.8, while designs 10.6 and 10.9 were worst.

When the researcher desires to have 3r d.f. in the second-stage, there are sixteen designs to choose from. All but two designs, 4.11 and 9.2 were found to be optimum by means of the trace criterion. Designs 1.9, 8.3, 10.10 and 10.12 were found to be worst. On the other hand, when the determinant criterion was used, designs 1.9 and 7.2 were found to be optimum with designs 2.3 and 10.10 being worst. The adjusted trace criterion determined that designs 1.7, 1.9, 2.10, 4.11, 5.3, 5.5 and 8.7 were optimum, while designs 10.10 and 10.12 were found to be worst.

If the experimenter decides to use a design with 4r d.f. in the second-stage, then there are eleven designs to choose from. Eight of these designs were found to be optimum by means of the trace criterion. The optimum designs were: 1.8, 2.4, 2.11, 2.12, 6.2, 8.4, 10.13 and 10.14, while the worst designs were 1.8, 8.4, 10.13 and 10.14. When the determinant criterion was used, designs 1.8 and 2.12 were found to be optimum, with design 10.13 always being worst. In terms of the adjusted trace criterion, designs 1.8, 2.11 and 2.12 were found to be optimum, while designs 10.13 and 10.14 were found to be worst.

There are only four designs to choose from if the experimenter wants to have 5r d.f. in the second-stage. Designs 1.5, 2.9 and 10.15 were found to be optimum, and designs 1.5 and 2.5 were found to be worst when the trace was the discriminating criterion. However, when the determinant and adjusted trace criteria were used, both found designs

1.5 and 2.9 to be optimum, whereas design 2.5 was always found to be worst by both criteria.

4.4.4 Comparisons within Design Classes with Equal Number of Third-Stage Degrees of Freedom

As with the last section, this section is not considered to be of much interest to a researcher that wants to use the designs in this study.

Table 4.4.4.1 was formed by grouping together all those designs that have the same number in the third position of the design codes found in the tables of section 2.5.

The detailed tables of the optimum designs for this classification were not entered into the appendix; however, a short discussion is given here in order to note which designs are optimum.

Table 4.4.4.1. Designs grouped by the number of third-stage degrees of freedom

Third-Stage Degrees of Freedom	Design Numbers
r	2.1;2.2;2.3;2.4;2.5;10.1;10.6;10.10;10.13;10.15
2r	1.1;1.2;1.3;1.4;1.5;2.6;2.7;2.8;2.9;4.2;10.2;10.7;10.11; 10.14
3r	2.10;2.11;3.2;4.3;4.7;5.2;8.4;9.3;10.3;10.8;10.12
4r	1.6;1.7;1.8;2.12;3.3;4.4;4.8;4.11;5.3;6.2;8.3;9.2;10.4; 10.9
5r	3.4;3.7;4.5;4.9;5.5;7.2;8.2;8.7;10.5
6r	1.9;3.5;3.8

There are ten designs to choose from when the experimenter wants to have r d.f. in the third-stage. Seven designs: 2.1, 2.2, 2.3, 2.4, 10.6, 10.10 and 10.15 were found to be optimum by the trace criterion, and designs 2.5 and 10.1 were found worst. The determinant and adjusted trace criterion both found similar results. Designs 2.4 and 10.15 were found to be optimum, while designs 2.5 and 10.1 were found to be worst.

When $2r$ d.f. are required in the third-stage, there are fourteen designs to choose from. Using the trace criterion, designs 1.1, 1.2, 1.3, 1.4, 1.5, 2.6, 2.9, 4.2, 10.7, 10.11 and 10.14 were found to be optimum, while designs 1.5 and 10.2 were found to be worst. On the other hand, by means of the determinant criterion, designs 1.5 and 2.9 were found to be optimum with design 10.2 always being worst. Similarly, when the adjusted trace criterion was employed, designs 1.1, 1.5 and 2.9 were found to be optimum, while again, design 10.2 was always found to be worst.

If the experimenter decides to use $3r$ d.f. in the third-stage, then there are eleven designs to choose from. The trace criteria found designs 2.11, 3.2, 4.3, 5.2, 8.4, 9.3, 10.8 and 10.12 as optimum, as well as finding designs 2.11 and 10.3 to be worst. However, using the determinant criterion, designs 2.11, 8.4 and 9.3 were found to be optimum, while design 10.3 was always found to be worst. Similarly, by means of the adjusted trace criterion, designs 2.11, 5.2, 8.4 and 9.3 were found to be optimum, while again, design 10.3 was always chosen as the worst design.

There are fourteen designs to choose from when the experimenter wants to have 4r d.f. in the third-stage. By means of the trace criterion, all the designs except 4.11 and 10.4 appeared as optimum designs, whereas designs 1.8 and 10.4 were determined to be worst designs. On the other hand, when the determinant was used, designs 1.8 and 2.12 were found to be optimum, but design 10.4 was always found to be worst. The adjusted trace criterion found designs 1.6, 1.8 and 2.12 as optimum, and design 10.4 was always the worst choice.

From the nine designs with 5r d.f. in the third-stage, the trace criterion found designs 3.4, 3.7, 4.5, 4.9, 5.5, 7.2, 8.2 and 8.7 to be optimum, while designs 5.5 and 10.5 were determined to be worst choices. However, when the determinant criterion was employed, designs 5.5, 7.2 and 8.7 were found to be optimum, and the worst choice was always design 10.5. Likewise, when the adjusted trace criterion was used, designs 5.5, 7.2 and 8.7 were found to be optimum, while design 10.5 was again always determined to be worst.

The last group, those designs with 6r d.f. in the third-stage, contains three designs. All three designs were determined to be optimum by the trace criterion, while designs 1.9 and 3.5 were also found to be worst designs. Both the determinant and adjusted trace criteria found design 1.9 to be optimum and design 3.5 to be worst for all of the cases studied.

4.5 Comparison of the Number of Fundamental Structures in a Design

When the designs were first enumerated in Chapter 2, the restriction of no more than three of the fundamental structures in a design was used. Using all three criteria in the case of comparing all of the

designs together, the number of structures contributing to the optimum and worst designs are found in tables 4.5.1 and 4.5.2, respectively. For each criterion, since there are 49 configurations and 10 sample sizes, there are a total of 490 possible cases to categorize.

Table 4.5.1. Number of fundamental structures in the optimum designs

Number of Structures	Criterion		
	Trace	Determinant	Adjusted Trace
1	151	483	342
2	282	7	147
3	57	0	1
Total	490	490	490

Table 4.5.2. Number of fundamental structures in the worst designs

Number of Structures	Criterion		
	Trace	Determinant	Adjusted Trace
1	48	0	0
2	113	0	130
3	329	490	360
Total	490	490	490

It is evident from tables 4.5.1 and 4.5.2 that optimum designs contain fewer fundamental structures than worst designs, regardless of which of the three criteria one uses to distinguish amongst the designs. From this viewpoint the choice of no more than three fundamental structures in a design was a reasonable one.

CHAPTER 5. SUMMARY AND SUGGESTIONS FOR FURTHER RESEARCH

5.1 Summary

Using a completely random three-stage nested model, five fundamental structures were identified. These fundamental structures were formed by allowing samples of size one or two to be selected from the second or third stages. Based on these five fundamental structures, ten classes of designs were enumerated such that each class contained no more than three fundamental structures and each design contained a multiple of twelve third-stage samples. The designs were also restricted to those that would permit the ANOVA estimation of all three variance components. This enumeration technique generated 61 designs, of which 11 have been discussed previously by other workers in the area. Each of the ten tables not only contains the number of replications of each fundamental structure, but also a design code for each design which can be used to identify the degrees of freedom structure for the design. Additionally, each design has a set of four coefficient numbers which simplify the computation of the expected mean square coefficients in the ANOVA table. An example is discussed to demonstrate how these tables can be used by the researcher.

General formulas for the variances and covariances of the ANOVA estimators of the variance components are given in Chapter 3. Conditional on the sample size and variance component configuration, where the specific sizes of the three variance components constitute

a configuration; the variances and covariances of all three variance component estimators were computed for each of the sixty-one designs. Three criteria: the trace, the determinant and the adjusted trace of the covariance matrix of the variance component estimators, were used to compare the designs. The adjusted trace criterion uses the covariance matrix when each element is scaled by the sizes of the variance components involved in the computation of the element. The smallest value of the criterion was used to indicate the optimum design. Optimum and worst designs were determined by means of the three criteria for five different classifications of the designs: (i) all designs together in a group, (ii) within the ten classes as they were enumerated, (iii) within classes of designs with the same number of first-stage samples, (iv) within classes of designs with the same number of second-stage degrees of freedom, and (v) within classes of designs with the same number of third-stage degrees of freedom. Detailed documentation of the optimum design, worst design and the criterion ratio for the worst to the optimum design was presented for the first three classifications over ten sample sizes and forty-nine configurations.

Of the fifty new designs found in this study, only six of them were never chosen to be optimum by at least one of the three criteria over the five classifications. These designs were 10.1, 10.2, 10.3, 10.4, 10.5 and 10.8. It is of interest to note that design 10.9, Prairie (4:D2) was the only previously discussed design that was not found to be optimum by at least one criterion over the five classifications. The appropriate choice of an optimum design depends on

the sample size and variance component configuration as well as other preliminary information that would enable the researcher to look in the appropriate table of the appendix, Chapter 7.

Sample size was a critical factor in the optimum design choice when it was small, i.e., in the neighborhood of 12, 24 or 36.

Since each of the three criteria seem to be reasonable measures of optimality, a choice of criterion will depend on intuitive background and how well the criterion performs in given situations.

Table 5.1.1. Optimum design codes for the trace criterion when comparing all designs

		ρ_1							
		0	1/8	1/4	1/2	1	2	4	8
ρ_2	0	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4)	(5,5,2)	(5,5,2)	
	1/8	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4)	(5,5,2)	(5,5,2)	
	1/4	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4)	(5,5,2)	(5,5,2)	
	1/2	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4)	(4,4,4) (5,5,2)	(5,5,2)	(5,5,2)	
	1	(6,2,4) (5,2,5)	(5,3,4) (4,4,4)	(4,4,4)	(4,4,4) (5,5,2)	(5,5,2)	(5,5,2)	(5,5,2)	
	2	(9,1,2)	(1,1,2) (6,3,3)	(6,4,2)	(6,4,2) (5,5,2)	(5,5,2)	(5,5,2)	(5,5,2)	
	4	(10,1,1)	(10,1,1) (8,2,2)	(9,2,1)	(8,3,1)	(7,4,1)	(6,5,1)	(6,5,1)	
	8								

Fundamentally, the trace criterion selects a design as optimum that concentrates the degrees of freedom where the variance component is largest. This can be readily seen from table 5.1.1. When both ρ_1 and ρ_2 are small, the balanced design was found to be optimum since

it concentrates the sampling in the third stage. Likewise, when ρ_1 is large, designs with the highest second-stage degrees of freedom are selected, and when ρ_2 is large, designs with high first-stage sampling are selected. It should be noted that 5 is the largest value of a design code's second position. This criterion is relatively sensitive to changes in sample size, especially when the sample size is small. This criterion is the best one in the opinion of the author.

Table 5.1.2. Optimum design codes for the determinant criterion when comparing all designs

		ρ_2							
		0	1/8	1/4	1/2	1	2	4	8
ρ_2	0	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4)	(3,3,6)
	1/8	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4)	(4,4,4)	(3,3,6)
	1/4	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4)	(4,4,4)	(3,3,6)
	1/2	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4)	(4,4,4)	(3,3,6)
	1	(3,3,6),	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4)	(4,4,4)	(3,3,6)
	2	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4)	(4,4,4)	(4,4,4)	(3,3,6)
	4	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4)	(4,4,4)	(4,4,4)	(3,3,6)
	8	(3,3,6)	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4)	(4,4,4)	(4,4,4)	(3,3,6)

The optimum design codes for the determinant criterion can be found in table 5.1.2. The determinant criterion is the least satisfactory of

the three criteria since it is relatively insensitive to changes in sample size and variance component configuration. When ρ_2 is large, the determinant still concentrates one-half of its sampling at the third-stage, which does not agree with intuition. Also, when ρ_1 is large, the determinant chooses a design that concentrates the sampling evenly or with predominance on the third-stage, and this does not seem reasonable.

Table 5.1.3. Optimum design codes for the adjusted trace criterion when comparing all designs

		ρ_1							
		0	1/8	1/4	1/2	1	2	4	8
ρ_2	0	(3,3,6)	(3,3,6)	(4,4,4) (3,3,6)	(5,5,2)	(5,5,2)	(5,5,2)	(5,5,2)	
	1/8	(3,3,6)	(3,3,6)	(4,4,4) (3,3,6)	(5,5,2)	(5,5,2)	(5,5,2)	(5,5,2)	
	1/4	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4)	(5,5,2)	(5,5,2)	(5,5,2)	
	1/2	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4)	(5,5,2)	(5,5,2)	(5,5,2)	
	1	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4) (3,3,6)	(4,4,4)	(5,5,2)	(5,5,2)	
	2	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4) (3,3,6)	(4,4,4)	(5,5,2) (4,4,4)	(5,5,2)	
	4	(3,3,6)	(3,3,6)	(3,3,6)	(4,4,4) (3,3,6)	(4,4,4)	(4,4,4)	(4,4,4)	
	8								

The optimum design codes for the adjusted trace criterion can be found in table 5.1.3. The adjusted trace criterion seems to be more satisfactory than the determinant criterion, but less satisfactory than the trace criterion. This criterion chose designs with increasing concentration of second-stage sampling as ρ_1 increased; however, it was not

sensitive to changes in ρ_2 . From this viewpoint, it was not determined to be as good a criterion as the trace. The adjusted trace criterion, however, was less sensitive to sample size changes than the trace criterion.

In practice, if negative variance component estimates are encountered, they are generally replaced by zero. This would have the effect of decreasing the variance of the variance component estimate, and would happen with high probability when the true value of the variance component was small. The effect that this would have on the trace criterion is anticipated to be marginal since it is dominated by the largest variance component estimate's variance. The effect the negative variance component estimates will have on the other criteria is expected to be more complex and was not considered in this study.

Each of the criteria discriminate amongst the designs in a classification, sometimes agreeing and sometimes disagreeing as to which design is indeed optimum. It is interesting to notice that design 1.9, the balanced design and the one most commonly used in practice, can be either an optimum design or a worst depending on the variance component configuration and the criterion used. Of the five fundamental structures, either structure 1 or structure 2 appeared in the optimum designs; structure 4 appeared most often in the worst designs, and the class X designs form the worst class. Although the designs were constructed to contain no more than three fundamental structures, most of the optimum designs contained no more than two of the fundamental structures.

Computational forms for all of the variance and covariances of the ANOVA estimators for each of the sixty-one designs can be found in the computer program that is available from the author on request.

5.2 Suggestions for Further Research

i) Permit samples of size three or more from one stage to another to see what is the optimum sample number for various variance component configurations and sample sizes.

ii) Study the distribution theory of the three judgment criteria to determine if there is one that is indeed best.

iii) Extend the model to four and more stages to see if similar results emerge and if the conjecture cited by Leone, et al. (1968), that results are unaffected by what happens more than two stages away, is in fact correct.

iv) Consider other models, such as the fixed or mixed model, as well as relaxing some of the covariance assumptions and study other underlying distributions rather than the normal.

v) Find out the effect of using some estimation technique other than the ANOVA estimators, i.e. maximum likelihood, Bayes, etc.

vi) Introduce some cost functions to the models, so that more realistic comparisons can be made.

vii) Study in detail the effect of optimum design choice by the three criteria when the ANOVA estimators take into account that negative estimates are replaced by zero.

CHAPTER 6. LIST OF REFERENCES

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CHAPTER 7. APPENDIX

This chapter contains those tables associated with the two major design classification - one associated with the ten classes of designs as they were enumerated, and the other, the one associated with the classes of designs with the same number of first-stage samples.

Table 7.1 Comparison of Class I Designs

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	1.9	1.1	3.33	1.9	1.1	15.03	1.9	1.1	3.53
	1/4	-	1.9	1.1	3.10	1.9	1.1	11.97	1.9	1.1	3.06
	1/2	-	1.9	1.1	2.62	1.9	1.1	8.83	1.9	1.2	2.99
	1	-	1.9	1.1	1.86	1.9	1.1	6.77	1.9	1.2	3.01
	2	1	1.6	1.9	1.29	1.9	1.1	4.05	1.9	1.3	2.98
		2	1.6	1.5	1.15	1.9	1.1	4.94	1.9	1.3	3.01
		others	1.6	1.1	1.19	1.9	1.1	5.59	1.9	1.3	3.03
	4	-	1.1	1.9	1.39	1.9	1.1	4.97	1.9	1.3	3.04
	8	-	1.1	1.9	1.77	1.9	1.1	4.66	1.9	1.3	3.04
1/4	1/8	-	1.9	1.1	3.35	1.9	1.1	14.14	1.9	1.1	3.96
	1/4	-	1.9	1.1	3.13	1.9	1.1	11.45	1.9	1.1	3.18
	1/2	-	1.9	1.1	2.68	1.9	1.1	8.83	1.9	1.1	2.82
	1	1	1.8	1.1	1.61	1.9	1.1	5.23	1.9	1.1	2.60
		others	1.9	1.1	1.95	1.9	1.1	6.86	1.9	1.1	2.75
	2	1	1.6	1.9	1.26	1.9	1.1	4.11	1.9	1.1	2.63
		2-4	1.6	1.1	1.23	1.9	1.1	5.42	1.9	1.1	2.74
		others	1.7	1.1	1.25	1.9	1.1	5.65	1.9	1.1	2.75
	4	1-5	1.1	1.9	1.37	1.9	1.1	4.85	1.9	1.1	2.76
		others	1.2	1.9	1.33	1.9	1.1	5.01	1.9	1.1	2.77
	8	-	1.1	1.9	1.73	1.9	1.1	4.68	1.9	1.1	2.78
1/2	1/8	1	1.9	1.1	3.26	1.9	1.1	10.76	1.8	1.1	3.81
		others	1.9	1.1	3.39	1.9	1.1	12.94	1.9	1.1	4.21
	1/4	1	1.9	1.1	3.01	1.9	1.1	9.00	1.8	1.1	3.05
		others	1.9	1.1	3.20	1.9	1.1	10.98	1.9	1.1	3.42

Table 7.1 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/2	1/2	-	1.9	1.1	2.80	1.9	1.1	8.83	1.9	1.1	2.77
	1	1	1.8	1.1	1.85	1.9	1.1	5.37	1.9	1.1	2.26
		others	1.9	1.1	2.13	1.9	1.1	7.01	1.9	1.1	2.52
	2	1	1.7	1.1	1.25	1.9	1.1	4.24	1.9	1.1	2.26
		others	1.8	1.1	1.42	1.9	1.1	5.79	1.9	1.1	2.49
	4	1-2	1.2	1.9	1.42	1.9	1.1	3.58	1.9	1.1	2.30
		others	1.3	1.9	1.28	1.9	1.1	5.09	1.9	1.1	2.51
	8	-	1.1	1.9	1.65	1.9	1.1	4.72	1.9	1.1	2.54
1	1/8	-	1.9	1.1	3.44	1.9	1.1	11.66	1.5	1.1	4.71
	1/4	-	1.9	1.1	3.29	1.9	1.1	10.41	1.5	1.1	3.94
	1/2	1	1.8	1.1	2.77	1.9	1.1	7.04	1.8	1.1	2.85
		others	1.9	1.1	2.99	1.9	1.1	8.83	1.8	1.1	3.07
	1	-	1.8	1.1	2.45	1.9	1.1	7.25	1.8	1.1	2.45
	2	1-2	1.5	1.1	1.70	1.9	1.1	5.36	1.8	1.1	2.14
		others	1.8	1.1	1.79	1.9	1.1	6.02	1.9	1.1	2.24
	4	1	1.3	1.9	1.59	1.9	1.1	3.73	1.8	1.1	2.02
		2-5	1.4	1.9	1.28	1.9	1.1	5.08	1.9	1.1	2.21
		6	1.5	1.9	1.27	1.9	1.1	5.14	1.9	1.1	2.22
		others	1.5	1.1	1.27	1.9	1.1	5.24	1.9	1.1	2.24
	8	1	1.2	1.9	2.11	1.9	1.1	3.32	1.8	1.1	2.05
		others	1.2	1.9	1.53	1.9	1.1	4.81	1.9	1.1	2.27
2	1/8	-	1.8	1.1	3.65	1.9	1.1	10.57	1.5	1.1	5.48
	1/4	-	1.8	1.1	3.56	1.9	1.1	9.87	1.5	1.1	4.97
	1/2	-	1.8	1.1	3.38	1.9	1.1	8.83	1.5	1.1	4.06
	1	1-5	1.5	1.1	3.00	1.9	1.1	7.41	1.5	1.1	2.94
		others	1.8	1.1	3.02	1.9	1.1	7.58	1.5	1.1	2.94
	2	-	1.5	1.1	2.50	1.9	1.1	6.39	1.8	1.1	2.34
	4	-	1.5	1.1	1.80	1.9	1.1	5.51	1.8	1.1	2.14
	8	1	1.3	1.9	1.87	1.8	1.1	3.48	1.8	1.1	1.94
		2-3	1.3	1.9	1.50	1.9	1.1	4.60	1.8	1.1	2.08
		others	1.4	1.9	1.43	1.9	1.1	4.97	1.8	1.1	2.12
4	1/8	-	1.5	1.1	4.35	1.9	1.1	9.81	1.5	1.1	6.09
	1/4	1	1.5	1.1	4.31	1.8	1.1	7.60	1.5	1.1	5.67
		others	1.5	1.1	4.30	1.9	1.1	9.44	1.5	1.1	5.77
	1/2	1	1.5	1.1	4.17	1.8	1.1	7.08	1.5	1.1	5.04
		others	1.5	1.1	4.18	1.9	1.1	8.83	1.5	1.1	5.15
	1	1	1.5	1.1	3.90	1.8	1.1	6.32	1.5	1.1	4.01
		others	1.5	1.1	3.94	1.9	1.1	7.95	1.5	1.1	4.11
	2	1	1.5	1.1	3.38	1.8	1.1	5.41	1.5	1.1	2.84
		others	1.5	1.1	3.48	1.9	1.1	6.91	1.5	1.1	2.92
	4	1	1.5	1.1	2.54	1.8	1.1	4.55	1.5	1.1	2.08

Table 7.1 Continued

Criterion											
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
4	4	others	1.5	1.1	2.70	1.9	1.1	5.96	1.8	1.1	2.26
		8	1.5	1.9	1.71	1.8	1.1	3.91	1.8	1.1	1.88
		others	1.5	1.1	1.79	1.9	1.1	5.26	1.8	1.1	2.08
8	1/8	1	1.5	1.1	4.90	1.8	1.1	7.88	1.5	1.1	6.42
		others	1.5	1.1	4.88	1.9	1.1	9.35	1.5	1.1	6.49
	1/4	1	1.5	1.1	4.86	1.8	1.1	7.70	1.5	1.1	6.24
		others	1.5	1.1	4.84	1.9	1.1	9.17	1.5	1.1	6.32
	1/2	1	1.5	1.1	4.77	1.8	1.1	7.40	1.5	1.1	5.87
		others	1.5	1.1	4.76	1.9	1.1	8.83	1.5	1.1	5.96
	1	1	1.5	1.1	4.59	1.8	1.1	6.88	1.5	1.1	5.16
		others	1.5	1.1	4.61	1.9	1.1	8.28	1.5	1.1	5.27
	2	1	1.5	1.1	4.23	1.8	1.1	6.15	1.5	1.1	4.03
		others	1.5	1.1	4.30	1.9	1.1	7.48	1.5	1.1	4.13
	4	1	1.5	1.1	3.57	1.8	1.1	5.28	1.5	1.1	2.81
		others	1.5	1.1	3.70	1.9	1.1	6.57	1.5	1.1	2.89
	8	1	1.5	1.1	2.58	1.8	1.1	4.48	1.5	1.1	2.05
		others	1.5	1.1	2.76	1.9	1.1	5.73	1.8	1.1	2.21

Table 7.2 Comparison of Class II Designs

Criterion											
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	2.12	2.5	4.82	2.12	2.5	27.24	2.12	2.5	5.25
	1/4	-	2.12	2.5	4.62	2.12	2.5	24.92	2.12	2.5	5.66
	1/2	-	2.12	2.5	4.18	2.12	2.5	22.42	2.12	2.5	5.84
	1	-	2.12	2.5	3.31	2.12	2.5	20.28	2.12	2.5	5.91
	2	1	2.10	2.5	1.94	2.12	2.5	17.04	2.12	2.5	5.91
		others	2.12	2.5	2.14	2.12	2.5	18.80	2.12	2.5	5.94
	4	-	2.6	2.5	1.44	2.12	2.5	17.92	2.12	2.5	5.95
	8	-	2.1	2.12	1.48	2.12	2.5	17.43	2.12	2.5	5.96
1/4	1/8	1-4	2.12	2.5	4.72	2.12	2.5	24.50	2.12	2.1	5.01
		others	2.12	2.5	4.74	2.12	2.1	24.92	2.12	2.1	5.05
	1/4	-	2.12	2.5	4.54	2.12	2.5	23.04	2.12	2.5	4.75
	1/2	-	2.12	2.5	4.12	2.12	2.5	21.06	2.12	2.5	5.19

Table 7.2 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/4	1	-	2.12	2.5	3.30	2.12	2.5	19.28	2.12	2.5	5.42
	2	1	2.10	2.5	1.95	2.12	2.5	16.30	2.12	2.5	5.44
		others	2.12	2.5	2.18	2.12	2.5	17.98	2.12	2.5	5.52
	4	-	2.6	2.5	1.44	2.12	2.5	17.18	2.12	2.5	5.57
	8	-	2.1	2.12	1.44	2.12	2.5	16.72	2.12	2.5	5.59
1/2	1/8	1-3	2.12	2.5	4.56	2.12	2.1	22.56	2.9	2.1	5.37
		others	2.12	2.5	4.59	2.12	2.1	23.23	2.12	2.1	5.47
	1/4	-	2.12	2.5	4.41	2.12	2.5	20.58	2.12	2.1	4.58
	1/2	-	2.12	2.5	4.04	2.12	2.5	19.26	2.12	2.5	4.04
	1	-	2.12	2.5	3.31	2.12	2.5	17.94	2.12	2.5	4.47
	2	1	2.11	2.5	2.02	2.12	2.5	15.32	2.12	2.5	4.54
		others	2.12	2.5	2.27	2.12	2.5	16.90	2.12	2.5	4.72
	4	1	2.6	2.5	1.47	2.12	2.5	14.64	2.12	2.5	4.69
		others	2.7	2.5	1.49	2.12	2.5	16.22	2.12	2.5	4.85
	8	1	2.1	2.12	1.68	2.12	2.5	14.24	2.12	2.5	4.77
		others	2.2	2.12	1.37	2.12	2.5	15.82	2.12	2.5	4.92
1	1/8	-	2.12	2.1	4.51	2.12	2.1	21.33	2.9	2.1	6.25
	1/4	-	2.12	2.1	4.36	2.12	2.1	19.38	2.9	2.1	5.28
	1/2	1	2.12	2.1	3.82	2.12	2.5	15.83	2.9	2.1	3.78
		others	2.12	2.1	4.04	2.12	2.5	17.30	2.12	2.1	4.08
	1	-	2.12	2.1	3.42	2.12	2.5	16.49	2.12	2.1	3.42
	2	1	2.9	2.5	2.22	2.12	2.5	14.28	2.12	2.5	3.44
		others	2.12	2.5	2.45	2.12	2.5	15.76	2.12	2.5	3.71
	4	1	2.4	2.5	1.58	2.12	2.5	13.75	2.12	2.5	3.70
		2-3	2.8	2.5	1.63	2.12	2.5	14.89	2.12	2.5	3.89
		others	2.9	2.5	1.66	2.12	2.5	15.23	2.12	2.5	3.95
	8	1-4	2.2	2.12	1.33	2.12	2.5	14.69	2.12	2.5	4.04
		5	2.3	2.12	1.32	2.12	2.5	14.76	2.12	2.5	4.06
2		others	2.3	2.5	1.31	2.12	2.5	14.90	2.12	2.5	4.08
	1/8	-	2.12	2.1	4.60	2.12	2.1	19.66	2.9	2.1	6.78
	1/4	-	2.12	2.1	4.49	2.12	2.1	18.51	2.9	2.1	6.16
	1/2	1	2.9	2.1	4.09	2.12	2.1	14.47	2.9	2.1	4.74
		others	2.12	2.1	4.26	2.12	2.1	16.76	2.9	2.1	5.07
	1	-	2.9	2.1	3.82	2.12	2.5	15.19	2.9	2.1	3.76
	2	-	2.9	2.1	3.12	2.12	2.5	14.75	2.12	2.1	3.09
	4	1	2.4	2.1	1.96	2.12	2.5	13.00	2.12	2.5	2.91
		others	2.9	2.1	2.14	2.12	2.5	14.40	2.12	2.5	3.19
	8	-	2.4	2.5	1.52	2.12	2.5	14.16	2.12	2.5	3.40
4	1/8	-	2.9	2.1	5.17	2.12	2.1	18.46	2.9	2.1	7.15
	1/4	-	2.9	2.1	5.10	2.12	2.1	17.84	2.9	2.1	6.79

Table 7.2 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
4	1/2	-	2.9	2.1	4.95	2.12	2.1	16.78	2.9	2.1	6.07
	1	-	2.9	2.1	4.65	2.12	2.1	15.21	2.9	2.1	4.89
	2	-	2.9	2.1	4.07	2.12	2.5	13.98	2.9	2.1	3.56
	4	1	2.9	2.1	2.83	2.12	2.5	12.45	2.9	2.1	2.59
		others	2.9	2.1	3.09	2.12	2.5	13.78	2.12	2.1	2.89
	8	-	2.4	2.1	2.04	2.12	2.5	13.64	2.12	2.5	2.92
8	1/8	-	2.9	2.1	5.57	2.12	2.1	17.72	2.9	2.1	7.40
	1/4	-	2.9	2.1	5.53	2.12	2.1	17.40	2.9	2.1	7.20
	1/2	-	2.9	2.1	5.43	2.12	2.1	16.81	2.9	2.1	6.79
	1	-	2.9	2.1	5.24	2.12	2.1	15.82	2.9	2.1	6.01
	2	-	2.9	2.1	4.86	2.12	2.1	14.37	2.9	2.1	4.76
	4	-	2.9	2.1	4.14	2.12	2.5	13.35	2.9	2.1	3.43
	8	1	2.4	2.1	2.76	2.12	2.5	11.97	2.11	2.1	2.50
		others	2.9	2.1	3.01	2.12	2.5	13.27	2.12	2.1	2.79

Table 7.3 Comparison of Class III Designs

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	3.9	3.1	3.33	3.9	3.1	15.03	3.9	3.1	3.53
	1/4	-	3.9	3.1	3.10	3.9	3.1	11.77	3.9	3.1	3.06
	1/2	-	3.9	3.1	2.62	3.9	3.1	8.83	3.9	3.1	2.93
	1	1	3.8	3.1	1.55	3.9	3.1	5.15	3.9	3.1	2.85
		others	3.9	3.1	1.86	3.9	3.1	6.77	3.9	3.1	2.91
	2	1	3.6	3.9	1.29	3.9	3.1	4.05	3.9	3.1	2.86
		2-5	3.6	3.1	1.18	3.9	3.1	5.43	3.9	3.1	2.91
		others	3.7	3.1	1.20	3.9	3.1	5.59	3.9	3.1	2.91
	4	1-9	3.1	3.9	1.39	3.9	3.1	4.95	3.9	3.1	2.92
		10	3.2	3.9	1.39	3.9	3.1	4.97	3.9	3.1	2.92
1/4	8	-	3.1	3.9	1.77	3.9	3.1	4.66	3.9	3.1	2.92
	1/8	-	3.9	3.1	3.35	3.9	3.1	14.14	3.9	3.1	3.96
	1/4	-	3.9	3.1	3.13	3.9	3.1	11.45	3.9	3.1	3.18
	1/2	-	3.9	3.1	2.68	3.9	3.1	8.83	3.9	3.1	2.82
	1	-	3.9	3.1	1.95	3.9	3.1	6.86	3.9	3.1	2.75
	2	1	3.6	3.9	3.26	3.9	3.1	4.11	3.9	3.1	2.63

Table 7.3 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/4	2	others	3.6	3.1	1.24	3.9	3.1	5.65	3.9	3.1	2.75
	4	-	3.1	3.9	1.33	3.9	3.1	5.01	3.9	3.1	2.77
	8	-	3.1	3.9	1.73	3.9	3.1	4.68	3.9	3.1	2.78
1/2	1/8	-	3.9	3.1	3.39	3.9	3.1	12.94	3.9	3.1	4.21
	1/4	-	3.9	3.1	3.20	3.9	3.1	10.98	3.9	3.1	3.42
	1/2	-	3.9	3.1	2.80	3.9	3.1	8.83	3.9	3.1	2.77
	1	-	3.9	3.1	2.13	3.9	3.1	7.01	3.9	3.1	2.52
	2	1	3.6	3.5	1.28	3.9	3.1	4.24	3.9	3.1	2.26
		2-6	3.6	3.1	1.34	3.9	3.1	5.68	3.9	3.1	2.47
		others	3.9	3.1	1.35	3.9	3.1	5.79	3.9	3.1	2.49
	4	-	3.1	3.9	1.20	3.9	3.1	5.09	3.9	3.1	2.51
	8	-	3.1	3.9	1.65	3.9	3.1	4.72	3.9	3.1	2.54
1	1/8	-	3.9	3.1	3.44	3.9	3.1	11.66	3.9	3.1	4.32
	1/4	-	3.9	3.1	3.29	3.9	3.1	10.41	3.9	3.1	3.73
	1/2	-	3.9	3.1	2.99	3.9	3.1	8.83	3.9	3.1	2.98
	1	-	3.9	3.1	2.43	3.9	3.1	7.25	3.9	3.1	2.43
	2	1	3.6	3.5	1.50	3.9	3.1	4.45	3.9	3.1	1.94
		others	3.9	3.5	1.67	3.9	3.1	6.02	3.9	3.1	2.24
	4	1	3.6	3.9	1.36	3.9	3.1	3.73	3.9	3.1	1.96
		others	3.6	3.5	1.24	3.9	3.1	5.24	3.9	3.1	2.24
	8	-	3.1	3.9	1.48	3.9	3.1	4.81	3.9	3.1	2.27
2	1/8	-	3.9	3.1	3.48	3.9	3.1	10.57	3.9	3.5	4.55
	1/4	-	3.9	3.1	3.39	3.9	3.1	9.87	3.9	3.5	4.17
	1/2	-	3.9	3.1	3.19	3.9	3.1	8.83	3.9	3.5	3.52
	1	-	3.9	3.5	2.81	3.9	3.1	7.58	3.9	3.5	2.72
	2	-	3.9	3.5	2.22	3.9	3.1	6.39	3.9	3.1	2.21
	4	1-5	3.6	3.5	1.50	3.9	3.1	5.36	3.9	3.1	2.04
		others	3.9	3.5	1.52	3.9	3.1	5.51	3.9	3.1	2.06
	8	1-4	3.1	3.9	1.22	3.9	3.1	4.74	3.9	3.1	2.03
		5-9	3.1	3.5	1.20	3.9	3.1	4.95	3.9	3.1	2.07
4		10	3.6	3.5	1.20	3.9	3.1	4.97	3.9	3.1	2.07
	1/8	-	3.9	3.5	3.56	3.9	3.1	9.81	3.9	3.5	4.73
	1/4	-	3.9	3.5	3.51	3.9	3.1	9.44	3.9	3.5	4.51
	1/2	-	3.9	3.5	3.40	3.9	3.1	8.83	3.9	3.5	4.09
	1	-	3.9	3.5	3.19	3.9	3.1	7.95	3.9	3.5	3.41
	2	-	3.9	3.5	2.79	3.9	3.1	6.91	3.9	3.5	2.60
	4	1	3.9	3.5	1.81	3.9	3.1	4.40	3.9	3.5	1.74
		others	3.9	3.5	2.14	3.9	3.1	5.96	3.9	3.1	2.09
	8	-	3.6	3.5	1.48	3.9	3.1	5.26	3.9	3.1	1.97

Table 7.3 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
8	1/8	-	3.9	3.5	3.62	3.9	3.1	9.35	3.9	3.5	4.84
	1/4	-	3.9	3.5	3.59	3.9	3.1	9.17	3.9	3.5	4.72
	1/2	-	3.9	3.5	3.53	3.9	3.1	8.83	3.9	3.5	4.49
	1	-	3.9	3.5	3.42	3.9	3.1	8.28	3.9	3.5	4.05
	2	-	3.9	3.5	3.19	3.9	3.1	7.48	3.9	3.5	3.35
	4	-	3.9	3.5	2.76	3.9	3.1	6.57	3.9	3.5	2.53
	8	1	3.6	3.5	1.78	3.9	3.1	4.19	3.9	3.5	1.69
	others		3.9	3.5	2.08	3.9	3.1	5.73	3.9	3.1	2.03

Table 7.4 Comparison of Class IV Designs

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	4.12	4.1	4.31	4.12	4.1	26.14	4.12	4.1	4.48
	1/4	-	4.12	4.1	4.14	4.12	4.1	21.49	4.12	4.1	4.07
	1/2	-	4.12	4.1	3.74	4.12	4.1	16.74	4.12	4.1	3.96
	1	-	4.12	4.1	2.94	4.12	4.1	12.98	4.12	4.1	3.94
	2	1	4.10	4.1	1.65	4.12	4.1	8.78	4.12	4.1	3.93
		2-5	4.11	4.1	1.83	4.12	4.1	10.40	4.12	4.1	3.95
		others	4.12	4.1	1.86	4.12	4.1	10.58	4.12	4.1	3.95
	4	1-3	4.6	4.12	1.18	4.12	4.1	8.83	4.12	4.1	3.95
		4	4.6	4.1	1.17	4.12	4.1	8.98	4.12	4.1	3.95
		others	4.7	4.1	1.18	4.12	4.1	9.25	4.12	4.1	3.95
1/4	8	-	4.1	4.12	1.48	4.12	4.1	8.55	4.12	4.1	3.96
	1/8	-	4.12	4.1	4.35	4.12	4.1	24.92	4.12	4.1	5.05
	1/4	-	4.12	4.1	4.17	4.12	4.1	21.00	4.12	4.1	4.24
	1/2	-	4.12	4.1	3.78	4.12	4.1	16.74	4.12	4.1	3.89
	1	-	4.12	4.1	3.01	4.12	4.1	13.14	4.12	4.1	3.82
	2	1	4.10	4.1	1.71	4.12	4.1	8.91	4.12	4.1	3.77
		others	4.12	4.1	1.94	4.12	4.1	10.73	4.12	4.1	3.83
	4	1	4.6	4.12	1.31	4.12	4.1	7.62	4.12	4.1	3.79
		2-7	4.6	4.1	1.21	4.12	4.1	9.26	4.12	4.1	3.84
		others	4.7	4.1	1.21	4.12	4.1	9.34	4.12	4.1	3.84
1/2	8	-	4.1	4.12	1.44	4.12	4.1	8.60	4.12	4.1	3.85
	1/8	-	4.12	4.1	4.42	4.12	4.1	23.23	4.12	4.1	5.47
	1/4	-	4.12	4.1	4.25	4.12	4.1	20.27	4.12	4.1	4.58

Table 7.4 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/2	1/2	-	4.12	4.1	3.88	4.12	4.1	16.74	4.12	4.1	3.86
	1	-	4.12	4.1	3.16	4.12	4.1	13.43	4.12	4.1	3.58
	2	1	4.10	4.1	1.82	4.12	4.1	9.17	4.12	4.1	3.41
		others	4.12	4.1	2.11	4.12	4.1	11.00	4.12	4.1	3.56
	4	1-2	4.6	4.5	1.27	4.12	4.1	8.78	4.12	4.1	3.54
		3-5	4.6	4.1	1.27	4.12	4.1	9.34	4.12	4.1	3.58
		others	4.10	4.1	1.28	4.12	4.1	9.51	4.12	4.1	3.59
	8	-	4.1	4.12	1.35	4.12	4.1	8.70	4.12	4.1	3.62
1	1/8	-	4.12	4.1	4.51	4.12	4.1	21.33	4.12	4.1	5.66
	1/4	-	4.12	4.1	4.36	4.12	4.1	19.38	4.12	4.1	4.98
	1/2	-	4.12	4.1	4.04	4.12	4.1	16.74	4.12	4.1	4.08
	1	-	4.12	4.1	3.42	4.12	4.1	13.89	4.12	4.1	3.42
	2	-	4.12	4.1	2.44	4.12	4.1	11.48	4.12	4.1	3.22
	4	1	4.6	4.5	1.44	4.12	4.1	8.09	4.12	4.1	3.02
		others	4.10	4.5	1.50	4.12	4.1	9.85	4.12	4.1	3.23
	8	1-8	4.6	4.12	1.25	4.12	4.1	8.85	4.12	4.1	3.27
		others	4.6	4.5	1.25	4.12	4.1	8.89	4.12	4.1	3.28
2	1/8	-	4.12	4.1	4.60	4.12	4.1	19.66	4.12	4.5	5.94
	1/4	-	4.12	4.1	4.49	4.12	4.1	18.51	4.12	4.5	5.46
	1/2	-	4.12	4.1	4.26	4.12	4.1	16.76	4.12	4.5	4.62
	1	-	4.12	4.1	3.79	4.12	4.1	14.51	4.12	4.1	3.68
	2	-	4.12	4.1	2.97	4.12	4.1	12.24	4.12	4.1	3.09
	4	1	4.10	4.5	1.83	4.12	4.1	8.64	4.12	4.1	2.66
		others	4.12	4.5	2.00	4.12	4.1	10.43	4.12	4.1	2.93
	8	-	4.6	4.5	1.42	4.12	4.1	9.25	4.12	4.1	2.96
4	1/8	-	4.12	4.1	4.67	4.12	4.1	18.46	4.12	4.5	6.14
	1/4	-	4.12	4.1	4.60	4.12	4.1	17.84	4.12	4.5	5.86
	1/2	-	4.12	4.1	4.45	4.12	4.1	16.78	4.12	4.5	5.32
	1	-	4.12	4.5	4.16	4.12	4.1	15.21	4.12	4.5	4.41
	2	-	4.12	4.5	3.64	4.12	4.1	13.26	4.12	4.1	3.42
	4	-	4.12	4.5	2.79	4.12	4.1	11.36	4.12	4.1	2.89
	8	-	4.10	4.5	1.84	4.12	4.1	9.89	4.12	4.1	2.77
8	1/8	-	4.12	4.5	4.78	4.12	4.1	17.72	4.12	4.5	6.27
	1/4	-	4.12	4.5	4.75	4.12	4.1	17.40	4.12	4.5	6.12
	1/2	-	4.12	4.5	4.67	4.12	4.1	16.81	4.12	4.5	5.82
	1	-	4.12	4.5	4.51	4.12	4.1	15.82	4.12	4.5	5.24
	2	-	4.12	4.5	4.20	4.12	4.1	14.37	4.12	4.5	4.28
	4	-	4.12	4.5	3.61	4.12	4.1	12.60	4.12	4.1	3.28
	8	-	4.12	4.5	2.70	4.12	4.1	10.90	4.12	4.1	2.79

Table 7.5 Comparison of Class V Designs

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	5.6	5.1	3.33	5.6	5.1	15.03	5.6	5.1	3.53
	1/4	-	5.6	5.1	3.10	5.6	5.1	11.77	5.6	5.1	3.06
	1/2	-	5.6	5.1	2.62	5.6	5.1	8.83	5.6	5.1	2.93
	1	1	5.5	5.1	1.54	5.6	5.1	5.15	5.6	5.1	2.85
		others	5.6	5.1	1.86	5.6	5.1	6.77	5.6	5.1	2.91
	2	1	5.2	5.6	1.29	5.6	5.1	4.05	5.6	5.1	2.86
		others	5.4	5.1	1.19	5.6	5.1	5.59	5.6	5.1	2.91
	4	-	5.1	5.6	1.39	5.6	5.1	4.97	5.6	5.1	2.92
1/4	8	-	5.1	5.6	1.77	5.6	5.1	4.66	5.6	5.1	2.92
	1/8	-	5.6	5.1	3.35	5.6	5.1	14.14	5.6	5.1	3.96
	1/4	-	5.6	5.1	3.13	5.6	5.1	11.45	5.6	5.1	3.18
	1/2	-	5.6	5.1	2.68	5.6	5.1	8.83	5.6	5.1	2.82
	1	1	5.5	5.1	1.63	5.6	5.1	5.23	5.6	5.1	2.60
		others	5.6	5.1	1.95	5.6	5.1	6.86	5.6	5.1	2.75
	2	1	5.4	5.6	1.26	5.6	5.1	4.11	5.6	5.1	2.63
		2-3	5.4	5.1	1.22	5.6	5.1	5.28	5.6	5.1	2.73
1/2		others	5.3	5.1	1.25	5.6	5.1	5.65	5.6	5.1	2.75
	4	-	5.1	5.6	1.33	5.6	5.1	5.01	5.6	5.1	2.77
	8	-	5.1	5.6	1.73	5.6	5.1	4.68	5.6	5.1	2.78
	1/8	-	5.6	5.1	3.39	5.6	5.1	12.94	5.6	5.1	4.21
	1/4	-	5.6	5.1	3.20	5.6	5.1	10.98	5.6	5.1	3.42
	1/2	-	5.1	5.1	2.80	5.6	5.1	8.83	5.6	5.1	2.77
	1	1	5.5	5.1	1.79	5.6	5.1	5.37	5.6	5.1	2.26
		others	5.6	5.1	2.13	5.6	5.1	7.01	5.6	5.1	2.52
1	2	-	5.3	5.1	1.39	5.6	5.1	5.79	5.6	5.1	2.49
	4	1	5.1	5.6	1.66	5.6	5.1	3.58	5.6	5.1	2.30
		others	5.2	5.6	1.25	5.6	5.1	5.09	5.6	5.1	2.51
	8	-	5.1	5.6	1.65	5.6	5.1	4.72	5.6	5.1	2.54
	1/8	-	5.6	5.1	3.44	5.6	5.1	11.66	5.6	5.1	4.32
	1/4	-	5.6	5.1	3.29	5.6	5.1	10.41	5.6	5.1	3.73
	1/2	-	5.6	5.1	2.99	5.6	5.1	8.83	5.6	5.1	2.98
	1	-	5.6	5.1	2.43	5.6	5.1	7.25	5.6	5.1	2.43
	2	1	5.3	5.1	1.51	5.6	5.1	4.45	5.5	5.1	1.95
		2-3	5.3	5.1	1.63	5.6	5.1	5.64	5.6	5.1	2.18
		others	5.5	5.1	1.68	5.6	5.1	6.02	5.6	5.1	2.24
	4	1-6	5.2	5.6	1.17	5.6	5.1	5.14	5.6	5.1	2.22
		others	5.3	5.1	1.17	5.6	5.1	5.24	5.6	5.1	2.24
	8	-	5.1	5.6	1.48	5.6	5.1	4.81	5.6	5.1	2.27

Table 7.5 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
2	1/8	-	5.6	5.1	3.48	5.6	5.1	10.57	5.6	5.1	4.41
	1/4	-	5.6	5.1	3.39	5.6	5.1	9.87	5.6	5.1	4.03
	1/2	-	5.6	5.1	3.19	5.6	5.1	8.83	5.6	5.1	3.42
	1	1	5.6	5.1	2.49	5.6	5.1	5.89	5.5	5.1	2.30
		others	5.6	5.1	2.80	5.6	5.1	7.58	5.6	5.1	2.70
	2	1	5.3	5.1	1.93	5.6	5.1	4.80	5.3	5.1	1.94
		2-3	5.5	5.1	2.07	5.6	5.1	6.01	5.6	5.1	2.13
		others	5.6	5.1	2.15	5.6	5.1	6.39	5.6	5.1	2.21
	4	1	5.3	5.1	1.36	5.6	5.1	3.98	5.3	5.1	1.83
		2	5.3	5.1	1.46	5.6	5.1	4.87	5.5	5.1	1.94
		others	5.3	5.1	1.53	5.6	5.1	5.51	5.6	5.1	2.06
	8	1	5.2	5.6	1.67	5.6	5.1	3.47	5.5	5.1	1.84
		others	5.2	5.6	1.28	5.6	5.1	4.97	5.6	5.1	2.07
4	1/8	-	5.6	5.1	3.51	5.6	5.1	9.81	5.6	5.1	4.49
	1/4	-	5.6	5.1	3.46	5.6	5.1	9.44	5.6	5.1	4.28
	1/2	-	5.6	5.1	3.35	5.6	5.1	8.83	5.6	5.1	3.87
	1	-	5.6	5.1	3.12	5.6	5.1	7.95	5.6	5.1	3.24
	2	1	5.3	5.1	2.39	5.6	5.1	5.28	5.3	5.1	2.22
		others	5.6	5.1	2.68	5.6	5.1	6.91	5.6	5.1	2.54
	4	1	5.3	5.1	1.88	5.6	5.1	4.40	5.3	5.1	1.89
		2-4	5.3	5.1	2.03	5.6	5.1	5.72	5.5	5.1	2.05
		others	5.3	5.1	2.05	5.6	5.1	5.96	5.6	5.1	2.09
	8	1	5.2	5.6	1.36	5.6	5.1	3.74	5.3	5.1	1.79
		2	5.3	5.1	1.37	5.6	5.1	4.62	5.3	5.1	1.87
		3-6	5.3	5.1	1.43	5.6	5.1	5.16	5.5	5.1	1.95
		others	5.3	5.1	1.45	5.6	5.1	5.26	5.6	5.1	1.97
8	1/8	-	5.6	5.1	3.53	5.6	5.1	9.35	5.6	5.1	4.55
	1/4	-	5.6	5.1	3.50	5.6	5.1	9.17	5.6	5.1	4.43
	1/2	-	5.6	5.1	3.44	5.6	5.1	8.83	5.6	5.1	4.21
	1	-	5.6	5.1	3.32	5.6	5.1	8.28	5.6	5.1	3.79
	2	1	5.6	5.1	2.81	5.6	5.1	5.81	5.5	5.1	2.69
		others	5.6	5.1	3.08	5.6	5.1	7.48	5.6	5.1	3.15
	4	1	5.3	5.1	2.40	5.6	5.1	4.96	5.3	5.1	2.20
		2	5.3	5.1	2.49	5.6	5.1	5.90	5.5	5.1	2.31
		others	5.6	5.1	2.61	5.6	5.1	6.57	5.6	5.1	2.46
	8	1-3	5.3	5.1	1.98	5.6	5.1	5.36	5.3	5.1	1.99
		others	5.3	5.1	2.03	5.6	5.1	5.73	5.5	5.1	2.04

Table 7.6 Comparison of Class VI Designs

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	6.4	6.1	2.64	6.4	6.1	4.85	6.4	6.1	2.42
	1/4	-	6.4	6.1	2.47	6.4	6.1	4.31	6.4	6.1	2.76
	1/2	-	6.4	6.1	2.14	6.4	6.1	3.78	6.4	6.1	2.91
	1	-	6.4	6.1	1.61	6.4	6.1	3.37	6.4	6.1	2.97
	2	1	6.2	6.4	1.18	6.4	6.1	2.56	6.4	6.1	2.95
		others	6.2	6.1	1.14	6.4	6.1	3.10	6.4	6.1	2.99
	4	-	6.1	6.4	1.23	6.4	6.1	2.95	6.4	6.1	3.01
	8	-	6.1	6.4	1.41	6.4	6.1	2.86	6.4	6.1	3.01
1/4	1/8	-	6.4	6.1	2.33	6.4	6.1	4.05	6.4	6.1	1.58
	1/4	-	6.4	6.1	2.21	6.4	6.1	3.68	6.4	6.1	1.91
	1/2	-	6.4	6.1	1.94	6.4	6.1	3.30	6.4	6.1	2.16
	1	1	6.3	6.1	1.28	6.4	6.1	2.49	6.4	6.1	2.18
		others	6.4	6.1	1.51	6.4	6.1	2.97	6.4	6.1	2.30
	2	1-2	6.2	6.4	1.09	6.4	6.1	2.56	6.4	6.1	2.32
		others	6.2	6.1	1.12	6.4	6.1	2.75	6.4	6.1	2.36
	4	-	6.1	6.4	1.23	6.4	6.1	2.62	6.4	6.1	2.38
1	1/8	-	6.1	6.4	1.41	6.4	6.1	2.55	6.4	6.1	2.40
	1/8	1	6.4	6.1	1.78	6.4	6.1	2.72	6.3	6.2	1.06
		2-3	6.4	6.1	1.86	6.4	6.1	3.07	6.4	6.2	1.10
		others	6.4	6.1	1.88	6.4	6.1	3.17	6.4	6.1	1.12
	1/4	1	6.4	6.1	1.68	6.4	6.1	2.54	6.3	6.1	1.12
		others	6.4	6.1	1.80	6.4	6.1	2.97	6.4	6.1	1.25
	1/2	-	6.4	6.1	1.64	6.4	6.1	2.74	6.4	6.1	1.44
	1	1	6.3	6.1	1.20	6.4	6.1	2.10	6.4	6.1	1.44
1		others	6.4	6.1	1.36	6.4	6.1	2.52	6.4	6.1	1.59
	2	1	6.1	6.4	1.18	6.4	6.1	1.95	6.4	6.1	1.55
		2-3	6.2	6.4	1.08	6.4	6.1	2.27	6.4	6.1	1.66
		others	6.2	6.1	1.08	6.4	6.1	2.36	6.4	6.1	1.68
	4	-	6.1	6.4	1.24	6.4	6.1	2.25	6.4	6.1	1.73
	8	-	6.1	6.4	1.41	6.4	6.1	2.19	6.4	6.1	1.75
	1/8	1-2	6.4	6.1	1.31	6.4	6.1	2.06	6.1	6.4	1.13
		others	6.4	6.1	1.38	6.4	6.1	2.42	6.1	6.2	1.11
1	1/4	1-5	6.4	6.1	1.34	6.4	6.1	2.31	6.1	6.4	1.07
		others	6.4	6.1	1.35	6.4	6.1	2.34	6.1	6.2	1.06
	1/2	1	6.3	6.1	1.18	6.4	6.1	1.87	6.3	6.4	1.12
		2-3	6.4	6.1	1.25	6.4	6.1	2.15	6.3	6.4	1.05
		others	6.4	6.1	1.27	6.4	6.1	2.23	6.3	6.1	1.06
	1	-	6.3	6.1	1.15	6.4	6.1	2.10	6.3	6.1	1.15

Table 7.6 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1	2	1-2	6.1	6.4	1.12	6.4	6.1	1.86	6.3	6.1	1.20
		others	6.3	6.4	1.07	6.4	6.1	2.00	6.4	6.1	1.24
	4	1	6.1	6.4	1.53	6.4	6.1	1.57	6.3	6.1	1.21
		others	6.1	6.4	1.26	6.4	6.1	1.92	6.4	6.1	1.30
	8	1	6.1	6.4	1.73	6.4	6.1	1.53	6.3	6.1	1.23
		others	6.1	6.4	1.41	6.4	6.1	1.87	6.4	6.1	1.33
2	1/8	1-4	6.3	6.4	1.05	6.4	6.2	2.04	6.1	6.4	1.26
		others	6.3	6.1	1.05	6.4	6.2	2.07	6.1	6.4	1.24
	1/4	-	6.3	6.4	1.05	6.4	6.2	1.93	6.1	6.4	1.23
	1/2	-	6.3	6.4	1.06	6.4	6.1	1.85	6.1	6.4	1.19
	1	1-5	6.1	6.4	1.08	6.4	6.1	1.76	6.1	6.4	1.11
		others	6.3	6.4	1.08	6.4	6.1	1.79	6.1	6.4	1.09
	2	1-4	6.1	6.4	1.18	6.4	6.1	1.69	6.3	6.4	1.07
		others	6.1	6.4	1.16	6.4	6.1	1.73	6.3	6.1	1.07
	4	1	6.1	6.4	1.55	6.4	6.1	1.37	6.2	6.4	1.13
		others	6.1	6.4	1.30	6.4	6.1	1.68	6.2	6.1	1.13
	8	1	6.1	6.4	1.73	6.2	6.1	1.35	6.2	6.1	1.12
		others	6.1	6.4	1.41	6.4	6.1	1.64	6.2	6.1	1.17
4	1/8	-	6.1	6.4	1.24	6.4	6.2	1.93	6.1	6.4	1.35
	1/4	1	6.1	6.4	1.31	6.3	6.2	1.60	6.1	6.4	1.52
		others	6.1	6.4	1.24	6.4	6.2	1.85	6.1	6.4	1.35
	1/2	1	6.1	6.4	1.33	6.3	6.2	1.49	6.1	6.4	1.52
		others	6.1	6.4	1.25	6.4	6.2	1.71	6.1	6.4	1.33
	1	1	6.1	6.4	1.36	6.3	6.2	1.35	6.1	6.4	1.46
		others	6.1	6.4	1.27	6.4	6.1	1.58	6.1	6.4	1.27
	2	1	6.1	6.4	1.44	6.3	6.1	1.30	6.1	6.4	1.34
		others	6.1	6.4	1.30	6.4	6.1	1.55	6.1	6.4	1.15
	4	1	6.1	6.4	1.56	6.3	6.1	1.29	6.1	6.4	1.20
		others	6.1	6.4	1.35	6.4	6.1	1.53	6.2	6.4	1.09
	8	1	6.1	6.4	1.72	6.3	6.1	1.28	6.2	6.4	1.17
		2	6.1	6.4	1.52	6.4	6.1	1.39	6.2	6.4	1.11
		others	6.1	6.4	1.43	6.4	6.1	1.50	6.2	6.1	1.12
8	1/8	1	6.1	6.4	1.45	6.3	6.2	1.67	6.1	6.4	1.61
		others	6.1	6.4	1.38	6.4	6.2	1.85	6.1	6.4	1.43
	1/4	1	6.1	6.4	1.45	6.3	6.2	1.63	6.1	6.4	1.61
		others	6.1	6.4	1.38	6.4	6.2	1.80	6.1	6.4	1.43
	1/2	1	6.1	6.4	1.46	6.3	6.2	1.56	6.1	6.4	1.61
		others	6.1	6.4	1.38	6.4	6.2	1.72	6.1	6.4	1.42
	1	1	6.1	6.4	1.48	6.3	6.2	1.46	6.1	6.4	1.59
		others	6.1	6.4	1.39	6.4	6.2	1.60	6.1	6.4	1.39

Table 7.6 Continued

Criterion											
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
8	2	1	6.1	6.4	1.51	6.3	6.2	1.32	6.1	6.4	1.52
		2-3	6.1	6.4	1.42	6.4	6.2	1.39	6.1	6.4	1.35
		others	6.1	6.4	1.40	6.4	6.1	1.45	6.1	6.4	1.31
	4	1	6.1	6.4	1.57	6.3	6.1	1.24	6.1	6.4	1.38
		others	6.1	6.4	1.42	6.4	6.1	1.43	6.1	6.4	1.18
	8	1	6.1	6.4	1.67	6.3	6.1	1.23	6.1	6.4	1.23
		others	6.1	6.4	1.45	6.4	6.1	1.42	6.2	6.4	1.11

Table 7.7 Comparison of Class VII Designs

Criterion											
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	7.4	7.1	2.19	7.4	7.1	3.90	7.4	7.1	2.91
		1/4	7.4	7.1	2.07	7.4	7.1	3.38	7.4	7.1	2.54
		1/2	7.4	7.1	1.82	7.4	7.1	2.88	7.4	7.1	2.44
	1	1	7.3	7.1	1.23	7.4	7.1	2.11	7.4	7.1	2.38
		others	7.4	7.1	1.42	7.4	7.1	2.50	7.4	7.1	2.43
	2	1	7.1	7.4	1.19	7.4	7.1	1.87	7.4	7.1	2.39
		2-4	7.2	7.4	1.07	7.4	7.1	2.20	7.4	7.1	2.42
		others	7.2	7.1	1.08	7.4	7.1	2.25	7.4	7.1	2.43
	4	-	7.1	7.4	1.26	7.4	7.1	2.12	7.4	7.1	2.43
	8	-	7.1	7.4	1.42	7.4	7.1	2.04	7.4	7.1	2.43
1/4	1/8	-	7.4	7.1	2.42	7.4	7.1	4.02	7.4	7.1	3.43
		1/4	7.4	7.1	2.29	7.4	7.1	3.52	7.4	7.1	2.77
		1/2	7.4	7.1	2.02	7.4	7.1	3.00	7.4	7.1	2.45
	1	-	7.4	7.1	1.58	7.4	7.1	2.58	7.4	7.1	2.38
		2	7.2	7.4	1.17	7.4	7.1	1.92	7.4	7.1	2.28
		others	7.2	7.1	1.16	7.4	7.1	2.31	7.4	7.1	2.38
	4	-	7.1	7.4	1.20	7.4	7.1	2.14	7.4	7.1	2.39
	8	-	7.1	7.4	1.39	7.4	7.1	2.06	7.4	7.1	2.40
1/2	1/8	-	7.4	7.1	2.76	7.4	7.1	4.20	7.4	7.1	3.91
		1/4	7.4	7.1	2.62	7.4	7.1	3.74	7.4	7.1	3.17
		1/2	7.4	7.1	2.35	7.4	7.1	3.21	7.4	7.1	2.55
	1	-	7.4	7.1	1.87	7.4	7.1	2.74	7.4	7.1	2.27
	2	-	7.2	7.1	1.33	7.4	7.1	2.41	7.4	7.1	2.22

Table 7.7 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/2	4	1	7.1	7.4	1.33	7.4	7.1	1.82	7.4	7.1	2.07
		others	7.2	7.4	1.15	7.4	7.1	2.20	7.4	7.1	2.23
		8	7.1	7.4	1.33	7.4	7.1	2.09	7.4	7.1	2.24
1	1/8	-	7.4	7.1	3.13	7.4	7.1	4.41	7.4	7.1	4.28
	1/4	-	7.4	7.1	3.01	7.4	7.1	4.02	7.4	7.1	3.69
	1/2	-	7.4	7.1	2.77	7.4	7.1	3.52	7.4	7.1	2.92
	1	-	7.4	7.1	2.31	7.4	7.1	3.00	7.4	7.1	2.31
	2	1-3	7.2	7.1	1.62	7.4	7.1	2.50	7.4	7.1	2.01
		others	7.4	7.1	1.67	7.4	7.1	2.58	7.4	7.1	2.06
	4	1	7.2	7.4	1.25	7.4	7.1	1.92	7.4	7.1	1.80
		others	7.2	7.1	1.25	7.4	7.1	2.31	7.4	7.1	2.01
	8	-	7.1	7.4	1.20	7.4	7.1	2.14	7.4	7.1	2.02
2	1/8	-	7.4	7.1	3.41	7.4	7.1	4.60	7.4	7.1	4.55
	1/4	-	7.4	7.1	3.33	7.4	7.1	4.32	7.4	7.1	4.17
	1/2	-	7.4	7.1	3.15	7.4	7.1	3.90	7.4	7.1	3.52
	1	-	7.4	7.1	2.81	7.4	7.1	3.38	7.4	7.1	2.72
	2	1	7.2	7.1	1.95	7.4	7.1	2.47	7.2	7.1	1.87
		others	7.4	7.1	2.22	7.4	7.1	2.88	7.4	7.1	2.13
	4	1	7.2	7.1	1.47	7.4	7.1	2.11	7.2	7.1	1.72
		others	7.2	7.1	1.61	7.4	7.1	2.50	7.4	7.1	1.89
	8	1	7.2	7.4	1.30	7.4	7.1	1.87	7.2	7.1	1.69
		2-4	7.2	7.4	1.18	7.4	7.1	2.20	7.4	7.1	1.82
		others	7.2	7.1	1.19	7.4	7.1	2.25	7.4	7.1	1.85
4	1/8	-	7.4	7.1	3.56	7.4	7.1	4.75	7.4	7.1	4.73
	1/4	-	7.4	7.1	3.51	7.4	7.1	4.57	7.4	7.1	4.51
	1/2	-	7.4	7.1	3.40	7.4	7.1	4.26	7.4	7.1	4.09
	1	-	7.4	7.1	3.19	7.4	7.1	3.82	7.4	7.1	3.41
	2	-	7.4	7.1	2.79	7.4	7.1	3.30	7.4	7.1	2.60
	4	1	7.2	7.1	1.95	7.4	7.1	2.41	7.2	7.1	1.84
		2-4	7.2	7.1	2.09	7.4	7.1	2.76	7.4	7.1	1.98
		others	7.4	7.1	2.14	7.4	7.1	2.81	7.4	7.1	2.02
	8	1	7.2	7.1	1.42	7.2	7.1	2.06	7.2	7.1	1.67
		2-5	7.2	7.1	1.55	7.4	7.1	2.42	7.2	7.1	1.77
		others	7.2	7.1	1.57	7.4	7.1	2.45	7.4	7.1	1.79
8	1/8	-	7.4	7.1	3.62	7.4	7.1	4.84	7.4	7.1	4.84
	1/4	-	7.4	7.1	3.59	7.4	7.1	4.74	7.4	7.1	4.72
	1/2	-	7.4	7.1	3.53	7.4	7.1	4.55	7.4	7.1	4.49
	1	-	7.4	7.1	3.42	7.4	7.1	4.23	7.4	7.1	4.05
	2	-	7.4	7.1	3.19	7.4	7.1	3.78	7.4	7.1	3.35
	4	1	7.4	7.1	2.45	7.4	7.1	2.83	7.2	7.1	2.20

Table 7.7 Continued

Criterion											
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
8	4	others	7.4	7.1	2.76	7.4	7.1	3.26	7.4	7.1	2.53
	8	1-3	7.2	7.1	2.07	7.4	7.1	2.69	7.2	7.1	1.91
		others	7.2	7.1	2.11	7.4	7.1	2.78	7.4	7.1	1.96

Table 7.8 Comparison of Class VIII Designs

Criterion											
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	8.9	8.5	2.64	8.9	8.4	6.39	8.9	8.1	2.91
	1/4	-	8.9	8.5	2.47	8.9	8.4	4.95	8.9	8.5	2.76
	1/2	-	8.9	8.5	2.14	8.9	8.5	3.78	8.9	8.5	2.91
	1	1	8.7	8.5	1.37	8.9	8.5	2.82	8.9	8.5	2.91
		others	8.9	8.5	1.61	8.9	8.5	3.37	8.9	8.5	2.97
	2	1	8.3	8.9	1.26	8.9	8.5	2.56	8.9	8.5	2.95
		2-3	8.3	8.5	1.13	8.9	8.5	2.98	8.9	8.5	2.99
		others	8.6	8.5	1.15	8.9	8.5	3.10	8.9	8.5	2.99
	4	-	8.3	8.9	1.29	8.9	8.5	2.95	8.9	8.5	3.01
	8	-	8.3	8.9	1.43	8.9	8.5	2.86	8.9	8.5	3.01
1/4	1/8	-	8.9	8.1	2.42	8.9	8.4	5.59	8.9	8.1	3.43
	1/4	-	8.9	8.1	2.29	8.9	8.4	4.45	8.9	8.1	2.77
	1/2	-	8.9	8.1	2.02	8.9	8.4	3.40	8.9	8.1	2.45
	1	1	8.7	8.1	1.37	8.9	8.5	2.49	8.9	8.1	2.27
		others	8.9	8.1	1.58	8.9	8.5	2.97	8.9	8.1	2.38
	2	1	8.3	8.9	1.24	8.9	8.5	2.28	8.9	8.1	2.28
		2	8.3	8.1	1.14	8.9	8.5	2.56	8.9	8.1	2.34
		others	8.7	8.1	1.16	8.9	8.5	2.75	8.9	8.1	2.38
	4	1	8.4	8.9	1.56	8.9	8.5	2.15	8.9	8.1	2.30
		others	8.3	8.9	1.27	8.9	8.5	2.62	8.9	8.1	2.39
	8	-	8.4	8.9	1.42	8.9	8.5	2.55	8.9	8.1	2.40
1/2	1/8	1	8.9	8.1	2.66	8.9	8.4	3.98	8.8	8.1	3.52
		others	8.9	8.1	2.76	8.9	8.4	4.71	8.9	8.1	3.91
	1/4	1	8.9	8.1	2.48	8.9	8.4	3.30	8.8	8.1	2.84
		others	8.9	8.1	2.62	8.9	8.4	3.91	8.9	8.1	3.17
	1/2	-	8.9	8.1	2.35	8.9	8.1	3.21	8.9	8.1	2.55
	1	1	8.8	8.1	1.67	8.9	8.1	2.34	8.9	8.1	2.07

Table 7.8 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/2	1	others	8.9	8.1	1.87	8.9	8.1	2.74	8.9	8.1	2.27
	2	1	8.4	8.1	1.32	8.9	8.1	2.02	8.9	8.1	2.04
		others	8.8	8.1	1.37	8.9	8.1	2.41	8.9	8.1	2.22
	4	-	8.4	8.9	1.26	8.9	8.5	2.25	8.9	8.1	2.23
	8	1	8.5	8.9	1.74	8.9	8.5	1.79	8.9	8.1	2.09
		others	8.4	8.9	1.41	8.9	8.5	2.19	8.9	8.1	2.24
1	1/8	-	8.9	8.1	3.13	8.9	8.1	4.41	8.5	8.1	4.66
	1/4	-	8.9	8.1	3.01	8.9	8.1	4.02	8.5	8.1	3.90
	1/2	1	8.8	8.1	2.59	8.9	8.1	3.08	8.8	8.1	2.83
		others	8.9	8.1	2.77	8.9	8.1	3.52	8.8	8.1	3.01
	1	-	8.8	8.1	2.32	8.9	8.1	3.00	8.8	8.1	2.32
	2	1-2	8.5	8.1	1.74	8.9	8.1	2.43	8.8	8.1	1.99
		others	8.8	8.1	1.79	8.9	8.1	2.58	8.9	8.1	2.06
	4	1	8.5	8.9	1.53	8.9	8.1	1.92	8.8	8.1	1.86
		others	8.5	8.1	1.41	8.9	8.1	2.31	8.9	8.1	2.01
	8	1	8.5	8.9	1.73	8.9	8.1	1.77	8.8	8.1	1.86
		others	8.5	8.9	1.41	8.9	8.1	2.14	8.9	8.1	2.02
2	1/8	-	8.8	8.1	3.58	8.9	8.1	4.60	8.5	8.1	5.65
	1/4	-	8.8	8.1	3.50	8.9	8.1	4.32	8.5	8.1	5.13
	1/2	-	8.8	8.1	3.34	8.9	8.1	3.90	8.5	8.1	4.18
	1	1-5	8.5	8.1	3.01	8.9	8.1	3.34	8.5	8.1	2.97
		others	8.8	8.1	3.02	8.9	8.1	3.38	8.5	8.1	2.97
	2	-	8.5	8.1	2.58	8.9	8.1	2.88	8.8	8.1	2.25
	4	-	8.5	8.1	1.97	8.9	8.1	2.50	8.8	8.1	1.96
	8	1	8.5	8.9	1.73	8.8	8.1	1.88	8.8	8.1	1.79
		2-4	8.5	8.9	1.45	8.9	8.1	2.20	8.8	8.1	1.87
		others	8.5	8.1	1.44	8.9	8.1	2.25	8.8	8.1	1.89
4	1/8	-	8.5	8.1	4.40	8.9	8.1	4.75	8.5	8.1	6.41
	1/4	1	8.5	8.1	4.39	8.8	8.1	4.05	8.5	8.1	6.03
		others	8.5	8.1	4.35	8.9	8.1	4.57	8.5	8.1	6.09
	1/2	1	8.5	8.1	4.28	8.8	8.1	3.79	8.5	8.1	5.39
		others	8.5	8.1	4.25	8.9	8.1	4.26	8.5	8.1	5.45
	1	1	8.5	8.1	4.04	8.8	8.1	3.41	8.5	8.1	4.30
		others	8.5	8.1	4.04	8.9	8.1	3.82	8.5	8.1	4.32
	2	1	8.5	8.1	3.58	8.8	8.1	2.94	8.5	8.1	3.00
		others	8.5	8.1	3.61	8.9	8.1	3.30	8.5	8.1	2.99
	4	1	8.5	8.1	2.83	8.8	8.1	2.49	8.5	8.1	2.10
		others	8.5	8.1	2.89	8.9	8.1	2.81	8.8	8.1	2.18
	8	1	8.5	8.1	1.99	8.8	8.1	2.15	8.4	8.1	1.79
		others	8.5	8.1	2.04	8.9	8.1	2.45	8.8	8.1	1.89

Table 7.8 Continued

Criterion											
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
8	1/8	1	8.5	8.1	5.05	8.8	8.1	4.51	8.5	8.1	6.88
		others	8.5	8.1	5.00	8.9	8.1	4.84	8.5	8.1	6.90
	1/4	1	8.5	8.1	5.01	8.8	8.1	4.41	8.5	8.1	6.70
		others	8.5	8.1	4.96	8.9	8.1	4.74	8.5	8.1	6.72
	1/2	1	8.5	8.1	4.93	8.8	8.1	4.23	8.5	8.1	6.33
		others	8.5	8.1	4.89	8.9	8.1	4.55	8.5	8.1	6.36
	1	1	8.5	8.1	4.78	8.8	8.1	3.94	8.5	8.1	5.60
		others	8.5	8.1	4.75	8.9	8.1	4.23	8.5	8.1	5.63
	2	1	8.5	8.1	4.46	8.8	8.1	3.51	8.5	8.1	4.38
		others	8.5	8.1	4.46	8.9	8.1	3.78	8.5	8.1	4.38
	4	1	8.5	8.1	3.86	8.8	8.1	3.01	8.5	8.1	3.00
		others	8.5	8.1	3.90	8.9	8.1	3.26	8.5	8.1	2.98
	8	1	8.5	8.1	2.94	8.8	8.1	2.54	8.5	8.1	2.07
		others	8.5	8.1	3.01	8.9	8.1	2.78	8.8	8.1	2.14

Table 7.9 Comparison of Class IX Designs

Criterion											
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	9.5	9.4	1.72	9.5	9.3	3.12	9.5	9.1	1.75
		-	9.5	9.4	1.68	9.5	9.3	2.80	9.5	9.4	1.70
	1/4	-	9.5	9.4	1.59	9.5	9.3	2.48	9.5	9.4	1.75
		-	9.5	9.4	1.40	9.5	9.3	2.24	9.5	9.4	1.76
	1/2	-	9.5	9.4	1.12	9.5	9.3	2.05	9.5	9.4	1.77
		-	9.5	9.4	1.15	9.5	9.3	2.10	9.5	9.4	1.77
	1	-	9.5	9.4	1.09	9.5	9.3	2.02	9.5	9.4	1.77
		-	9.2	9.5	1.16	9.5	9.3	1.99	9.5	9.4	1.77
	2	-	9.2	9.5	1.16	9.5	9.3	1.99	9.5	9.4	1.77
		-	9.2	9.5	1.16	9.5	9.3	1.99	9.5	9.4	1.77
1/4	1/8	-	9.5	9.4	1.62	9.5	9.3	2.74	9.5	9.1	2.12
		-	9.5	9.4	1.58	9.5	9.3	2.51	9.5	9.1	1.75
	1/4	-	9.5	9.4	1.51	9.5	9.3	2.26	9.5	9.1	1.57
		-	9.5	9.4	1.35	9.5	9.3	2.05	9.5	9.4	1.60
	1/2	-	9.5	9.4	1.07	9.5	9.3	1.80	9.5	9.4	1.60
		-	9.5	9.4	1.13	9.5	9.3	1.92	9.5	9.4	1.62
	1	-	9.3	9.5	1.08	9.5	9.3	1.85	9.5	9.4	1.63
		-	9.3	9.5	1.16	9.5	9.3	1.82	9.5	9.4	1.63
	2	-	9.3	9.5	1.16	9.5	9.3	1.82	9.5	9.4	1.63
		-	9.3	9.5	1.16	9.5	9.3	1.82	9.5	9.4	1.63

Table 7.9 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/2	1/8	1-3	9.5	9.1	1.68	9.5	9.3	2.26	9.4	9.1	2.38
		others	9.5	9.1	1.68	9.5	9.3	2.29	9.5	9.1	2.41
	1/4	-	9.5	9.1	1.65	9.5	9.3	2.15	9.5	9.1	2.05
	1/2	-	9.5	9.1	1.58	9.5	9.3	1.98	9.5	9.1	1.72
	1	-	9.5	9.1	1.43	9.5	9.3	1.82	9.5	9.1	1.56
	2	1	9.3	9.1	1.15	9.5	9.3	1.61	9.5	9.1	1.48
		others	9.5	9.1	1.21	9.5	9.3	1.71	9.5	9.1	1.51
	4	1-4	9.3	9.5	1.08	9.5	9.4	1.65	9.5	9.1	1.50
		others	9.3	9.1	1.08	9.5	9.4	1.67	9.5	9.1	1.51
	8	1	9.4	9.5	1.23	9.5	9.4	1.54	9.5	9.1	1.48
		others	9.3	9.5	1.15	9.5	9.4	1.65	9.5	9.1	1.51
1	1/8	-	9.5	9.1	1.90	9.5	9.1	2.16	9.4	9.1	2.84
	1/4	-	9.5	9.1	1.86	9.5	9.1	2.05	9.4	9.1	2.46
	1/2	1	9.5	9.1	1.74	9.5	9.1	1.82	9.4	9.1	1.89
		others	9.5	9.1	1.78	9.5	9.1	1.90	9.5	9.1	1.95
	1	-	9.5	9.1	1.63	9.5	9.1	1.73	9.5	9.1	1.63
	2	1	9.4	9.1	1.34	9.5	9.1	1.49	9.5	9.1	1.43
		others	9.5	9.1	1.38	9.5	9.4	1.59	9.5	9.1	1.49
	4	-	9.4	9.1	1.20	9.5	9.4	1.57	9.5	9.1	1.45
	8	-	9.4	9.5	1.15	9.5	9.4	1.56	9.5	9.1	1.45
2	1/8	-	9.5	9.1	2.10	9.5	9.1	2.29	9.4	9.1	3.15
	1/4	-	9.5	9.1	2.07	9.5	9.1	2.20	9.4	9.1	2.92
	1/2	1	9.4	9.1	1.98	9.5	9.1	1.99	9.4	9.1	2.44
		others	9.5	9.1	2.00	9.5	9.1	2.06	9.4	9.1	2.48
	1	-	9.4	9.1	1.89	9.5	9.1	1.89	9.4	9.1	1.89
	2	-	9.4	9.1	1.72	9.5	9.1	1.70	9.5	9.1	1.54
	4	-	9.4	9.1	1.46	9.5	9.1	1.55	9.5	9.1	1.40
	8	1	9.4	9.5	1.25	9.5	9.4	1.38	9.5	9.1	1.31
		others	9.4	9.1	1.23	9.5	9.4	1.48	9.5	9.1	1.36
4	1/8	-	9.4	9.1	2.46	9.5	9.1	2.39	9.4	9.1	3.37
	1/4	-	9.4	9.1	2.44	9.5	9.1	2.33	9.4	9.1	3.23
	1/2	-	9.4	9.1	2.40	9.5	9.1	2.23	9.4	9.1	2.96
	1	-	9.4	9.1	2.32	9.5	9.1	2.07	9.4	9.1	2.46
	2	-	9.4	9.1	2.15	9.5	9.1	1.88	9.4	9.1	1.84
	4	1	9.4	9.1	1.82	9.5	9.1	1.61	9.4	9.1	1.40
		others	9.4	9.1	1.86	9.5	9.1	1.69	9.5	9.1	1.47
	8	-	9.4	9.1	1.49	9.5	9.1	1.54	9.5	9.1	1.34
8	1/8	-	9.4	9.1	2.70	9.5	9.1	2.45	9.4	9.1	3.51
	1/4	-	9.4	9.1	2.69	9.5	9.1	2.42	9.4	9.1	3.44
	1/2	-	9.4	9.1	2.66	9.5	9.1	2.35	9.4	9.1	3.29

Table 7.9 Continued

Criterion											
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
8	1	-	9.4	9.1	2.61	9.5	9.1	2.24	9.4	9.1	2.98
	2	-	9.4	9.1	2.49	9.5	9.1	2.08	9.4	9.1	2.44
	4	-	9.4	9.1	2.26	9.5	9.1	1.88	9.4	9.1	1.80
	8	1	9.4	9.1	1.86	9.5	9.1	1.60	9.4	9.1	1.37
	others		9.4	9.1	1.90	9.5	9.1	1.68	9.5	9.1	1.43

Table 7.10 Comparison of Class X Designs

Criterion											
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	10.12	10.1	3.11	10.12	10.1	12.00	10.14	10.1	3.26
	1/4	-	10.12	10.1	3.06	10.12	10.1	10.90	10.12	10.1	3.02
	1/2	-	10.12	10.1	2.94	10.12	10.1	9.56	10.12	10.1	2.99
	1	-	10.12	10.1	2.66	10.12	10.1	8.28	10.12	10.1	2.99
	2	-	10.12	10.1	2.12	10.12	10.1	7.30	10.12	10.1	2.99
	4	1	10.11	10.1	1.31	10.12	10.1	6.18	10.12	10.1	2.99
	others		10.12	10.1	1.42	10.12	10.1	6.68	10.12	10.1	2.99
	8	1-3	10.6	10.5	1.22	10.12	10.1	6.21	10.12	10.1	2.99
	others		10.7	10.5	1.20	10.12	10.1	6.33	10.12	10.1	2.99
1/4	1/8	-	10.12	10.1	3.12	10.12	10.1	11.48	10.14	10.1	4.04
	1/4	-	10.12	10.1	3.06	10.12	10.1	10.59	10.14	10.1	3.29
	1/2	-	10.12	10.1	2.95	10.12	10.1	9.44	10.12	10.1	2.98
	1	-	10.12	10.1	2.68	10.12	10.1	8.27	10.12	10.1	2.96
	2	-	10.12	10.1	2.16	10.12	10.1	7.33	10.12	10.1	2.96
	4	1	10.11	10.1	1.36	10.12	10.1	6.21	10.12	10.1	2.96
	others		10.12	10.1	1.47	10.12	10.1	6.71	10.12	10.1	2.96
	8	1-5	10.6	10.5	1.23	10.12	10.1	6.30	10.12	10.1	2.97
	others		10.7	10.5	1.23	10.12	10.1	6.35	10.12	10.1	2.97
1/2	1/8	-	10.14	10.1	3.20	10.12	10.1	10.75	10.15	10.1	5.66
	1/4	-	10.14	10.1	3.15	10.12	10.1	10.12	10.14	10.1	3.96
	1/2	-	10.14	10.1	3.04	10.12	10.1	9.24	10.14	10.1	3.27
	1	-	10.14	10.1	2.79	10.12	10.1	8.26	10.14	10.1	2.94
	2	-	10.14	10.1	2.29	10.12	10.1	7.39	10.12	10.1	2.88
	4	1	10.11	10.5	1.46	10.12	10.1	6.26	10.12	10.1	2.87
	others		10.14	10.1	1.59	10.12	10.1	6.77	10.12	10.1	2.89

Table 7.10 Continued

Criterion											
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/2	8	-	10.6	10.5	1.27	10.12	10.1	6.39	10.12	10.1	2.90
1	1/8	-	10.14	10.1	3.54	10.14	10.1	9.95	10.15	10.5	6.72
	1/4	-	10.14	10.1	3.49	10.12	10.1	9.52	10.15	10.1	5.47
	1/2	-	10.14	10.1	3.37	10.12	10.1	8.95	10.15	10.1	3.83
	1	-	10.14	10.1	3.11	10.12	10.1	8.22	10.14	10.1	3.11
	2	-	10.14	10.1	2.61	10.12	10.1	7.47	10.14	10.1	2.84
	4	1-2	10.14	10.5	1.79	10.12	10.1	6.65	10.14	10.1	2.76
	8	others	10.14	10.1	1.85	10.12	10.1	6.86	10.14	10.1	2.78
2	1/8	-	10.14	10.1	3.92	10.14	10.1	9.50	10.15	10.5	7.16
	1/4	-	10.14	10.1	3.87	10.14	10.1	9.23	10.15	10.5	6.49
	1/2	1	10.15	10.1	3.68	10.14	10.1	8.27	10.15	10.5	4.91
		others	10.14	10.1	3.76	10.14	10.1	8.77	10.15	10.5	5.16
	1	-	10.15	10.1	3.54	10.12	10.1	8.14	10.15	10.1	3.52
	2	-	10.15	10.1	3.11	10.12	10.1	7.57	10.14	10.1	2.88
	4	-	10.15	10.5	2.41	10.12	10.1	7.00	10.14	10.1	2.65
4	1/8	-	10.15	10.5	1.71	10.12	10.1	6.07	10.14	10.1	2.54
	1/4	1	10.13	10.5	1.74	10.12	10.1	6.57	10.12	10.1	2.61
	1/2	-	10.15	10.1	4.89	10.14	10.1	9.14	10.15	10.5	7.35
	1	-	10.15	10.1	4.84	10.14	10.1	9.00	10.15	10.5	7.02
	1/2	-	10.15	10.1	4.74	10.14	10.1	8.74	10.15	10.5	6.27
	1	1	10.15	10.5	4.42	10.14	10.1	7.83	10.15	10.5	4.62
		others	10.15	10.1	4.54	10.14	10.1	8.32	10.15	10.5	4.84
8	2	-	10.15	10.5	4.15	10.14	10.1	7.71	10.14	10.1	3.23
	4	-	10.15	10.5	3.46	10.12	10.1	7.19	10.14	10.1	2.68
	8	-	10.15	10.5	2.50	10.12	10.1	6.75	10.14	10.1	2.48
	1/8	-	10.15	10.5	5.57	10.14	10.1	8.91	10.15	10.5	7.44
	1/4	-	10.15	10.5	5.54	10.14	10.1	8.84	10.15	10.5	7.28
	1/2	-	10.15	10.5	5.47	10.14	10.1	8.70	10.15	10.5	6.92
	1	-	10.15	10.5	5.34	10.14	10.1	8.46	10.15	10.5	6.11
	2	-	10.15	10.5	5.07	10.14	10.1	8.05	10.15	10.5	4.61
	4	-	10.15	10.5	4.53	10.14	10.1	7.49	10.14	10.1	3.08
	8	-	10.15	10.5	3.60	10.12	10.1	7.00	10.14	10.1	2.56

Table 7.11 Comparison of Designs with 4r First-Stage Samples

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	1-2	5.5	2.12	1.30	3.8	6.2	1.85	5.5	2.12	1.33
		others	5.5	2.12	1.32	5.5	6.2	2.02	5.5	2.12	1.35
	1/4	1	5.5	2.12	1.25	3.8	6.2	1.59	5.5	2.12	1.40
		others	5.5	2.12	1.29	5.5	6.2	1.77	5.5	2.12	1.43
	1/2	1	5.5	2.12	1.18	3.8	6.2	1.44	5.5	2.12	1.45
		others	5.5	2.12	1.23	5.5	6.2	1.53	5.5	2.12	1.46
	1	1	8.7	2.12	1.08	3.8	2.12	1.35	5.5	2.12	1.47
		2	5.5	2.12	1.10	3.8	2.12	1.36	5.5	2.12	1.47
		others	5.5	2.12	1.12	5.5	2.12	1.38	5.5	2.12	1.48
	2	1-2	7.2	5.5	1.03	3.8	2.12	1.35	5.5	2.12	1.48
		3-9	7.2	1.8	1.02	3.8	2.12	1.35	5.5	2.12	1.48
		10	7.2	1.8	1.02	5.5	2.12	1.35	5.5	2.12	1.48
	4	-	2.12	5.5	1.09	3.8	2.12	1.34	5.5	2.12	1.48
	8	-	2.12	5.5	1.14	3.8	2.12	1.33	5.5	2.12	1.48
1/4	1/8	1	5.5	2.12	1.22	8.7	6.2	1.63	1.8	3.8	1.38
		others	5.5	2.12	1.25	5.5	6.2	1.91	1.8	3.8	1.37
	1/4	1	5.5	2.12	1.20	8.7	6.2	1.48	5.5	3.8	1.19
		others	5.5	2.12	1.23	5.5	6.2	1.69	5.5	2.12	1.23
	1/2	-	5.5	2.12	1.19	5.5	6.2	1.48	5.5	2.12	1.29
		1	5.5	2.12	1.10	5.5	6.2	1.32	5.5	2.12	1.32
	2	1	2.12	5.5	1.05	5.5	2.12	1.23	5.5	2.12	1.32
		others	2.12	3.8	1.02	5.5	2.12	1.26	5.5	2.12	1.33
	4	1	2.12	5.5	1.12	3.8	2.12	1.23	5.5	2.12	1.32
		others	2.12	5.5	1.09	5.5	2.12	1.24	5.5	2.12	1.33
	8	1	2.12	5.5	1.15	3.8	2.12	1.23	5.5	2.12	1.33
		others	2.12	5.5	1.14	5.5	2.12	1.23	5.5	2.12	1.34
1/2	1/8	1-3	5.5	3.8	1.21	1.8	6.2	1.69	1.8	3.8	1.67
		others	5.5	3.8	1.21	5.5	6.2	1.78	1.8	3.8	1.67
	1/4	1-2	5.5	3.8	1.20	1.8	6.2	1.50	1.8	3.8	1.46
		others	5.5	3.8	1.20	5.5	6.2	1.60	1.8	3.8	1.46
	1/2	1	1.8	3.8	1.18	1.8	6.2	1.29	1.8	3.8	1.27
		others	5.5	3.8	1.18	5.5	6.2	1.41	1.8	3.8	1.26
	1	1	1.8	3.8	1.14	8.7	6.2	1.17	5.5	3.8	1.16
		others	1.8	3.8	1.13	5.5	6.2	1.26	5.5	3.8	1.17
	2	-	6.2	3.8	1.09	5.5	6.2	1.17	5.5	3.8	1.16
	4	1-6	2.12	5.5	1.10	5.5	2.12	1.13	5.5	3.8	1.15
		others	2.12	3.8	1.10	5.5	2.12	1.13	5.5	3.8	1.15
	8	1	2.12	5.5	1.15	8.7	2.12	1.11	5.5	2.12	1.14
		others	2.12	5.5	1.14	5.5	2.12	1.12	5.5	2.12	1.16

Table 7.11 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1	1/8	-	1.8	3.8	1.43	1.8	6.2	1.71	1.8	3.8	1.91
	1/4	-	1.8	3.8	1.42	1.8	6.2	1.56	1.8	3.8	1.75
	1/2	-	1.8	3.8	1.39	1.8	6.2	1.37	1.8	3.8	1.52
	1	1-3	1.8	3.8	1.32	1.8	3.8	1.19	1.8	3.8	1.32
		others	1.8	3.8	1.32	5.5	6.2	1.21	1.8	3.8	1.32
	2	1	1.8	3.8	1.22	1.8	3.8	1.15	1.8	3.8	1.23
		others	1.8	3.8	1.22	5.5	3.8	1.16	1.8	3.8	1.22
	4	1	2.12	3.8	1.15	8.7	3.8	1.11	1.8	3.8	1.19
		others	2.12	3.8	1.16	5.5	3.8	1.12	1.8	3.8	1.19
	8	1	2.12	5.5	1.15	8.7	3.8	1.09	1.8	3.8	1.18
		others	2.12	5.5	1.14	5.5	3.8	1.10	1.8	3.8	1.18
2	1/8	1	1.8	3.8	1.72	1.8	3.8	1.67	1.8	3.8	2.15
		others	1.8	3.8	1.68	1.8	6.2	1.71	1.8	3.8	2.10
	1/4	1	1.8	3.8	1.70	1.8	3.8	1.62	1.8	3.8	2.03
		others	1.8	3.8	1.67	1.8	6.2	1.60	1.8	3.8	1.99
	1/2	-	1.8	3.8	1.63	1.8	3.8	1.45	1.8	3.8	1.80
	1	-	1.8	3.8	1.56	1.8	3.8	1.36	1.8	3.8	1.54
	2	-	1.8	3.8	1.42	2.12	3.8	1.29	1.8	3.8	1.34
	4	-	2.12	3.8	1.29	2.12	3.8	1.26	2.12	3.8	1.26
	8	-	2.12	3.8	1.19	2.12	3.8	1.24	2.12	3.8	1.23
4	1/8	1	1.8	3.8	1.89	1.8	3.8	1.86	1.8	3.8	2.30
		others	2.12	3.8	1.85	1.8	3.8	1.72	1.8	3.8	2.23
	1/4	1	1.8	3.8	1.88	1.8	3.8	1.81	1.8	3.8	2.23
		others	2.12	3.8	1.84	1.8	3.8	1.68	1.8	3.8	2.17
	1/2	1	1.8	3.8	1.85	1.8	3.8	1.74	1.8	3.8	2.09
		others	2.12	3.8	1.82	1.8	3.8	1.62	1.8	3.8	2.04
	1	1	1.8	3.8	1.78	1.8	3.8	1.63	1.8	3.8	1.86
		2-3	2.12	3.8	1.77	1.8	3.8	1.55	1.8	3.8	1.83
		others	2.12	3.8	1.77	2.12	3.8	1.54	1.8	3.8	1.82
	2	-	2.12	3.8	1.67	2.12	3.8	1.47	1.8	3.8	1.55
	4	-	2.12	3.8	1.50	2.12	3.8	1.42	2.12	3.8	1.36
	8	-	2.12	3.8	1.32	2.12	3.8	1.38	2.12	3.8	1.28
8	1/8	-	2.12	3.8	2.00	1.8	3.8	1.82	1.8	3.8	2.31
	1/4	-	2.12	3.8	1.99	1.8	3.8	1.79	1.8	3.8	2.28
	1/2	-	2.12	3.8	1.97	1.8	3.8	1.75	1.8	3.8	2.21
	1	1	2.12	3.8	1.95	1.8	3.8	1.81	1.8	3.8	2.13
		others	2.12	3.8	1.94	2.12	3.8	1.70	1.8	3.8	2.07
	2	-	2.12	3.8	1.87	2.12	3.8	1.63	1.8	3.8	1.83
	4	1	2.12	3.8	1.73	2.12	3.8	1.62	1.8	3.8	1.58
		others	2.12	3.8	1.74	2.12	3.8	1.56	2.12	3.8	1.56
	8	-	2.12	3.8	1.54	2.12	3.8	1.50	2.12	3.8	1.37

Table 7.12 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1	8	others	2.9	3.7	1.24	5.3	3.5	1.54	1.7	3.5	1.79
2	1/8	-	1.5	3.5	3.40	1.7	3.5	2.73	1.5	3.5	5.65
	1/4	-	1.5	3.5	3.35	1.7	3.5	2.63	1.5	3.5	5.13
	1/2	-	1.5	3.5	3.24	1.7	3.5	2.46	1.5	3.5	4.18
	1	-	1.5	3.5	3.01	1.7	3.5	2.23	1.5	3.5	2.97
	2	1	1.5	3.5	2.55	1.7	3.5	1.99	1.5	3.5	2.11
		2-5	1.5	3.5	2.58	1.7	3.5	1.99	2.11	3.5	2.13
		others	1.5	3.5	2.58	5.3	3.5	1.99	2.11	3.5	2.14
	4	1	2.9	3.5	1.93	2.11	3.5	1.80	2.11	3.5	1.82
		others	2.9	3.5	2.00	5.3	3.5	1.82	2.11	3.5	1.86
	8	-	2.9	3.5	1.50	4.11	3.5	1.74	2.11	3.5	1.78
4	1/8	1-2	2.9	3.5	4.46	1.5	3.5	3.00	1.5	3.5	6.37
		others	2.9	3.5	4.45	1.7	3.5	2.99	1.5	3.5	6.41
	1/4	1	2.9	3.5	4.42	1.5	3.5	3.03	1.5	3.5	6.03
		others	2.9	3.5	4.40	1.7	3.5	2.91	1.5	3.5	6.09
	1/2	1	2.9	3.5	4.30	1.5	3.5	2.85	1.5	3.5	5.39
		others	2.9	3.5	4.30	1.7	3.5	2.78	1.5	3.5	5.45
	1	1-5	2.9	3.5	4.08	2.11	3.5	2.57	1.5	3.5	4.32
		others	2.9	3.5	4.08	1.7	3.5	2.57	1.5	3.5	4.32
	2	1-2	2.9	3.5	3.63	2.11	3.5	2.39	1.5	3.5	2.99
		others	2.9	3.5	3.66	2.11	3.5	2.38	2.9	3.5	3.00
	4	1	2.9	3.5	2.84	2.11	3.5	2.19	2.9	3.5	2.10
		others	2.9	3.5	2.95	2.11	3.5	2.17	2.11	3.5	2.15
	8	1	2.9	3.5	2.01	2.11	3.5	1.99	9.3	3.5	1.82
		others	2.9	3.5	2.10	2.11	3.5	2.00	2.11	3.5	1.86
8	1/8	-	2.9	3.5	5.17	1.5	3.5	3.29	1.5	3.5	6.90
	1/4	1-3	2.9	3.5	5.13	1.5	3.5	3.28	1.5	3.5	6.71
		others	2.9	3.5	5.13	2.9	3.5	3.23	1.5	3.5	6.72
	1/2	1	2.9	3.5	5.06	1.5	3.5	3.36	1.5	3.5	6.33
		others	2.9	3.5	5.06	2.9	3.5	3.13	1.5	3.5	6.36
	1	-	2.9	3.5	4.90	2.9	3.5	2.96	1.5	3.5	5.63
	2	1-3	2.9	3.5	4.58	2.11	3.5	2.80	1.5	3.5	4.38
		others	2.9	3.5	4.60	2.11	3.5	2.78	2.9	3.5	4.40
	4	1	2.9	3.5	3.92	2.11	3.5	2.60	1.5	3.5	3.00
		others	2.9	3.5	4.02	2.11	3.5	2.55	2.9	3.5	3.01
	8	1	2.9	3.5	2.97	9.3	3.5	2.35	2.11	3.5	2.09
		others	2.9	3.5	3.10	2.11	3.5	2.32	2.11	3.5	2.14

Table 7.13 Comparison of Designs with 6r First-Stage Samples

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	1.6	2.5	4.98	1.6	2.5	25.86	1.6	10.5	5.98
	1/4	-	1.6	2.5	4.77	1.6	2.5	23.96	1.6	2.5	6.76
	1/2	-	1.6	2.5	4.30	1.6	2.5	21.81	1.6	2.5	7.31
	1	-	1.6	2.5	3.38	1.6	2.5	19.91	1.6	2.5	7.51
	2	-	1.6	2.5	2.17	1.6	2.5	18.55	1.6	2.5	7.58
	4	-	10.12	2.5	1.45	1.6	2.5	17.72	1.6	2.5	7.61
	8	-	10.12	1.6	1.34	1.6	2.5	17.26	1.6	2.5	7.62
1/4	1/8	1	1.6	2.5	4.32	1.6	2.5	18.39	1.4	10.5	5.01
		others	1.6	2.5	4.58	1.6	2.5	20.83	1.6	10.5	5.73
	1/4	-	1.6	2.5	4.40	1.6	2.5	19.74	1.6	10.5	5.09
	1/2	-	1.6	2.5	4.01	1.6	2.5	18.40	1.6	2.5	5.51
	1	-	1.6	2.5	3.24	1.6	2.5	17.08	1.6	2.5	5.99
	2	-	1.6	2.5	2.16	1.6	2.5	16.06	1.6	2.5	6.18
	4	-	10.12	2.5	1.46	1.6	2.5	15.40	1.6	2.5	6.26
	8	-	10.12	1.6	1.33	1.6	2.5	15.02	1.6	2.5	6.29
1/2	1/8	-	1.6	10.5	3.95	1.6	2.5	15.63	1.4	10.5	6.54
	1/4	-	1.6	10.5	3.83	1.6	2.5	15.19	1.4	10.5	5.01
	1/2	-	1.6	10.5	3.56	1.6	2.5	14.57	1.6	10.5	4.00
	1	-	1.6	10.5	3.02	1.6	2.5	13.86	1.6	2.5	4.13
	2	1	2.10	10.5	2.01	1.6	2.5	13.16	1.6	2.5	4.37
		others	1.6	10.5	2.16	1.6	2.5	13.22	1.6	2.5	4.48
	4	-	10.14	10.5	1.51	1.6	2.5	12.76	1.6	2.5	4.63
	8	-	10.14	1.6	1.32	1.6	2.5	12.48	1.6	2.5	4.70
1	1/8	1-3	1.4	10.5	3.70	1.6	2.5	11.24	10.15	10.5	6.63
		others	1.4	10.5	3.72	1.6	2.5	11.49	1.4	10.5	6.77
	1/4	-	1.4	10.5	3.65	1.6	2.5	11.39	1.4	10.5	5.85
	1/2	-	1.4	10.5	3.50	1.6	2.5	11.22	1.4	10.5	4.42
	1	-	1.4	10.5	3.19	1.6	2.5	10.96	1.4	10.5	3.19
	2	-	1.4	10.5	2.58	1.6	2.5	10.67	1.6	2.5	3.08
	4	-	10.14	10.5	1.83	1.6	2.5	10.41	1.6	2.5	3.31
	8	-	10.14	10.5	1.37	1.6	2.5	10.23	1.6	2.5	3.42
2	1/8	-	1.4	10.5	4.31	1.6	2.5	8.84	10.15	10.5	7.16
	1/4	1	1.4	10.5	4.13	2.10	2.5	8.17	10.15	10.5	6.14
		others	1.4	10.5	4.25	1.6	2.5	8.86	10.15	10.5	6.49
	1/2	1	1.4	10.5	3.99	2.10	2.5	8.33	10.15	10.5	4.91
		2	1.4	10.5	4.06	1.6	2.5	8.63	10.15	10.5	5.05
		others	1.4	10.5	4.11	1.6	2.5	8.86	1.4	10.5	5.21
	1	1-4	1.4	10.5	3.81	2.10	2.5	8.79	1.4	10.5	3.79

Table 7.13 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
2	1	others	1.4	10.5	3.83	1.6	2.5	8.85	1.4	10.5	3.82
	2	-	1.4	10.5	3.26	2.10	2.5	9.02	1.4	10.5	2.70
	4	1-2	10.15	10.5	2.36	2.10	2.5	9.14	2.10	2.5	2.66
		others	2.8	10.5	2.42	2.10	2.5	9.19	2.10	2.5	2.70
	8	-	10.15	10.5	1.73	2.10	2.5	9.31	2.10	2.5	2.86
4	1/8	1	10.15	10.5	4.76	1.4	2.5	7.46	10.15	10.5	6.96
		others	10.15	10.5	4.83	2.10	2.5	7.79	10.15	10.5	7.35
	1/4	-	10.15	10.5	4.79	2.10	2.5	7.85	10.15	10.5	7.02
	1/2	-	10.15	10.5	4.70	2.10	2.5	7.96	10.15	10.5	6.27
	1	-	10.15	10.5	4.52	2.10	2.5	8.13	10.15	10.5	4.84
	2	1	10.15	10.5	4.02	2.10	2.5	8.13	1.4	10.5	3.24
		others	10.15	10.5	4.15	2.10	2.5	8.36	2.8	10.5	3.42
	4	-	10.15	10.5	3.46	2.10	2.5	8.61	2.8	10.5	2.39
	8	-	10.15	10.5	2.50	2.10	2.5	8.82	2.10	2.5	2.53
8	1/8	1-3	10.15	10.5	5.55	2.8	2.5	7.36	10.15	10.5	7.34
		others	10.15	10.5	5.57	2.10	2.5	7.43	10.15	10.5	7.44
	1/4	1-2	10.15	10.5	5.50	2.8	2.5	7.36	10.15	10.5	7.12
		others	10.15	10.5	5.54	2.10	2.5	7.46	10.15	10.5	7.28
	1/2	1	10.15	10.5	5.37	2.8	2.5	7.33	10.15	10.5	6.56
		others	10.15	10.5	5.47	2.10	2.5	7.53	10.15	10.5	6.92
	1	1	10.15	10.5	5.22	2.8	2.5	7.41	10.15	10.5	5.80
		others	10.15	10.5	5.34	2.10	2.5	7.65	10.15	10.5	6.11
	2	-	10.15	10.5	5.07	2.10	2.5	7.84	10.15	10.5	4.61
	4	-	10.15	10.5	4.53	2.10	2.5	8.10	2.8	10.5	3.20
	8	-	10.15	10.5	3.60	2.10	2.5	8.38	2.8	2.5	2.27

Table 7.14 Comparison of Designs with 7r First-Stage Samples

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	5.2	10.4	3.09	5.2	10.4	3.75	5.2	10.4	4.31
	1/4	-	5.2	10.4	3.01	5.2	10.4	3.65	5.2	10.4	3.99
	1/2	-	5.2	10.4	2.83	5.2	10.4	3.47	5.2	10.4	3.89
	1	-	5.2	10.4	2.45	5.2	10.4	3.24	5.2	10.4	3.88
	2	-	5.2	10.4	1.82	5.2	10.4	3.02	5.2	10.4	3.88
	4	1	3.3	10.4	1.24	5.2	10.4	3.00	5.2	10.4	3.88
		others	4.7	10.4	1.26	5.2	10.4	2.87	5.2	10.4	3.88
	8	-	10.11	5.2	1.21	5.2	10.4	2.77	5.2	10.4	3.88
1/4	1/8	-	5.2	10.4	3.12	5.2	10.4	3.52	1.3	10.4	4.48
	1/4	-	5.2	10.4	3.05	5.2	10.4	3.41	5.2	10.4	3.88
	1/2	-	5.2	10.4	2.88	5.2	10.4	3.23	5.2	10.4	3.61
	1	-	5.2	10.4	2.52	5.2	10.4	3.00	5.2	10.4	3.54
	2	-	5.2	10.4	1.91	5.2	10.4	2.77	5.2	10.4	3.54
	4	1	1.3	10.4	1.29	5.2	10.4	2.72	5.2	10.4	3.54
		others	4.7	10.4	1.32	5.2	10.4	2.61	5.2	10.4	3.54
	8	-	10.11	5.2	1.20	5.2	10.4	2.50	5.2	10.4	3.55
1/2	1/8	-	5.2	10.4	3.12	5.2	10.4	3.23	2.4	10.4	5.59
	1/4	-	5.2	10.4	3.05	5.2	10.4	3.12	1.3	10.4	4.11
	1/2	1	5.2	10.4	2.79	5.2	10.4	2.80	1.3	10.4	3.12
		others	5.2	10.4	2.92	5.2	10.4	2.94	5.2	10.4	3.28
	1	-	5.2	10.4	2.61	5.2	10.4	2.70	5.2	10.4	3.03
	2	-	5.2	10.4	2.05	5.2	10.4	2.47	5.2	10.4	2.97
	4	1	1.3	10.4	1.43	5.2	10.4	2.38	5.2	10.4	2.96
		others	2.7	10.4	1.45	5.2	10.4	2.29	5.2	10.4	2.98
	8	1	10.11	10.4	1.17	5.2	10.4	2.30	5.2	10.4	2.98
		others	10.11	5.2	1.20	5.2	10.4	2.18	5.2	10.4	2.99
1	1/8	-	1.3	10.4	3.18	5.2	10.4	2.93	2.4	10.4	6.55
	1/4	-	1.3	10.4	3.13	5.2	10.4	2.83	2.4	10.4	5.42
	1/2	-	1.3	10.4	3.03	5.2	10.4	2.67	2.4	10.4	3.75
	1	-	1.3	10.4	2.81	5.2	10.4	2.45	1.3	10.4	2.81
	2	1-2	1.3	10.4	2.33	5.2	10.4	2.21	1.3	10.4	2.39
		others	1.3	10.4	2.36	5.2	10.4	2.21	5.2	10.4	2.41
	4	1	1.3	10.4	1.69	5.2	10.4	2.07	5.2	10.4	2.33
		others	2.4	10.4	1.73	5.2	10.4	2.01	5.2	10.4	2.35
	8	-	10.11	10.4	1.32	5.2	10.4	1.87	5.2	10.4	2.35
2	1/8	-	2.4	10.4	3.76	5.2	10.4	2.69	2.4	10.4	6.71
	1/4	-	2.4	10.4	3.72	5.2	10.4	2.61	2.4	10.4	6.14
	1/2	-	2.4	10.4	3.64	5.2	10.4	2.48	2.4	10.4	4.97

Table 7.14 Continued

Criterion											
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
2	1	-	2.4	10.4	3.46	5.2	10.4	2.29	2.4	10.4	3.36
	2	-	2.4	10.4	3.07	5.2	10.4	2.06	1.3	10.4	2.38
	4	-	2.4	10.4	2.42	5.2	10.4	1.83	1.3	10.4	2.01
	8	1-5	2.4	10.4	1.68	5.2	10.4	1.67	1.3	10.4	1.90
		others	2.4	10.4	1.68	4.7	10.4	1.67	1.3	10.4	1.91
4	1/8	-	2.4	10.4	4.63	1.3	10.4	2.56	2.4	10.4	6.56
	1/4	-	2.4	10.4	4.59	1.3	10.4	2.51	2.4	10.4	6.30
	1/2	-	2.4	10.4	4.51	1.3	10.4	2.41	2.4	10.4	5.70
	1	1	2.4	10.4	4.21	1.3	10.4	2.20	2.4	10.4	4.26
		others	2.4	10.4	4.34	2.4	10.4	2.26	2.4	10.4	4.50
	2	-	2.4	10.4	3.99	2.7	10.4	2.06	2.4	10.4	2.96
	4	-	2.4	10.4	3.33	2.7	10.4	1.84	2.7	10.4	2.11
	8	-	2.4	10.4	2.40	10.11	10.4	1.69	2.7	10.4	1.77
8	1/8	-	2.4	10.4	4.92	2.7	10.4	2.56	2.4	10.4	6.37
	1/4	-	2.4	10.4	4.90	2.7	10.4	2.53	2.4	10.4	6.26
	1/2	-	2.4	10.4	4.85	2.7	10.4	2.47	2.4	10.4	5.99
	1	-	2.4	10.4	4.74	2.7	10.4	2.37	2.4	10.4	5.38
	2	-	2.4	10.4	4.53	2.7	10.4	2.22	2.4	10.4	4.18
	4	-	2.4	10.4	4.09	2.7	10.4	2.01	2.4	10.4	2.69
	8	-	2.4	10.4	3.31	10.11	10.4	1.80	2.7	10.4	1.96

Table 7.15 Comparison of Designs with 8r First-Stage Samples

Criterion											
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	3.2	10.3	2.59	3.2	10.3	3.26	1.2	10.3	3.33
	1/4	-	3.2	10.3	2.54	3.2	10.3	3.21	3.2	10.3	3.25
	1/2	-	3.2	10.3	2.44	3.2	10.3	3.11	3.2	10.3	3.30
	1	-	3.2	10.3	2.20	3.2	10.3	2.96	3.2	10.3	3.32
	2	-	3.2	10.3	1.76	3.2	10.3	2.80	3.2	10.3	3.32
	4	-	3.2	10.3	1.26	3.2	10.3	2.68	3.2	10.3	3.33
	8	1	10.7	10.3	1.07	3.2	10.3	2.83	3.2	10.3	3.33
		others	10.7	1.2	1.14	3.2	10.3	2.60	3.2	10.3	3.33

Table 7.15 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
8	1/8	-	2.3	10.3	3.32	2.6	10.3	1.98	2.3	10.3	4.02
	1/4	-	2.3	10.3	3.31	2.6	10.3	1.97	2.3	10.3	3.98
	1/2	-	2.3	10.3	3.29	2.6	10.3	1.94	2.3	10.3	3.88
	1	-	2.3	10.3	3.25	2.6	10.3	1.88	2.3	10.3	3.63
	2	-	2.3	10.3	3.16	2.6	10.3	1.79	2.3	10.3	3.07
	4	-	2.3	10.3	2.97	2.6	10.3	1.67	2.3	10.3	2.22
	8	-	2.3	10.3	2.59	2.6	10.3	1.52	2.6	10.3	1.65

Table 7.16 Comparison of Designs with 9r First-Stage Samples

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	1.1	10.2	2.37	4.2	10.2	4.34	1.1	10.2	3.08
	1/4	1-4	1.1	10.2	2.33	4.2	10.2	3.19	1.1	10.2	3.23
		others	1.1	10.2	2.34	1.1	10.2	3.21	1.1	10.2	3.24
	1/2	-	1.1	10.2	2.27	1.1	10.2	3.11	1.1	10.2	3.32
	1	-	1.1	10.2	2.09	1.1	10.2	2.95	1.1	10.2	3.35
	2	-	1.1	10.2	1.74	1.1	10.2	2.78	1.1	10.2	3.37
	4	-	1.1	10.2	1.28	1.1	10.2	2.65	1.1	10.2	3.37
	8	1	4.2	10.2	1.04	1.1	10.2	2.85	1.1	10.2	3.37
		others	4.2	1.1	1.07	1.1	10.2	2.56	1.1	10.2	3.37
1/4	1/8	-	1.1	10.2	2.27	4.2	10.2	3.12	1.1	10.2	2.60
	1/4	-	1.1	10.2	2.25	1.1	10.2	2.80	1.1	10.2	2.68
	1/2	-	1.1	10.2	2.18	1.1	10.2	2.71	1.1	10.2	2.78
	1	-	1.1	10.2	2.04	1.1	10.2	2.57	1.1	10.2	2.84
	2	-	1.1	10.2	1.73	1.1	10.2	2.42	1.1	10.2	2.86
	4	-	1.1	10.2	1.30	1.1	10.2	2.29	1.1	10.2	2.87
	8	1	1.1	10.2	1.05	1.1	10.2	2.45	1.1	10.2	2.88
		2	10.6	10.2	1.05	1.1	10.2	2.29	1.1	10.2	2.88
		others	10.6	1.1	1.07	1.1	10.2	2.21	1.1	10.2	2.88
1/2	1/8	-	1.1	10.2	2.08	1.1	10.2	2.37	2.2	10.2	2.53
	1/4	-	1.1	10.2	2.06	1.1	10.2	2.33	2.2	10.2	2.20
	1/2	-	1.1	10.2	2.02	1.1	10.2	2.26	1.1	10.2	2.18
	1	-	1.1	10.2	1.92	1.1	10.2	2.14	1.1	10.2	2.20

Table 7.16 Continued

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/2	2	-	1.1	10.2	1.70	1.1	10.2	2.00	1.1	10.2	2.22
	4	-	1.1	10.2	1.34	1.1	10.2	1.89	1.1	10.2	2.23
	8	1	2.2	10.2	1.09	1.1	10.2	2.00	1.1	10.2	2.25
		others	10.6	10.2	1.10	1.1	10.2	1.81	1.1	10.2	2.24
1	1/8	1	2.2	10.2	1.76	1.1	10.2	1.83	2.2	10.2	2.45
		others	1.1	10.2	1.81	1.1	10.2	1.95	2.2	10.2	2.56
	1/4	1	2.2	10.2	1.75	1.1	10.2	1.82	2.2	10.2	2.32
		others	1.1	10.2	1.80	1.1	10.2	1.91	2.2	10.2	2.41
	1/2	1	2.2	10.2	1.74	1.1	10.2	1.80	2.2	10.2	2.03
		others	1.1	10.2	1.77	1.1	10.2	1.85	2.2	10.2	2.09
	1	1-3	2.2	10.2	1.71	1.1	10.2	1.76	2.2	10.2	1.71
		others	1.1	10.2	1.72	1.1	10.2	1.76	1.1	10.2	1.72
	2	-	2.2	10.2	1.62	1.1	10.2	1.65	1.1	10.2	1.68
	4	-	2.2	10.2	1.44	1.1	10.2	1.54	1.1	10.2	1.67
	8	1-2	2.2	10.2	1.19	1.1	10.2	1.51	1.1	10.2	1.68
		others	10.6	10.2	1.19	1.1	10.2	1.46	1.1	10.2	1.68
2	1/8	-	2.2	10.2	2.01	1.1	10.2	1.64	2.2	10.2	2.51
	1/4	-	2.2	10.2	2.00	1.1	10.2	1.62	2.2	10.2	2.45
	1/2	-	2.2	10.2	1.98	1.1	10.2	1.58	2.2	10.2	2.28
	1	-	2.2	10.2	1.94	1.1	10.2	1.51	2.2	10.2	1.94
	2	-	2.2	10.2	1.86	1.1	10.2	1.41	2.2	10.2	1.56
	4	-	2.2	10.2	1.68	1.1	10.2	1.31	2.2	10.2	1.36
	8	-	2.2	10.2	1.39	1.1	10.2	1.22	1.1	10.2	1.33
4	1/8	-	2.2	10.2	2.13	1.1	10.2	1.46	2.2	10.2	2.47
	1/4	-	2.2	10.2	2.13	1.1	10.2	1.44	2.2	10.2	2.44
	1/2	-	2.2	10.2	2.12	1.1	10.2	1.41	2.2	10.2	2.36
	1	-	2.2	10.2	2.09	1.1	10.2	1.37	2.2	10.2	2.17
	2	-	2.2	10.2	2.03	1.1	10.2	1.29	2.2	10.2	1.81
	4	-	2.2	10.2	1.91	1.1	10.2	1.20	2.2	10.2	1.43
	8	-	2.2	10.2	1.66	1.1	10.2	1.11	2.2	10.2	1.24
8	1/8	-	2.2	10.2	2.17	1.1	10.2	1.35	2.2	10.2	2.43
	1/4	-	2.2	10.2	2.17	1.1	10.2	1.35	2.2	10.2	2.42
	1/2	-	2.2	10.2	2.16	1.1	10.2	1.33	2.2	10.2	2.39
	1	-	2.2	10.2	2.15	1.1	10.2	1.30	2.2	10.2	2.31
	2	-	2.2	10.2	2.12	1.1	10.2	1.25	2.2	10.2	2.10
	4	1-4	2.2	10.2	2.04	1.1	10.2	1.17	2.2	10.2	1.71
		others	2.2	10.2	2.05	2.2	10.2	1.19	2.2	10.2	1.72
	8	1	2.2	10.2	1.86	1.1	10.2	1.13	2.2	10.2	1.34
		2-8	2.2	10.2	1.90	10.6	10.2	1.11	2.2	10.2	1.35
		others	2.2	10.2	1.90	2.2	10.2	1.11	2.2	10.2	1.35

Table 7.17 Comparison of Designs with 10r First-Stage Samples

			Criterion								
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
1/8	1/8	-	2.1	10.1	1.45	2.1	10.1	1.66	2.1	10.1	1.66
	1/4	-	2.1	10.1	1.44	2.1	10.1	1.65	2.1	10.1	1.68
	1/2	-	2.1	10.1	1.43	2.1	10.1	1.63	2.1	10.1	1.69
	1	-	2.1	10.1	1.40	2.1	10.1	1.60	2.1	10.1	1.69
	2	-	2.1	10.1	1.33	2.1	10.1	1.56	2.1	10.1	1.70
	4	-	2.1	10.1	1.20	2.1	10.1	1.53	2.1	10.1	1.70
	8	-	2.1	10.1	1.04	2.1	10.1	1.51	2.1	10.1	1.70
1/4	1/8	-	2.1	10.1	1.43	2.1	10.1	1.58	2.1	10.1	1.55
	1/4	-	2.1	10.1	1.42	2.1	10.1	1.57	2.1	10.1	1.57
	1/2	-	2.1	10.1	1.41	2.1	10.1	1.55	2.1	10.1	1.59
	1	-	2.1	10.1	1.39	2.1	10.1	1.52	2.1	10.1	1.60
	2	-	2.1	10.1	1.33	2.1	10.1	1.48	2.1	10.1	1.61
	4	-	2.1	10.1	1.20	2.1	10.1	1.45	2.1	10.1	1.61
	8	-	2.1	10.1	1.05	2.1	10.1	1.43	2.1	10.1	1.61
1/2	1/8	-	2.1	10.1	1.39	2.1	10.1	1.47	2.1	10.1	1.44
	1/4	-	2.1	10.1	1.38	2.1	10.1	1.46	2.1	10.1	1.44
	1/2	-	2.1	10.1	1.37	2.1	10.1	1.44	2.1	10.1	1.44
	1	-	2.1	10.1	1.35	2.1	10.1	1.42	2.1	10.1	1.45
	2	-	2.1	10.1	1.31	2.1	10.1	1.38	2.1	10.1	1.46
	4	-	2.1	10.1	1.20	2.1	10.1	1.35	2.1	10.1	1.46
	8	-	2.1	10.1	1.06	2.1	10.1	1.32	2.1	10.1	1.46
1	1/8	-	2.1	10.1	1.31	2.1	10.1	1.35	2.1	10.1	1.34
	1/4	-	2.1	10.1	1.31	2.1	10.1	1.34	2.1	10.1	1.33
	1/2	-	2.1	10.1	1.31	2.1	10.1	1.33	2.1	10.1	1.32
	1	-	2.1	10.1	1.29	2.1	10.1	1.30	2.1	10.1	1.29
	2	-	2.1	10.1	1.29	2.1	10.1	1.27	2.1	10.1	1.28
	4	-	2.1	10.1	1.20	2.1	10.1	1.24	2.1	10.1	1.28
	8	-	2.1	10.1	1.08	2.1	10.1	1.21	2.1	10.1	1.28
2	1/8	-	2.1	10.1	1.23	2.1	10.1	1.25	2.1	10.1	1.25
	1/4	-	2.1	10.1	1.22	2.1	10.1	1.25	2.1	10.1	1.25
	1/2	-	2.1	10.1	1.22	2.1	10.1	1.24	2.1	10.1	1.23
	1	-	2.1	10.1	1.22	2.1	10.1	1.21	2.1	10.1	1.21
	2	-	2.1	10.1	1.20	2.1	10.1	1.18	2.1	10.1	1.17
	4	-	2.1	10.1	1.17	2.1	10.1	1.15	2.1	10.1	1.15
	8	-	2.1	10.1	1.10	2.1	10.1	1.12	2.1	10.1	1.14
4	1/8	-	2.1	10.1	1.15	2.1	10.1	1.19	2.1	10.1	1.19
	1/4	-	2.1	10.1	1.15	2.1	10.1	1.18	2.1	10.1	1.18
	1/2	-	2.1	10.1	1.15	2.1	10.1	1.17	2.1	10.1	1.18

7.17 Continued

Criterion											
			Trace			Determinant			Adjusted Trace		
ρ_1	ρ_2	r	Opt.	Worst	Ratio	Opt.	Worst	Ratio	Opt.	Worst	Ratio
	1	-	2.1	10.1	1.15	2.1	10.1	1.16	2.1	10.1	1.17
	2	-	2.1	10.1	1.14	2.1	10.1	1.13	2.1	10.1	1.14
	4	-	2.1	10.1	1.13	2.1	10.1	1.10	2.1	10.1	1.10
	8	-	2.1	10.1	1.10	2.1	10.1	1.06	2.1	10.1	1.07
8	1/8	-	2.1	10.1	1.10	2.1	10.1	1.15	2.1	10.1	1.14
	1/4	-	2.1	10.1	1.10	2.1	10.1	1.15	2.1	10.1	1.14
	1/2	-	2.1	10.1	1.10	2.1	10.1	1.14	2.1	10.1	1.14
	1	-	2.1	10.1	1.10	2.1	10.1	1.13	2.1	10.1	1.14
	2	-	2.1	10.1	1.10	2.1	10.1	1.11	2.1	10.1	1.12
	4	-	2.1	10.1	1.09	2.1	10.1	1.08	2.1	10.1	1.10
	8	-	2.1	10.1	1.08	2.1	10.1	1.05	2.1	10.1	1.06