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This research was supported in part by the U.S. Army Research Office, Durham, under Contract No. DAHC04-71-C-0042.

¹

Ph.D. dissertation written under the direction of Professor N.L. Johnson

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SEQUENTIAL TESTS OF COMPOSITE HYPOTHESES^{*,1}

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Institute of Statistics Mimeo Series No. 848

October, 1972

NANAK CHAND. Sequential Tests of Composite Hypotheses. (Under the direction of Professor NORMAN L. JOHNSON.)

Chapter II contains the derivation and properties of sequential tests of composite hypotheses for families of distributions satisfying certain conditions which are discussed in section 2.1. Discrimination among $k \geq 3$ composite hypotheses is considered in Chapter III.

Some of the cases in which none of the conditions of Chapter II is satisfied have been studied in Chapter IV. Procedures in sections 4.2, 4.4, 4.8 and 4.9 give tests whose power functions as well as A.S.N. functions are independent of the nuisance parameters. An outline of a general procedure for obtaining such a test is contained in Chapter I. Sequential procedures for analysis of variance under random and mixed models have been developed in Chapter V. We consider a composite hypothesis in the general random model and derive a sequential test through the principle of invariance. Unlike the current sequential test procedures for this problem, most of our tests reduce to the form in which their properties can be studied by standard methods.

Exact properties of tests of composite hypotheses in some special cases have been obtained in Chapter VI. Tests of sections 6.2 and 6.3 may be chosen so as to achieve a desired strength $(\alpha, 0)$. Procedures to choose one of the two and one of the three hypotheses about the mean of a rectangular distribution with unknown variance have been constructed in sections 6.4 and 6.5 respectively.

TABLE OF CONTENTS

CHAPTER		PAGE
	ACKNOWLEDGEMENTS	v
I	INTRODUCTION	
	1.1 Outline	1
	1.2 A Sequential Analog of Stein's Two Stage Test	2
	1.3 An Outline of a General Procedure for Deriving a Sequential Test with Power Function Independent of the Nuisance Parameters	4
	1.4 A Property of Maximum Likelihood Estimators of Location and Scale Parameters	6
	1.5 A Summary of the Results in Chapters II-VI	7
II	SEQUENTIAL TEST 'S'	
	2.1 Determining the Decision Boundaries	9
	2.2 Strength of the Test 'S'	11
	2.3 Operating Characteristic and Average Sample Number	14
	2.4 A Class of Continuous Distributions Whose Density Function Satisfied (A1)	15
	2.5 Laplace Distribution: Inferences on the Location Parameter When the Scale Parameter is Not Known	16
	2.6 An Alternative (Approximate) Form of Results in 2.5	21
	2.7 Inferences on the Mean of Inverse Gaussian Distribution	23
	2.8 Testing a Parameter of Lognormal Distribution	26
	2.9 Numerical Results	28
III	DISCRIMINATION AMONG $k > 2$ HYPOTHESES IN THE PRESENCE OF NUISANCE PARAMETERS	
	3.1 Introduction	37
	3.2 An Extension of a Sequential Discrimination Procedure	37
	3.3 Discriminating Three Hypotheses About a Normal Mean, the Variance Being Unknown	44

IV	SOME TESTS WITH THE USUAL DECISION BOUNDARIES	
	4.1 Introduction	56
	4.2 A Test of Randomness with Power Function and A.S.N. Function Independent of the Nuisance Parameter	57
	4.3 Testing Scale Parameter of a Gamma Distribution	59
	4.4 Testing the Scale Parameter of Laplace Distribution; the Power Function and A.S.N. Function of the Test Being Independent of the Location Parameter	60
	4.5 Testing an Inverse Gaussian Parameter	63
	4.6 Inferences on a Parameter of a Singly Truncated Normal Distribution	64
	4.7 Testing Equality of Two Poisson Means	66
	4.8 Inferences on the Shape Parameter of Pareto Distribution; the Power Function and A.S.N. Function of the Test Being Independent of the Nuisance Parameter	68
	4.9 Testing the Scale Parameter of Exponential Distribution; the Power Function and the Average Sample Number of the Test Being Independent of the Location Parameter	70
	4.10 Numerical Results	72
V	SEQUENTIAL TESTS FOR ANALYSIS OF VARIANCE UNDER RANDOM AND MIXED MODELS	
	5.1 Introduction	83
	5.2 A General Problem	83
	5.3 A General Procedure for Deriving an Invariant S.P.R.T.	84
	5.4 Definition of Sequential χ^2 -Test	86
	5.5 Application of the χ^2 -Test for Testing Hypotheses About the Ratio of Variances of Two Normal Populations	88
	5.6 One Way Classification	92
	5.7 The Randomized Block Design	98
	5.8 Testing Interaction Hypotheses in a Two-Way Classification with Balanced Replications	100
	5.9 The Two Stage Nested Design	103
	5.10 Randomized Block Design (Tests Under Mixed Models)	104

CHAPTER		PAGE
	5.11 Two-Way Classification with Balanced Replications (Tests Under Mixed Models)	105
	5.12 The Two Stage Nested Design (Tests Under Mixed Models)	106
	5.13 Numerical Evaluation and Comparison	107
VI	SOME EXACT RESULTS	
	6.1 Introduction	110
	6.2 Testing Location Parameter of Exponential Distribution	110
	6.3 Testing a Parameter of Pareto Distribution	115
	6.4 Testing the Mean of the Rectangular Distribution	118
	6.5 Choosing One of the Three Hypotheses about the Mean of the Rectangular Distribution with Unknown Variance	124
	BIBLIOGRAPHY	127

ACKNOWLEDGEMENTS

I am deeply indebted to Professor Norman L. Johnson, not only for suggesting the problem, but also for his guidance and inspiration throughout the course of this investigation. His illuminating comments and suggestions were a constant encouragement; it has been an invaluable experience to have had the opportunity to work under his direction.

I would also like to thank Professor Wassily Hoeffding, the Chairman of my Dissertation Committee, for his helpful suggestions and also for his encouragement during my graduate study in Chapel Hill. Thanks are also due to Professors I. M. Chakravarti, Gordon D. Simons, and R. L. Davis for their interest and suggestions as well as to the faculty of the Departments of Statistics and Mathematics for their teaching.

The financial support provided by the Department of Statistics during my period of study and work at Chapel Hill is greatly appreciated.

Finally, I wish to thank Mrs. Mary Jeffcoat for her excellent typing of the final manuscript and Mrs. Eloise Walker, Mr. James O. Kitchen and the secretarial staff of the Department of Statistics for their generous help throughout my graduate study and work in this Department.

CHAPTER I

INTRODUCTION

1.1. *Outline*

Sequential t-test procedures ([6],[7],[11],[13],[16],[22],[25],[26],[27],[32]) to discriminate between two possible values μ_0 and μ_1 of the mean μ of a normal population with unknown variance σ^2 necessitate expressing the difference $\mu_1 - \mu_0$ in (unknown) standard deviation units or equivalently allowing the upper bound of the 'second kind of error probability' to depend upon σ . Neither of these reformulations may be completely satisfactory.

One solution to the problem is Stein's two stage test (Stein [29]; Moshman [21]) in which a first stage sample of fixed size n_0 is taken to estimate σ^2 and a second sample of size depending upon this first stage estimate is then taken, if necessary, to reach the terminal decision.

A second solution, which may be regarded as a generalization to Stein's procedure, is due to Baker [5]. Hall [14] developed the same procedure in somewhat modified form. A summary of the procedure is given in section 1.2.

In this dissertation we shall study similar problems in a number of different circumstances. An outline of a fairly general procedure to derive a sequential test with power function independent of nuisance

parameters is contained in section 1.3. Later chapters are concerned with some specific applications. A general result of Antle and Bain [1] on distribution of maximum likelihood estimators of location and scale parameters, which we describe in section 1.4, is of some use in these applications. Section 1.5 contains a summary of the results of Chapters II-VI.

1.2. A Sequential Analog of Stein's Two Stage Test

This section summarizes the work of Baker [5] and Hall [14].

Let $X_1, X_2 \dots$ be independent $N(\mu, \sigma^2)$. The null and the alternative hypotheses are $H_0 : \mu = \mu_0$ vs. $H_1 : \mu = \mu_1 ; \sigma > 0$. Baker [5] developed the following procedure:

Following Stein [29], a preliminary sample $\{x_1, \dots, x_{n_0}\}$ of fixed size $n_0 (> 1)$ is taken to estimate σ^2 by $s_{n_0}^2 = (n_0 - 1)^{-1} \sum_{i=1}^{n_0} (x_i - \bar{x}_{n_0})^2$ (where $\bar{x}_{n_0} = n_0^{-1} \sum_{i=1}^{n_0} x_i$) and sampling is then continued, one observation at a time, until the sequential probability ratio test (S.P.R.T.), in which $s_{n_0}^2$ replaces σ^2 in the likelihood ratio decides one of the hypotheses to be true. In Stein's procedure, the size of the second sample depends only upon $s_{n_0}^2$, so its distribution depends only upon σ^2 ; but in the above procedure, the distribution of the sample size depends upon μ as well as upon σ^2 ; however the power function is independent of σ^2 as in Stein's procedure. The usual termination boundaries are modified so as to give the desired strength (α, β) (bounds on the error probabilities). Since it is a conditional S.P.R.T., given s_{n_0} and σ its behaviour was studied by taking expectations with respect to the random variable S_{n_0} . The following expressions were obtained:

$$\Pr\{\text{accepting } H_0 | \mu\}$$

$$= \left(\frac{1}{2}v\right)^{\frac{v}{2}} \sum_{r=0}^{\infty} \left\{ \left(\frac{1}{2}v + rh(\mu)(C + D)\right)^{-\frac{v}{2}} - \left(\frac{1}{2}v + Ch(\mu) + rh(\mu)(C + D)\right)^{-\frac{v}{2}} \right\},$$

if $h(\mu) > 0$;

$$= \left(\frac{1}{2}v\right)^{\frac{v}{2}} \sum_{r=0}^{\infty} \left\{ \left(\frac{1}{2}v - Dh(\mu) - rh(\mu)(C + D)\right)^{-\frac{v}{2}} - \left(\frac{1}{2}v - (r + 1)h(\mu)(C + D)\right)^{-\frac{v}{2}} \right\},$$

if $h(\mu) < 0$ (1.2.1)

$$\begin{aligned} E(n|\mu, \sigma) &= \frac{\sigma^2 \left(\frac{v}{2}\right)^{\frac{v}{2}+1}}{\Delta(\mu - \bar{\mu})} \sum_{r=0}^{\infty} \left\{ -D(rh(\mu)(C + D) + \frac{v}{2})^{-\frac{(v+2)}{2}} \right. \\ &\quad \left. + (C + D)(Ch(\mu) + rh(\mu)(C + D) + \frac{v}{2})^{-\frac{(v+2)}{2}} \right. \\ &\quad \left. - C((r + 1)h(\mu)(C + D) + \frac{v}{2})^{-\frac{(v+2)}{2}} \right\}, \quad \text{if } h(\mu) > 0 \\ &= \frac{\sigma^2 \left(\frac{v}{2}\right)^{\frac{v}{2}+1}}{\Delta(\mu - \bar{\mu})} \sum_{r=0}^{\infty} \left\{ D\left(\frac{v}{2} - (r + 1)h(\mu)(C + D)\right)^{-\frac{(v+2)}{2}} \right. \\ &\quad \left. - (C + D)\left(\frac{v}{2} - Dh(\mu) - rh(\mu)(C + D)\right)^{-\frac{(v+2)}{2}} + C\left(\frac{v}{2} - rh(\mu)(C + D)\right)^{-\frac{(v+2)}{2}} \right\}, \end{aligned}$$

if $h(\mu) < 0$ (1.2.2)

where $h(\mu) = 2(\bar{\mu} - \mu)/\Delta$, $\Delta = \mu_1 - \mu_0$, $\bar{\mu} = (\mu_1 + \mu_0)/2$, $v = n_0 - 1$ and where C and D were used in place of $\ln \frac{1-\beta}{\alpha}$ and $-\ln \frac{\beta}{1-\alpha}$ respectively in the Wald sequential probability ratio test and were determined by the following equations:

$$\begin{aligned} \left(\frac{1}{2}v\right)^{\frac{v}{2}} \sum_{r=0}^{\infty} \left\{ \left(\frac{v}{2} + C + r(C + D)\right)^{-\frac{v}{2}} - \left(\frac{v}{2} + (r + 1)(C + D)\right)^{-\frac{v}{2}} \right\} &= \alpha \\ \left(\frac{1}{2}v\right)^{\frac{v}{2}} \sum_{r=0}^{\infty} \left\{ \left(\frac{v}{2} + D + r(C + D)\right)^{-\frac{v}{2}} - \left(\frac{v}{2} + (r + 1)(C + D)\right)^{-\frac{v}{2}} \right\} &= \beta \end{aligned} \quad (1.2.3)$$

Hall [14] obtained expressions (1.2.1) and (1.2.2) but instead of

using C and D given by (1.2.3), he used (following Paulson [23]),

$$C = \frac{\nu}{2}(\alpha^{\frac{-2}{\nu}} - 1) = (-\ln \alpha)[1 + (-\ln \alpha)/\nu + o(\frac{1}{\nu})],$$

$$D = \frac{\nu}{2}(\beta^{\frac{-2}{\nu}} - 1) = (-\ln \beta)[1 + (-\ln \beta)/\nu + o(\frac{1}{\nu})] \quad (1.2.4)$$

Since the decision boundaries given by (1.2.4) correspond to conservative bounds $(-\ln \alpha, -\ln \beta)$ of Wald, in case σ were known, a test using C and D given by (1.2.4) will require, on the average, more observations to reach a decision than the one using the decision boundaries given by (1.2.3).

1.3. *An Outline of a General Procedure for Deriving a Sequential Test with Power Function Independent of the Nuisance Parameters*

Let $X_1, X_2, \dots$ be independent with a common probability density function (probability function in case of discrete X_j) $g(x, \theta_1, \theta_2)$.

We consider testing the hypothesis $H_0 : \theta_1 = \theta_{10}$ against

$$H_1 : \theta_1 = \theta_{11} ; \theta_2 \text{ (possibly vector valued) is unknown.} \quad (1.3.1)$$

Let $\hat{\theta}_2$ be an estimator of θ_2 obtained from a preliminary sample (independent of $X_1, X_2, \dots$) of fixed size n_0 . If $y_j(\hat{\theta}_2) = g(x_j, \theta_{11}, \hat{\theta}_2)/g(x_j, \theta_{10}, \hat{\theta}_2)$; $j = 1, 2, \dots$, then the test procedure at the n^{th} stage is:

$$(H_0) : B < \prod_{j=1}^n y_j(\hat{\theta}_2) < A : (H_1) \quad (1.3.2)$$

i.e., stop sampling and accept H_0 if $\prod_{j=1}^n y_j(\hat{\theta}_2) \leq B$,
stop sampling and accept H_1 if $\prod_{j=1}^n y_j(\hat{\theta}_2) \geq A$,

otherwise continue sampling;

where A and B are constants chosen suitably to achieve the desired

strength (α, β) .

Assuming the random variable $Z(\hat{\theta}_2) = \ln Y(\hat{\theta}_2)$ (where $Y(\hat{\theta}_2) = g(X, \theta_{11}, \hat{\theta}_2)/g(X, \theta_{10}, \hat{\theta}_2)$) satisfies conditions (I - IV) of Wald [31] Lemma 2, for each fixed $\hat{\theta}_2$; then the equation

$$\int (y(\hat{\theta}_2))^h g(x, \theta_1, \theta_2) dx = 1 \quad (1.3.3)$$

has a unique nonzero solution $h = h(\theta_1, \hat{\theta}_2, \theta_2)$ for $E Z(\hat{\theta}_2) = \int z(\hat{\theta}_2) g(x, \theta_1, \theta_2) dx \neq 0$. For $E Z(\hat{\theta}_2) = 0$, the only solution is $h = 0$.

We assume:

(C1) h depends upon $(\hat{\theta}_2, \theta_2)$ only through a function $t = t(\hat{\theta}_2, \theta_2)$ ($h = h(\theta_1, t)$ say) where t is distributed independently of θ_2 .

Let F_{θ_1} be the cumulative distribution function of t . Using Wald theory and making usual assumptions about negligible boundary overlap, we obtain,

$$L(\theta_1, A, B) = \Pr \{H_0 \text{ is accepted} | \theta_1\} = \int L(\theta_1, A, B | t) dF_{\theta_1}(t)$$

where $L(\theta_1, A, B | t) = \Pr \{H_0 \text{ is accepted} | \theta_1, t\}$

$$= (A^{h(\theta_1, t)} - 1) / (A^{h(\theta_1, t)} - B^{h(\theta_1, t)}), \text{ for } E Z(\hat{\theta}_2) \neq 0$$

$$\text{and } L(\theta_1, A, B | t) = \log A / (\log A + |\log B|), \text{ for } E Z(\hat{\theta}_2) = 0 \quad (1.3.4)$$

To achieve strength (α, β) , A and B are to be obtained from

$$L(\theta_{10}, A, B) = 1 - \alpha; \quad L(\theta_{11}, A, B) = \beta \quad (1.3.5)$$

If, in addition, we assume:

(C2) $E Z(\hat{\theta}_1)$ and $E Z^2(\hat{\theta}_1)$ depend upon $(\hat{\theta}_2, \theta_2)$ only through $t = t(\hat{\theta}_2, \theta_2)$; then the average sample number (A.S.N.) function of the test will also be independent of θ_2 and will be obtained from the formula

$$E(n | \theta_1) = \int E(n | \theta_1, t) dF_{\theta_1}(t)$$

where $E(n|\theta_1, t) = \frac{\log B L(\theta_1, A, B|t) + \log A(1 - L(\theta_1, A, B|t))}{E Z(\hat{\theta}_1)}$,

for $E Z(\hat{\theta}_1) \neq 0$

$$= \frac{-\log A \log B}{E Z^2(\hat{\theta}_1)}, \text{ for } E Z(\hat{\theta}_1) = 0 \quad (1.3.6)$$

If for each fixed $\hat{\theta}_2$ and for each n , $\prod_{j=1}^n g(x_j, \theta_1, \hat{\theta}_2)$ possesses the monotone likelihood ratio property in θ_1 , then by using Theorem 2 of Ghosh [12], we conclude that the above procedure also provides a solution for testing $H_0 : \theta_1 \leq \theta_{10}$ against $H_1 : \theta_1 \geq \theta_{11}$ ($\theta_{10} < \theta_{11}$).

1.4. A Property of Maximum Likelihood Estimators of Location and Scale Parameters

If $X_1, \dots, X_n$ have joint probability density function of the form $f(x_1, \dots, x_n) = \eta^{-n} g(\frac{x_1 - \xi}{\eta}, \dots, \frac{x_n - \xi}{\eta})$, $\xi, \eta \in \Theta$,

$\Theta = \{(\xi, \eta) : -\infty < \xi < \infty, \eta > 0\}$, then Antle and Bain [1] have shown that, under very general conditions, the maximum likelihood estimators $\hat{\xi}$ and $\hat{\eta}$ have the property that $\hat{\eta}/\eta$, $(\hat{\xi} - \xi)/\eta$ and $(\hat{\xi} - \xi)/\hat{\eta}$ are each distributed independently of ξ and η .

We note that it is not essential that $x_1, \dots, x_n$ be independent.

Thus it follows that under the procedure of section 1.3, the power function as well as the A.S.N. function will be independent of the nuisance parameter θ_2 when t is of the form $t_1(\hat{\theta}_2/\theta_2)$ in case θ_2 is the scale parameter and also when t is of the form $t_2(\hat{\theta}_2 - \theta_2)$ in case θ_2 is the location parameter.

The results apply to other types of parameters if they are appropriately related to location and scale parameters under some change of variable. This includes, in particular, Gamma, Pareto and Weibull distributions.

1.5. *A Summary of the Results in Chapters II-VI*

First we consider the problem (1.3.1). In most situations, it is not practicable to express (1.3.5) explicitly when (1.3.3) does not give an explicit expression for h . It is easier to determine termination boundaries to achieve a desired strength (α, β) for families satisfying certain assumptions which will be discussed in Chapter II. The numerical results of that Chapter show that in such cases we can achieve an approximate strength (α, β) even when we use the usual termination boundaries. Later chapters show that similar results hold rather more generally.

Chapter III is devoted to methods of choosing one of three (or more) hypotheses when nuisance parameters are present. In section 3.2, we restrict ourselves to families of distributions satisfying the special conditions similar to those discussed in Chapter II, in which case it is possible to determine values of the decision constants so as to satisfy the requirements on the operating characteristics while discriminating among one of the k hypotheses about a parameter. In section 3.3, Sobel and Wald's [28] method for discriminating among three possible values of a normal mean has been extended to the case when the variance is unknown.

Some cases where none of the assumptions discussed in Chapter II are satisfied are studied in Chapter IV. The usual decision boundaries have been used to make terminal decisions. This includes cases of random sampling from Gamma, Laplace, inverse Gaussian, Pareto and exponential distributions. Procedures in sections 4.2, 4.4, 4.8 and 4.9 give tests whose power functions as well as A.S.N. functions are independent of nuisance parameters.

Sequential procedures for analysis of variance under random and mixed models are studied in Chapter V. We consider a composite hypothesis in the

general random model and develop a sequential test through the principle of invariance. Then we apply this test to solve certain standard problems in the one-way classification, the randomized block design, the two-way classification with balanced replications and the two-stage nested design. We discuss similar procedures for three specific cases of the mixed model. The properties of the test have been compared with those of the sequential F-test and the fixed sample F-test in section 5.13.

Exact results for some special cases have been obtained in Chapter VI. Tests in sections 6.2 and 6.3 may be chosen to achieve a predetermined strength $(\alpha, 0)$. We study procedures for choosing one of two and one of three hypotheses about the mean of a rectangular distribution in sections 6.3 and 6.4 respectively; an estimator of the nuisance parameter though having been obtained from a first stage sample.

CHAPTER II

SEQUENTIAL TEST 'S'

2.1. *Determining the Decision Boundaries*

Lest $X_1, X_2, \dots$ be independent and identically distributed with probability density function (probability function in case of discrete r.v.) $g(x, \theta_1, \theta_2)$. The null and the alternative hypotheses are

$$H_0 : \theta_1 = \theta_{10} \text{ vs. } H_1 : \theta_1 = \theta_{11}, \theta_2 \text{ being unknown.}$$

An estimator $\hat{\theta}_2$ of θ_2 is obtained from a preliminary sample of fixed size n_0 .

Let $y_j(\theta_2) = g(x_j, \theta_{11}, \theta_2)/g(x_j, \theta_{10}, \theta_2)$ for $j = 1, 2, \dots$
and let $Z_j(\theta_2) = \ln y_j(\theta_2)$

Assuming for each $n \geq 1$,

$$(A1) \quad \sum_{j=1}^n Z_j(\theta_2) / \sum_{j=1}^n Z_j(\hat{\theta}_2) = f(\theta_{11}, \theta_{10}, \hat{\theta}_2, \theta_2) = f_1(t), \text{ say (for some}$$

function $t = t(\hat{\theta}_2, \theta_2)$) where f is independent of n and x_j 's,

$$j = 1, 2, \dots, n \tag{2.1.1};$$

we define 'S' to be the Sequential Probability Ratio Test to discriminate $H_1, i = 0, 1$ with termination boundaries A and B, with θ_2 replaced by $\hat{\theta}_2$ in the probability ratio. A and B may be determined from the desired error probabilities α and β , by the equations:

$$\int_{[f_1(t) \leq 0]} e^{Df_1(t)} dF(t) + \int_{[f_1(t) > 0]} e^{-Cf_1(t)} dF(t) = \alpha,$$

$$\int_{[f_1(t) \leq 0]} e^{Cf_1(t)} dF(t) + \int_{[f_1(t) > 0]} e^{-Df_1(t)} dF(t) = \beta \quad (2.1.2)$$

or

$$\int_{[f_1(t) \leq 0]} \frac{e^{Cf_1(t)} - 1}{Cf_1(t) - e^{-Df_1(t)}} dF(t) + \int_{[f_1(t) > 0]} \frac{1 - e^{-Df_1(t)}}{Cf_1(t) - e^{-Df_1(t)}} dF(t) = \alpha,$$

$$\int_{[f_1(t) \leq 0]} \frac{e^{Cf_1(t)} (1 - e^{-Df_1(t)})}{Cf_1(t) - e^{-Df_1(t)}} dF(t) +$$

$$\int_{[f_1(t) > 0]} \frac{e^{-Df_1(t)} (e^{Cf_1(t)} - 1)}{Cf_1(t) - e^{-Df_1(t)}} dF(t) = \beta \quad (2.1.3),$$

where $C = \ln A$, $D = -\ln B$ and F is the cumulative distribution function of t , assumed independent of θ_2 .

Assuming values of C and D satisfying (2.1.2) or (2.1.3) exist and are positive, the decision rule of S at the n th stage is given by:

stop sampling and accept H_0 if $\prod_{j=1}^n y_j(\hat{\theta}_2) \leq B$,

stop sampling and accept H_1 if $\prod_{j=1}^n y_j(\hat{\theta}_2) \geq A$,

otherwise continue sampling

(2.1.4)

2.2 Strength of the Test S

Theorem 2.1 Let $X_1, X_2, \dots$ be i.i.d. with p.d.f.(p.f.) $g(x, \theta_1, \theta_2)$ satisfying (A1). Let the sequential test S be defined by (2.1.4). If A and B are determined by (2.1.2), then S has strength at least (α, β) for testing the composite hypothesis $H_0 : \theta_1 = \theta_{10}$ vs. $H_1 : \theta_1 = \theta_{11}$; θ_2 unknown. If A and B are determined by (2.1.3) and if the excess of the cumulative sum $\sum_{j=1}^n Z_j(\hat{\theta}_2)$ over the boundaries is neglected, then S has strength (α, β) for testing the same hypothesis.

Proof: For given $(\hat{\theta}_2, \theta_2)$, we consider the conditional S.P.R.T. $S(\hat{\theta}_2, \theta_2)$ for testing $\theta_1 = \theta_{10}$ against $\theta_1 = \theta_{11}$ with termination boundaries $\bar{A}$ and $\bar{B}$ where

$$\ln \bar{A} = \bar{C} = \begin{cases} Cf_1(t) & \text{if } f_1(t) > 0 \\ -Df_1(t) & \text{if } f_1(t) \leq 0 \end{cases}$$

and

$$-\ln \bar{B} = \bar{D} = \begin{cases} Df_1(t) & \text{if } f_1(t) > 0 \\ -Cf_1(t) & \text{if } f_1(t) \leq 0 \end{cases} \quad (2.2.1)$$

Then $S(\hat{\theta}_2, \theta_2)$ is given by the decision rule

$$(H_0) : \bar{B} < \prod_{j=1}^n y_j(\theta_2) < \bar{A} : (H_1)$$

Using conservative bounds of Wald on error probabilities, we have

$$\Pr\{S(\hat{\theta}_2, \theta_2) \text{ accepts } H_1 | \theta_{10}, \hat{\theta}_2, \theta_2\} < \frac{1}{\bar{A}},$$

$$\Pr\{S(\hat{\theta}_2, \theta_2) \text{ accepts } H_0 | \theta_{11}, \hat{\theta}_2, \theta_2\} < \bar{B} \quad (2.2.2)$$

Also, if we denote the associated nominal risks of the conditional test $S(\hat{\theta}_2, \theta_2)$ by $\alpha(t)$ and $\beta(t)$, we have, making the usual

assumption of negligible boundary overlap,

$$\begin{aligned} \Pr\{S(\hat{\theta}_2, \theta_2) \text{ accepts } H_1 | \theta_{10}, \hat{\theta}_2, \theta_2\} &= \alpha(t), \\ \Pr\{S(\hat{\theta}_2, \theta_2) \text{ accepts } H_0 | \theta_{11}, \hat{\theta}_2, \theta_2\} &= \beta(t) \end{aligned} \quad (2.2.3)$$

where $\alpha(t)$ and $\beta(t)$ satisfy

$$\bar{A} = (1 - \beta(t))/\alpha(t), \quad \bar{B} = \beta(t)/(1 - \alpha(t)) \quad (2.2.4)$$

Since $\sum_{j=1}^n Z_j(\theta_2) = f_1(t) \sum_{j=1}^n Z_j(\hat{\theta}_2)$, $n \geq 1$; $S(\hat{\theta}_2, \theta_2)$ follows

exactly the same decision rule as S .

Thus we have,

$$\begin{aligned} E \Pr\{S(\hat{\theta}_2, \theta_2) \text{ accepts } H_1 | \theta_1, \hat{\theta}_2, \theta_2\} &= E \Pr\{S \text{ accepts } H_1 | \theta_1, \hat{\theta}_2, \theta_2\} \\ &= \Pr\{S \text{ accepts } H_1 | \theta_1, \theta_2\}, \text{ expectations being taken over the distribution of } t. \end{aligned} \quad (2.2.5)$$

(i) Let A and B be determined from (2.1.2)

From (2.2.2) and (2.2.5), we have,

$$\begin{aligned} \Pr\{S \text{ accepts } H_1 | \theta_{10}, \theta_2\} &< E [e^{-\bar{C}}] = \\ &\int_{[f_1(t) \leq 0]} e^{Df_1(t)} dF(t) + \int_{[f_1(t) > 0]} e^{-Cf_1(t)} dF(t) = \alpha \text{ by (2.1.2)} \end{aligned}$$

$$\begin{aligned} \Pr\{S \text{ accepts } H_0 | \theta_{11}, \theta_2\} &< E [e^{-\bar{D}}] = \\ &\int_{[f_1(t) \leq 0]} e^{Cf_1(t)} dF(t) + \int_{[f_1(t) > 0]} e^{-Df_1(t)} dF(t) = \beta \text{ by (2.1.2)} \end{aligned}$$

(ii) Let A and B be determined from (2.1.3)

From (2.2.3) and (2.2.5), we have

$$\Pr\{S \text{ accepts } H_1 | \theta_{10}, \theta_2\} = E \alpha(t) ,$$

$$\Pr\{S \text{ accepts } H_0 | \theta_{11}, \theta_2\} = E \beta(t) \quad (2.2.6)$$

Also by (2.2.1) and (2.2.4), we have

$$\alpha(t) = \frac{1 - e^{-Df_1(t)}}{e^{Cf_1(t)} - e^{-Df_1(t)}} ,$$

$$\beta(t) = \frac{e^{-Df_1(t)} Cf_1(t)}{e^{Cf_1(t)} - e^{-Df_1(t)}} ; \text{ if } f_1(t) > 0$$

whereas

$$\alpha(t) = \frac{e^{Cf_1(t)} - 1}{e^{Cf_1(t)} - e^{-Df_1(t)}} ,$$

$$\beta(t) = \frac{e^{Cf_1(t)} (1 - e^{-Df_1(t)})}{e^{Cf_1(t)} - e^{-Df_1(t)}} ; \text{ if } f_1(t) \leq 0 \quad (2.2.7)$$

Thus, we have, by (2.2.6)

$$\Pr\{S \text{ accepts } H_1 | \theta_{10}, \theta_2\} =$$

$$\int_{[f_1(t) \leq 0]} \frac{e^{Cf_1(t)} - 1}{e^{Cf_1(t)} - e^{-Df_1(t)}} dF(t) + \int_{[f_1(t) > 0]} \frac{1 - e^{-Df_1(t)}}{e^{Cf_1(t)} - e^{-Df_1(t)}} dF(t) = \alpha$$

by (2.1.3)

Similarly (2.1.3), (2.2.6) and (2.2.7) imply

$$\Pr\{S \text{ accepts } H_0 | \theta_{11}, \theta_2\} = \beta$$

Thus S has strength (α, β) .

2.3. Operating Characteristic and Average Sample Numbers

Assuming $Z(\theta_2) = \ln \frac{g(x, \theta_{11}, \theta_2)}{g(x, \theta_{10}, \theta_2)}$ satisfies conditions (I - IV)

of Wald [31] Lemma 2, the equation

$$\int e^{hZ(\theta_2)} g(x, \theta_1, \theta_2) dx = 1 \quad (2.3.1)$$

has a unique nonzero solution $h = h(\theta_1, \theta_2)$. Neglecting excess of the cumulative sum over the boundaries,

$$\begin{aligned} L(\theta_1, \hat{\theta}_2, \theta_2) &= \Pr\{S(\hat{\theta}_2, \theta_2) \text{ accepts } H_0 | \theta_1, \hat{\theta}_2, \theta_2\} \\ &= (\bar{A}^h - 1)/(\bar{A}^h - \bar{B}^h) \text{ for } E Z(\theta_2) \neq 0, \\ &= \ln \bar{A}/(\ln \bar{A} - \ln \bar{B}) \text{ for } E Z(\theta_2) = 0 \end{aligned}$$

and A.S.N. function of $S(\hat{\theta}_2, \theta_2)$ is

$$E(n | \theta_1, \hat{\theta}_2, \theta_2) = \{(\bar{A}^h - 1) \ln \bar{B} + (1 - \bar{B}^h) \ln \bar{A}\} / (\bar{A}^h - \bar{B}^h) E Z(\theta_2) \quad \text{for}$$

$$E Z(\theta_2) \neq 0; \quad = -\ln \bar{A} \ln \bar{B} / E Z^2(\theta_2) \text{ for } E Z(\theta_2) = 0$$

Thus $L(\theta_1, \theta_2) = \Pr\{S \text{ accepts } H_0 | \theta_1, \theta_2\} =$

$$\begin{aligned} &\int \frac{e^{Chf_1(t)} - 1}{e^{Chf_1(t)} - e^{-Dhf_1(t)}} dF(t) + \int \frac{1 - e^{-Dhf_1(t)}}{e^{Chf_1(t)} - e^{-Dhf_1(t)}} dF(t) \\ &[f_1(t) > 0] \quad [f_1(t) \leq 0] \end{aligned}$$

$$\text{for } E Z(\theta_2) \neq 0, \quad = C/(C + D) \text{ for } E Z(\theta_2) = 0 \quad (2.3.2)$$

$$E(n | \theta_1, \theta_2) = \int \frac{D(1 - e^{Chf_1(t)})f_1(t) + C(1 - e^{-Dhf_1(t)})f_1(t)}{(e^{Chf_1(t)} - e^{-Dhf_1(t)})E Z(\theta_2)} dF(t)$$

for $E Z(\theta_2) \neq 0,$

$$= \int \frac{CD(f_1(t))^2}{E Z^2(\theta_2)} dF(t) \text{ for } E Z(\theta_2) = 0 \quad (2.3.3)$$

2.4. A Class of Continuous Distributions Whose Density Function Satisfies (A1)

We consider the class of Subbotin distributions with p.d.f.

$$p_\delta(x|\theta, \phi) = [2^{\frac{\delta}{2}+1} \Gamma(\frac{\delta}{2} + 1)]^{-1} \phi^{-1} \exp[-\frac{1}{2} \left| \frac{x - \theta}{\phi} \right|^{\frac{2}{\delta}}]; \quad \delta, \phi > 0; \quad \delta \text{ being known. ([19], Vol. 2, p. 33)}$$

It is desired to discriminate between $H_1 : \theta = \theta_1, \quad i = 0, 1;$
 ϕ being unknown.

$$\text{In the notation of section 2.1., } Z_j(\phi) = \frac{1}{2} \left(\left| \frac{x_1 - \theta_0}{\phi} \right|^{\frac{2}{\delta}} - \left| \frac{x_1 - \theta_1}{\phi} \right|^{\frac{2}{\delta}} \right),$$

$$\frac{\sum_{j=1}^n Z_j(\phi)}{\sum_{j=1}^n Z_j(\hat{\phi})} = (\hat{\phi}/\phi)^{\frac{2}{\delta}} = t^{\frac{2}{\delta}}. \text{ Thus (A1) is satisfied with } t = \hat{\phi}/\phi,$$

$f_1(t) = t^{\frac{2}{\delta}}.$ By Antle and Bain [1], t is distributed independently of θ and ϕ .

The decision rule of the test is

$$(H_0) : -D < \frac{1}{2} \sum_{j=1}^n \left(\left| \frac{x_1 - \theta_0}{\hat{\phi}} \right|^{\frac{2}{\delta}} - \left| \frac{x_1 - \theta_1}{\hat{\phi}} \right|^{\frac{2}{\delta}} \right) < C : (H_1) \text{ where } C \text{ and } D \text{ are}$$

given by (2.1.2) or (2.1.3)

In the notation of section (2.3), $L(\theta, \phi)$ and $E(n|\theta, \phi)$ are given by the right sides of (2.3.2) and (2.3.3) respectively where $h = h(\theta, \phi)$ is determined from

$$[2^{\frac{\delta}{2}+1} \Gamma(\frac{\delta}{2} + 1)]^{-1} \phi^{-1} \int_{-\infty}^{\infty} \exp[\frac{h}{2} \left| \frac{x - \theta_0}{\phi} \right|^{\frac{2}{\delta}} - \frac{h}{2} \left| \frac{x - \theta_1}{\phi} \right|^{\frac{2}{\delta}} - \frac{1}{2} \left| \frac{x - \theta}{\phi} \right|^{\frac{2}{\delta}}] dx = 1$$

2.5. *Laplace Distribution: Inferences on the Location Parameter When the Scale Parameter is Not Known*

$x_1, x_2, \dots$, are independent with a common p.d.f.

$$(2\theta_2)^{-1} \exp\left\{-\frac{|x - \theta_1|}{\theta_2}\right\}; \quad -\infty < \theta_1 < \infty, \quad \theta_2 > 0 \quad (2.5.1)$$

The hypothesis to be tested is $H_0 : \theta_1 = \theta_{10}$ against $H_1 : \theta_1 = \theta_{11}$;

$\theta_2 > 0$. The density belongs to the class of section 2.4 with $\delta = 2$,

$$\theta = \theta_1, \quad \phi = \theta_2/2$$

If we take $\hat{\theta}_2$ as the maximum likelihood estimator of θ_2 given by

a first stage sample $\{x_1, \dots, x_{n_0}\}$, then $\hat{\theta}_2 = \frac{n_0}{\sum_{i=1}^{n_0} |x_i' - \tilde{x}|}$, $\tilde{x}$ being the

median of this sample. Let $n_0 = 2m + 1$, m being a positive integer.

$$z_j(\theta_2) = \frac{|x_j - \theta_{10}|}{\theta_2} - \frac{|x_j - \theta_{11}|}{\theta_2}; \quad j = 1, 2, \dots;$$

$\frac{\sum_{j=1}^n z_j(\theta_2)}{\sum_{j=1}^n z_j(\hat{\theta}_2)} = \hat{\theta}_2/\theta_2 = f_1(t) = t$. The decision rule of the test is

$$(H_0) : -D < \sum_{j=1}^n \left\{ \frac{|x_j - \theta_{10}|}{\hat{\theta}_2} - \frac{|x_j - \theta_{11}|}{\hat{\theta}_2} \right\} < C : (H_1)$$

The probability density function of t is independent of θ_1 and θ_2 and is given by Karst and Polowy [20] as

$$p(t) = n_0 \left[\sum_{i=1}^{2m} \frac{a_i}{(i-1)!} (n_0 t)^{i-1} e^{-n_0 t} + \sum_{i=1}^m b_i e^{-n_0 t C_i} \right], \quad t > 0; \quad \text{where}$$

$C_i = (m+1)/(m-i+1)$ and a_i and b_i are obtained from

$$m(\theta) = \frac{2n_0!}{2^{n_0}} \sum_{r=0}^m \frac{1}{r! (m-r)! (1-\theta)^{2m-r}} \prod_{j=0}^r \frac{1}{(m+1+j)[1-(m-j)\theta/m+j+1]}$$

$$\sum_{i=1}^{2m} \frac{a_i}{(1-\theta)^i} + \sum_{i=1}^m \frac{b_i}{C_i - \theta} \quad (2.5.2)$$

$$(2.1.2) \text{ implies } \int_0^{\infty} e^{-Ct} p(t) dt = \alpha, \quad \int_0^{\infty} e^{-Dt} p(t) dt = \beta$$

After integration and simplification, this gives

$$\sum_{i=1}^{2m} \left(\frac{n_0}{n_0 + C} \right)^i a_i + \sum_{i=1}^m \frac{b_i n_0}{C + n_0 C_i} = \alpha,$$

$$\sum_{i=1}^{2m} \left(\frac{n_0}{n_0 + D} \right)^i a_i + \sum_{i=1}^m \frac{b_i n_0}{D + n_0 C_i} = \beta \quad (2.5.3)$$

$$(2.1.3) \text{ implies } \int_0^{\infty} \frac{1 - e^{-Dt}}{e^{Ct} - e^{-Dt}} p(t) dt = \alpha, \quad \int_0^{\infty} \frac{e^{-Dt}(e^{Ct} - 1)}{e^{Ct} - e^{-Dt}} p(t) dt = \beta$$

Expanding the denominator in series, applying Fubini's Theorem, integrating and simplifying,

$$\begin{aligned} & \sum_{i=1}^{2m} a_i n_0^i \sum_{r=0}^{\infty} \{ (n_0 + C + r(C + D))^{-i} - (n_0 + (r+1)(C + D))^{-i} \} + \\ & \sum_{i=1}^m b_i n_0^i \sum_{r=0}^{\infty} \{ (n_0 C_i + C + r(C + D))^{-1} - (n_0 C_i + (r+1)(C + D))^{-1} \} = \alpha, \\ & \sum_{i=1}^{2m} a_i n_0^i \sum_{r=0}^{\infty} \{ (n_0 + D + r(C + D))^{-i} - (n_0 + (r+1)(C + D))^{-i} \} + \\ & \sum_{i=1}^m b_i n_0^i \sum_{r=0}^{\infty} \{ (n_0 C_i + D + r(C + D))^{-1} - (n_0 C_i + (r+1)(C + D))^{-1} \} = \beta \quad (2.5.4) \end{aligned}$$

By Theorem 2.1, decision boundaries given by (2.5.3) will ensure a minimum strength (α, β) while those determined from (2.5.4) will give a test with approximate strength (α, β) , approximation being due to the excess over the boundaries.

The Operating Characteristic

$$E e^{hZ(\theta_2)} = (2\theta_2)^{-1} \int_{-\infty}^{\infty} \exp\left[-\frac{h|x - \theta_{10}|}{\theta_2} - \frac{h|x - \theta_{11}|}{\theta_2} - \frac{|x - \theta_1|}{\theta_2}\right] dx$$

After integration and simplification, (2.3.1) implies:

$$(i) \text{ for } -\infty < \theta_1 \leq \theta_{10} \leq \theta_{11} < \infty,$$

$$w^2 \frac{h}{2h-1} e^{-v} - w + 1 - \frac{h}{2h-1} e^{u-v} = 0, \text{ where } u = \frac{\theta_{11} - \theta_{10}}{\theta_2},$$

$$v = \frac{\theta_{11} - \theta_1}{\theta_2}, \quad w = e^{hu} \quad (2.5.5)$$

$$(ii) \text{ for } -\infty < \theta_{10} \leq \theta_{11} \leq \theta_1 < \infty,$$

$$w^2 \left(1 - \frac{h}{2h+1} e^v\right) - w + \frac{h}{2h+1} e^{-(u-v)} = 0 \quad (2.5.6)$$

$$(iii) \text{ for } -\infty < \theta_{10} \leq \theta_1 \leq \theta_{11} < \infty,$$

$$\frac{h}{w(2h+1)} e^{-u+v} - \frac{w}{(2h-1)(2h+1)} e^{-2hv} + \frac{hw}{2h-1} e^{-v} - 1 = 0 \quad (2.5.7)$$

With h determined from (2.5.5) - (2.5.7) and with C and D given by (2.5.3) or (2.5.4), we have, by (2.3.2), in notation of section 2.3, the following value of the operating characteristic:

$$L(\theta_1, \theta_2) = \int_0^{\infty} \frac{e^{hCt} - 1}{e^{hCt} - e^{-hDt}} n_0 \left(\sum_{i=1}^{2m} \frac{a_i}{(i-1)!} (n_0 t)^{i-1} e^{-n_0 t} + \sum_{i=1}^m b_i e^{-n_0 t C_i} \right) dt$$

Expanding the denominator in series, applying Fubini's Theorem twice, integrating and simplifying, we have,

(i) for $h > 0$,

$$L(\theta_1, \theta_2) = \sum_{i=1}^{2m} a_i n_0^i \sum_{r=0}^{\infty} \{ (n_0 + hr(C+D))^{-i} - (n_0 + hC + hr(C+D))^{-i} \} \\ + \sum_{i=1}^m b_i n_0^i \sum_{r=0}^{\infty} \{ (n_0 C_i + hr(C+D))^{-1} - (n_0 C_i + hC + hr(C+D))^{-1} \} \quad (2.5.8)$$

$$(ii) \text{ for } h = 0, \quad E Z(\theta_2) = 0, \quad L(\theta_1, \theta_2) = C/(C+D) \quad (2.5.9)$$

(iii) for $h < 0$,

$$L(\theta_1, \theta_2) = \sum_{i=1}^{2m} a_i n_0^i \sum_{r=0}^{\infty} \{ (n_0 - hD - hr(C+D))^{-i} - \\ (n_0 - h(r+1)(C+D))^{-i} \} + \sum_{i=1}^m b_i n_0^i \sum_{r=0}^{\infty} \{ (n_0 C_i - hD - hr(C+D))^{-1} - \\ (n_0 C_i - h(r+1)(C+D))^{-1} \} \quad (2.5.10)$$

The A.S.N. Function

To obtain A.S.N. from (2.3.3), we need $E Z(\theta_2) =$

$$Q\theta_2^2)^{-1} \int_{-\infty}^{\infty} \{ |x - \theta_{10}| - |x - \theta_{11}| \} \exp\{-|x - \theta_1|/\theta_2\} dx. \text{ After integration}$$

and simplification, we obtain

$$(i) \text{ for } -\infty < \theta_1 \leq \theta_{10} \leq \theta_{11} < \infty,$$

$$E Z(\theta_2) = -u + e^{u-v} - e^{-v} \text{ where } u \text{ and } v \text{ are as in (2.5.5)} \quad (2.5.11)$$

$$(ii) \text{ for } -\infty < \theta_{10} \leq \theta_{11} < \theta_1 < \infty,$$

$$E Z(\theta_2) = u + e^{-u+v} - e^v \quad (2.5.12)$$

$$(iii) \text{ for } -\infty < \theta_{10} \leq \theta_1 \leq \theta_{11} < \infty,$$

$$E Z(\theta_2) = u - 2v + e^{-u+v} - e^{-v} \quad (2.5.13)$$

We also need $E Z^2(\theta_2) = (2\theta_2^3)^{-1} \int_{-\infty}^{\infty} (|x - \theta_{10}| -$

$|x - \theta_{11}|)^2 \exp\{-|x - \theta_1|/\theta_2\} dx$. After integration and simplification, we have

(i) for $-\infty < \theta_1 \leq \theta_{10} \leq \theta_{11} < \infty$,

$$E Z^2(\theta_2) = -2(u+2)e^{-v} - 2(u-2)e^{u-v} + u^2 \quad (2.5.14)$$

(ii) for $-\infty < \theta_{10} \leq \theta_{11} \leq \theta_1 < \infty$,

$$E Z^2(\theta_2) = -2(u+2)e^{-u+v} - 2(u-2)e^v + u^2 \quad (2.5.15)$$

(iii) for $-\infty < \theta_{10} \leq \theta_1 \leq \theta_{11} < \infty$,

$$E Z^2(\theta_2) = -2(u+2)e^{-u+v} + (u-2v)^2 + 8 - 2(u+2)e^{-v} \quad (2.5.16)$$

With h , $E Z(\theta_2)$ and $E Z^2(\theta_2)$ given by (2.5.5) - (2.5.7), (2.5.11) - (2.5.13) and (2.5.14) - (2.5.16) respectively and with C and D determined from (2.5.3) or (2.5.4), (2.3.3) gives the following expression for the A.S.N. function:

$$E(n|\theta_1, \theta_2) = \int_0^\infty \frac{Dt(1 - e^{Cht}) + Ct(1 - e^{-Dht})}{(e^{Cht} - e^{-Dht})E Z(\theta_2)} p(t) dt \quad \text{for } E Z(\theta_2) \neq 0$$

$$\text{and } E(n|\theta_1, \theta_2) = \frac{CD}{E Z^2(\theta_2)} \int_0^\infty t^2 p(t) dt \quad \text{for } E Z(\theta_2) = 0, \text{ where } p(t) \text{ is}$$

as in (2.5.2). Expanding the denominator in series (for the case $E Z(\theta_2) \neq 0$), applying Fubini's Theorem, integrating and rearranging the terms, we obtain:

(i) for $h > 0$,

$$\begin{aligned}
E(n|\theta_1, \theta_2) &= \frac{1}{E Z(\theta_2)} \sum_{i=1}^{2m} i a_i n_0 \sum_{r=0}^{\infty} \{-D(n_0 + r(C+D)h)^{-(i+1)} + \\
&\quad (C+D)(n_0 + Ch + r(C+D)h)^{-(i+1)} - C(n_0 + (C+D)(r+1)h)^{-(i+1)}\} + \\
&\quad \frac{1}{E Z(\theta_2)} \sum_{i=1}^m n_0 b_i \sum_{r=0}^{\infty} \{-D(n_0 C_i + r(C+D)h)^{-2} + (C+D)(n_0 C_i + Ch + r(C+D)h)^{-2} \\
&\quad - C(n_0 C_i + (r+1)(C+D)h)^{-2}\} \quad (2.5.17)
\end{aligned}$$

(ii) for $h = 0$

$$E(n|\theta_1, \theta_2) = \frac{CDn_0^{-2}}{E Z^2(\theta_2)} \left\{ \sum_{i=1}^{2m} i(i+1)a_i + 2 \sum_{i=1}^m \frac{b_i}{C_i^3} \right\} \quad (2.5.18)$$

(iii) for $h < 0$,

$$\begin{aligned}
E(n|\theta_1, \theta_2) &= \frac{1}{E Z(\theta_2)} \sum_{i=1}^{2m} i a_i n_0 \sum_{r=0}^{\infty} \{-(C+D)(n_0 - hD - hr(C+D))^{-(i+1)} \\
&\quad + C(n_0 - hr(C+D))^{-(i+1)} + D(n_0 - h(r+1)(C+D))^{-(i+1)}\} + \\
&\quad \frac{1}{E Z(\theta_2)} \sum_{i=1}^m n_0 b_i \sum_{r=0}^{\infty} \{-(C+D)(n_0 C_i - hD - hr(C+D))^{-2} + C(n_0 C_i - hr(C+D))^{-2} \\
&\quad + D(n_0 C_i - h(r+1)(C+D))^{-2}\} \quad (2.5.19)
\end{aligned}$$

2.6. An Alternative (Approximate) Form of the Results in 2.5.

The numerical evaluation of operating characteristic and average sample number of the test S in section 2.5. amounts to finding a_i , $i = 1, 2, \dots, 2m$; b_i , $i = 1, 2, \dots, m$ from the relation (2.5.2), then solving (2.5.3) or (2.5.4) for C and D and summing the infinite series on the right sides of (2.5.8), (2.5.10) and (2.5.17) - (2.5.19). The first two steps are

laborious for large n_0 , even for $n_0 > 5$. It is convenient to use an approximate distribution of $t = \hat{\theta}_2 / \theta_2$ in place of (2.5.2). Bain and Antle [4] have found that $2n_0 t$ is approximately a $\chi^2_{2n_0-1}(0)$ variable, the approximation is good for small n_0 and approaches the asymptotic result for large n_0 . Thus

$$p(t) \doteq (n_0)^{(2n_0-1)/2} t^{(2n_0-1)/2-1} \exp(-n_0 t) / \Gamma\left(\frac{2n_0-1}{2}\right); \quad t > 0 \quad (2.6.1)$$

After integration, solution of (2.1.2) may now be written as:

$$C = n_0(\alpha^{-2/(2n_0-1)} - 1), \quad D = n_0(\beta^{-2/(2n_0-1)} - 1) \quad (2.6.2)$$

(2.1.3) along with (2.6.1) gives after expanding the denominator in series, applying Fubini's Theorem, integrating and simplifying,

$$\begin{aligned} n_0^{(2n_0-1)/2} \sum_{r=0}^{\infty} \{ (C + n_0 + r(C + D))^{-(2n_0-1)/2} - \\ (n_0 + (r+1)(C + D))^{-(2n_0-1)/2} \} &= \alpha, \\ n_0^{(2n_0-1)/2} \sum_{r=0}^{\infty} \{ (D + n_0 + r(C + D))^{-(2n_0-1)/2} - \\ (n_0 + (r+1)(C + D))^{-(2n_0-1)/2} \} &= \beta \end{aligned} \quad (2.6.3)$$

With h given by (2.5.5) - (2.5.7), we have, from (2.3.2) and (2.6.1), after expanding suitably, applying Fubini's Theorem, integrating and simplifying,

$$\begin{aligned} (1) \quad \text{for } h > 0, \\ L(\theta_1, \theta_2) = n_0^{(2n_0-1)/2} \sum_{r=0}^{\infty} \{ (n_0 + r(C + D)h)^{-(2n_0-1)/2} - \\ (n_0 + Ch + r(C + D)h)^{-(2n_0-1)/2} \}, \end{aligned} \quad (2.6.4)$$

$$(ii) \text{ for } h = 0, L(\theta_1, \theta_2) = C/(C + D) \quad (2.6.5)$$

$$(iii) \text{ for } h < 0,$$

$$L(\theta_1, \theta_2) = n_0 \sum_{r=0}^{(2n_0-1)/2} \{ (n_0 - Dh - r(C + D)h)^{-(2n_0-1)/2} - (n_0 - (r + 1)(C + D)h)^{-(2n_0-1)/2} \} \quad (2.6.6)$$

From (2.3.3) and (2.6.1), we obtain, by similar operations,

$$(i) \text{ for } h > 0, E(n|\theta_1, \theta_2) = \frac{n_0}{2E Z(\theta_2)} \sum_{r=0}^{(2n_0-1)/2} \{ (C + D)(n_0 + Ch + r(C + D)h)^{-(2n_0+1)/2} - D(n_0 + r(C + D)h)^{-(2n_0+1)/2} - C(n_0 + (r + 1)(C + D)h)^{-(2n_0+1)/2} \} \quad (2.6.7)$$

$$(ii) \text{ for } h = 0, E(n|\theta_1, \theta_2) = (1 - \frac{1}{4n_0^2}) \frac{CD}{E Z^2(\theta_2)} \quad (2.6.8)$$

$$(iii) \text{ for } h < 0, E(n|\theta_1, \theta_2) = \frac{n_0}{2E Z(\theta_2)} \sum_{r=0}^{(2n_0-1)/2} \{ C(n_0 - rh(C + D))^{-(2n_0+1)/2} + D(n_0 - (r + 1)(C + D)h)^{-(2n_0+1)/2} - (C + D)(n_0 - Dh - r(C + D)h)^{-(2n_0+1)/2} \} \quad (2.6.9)$$

2.7. Inferences on the Mean of Inverse Gaussian Distribution

Let $X_1, X_2, \dots$ be independent with a common p.d.f.

$$p_x(x|\mu, \lambda) = (2\pi x^3 \lambda)^{-\frac{1}{2}} \exp(-(x - \mu)^2 / 2\lambda \mu^2 x); \quad x > 0; \quad \lambda > 0, \quad \mu > 0 \quad (2.7.1)$$

The null and the alternative hypotheses are

$$H_0 : \mu \leq \mu_0 \text{ vs. } H_1 : \mu \geq \mu_1 ; \mu_0 < \mu_1 ; 0 < \lambda < \infty \quad (2.7.2)$$

The first stage sample $\{x_1, \dots, x_{n_0}\}$ determines $\hat{\lambda}$, the maximum likelihood estimator of λ as $\hat{\lambda} = n_0^{-1} \sum_{i=1}^{n_0} (x_i^{-1} - \bar{x}^{-1})$ where $\bar{x} = n_0^{-1} \sum_{i=1}^{n_0} x_i$

and $\hat{\lambda}$ is distributed as $(\lambda/n_0) \chi_{n_0-1}^2(0)$ (Tweedie [30]). $Z_j(\lambda) = \lambda^{-1} \{1 - \frac{x_j}{2} (\frac{1}{\mu_1} + \frac{1}{\mu_0})\} (\frac{1}{\mu_1} - \frac{1}{\mu_0})$. Thus (A1) is satisfied with $f_1(t) = t = \hat{\lambda}/\lambda$. The p.d.f. of t is independent of λ and μ and is given by

$$p(t) = \left(\frac{n_0}{2}\right)^{\frac{n_0-1}{2}} \frac{n_0^{-3}}{t^2} e^{-\frac{n_0 t}{2}} / \Gamma\left(\frac{n_0-1}{2}\right) ; t > 0 \quad (2.7.3)$$

The decision rule of the test for testing $\mu = \mu_0$ against $\mu = \mu_1$ is

$$(H_0) : -D < \sum_{j=1}^n Z_j(\hat{\lambda}) < C : (H_1) \quad (2.7.4)$$

$$E e^{hZ(\lambda)} = \int_0^\infty (2\pi\lambda x^3)^{-\frac{1}{2}} \exp\{-(2\lambda x)^{-1} \left\{ \frac{(x-\mu)^2}{\mu^2} + \frac{h(x-\mu_1)^2}{\mu_1^2} - \frac{h(x-\mu_0)^2}{\mu_0^2} \right\}\} dx,$$

after some rearrangements, the right side can be written as

$$\exp\left[\lambda^{-1} \left\{ \frac{1}{\mu} + \frac{h}{\mu_1} - \frac{h}{\mu_0} - \left(\frac{1}{\mu^2} + \frac{h}{\mu_1^2} - \frac{h}{\mu_0^2} \right) \frac{1}{2} \right\}\right] \int_0^\infty (2\pi\theta_2 x^3)^{-\frac{1}{2}} \exp\{-(x-\theta_1)^2 / 2x\theta_1^2\theta_2\} dx,$$

where $\theta_1 = \left(\frac{1}{\mu^2} + \frac{h}{\mu_1^2} - \frac{h}{\mu_0^2} \right)^{-\frac{1}{2}}$, $\theta_2 = \lambda$. Thus, after simplification (2.3.1)

implies

$$h = \left(\frac{1}{\mu_1} + \frac{1}{\mu_0} - \frac{2}{\mu} \right) / \left(\frac{1}{\mu_1} - \frac{1}{\mu_0} \right) \quad (2.7.5)$$

Since h is a decreasing function of μ , power function of the conditional test with decision rule (2.7.4) increases as μ increases, for each fixed $\hat{\lambda}$. Thus (2.7.4) also provides a solution to the problem

(2.7.2).

Since $E X = \mu$, $E X^2 = \mu^3 \lambda + \mu^2$; we obtain, after simplifying,

$$E Z(\lambda) = \lambda^{-1} \left\{ \frac{\mu}{2} \left(\frac{1}{\mu_0} + \frac{1}{\mu_1} \right) - 1 \right\} \left(\frac{1}{\mu_0} - \frac{1}{\mu_1} \right), \quad (2.7.6)$$

$$E Z^2(\lambda) = \lambda^{-2} \left(\frac{1}{\mu_0} - \frac{1}{\mu_1} \right)^2 \left\{ \left(1 - \frac{\mu}{2} \left(\frac{1}{\mu_1} + \frac{1}{\mu_0} \right) \right)^2 + \frac{\mu^3 \lambda}{4} \left(\frac{1}{\mu_1} + \frac{1}{\mu_0} \right)^2 \right\} \quad (2.7.7)$$

We may obtain (2.7.3) by replacing n_0 by $n_0/2$ on the right side of (2.6.1). Thus by (2.6.2) and (2.6.3),

$$\begin{aligned} C &= (n_0/2) (\alpha^{-2/(n_0-1)} - 1), \quad D = (n_0/2) (\beta^{-2/(n_0-1)} - 1) \quad (2.7.8); \\ (n_0/2)^{(n_0-1)/2} \sum_{r=0}^{\infty} \{ (C + (n_0/2) + r(C + D))^{-(n_0-1)/2} - \\ &((n_0/2) + (r+1)(C + D))^{-(n_0-1)/2} \} = \alpha, \\ (n_0/2)^{(n_0-1)/2} \sum_{r=0}^{\infty} \{ (C + (n_0/2) + r(C + D))^{-(n_0-1)/2} - \\ &((n_0/2) + (r+1)(C + D))^{-(n_0-1)/2} \} = \beta \end{aligned} \quad (2.7.9)$$

The form of h implies that operating characteristic is independent of λ ; so, we denote it by $L(\mu)$. From section 2.6:

(i) for $h > 0$,

$$\begin{aligned} L(\mu) &= \left(\frac{n_0}{2} \right)^{\frac{n_0-1}{2}} \sum_{r=0}^{\infty} \left\{ \left(\frac{n_0}{2} + rh(C + D) \right)^{-(n_0-1)/2} - \right. \\ &\left. \left(\frac{n_0}{2} + Ch + (r+1)(C + D) \right)^{-(n_0-1)/2} \right\} \end{aligned} \quad (2.7.10)$$

(ii) for $h = 0$, $L(\mu) = C/(C + D)$ (2.7.11)

(iii) for $h < 0$,

$$L(\mu) = \left(\frac{n_0}{2}\right)^{\frac{n_0-1}{2}} \sum_{r=0}^{\infty} \left\{ \left(\frac{n_0}{2} - Dh - rh(C+D)\right)^{-(n_0-1)/2} - \left(\frac{n_0}{2} - (r+1)h(C+D)\right)^{-(n_0-1)/2} \right\} \quad (2.7.12)$$

and (i) for $h > 0$,

$$E(n|\mu, \lambda) = \frac{\left(\frac{n_0}{2}\right)^{\frac{n_0-1}{2}} (n_0 - 1)}{2E Z(\lambda)} \sum_{r=0}^{\infty} \left\{ (C+D) \left(\frac{n_0}{2} + Ch + rh(C+D)\right)^{-(n_0+1)/2} - D \left(\frac{n_0}{2} + rh(C+D)\right)^{-(n_0+1)/2} - C \left(\frac{n_0}{2} + (r+1)h(C+D)\right)^{-(n_0+1)/2} \right\} \quad (2.7.13)$$

$$(ii) \text{ for } h=0, E Z(\lambda) = 0, E(n|\mu, \lambda) = \left(1 - \frac{1}{2}\right) \frac{CD}{n_0 E Z^2(\lambda)} \quad (2.7.14)$$

(iii) for $h < 0$,

$$E(n|\mu, \lambda) = \frac{\left(\frac{n_0}{2}\right)^{\frac{n_0-1}{2}} (n_0 - 1)}{2E Z(\lambda)} \sum_{r=0}^{\infty} \left\{ C \left(\frac{n_0}{2} - rh(C+D)\right)^{-(n_0+1)/2} + D \left(\frac{n_0}{2} - (r+1)h(C+D)\right)^{-(n_0+1)/2} - (C+D) \left(\frac{n_0}{2} - Dh - rh(C+D)\right)^{-(n_0+1)/2} \right\} \quad (2.7.15)$$

2.8. Testing a Parameter of Lognormal Distribution

The p.d.f. of X is

$$g(x, \zeta, \sigma) = \frac{1}{(2\pi)^{\frac{1}{2}}(\sigma x)} \exp^{-\frac{1}{2}\sigma^{-2}(\ln x - \zeta)^2}; \quad x > 0 \quad (2.8.1)$$

The problem is to test

$$H_0 : \zeta \leq \zeta_0 \text{ vs. } H_1 : \zeta \geq \zeta_1; \quad \zeta_1 > \zeta_0; \quad \sigma > 0 \quad (2.8.2)$$

The maximum likelihood estimator $\hat{\sigma}$ of σ given by the first stage

$$\text{sample of size } n_0 \text{ is } \hat{\sigma} = \left\{ n_0^{-1} \sum_{i=1}^{n_0} (z_i - \bar{z})^2 \right\}^{\frac{1}{2}} \text{ where } \bar{z} = n_0^{-1} \sum_{i=1}^{n_0} z_i,$$

$$z_i = \log x_i ; \quad i = 1, 2, \dots, n_0.$$

$Z_j(\sigma) = \sigma^{-2}(\log x_j - \frac{\zeta_1 + \zeta_0}{2})(\zeta_1 - \zeta_0)$. Thus we easily see that (A1) is satisfied with $f_1(t) = t = \hat{\sigma}^2/\sigma^2$. The decision rule of the test for testing $\zeta = \zeta_0$ against $\zeta = \zeta_1$ is

$$(H_0) : -D < \hat{\sigma}^{-2} \sum_{j=1}^n \{ \log x_j - \frac{\zeta_1 + \zeta_0}{2} \} (\zeta_1 - \zeta_0) < C : (H_1) \quad (2.8.3)$$

Corresponding to (2.3.1), we have, after integration and simplification;

$$h = (\zeta_1 + \zeta_0 - 2\zeta)/(\zeta_1 - \zeta_0) \quad (2.8.4)$$

Also, it is easily seen that

$$E Z(\sigma) = (\zeta_1 - \zeta_0) \left(\mu - \frac{\zeta_1 + \zeta_0}{2} \right) \sigma^{-2} \quad (2.8.5),$$

$$E Z^2(\sigma) = \sigma^{-4} (\zeta_1 - \zeta_0)^2 \left\{ \sigma^2 + \left(\zeta - \frac{\zeta_1 + \zeta_0}{2} \right)^2 \right\} \quad (2.8.6)$$

(2.8.4) shows that the power of the test is an increasing function of ζ and is independent of σ . Thus (2.8.3) will also be a test for the problem (2.8.2).

Since t is distributed as $n_0^{-1} \chi_{n_0-1}^2(0)$, the p.d.f. of t is

given by (2.7.3). Thus the operating characteristic and the A.S.N. are given by (2.7.10) - (2.7.12) and (2.7.13) - (2.7.15) respectively, with C and D determined from (2.7.8) or (2.7.9), with h , $E Z(\sigma)$ and $E Z^2(\sigma)$ given by (2.8.4), (2.8.5) and (2.8.6) respectively and with L.H.S.'s of (2.7.10) - (2.7.12) and (2.7.13) - (2.7.15) replaced by $L(\zeta)$ and $E(n|\zeta, \sigma)$ respectively.

2.9. Numerical Results

Tables 2.9.1 - 2.9.6 follow. The notation used is: $OC_{(1)}$ = OC of the test S when adjusted decision boundaries are used, adjustment being made according to the formula (2.1.2); $OC_{(2)}$ = OC of the test which uses the usual decision boundaries, i.e., $C = \ln(1 - \beta)/\alpha$, $D = -\ln \beta/(1 - \alpha)$; $OC_{(3)}$ = OC when the second parameter is known.

$OC_{(3)}$ and $A.S.N._{(3)}$ in Tables 2.9.1 - 2.9.6 were obtained from the standard approximate formulae for OC and A.S.N. functions of the Wald S.P.R.T. $OC_{(1)}$ - $OC_{(2)}$ in Tables 2.9.1 and 2.9.3 were calculated from (2.6.4) - (2.6.6) and (2.7.10) - (2.7.12) respectively whereas $A.S.N._{(1)}$ - $A.S.N._{(2)}$ of Tables 2.9.2 and 2.9.4 were calculated from (2.6.7) - (2.6.9) and (2.7.13) - (2.7.15) respectively. Using h , $E Z$ and $E Z^2$ of section 2.8; $OC_{(1)}$ - $OC_{(2)}$ of Table 2.9.5 and $A.S.N._{(1)}$ - $A.S.N._{(2)}$ of Table 2.6.9 were also obtained from (2.7.10) - (2.7.12) and (2.7.13) - (2.7.15) respectively.

Tables 2.9.1, 2.9.3 and 2.9.5 show that with unadjusted boundaries, we get probabilities of both kinds of error slightly more than the desired error probabilities. According to Tables 2.9.2, 2.9.4 and 2.9.6, the expected sample size in case of unadjusted boundaries is always less than that in case of adjusted boundaries and even less than that of the case when the second parameter is known, in most cases. Thus, depending upon the practical situation, it may even be better to use the usual decision boundaries than the adjusted ones. The results agree in all the three cases: Laplace, Inverse Gaussian and Lognormal. A comparison of Tables 2.9.3 (i) and 2.9.3 (ii) shows that an increased initial sample size tends to decrease probabilities of both kinds of error in case of usual decision boundaries.

Table 2.9.1

Values of Operating Characteristic of S for Testing Location Parameter of
Laplace Distribution ($n_0 = 30$, $\alpha = \beta = .05$)

v	$u = .5$			$u = 1$		
	OC(1)	OC(2)	OC(3)	OC(1)	OC(2)	OC(3)
0.7	0.9926	0.9893	0.9931	0.991	0.9872	0.9915
0.6	0.982	0.9758	0.9818	0.9804	0.9738	0.9801
0.5	0.953	0.9414	0.95	0.953	0.9414	0.95
0.4	0.869	0.852	0.86	0.875	0.857	0.865
0.3	0.658	0.646	0.649	0.664	0.652	0.655
0.2	0.342	0.354	0.351	0.336	0.348	0.345
0.1	0.131	0.148	0.14	0.125	0.143	0.135
0	0.047	0.0586	0.05	0.047	0.0586	0.05
-0.1	0.018	0.0242	0.0182	0.0196	0.0262	0.0193
-0.2	0.0074	0.0107	0.0069	0.009	0.0128	0.0085

v	$u = 1.5$			$u = 2$		
	OC(1)	OC(2)	OC(3)	OC(1)	OC(2)	OC(3)
2.1	0.9898	0.9857	0.9902	0.9889	0.9844	0.9892
1.8	0.9791	0.9722	0.9787	0.9781	0.9709	0.9775
1.5	0.953	0.9414	0.95	0.953	0.9414	0.95
1.2	0.879	0.862	0.87	0.884	0.866	0.875
0.9	0.671	0.658	0.661	0.678	0.664	0.667
0.6	0.329	0.342	0.339	0.322	0.336	0.333
0.3	0.121	0.138	0.13	0.116	0.114	0.125
0	0.047	0.0586	0.05	0.047	0.0586	0.05
-0.3	0.0209	0.0278	0.0213	0.0219	0.0291	0.0225
-0.6	0.0102	0.0143	0.0098	0.0111	0.0156	0.0108

v	u = 2.5			v	u = 3		
	OC(1)	OC(2)	OC(3)		OC(1)	OC(2)	OC(3)
3.5	0.9881	0.9835	0.9884	4.2	0.9876	0.9827	0.9878
3	0.9773	0.9699	0.9767	3.6	0.9766	0.9692	0.976
2.5	0.953	0.9414	0.95	3	0.953	0.9414	0.95
2	0.888	0.87	0.879	2.4	0.891	0.874	0.883
1.5	0.685	0.671	0.674	1.8	0.692	0.678	0.681
1	0.315	0.329	0.326	1.2	0.308	0.322	0.319
0.5	0.112	0.13	0.121	0.6	0.109	0.126	0.117
0	0.047	0.0586	0.05	0	0.047	0.0586	0.05
-0.5	0.0227	0.0301	0.0233	-0.6	0.0234	0.0308	0.024
-1	0.0119	0.0165	0.0116	-1.2	0.0124	0.0173	0.0122

v	u = 3.5			v	u = 4		
	OC(1)	OC(2)	OC(3)		OC(1)	OC(2)	OC(3)
4.9	0.9871	0.9822	0.9873	5.6	0.9868	0.9817	0.9869
4.2	0.9761	0.9685	0.9754	4.8	0.9757	0.968	0.975
3.5	0.953	0.9414	0.95	4	0.953	0.9414	0.95
2.8	0.894	0.877	0.886	3.2	0.897	0.88	0.883
2.1	0.699	0.685	0.688	2.4	0.707	0.692	0.696
1.4	0.301	0.315	0.312	1.6	0.293	0.308	0.304
0.7	0.106	0.123	0.114	0.8	0.103	0.12	0.111
0	0.047	0.0586	0.05	0	0.047	0.0586	0.05
-0.7	0.0239	0.0315	0.0246	-0.8	0.0243	0.032	0.025
-1.4	0.0129	0.0178	0.0127	-1.6	0.0132	0.0183	0.0131

v	u = 5			v	u = 10		
	OC (1)	OC (2)	OC (3)		OC (1)	OC (2)	OC (3)
7	0.9862	0.9811	0.9864	14	0.9852	0.9797	0.9852
6	0.9751	0.9673	0.9743	12	0.9739	0.9658	0.9729
5	0.953	0.9414	0.95	10	0.953	0.9414	0.95
4	0.9012	0.885	0.894	8	0.9114	0.896	0.9045
3	0.772	0.706	0.71	6	0.781	0.763	0.769
2	0.278	0.294	0.29	4	0.219	0.237	0.231
1	0.0988	0.115	0.106	2	0.0886	0.104	0.0955
0	0.047	0.0586	0.05	0	0.047	0.0586	0.05
-1	0.0249	0.0327	0.0257	-2	0.0261	0.0342	0.0271
-2	0.0138	0.0189	0.0136	-4	0.0148	0.0203	0.0148

Table 2.9.2

Average Sample Number of S for Testing Location Parameter of Laplace

Distribution ($n_0 = 30$, $\alpha = \beta = .05$)

v	u = .5			v	u = 1		
	ASN (1)	ASN (2)	ASN (3)		ASN (1)	ASN (2)	ASN (3)
0.7	17.5	15.98	16.33	1.4	5.39	4.91	5.02
0.6	21.21	19.25	19.71	1.2	6.31	5.72	5.86
0.5	27.07	24.26	24.88	1	7.84	7.03	7.2
0.4	36.26	31.72	32.36	0.8	10.43	9.14	9.33
0.3	46.53	39.48	39.64	0.6	13.6	11.55	11.6
0.2	46.53	39.48	39.64	0.4	13.6	11.55	11.6
0.1	26.26	31.72	32.36	0.2	10.43	9.14	9.33
0	27.07	24.26	24.88	0	7.84	7.03	7.2
-0.1	21.21	19.25	19.71	-0.2	6.31	5.72	5.86
-0.2	17.5	15.98	16.31	-0.4	5.39	4.91	5.02

v	u = 1.5			v	u = 2		
	ASN (1)	ASN (2)	ASN (3)		ASN (1)	ASN (2)	ASN (3)
2.1	2.88	2.63	2.69	2.8	1.92	1.75	1.79
1.8	3.28	2.98	3.05	2.4	2.13	1.93	1.98
1.5	3.99	3.57	3.66	2	2.54	2.28	2.33
1.2	5.29	4.64	4.74	1.6	3.37	2.95	3.02
0.9	7.04	5.98	6.01	1.2	4.58	3.89	3.91
0.6	7.04	5.98	6.01	0.8	4.58	3.89	3.91
0.3	5.29	4.64	4.74	0.4	3.37	2.95	3.02
0	3.99	3.57	3.66	0	2.54	2.28	2.33
-0.3	3.28	2.98	3.05	-0.4	2.13	1.93	1.98
-0.6	2.88	2.63	2.69	-0.8	1.92	1.75	1.79

v	u = 2.5			v	u = 3		
	ASN (1)	ASN (2)	ASN (3)		ASN (1)	ASN (2)	ASN (3)
3.5	1.43	1.3	1.33	4.2	1.14	1.03	1.06
3	1.56	1.41	1.44	3.6	1.22	1.1	1.13
2.5	1.82	1.63	1.67	3	1.41	1.26	1.29
2	2.42	2.12	2.17	2.4	1.87	1.64	1.68
1.5	3.38	2.87	2.89	1.8	2.69	2.29	2.3
1	3.38	2.87	2.89	1.2	2.69	2.29	2.3
0.5	2.42	2.12	2.17	0.6	1.87	1.64	1.68
0	1.82	1.63	1.67	0	1.41	1.26	1.29
-0.5	1.56	1.41	1.44	-0.6	1.22	1.1	1.13
-1	1.43	1.3	1.33	-1.2	1.14	1.03	1.06

Table 2.9.3 (1)

OC Function of S for Testing the Inverse Gaussian Mean ($\alpha = \beta = .05$)

μ	$\mu_0 = 1, \mu_1 = 2, n_0 = 40$			$\mu_0 = .5, \mu_1 = 2.5, n_0 = 40$		
	OC(1)	OC(2)	OC(3)	OC(1)	OC(2)	OC(3)
0.8	0.9963	0.9936	0.9972	1	1	1
0.9	0.9853	0.9777	0.986	0.9992	0.9984	0.9996
1	0.9534	0.9372	0.95	0.9534	0.9372	0.95
1.1	0.879	0.854	0.867	0.714	0.693	0.699
1.2	0.744	0.72	0.727	0.411	0.421	0.419
1.3	0.562	0.555	0.556	0.239	0.263	0.255
1.4	0.387	0.399	0.396	0.156	0.181	0.17
1.5	0.256	0.28	0.273	0.112	0.136	0.123
1.6	0.171	0.197	0.187	0.0866	0.109	0.0952
1.7	0.118	0.142	0.13	0.0707	0.0908	0.0773
1.8	0.0837	0.1054	0.0919	0.0599	0.0785	0.0651
1.9	0.0615	0.0804	0.067	0.0523	0.0696	0.0554
2	0.0466	0.0628	0.05	0.0466	0.0628	0.05
2.1	0.0363	0.0503	0.0382	0.0423	0.0576	0.0451
2.2	0.0289	0.0411	0.0299	0.0389	0.0535	0.0412

$\mu_0 = 1.5, \mu_1 = 4, \mu_0 = 40$	$\mu_0 = 2, \mu_1 = 5, \mu_0 = 40$
μ	μ
OC(1)	OC(1)
OC(2)	OC(2)
OC(3)	OC(3)
0.9991	0.9987
0.9911	0.9975
0.9534	0.9898
0.846	0.9534
0.656	0.855
0.446	0.677
0.289	0.472
0.192	0.309
0.133	0.204
0.0966	0.14
0.0733	0.101
0.0576	0.0753
0.0466	0.0583
0.0387	0.0468
0.0328	0.0382
	0.032

$\mu_0 = 1.5, \mu_1 = 4, \mu_0 = 40$	$\mu_0 = 2, \mu_1 = 5, \mu_0 = 40$
μ	μ
OC(1)	OC(1)
OC(2)	OC(2)
OC(3)	OC(3)
0.9991	0.9982
0.9911	0.9857
0.9534	0.9372
0.846	0.821
0.656	0.639
0.446	0.452
0.289	0.31
0.192	0.217
0.133	0.158
0.0966	0.1196
0.0733	0.0937
0.0576	0.0758
0.0466	0.0628
0.0387	0.0533
0.0328	0.046

Table 2.9.3 (11)

$\mu_0 = .5, \mu_1 = 2.5, \mu_0 = 60$	$\mu_0 = .5, \mu_1 = 2.5, \mu_0 = 60$
μ	μ
OC(1)	OC(1)
OC(2)	OC(2)
OC(3)	OC(3)
0.1144	0.1144
0.0884	0.1318
0.0719	0.1042
0.0607	0.0864
0.0528	0.0741
0.047	0.0652
0.0425	0.0586
	0.0534

$\mu_0 = .5, \mu_1 = 2.5, \mu_0 = 60$	$\mu_0 = .5, \mu_1 = 2.5, \mu_0 = 60$
μ	μ
OC(1)	OC(1)
OC(2)	OC(2)
OC(3)	OC(3)
1	1
0.9994	0.999
0.953	0.9414
0.7099	0.6948
0.4134	0.4203
0.2432	0.2604
0.159	0.1776

Table 2.9.4

ASN Function of the Test S for Testing Inverse Gaussian Mean (Table Gives

 $\lambda^{-1} \times \text{ASN}(i); i = 1, 2, 3 \text{ for } \alpha = \beta = .05, n_0 = 40$

$\mu / \lambda^{-1} \times$	$\mu_0 = 1, \mu_1 = 2$			$\mu_0 = 1.5, \mu_1 = 4$		
	ASN(1)	ASN(2)	ASN(3)	ASN(1)	ASN(2)	ASN(3)
0.8	16.1	14.21	14.64	14.33	12.69	13.03
0.9	19.45	17.01	17.61	17.94	15.76	16.28
1	23.82	20.42	21.2	22.86	19.61	20.35
1.1	28.8	23.95	24.69	28.04	23.08	23.69
1.2	32.87	26.41	26.78	30.42	24.12	24.27
1.3	33.7	26.52	26.56	28.08	22.09	22.12
1.4	30.8	24.32	24.41	23.42	18.71	18.91
1.5	26.3	21.13	21.43	19.05	15.52	15.85
1.6	22	18.02	18.46	15.65	12.96	13.34
1.7	18.48	15.39	15.87	13.1	10.99	11.35
1.8	15.73	13.26	13.73	11.19	9.48	9.82
1.9	13.59	11.57	12	9.72	8.29	8.6
2	11.91	10.21	10.6	8.57	7.35	7.63
2.1	10.57	9.11	9.46	7.65	6.59	6.84
2.2	9.48	8.2	8.52	6.91	5.97	6.19

$\mu / \lambda^{-1} \times$	$\mu_0 = 2, \mu_1 = 5$			$\mu_0 = 2, \mu_1 = 5$		
	ASN(1)	ASN(2)	ASN(3)	ASN(1)	ASN(2)	ASN(3)
1.4	21.12	18.69	19.21	29.41	23.89	24.37
1.7	26.22	23.01	23.79	24.23	20.02	20.59
2	33.08	28.37	29.44	20.29	17	17.56
2.3	40.45	33.38	34.31	17.31	14.64	15.18
2.6	44.53	35.41	35.69	15.02	12.81	13.29
2.9	42.18	33.16	33.19	13.23	11.35	11.78
3.2	35.85	28.56	28.81	11.8	10.16	10.55

Table 2.9.5

Operating Characteristic of the Test S for Testing the Lognormal Parameter $(\alpha = \beta = .05, n_0 = 60)$ $\zeta_0 = 0, \quad \zeta_1 = 1$

ζ	OC(1)	OC(2)	OC(3)	ζ	OC(1)	OC(2)	OC(3)
-0.2	0.9841	0.9783	0.984	0.6	0.348	0.36	0.357
-0.1	0.9727	0.9644	0.9716	0.7	0.223	0.241	0.235
0	0.953	0.9414	0.95	0.8	0.136	0.154	0.146
0.1	0.9196	0.9043	0.9134	0.9	0.0804	0.0957	0.0866
0.2	0.864	0.846	0.854	1	0.047	0.0586	0.05
0.3	0.777	0.759	0.765	1.1	0.0273	0.0356	0.0284
0.4	0.652	0.64	0.643	1.2	0.0159	0.0217	0.016

Table 2.9.6

The Table Gives $(\sigma^2/\Delta) E(n|\zeta, \sigma)$ of S for Testing the Lognormal Parameter ζ ($\alpha = \beta = .05, n_0 = 40$) $\zeta_0 = 0, \quad \zeta_1 = 1$

$\zeta \sqrt{(\sigma^2/\Delta) \times}$	ASN(1)	ASN(2)	ASN(3)	$\zeta \sqrt{(\sigma^2/\Delta) \times}$	ASN(1)	ASN(2)	ASN(3)
-0.2	4.5	3.93	4.07	0.6	10.56	8.38	8.43
-0.1	5.15	4.46	4.63	0.7	9.45	7.65	7.79
0	5.95	5.11	5.3	0.8	8.15	6.75	6.95
0.1	6.95	5.88	6.09	0.9	6.95	5.88	6.09
0.2	8.15	6.75	6.95	1	5.95	5.11	5.3
0.3	9.45	7.65	7.79	1.1	5.15	4.46	4.63
0.4	10.56	8.38	8.43	1.2	4.5	3.93	4.07

CHAPTER III

DISCRIMINATION AMONG $k > 2$ HYPOTHESES IN THE PRESENCE OF NUISANCE PARAMETERS

3.1. *Introduction*

Sequential procedures to choose one of the k simple hypotheses have been extended to the case when nuisance parameters are present. Section 3.2 contains an extension of Armitage's method [3] in case the distribution of the observed random variable satisfies a condition corresponding to condition (A1) of Chapter II. Although it was only practicable to deal with the case of a single nuisance parameter, the theory holds for multi-dimensional nuisance parameters for which an estimator is available either by previous experiments or by a first stage sample. Section 3.3 is a generalization of Sobel Wald test [28] for choosing one of the three hypotheses of a normal mean, to the case when the variance is unknown. We shall note that for $k = 2$, the test of section 3.2 is identical with the test S of Chapter II for testing H_1 against H_2 ; but for $k = 3$, it is not generally the same as the test of section 3.3 for discriminating among H_1 , H_2 and H_3 .

3.2. *An Extension of a Sequential Discrimination Procedure*

Let $X_1, X_2, \dots$ be independent and identically distributed with probability density function (or probability function) $g(x, \theta_1, \theta_2)$.

The problem is to choose one of the k hypotheses

$$H_r : \theta_1 = \theta_{1r}, \quad r = 1, 2, \dots, k; \quad \theta_2 \text{ unknown} \quad (3.2.1)$$

The test should be such that

$$\Pr(H_r \text{ is accepted} | \theta_1 = \theta_{1r}, \theta_2) \geq 1 - \gamma_r; \quad r = 1, 2, \dots, k \quad (3.2.2)$$

We assume $g(x, \theta_1, \theta_2)$ satisfies for every r, s ($r \neq s$) $= 1, 2, \dots, k$ and for every $n \geq 1$,

$$(A1^{rs}) : \frac{\sum_{j=1}^n z_j^{rs}(\theta_2)}{\sum_{j=1}^n z_j^{rs}(\hat{\theta}_2)} = f^{rs}(\theta_{1r}, \theta_{1s}, \hat{\theta}_2, \theta_2) = f_2^{rs}(t), \quad (3.2.3)$$

say for some $t = t(\hat{\theta}_2, \theta_2)$ where f is independent of x_j, s and n ,

$z_j^{rs}(\theta_2) = \ln y_j^{rs}(\theta_2)$, $y_j^{rs}(\theta_2) = g(x_j, \theta_{1s}, \theta_2) / g(x_j, \theta_{1r}, \theta_2)$ and where $\hat{\theta}_2$ is an estimator of θ_2 obtained from a preliminary sample of fixed size n_0 .

Assuming $f_2^{rs}(t)$ is positive a.s., let $k(k-1)$ constants

$B_{sr}(s, r = 1, 2, \dots, k; s \neq r)$ satisfy the equations:

$$\gamma_r = \sum_{\substack{s=1 \\ s \neq r}}^k E(e^{-B_{sr} f_2^{sr}(t)}), \quad r = 1, 2, \dots, k \quad (3.2.4)$$

where the expectation is taken over the distribution of $t = t(\hat{\theta}_2, \theta_2)$ assumed independent of θ_2 .

Writing $B_{sr} = \ln A_{sr}$; the following procedure S provides a solution to the problem (3.2.1):

(i) Accept $H_r : \theta_1 = \theta_{1r}$, $r = 1, 2, \dots, k-1$ if

$$\prod_{j=1}^n y_j^{1r}(\hat{\theta}_2) \geq A_{r1}, \quad 2 = 1, 2, \dots, r-1, r+1, \dots, k$$

(ii) Accept $H_k : \theta_1 = \theta_{1k}$ if

$$\prod_{j=1}^n y_j^{kl}(\hat{\theta}_2) \leq (A_{kl})^{-1}, \quad l = 1, 2, \dots, k-1$$

(iii) If neither (i) nor (ii) holds at stage n , then continue by taking the $(n+1)$ st observation (3.2.5)

If the procedure does not terminate at stage $n \leq \bar{n}$, then at least one of the following $\binom{k}{2}$ relations must hold for every $n \leq \bar{n}$:

$$(A_{rs})^{-1} < \prod_{j=1}^n y_j^{rs}(\hat{\theta}_2) < A_{sr}, \quad r < s = 2, 3, \dots, k \quad (3.2.6)$$

Thus $\Pr\{n > \bar{n} | \theta_1, \hat{\theta}_2, \theta_2\} \leq \Pr\{\text{at least one of the } \binom{k}{2} \text{ relations in (3.2.6) hold for } n = \bar{n} \text{ irrespective of the previous stages}\}$

$$\leq \sum_{\substack{r,s=1 \\ r < s}}^k \Pr\{(A_{rs})^{-1} < \prod_{j=1}^{\bar{n}} y_j^{rs}(\hat{\theta}_2) < A_{sr}\}$$

But the event inside the brackets gives the continuation region of the conditional S.P.R.T. (given $\hat{\theta}_2$) with decision boundaries $((A_{rs})^{-1}, A_{sr})$ whose likelihood ratio is a product of $\bar{n}$ identical p.d.f.'s. Thus the procedure terminates with probability one. Thus

$$\sum_{\substack{s=1 \\ s \neq r}}^k \Pr\{H_s \text{ is accepted} | H_r, \hat{\theta}_2, \theta_2\} = 1 - \Pr\{H_r \text{ is accepted} | H_r, \hat{\theta}_2, \theta_2\},$$

(3.2.7)

Now if we consider the conditional test $S(\hat{\theta}_2, \theta_2)$ defined by the decision rule:

(1) Accept $H_r : \theta_1 = \theta_{1r}$, $r = 1, 2, \dots, k-1$ if

$$\sum_{j=1}^n z_j^{sr}(\theta_2) \geq B_{rs} f_2^{rs}(t), \quad s = 1, 2, \dots, r-1, r+1, \dots, k$$

(ii) Accept $H_k : \theta_1 = \theta_{1k}$ if

$$\sum_{j=1}^n z_j^{ks}(\theta_2) \leq -B_{ks} f_2^{ks}(t), \quad s = 1, 2, \dots, k-1$$

(iii) Otherwise proceed to next stage of sampling, (3.2.8)

then the decision rules in (3.2.5) and (3.2.8) are the same. (3.2.8)

implies

$$\begin{aligned} \Pr\{S(\hat{\theta}_2, \theta_2) \text{ accepts } H_s | H_r, \hat{\theta}_2, \theta_2\} &\leq \Pr\{S(\hat{\theta}_2, \theta_2) \text{ accepts } H_s | H_s, \\ \hat{\theta}_2, \theta_2\} \exp(-B_{sr} f_2^{sr}(t)) &\leq \exp(-B_{sr} f_2^{sr}(t)) \text{ for } s, r(s \neq r) = 1, \dots, k \end{aligned} \quad (3.2.9)$$

Summing over s , we obtain,

$$\sum_{\substack{s=1 \\ s \neq r}}^k \Pr\{S(\hat{\theta}_2, \theta_2) \text{ accepts } H_s | H_r, \hat{\theta}_2, \theta_2\} \leq \sum_{\substack{s=1 \\ s \neq r}}^k \exp(-B_{sr} f_2^{sr}(t))$$

Thus using (3.2.7), $1 - \Pr\{S(\hat{\theta}_2, \theta_2) \text{ accepts } H_r | H_r, \hat{\theta}_2, \theta_2\} \leq$

$$\sum_{\substack{s=1 \\ s \neq r}}^k \exp(-B_{sr} f_2^{sr}(t)), \text{ which implies after taking expectations and using}$$

(3.2.4),

$$1 - \Pr\{S \text{ accepts } H_r | H_r\} \leq \gamma_r, \quad r = 1, 2, \dots, k; \text{ which gives (3.2.2)}$$

Some Special Cases

I) Testing the Mean of a Normal Distribution

$X_1, X_2, \dots$ are independent $N(\mu, \sigma^2)$.

$H_r : \mu = \mu_r, r = 1, 2, \dots, k, \sigma$ unspecified.

The requirements on the operating characteristic are

$$\Pr\{H_r \text{ is accepted} | \mu = \mu_r\} \geq 1 - \gamma_r, \quad r = 1, 2, \dots, k.$$

$\hat{\sigma}^2 = (n_0 - 1)^{-1} \sum_{i=1}^{n_0} (x_i' - \bar{x}_{n_0}')^2$ is obtained from the first stage

sample $\{x_1', \dots, x_{n_0}'\}$, $\bar{x}_{n_0}'$ being the mean of this sample. Then

$$z_j^{rs}(\hat{\sigma}) = \Delta_{sr}(x_j - \bar{\mu}_{sr})\hat{\sigma}^{-2} \text{ where } \Delta_{sr} = \mu_s - \mu_r = -\Delta_{rs} \text{ and } \bar{\mu}_{sr} =$$

$(\mu_s + \mu_r)/2 = \bar{\mu}_{rs}$; $f_2^{rs}(t) = \hat{\sigma}^2/\sigma^2 = t$, say. Density function of t is

$$(\nu_0/2)^{\nu_0/2} e^{-(\nu_0/2)t} t^{\nu_0/2-1} / \Gamma(\nu_0/2), \quad t > 0, \quad \nu_0 = n_0 - 1. \text{ After integra-}$$

tion, (3.2.4) is simplified to

$$\gamma_r = \sum_{\substack{s=1 \\ s \neq r}}^k (1 + \frac{2B_{sr}}{\nu_0})^{-\nu_0/2}, \quad r = 1, 2, \dots, k. \text{ In the special case}$$

$\gamma_r = \gamma$, $r = 1, \dots, k$, we may choose, in particular,

$$B_{sr} = (\nu_0/2) \left(\left(\frac{\gamma}{k-1} \right)^{-2/\nu_0} - 1 \right) \text{ for all } s, r = 1, 2, \dots, k, \quad s \neq r$$

In this case we also obtain by taking expectations in (3.2.9),

$$\Pr(H_s \text{ is accepted} | H_r) \leq (1 + \frac{2B_{sr}}{\nu_0})^{-\nu_0/2} = \frac{\gamma}{k-1} \text{ for all } s \neq r,$$

$s, r = 1, \dots, k$

The test follows the decision rule:

(i) Accept $H_r : \mu = \mu_r$, $r = 1, 2, \dots, k-1$ if

$$\Delta_{ri} \sum_{j=1}^n x_j \geq n \Delta_{ri} \bar{\mu}_{ri} + \hat{\sigma}^2 B_{ri}, \quad i = 1, 2, \dots, r-1, r+1, \dots, k$$

(ii) Accept $H_k : \mu = \mu_k$ if

$$\Delta_{ik} \sum_{j=1}^n x_j \leq n \Delta_{ik} \bar{\mu}_{ik} - \hat{\sigma}^2 B_{ki}, \quad i = 1, \dots, k-1$$

(iii) Otherwise proceed to $(n+1)$ st stage.

II) Testing Location Parameter of Laplace Distribution

$X_1, X_2, \dots$ are independent with a common p.d.f. (2.5.1).

$$H_r : \theta_1 = \theta_{1r}, \quad r = 1, \dots, k, \quad 0 < \theta_2 < \infty$$

Taking $n_0 = 2m + 1$, m being a positive integer, then as in section 2.5,

$$\hat{\theta}_2 = \frac{n_0}{\sum_{i=1}^{n_0} |x_i' - \tilde{x}| / n_0}, \quad \tilde{x}' \text{ being the median of this sample.}$$

Since $z_j^{rs}(\theta_2) = (|x_j - \theta_{1r}| - |x_j - \theta_{1s}|) / \theta_2$, $(A1^{rs})$ is satisfied with $f_2^{rs}(t) = t = \hat{\theta}_2 / \theta_2$, all r, s ; $r \neq s$. The exact distribution of t is contained in (2.5.2). Using this distribution, we obtain from (3.2.4), after integration and simplification, in notations of (2.5.2),

$$\gamma_r = \sum_{\substack{s=1 \\ s \neq r}}^k \sum_{i=1}^{2m} a_i \left(1 + \frac{B_{sr}}{n_0}\right)^{-1} + \sum_{i=1}^m b_i \left(C_i + \frac{B_{sr}}{n_0}\right)^{-1}, \quad r = 1, 2, \dots, k.$$

This will determine B_{sr} for any $s \neq r$. Using the approximate distribution of t given by (2.6.1), the equations satisfied by $B_{sr}(s, r = 1, 2, \dots, k, s \neq r)$ are now simplified to

$$\gamma_r = \sum_{\substack{s=1 \\ s \neq r}}^k \left(1 + \frac{B_{sr}}{n_0}\right)^{-(2n_0-1)/2}, \quad r = 1, 2, \dots, k. \quad \text{If } \gamma_r = \gamma \text{ for all}$$

r , we may take

$$B_{sr} = n_0 \left(\left(\frac{\gamma}{k-1} \right)^{-2/(2n_0-1)} - 1 \right) \text{ independently of } s \text{ and } r. \text{ In this}$$

case, it is also insured that,

$$\Pr\{H_s \text{ is accepted} | H_r\} \leq \frac{\gamma}{k-1} \text{ for all } s, r, s \neq r.$$

The test follows the decision rule:

- (i) Accept $H_r : \theta_1 = \theta_{1r}$, $r = 1, 2, \dots, k-1$ if

$$\sum_{j=1}^n (|x_j - \theta_{1i}| - |x_j - \theta_{1r}|) \geq \hat{\theta}_2 B_{ri}, \quad i = 1, 2, \dots, r-1, r+1, \dots, k$$

(ii) Accept $H_k: \theta_1 = \theta_{1k}$ if

$$\sum_{j=1}^n (|x_j - \theta_{1k}| - |x_j - \theta_{1i}|) \leq -\hat{\theta}_2 B_{ki}, \quad i = 1, 2, \dots, k-1$$

(iii) Otherwise continue by taking an additional observation.

III) Testing Inverse Gaussian Mean

The probability density function of X is given by (2.7.1)

$H_r: \mu = \mu_r, \quad r = 1, 2, \dots, k; \quad \lambda$ being unknown.

$$Z_j^{rs}(\lambda) = (2\lambda x_j)^{-1} \left\{ \frac{(x_j - \mu_r)^2}{\mu_r^2} - \frac{(x_j - \mu_s)^2}{\mu_s^2} \right\}, \quad j = 1, 2, \dots \quad (A1^{rs}) \text{ is}$$

satisfied with $f_z^{rs}(t) = t = \hat{\lambda}/\lambda$; for all $r, s, r \neq s$; where

$\hat{\lambda} = n_0^{-1} \sum_{i=1}^{n_0} (x_i^{-1} - \bar{x}_{n_0}^{-1})$ is the maximum likelihood estimator of λ obtained

from a first stage sample, $\bar{x}_{n_0}$ being the mean of this sample. Using

p.d.f. of t given in (2.7.3), the equations satisfied by the decision boundaries are reduced to

$$\gamma_r = \sum_{\substack{s=1 \\ s \neq r}}^k \left(1 + \frac{2B_{sr}}{n_0}\right)^{-(n_0-1)/2}, \quad r = 1, 2, \dots, k. \quad \text{A particular solution}$$

for the case $\gamma_r = \gamma$, all r , is

$$B_{sr} = (n_0/2) \left(\left(\frac{\gamma}{k-1} \right)^{-2/(n_0-1)} - 1 \right), \quad \text{for any } s \neq r. \quad \text{In this case,}$$

we also have, as above,

$$\Pr\{H_s \text{ is accepted} | H_r\} \leq \frac{\gamma}{k-1} \quad \text{for any } s \neq r; \quad s, r = 1, 2, \dots, k.$$

The test will follow the decision rule:

(i) Accept $H_r : \mu = \mu_r$, $r = 1, 2, \dots, k-1$ if

$$\sum_{j=1}^n x_j \bar{\mu}_{ir}^{-1} \Delta_{ir}^{-1} \geq n \Delta_{ir}^{-1} + \hat{\lambda} B_{ri}, \quad i = 1, \dots, r-1, r+1, \dots, k \text{ where}$$

$$\bar{\mu}_{sr}^{-1} = (\mu_s^{-1} + \mu_r^{-1})/2 = \bar{\mu}_{rs}^{-1} \quad \text{and} \quad \Delta_{sr}^{-1} = \mu_s^{-1} - \mu_r^{-1} = -\Delta_{rs}^{-1}$$

(ii) Accept $H_k : \mu = \mu_k$ if

$$\sum_{j=1}^n x_j \bar{\mu}_{ki}^{-1} \Delta_{ki}^{-1} \leq n \Delta_{ki}^{-1} - \hat{\lambda} B_{ki}, \quad i = 1, \dots, k-1$$

(iii) Otherwise continue by observing x_{n+1}

IV) Testing a Parameter of Lognormal Distribution

The p.d.f. of X is given by (2.8.1).

$H_r : \zeta = \zeta_r$, $r = 1, 2, \dots, k$; σ not known.

Let $\hat{\sigma}$ be the maximum likelihood estimator of σ obtained from a first stage sample of size n_0 . (A_1^{rs}) is satisfied with $f_2^{rs}(t) = t = \hat{\sigma}^2/\sigma^2$.

Density function of t is as in (2.7.3) and the equations determining the decision boundaries are the same as in case III). The decision rule of the test is the same as in case I) with μ and x_j replaced by ζ and $\ln x_j$ ($j = 1, 2, \dots$) respectively.

3.3. Discriminating Three Hypotheses About a Normal Mean, the Variance Being Unknown

$X \sim N(\theta, \sigma^2)$; (θ, σ^2) is unknown. The problem is to discriminate among three mutually exclusive and exhaustive hypotheses: $H_1 : \theta < a_1$, $H_2 : a_1 \leq \theta \leq a_2$, $H_3 : \theta > a_2$; $-\infty < a_1 < a_2 < \infty$. The preliminary sample $\{x_1, \dots, x_{n_0}\}$ gives $\hat{\sigma}^2 = (n_0 - 1)^{-1} \sum_{i=1}^{n_0} (x_i - \bar{x}_{n_0})^2$. Our

conditional tests for given $\hat{\sigma}^2$ will be based upon the method developed by Sobel and Wald [28] and the method of Chapter II will be modified to determine the decision constants.

Let (θ_1, θ_2) and (θ_3, θ_4) be intervals satisfying

$$\begin{aligned} \text{i) } \theta_1 < a_1 < \theta_2 \leq \theta_3 < a_2 < \theta_4 \quad \text{ii) } a_1 &= \frac{\theta_1 + \theta_2}{2}, \quad a_2 = \frac{\theta_3 + \theta_4}{2} \\ \text{iii) } \Delta = \theta_2 - \theta_1 = \theta_4 - \theta_3 \end{aligned} \quad (3.3.1)$$

A wrong decision is defined as acceptance of H_2 or H_3 for $\theta \leq \theta_1$, acceptance of H_3 for $\theta_1 < \theta < \theta_2$, acceptance of H_1 or H_3 for $\theta_2 \leq \theta \leq \theta_3$, acceptance of H_1 for $\theta_3 < \theta < \theta_4$ and acceptance of H_1 or H_2 for $\theta \geq \theta_4$. The test is to be such that

$$\begin{aligned} \text{i) } \Pr\{\text{Wrong decision} | \theta \leq \theta_1\} &\leq \gamma_1 \quad \text{ii) } \Pr\{\text{Wrong decision} | \theta_1 < \theta < \theta_4\} \leq \gamma_2 \\ \text{iii) } \Pr\{\text{Wrong decision} | \theta \geq \theta_4\} &\leq \gamma_3. \end{aligned}$$

Let R_1 be the S.P.R.T. for testing $H_{\theta_1} : \theta = \theta_1$ against $H_{\theta_2} : \theta = \theta_2$, based upon $\bar{x}_{n_0}, x_{n_0+1}, x_{n_0+2}, \dots$ with termination boundaries A_1 and B_1 and with σ replaced by $\hat{\sigma}$ where $A_1(>1), B_1(<1)$ are to be suitably determined. R_1 follows the decision rule:

$$(H_{\theta_1}) : \frac{\hat{\sigma}^2}{\Delta} \ln B_1 + na_1 < \sum_{j=1}^n x_j < \frac{\hat{\sigma}^2}{\Delta} \ln A_1 + na_1 : (H_{\theta_2})$$

For given $(\hat{\sigma}, \sigma)$, the conditional S.P.R.T. $R_1(\hat{\sigma}, \sigma)$ of H_{θ_1} against H_{θ_2} based upon the observations $\bar{x}_{n_0}, x_{n_0+1}, \dots$ with termination boundaries $\bar{A}_1$ and $\bar{B}_1$ (where $\ln \bar{A}_1 = (\ln A_1) \frac{\hat{\sigma}^2}{\sigma^2}$, $\ln \bar{B}_1 = (\ln B_1) \frac{\hat{\sigma}^2}{\sigma^2}$), follows exactly the same decision rule as R_1 . Similarly we define R_2 as the S.P.R.T. for testing $H_{\theta_3} : \theta = \theta_3$ against $H_{\theta_4} : \theta = \theta_4$ with termination boundaries A_2 and B_2 where, for the present A_2, B_2 satisfy

$$A_2 \geq A_1 > 1, \quad B_1 \leq B_2 < 1 \quad (3.3.2)$$

$R_2(\hat{\sigma}, \sigma)$ is defined analogously with boundaries A_2, B_2 . As in [28], R is defined as:

R_1	R_2	R
accepts θ_1	accepts θ_3	accepts H_1
accepts θ_2	accepts θ_3	accepts H_2
accepts θ_2	accepts θ_4	accepts H_3

R is well defined if R_1 accepts θ_1 implies R_2 accepts θ_3 and R_2 accepts θ_4 implies R_1 accepts θ_2 . (3.3.3)

A necessary and sufficient condition for (3.3.3) to hold is that

$$\frac{\hat{\sigma}^2}{\Delta} \ln A_1 + a_1 \leq \frac{\hat{\sigma}^2}{\Delta} \ln A_2 + a_2 \quad \text{and} \quad \frac{\hat{\sigma}^2}{\Delta} \ln B_1 + a_1 \leq \frac{\hat{\sigma}^2}{\Delta} \ln B_2 + a_2$$

i.e. $A_1 \leq A_2$ and $B_1 \leq B_2$ which is implied by (3.3.2).

OC Functions

Let $L(H_{\theta_j} | \theta, R_1(\hat{\sigma}, \sigma))$ denote the probability of accepting θ_j in $R_1(\hat{\sigma}, \sigma)$ when θ is the true mean, $(1, j) = (1, 1), (1, 2), (2, 3), (2, 4)$. Writing $C_1 = \ln A_1$, $D_1 = -\ln B_1$, $\bar{C}_1 = \ln \bar{A}_1$, $\bar{D}_1 = -\ln \bar{B}_1$, $h_1 = \frac{2(a_1 - \theta)}{\Delta}$; $i = 1, 2$; $t = \hat{\sigma}^2/\sigma^2$, we have, $\bar{C}_1 = C_1 t$, $\bar{D}_1 = D_1 t$ and so $L(H_{\theta_j} | \theta, R_1(\hat{\sigma}, \sigma)) = (e^{C_1 t h_1} - 1) / (e^{C_1 t h_1} - e^{-D_1 t h_1})$; $(1, j) = (1, 1), (2, 3)$.

Let $L(H_1 | \theta, R(\hat{\sigma}, \sigma))$ denote the probability of accepting H_1 ; $i = 1, 2, 3$; when θ is the true mean and $R(\hat{\sigma}, \sigma)$ is the procedure used and let $L(H_1 | \theta, R) = E L(H_1 | \theta, R(\hat{\sigma}, \sigma))$; $i = 1, 2, 3$;
 $L(H_{\theta_j} | \theta, R_1) = E L(H_{\theta_j} | \theta, R_1(\hat{\sigma}, \sigma))$; $(1, j) = (1, 1), (1, 2), (2, 3)$.

(2, 4). By (3.3.3), we have, $L(H_1|\theta, R) = L(H_{\theta_1}|\theta, R_1)$, $L(H_3|\theta, R) = L(H_{\theta_4}|\theta, R_2)$ and $L(H_2|\theta, R) = 1 - L(H_{\theta_1}|\theta, R_1) - L(H_{\theta_4}|\theta, R_2)$.

Integrating over the density of t and simplifying we have for $h_1 > 0$,

$$L(H_{\theta_j}|\theta, R_1) = \sum_{r=0}^{\infty} \left\{ \left(1 + \frac{2r(C_1 + D_1)h_1}{n_0 - 1} \right)^{-\frac{n_0-1}{2}} \left(1 + \frac{2h_1(C_1 + r(C_1 + D_1))}{n_0 - 1} \right)^{-\frac{n_0-1}{2}} \right\} = (s_{1j})_1, \text{ say; } (i, j) = (1, 1), (2, 3)$$

$$\text{and for } h_1 < 0, L(H_{\theta_j}|\theta, R_1) = \sum_{r=0}^{\infty} \left\{ \left(1 - \frac{2h_1(D_1 + r(C_1 + D_1))}{n_0 - 1} \right)^{-\frac{n_0-1}{2}} \left(1 - \frac{2h_1(r+1)(C_1 + D_1)}{n_0 - 1} \right)^{-\frac{n_0-1}{2}} \right\} = (s_{1j})_2, \text{ say; } (i, j) = (1, 1), (2, 3).$$

$$\text{Thus } L(H_1|\theta, R) = (s_{11})_1 \text{ for } h_1 > 0$$

$$= (s_{11})_2 \text{ for } h_1 < 0$$

$$= C_1/(C_1 + D_1) \text{ for } h_1 = 0$$

$$L(H_3|\theta, R) = 1 - (s_{23})_1 \text{ for } h_2 > 0$$

$$= 1 - (s_{23})_2 \text{ for } h_2 < 0$$

$$= D_2/(D_2 + C_2) \text{ for } h_2 = 0$$

$$\text{and } L(H_2|\theta, R) = (s_{23})_1 - (s_{11})_1 \text{ for } h_1 > 0, h_2 > 0$$

$$= (s_{23})_1 - (s_{11})_2 \text{ for } h_1 < 0, h_2 > 0$$

$$= (s_{23})_2 - (s_{11})_1 \text{ for } h_1 > 0, h_2 < 0$$

$$= (s_{23})_2 - (s_{11})_2 \text{ for } h_1 < 0, h_2 < 0;$$

expressions for $L(H_2|\theta, R)$ for $h_1 = 0$ will be obtained from this formula by replacing $(s_{11})_1$ by $\frac{C_1}{C_1 + D_1}$, and for $h_2 = 0$ by replacing

$(s_{23})_1$ by $\frac{C_2}{C_2 + D_2}$, $i = 1, 2$.

Monotonicity Properties of Probabilities of Correct Decision

By the above definitions, probability of a correct decision (to be denoted by $L(\theta|R)$) equals $L(H_{\theta_1}|\theta, R_1)$ for $\theta \leq \theta_1$, $L(H_{\theta_3}|\theta, R_2)$ for $\theta_1 < \theta < \theta_2$, $1 - L(H_{\theta_1}|\theta, R_1) - L(H_{\theta_4}|\theta, R_2)$ for $\theta_2 \leq \theta \leq \theta_3$, $1 - L(H_{\theta_1}|\theta, R_1)$ for $\theta_3 < \theta < \theta_4$, $L(H_{\theta_4}|\theta, R_2)$ for $\theta \geq \theta_4$; where, at the points of discontinuity, $L(\theta|R)$ is defined to be the smaller of two limiting values. If $L(\theta|R(\hat{\sigma}, \sigma))$ is obtained from this definition by replacing R_i by $R_i(\hat{\sigma}, \sigma)$, $i = 1, 2$; then since for each fixed $(\hat{\sigma}, \sigma)$, $R_i(\hat{\sigma}, \sigma)$ is a normal mean S.P.R.T.; $L(H_{\theta_1}|\theta, R_1(\hat{\sigma}, \sigma))$ and $L(H_{\theta_3}|\theta, R_2(\hat{\sigma}, \sigma))$ are continuous in θ with continuous first and second derivatives and are monotonically decreasing for all θ with a single point of inflexion in the intervals $\theta_1 < \theta < \theta_2$ and $\theta_3 < \theta < \theta_4$ respectively. Thus for each fixed $(\hat{\sigma}, \sigma)$, $L(\theta|R(\hat{\sigma}, \sigma))$ and therefore $L(\theta|R)$ satisfies properties (i) - (v) of [28] (p.508). In this chapter, we shall refer to them as (i) - (v).

Choice of A_i, B_i ; $i = 1, 2$ to Satisfy

Requirements on the OC Functions

Since $L(\theta|R)$ satisfies (i) - (v); A_i, B_i , $i = 1, 2$; are to be chosen such that $\gamma_1 = 1 - L(H_{\theta_1}|\theta_1, R_1)$, $\gamma_2 = L(H_{\theta_1}|\theta_2, R_1) + L(H_{\theta_4}|\theta_2, R_2)$, $\gamma_2 = L(H_{\theta_1}|\theta_3, R_1) + L(H_{\theta_4}|\theta_3, R_2)$, $\gamma_3 = 1 - L(H_{\theta_4}|\theta_4, R_2)$. Since $h_1(\theta_1) = 1$, $h_1(\theta_2) = -1$, $h_1(\theta_3) = -2(\theta_3 - a_1)/\Delta = -p$, say ($p > 0$), $h_2(\theta_2) = p$, $h_2(\theta_3) = 1$, $h_2(\theta_4) = -1$; we have after integration and simplification,

$$\gamma_1 = \sum_{r=0}^{\infty} \left\{ \left(1 + \frac{2(C_1 + r(C_1 + D_1))}{n_0 - 1} \right) - \frac{n_0 - 1}{2} \right\} - \left\{ \left(1 + \frac{2(r + 1)(C_1 + D_1)}{n_0 - 1} \right) - \frac{n_0 - 1}{2} \right\} \quad (3.3.4),$$

$$\gamma_2 = \sum_{r=0}^{\infty} \left[\left\{ 1 + \frac{2(D_1 + r(C_1 + D_1))}{n_0 - 1} \right\}^{-\frac{n_0-1}{2}} - \left\{ 1 + \frac{2(r+1)(C_1 + D_1)}{n_0 - 1} \right\}^{-\frac{n_0-1}{2}} \right] + \sum_{r=0}^{\infty} \left[\left\{ 1 + \frac{2p(C_2 + r(C_2 + D_2))}{n_0 - 1} \right\}^{-\frac{n_0-1}{2}} - \left\{ 1 + \frac{2p(r+1)(C_2 + D_2)}{n_0 - 1} \right\}^{-\frac{n_0-1}{2}} \right] \quad (3.3.5),$$

$$\gamma_2 = \sum_{r=0}^{\infty} \left[\left\{ 1 + \frac{2(C_2 + r(C_2 + D_2))}{n_0 - 1} \right\}^{-\frac{n_0-1}{2}} - \left\{ 1 + \frac{2(r+1)(C_2 + D_2)}{n_0 - 1} \right\}^{-\frac{n_0-1}{2}} \right] + \sum_{r=0}^{\infty} \left[\left\{ 1 + \frac{2p(D_1 + r(C_1 + D_1))}{n_0 - 1} \right\}^{-\frac{n_0-1}{2}} - \left\{ 1 + \frac{2p(r+1)(C_1 + D_1)}{n_0 - 1} \right\}^{-\frac{n_0-1}{2}} \right] \quad (3.3.6),$$

$$\gamma_3 = \sum_{r=0}^{\infty} \left[\left\{ 1 + \frac{2(D_2 + r(C_2 + D_2))}{n_0 - 1} \right\}^{-\frac{n_0-1}{2}} - \left\{ 1 + \frac{2(r+1)(C_2 + D_2)}{n_0 - 1} \right\}^{-\frac{n_0-1}{2}} \right] \quad (3.3.7)$$

We may write (3.3.4) - (3.3.7) as $\gamma_1 = t_1(C_1, D_1)$, $\gamma_2 = t_4(C_1, D_1) + t_2(p, C_2, D_2)$, $\gamma_2 = t_1(C_2, D_2) + t_2'(p, C_1, D_1)$, $\gamma_3 = t_4(C_2, D_2)$. Thus for $C_1 = C_2 = C$, $D_1 = D_2 = D$; we obtain $\gamma_1 = t_1$, $\gamma_2 = t_4 + t_2$, $\gamma_2 = t_1 + t_2'$, $\gamma_3 = t_4$.

We note that the equations $\gamma_1 = t_1$ and $\gamma_3 = t_4$ are the same as equations (26) and (27) of Baker [5]. Solutions for these equations in the special case $C = D$ for different values of n_0 and for $\gamma_1 = \gamma_3 = .01$, $\gamma_1 = \gamma_3 = .05$ are given in Table 4 of [5]. Since for $C = D$, $t_4(C, D) = t_1(C, D)$ and $t_2(p, C, D) = t_2'(p, C, D) = f(p)$, say; (3.3.4) - (3.3.7) further simplify to $\gamma_1 = t_1$, $\gamma_2 = t_1 + t_2$, $\gamma_3 = t_1$. Thus for $\gamma_1 = \gamma_2 = \gamma_3$, we have, $\gamma = t_1$, $\gamma = t_1 + t_2$. Thus if we use the value of $C(=D)$ as given by $\gamma = t_1$, we are neglecting the term $f(p)$ on the right side of $\gamma = t_1 + t_2$. Considering $f(p)$ negligible if the result

of neglecting produces a change of less than 20% in $(1 - L(\theta|R))$ at $\theta =$

θ_2, θ_3 , i.e., if

$$f(p) = \sum_{r=0}^{\infty} \left[\left(1 + \frac{2pC(1+2r)}{n_0-1}\right)^{-\frac{n_0-1}{2}} - \left(1 + \frac{4pC(1+r)}{n_0-1}\right)^{-\frac{n_0-1}{2}} \right] \leq \gamma/5,$$

we see that in all such cases, we may safely use C given by $\gamma = t_1$;

not effecting much the probability of a wrong decision in $\theta_1 < \theta < \theta_4$.

We shall examine the effect of choosing $A_1 = A_2 = A$, and $B_1 = B_2 = B$ satisfying:

$$C = \ln A = \frac{n_0-1}{2}(\gamma_1^{-\frac{2}{n_0-1}} - 1), \quad D = -\ln B = \frac{n_0-1}{2}(\gamma_3^{-\frac{2}{n_0-1}} - 1) \quad (3.3.8).$$

$L(\theta_1|R(\hat{\sigma}, \sigma)) = L(H_{\theta_1}|\theta_1, R_1(\hat{\sigma}, \sigma)) > 1 - \frac{1}{A}$ (by using Wald's conservative bounds); we have, after taking expectations, $L(\theta_1|R) > 1 - (1 + \frac{2C}{n_0-1})^{-\frac{n_0-1}{2}}$

$= 1 - \gamma_1$. Thus since $L(\theta|R)$ satisfies (i)

$$L(\theta|R) > 1 - \gamma_1 \quad \text{for } \theta \leq \theta_1 \quad (3.3.9)$$

$L(\theta_2|R(\hat{\sigma}, \sigma)) = 1 - L(H_{\theta_1}|\theta_2, R_1(\hat{\sigma}, \sigma)) - L(H_{\theta_4}|\theta_2, R_2(\hat{\sigma}, \sigma))$ (3.3.10)
 $\geq 1 - L(H_{\theta_1}|\theta_2, R_1(\hat{\sigma}, \sigma)) - L(H_{\theta_4}|\theta_3, R_2(\hat{\sigma}, \sigma)) > 1 - \bar{B} - \frac{1}{A}$ and after
 taking expectations, $L(\theta_2|R) > 1 - (1 + \frac{2D}{n_0-1})^{-\frac{n_0-1}{2}} - (1 + \frac{2C}{n_0-1})^{-\frac{n_0-1}{2}} =$
 $1 - \gamma_1 - \gamma_3$.

$L(\theta_3|R(\hat{\sigma}, \sigma)) = 1 - L(H_{\theta_1}|\theta_3, R_1(\hat{\sigma}, \sigma)) - L(H_{\theta_4}|\theta_3, R_2(\hat{\sigma}, \sigma))$ (3.3.11)
 $\geq 1 - L(H_{\theta_1}|\theta_2, R_1(\hat{\sigma}, \sigma)) - L(H_{\theta_4}|\theta_3, R_2(\hat{\sigma}, \sigma)) > 1 - \frac{1}{A} - \bar{B}$;
 which implies, $L(\theta_3|R) > 1 - (1 + \frac{2C}{n_0-1})^{-\frac{n_0-1}{2}} - (1 + \frac{2D}{n_0-1})^{-\frac{n_0-1}{2}} =$

$1 - \gamma_1 - \gamma_3$ and since $L(\theta|R)$ satisfies (iii) - (v), we have,

$$L(\theta|R) > 1 - \gamma_1 - \gamma_3 \quad \text{for } \theta_1 < \theta < \theta_4 \quad (3.3.12)$$

$L(\theta_4|R(\hat{\sigma}, \sigma)) = 1 - L(H_{\theta_3}|\theta_4, R_2(\hat{\sigma}, \sigma)) > 1 - \bar{B}$, which implies,

$L(\theta_4|R) > 1 - (1 + \frac{2D}{n_0-1})^{-\frac{n_0-1}{2}} = 1 - \gamma_3$ and since $L(\theta|R)$ satisfies (ii),

$$L(\theta_4|R) > 1 - \gamma_3 \text{ for } \theta \geq \theta_4 \quad (3.3.13)$$

From (3.3.9), (3.3.12) and (3.3.13) we see that the choice given by

(3.3.8) will satisfy requirements if $\gamma_2 \geq \gamma_1 + \gamma_3$. Also from (3.3.10) -

(3.3.11), we have, by using Wald's conservative bounds, $L(\theta_2|R(\hat{\sigma}, \sigma)) >$

$1 - \bar{B} - L(H_{\theta_4}|\theta_2, R_2(\hat{\sigma}, \sigma)), L(\theta_3|R(\hat{\sigma}, \sigma)) > 1 - \frac{1}{A} - L(H_{\theta_1}|\theta_3, R_1(\hat{\sigma}, \sigma))$

and after taking expectations; $L(\theta_2|R) > 1 - \gamma_3 - t_2$, $L(\theta_3|R) >$

$1 - \gamma_1 - t'_2$.

Since for $C = D$ ($\gamma_1 = \gamma_3 = \gamma$, say), $t_2 = t'_2 = f(p)$, we have, by considering $f(p)$ negligible if result of neglecting produces a change of less than 20% in $(1 - L(\theta|R))$ at $\theta = \theta_2, \theta_3$; for $f(p) \leq \gamma/5$; using (i) - (v);

$$L(\theta|R) > 1 - \gamma \text{ for } \theta \leq \theta_1, \theta \geq \theta_4; L(\theta|R) \geq 1 - \gamma \text{ for } \theta_1 < \theta < \theta_4 \quad (3.3.14)$$

While it is easier to compute C and D by this method, the inequalities in (3.3.14) may not be close and this choice may result in a larger sample size.

Bounds for the A.S.N. Function

Let $\tilde{R}$ be the procedure which continues to take observations until R_1 accepts θ_1 and let $\tilde{R}$ be the procedure which continues to take observations until R_2 accepts θ_4 . Then the following result holds:

$$\min[E(\tilde{N}; \theta, \sigma), E(\tilde{N}; \theta, \sigma)] \geq E(N; \theta, \sigma) \geq \max[E(N_1; \theta, \sigma), E(N_2; \theta, \sigma)] \quad (3.3.15)$$

where $E(N; \theta, \sigma)$ is the A.S.N. function of the procedure R and

$E(\tilde{N}; \theta, \sigma)$, $E(\tilde{N}^*; \theta, \sigma)$, $E(N_1; \theta, \sigma)$ and $E(N_2; \theta, \sigma)$ are the A.S.N. functions of the procedures $\tilde{R}$, $\tilde{R}^*$, R_1 and R_2 respectively and, neglecting excess of the cumulative sum over the boundaries, are given by,

$$E(\tilde{N}; \theta, \sigma) = \frac{D_1}{\frac{\Delta}{\sigma^2}(a_1 - \theta)} \quad (3.3.16)$$

$$E(\tilde{N}^*; \theta, \sigma) = \frac{C_2}{\frac{\Delta}{\sigma^2}(\theta - a_2)} \quad (3.3.17)$$

$$E(N_1; \theta, \sigma) = \frac{\sigma^2}{\Delta(\theta - a_1)} \sum_{r=0}^{\infty} \left[C_1 \left(1 - \frac{2h_1}{n_0-1} (C_1 + D_1)r \right)^{-\frac{n_0+1}{2}} - \right. \\ \left. (C_1 + D_1) \left(1 - \frac{2h_1}{n_0-1} (D_1 + r(C_1 + D_1)) \right)^{-\frac{n_0+1}{2}} + \right. \\ \left. D_1 \left(1 - \frac{2h_1}{n_0-1} (r+1)(C_1 + D_1) \right)^{-\frac{n_0+1}{2}} \right], \text{ for } h_1 < 0 \quad (3.3.18)$$

$$E(N_1; \theta, \sigma) = C_1 D_1 \left(1 + \frac{2}{n_0-1} \frac{\sigma^4}{\Delta^2(\sigma^2 + (\theta - a_1)^2)} \right), \text{ for } h_1 = 0 \quad (3.3.19)$$

$$E(N_1; \theta, \sigma) = \frac{\sigma^2}{\Delta(\theta - a_1)} \sum_{r=0}^{\infty} \left[(C_1 + D_1) \left(1 + \frac{2h_1}{n_0-1} (C_1 + r(C_1 + D_1)) \right)^{-\frac{n_0+1}{2}} - \right. \\ \left. D_1 \left(1 + \frac{2h_1}{n_0-1} r(C_1 + D_1) \right)^{-\frac{n_0+1}{2}} - C_1 \left(1 + \frac{2h_1}{n_0-1} (r+1)(C_1 + D_1) \right)^{-\frac{n_0+1}{2}} \right], \text{ for } \\ h_1 > 0; \quad i = 1, 2 \quad (3.3.20)$$

Moreover,

$$\begin{aligned} & \text{i) } E(N; \theta, \sigma) = E(\tilde{N}; \theta, \sigma) \text{ for } \theta \leq \theta_1 \quad \text{ii) } E(N; \theta, \sigma) = \\ & E(\tilde{N}^*; \theta, \sigma) \text{ for } \theta \geq \theta_4 \quad \text{iii) } E(N; \theta, \sigma) = E(N_1; \theta, \sigma) \text{ for } \theta < \theta_2 \\ & \text{iv) } E(N; \theta, \sigma) = E(N_2; \theta, \sigma) \text{ for } \theta > \theta_3 \end{aligned} \quad (3.3.21)$$

Proof: Let $E(n; \theta, \sigma | \hat{\sigma})$ denote the expected sample size in the corresponding conditional test for $n = N, N_1, N_2, \tilde{N}, \tilde{N}^*$.

Using Wald's first identity, we have, under the above assumption,

$E(\tilde{N}, \theta, \sigma | \hat{\sigma}) = \frac{-D_1 t \sigma^2}{\Delta(\theta - a_1)}$: Multiplying by the density of t and integrating, we get (3.3.16). (3.3.17) follows similarly. (3.3.21) follows directly from [28].

Left hand inequality of (3.3.15) follows from the definition of $\tilde{R}$ and $\tilde{R}^*$ and the right hand inequality is implied by the relation

$$\begin{aligned} \Pr\{N > n | \theta, \sigma\} &= \Pr\{N_1 > n | \theta, \sigma\} + \Pr\{N_2 > n | \theta, \sigma\} - \\ &\Pr\{N_1 > n, N_2 > n | \theta, \sigma\}. \\ E(N_1, \theta, \sigma | \hat{\sigma}) &= \frac{\frac{C_1 h_1 t}{D_1 t(1 - e^{-C_1 h_1 t})} + C_1 t(1 - e^{-D_1 h_1 t})}{(e^{-C_1 h_1 t} - e^{-D_1 h_1 t}) \Delta(\theta - a_1) \sigma^{-2}}. \end{aligned}$$

Rearranging, expanding the denominator in series, applying Fubini's

Theorem and using $E(te^{-\theta t}) = (1 + \frac{2\theta}{n_0-1})^{-\frac{n_0+1}{2}}$ ($\theta > 0$);

we get (3.3.20) for $h_1 > 0$. (3.3.18) is obtained similarly.

$$\text{For } h_1 = 0, E(N_1, \theta, \sigma) = \frac{C_1 D_1 E t^2}{\frac{\Delta^2}{\sigma^4}(\sigma^2 + (\theta - a_1)^2)} = \frac{C_1 D_1 \{1 + \frac{2}{n_0-1}\} \sigma^4}{\Delta^2(\sigma^2 + (\theta - a_1)^2)}$$

which is (3.3.19).

Example: Let $\theta_1 = -\frac{5}{16}$, $\theta_2 = -\frac{3}{16}$, $\theta_3 = \frac{3}{16}$, $\theta_4 = \frac{5}{16}$; $\gamma_1 = \gamma_2 =$

$\gamma_3 = \gamma = .05$; $\sigma = 1$, $n_0 = 31$; then $\Delta = \frac{1}{8}$, $p = 7$, $a_1 = -\frac{1}{4}$, $a_2 = \frac{1}{4}$.

From $\gamma = t_1$, $C = 3.22$; $f(p) = 1.05580251 \times 10^{-6} < \frac{\gamma}{5} = .01$. From

(3.3.8), $C = D = 3.315829502$; $f(p) = 8.095191134 \times 10^{-7} < \frac{\gamma}{5}$. For σ -

known, we take $C = D = \ln \frac{1-\gamma}{\gamma}$ since $k = (\ln 5/\gamma)/\ln \frac{1-\gamma}{\gamma} =$

$1.564022966 < p$ (see p.511 of [28]). (3.3.15) gives following upper and

lower bounds on the A.S.N. of R : Notation: UB_1 = Upper bound using $C = D$

given by $\gamma = t_1$, UB_2 = Upper bound using $C = D$ given by (3.3.8),

UB_3 = Upper bound for Sobel Wald test (σ - known). Similarly LB_i ,
 $i = 1, 2, 3$; are the corresponding lower bounds.

Table 3.3.1

θ	UB1	LB1	UB2	LB2	UB3	LB3
5/16	412.16	377.91	424.43	391.87	376.89	339.2
6/16	206.08	204.74	212.21	211.02	188.44	187.4
7/16	137.39	137.29	141.48	141.39	125.63	125.59
8/16	103.04	103.03	106.11	106.1	94.22	94.22
9/16	82.43	82.43	84.88	84.88	75.38	75.38
10/16	68.69	68.69	70.74	70.74	62.81	62.81
12/16	51.52	51.52	53.05	53.05	47.11	47.11
14/16	41.22	41.22	42.44	42.44	37.69	37.69
16/16	34.35	34.35	35.37	35.37	31.41	31.41
18/16	29.44	29.44	30.32	30.32	26.92	26.92
20/16	25.76	25.76	26.53	26.53	23.56	23.56

CHAPTER IV

SOME TESTS WITH THE USUAL DECISION BOUNDARIES

4.1. Introduction

In this chapter, we study some of the cases when the probability density function $g(x, \theta_1, \theta_2)$ of the random variable X does not satisfy the assumption (A1) of Chapter II. For the problem (1.3.1), the decision rule of our test is given by (1.3.2) with $A = (1 - \beta)/\alpha$, $B = \beta/(1 - \alpha)$. The actual strength of the test in this case is (α', β') where $\alpha' = 1 - L(\theta_{10}, \theta_2)$, $\beta' = L(\theta_{11}, \theta_2)$, $L(\theta_1, \theta_2) = E L(\theta_1, \hat{\theta}_2, \theta_2)$ where under the assumption of negligible boundary overlap,

$$\begin{aligned} L(\theta_1, \hat{\theta}_2, \theta_2) &= (e^{Ch} - 1)/(e^{Ch} - e^{-Dh}) \quad \text{if } h \neq 0 \\ &= C/(C + D) \quad \text{if } h = 0 \end{aligned} \quad (4.1.1),$$

with $C = \ln A$, $D = -\ln B$; $h = h(\theta_1, \hat{\theta}_2, \theta_2)$ being the unique solution of (1.3.3).

In the notation of section 1.3 and under the above assumption, the A.S.N. function is $E E(n|\theta_1, \hat{\theta}_2, \theta_2)$ where

$$\begin{aligned} E(n|\theta_1, \hat{\theta}_2, \theta_2) &= (D(1 - e^{Ch}) + C(1 - e^{-Dh}))/((e^{Ch} - e^{-Dh})E Z(\hat{\theta}_2)), \\ \text{if } E Z(\hat{\theta}_2) &\neq 0 \\ &= -CD/E Z^2(\hat{\theta}_2) \quad \text{if } E Z(\hat{\theta}_2) = 0 \end{aligned} \quad (4.1.2)$$

Conditions (C1) and (C2) of Chapter I are satisfied by the tests of sections 4.2, 4.4, 4.8, and 4.9; consequently the power functions

as well as the A.S.N. functions of these tests are independent of the nuisance parameter.

4.2. A Test of Randomness Whose Power Function and A.S.N. Function are Independent of the Nuisance Parameter

Let $X_1, X_2, \dots$ be independent with a common probability density function

$$g(x, \lambda, \nu) = e^{-x/\lambda} x^{\nu-1} / \lambda^\nu \Gamma \nu, \quad \lambda > 0; \quad \nu > 0; \quad x \geq 0 \quad (4.2.1)$$

We consider testing

$$H_0 : \nu = \nu_0 \text{ against } H_1 : \nu = \nu_1; \quad \lambda \text{ being unknown} \quad (4.2.2)$$

The hypothesis of randomness of a series of events against the alternative H_1 is a particular case of H_0 . Under the hypothesis of randomness, the probability of an event in any time interval δx is $\lambda^{-1} \delta x$, independently of other events for some $\lambda > 0$. Thus the intervals $x_1, \dots, x_n$ between successive events are distributed independently with a probability density function $\lambda^{-1} \exp -x/\lambda$, $x \geq 0$. Thus (4.2.2) reduces to randomness hypothesis for $\nu_0 = 1$.

After determining $\hat{\lambda}$, the maximum likelihood estimator of λ from a first stage sample of size n_0 ; the test procedure at the n th stage of sequential sampling is:

$$(H_0) : -D < \sum_{j=1}^n Z_j(\hat{\lambda}) < C : (H_1), \text{ where conditionally on } \hat{\lambda} \text{ fixed, } Z_j(\hat{\lambda}), \quad j = 1, 2, \dots \text{ are independent and identically distributed as}$$

$$Z(\hat{\lambda}) = \ln \frac{\Gamma \nu_0}{\Gamma \nu_1} + (\nu_1 - \nu_0)(\ln X - \ln \hat{\lambda}) \quad (4.2.3)$$

and where C and D are as in section 4.1.

$$E e^{hZ(\hat{\lambda})} = \left(\frac{\Gamma \nu_0}{\Gamma \nu_1} \right)^{h(\nu_1 - \nu_0)} E X^{h(\nu_1 - \nu_0)} \text{ exists for } h(\nu_1 - \nu_0) + \nu > 0.$$

After integrating and simplifying, equation determining h reduces to

$$\left[\frac{\Gamma v_0}{\Gamma v_1} \left(\frac{\hat{\lambda}}{\lambda} \right)^{v_0 - v_1} \right]^h = \frac{\Gamma v}{\Gamma(v + h(v_1 - v_0))} \quad (4.2.4)$$

Since h depends upon $(\hat{\lambda}, \lambda)$ only through the ratio $t = \hat{\lambda}/\lambda$ and since by Theorem 1 (p.82 of [4]), t is distributed independently of λ ; the power function does not depend upon λ . Let F_v be the distribution function of t . Then the operating characteristic is:

$$L(v) = \int L(v|t) dF_v(t) \quad (4.2.5)$$

where $L(v|t)$ is given by the right side of (4.1.1), $h = h(v, t)$ being as in this section.

Using $E(\ln X) = \ln \lambda + \psi(v)$ and $E(\ln X)^2 = \psi^{(1)}(v) + (\ln \lambda + \psi(v))^2$ where $\psi(v) = \frac{d}{dv} \ln \Gamma v$; we obtain, after simplifying,

$$E Z(\hat{\lambda}) = \ln \frac{\Gamma v_0}{\Gamma v_1} + (v_0 - v_1) \ln \frac{\hat{\lambda}}{\lambda} + (v_1 - v_0) \psi(v) \quad (4.2.6)$$

$$\begin{aligned} \text{and } E Z^2(\hat{\lambda}) &= \left(\ln \frac{\Gamma v_0}{\Gamma v_1} \right)^2 + (v_1 - v_0)^2 \left(\ln \frac{\hat{\lambda}}{\lambda} \right)^2 + (v_1 - v_0)^2 \psi^{(1)}(v) + \\ &+ (\psi(v))^2 - 2(v_1 - v_0) \left(\ln \frac{\Gamma v_0}{\Gamma v_1} \right) \left(\ln \frac{\hat{\lambda}}{\lambda} \right) + 2(v_1 - v_0) \left(\ln \frac{\Gamma v_0}{\Gamma v_1} \right) \psi(v) - \\ &- 2(v_1 - v_0)^2 \left(\ln \frac{\hat{\lambda}}{\lambda} \right) \psi(v). \end{aligned} \quad (4.2.7)$$

Since $E Z(\hat{\lambda})$ and $E Z^2(\hat{\lambda})$ depend on $(\hat{\lambda}, \lambda)$ only through $t = \hat{\lambda}/\lambda$, the A.S.N. function of the test is independent of λ and is obtained from

$$E(n|v) = \int E(n|v, t) dF_v(t) \quad (4.2.8)$$

where $E(n|v, t)$ is given by the right side of (4.1.2), with $\hat{\theta}_2$ replaced by $\hat{\lambda}$.

Tables of cumulative probability of t are given by Bain and Antle [4]. For large n_0 , we may use asymptotic distribution of t given by

$$t \sim N(1, \sigma^2) \text{ where } \sigma^2 = n_0^{-1} \psi^{(1)}(v) (\psi^{(1)}(v) - 1) \quad (4.2.9)$$

4.3. Testing Scale Parameter of a Gamma Distribution

$X_1, X_2, \dots$ are independent with a common probability density given by (4.2.1).

The null and the alternative hypotheses are

$$H_0 : \lambda = \lambda_0 \text{ and } H_1 : \lambda = \lambda_1, \quad v > 0 \text{ being unknown} \quad (4.3.1)$$

$\hat{v}$, the maximum likelihood estimator of v is obtained from a first stage sample of size n_0 . The decision rule of the test at the n th stage of sampling is:

$$(H_0) : -D < \sum_{j=1}^n Z_j(\hat{v}) < C : (H_1), \text{ where conditionally on } \hat{v} \text{ fixed, } Z_j(\hat{v}), j = 1, 2, \dots \text{ are independent and identically distributed as}$$

$$Z(\hat{v}) = \hat{v} \ln(\lambda_0/\lambda_1) + X(\lambda_0^{-1} - \lambda_1^{-1}) \quad (4.3.2)$$

and where C and D are as before. For $\lambda_1 > \lambda_0$

$$E e^{hZ(\hat{v})} = (\lambda_0/\lambda_1)^{h\hat{v}} E e^{h(\lambda_0^{-1} - \lambda_1^{-1})X} \text{ exists for } h < \lambda^{-1}/(\lambda_0^{-1} - \lambda_1^{-1})$$

and conditions (I - IV) of Wald [31] Lemma 2 are satisfied for these values of h . After integration and simplification, equation determining h reduces to

$$(\lambda_0/\lambda_1)^{h\hat{v}/v} = 1 - h\lambda(\lambda_0^{-1} - \lambda_1^{-1}) \quad (4.3.3)$$

With $h = h(\lambda, v, \hat{v})$ given by (4.3.3), the operating characteristic is

$$L(\lambda, v) = \int L(\lambda, v, \hat{v}) dF_v(\hat{v}) \quad (4.3.4)$$

where F_v is the c.d.f. of $\hat{v}$ (which is independent of λ by Theorem 1 of [4]) and where $L(\lambda, v, \hat{v})$ is obtained from the right side of (4.1.1).

Also,

$$E Z(\hat{v}) = \hat{v} \ln (\lambda_0 / \lambda_1) + (\lambda_0^{-1} - \lambda_1^{-1}) \lambda v ; \quad (4.3.5)$$

$$E Z^2(\hat{v}) = (\hat{v} \ln (\lambda_0 / \lambda_1))^2 + 2 \lambda v \hat{v} (\lambda_0^{-1} - \lambda_1^{-1}) \ln (\lambda_0 / \lambda_1) + v(v+1) \lambda^2 (\lambda_0^{-1} - \lambda_1^{-1})^2 \quad (4.3.6)$$

The A.S.N. function of the test is

$$E (n | \lambda, v) = \int E (n | \lambda, v, \hat{v}) dF_v(\hat{v}) \quad (4.3.7)$$

where $E (n | \lambda, v, \hat{v})$ is obtained from the right side of (4.1.2) by replacing $\hat{\theta}_2$ by $\hat{v}$.

The tables of the cumulative distribution function of $\hat{v}$ given by Bain and Antle [4] are inadequate for the present purpose. For large n_0 , we may, however, use asymptotic distribution of $t = \sqrt{n_0} \hat{v}$ which is given by

$$t \sim N(\sqrt{n_0} v, \sigma^2) \text{ where } \sigma^2 = v(v\psi^{(1)}(v) - 1)^{-1} \quad (4.3.8)$$

4.4. Testing the Scale Parameter of Laplace Distribution; the Power Function and A.S.N. Function of the Test Being Independent of the Location Parameter.

$X_1, X_2, \dots$ are independent with a common density function $g(x, \theta_1, \theta_2)$ given by (2.5.1). We consider testing

$$H_0 : \theta_2 = \theta_{20} \text{ against } H_1 : \theta_2 = \theta_{21} ; -\infty < \theta_1 < \infty \quad (4.4.1)$$

Let $\hat{\theta}_1$ be the median of the first stage sample $\{x'_1, x'_2, \dots, x'_{n_0}\}$.

The test procedure at the n th stage of second sampling is

$$(H_0) : -D < \sum_{j=1}^n Z_j(\hat{\theta}_1) < C : (H_1) \text{ where } Z_j(\hat{\theta}_1)'s \text{ are independent}$$

and have a p.d.f. common with

$$Z(\hat{\theta}_1) = \ln (\theta_{20} / \theta_{21}) + (\theta_{20}^{-1} - \theta_{21}^{-1}) |x - \hat{\theta}_1| \quad (4.4.2)$$

where $\hat{\theta}_1$ is fixed.

$$\phi(h) = E e^{hZ(\hat{\theta}_1)} = \left(\frac{\theta_{20}}{\theta_{21}}\right)^h (2\theta_2)^{-1} \int_{-\infty}^{\infty} \exp\left\{-\frac{|x - \theta_1|}{\theta_2} + hq \frac{|x - \hat{\theta}_1|}{\theta_2}\right\} dx$$

where $q = \theta_2(\theta_{20}^{-1} - \theta_{21}^{-1})$. After integration and certain other operations, we see that $\phi(h)$ exists for $qh < 1$, the conditions of Lemma 2 [31] are satisfied for these values of h and $\phi(h) = 1$ can be written as:

$$\left(\frac{\theta_{20}}{\theta_{21}}\right)^h \left\{ e^{\frac{qh(\hat{\theta}_1 - \theta_1)}{\theta_2}} + qhe^{-\frac{(\hat{\theta}_1 - \theta_1)}{\theta_2}} \right\} = 1 - q^2 h^2 \quad \text{if } -\infty < \theta_1 \leq \hat{\theta}_1 < \infty$$

$$\text{and } \left(\frac{\theta_{20}}{\theta_{21}}\right)^h \left\{ qhe^{\frac{\hat{\theta}_1 - \theta_1}{\theta_2}} + e^{-\frac{qh(\hat{\theta}_1 - \theta_1)}{\theta_2}} \right\} = 1 - q^2 h^2 \quad \text{if } -\infty < \hat{\theta}_1 \leq \theta_1 < \infty.$$

Thus $h = h(\theta_2, t)$ may be obtained from (writing $t = (\hat{\theta}_1 - \theta_1)/\theta_2$)

$$\left(\frac{\theta_{20}}{\theta_{21}}\right)^h \{ qhe^{-|t|} + e^{qh|t|} \} = 1 - q^2 h^2 \quad (4.4.3)$$

Supposing $t = y_r$ where $r = (n_0 + 1)/2$ for n_0 odd and $r = n_0/2$ for n_0 even, then the exact probability density function of y_r is

$$p(t) = \begin{cases} p_1(t) & \text{if } t \leq 0 \\ p_2(t) & \text{if } t > 0 \end{cases} \quad (4.4.4)$$

$$\text{where } p_1(t) = \frac{n_0! 2^{-r}}{(r-1)!(n_0-r)!} e^{rt} \left(1 - \frac{e^t}{2}\right)^{n_0-r} \quad \text{and } p_2(t) = \frac{n_0! 2^{-n_0+r-1}}{(r-1)!(n_0-r)!} e^{-(n_0-r+1)t} \left(1 - \frac{e^{-t}}{2}\right)^{r-1}. \quad \text{Thus if we take } n_0 = 2m+1,$$

for some positive integer m , then writing $t = y_{m+1}$, the exact probability density of y_{m+1} is simplified to

$$p(t) = \frac{n_0!}{(m!)^2} 2^{-(m+1)} e^{-(m+1)|t|} \left(1 - \frac{e^{-|t|}}{2}\right)^m; \quad -\infty < t < \infty \quad (4.4.5)$$

Also, irrespective of whether n_0 is an odd integer or not

$$\sqrt{n_0}t \sim N(0, 1) \text{ (asymptotically)} \quad (4.4.6)$$

Since by (4.4.3), h depends on $(\hat{\theta}_1, \theta_1)$ only through $t = (\hat{\theta}_1 - \theta_1)/\theta_2$ and exact and asymptotic distributions of t given by (4.4.4) - (4.4.6) are independent of θ_1 ; the power function of the test will be independent of θ_1 . We may denote the operating characteristic by $L(\theta_2)$. Taking $n_0 = 2m + 1$, we have, using exact distribution of t ;

$$L(\theta_2) = \frac{n_0!}{(m!)^2} 2^{-(m+1)} \int_{-\infty}^{\infty} L(\theta_2, t) e^{-(m+1)|t|} \left(1 - \frac{e^{-|t|}}{2}\right)^m dt \quad (4.4.7)$$

whereas using (4.4.6);

$$L(\theta_2) = \sqrt{n_0/2\pi} \int_{-\infty}^{\infty} L(\theta_2, t) e^{-\frac{n_0 t^2}{2}} dt \quad (4.4.8)$$

where $L(\theta_2, t)$ equals the right side of (4.1.1).

After multiplying the right side of (4.4.2) (with X replaced by x) and its square by $g(x, \theta_1, \theta_2)$ and integrating, we obtain (after simplification),

$$E Z(\hat{\theta}_1) = \ln \frac{\theta_{20}}{\theta_{21}} + q(|t| + e^{-|t|}) \quad (4.4.9),$$

$$E Z^2(\hat{\theta}_1) = \left(\ln \frac{\theta_{20}}{\theta_{21}}\right)^2 + q^2(t^2 + 2) + 2q\left(\ln \frac{\theta_{20}}{\theta_{21}}\right)(|t| + e^{-|t|}) \quad (4.4.10)$$

where t and q have been defined.

The right sides of (4.4.9) - (4.4.10) and the distributional forms of t show that the A.S.N. function does not depend on θ_1 . Denoting it by $E(n|\theta_2)$, we have, for $n_0 = 2m + 1$, in case of exact distribution of t ;

$$E(n|\theta_2) = \frac{n_0!}{(m!)^2} 2^{-(m+1)} \int_{-\infty}^{\infty} E(n|\theta_2, t) e^{-(m+1)|t|} \left(1 - \frac{e^{-|t|}}{2}\right)^m dt \quad (4.4.11)$$

whereas in case of asymptotic distribution of t ,

$$E(n|\theta_2) = \sqrt{n_0/2\pi} \int_{-\infty}^{\infty} E(n|\theta_2, t) e^{-\frac{n_0 t^2}{2}} dt \quad (4.4.12)$$

where $E(n|\theta_2, t)$ is as on the right side of (4.1.2), with $\hat{\theta}_2$ replaced by $\hat{\theta}_1$.

4.5. Testing an Inverse Gaussian Parameter

The probability density function of X is given by (2.7.1).

The null and the alternative hypotheses are

$$H_0 : \lambda = \lambda_0 \text{ vs. } H_1 : \lambda = \lambda_1 ; 0 < \mu < \infty (\lambda_0 < \lambda_1) \quad (4.5.1)$$

$\hat{\mu} = n_0^{-1} \sum_{i=1}^{n_0} x_i$ is obtained from the preliminary sample $\{x_1, \dots, x_{n_0}\}$.

The distribution of $\hat{\mu}$ is inverse Gaussian with parameters λ/n_0 and μ .

Writing $Z_j(\hat{\mu}) = \frac{1}{2} \ln \frac{\lambda_0}{\lambda_1} + (\frac{1}{\lambda_0} - \frac{1}{\lambda_1}) (\frac{x_j}{2\hat{\mu}^2} + \frac{1}{2x_j} - \frac{1}{\hat{\mu}})$; $j = 1, 2, \dots$,

the decision rule of the test at the n th stage is

$$(H_0) : -D < \sum_{j=1}^n Z_j(\hat{\mu}) < C : (H_1) \text{ and conditionally on } \hat{\mu} \text{ fixed}$$

$Z_j(\hat{\mu})$'s are i.i.d. Writing $d = (\lambda_0^{-1} - \lambda_1^{-1})$, $q = -d/\hat{\mu}^2$, we have,

$$\phi(h) = E e^{hZ(\hat{\mu})} = \left(\frac{\lambda_0}{\lambda_1}\right)^{\frac{h}{2}} \int_0^{\infty} (2\pi\lambda x^3)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}x\left(\frac{(x-\mu)^2}{\lambda\mu^2} + qh(x-\hat{\mu})^2\right)\right\} dx$$

After several rearrangements, the right side may be written as,

$$\left(\lambda_0/\lambda_1\right)^{\frac{h}{2}} \frac{1}{\sqrt{\theta_2/\lambda}} \exp\left(\frac{\theta_3}{\lambda\mu} - \frac{1}{\theta_1\theta_2}\right) \int_0^{\frac{1}{\lambda}} (2\pi x^3 \theta_2)^{-\frac{1}{2}} \exp\left\{-(x-\theta_1)^2/2\theta_1^2\theta_2 x\right\} dx \text{ where}$$

$$\theta_1 = \mu\{(1+qh\lambda\hat{\mu}^2)/(1+qh\lambda\mu^2)\}^{\frac{1}{2}}, \quad \theta_2 = \lambda/(1+qh\lambda\hat{\mu}^2), \quad \theta_3 = 1+qh\lambda\mu\hat{\mu}$$

Thus $\phi(h)$ exists for $\theta_1 > 0$, $\theta_2 > 0$ and Lemma 2 [31] is applied

for these values of h . The nonzero values of $h = h(\lambda, \hat{\mu}, \mu)$ will be

obtained from

$$h \ln (\lambda_0/\lambda_1) = \frac{2}{\theta_1 \theta_2} - \ln \frac{\theta_2}{\lambda} - \frac{2\theta_3}{\lambda \mu} \quad (4.5.2)$$

The probability of accepting H_0 is

$$L(\lambda, \mu) = \frac{\sqrt{n_0}}{\sqrt{2\pi\lambda}} \int_0^\infty L(\lambda, \hat{\mu}, \mu | \hat{\mu} = t) t^{-\frac{3}{2}} \exp\left\{-\frac{n_0(t-\mu)^2}{2\lambda\mu^2 t}\right\} dt \quad (4.5.3)$$

where $L(\lambda, \hat{\mu}, \mu)$ is as on the right side of (4.1.1). Writing $l = \ln (\lambda_0/\lambda_1)$ and using $E X = \mu$, $E X^{-1} = \lambda + \mu^{-1}$, $E X^2 = \lambda\mu^3 + \mu^2$ and $E X^{-2} = 3\lambda^2 + 3\lambda\mu^{-1} + \mu^{-2}$, we obtain, after simplifying,

$$E Z(\hat{\mu}) = \frac{1}{2}l + d\left(-\frac{\mu}{\hat{\mu}^2} - \frac{1}{\hat{\mu}} + \frac{1}{2}(\mu^{-1} + \lambda)\right) \quad (4.5.4)$$

$$\text{and } E Z^2(\hat{\mu}) = d^2\left\{\frac{3\lambda^2}{4} + \frac{\lambda}{4}\left(\frac{\mu^3}{\hat{\mu}^4} - \frac{4}{\hat{\mu}} + \frac{3}{\mu}\right) + \frac{\mu^2}{4\hat{\mu}^4} - \frac{\mu}{\hat{\mu}^3} + \frac{3}{2\hat{\mu}^2} - \frac{1}{\mu\hat{\mu}} + \frac{1}{4\mu^2}\right\} + \\ d^2\left(\frac{\lambda}{2} - \frac{1}{\hat{\mu}} + \frac{\mu}{2\hat{\mu}^2} + \frac{1}{2\mu}\right) + \frac{l^2}{4} \quad (4.5.5)$$

In the usual notation,

$$E(n|\lambda, \mu) = \frac{\sqrt{n_0}}{\sqrt{2\pi\lambda}} \int_0^\infty E(n|\lambda, \hat{\mu} = t, \mu) t^{-\frac{3}{2}} \exp\left\{-\frac{n_0(t-\mu)^2}{2\lambda\mu^2 t}\right\} dt \quad (4.5.6)$$

where $E(n|\lambda, \hat{\mu}, \mu)$ is obtained from the right side of (4.1.2) by replacing $\hat{\theta}_2$ by $\hat{\mu}$.

4.6. Inferences on a Parameter of a Singly Truncated Normal Distribution

Suppose $X_1, X_2, \dots$ are independent with a common probability density function

$$g(x, \mu, \sigma) = \left(\frac{\mu - A}{\sigma}\right)^{-1} \sqrt{2\pi\sigma}^{-1} \exp\left\{-(x - \mu)^2/2\sigma^2\right\}; \quad A \leq x < \infty; \\ -\infty < A < \infty; \quad A \text{ being known.} \quad (4.6.1)$$

The null and the alternative hypotheses are

$$H_0 : \mu = \mu_0 \text{ vs. } H_1 : \mu = \mu_1 ; 0 < \sigma < \infty$$

After obtaining $\hat{\sigma}$, the maximum likelihood estimator of σ , from an initial sample of size n_0 , the test procedure may be expressed as

$$(H_0) : -D < \sum_{j=1}^n Z_j(\hat{\sigma}) < C : (H_1) \text{ where, writing } \Delta = \mu_1 - \mu_0,$$

$$\bar{\mu} = (\mu_1 + \mu_0)/2 \text{ and } r(\hat{\sigma}) = \phi\left(\frac{\mu_0 - A}{\hat{\sigma}}\right) / \phi\left(\frac{\mu_1 - A}{\hat{\sigma}}\right); \quad Z_j(\hat{\sigma}) = \ln r(\hat{\sigma}) + \Delta \hat{\sigma}^{-2} (x_j - \bar{\mu}); \quad j = 1, 2, \dots \quad E e^{hZ(\hat{\sigma})} = (r(\hat{\sigma}))^h \exp(-h\Delta \bar{\mu} \hat{\sigma}^{-2}) E \exp(h\Delta \hat{\sigma}^{-2} X).$$

After integrating and simplifying, equation determining $h = h(\mu, \hat{\sigma}, \sigma)$ may be expressed as

$$h \ln r(\hat{\sigma}) + \ln \left(\phi(q) / \phi\left(\frac{\mu - A}{\sigma}\right) \right) + \frac{1}{2} h^2 \Delta^2 \sigma^2 \hat{\sigma}^{-4} + h\Delta(\mu - \bar{\mu}) \hat{\sigma}^{-2} = 0 \quad (4.6.2)$$

where $q = q(\hat{\sigma}) = (\mu + h\Delta \sigma^2 \hat{\sigma}^{-2} - A)/\sigma$. Also since $E X = \mu +$

$$\sigma Z_0\left(\frac{A - \mu}{\sigma}\right) / \phi\left(\frac{\mu - A}{\sigma}\right) \text{ and } E X^2 = \sigma^2 + \mu^2 + (A + \mu) \sigma Z_0\left(\frac{A - \mu}{\sigma}\right) / \phi\left(\frac{\mu - A}{\sigma}\right)$$

(where $Z_0(\mu) = (1/\sqrt{2\pi}) \exp(-\frac{\mu^2}{2})$); we obtain, after simplifying;

$$E Z(\hat{\sigma}) = \ln r(\hat{\sigma}) - \bar{\mu} \Delta \hat{\sigma}^{-2} + \Delta \hat{\sigma}^{-2} \left\{ \mu + \sigma Z_0\left(\frac{A - \mu}{\sigma}\right) / \phi\left(\frac{\mu - A}{\sigma}\right) \right\} \quad (4.6.3)$$

$$\text{and } E Z^2(\hat{\sigma}) = (\ln r(\hat{\sigma}))^2 + \Delta^2 \hat{\sigma}^{-4} (\bar{\mu}^2 + \mu^2 + \sigma^2) + 2\Delta \hat{\sigma}^{-2} ((\mu - \bar{\mu}) \ln r(\hat{\sigma}) -$$

$$\Delta \bar{\mu} \hat{\sigma}^{-2}) + \sigma \hat{\sigma}^{-4} Z_0\left(\frac{A - \mu}{\sigma}\right) (\Delta^2 (A + \mu) + 2\sigma^2 \Delta \ln r(\hat{\sigma}) - 2\Delta^2 \bar{\mu}) / \phi\left(\frac{\mu - A}{\sigma}\right) \quad (4.6.4)$$

With h , $E Z(\hat{\sigma})$ and $E Z^2(\hat{\sigma})$ obtained from (4.6.2) - (4.6.4), the probability of accepting H_0 and the A.S.N. function are given by

$$L(\mu, \sigma) = E L(\mu, \hat{\sigma}, \sigma) \quad (4.6.5)$$

$$\text{and } E(n|\mu, \sigma) = E E(n|\mu, \hat{\sigma}, \sigma) \quad (4.6.6)$$

respectively where $L(\mu, \hat{\sigma}, \sigma)$ and $E(n|\mu, \hat{\sigma}, \sigma)$ may be determined from the right hand sides of (4.1.1) and (4.1.2) respectively by replacing $\hat{\theta}_2$ by $\hat{\sigma}$ determined from the first stage sample $\{x'_1, x'_2, \dots, x'_{n_0}\}$ by

the relations:

$$(1) \quad \hat{\sigma} = \left(\frac{Z_0(\hat{\delta})}{1 - \Phi(\hat{\delta})} - \hat{\delta} \right)^{-1} \bar{x}_{n_0}^{-2} \quad (11) \quad (\hat{\sigma} / \bar{x}_{n_0}') (\hat{\sigma} \bar{x}_{n_0}'^{-1} - \hat{\delta}) = n_0^{-1} \sum_{i=1}^{n_0} x_i^2 \bar{x}_{n_0}'^{-2} \quad (4.6.7)$$

where $\bar{x}_{n_0}'$ is the mean of the sample and $\hat{\delta} = (A - \hat{\mu}) / \hat{\sigma}$ is the estimate of the lower truncation point. While no explicit form of the distribution of $\hat{\sigma}$ is available, tables of its cumulative distribution function are given by Francis [9].

4.7. Testing Equality of Two Poisson Means

The distribution of two independent random variables X and Y is Poisson with means θ_1 and θ_2 respectively. The hypotheses to be discriminated are

$$H_0 : \theta_1 = \theta_2 \quad \text{and} \quad H_1 : \theta_1 - \theta_2 = d > 0 \quad (4.7.1)$$

Let $\{x_1', x_2', \dots, x_{n_0}'\}$ and $\{y_1', y_2', \dots, y_{n_0}'\}$ be the first stage samples from the two populations and let $\hat{\theta} = (\hat{\theta}_1 + \hat{\theta}_2) / 2$, where $\hat{\theta}_1 = n_0^{-1} \sum_{i=1}^{n_0} x_i'$ and $\hat{\theta}_2 = n_0^{-1} \sum_{i=1}^{n_0} y_i'$. The likelihood ratio for testing

$\bar{H}_0 : \theta_1 = \theta_2 = \bar{\theta}$ against $\bar{H}_1 : \theta_1 = \hat{\theta} + \frac{1}{2}d, \theta_2 = \hat{\theta} - \frac{1}{2}d$ at the n th stage of sequential sampling is

$$\prod_{j=1}^n \left[\left(1 + \frac{d}{\hat{\theta}}\right)^{\frac{x_j}{2\hat{\theta}}} \left(1 - \frac{d}{\hat{\theta}}\right)^{\frac{y_j}{2\hat{\theta}}} \right] \quad (4.7.2)$$

For $\hat{\theta} > \frac{1}{2}d$, we write $Z(\hat{\theta}) = X \ln \left(1 + \frac{d}{\hat{\theta}}\right) + Y \ln \left(1 - \frac{d}{\hat{\theta}}\right)$ and follow the following decision rule for testing H_0 against H_1 :

Accept H_0 without taking further observations whenever $\hat{\theta} \leq \frac{1}{2}d$.

Otherwise accept H_0 or H_1 according as the lower or the upper

inequality in $-D < \sum_{j=1}^n Z_j(\hat{\theta}) < C$ is first violated for $n \geq 1$ (4.7.3)

where $Z_j(\hat{\theta})$, $j = 1, 2, \dots$ are independent and have a probability function common with $Z(\hat{\theta})$.

Since $E \exp h(t_1 X + t_2 Y) = (\exp(-\theta_1 + \theta_2)) \exp\{\theta_1 e^{ht_1} + \theta_2 e^{ht_2}\}$,
 it follows that $E \exp(hZ(\hat{\theta})) = (\exp(-\theta_1 + \theta_2)) \exp\{\theta_1 (1 + \frac{d}{2\theta})^h + \theta_2 (1 - \frac{d}{2\theta})^h\}$. Thus $E \exp(hZ(\hat{\theta})) = 1$ implies

$$1 + w = (1 + v)^h + w(1 - v)^h \quad (4.7.4)$$

where $w = \theta_2/\theta_1$ and $v = \frac{d}{2\theta}$.

Also, since the distribution of $n_0(\hat{\theta}_1 + \hat{\theta}_2)$ is Poisson with mean $2n_0\theta$ (where $\theta = (\theta_1 + \theta_2)/2$), it is verified that

$$\Pr\{(d/2\hat{\theta}) = v\} = \frac{e^{-2n_0\theta} (2n_0\theta)^{n_0 d/v}}{(n_0 d/v)!} ; \quad v = n_0 d/r \quad (r = 0, 1, 2, \dots)$$

Thus the probability of accepting H_0 is given by

$$L(\theta_1, \theta_2) = \sum_{\substack{r=0 \\ \{v=n_0 d/r\}}}^{\infty} \frac{L(\hat{\theta}, \theta_1, \theta_2 | \frac{d}{2\hat{\theta}} = v) e^{-2n_0\theta} (2n_0\theta)^{n_0 d/v}}{(n_0 d/v)!} \quad (4.7.5)$$

$$\text{where } L(\hat{\theta}, \theta_1, \theta_2) = \begin{cases} 1 & \text{if } v \geq 1 \\ \text{right side of (4.1.1)} & \text{if } v < 1, \end{cases}$$

$h = h(\hat{\theta}, \theta_2/\theta_1)$ being as in this section.

Since, for $\theta_1 = \theta_2$, the nonzero solution of (4.7.4) is $h = 1$, independently of $\hat{\theta}$; neglecting excess over the boundaries, we see that

$$\Pr\{H_0 \text{ is accepted} | \theta_1 = \theta_2\} = 1 - \sum_{\substack{r=[n_0 d]+1 \\ \{v=n_0 d/r\}}}^{\infty} \frac{e^{-2n_0\theta} (2n_0\theta)^{n_0 d/v}}{(n_0 d/v)!} > 1 - \alpha,$$

so that the first kind of error probability is always less than α .

The A.S.N. function of the test is obtained from

$$E(n|\hat{\theta}_1, \theta_2) = \sum_{r=0}^{\infty} \frac{E(n|\hat{\theta}_1, \theta_1, \theta_2, v = \frac{d}{2\hat{\theta}}) e^{-2n_0\hat{\theta}} (2n_0\hat{\theta})^{n_0 d/v}}{(n_0 d/v)!} \quad (4.7.6)$$

where

$$E(n|\hat{\theta}_1, \theta_1, \theta_2) = \begin{cases} 0 & \text{if } v \geq 1 \\ \text{right side of (4.1.2) with } \hat{\theta}_2 \text{ replaced by } \hat{\theta} & \text{if } v < 1 \end{cases}$$

and where

$$E Z(\hat{\theta}) = \theta_1 \ln(1+v) + \theta_2 \ln(1-v)$$

$$\text{and } E Z^2(\hat{\theta}) = (\theta_1 \ln(1+v) + \theta_2 \ln(1-v))^2 + \theta_1 (\ln(1+v))^2 + \theta_2 (\ln(1-v))^2$$

4.8. Inferences on the Shape Parameter of Pareto Distribution; the Power Function and A.S.N. Function of the Test Being Independent of the Nuisance Parameter

Let $X_1, X_2, \dots$ be independent with a common probability density function

$$g(x, a, k) = ak^a/x^{a+1}; \quad k > 0; \quad a > 0; \quad x \geq k \quad (4.8.1)$$

The null and the alternative hypotheses are

$$H_0: a = a_0 \text{ vs. } H_1: a = a_1; \quad (a_0 < a_1); \quad k \text{ unknown.}$$

$\hat{k}$, the maximum likelihood estimator of k is given by the preliminary sample $\{x_1, \dots, x_{n_0}\}$ as $\hat{k} = \min(x_1, \dots, x_{n_0})$. Let $\tilde{k} = pk$ where

$0 < p < 1$ is to be suitably chosen. The probability density function of $\tilde{k}$ is $(an_0)(pk)^{an_0}/y^{an_0+1}; \quad y \geq pk > 0$.

$$g(x, a, \tilde{k}) = ak/x^{a+1} \text{ for } x \geq \tilde{k}. \text{ Also,}$$

$$\Pr\{X < \tilde{k}\} = \int_{pk}^{\infty} \Pr\{X < \tilde{k} | \tilde{k} = y\} dF(y), \quad F \text{ being the c.d.f. of } \tilde{k}.$$

Since $\Pr\{X < y | y < k\} = 0$, we have

$$\Pr(X < \tilde{k}) = \int_k^\infty \Pr(X < \tilde{k} | \tilde{k} = y \geq k) dF(y) =$$

$$\int_k^\infty \left(\int_k^y (ak^a/x^{a+1}) dx \right) (an_0(pk)^{an_0}/y^{an_0+1}) dy = p^{an_0}/(n_0+1) = I(p, a), \text{ say.}$$

Thus for any preassigned positive ϵ ; we have,

$$\max_{i=0,1} I(p, a_i) < \epsilon \text{ if } p < \min_{i=0,1} \{\epsilon(n_0 + 1)\}^{1/a_i n_0}. \text{ Thus for small}$$

values of ϵ and for such a choice of p ; we base our test procedure upon those observations $\{x_j\}$ for which $g(x_j, a_i, \tilde{k})$ is positive for $i = 0, 1$; $j = 1, 2, \dots$

Thus the test procedure is

$$(H_0) : -D < \sum_{j=1}^n \{\ln(a_1/a_0) + (a_1 - a_0) \ln(\tilde{k}/x_j)\} < C : (H_1) \quad (4.8.2)$$

Writing $Z(\tilde{k}) = \ln(a_1/a_0) + (a_1 - a_0) \ln(\tilde{k}/X)$, we have,

$$E \exp(hZ(\tilde{k})) = (a_1/a_0)^{h(\tilde{k})} \frac{h(a_1 - a_0)}{E X} \frac{h(a_0 - a_1)}{X} \text{ exists for}$$

$$h > a/(a_0 - a_1) \text{ and equals } \{(a_1/a_0)(\tilde{k}/k)^{a_1 - a_0}\}^h (1 + h(a_1 - a_0)/a)^{-1}.$$

Lemma 2 [31] is applied for $h > a/(a_0 - a_1)$ and nonzero solution for $h = h(a, \tilde{k}, k)$ is obtained from

$$h \ln \left((a_1/a_0)(\tilde{k}/k)^{a_1 - a_0} \right) - \ln(1 + h(a_1 - a_0)/a) = 0 \quad (4.8.3)$$

Since h depends upon $(\tilde{k}, k)$ only through the ratio $\tilde{k}/k$ and since $\tilde{k}/k$ is distributed independently of k ; the power function of the test (given by (4.8.2)) does not depend upon k . We may denote the probability of accepting H_0 by $L(a)$. Thus

$$L(a) = an_0 p^{an_0} \int_p^\infty L(a, \tilde{k}, k | \tilde{k}/k = y) y^{-(an_0+1)} dy \quad (4.8.4)$$

where $L(a, \tilde{k}, k)$ is as on the right side of (4.1.1).

Using $E \ln X = \ln k + a^{-1}$ and $E (\ln X)^2 = (\ln k)^2 + 2a^{-2} + 2a^{-1} \ln k$, we obtain, after simplifying,

$$E Z(\tilde{k}) = \ln (a_1/a_0) + (a_1 - a_0) \ln (\tilde{k}/k) - (a_1 - a_0)/a \text{ and } E Z^2(\tilde{k}) = (\ln (a_1/a_0))^2 + (a_1 - a_0)^2 \{ (\ln (\tilde{k}/k))^2 - 2a^{-1} \ln (\tilde{k}/k) + 2a^{-2} \} + 2(a_1 - a_0) (\ln (a_1/a_0)) (\ln (\tilde{k}/k) - a^{-1}).$$

The forms of $E Z(\tilde{k})$ and $E Z^2(\tilde{k})$ and the distributional form of $\tilde{k}/k$ imply that the A.S.N. function is also independent of k . We may denote it by $E (n|a)$. Thus

$$E (n|a) = a n_0 p^{a n_0} \int_p^\infty E (n|a, \tilde{k}, k, \tilde{k}/k = y) y^{-(a n_0 + 1)} dy \quad (4.8.5)$$

where $E (n|a, \tilde{k}, k)$ is determined from (4.1.2) by replacing $\hat{\theta}_2$ by $\tilde{k}$.

4.9. Testing the Scale Parameter of Exponential Distribution; the Power Function and the Average Sample Number Being Independent of the Location Parameter

The probability density function of X is

$$g(x, \theta_1, \theta_2) = \theta_2^{-1} \exp\{-(x - \theta_1)/\theta_2\} ; x > \theta_1 ; \theta_2 > 0 \quad (4.9.1)$$

and the hypotheses are

$$H_0 : \theta_2 = \theta_{20} \text{ against } H_1 : \theta_2 = \theta_{21} ; -\infty < \theta_1 < \infty.$$

$\hat{\theta}_1$, the maximum likelihood estimator of θ_1 based upon the initial sample $\{x'_1, \dots, x'_{n_0}\}$ is given by $\hat{\theta}_1 = \min(x'_1, \dots, x'_{n_0})$. Let $\tilde{\theta}_1 = \hat{\theta}_1 - p$ where $p > 0$ is to be suitably chosen. The probability density function of $\tilde{\theta}_1$ is $n_0 \hat{\theta}_2^{-1} \exp\{-n_0(y - \theta_1 + p)/\theta_2\} ; y > \theta_1 - p$ and $g(x, \tilde{\theta}_1, \theta_2) = \theta_2^{-1} \exp\{-(x - \tilde{\theta}_1)/\theta_2\}$ for $x > \tilde{\theta}_1$.

$$\Pr(X \leq \tilde{\theta}_1) = \int_{\theta_1 - p}^\infty \Pr(X \leq \tilde{\theta}_1 | \tilde{\theta}_1 = y) dF(y), \quad F \text{ being the c.d.f. of } \tilde{\theta}_1.$$

Since $\Pr\{X \leq y | y \leq \theta_1\} = 0$, we have,

$$\Pr\{X \leq \tilde{\theta}_1\} = \int_{\theta_1}^{\infty} \Pr\{X \leq \tilde{\theta}_1 | \tilde{\theta}_1 = y > \theta_1\} dF(y) =$$

$$\int_{\theta_1}^{\infty} \left\{ \int_{\theta_1}^y \theta_2^{-1} \exp(-(x - \theta_1)/\theta_2) dx \right\} n_0 \theta_2^{-1} \exp(-n_0(y - \theta_1 + p)/\theta_2) dy =$$

$$(n_0 + 1)^{-1} \exp(-n_0 p / \theta_2) = I(p, \theta_2), \text{ say. Given } \epsilon > 0, \text{ we can have}$$

$\max_{i=0,1} I(p, \theta_{2i}) < \epsilon$ by choosing

$$p > \max_{i=0,1} \{-n_0^{-1} \theta_{2i} \ln(\epsilon(1 + n_0))\}$$

Thus for small values of ϵ ; we may base our test upon those $\{x_j\}$ for which both $g(x_j, \tilde{\theta}_1, \theta_{20})$ and $g(x_j, \tilde{\theta}_1, \theta_{21})$ are positive; $j = 1, 2, \dots$

The test procedure at the n th stage of sequential sampling is

$$(H_0) : -D < \sum_{j=1}^n \{\ln(\theta_{20}/\theta_{21}) + d(x_j - \tilde{\theta}_1)\} < C : (H_1) \quad (4.9.2)$$

where $d = (\theta_{20}^{-1} - \theta_{21}^{-1})$. If, conditionally on $\tilde{\theta}_1$ fixed, $Z(\tilde{\theta}_1)$ is distributed as $Z_j(\tilde{\theta}_1)$, $j = 1, 2, \dots$, then

$E \exp(hZ(\tilde{\theta}_1)) = (\theta_{20}/\theta_{21})^h \exp(-dh\tilde{\theta}_1) E \exp(dhX)$ exists for $dh\theta_2 < 1$ and equals $(\theta_{20}/\theta_{21})^h \{\exp(-dh(\tilde{\theta}_1 - \theta_1))\} / (1 - dh\theta_2)$. Lemma 2 [31] is applicable for $dh\theta_2 < 1$ and $h = h(\theta_2, \tilde{\theta}_1, \theta_1)$ is given by

$$h \ln(\theta_{20}/\theta_{21}) - \ln(1 - dh\theta_2) - dh(\tilde{\theta}_1 - \theta_1) = 0 \quad (4.9.3)$$

Since h depends upon $(\tilde{\theta}_1, \theta_1)$ only through the difference $\tilde{\theta}_1 - \theta_1$ and since the distribution of $\tilde{\theta}_1 - \theta_1$ does not involve θ_1 , the operating characteristic (to be denoted by $L(\theta_2)$) is independent of θ_1 and is obtained from

$$L(\theta_2) = n_0 \theta_2^{-1} \int_{-p}^{\infty} L(\theta_2, \tilde{\theta}_1, \theta_1 | \tilde{\theta}_1 - \theta_1 = y) \exp(-n_0 \theta_2^{-1}(y + p)) dy \quad (4.9.4)$$

where $L(\theta_2, \tilde{\theta}_1, \theta_1)$ is given by the right side of (4.1.1).

Also, since $E X = \theta_1 + \theta_2$, $E X^2 = (\theta_1 + \theta_2)^2 + \theta_2^2$, we obtain, after some readjustments,

$$E Z(\tilde{\theta}_1) = \ln(\theta_{20}/\theta_{21}) - d(\tilde{\theta}_1 - \theta_1) + d\theta_2 \quad \text{and} \quad E Z^2(\tilde{\theta}_1) = (\ln(\theta_{20}/\theta_{21}) + d\theta_2)^2 + d^2(\tilde{\theta}_1 - \theta_1 - \theta_2)^2 - 2(\tilde{\theta}_1 - \theta_1)d \ln(\theta_{20}/\theta_{21}).$$

The right sides of above two equations and the distributional form of $\tilde{\theta}_1 - \theta_1$ show that the A.S.N. function is also independent of θ_1 .

This is determined from

$$E(n|\theta_2) = n_0 \theta_2^{-1} \int_{-p}^{\infty} E(n|\theta_2, \tilde{\theta}_1, \theta_1, \tilde{\theta}_1 - \theta_1 = y) \exp(-n_0 \theta_2^{-1}(y + p)) dy \quad (4.9.5)$$

where $E(n|\theta_2, \tilde{\theta}_1, \theta_1)$ is obtained from the right side of (4.1.2) by replacing $\hat{\theta}_2$ by $\tilde{\theta}_1$.

4.10. Numerical Results

For studying the properties of the tests presented in this chapter, we are to evaluate quantities of the form $E f(Y)$ where the expectation is taken over the distribution of the random variable Y . Whenever the exact evaluation of such expectations presents analytical difficulties, a very close approximation may be obtained by evaluating $E f(Y)$ in the form

$$\sum_{y=y_1}^{y_2} \frac{1}{l} f(y + l\theta) p(y + l\theta) \quad (4.10.1)$$

($p(y)$ being the p.d.f. of Y); summed at intervals of length l where l and θ ($0 \leq \theta < 1$) are chosen suitably according to the shape of the curve $p(y)$ and the truncation points y_1 and y_2 are chosen such that

$\sum_{y=y_1}^{y_2} p(y + l\theta)$ is extremely close to one. If for each fixed y , $f(y)$ is

not very large, it would seem reasonable that (4.10.1) gives a close fit to the desired expectation.

Notation: (Table 4.10.7) $OC(n_0, p) = OC$ when k is estimated by $\bar{k} = p\hat{k}$, $\hat{k}$ being the m.l.e. of k given by the initial sample of size n_0 .
 (Table 4.10.9) $OC(n_0, p) = OC$ when θ_1 is estimated by $\bar{\theta}_1 = \hat{\theta}_1 - p$, $\hat{\theta}_1$ being the m.l.e. of θ_1 obtained from the initial sample of size n_0 .
 A.S.N. (n_0, p) of Tables 4.10.8 and 4.10.10 have analogous meanings. In the notation of sections 4.8-4.9, values of $I(p, a)$ and $I(p, \theta_2)$ for $(a_0, a_1) = (1, 2) = (\theta_{20}, \theta_{21})$ are as below:

n_0	50		25		15	
p	$I(p, a_0)$	$I(p, a_1)$	$I(p, a_0)$	$I(p, a_1)$	$I(p, a_0)$	$I(p, a_1)$
$1-1/n_0$	.00714	.00260	.01386	.00499	.02220	.00789
$1-1/1.3n_0$	.00903	.00416	.01761	.00806	.02837	.01288
n_0	40		25		15	
p	$I(p, \theta_{20})$	$I(p, \theta_{21})$	$I(p, \theta_{20})$	$I(p, \theta_{21})$	$I(p, \theta_{20})$	$I(p, \theta_{21})$
$2/n_0$	.00330	.00897	.00520	.01415	.00846	.02299
$1.5/n_0$	.00544	.01152	.00858	.01817	.01394	.00957

Probability of accepting H_0 and A.S.N. function of Tables 4.10.1 - 4.10.2 for unknown λ were calculated from (4.2.5) and (4.2.8) respectively using the distribution (4.2.9) whereas those of Tables 4.10.3 - 4.10.4 were obtained from (4.3.4) and (4.3.7) respectively using the distribution (4.3.8). OC and A.S.N. functions of Table 4.10.5 for unknown θ_1 were obtained from the formulae (4.4.8) and (4.4.12) respectively.

The results on the OC function agrees with those of Chapter II (cases

of unadjusted decision boundaries). Tables 4.10.7 and 4.10.9 show that for the same choice of p (as a suitable function of n_0) a smaller n_0 will tend to increase both kinds of error probabilities (e.g. in addition to Table 4.10.7, $OC(15, 14/15|a = 1) = .9371$ and $OC(15, 14/15|a = 2) = .0661$) though according to Tables 4.10.8 and 4.10.10, the A.S.N. is only slightly affected by a change in n_0 . A right choice of p appears to be $1 - 1/a_0 n_0$ in case of Pareto distribution and θ_{21}/n_0 in case of exponential distribution.

Table 4.10.1

Probability of Accepting the Null Hypotheses ($H_0 : v = 1.5$)
 Against the Alternative ($H_1 : v = 2.5$) for Testing
 the Parameter v of the Gamma Distribution
 ($\alpha = 0.05 = \beta, n_0 = 50$)

v	<u>Pr(H_0 is accepted v)</u>		h
	(λ -unknown)	(λ -known)	(λ -known)
1.25	0.9883	0.9906	1.58056
1.375	0.9726	0.9778	1.28489
1.5	0.944	0.95	1
1.625	0.886	0.894	0.72466
1.75	0.783	0.794	0.45787
1.875	0.625	0.642	0.19884
2	0.486	0.461	-0.05311
2.125	0.316	0.293	-0.29853
2.25	0.185	0.17	-0.53789
2.375	0.1006	0.0935	-0.7716
2.5	0.056	0.05	-1
2.625	0.0287	0.0265	-1.22339
2.75	0.0154	0.0141	-1.44205

Table 4.10.2

A.S.N. Function for Testing the Parameter v of the Gamma Distribution
 ($H_0 : v = 1.5$ vs. $H_1 : v = 2.5; \alpha = 0.05 = \beta, n_0 = 50$)

v	<u>A.S.N.</u>	
	(λ -unknown)	(λ -known)
1.25	4.83	4.56
1.375	6.15	5.71
1.5	8.5	7.18
1.625	10.87	8.99
1.75	11.23	10.95
1.875	12.7	12.58
2	13.25	13.27
2.125	12.91	12.87
2.25	12.03	11.62
2.375	10.78	10.21
2.5	9.91	8.9
2.625	9.03	7.81
2.75	7.52	6.92

Table 4.10.3

Probability of Accepting the Null Hypotheses ($H_0 : \lambda = .1$) Against
the Alternative ($H_1 : \lambda = 1.1$) for Testing the Parameter λ of
the Gamma Distribution ($v = 1, n_0 = 50, \alpha = 0.05 = \beta$)

λ	$\Pr\{H_0 \text{ is accepted} \lambda\}$		h
	(v-unknown)	(v-known)	(v-known)
0.05	0.9961	0.9984	2.18843
0.1	0.9457	0.95	1
0.2	0.644	0.671	0.24255
0.3	0.403	0.423	-0.10514
0.35	0.326	0.34	-0.22576
0.4	0.269	0.277	-0.32615
0.5	0.192	0.193	-0.48662
0.7	0.113	0.109	-0.71444
0.9	0.0765	0.0705	-0.8757
1.1	0.054	0.05	-1
1.3	0.0434	0.0376	-1.10088

Table 4.10.4

A.S.N. Function for Testing the Parameter λ of the Gamma Distribution
($H_0 : \lambda = 1$ vs. $H_1 : \lambda = 2$; $n_0 = 50, \alpha = 0.05 = \beta$)

λ	A.S.N.		h
	(v-unknown)	(v-known)	(v-known)
0.5	7.28	6.64	3.69
0.75	11.2	9.2	2
0.875	13.81	11.2	1.44784
1	16.05	13.72	1
1.125	17.23	16.36	0.62516
1.25	17.94	18.14	0.30395
1.3125	18.21	18.38	0.1593
1.375	17.67	18.14	0.02364
1.4375	16.89	17.48	-0.10403
1.5	15.41	16.52	-0.22455
1.625	13.64	14.23	-0.44689
1.75	11.98	12.01	-0.64803
2	8.67	8.64	-1
2.125	7.49	7.46	-1.15571
2.25	6.7	7.53	-1.30036
2.5	5.43	5.18	-1.56184

Table 4.10.5

Probability of Accepting the Null Hypothesis and the A.S.N. Function
for Testing the Scale Parameter θ_2 of Laplace Distribution
($H_0 : \theta_2 = 1.5$ vs. $H_1 : \theta_2 = 4.5$; $n_0 = 50$, $\alpha = 0.05 = \beta$)

θ_2	<u>Pr(H_0 is accepted θ_2)</u>		<u>A.S.N.</u>	
	(θ_1 -unknown)	(θ_1 -known)	(θ_1 -unknown)	(θ_1 -known)
1.2	0.9886	0.9888	5.14	5.09
1.5	0.9489	0.95	6.21	6.13
1.8	0.854	0.858	7.17	7.06
1.95	0.784	0.79	7.48	7.37
2.1	0.702	0.711	7.62	7.51
2.25	0.615	0.625	7.57	7.48
2.4	0.529	0.54	7.49	7.31

Table 4.10.6

Probability of Accepting the Null Hypothesis and the A.S.N. Function for Testing the

Parameter λ of the Inverse Gaussian Distribution

$(H_0 : \lambda = 2 \text{ vs. } H_1 : \lambda = 3.5, n_0 = 50, \alpha = 0.05 = \beta), \mu = 1$

λ	$\Pr\{H_0 \text{ is accepted} \lambda\}$		A.S.N.		h
	(μ -unknown)	(μ -known)	(μ -unknown)	(μ -known)	
1.5	0.9907	0.9985	24.93	24.65	2.2057
1.75	0.988	0.9894	28.7	31.22	1.54074
2	0.937	0.95	36.47	40.44	1
2.25	0.802	0.833	45.71	50.67	0.54646
2.375	0.701	0.734	52.13	54.39	0.34486
2.5	0.585	0.614	55.78	56.01	0.15715
2.625	0.517	0.487	53.4	55.16	-0.01836
2.75	0.402	0.368	49.94	52.23	-0.18307
2.875	0.303	0.27	45.52	48.03	-0.33816
3	0.219	0.194	40.88	43.36	-0.48465
3.25	0.108	0.0977	32.55	34.64	-0.75511
3.5	0.063	0.05	26.23	27.84	-1
3.75	0.0376	0.0265	21.75	22.86	-1.22354
4	0.0265	0.0147	18.79	19.21	-1.42901

Table 4.10.7

Probability of Accepting the Null Hypothesis ($H_0 : a = 1$) Against the Alternative ($H_1 : a = 2$)
for Testing the Shape Parameter of Pareto Distribution ($\alpha = 0.05 = \beta$)

a	0.5	0.75	1	1.125	1.25	1.3125
OC(50,49/50)	0.9989	0.9921	0.9487	0.886	0.776	0.702
OC(50,64/65)	0.9974	0.9902	0.9464	0.878	0.764	0.688
OC(25,24/25)	0.9975	0.9889	0.9449	0.886	0.784	0.716
OC(k-known)	0.9996	0.9937	0.95	0.884	0.767	0.688
h(k-known)	2.6599	1.7189	1	0.69022	0.40424	0.26873
a	1.5	1.625	1.75	2	2.25	2.5
OC(50,49/50)	0.618	0.282	0.169	0.0552	0.0182	0.0063
OC(50,64/65)	0.603	0.271	0.161	0.0512	0.0172	0.0059
OC(25,24/25)	0.637	0.307	0.187	0.0622	0.0205	0.0071
OC(k-known)	0.6	0.263	0.155	0.05	0.0162	0.0055
h(k-known)	0.13758	-0.35044	-0.57634	-1	-1.3937	-1.7638

Table 4.10.8

A.S.N. Function for Testing the Shape Parameter of Pareto Distribution

($H_0 : a = 1$ vs. $H_1 : a = 2$; $\alpha = 0.05 = \delta$)

a	0.5	0.75	1	1.125	1.25	1.3125	1.375
ASN(50,49/50)	2.29	4.58	8.62	11.49	14.63	16.08	17.29
ASN(50,64/65)	2.3	4.61	8.68	11.56	14.65	16.06	17.21
ASN(25,24/25)	2.32	4.61	8.57	11.36	14.47	15.96	17.26
ASN(k-known)	2.25	4.54	8.64	11.56	14.7	16.11	17.24
a	1.5	1.625	1.75	2	2.25	2.5	
ASN(50,49/50)	18.64	18.49	17.34	14.34	11.97	10.36	
ASN(50,64/65)	18.43	18.18	17	14.06	11.75	10.19	
ASN(25,24/25)	18.91	19.07	18.09	15.09	12.58	10.86	
ASN(k-known)	18.35	17.97	16.7	13.72	11.45	9.93	

Table 4.10.9

Probability of Accepting the Null Hypothesis ($H_0 : \theta_2 = 1$) Against the Alternative
 $(H_1 : \theta_2 = 2)$ for Testing the Scale Parameter of the
 Exponential Distribution ($\alpha = 0.05 = \beta$)

θ_2	0.75	1	1.125	1.1875	1.25	1.3125	1.375
OC(40, .05)	0.9962	0.9434	0.847	0.773	0.648	0.588	0.491
OC(40, .0375)	0.9964	0.9466	0.856	0.785	0.7	0.606	0.51
OC(25, .08)	0.9959	0.9383	0.835	0.757	0.666	0.569	0.478
OC(25, .06)	0.9961	0.944	0.849	0.778	0.692	0.599	0.504
OC(15, 2/15)	0.9952	0.9335	0.808	0.723	0.628	0.531	0.44
OC(15, .1)	0.9957	0.9383	0.836	0.761	0.674	0.581	0.491
OC(θ_1 -known)	0.9972	0.95	0.863	0.794	0.71	0.615	0.517
h(θ_1 -known)	2	1	0.62516	0.45876	0.30394	0.1593	0.02364
θ_2	1.4375	1.5	1.625	1.75	2	2.25	2.5
OC(40, .05)	0.401	0.322	0.203	0.127	0.0523	0.0243	0.0126
OC(40, .0375)	0.42	0.34	0.217	0.137	0.0567	0.0265	0.0141
OC(25, .08)	0.387	0.312	0.199	0.127	0.0558	0.0277	0.0153
OC(25, .06)	0.417	0.339	0.219	0.142	0.057	0.0312	0.0173
OC(15, 2/15)	0.36	0.292	0.192	0.129	0.0637	0.035	0.0228
OC(15, .1)	0.408	0.336	0.226	0.151	0.067	0.0381	0.026
OC(θ_1 -known)	0.424	0.34	0.211	0.129	0.05	0.0213	0.01
h(θ_1 -known)	-0.10403	-0.22455	-0.44689	-0.64803	-1	-1.3004	-1.5618

Table 4.10.10

A.S.N. Function for Testing the Scale Parameter of the Exponential Distribution

($H_0 : \theta_2 = 1$ vs. $H_1 : \theta_2 = 2$; $\alpha = 0.05 = \beta$)

θ_2	θ_1	1.125	1.1875	1.25	1.3125	1.375
ASN(40, .05)	14.5	17.17	18.17	18.73	18.79	18.37
ASN(40, .0375)	14.11	16.72	17.73	18.34	18.48	18.15
ASN(25, .08)	15.04	17.73	18.65	19.1	19.02	18.49
ASN(25, .06)	14.39	16.98	17.94	18.48	18.54	18.14
ASN(15, 2/15)	16.14	18.81	19.55	19.73	19.34	18.52
ASN(15, .1)	14.96	17.51	18.36	18.75	18.63	18.07
ASN(θ_1 -known)	13.72	16.36	17.43	18.14	18.38	18.14
	1.5	1.625	1.75	2	2.25	2.5
ASN(40, .05)	16.5	14.12	11.91	8.61	6.55	5.23
ASN(40, .0375)	16.45	14.18	12	8.7	6.62	5.27
ASN(25, .08)	16.42	14	11.81	8.58	6.56	5.25
ASN(25, .06)	16.36	14.09	11.95	8.72	6.66	5.33
ASN(15, 2/15)	16.18	13.71	11.57	8.48	6.54	5.27
ASN(15, .1)	16.14	13.87	11.8	8.69	6.7	5.39
ASN(θ_1 -known)	16.52	14.23	12.01	8.64	6.53	5.18

CHAPTER V

SEQUENTIAL TESTS FOR ANALYSIS OF VARIANCE UNDER RANDOM AND MIXED MODELS

5.1. *Introduction*

Sequential methods for analysis of variance under fixed and random effects models have been developed by Johnson ([17] and [18]). Procedures for discriminating between two values for the ratio of variance components in a single one way classification have been discussed in [18]. Hall et al [15] and Ghosh [10] have studied the problem through the principle of invariance. Our method reduces the problem from the sequential F-test to a sequential χ^2 -test and in most cases our tests reduce to the form in which their properties can be studied by standard methods unlike the sequential F-test where it is possible only to make a conjectural study (as in [2], [8], [10], and [24]) of the A.S.N. function when either the null or the alternative hypothesis is true. The conditional tests have been developed through the invariance principle except the alternative procedures of sections 5.5 and 5.6 in which cases the power functions coincide with those of the corresponding invariant tests.

5.2. *A General Problem*

The assumed model is the general linear mixed model

$$\underline{x}_{t \times 1} = \mu \underline{1}_{t \times 1} + C_{t \times s} \underline{y}_{s \times 1} + \underline{z}_{t \times 1} \quad (5.2.1)$$

where (i) $\underline{x}'_{1 \times t} = (x_1, \dots, x_t)$ are t observations available from a population; (ii) $\underline{J}'_{1 \times t} = (1, 1, \dots, 1)$; (iii) $\underline{y}'_{1 \times s} = (y_1, \dots, y_s)$ are independent random variables acting upon the population, affecting x_1 's, $Y_j \sim N(0, \sigma_j^2)$ $j = 1, 2, \dots, s$; (iv) $\underline{z}'_{1 \times t} = (z_1, \dots, z_t)$ are error components, $z_1 \sim N(0, \sigma_0^2)$ all $i, \{z_i\}$ are mutually independent and independent of $\{y_j\}$; (v) $\{c_{ij}\}$ are ts known constants specified by the design of experiment; (vi) μ is an unknown parameter.

The assumption $t > p + 1$ is required of $p(1 \leq p \leq s)$ is the number of distinct σ_j 's.

The parameter set is $\theta = (\mu, \sigma_0, \sigma_1, \dots, \sigma_s) \in \theta = \{\mu, \sigma_0, \sigma_j; j = 1, \dots, s \mid -\infty < \mu < \infty, \sigma_0 > 0, \sigma_j \geq 0; j = 1, \dots, s\}$.

The problem is to test, for a fixed j ,

$$H_0: \sigma_j^2 / \sigma_0^2 = \delta_0 \text{ vs. } H_1: \sigma_j^2 / \sigma_0^2 = \delta_1 \quad (5.2.2)$$

where $0 \leq \delta_0 < \delta_1$ are specified real numbers. The hypotheses are equivalent to $H_1: \tau^2 = \sigma_0^2(1 + c\delta_1)$; $i = 0, 1$ where $\tau^2 = c\sigma_j^2 + \sigma_0^2$ and c is a nonzero real number. We have an estimator s^2 of σ_0^2 from a first stage sample of appropriate size (to be determined in special cases) and we write the problem as

$$\bar{H}_1: \tau^2 = s^2(1 + c\delta_1); i = 0, 1 \quad (5.2.3)$$

5.3. A General Procedure for Deriving an Invariant S.P.R.T.

We consider a sequential form of experiment in which we draw $\{n_v - n_{v-1}\}$ observations at the v th stage where $n_0 = 0$. The total data at stage v is $\{x_1, \dots, x_{n_v}\}$. The test of (5.2.3) will be based on a sequence $g_v = g_v(x_1, \dots, x_{n_v})$ whose density depends only on τ^2 .

We note that for every fixed s^2 , the problem (5.2.3) remains invariant under the group G of one-to-one location transformations on

the sample space of $\{x_i\}$ such that every $g \in G$ transforms each x_i to $x_i + a$ for some a , $-\infty < a < \infty$.

We assume that

(C) for every $v \geq 1$, there exists a statistic $t_v = t_v(x_1, \dots, x_{n_v})$ which is sufficient for the underlying parameter set and is such that $\{t_v\}$ is a transitive sequence, that is, the conditional density of t_{v+1} given $(x_1, \dots, x_{n_v})$ is identical with conditional density of t_{v+1} given t_v , for each value of the parameter set.

Thus the results in [15] (pp. 583-84) imply that if $g_v = g_v(x_1, \dots, x_{n_v})$ is a maximal invariant under the group induced by G on the sample space of $\{t_v\}$, then $\{g_v\}$ is an invariantly sufficient and transitive sequence with some τ^2 which is a maximal invariant for the group induced by G on the parameter space. Thus an invariant test of (5.2.3) will be based upon the sequence $\{g_v\}$. Also, by transitivity of $\{t_v\}$, the joint density of $(g_1, \dots, g_v)$ factorizes into

$$p_v(g_1, \dots, g_v | \tau^2) = f_v(g_v | \tau^2) h_v(g_1, \dots, g_v), \text{ for every } v \geq 1$$

where $f_v(g_v | \tau^2)$ is the density function of g_v . Thus any S.P.R.T. of (5.2.3) based upon the observations $\{g_v\}$ will be specified by decision rules of the form:

$$(\bar{H}_0) : B < G_v(g_v) = f_v(g_v | s^2(1 + c\delta_1)) / f_v(g_v | s^2(1 + c\delta_0)) < A : (\bar{H}_1) \quad (5.3.1)$$

Continuation amounts to taking a further sample of $n_{v+1} - n_v$ observations. We shall take $A = (1 - \beta)/\alpha$, $B = \beta/(1 - \alpha)$ (we may, at the cost of larger sample size, take $A = \alpha^{-1}$, $B = \beta$; to reduce probabilities of wrong decisions). We shall consider the case $n_v = vN$, that is, sampling which takes a constant number N of observations at each stage. In special cases, we shall verify (C) and find out the

sequence $\{g_v\}$.

5.4. Definition of Sequential χ^2 -Test

We suppose that g_v/τ^2 follows, for every $v \geq 1$, a central χ^2 distribution with q degrees of freedom where q is an integral multiple of v ; $q = kv$, say, k being a positive integer.

Thus $f_v(g_v|\tau^2) = e^{-g_v/2\tau^2} \frac{g_v^{q/2-1}}{g_v^2} / ((2\tau^2)^{q/2} \Gamma(q/2))$. Consequently

$$G_v(g_v) = \left(\frac{1 + c\delta_0}{1 + c\delta_1} \right)^{\frac{q}{2}} \exp\left\{ -\frac{g_v}{2s^2} \left(\frac{1}{1 + c\delta_1} - \frac{1}{1 + c\delta_0} \right) \right\}. \text{ Thus the decision rule}$$

of our test is

$$(H_0) : \ln B < \frac{q}{2} \ln \frac{1 + c\delta_0}{1 + c\delta_1} - \frac{g_v}{2s^2} \left(\frac{1}{1 + c\delta_1} - \frac{1}{1 + c\delta_0} \right) < \ln A : (H_1)$$

In all practical cases we shall have $c > 0$ and without loss of generality we may assume $\delta_1 > \delta_0$. Thus $\ln G_v(g_v)$ (as a function of g_v) strictly increases from $\ln G_v(0) = \frac{q}{2} \ln \frac{1 + c\delta_0}{1 + c\delta_1}$ to $\ln G_v(\infty) = \infty$.

We may thus express the decision rule of the test as

$$(H_0) : g_v < g_v < \bar{g}_v : (H_1) \quad (5.4.1)$$

where $g_v = (\ln B - \frac{q}{2} \ln \frac{1 + c\delta_0}{1 + c\delta_1}) / \frac{1}{2s^2} \left(\frac{1}{1 + c\delta_0} - \frac{1}{1 + c\delta_1} \right)$ and

$$\bar{g}_v = (\ln A - \frac{q}{2} \ln \frac{1 + c\delta_0}{1 + c\delta_1}) / \frac{1}{2s^2} \left(\frac{1}{1 + c\delta_0} - \frac{1}{1 + c\delta_1} \right)$$

If c is independent of v , we can write

$$\ln G_v(g_v) = \sum_{i=1}^v \left[\frac{k}{2} \ln \frac{1 + c\delta_0}{1 + c\delta_1} - \frac{V_i}{2s^2} \left(\frac{1}{1 + c\delta_1} - \frac{1}{1 + c\delta_0} \right) \right] \quad (5.4.2)$$

where V_i 's are independent and are distributed as $\tau^2 \chi_k^2(0)$ for each $i = 1, \dots, v$. Thus (5.4.2) may be written as $\sum_{i=1}^v Z_i$ where, conditionally

on s^2 fixed, Z_1 's are i.i.d.

We shall find expressions for the operating characteristic and A.S.N. functions (to be denoted by $L(\tau^2, \sigma_0^2)$ and $E(v|\tau^2, \sigma_0^2)$ respectively).

We see that, for s^2 fixed, $E e^{hZ}$ exists for

$$\frac{h\tau^2}{t\sigma_0^2} \left(\frac{1}{1+c\delta_0} - \frac{1}{1+c\delta_1} \right) < 1 \quad (\text{where } t = s^2/\sigma_0^2) \quad \text{and equals} \\ \left(\frac{1+c\delta_0}{1+c\delta_1} \right)^{hk/2} \left\{ 1 + \frac{h\tau^2}{t\sigma_0^2} \left(\frac{1}{1+c\delta_1} - \frac{1}{1+c\delta_0} \right) \right\}^{-k/2}. \quad \text{Conditions of Lemma 2 [31]}$$

are satisfied for these values of h . Thus $h = h(\tau^2, t, \sigma_0^2) = h(\tau^2/\sigma_0^2, t)$ may be obtained from

$$\left(\frac{1+c\delta_0}{1+c\delta_1} \right)^h = 1 + \frac{h\tau^2}{t\sigma_0^2} \left(\frac{1}{1+c\delta_1} - \frac{1}{1+c\delta_0} \right) \quad (5.4.3)$$

With h determined from (5.4.3), we have, assuming excess of $\ln G_v(g_v)$ over the decision boundaries is negligible,

$$L(\tau^2, \sigma_0^2) = E L(\tau^2/\sigma_0^2, t) \quad \text{where} \\ L(\tau^2/\sigma_0^2, t) = (A^h - 1)/(A^h - B^h) \quad \text{for } EZ \neq 0 \\ = \ln A / (\ln A - \ln B) \quad \text{for } EZ = 0 \quad (5.4.4)$$

where $EZ = E(Z|t)$.

Also, under the same assumption,

$$E(v|\tau^2, \sigma_0^2) = E E(v|\tau^2/\sigma_0^2, t) \quad \text{where} \\ E(v|\tau^2/\sigma_0^2, t) = \{\ln B(A^h - 1) + (1 - B^h) \ln A\} / (A^h - B^h) EZ \quad \text{for } EZ \neq 0 \\ = -(\ln A \ln B) / EZ^2 \quad \text{for } EZ = 0 \quad (5.4.5)$$

where

$$E(Z|t) = \frac{k}{2} \ln \frac{1+c\delta_0}{1+c\delta_1} - \frac{\tau^2 k}{2t\sigma_0^2} \left(\frac{1}{1+c\delta_1} - \frac{1}{1+c\delta_0} \right) \quad (5.4.6)$$

and

$$E(Z^2|t) = \frac{k^2}{4} \left(\ln \frac{1 + c\delta_0}{1 + c\delta_1} \right)^2 + \frac{\tau^4 k(k+2)}{4t^2 \sigma_0^4} \left(\frac{1}{1 + c\delta_1} - \frac{1}{1 + c\delta_0} \right)^2 - \frac{k^2 \tau^2}{2t \sigma_0^2} \left(\ln \frac{1 + c\delta_0}{1 + c\delta_1} \right) \left(\frac{1}{1 + c\delta_1} - \frac{1}{1 + c\delta_0} \right) \quad (5.4.7)$$

5.5. Application of the χ^2 -Test for Testing Hypotheses About the Ratio of Variances of Two Normal Populations

Let Π_1 and Π_2 be two normal populations with variances τ^2 and σ_0^2 respectively. We want to discriminate between $H_i : \tau^2/\sigma_0^2 = \theta_i$, $i = 0, 1$. A first stage sample $\{x_{i2}, i = 1, 2, \dots, n_0\}$ from Π_2 determines $s^2 = (n_0 - 1)^{-1} \sum_{i=1}^{n_0} (x_{i2} - \bar{x}_2)^2$, $\bar{x}_2$ being the mean of this sample. Corresponding to (5.2.3), we have, $\bar{H}_i : \tau^2 = s^2 \theta_i$, $i = 0, 1$; ($\theta_0 < \theta_1$).

Procedure I: Start by taking sample of two (x_{11}, x_{21}) from Π_1 . At each subsequent stage (if required) a further individual is taken from Π_1 .

From the joint density function of $\{x_{i1}, i = 1, \dots, n\}$, we see that $t_{n-1} = (\bar{x}_1(n), \sum_{i=1}^n (x_{i1} - \bar{x}_1(n))^2)$ (where $\bar{x}_1(n) = n^{-1} \sum_{i=1}^n x_{i1}$) forms a sufficient sequence for (μ_1, τ^2) where μ_1 is the mean of Π_1 . The group G of location transformations leaves the problem invariant with τ^2 as a maximal invariant on $\Theta = \{\mu_1, \tau | -\infty < \mu_1 < \infty, \tau > 0\}$ under the induced transformation $(\mu_1 + a, \tau)$. The group G induces the transformation $(x_1(n) + a, \sum_{i=1}^n (x_{i1} - \bar{x}_1(n))^2)$ on the sample space of t_{n-1} with $g_{n-1} = \sum_{i=1}^n (x_{i1} - \bar{x}_1(n))^2$ as a maximal invariant.

$$\text{Writing } \bar{x}_1(n+1) = \frac{n}{n+1} \bar{x}_1(n) + \frac{1}{n+1} x_{n+1}, \quad 1$$

$$\text{and } \sum_{i=1}^{n+1} (x_{i1} - \bar{x}_1(n+1))^2 = \sum_{i=1}^n (x_{i1} - \bar{x}_1(n))^2 + \frac{n}{(n+1)^2} (x_{n+1,1} - \bar{x}_1(n))^2,$$

we have, from Theorem 4.3 of [15] that $\{\tau_{n-1}\}$ is a transitive sequence.

Thus invariant S.P.R.T. is given by the rule

$$(H_0) : B < f_{n-1}(g_{n-1} | \tau^2 = s^2 \theta_1) / f_{n-1}(g_{n-1} | \tau^2 = s^2 \theta_0) < A : (H_1)$$

Since $g_{n-1} \sim \tau^2 \chi_{n-1}^2(0)$, the decision rule may be expressed as

$$(H_0) : g_{n-1} < g_{n-1} < \bar{g}_{n-1} : (H_1) \quad (5.5.1)$$

where $g_{n-1} = (\ln B - \frac{n-1}{2} \ln \frac{\theta_0}{\theta_1}) / \frac{1}{2s^2} (\frac{1}{\theta_0} - \frac{1}{\theta_1})$ and $\bar{g}_{n-1} =$

$$(\ln A - \frac{n-1}{2} \ln \frac{\theta_0}{\theta_1}) / \frac{1}{2s^2} (\frac{1}{\theta_0} - \frac{1}{\theta_1}).$$

In the notation of section (5.4)

$$\ln G_{n-1}(g_{n-1}) = \sum_{i=1}^{n-1} Z_i = \sum_{i=1}^{n-1} [\frac{1}{2} \ln \frac{\theta_0}{\theta_1} - \frac{V_i}{2t\sigma_0^2} (\frac{1}{\theta_1} - \frac{1}{\theta_0})] \quad (5.5.2)$$

where $t = s^2/\sigma_0^2$ and V_i is distributed as $\tau^2 \chi_1^2(0)$ for each i .

Thus $h = h(\tau^2, \sigma_0^2, t)$ is given by

$$\left(\frac{\theta_0}{\theta_1}\right)^h = 1 + \frac{h\tau^2}{t\sigma_0^2} \left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) \quad (5.5.3)$$

Since t is distributed independently of σ_0^2 and h depends upon (τ^2, σ_0^2) only through the ratio τ^2/σ_0^2 ($h = h(\tau^2/\sigma_0^2, t)$, say), the operating characteristic will depend upon (τ^2, σ_0^2) only through τ^2/σ_0^2 . A similar conclusion will follow for A.S.N. We shall, therefore, deviate from notation of section 5.4 and shall denote them by $L(\tau^2/\sigma_0^2)$ and $E(v|\tau^2/\sigma_0^2)$ respectively in this section (and in all the following sections). So, we have,

$$L(\tau^2/\sigma_0^2) = \frac{((n_0 - 1)/2)^{(n_0-1)/2}}{\Gamma((n_0 - 1)/2)} \int_0^\infty L(\tau^2/\sigma_0^2, t) t^{(n_0-3)/2} \exp -((n_0-1)/2)t \, dt \quad (5.5.4)$$

where $L(\tau^2/\sigma_0^2, t)$ is as in (5.4.4). Also, for $v = n - 1$,

$$E(v|\tau^2/\sigma_0^2) = \frac{((n_0-1)/2)^{(n_0-1)/2}}{\Gamma((n_0-1)/2)} \int_0^\infty E(v|\tau^2/\sigma_0^2, t) t^{(n_0-3)/2} \exp -((n_0-1)/2)t \, dt \quad (5.5.5)$$

where $E(v|\tau^2/\sigma_0^2, t)$ is determined from (5.4.5) with

$$E Z = \frac{1}{2} \ln \frac{\theta_0}{\theta_1} - \frac{\tau^2}{2\sigma_0^2 t} \left(\frac{1}{\theta_1} - \frac{1}{\theta_0} \right) \quad (5.5.6)$$

and

$$E Z^2 = \frac{1}{4} \left(\ln \frac{\theta_0}{\theta_1} \right)^2 + \frac{3\tau^4}{4\sigma_0^4 t^2} \left(\frac{1}{\theta_1} - \frac{1}{\theta_0} \right)^2 - \left(\ln \frac{\theta_0}{\theta_1} \right) \frac{\tau^2}{2t\sigma_0^2} \left(\frac{1}{\theta_1} - \frac{1}{\theta_0} \right) \quad (5.5.7)$$

Procedure II: At each stage take a sample of r (≥ 2) individuals from Π_1 . Calculate $g_r = \sum_{i=1}^r (x_{i1} - \bar{x}_1)^2$ ($\bar{x}_1 = r^{-1} \sum_{i=1}^r x_{i1}$) and call it $g_{r,j}$ if based upon sample at the j th stage.

At the n th stage, the test procedure to be followed is:

$$(H_0) : B < \prod_{j=1}^n [p(g_{r,j}|\tau^2 = s^2\theta_1)/p(g_{r,j}|\tau^2 = s^2\theta_0)] < A : (H_1)$$

where $p(g_{r,j}|\tau^2) = \{(2\tau^2)^{-(r-1)/2} g_{r,j}^{(r-3)/2} \exp -(g_{r,j}/2\tau^2)\} / \Gamma((r-1)/2)$

After simplifying, the decision rule may be written as

$$(H_0) : g_n < \sum_{j=1}^n g_{r,j} < \bar{g}_n : (H_1) \quad (5.5.1) \text{ II}$$

where $g_n = (\ln B - \frac{n(r-1)}{2} \ln \frac{\theta_0}{\theta_1}) / \frac{1}{2s^2} (\frac{1}{\theta_0} - \frac{1}{\theta_1})$ and $\bar{g}_n =$

$$(\ln A - \frac{n(r-1)}{2} \ln \frac{\theta_0}{\theta_1}) / \frac{1}{2s^2} (\frac{1}{\theta_0} - \frac{1}{\theta_1})$$

An expression corresponding to the right side of (5.4.2) is

$$\sum_{j=1}^n Z_j = \sum_{j=1}^n \left[\frac{r-1}{2} \ln \frac{\theta_0}{\theta_1} - \frac{g_{r,j}}{2s^2} \left(\frac{1}{\theta_1} - \frac{1}{\theta_0} \right) \right] \quad (5.5.2) \text{ II}$$

A comparison between (5.4.2) and (5.5.2) II implies that formulae giving $h = h(\tau^2/\sigma_0^2, t)$, $L(\tau^2/\sigma_0^2)$ and $E(v|\tau^2/\sigma_0^2)$ ($v = n$ here) are the same as for Procedure I where $t = s^2/\sigma_0^2$ and with

$$E Z = \frac{r-1}{2} \left\{ \ln \frac{\theta_0}{\theta_1} - \frac{\tau^2}{\sigma_0^2 t} \left(\frac{1}{\theta_1} - \frac{1}{\theta_0} \right) \right\} \quad (5.5.6) \text{ II}$$

and

$$E Z^2 = \frac{(r-1)^2}{4} \left(\ln \frac{\theta_0}{\theta_1} \right)^2 + \frac{\tau^4 (r^2 - 1)}{4t^2 \sigma_0^4} \left(\frac{1}{\theta_1} - \frac{1}{\theta_0} \right)^2 - \frac{(r-1)^2 \tau^2}{2t \sigma_0^2} \left(\ln \frac{\theta_0}{\theta_1} \right) \left(\frac{1}{\theta_1} - \frac{1}{\theta_0} \right) \quad (5.5.7) \text{ II}$$

Procedure III: Start by taking sample of two (x_{11}, x_{21}) from Π_1 . At each subsequent stage (if required) one further individual is chosen from Π_1 .

At the $(n-1)$ st stage, calculate

$$g_{n-1} = (x_{11} + x_{21} + \dots + x_{n-1,1} - (n-1)x_{n,1})^2 / n(n-1) \text{ and}$$

follow the decision rule

$$(H_0) : B < \prod_{j=2}^n [p(g_{j-1}|\tau^2 = s^2\theta_1) / p(g_{j-1}|\tau^2 = s^2\theta_0)] < A : (H_1)$$

where $p(g_{j-1}|\tau^2) = g_{j-1}^{-1/2} \exp(-g_{j-1}/2\tau^2) / \sqrt{2\pi} \Gamma(\frac{1}{2})$.

$$\text{Since } \prod_{j=2}^n [p(g_{j-1}|\tau^2 = s^2\theta_1) / p(g_{j-1}|\tau^2 = s^2\theta_0)] =$$

$$\left(\frac{\theta_0}{\theta_1} \right)^{(n-1)/2} \exp \left\{ - \sum_{j=2}^n \frac{g_{j-1}}{2s^2} \left(\frac{1}{\theta_1} - \frac{1}{\theta_0} \right) \right\}, \text{ the decision rule may be written as}$$

$$(H_0) : g_{n-1} < \sum_{j=2}^n g_{j-1} < \bar{g}_{n-1} : (H_1) \quad (5.5.1) \text{ III}$$

where g_{n-1} and $\bar{g}_{n-1}$ are the same as for Procedure I.

Since corresponding to (5.4.2), we have,

$$\sum_{j=2}^n Z_j = \sum_{j=2}^n \left[\ln \frac{\theta_0}{\theta_1} - \frac{8j-1}{2s^2} \left(\frac{1}{\theta_1} - \frac{1}{\theta_0} \right) \right], \quad (5.5.2) \text{ III}$$

a comparison implies that the formulae giving $h(\tau^2/\sigma_0^2, t)$, $L(\tau^2/\sigma_0^2)$, $E(v|\tau^2/\sigma_0^2)$, $E(Z|t)$ and $E(Z^2|t)$ are exactly the same as for Procedure I.

Similar procedures may be developed by interchanging Π_1 and Π_2 .

5.6. One Way Classification

We consider a one way classification by groups and denote the internal (within group) variance by σ_0^2 and the external (between group) variance by σ_u^2 . If x_{ij} denotes the j th observation ($j = 1, \dots, m$) in the i th group ($i = 1, \dots, n$) in a randomly chosen set of $n > 1$ groups, then

$$x_{ij} = \mu + u_i + z_{ij} \quad (i = 1, \dots, n; \quad j = 1, \dots, m) \quad (5.6.0)$$

We have a special case of (5.2.1) where $\{c_{ij}\}$ have values 0 and 1,

$t = nm$, $s = n$, $p = 1$. A problem corresponding to (5.2.2) is

$$H_0 : \sigma_u^2 = \delta_0 \sigma_0^2 \text{ vs. } H_1 : \sigma_u^2 = \delta_1 \sigma_0^2. \text{ If } s^2 =$$

$$(n_0(m_0 - 1))^{-1} \sum_{i=1}^{n_0} \sum_{j=1}^{m_0} (x_{ij} - \bar{x}_i)^2 \text{ (where } \bar{x}_i = \sum_{j=1}^{m_0} x_{ij} / m_0 \text{)} \text{ is determined}$$

from a preliminary sample $\{x_{ij}; \quad i = 1, \dots, n_0; \quad j = 1, \dots, m_0\}$, then

$n_0(m_0 - 1)t$ is distributed as $\chi_{n_0(m_0-1)}^2(0)$ where $t = s^2/\sigma_0^2$. Thus

corresponding to (5.2.3), we have, $\bar{H}_i : \tau^2 = s^2(1 + m\delta_i)$; $i = 0, 1$ with $\tau^2 = \sigma_0^2 + m\sigma_u^2$.

There can be two possible kinds of sampling schemes: (a) At each stage selecting (at random) a further group (or a set of r groups) and taking a fixed number m of observations from each group. (b) $n > 1$

groups may be selected at once and for all, and an observation (or a set of m observations) be drawn from each at successive stages.

Procedures corresponding to II and III of section 5.5 are not applicable in scheme (b) as the successive sets of observations are not independent of each other.

Scheme (a): (To be preferred if availability of observations inside a group is limited.)

Procedure I: Start by taking two groups and m observations in each group. At each subsequent stage (if required) one further group is chosen and m observations taken in it.

Total data at stage $(n - 1)$ is $\{x_{ij}; i = 1, \dots, n; j = 1, \dots, m\}$.

For s^2 fixed, the joint density of $\{x_{ij}; i = 1, \dots, n; j = 1, \dots, m\}$ may be written as

$$f(\{x_{ij}\} | \mu, \tau^2, s^2) = (\sqrt{2\pi})^{-nm} (s^2)^{-n(m-1)/2} (\tau^2)^{-n/2} \exp\{-(V/2s^2) - (U + nm(\bar{x} - \mu)^2)/2\tau^2\}$$

$$\text{where } V = \sum_{i=1}^n \sum_{j=1}^m (x_{ij} - \bar{x}_i)^2, \quad U = m \sum_{i=1}^n (\bar{x}_i - \bar{x})^2, \quad \bar{x}_i = \sum_{j=1}^m x_{ij}/m$$

$$\text{and } \bar{x} = \sum_{i=1}^n \sum_{j=1}^m x_{ij}/nm.$$

Thus the set of sufficient statistics for $\theta = (\mu, \tau)$ is $t_{n-1} = (\bar{x}, U)$ (to be denoted by $(\bar{x}(n), U(n))$ to indicate $(n-1)$ st stage). The group G of transformations on the sample space of $\{x_{ij}\}$ involving any common change of location in $\{x_{ij}\}$ leaves the problem invariant with τ^2 as a maximal invariant on $\theta = \{\mu, \tau | -\infty < \mu < \infty, \tau > 0\}$ under the induced transformation $(\mu + a, \tau)$. The group G induces the transformation $(\bar{x} + a, U)$ on the sample space of $\{t_{n-1}\}$ with

$$g_{n-1}(m) = U = m \sum_{i=1}^n (\bar{x}_i - \bar{x})^2$$

as a maximal invariant. It remains to establish the transitivity of t_{n-1} , i.e., to show that for every $\theta = (u, \tau)$, the conditional density of $t_n = (\bar{x}(n+1), U(n+1))$ given $\{x_{ij}; i = 1, \dots, n; j = 1, \dots, m\}$ depends on $\{x_{ij}\}$ only through $\bar{x}(n)$ and $U(n)$. Writing

$$\bar{x}(n+1) = n\bar{x}(n)/(n+1) + \sum_{j=1}^m x_{n+1, j}/m(n+1) \quad \text{and} \quad U(n+1) = U(n) + nm \sum_{j=1}^m (x_{n+1, j}/m - \bar{x}(n))^2/(n+1) \quad \text{and using Theorem 4.3 in [15],}$$

we see that $\{t_n\}$ is a transitive sequence.

Thus the decision rule at stage $(n-1)$ will be

$$(H_0) : B < \frac{f_{n-1}(g_{n-1}(m) | \tau^2 = s^2(1+m\delta_1))}{f_{n-1}(g_{n-1}(m) | \tau^2 = s^2(1+m\delta_0))} < A : (H_1)$$

where $g_{n-1}(m) = \tau^2 \chi_{n-1}^2(0)$. Thus the decision rule becomes

$$(H_0) : g_{n-1} < g_{n-1}(m) < \bar{g}_{n-1} : (H_1) \quad (5.6.1)$$

where $g_{n-1} = (\ln B - \frac{n-1}{2} \ln \frac{1+m\delta_0}{1+m\delta_1}) / \frac{1}{2s^2} (\frac{1}{1+m\delta_0} - \frac{1}{1+m\delta_1})$ and

$$\bar{g}_{n-1} = (\ln A - \frac{n-1}{2} \ln \frac{1+m\delta_0}{1+m\delta_1}) / \frac{1}{2s^2} (\frac{1}{1+m\delta_0} - \frac{1}{1+m\delta_1}).$$

$$\sum_{i=1}^{n-1} Z_i = \sum_{i=1}^{n-1} [\frac{1}{2} \ln \frac{1+m\delta_0}{1+m\delta_1} + \frac{V_i}{2s^2} (\frac{1}{1+m\delta_0} - \frac{1}{1+m\delta_1})] \quad (5.6.2)$$

where $V_i = \tau^2 \chi_1^2(0)$ for each i .

With h determined from

$$\left(\frac{1+m\delta_0}{1+m\delta_1}\right)^h = 1 + \frac{h\tau^2}{t\sigma_0^2} \left(\frac{1}{1+m\delta_1} - \frac{1}{1+m\delta_0}\right), \quad t = s^2/\sigma_0^2; \quad (5.6.3)$$

$$L(\tau^2/\sigma_0^2) = \frac{(n_0(m_0-1)/2)^{n_0(m_0-1)/2}}{\Gamma(n_0(m_0-1)/2)} \int_0^\infty L(\tau^2/\sigma_0^2, t) e^{-n_0 t(m_0-1)/2} t^{(n_0(m_0-1)/2)-1} dt \quad (5.6.4)$$

where $L(\tau^2/\sigma_0^2, t)$ is obtained from (5.4.4) and with $v = n-1$

$$E(v|\tau^2/\sigma_0^2) =$$

$$\frac{(n_0(m_0-1)/2)}{\Gamma(n_0(m_0-1)/2)} \int_0^\infty E(v|\tau^2/\sigma_0^2, t) e^{-n_0 t(m_0-1)/2} t^{(n_0(m_0-1)/2)-1} dt \quad (5.6.5)$$

where $E(v|\tau^2/\sigma_0^2, t)$ is as in (5.4.5) with

$$E Z = \frac{1}{2} \ln \frac{1+m\delta_0}{1+m\delta_1} - \frac{\tau^2}{2\sigma_0^2} \left(\frac{1}{1+m\delta_1} - \frac{1}{1+m\delta_0} \right), \quad (5.6.6)$$

$$E Z^2 = \frac{1}{4} \left(\ln \frac{1+m\delta_0}{1+m\delta_1} \right)^2 + \frac{3\tau^4}{4t^2\sigma_0^4} \left(\frac{1}{1+m\delta_1} - \frac{1}{1+m\delta_0} \right)^2 - \frac{\tau^2}{2t\sigma_0^2} \left(\ln \frac{1+m\delta_0}{1+m\delta_1} \right) \left(\frac{1}{1+m\delta_1} - \frac{1}{1+m\delta_0} \right) \quad (5.6.7)$$

Procedure II: At each stage take a sample of r groups and take m observations in each group. Calculate $g_r = m \sum_{t=1}^r (\bar{x}_t - \bar{x})^2$ and call it $g_{r,j}$ if based upon the observations at the j th stage.

At the n th stage, the test procedure is

$$(H_0) : B < \prod_{j=1}^n [p(g_{r,j}|\tau^2 = s^2(1+m\delta_1)) / p(g_{r,j}|\tau^2 = s^2(1+m\delta_0))] < A : (H_1)$$

$$\text{where } p(g_{r,j}|\tau^2) = \frac{((2\tau^2)^{-(r-1)/2} g_{r,j}^{(r-3)/2} \exp -(g_{r,j}/2\tau^2)) / \Gamma((r-1)/2)}$$

After simplifying, we may write the decision rule as

$$g_n < \sum_{j=1}^n g_{r,j} < \bar{g}_n \quad (5.6.1.) \text{ II}$$

$$\text{where } g_n = \left\{ \ln B - \frac{n(r-1)}{2} \ln \frac{1+m\delta_0}{1+m\delta_1} \right\} / \frac{1}{2s^2} \left(\frac{1}{1+m\delta_0} - \frac{1}{1+m\delta_1} \right) \text{ and } \bar{g}_n = \left\{ \ln A - \frac{n(r-1)}{2} \ln \frac{1+m\delta_0}{1+m\delta_1} \right\} / \frac{1}{2s^2} \left(\frac{1}{1+m\delta_0} - \frac{1}{1+m\delta_1} \right). \text{ Also, since}$$

$$\sum_{j=1}^n Z_j = \sum_{j=1}^n \left[\frac{r-1}{2} \ln \frac{1+m\delta_0}{1+m\delta_1} - \frac{g_{r,j}}{2s^2} \left(\frac{1}{1+m\delta_1} - \frac{1}{1+m\delta_0} \right) \right], \quad (5.6.2) \text{ II}$$

the formulae giving h , $L(\tau^2/\sigma_0^2)$ and $E(v|\tau^2/\sigma_0^2)$ (with $v = n$) are the same as for Procedure I with

$$E Z = \frac{r-1}{2} \left\{ \ln \frac{1+m\delta_0}{1+m\delta_1} - \frac{\tau^2}{\sigma_0^2} \left(\frac{1}{1+m\delta_1} - \frac{1}{1+m\delta_0} \right) \right\}, \quad (5.6.6) \text{ II}$$

$$E Z^2 = \frac{(r-1)^2}{4} \left(\ln \frac{1+m\delta_0}{1+m\delta_1} \right)^2 - \frac{(r-1)^2 \tau^2}{2\sigma_0^2} \left(\ln \frac{1+m\delta_0}{1+m\delta_1} \right) \left(\frac{1}{1+m\delta_1} - \frac{1}{1+m\delta_0} \right) + \frac{\tau^4(r^2-1)}{4\sigma_0^4} \left(\frac{1}{1+m\delta_1} - \frac{1}{1+m\delta_0} \right)^2 \quad (5.6.7) \text{ II}$$

Procedure III: Start by taking two groups and m observations in each group. At each subsequent stage (if required) one further group is chosen and m observations taken in it. At the $(n-1)$ st stage calculate

$$g_{n-1} = m(\bar{x}_1 + \bar{x}_2 + \dots + \bar{x}_{n-1} - (n-1)\bar{x}_n)^2 / n(n-1) \text{ and follow}$$

the decision rule

$$(H_0) : B < \prod_{j=2}^n [p(g_{j-1}|\tau^2 = s^2(1+m\delta_1)) / p(g_{j-1}|\tau^2 = s^2(1+m\delta_0))] < A : (H_1)$$

where $p(g_{j-1}|\tau^2) = g_{j-1}^{-1/2} \exp(-g_{j-1}/2\tau^2) / \sqrt{2\tau} \Gamma(1/2)$. After simplifying, the decision rule is

$$(H_0) : g_{n-1} < \sum_{j=2}^n g_{j-1} < \bar{g}_{n-1} : (H_1) \quad (5.6.1) \text{ III}$$

where g_{n-1} and $\bar{g}_{n-1}$ are the same as for Procedure I. In the notation of section 5.4,

$$\sum_{j=2}^n Z_j = \ln G_{n-1}(g_{n-1}) = \sum_{j=2}^n \left[\ln \frac{1+m\delta_0}{1+m\delta_1} - \frac{g_{j-1}}{2s^2} \left(\frac{1}{1+m\delta_1} - \frac{1}{1+m\delta_0} \right) \right] \quad (5.6.2) \text{ III}$$

where g_{j-1} is distributed as $\tau^2 \chi_1^2(0)$. Thus the formulae giving h ,

$L(\tau^2/\sigma_0^2)$, $E(v|\tau^2/\sigma_0^2)$, $E(Z|t)$ and $E(Z^2|t)$ are exactly the same as for Procedure 1.

Scheme (b): (To be preferred when the cost or waiting time for drawing a group is much higher than that of a single observation.)

Procedure: A fixed number $n > 1$ of groups is selected in the beginning and an observation is drawn from each at each stage. If we proceed as in Scheme (a) (Procedure I), we find that $U = m \sum_{i=1}^n (\bar{x}_i - \bar{x})^2$ (which will now be denoted by $g_m(n)$) is a maximal invariant on the sample space of $t_m = (X, U)$ (which will now be denoted by $(X(m), U(m))$ to indicate m th stage) under the group G of location transformations.

To show the transitivity of the sequence $\{t_m\}$, we write

$$\bar{x}(m+1) = m\bar{x}(m)/(m+1) + v_1/n(m+1),$$

$$U(m+1) = mU(m)/(m+1) - 2m\bar{x}(m)v_1/(m+1) - v_1^2/n(m+1) + v_2/(m+1) + 2mv_3/(m+1)$$

$$\text{where } v_1 = \sum_{i=1}^n x_{i,m+1}; \quad v_2 = \sum_{i=1}^n x_{i,m+1}^2; \quad v_3 = \sum_{i=1}^n \bar{x}_i(m)x_{i,m+1}$$

and $\bar{x}_i(m) = \sum_{j=1}^m x_{ij}/m$. It has been proved in [10] that the characteristic function of the conditional joint distribution of (v_1, v_2, v_3) given $\{x_{ij}; i = 1, \dots, n; j = 1, \dots, m\}$ depends only on $\bar{x}(m)$ and $U(m)$. Thus the conditional joint density of $t_{m+1} = (\bar{x}(m+1), U(m+1))$ depends on $\{x_{ij}\}$ only through $t_m = (\bar{x}(m), U(m))$. Thus $\{t_m\}$ is a transitive sequence and consequently $\{g_m(n)\}$ is an invariantly sufficient and transitive sequence.

The decision rule at stage m is

$$(H_0) : g_m < g_m(n) < \bar{g}_m : (H_1) \quad (5.6.1)(b)$$

where g_m and $\bar{g}_m$ are the same as g_{n-1} and $\bar{g}_{n-1}$ of Procedure I,

Scheme (a). Since

$$\ln G_m(g_m) = \frac{n-1}{2} \ln \frac{1+m\delta_0}{1+m\delta_1} - \frac{g_m(n)}{2s^2} \left(\frac{1}{1+m\delta_1} - \frac{1}{1+m\delta_0} \right) \quad (5.6.2)(b)$$

is not expressible as sum of m i.i.d. random variables, the properties of the test cannot be studied by the above method. The conjectural formula (first introduced in [8] and later used in [2], [10] and [24]) finds $E(m|H_1, t)$; $i = 0, 1$ as values of m given by the relations $E(\ln G_m(g_m)|H_0) \approx \alpha \ln A + (1 - \alpha) \ln B$ and $E(\ln G_m(g_m)|H_1) \approx \beta \ln B + (1 - \beta) \ln A$ respectively where the approximation is due to the excess of the cumulative sum over the boundaries and the deviation of the actual error probabilities from the desired ones. Thus $E(m|H_0, t)$ and $E(m|H_1, t)$ may be obtained from

$$\frac{n-1}{2} \left(\ln \frac{1+m\delta_0}{1+m\delta_1} + \frac{m(\delta_1 - \delta_0)}{t(1+m\delta_1)} \right) \approx \alpha \ln A + (1 - \alpha) \ln B, \quad (5.6.3)(b)$$

$$\frac{n-1}{2} \left(\ln \frac{1+m\delta_0}{1+m\delta_1} + \frac{m(\delta_1 - \delta_0)}{t(1+m\delta_0)} \right) \approx \beta \ln B + (1 - \beta) \ln A \quad (5.6.4)(b)$$

5.7. The Randomized Block Design

We have two factors 'variety' (V) and 'block' (B), each of which can have infinitely many levels. In a random sample of $n > 1$ varieties and $m > 1$ blocks, x_{ij} is observed under the influence of i th variety and j th block; $i = 1, \dots, n$; $j = 1, \dots, m$. The model

$$x_{ij} = \mu + u_i + w_j + z_{ij}; \quad i = 1, \dots, n; \quad j = 1, \dots, m \quad (5.7.0)$$

is a special case of (5.2.1) with $\{c_{ij}\}$ suitably chosen, with $t = nm$, $s = n + m$, $p = 2$ and with components of variance $\sigma_0^2 > 0$, $\sigma_u^2 \geq 0$,

$\sigma_w^2 \geq 0$. The preliminary sample of n_0 varieties and m_0 blocks gives

$$s^2 = ((n_0 - 1)(m_0 - 1))^{-1} \sum_{i=1}^{n_0} \sum_{j=1}^{m_0} (x_{ij} - \bar{x}_{i.} - \bar{x}_{.j} - \bar{x}_{..})^2. \quad \text{One problem is}$$

to test $H_0 : \sigma_u^2/\sigma_0^2 = \delta_0$ vs. $H_1 : \sigma_u^2/\sigma_0^2 = \delta_1$. Corresponding to (5.2.3), we have, $\bar{H}_1 : \tau^2 = s^2(1 + m\delta_1)$ $i = 0, 1$; $\tau^2 = m\sigma_u^2 + \sigma_0^2$.

Procedure under Scheme (a): A set of $m > 1$ blocks is chosen in the beginning and a new variety is included in each block at each stage (two in the first stage).

Total data at stage $n-1$ is $\{x_{ij}; i = 1, \dots, n; j = 1, \dots, m\}$. The reduced parameter space is $\Theta = \{\mu, \tau, \tau_1 | -\infty < \mu < \infty, \tau > 0, \tau_1 > 0\}$ where $\tau_1^2 = m\sigma_w^2 + \sigma_0^2$. The group G of location transformations leaves the problem invariant and a maximal invariant on Θ under the induced transformation $(\mu + a, \tau, \tau_1)$ is τ^2 . A set of sufficient statistics for $\Theta = (\mu, \tau, \tau_1)$ is $t_{n-1} = (x.., \text{variety sum of squares (V.S.S.)}, \text{block sum of squares (B.S.S.)})$. The group G induces the transformation $(x.. + a, \text{V.S.S.}, \text{B.S.S.})$ on the sample space of t_{n-1} with $g_{n-1}(m) = \text{V.S.S.} = m \sum_{i=1}^n (x_{i.} - x..)^2$ as a maximal invariant. Transitivity of $\{t_{n-1}\}$ in this case as well as in the following cases is verified similarly as in the one way classification. The test procedure at stage $n-1$ is

$$(H_0) : B < f_{n-1}(g_{n-1}(m) | \tau^2 = s^2(1+m\delta_1)) / f_{n-1}(g_{n-1}(m) | \tau^2 = s^2(1+m\delta_0)) < A : (H_1)$$

where $g_{n-1}(m) \sim \tau^2 \chi_{n-1}^2(0)$. This may be expressed as

$$(H_0) : g_{n-1} < g_{n-1}(m) < \bar{g}_{n-1} : (H_1) \quad (5.7.1)$$

where g_{n-1} and $\bar{g}_{n-1}$ are the same as in (5.6.1). Formulae giving $\ln G_{n-1}(g_{n-1}(m))$ and h are the same as in (5.6.2) and (5.6.3) respectively (s^2 being as in this section) and

$$L(\tau^2/\sigma_0^2) = k \int_0^\infty L(\tau^2/\sigma_0^2, t) t^{l-1} \exp(-lt) dt, \quad (5.7.4)$$

$$E(v | \tau^2/\sigma_0^2) = k \int_0^\infty E(v | \tau^2/\sigma_0^2, t) t^{l-1} \exp(-lt) dt \quad (5.7.5)$$

where $L(\tau^2/\sigma_0^2, t)$ and $E(v|\tau^2/\sigma_0^2, t)$ are obtained from (5.4.4) and (5.4.5) respectively; $E(Z|t)$, $E(Z^2|t)$ being as in section 5.6 and where $\ell = (n_0-1)(m_0-1)/2$, $k = \ell^2/\Gamma\ell$.

Procedure under Scheme (b): A block containing the same fixed number $n > 1$ of varieties is chosen at each stage.

$t_m = (x, \dots, V.S.S., B.S.S.)$ is a sufficient statistic for (μ, τ, τ_1) and $g_m(n) = m \sum_{i=1}^n (x_i - \bar{x})^2$ is a maximal invariant for the transformation on the sample space of t_m with τ^2 as a maximal invariant on θ .

The decision rule at stage m is

$$(H_0) : g_m < g_m(n) < \bar{g}_m : (H_1) \quad (5.7.1)(b)$$

where g_m and $\bar{g}_m$ are the same as g_{n-1} and $\bar{g}_{n-1}$ of (5.7.1). Since $\ln(G_m(g_m))$ is the same as right side of (5.6.2)(b), $E(m|H_0)$ and $E(m|H_1)$ may be obtained from (5.6.3)(b) and (5.6.4)(b) respectively.

Similar hypotheses about σ_w^2/σ_0^2 are tested exactly similarly by interchanging V and B .

5.8. Testing Interaction Hypotheses in a Two-Way Classification With Balanced Replications

In case of interaction between varieties and blocks, we modify the model (5.7.0) by adding an additional random component I_{ij} . We also assume availability of $r > 1$ replications corresponding to each of nm (variety - block) combination. Thus if x_{ijk} ($k = 1, \dots, r$) denotes the observation from k th replication, i th variety and j th block, the model is, $x_{ijk} = \mu + u_i + w_j + I_{ij} + Z_{ijk}$; $i = 1, \dots, n$; $j = 1, \dots, m$; $k = 1, \dots, r$ (a special case of (5.2.1) with $t = nmr$, $s = n + m + nm$, $p = 3$ and components of variance $\sigma_0^2 > 0$, $\sigma_u^2 \geq 0$, $\sigma_w^2 \geq 0$, $\sigma_I^2 \geq 0$).

Hypotheses is of detecting the presence of interaction (if any) between varieties and blocks. $H_1 : \sigma_1^2/\sigma_0^2 = \delta_1$ ($i = 0, 1$) A first stage sample of n_0 varieties, m_0 blocks and r_0 replications determines $s^2 = (n_0 m_0 (r_0 - 1))^{-1} \sum_{i=1}^{n_0} \sum_{j=1}^{m_0} \sum_{k=1}^{r_0} (x_{ijk} - x'_{ij})^2$ so that $n_0 m_0 (r_0 - 1) s^2 = \sigma_0^2 \chi_{n_0 m_0 (r_0 - 1)}^2(0)$. Here, we have $\bar{H}_1 : \tau^2 = s^2(1 + r\delta_1)$; $i = 0, 1$; with $\tau^2 = r\sigma_1^2 + \sigma_0^2$.

Procedure (i): We consider a sampling scheme where an additional variety is introduced at subsequent stages (two in the first stage) for each of a fixed number rm of (replication - block) combinations.

At the $(n-1)$ st stage, a set of sufficient statistics for $\Theta = \{\mu, \tau, \tau_1, \tau_2 | -\infty < \mu < \infty, \tau > 0, \tau_1 > 0, \tau_2 > 0\}$ (where $\tau_1^2 = nr\sigma_u^2 + r\sigma_1^2 + \sigma_0^2$, $\tau_2^2 = nr\sigma_w^2 + r\sigma_1^2 + \sigma_0^2$) is $t_{n-1} = (x, \dots, V.S.S., B.S.S., \text{Interaction sum of squares (I.S.S.)})$. A maximal invariant on Θ under the induced transformation $(\mu + a, \tau, \tau_1, \tau_2)$ is τ^2 with corresponding maximal invariant

$$g_{n-1}(rm) = r \sum_{i=1}^n \sum_{j=1}^m (x_{ij} - x_{i..} - x_{.j} - x_{...})^2$$

on the sample space of $\{t_{n-1}\}$ under the induced transformation $(x, \dots + a, V.S.S., B.S.S., I.S.S.)$

Thus $\{g_{n-1}(rm)\}$ is an invariantly sufficient and transitive sequence and $g_{n-1}(rm) = \tau^2 \chi_{(n-1)(m-1)}^2(0)$. The decision rule is

$$(H_0) : g_{n-1}(rm) < g_{n-1}(rm) < \bar{g}_{n-1}(rm) : (H_1) \quad (5.8.1)$$

where $g_{n-1}(rm) = (\ln B - \frac{(n-1)(m-1)}{2} \ln \frac{1+r\delta_0}{1+r\delta_1}) / \frac{1}{2s^2} (\frac{1}{1+r\delta_0} - \frac{1}{1+r\delta_1})$

and $\bar{g}_{n-1}(rm)$ is obtained by replacing B by A .

$$\ln G_{n-1}(g_{n-1}(rm)) = \sum_{i=1}^{n-1} \left[\frac{m-1}{2} \ln \frac{1+r\delta_0}{1+r\delta_1} - \frac{v_i}{2} \left(\frac{1}{1+r\delta_1} - \frac{1}{1+r\delta_0} \right) \right] \quad (5.8.2)$$

where $V_1 \sim \tau^2 \chi_{m-1}^2(0)$. Thus with h determined from

$$\left(\frac{1+r\delta_0}{1+r\delta_1} \right)^h = 1 + \frac{hr^2}{\tau_0^2} \left(\frac{1}{1+r\delta_0} - \frac{1}{1+r\delta_1} \right) \quad (5.8.3)$$

$$L(\tau^2/\sigma_0^2) = k_0 \int_0^\infty L(\tau^2/\sigma_0^2, t) t^{(v_0-2)/2} \exp\{-(v_0/2)t\} dt \quad (5.8.4),$$

and for $n = v + 1$,

$$E(v|\tau^2/\sigma_0^2) = k_0 \int_0^\infty E(v|\tau^2/\sigma_0^2, t) t^{(v_0-2)/2} \exp\{-(v_0/2)t\} dt \quad (5.8.5)$$

where $L(\tau^2/\sigma_0^2, t)$ and $E(v|\tau^2/\sigma_0^2, t)$ are as in (5.4.4) and (5.4.5)

respectively; $E(Z|t)$, $E(Z^2|t)$ being determined from (5.4.6) -

(5.4.7) by changing k to $m-1$, c to r and where $v_0 = n_0 m_0 (r_0 - 1)$,

$$k_0 = (v_0/2)^{v_0/2} / \Gamma(v_0/2).$$

Procedure (ii): Here the sampling scheme consists of introducing an additional block at subsequent stages (two in the first stage) for each of a fixed number rn of (replication-variety) combination.

All formulae are obtained from those of Procedure (i) by interchanging m and n .

Procedure (iii): If we let the number of replications vary with each stage, the test statistic (which we now denote by $g_r(nm)$) is the same as for Procedure (i). The decision rule at the r th stage is

$$(H_0) : g_r(nm) < g_r(nm) < \bar{g}_r(nm) : (H_1) \quad (5.8.1)(iii)$$

where $\underline{g}_r(nm)$ and $\bar{g}_r(nm)$ are the same as $\underline{g}_{n-1}(rm)$ and $\bar{g}_{n-1}(rm)$ of (5.8.1).

$$\ln G_r(g_r) = \frac{k_2}{2} \ln \frac{1 + r\delta_0}{1 + r\delta_1} - \frac{g_r(nm)}{2s^2} \left(\frac{1}{1 + r\delta_1} - \frac{1}{1 + r\delta_0} \right) \quad (5.8.2)(iii)$$

where $k_2 = (n-1)(m-1)$. The equations giving $E(r|H_0, t)$ and $E(r|H_1, t)$ are obtained from (5.6.3)(b) and (5.6.4)(b) respectively by changing m to r and $(n-1)$ to k_2 .

5.9. The Two-Stage Nested Design

We get this design if the blocks ('subgroups') in the model of section 5.8 are nested within varieties ('main groups'). The general model (5.2.1) now becomes

$$x_{ijk} = \mu + u_i + w_{ij} + z_{ijk} \quad (i = 1, \dots, n; \quad j = 1, \dots, m; \quad k = 1, \dots, r) \quad (5.9.0)$$

with $t = nmr$, $s = n(m+1)$, $p = 2$ and components of variance $\sigma_0^2 > 0$, $\sigma_u^2 \geq 0$, $\sigma_w^2 \geq 0$).

One hypothesis of interest is $H_0 : \sigma_w^2/\sigma_0^2 = \delta_0$ vs. $H_1 : \sigma_w^2/\sigma_0^2 = \delta_1$. Let $s^2 = (n_0 m_0 (r_0 - 1))^{-1} \sum_{i=1}^{n_0} \sum_{j=1}^{m_0} \sum_{k=1}^{r_0} (x_{ijk} - \bar{x}_{ij.})^2$ be given by a first stage sample of size $n_0 m_0 r_0$. Then, corresponding to (5.2.3), we have $\bar{H}_1 : \tau^2 = s^2(1 + r\delta_1)$; $i = 0, 1$; $0 \leq \delta_0 < \delta_1$; $\tau^2 = r\sigma_w^2 + \sigma_0^2$.

We use a procedure which chooses $n > 1$ main groups and $m > 1$ subgroups within each main group in the beginning. We start with two observations from each of the nm subgroups and an additional observation (if required) is drawn from each subgroup at subsequent stages. Under the group G of location transformations, the problem remains invariant, with maximal invariant

$$g_{r-1}(nm) = r \sum_{i=1}^n \sum_{j=1}^m (x_{ij.} - \bar{x}_{i..})^2 \quad \text{on the sample space of } t_{r-1} =$$

($x, \dots$, main group sum of squares (M.S.S.), sub-group sum of squares

(S.S.S.)) under the induced transformation $(x... + a, M.S.S., S.S.S.)$.
 t_{r-1} is sufficient for $\theta = (\mu, \tau, \tau_1)$ where $\tau_1^2 = m\tau\sigma_u^2 + r\sigma_w^2 + \sigma_0^2$.
A maximal invariant on $\theta = \{\mu, \tau, \tau_1 | -\infty < \mu < \infty, \tau > 0, \tau_1 > 0\}$
is τ^2 under the induced transformation. The decision rule at $(r-1)$ st
stage may be expressed as

$$(H_0) : g_{r-1}(nm) < g_{r-1}(nm) < \bar{g}_{r-1}(mn) : (H_1) \quad (5.9.1)$$

where $g_{r-1}(nm)$ and $\bar{g}_{r-1}(nm)$ are obtained from g_v and $\bar{g}_v$ of
(5.4.1) by replacing q and c by $n(m-1)$ and r respectively.

Also, in the notation of section 5.4,

$$\ln G_{r-1}(g_{r-1}) = \frac{n(m-1)}{2} \ln \frac{1 + r\delta_0}{1 + r\delta_1} - \frac{g_{r-1}(nm)}{2s^2} \left(\frac{1}{1 + r\delta_1} - \frac{1}{1 + r\delta_0} \right) \quad (5.9.2)$$

A comparison shows that for $r = v + 1$, equations giving $E(v|H_0, t)$
and $E(v|H_1, t)$ are obtained from (5.6.3)(b) and (5.6.4)(b) respec-
tively by replacing m by r and $n-1$ by $n(m-1)$.

Tests Under Mixed Models

5.10. Randomized Block Design

We consider the model (5.7.0) where w_j 's are now m constants
satisfying $\sum_{j=1}^m w_j = 0$.

The problem is to test $H_0 : \sigma_u^2/\sigma_0^2 = \delta_0$ against $H_1 : \sigma_u^2/\sigma_0^2 = \delta_1$.

The procedures under schemes (a) and (b) of section 5.7 may be used
without any change. The derivation of the test statistics $g_{n-1}(m)$
and $g_m(n)$ under the respective schemes and the proof of their being
invariantly sufficient and transitive are easily obtained from that
section.

5.11. Two-Way Classification with Balanced Replications

The model of section 5.8 is further restricted to $\sum_{j=1}^m w_j = 0$,
 $\sum_{j=1}^m I_{ij} = 0$ ($i = 1, \dots, n$) where w_j 's are m constants.

For testing $H_0 : \sigma_I^2/\sigma_0^2 = \delta_0$ against $H_1 : \sigma_I^2/\sigma_0^2 = \delta_1$, the
 Procedures (i) - (iii) of section 5.8 give exactly the same tests.

In this case, we can also test $H_0 : \sigma_u^2/\sigma_0^2 = \delta_0$ vs. $H_1 : \sigma_u^2/\sigma_0^2 = \delta_1$.
 Writing $\tau^2 = \sigma_0^2 + m\sigma_u^2$; (5.2.3) now becomes $\bar{H}_1 : \tau^2 = s^2(1 + m\delta_1)$;
 $i = 0, 1$ where s^2 is as in section 5.8.

We consider the sampling scheme when an additional replication is
 introduced at successive stages for each of a fixed number of nm
 (variety-block) combinations.

Under the group G of location transformations, a maximal invariant
 on the sample space of $t_r = (x_{.1}, \dots, x_{.m}, \text{V.S.S.}, \text{I.S.S.})$ under the
 induced transformation is

$$g_r(nm) = m\tau \sum_{i=1}^n (x_{i.} - \bar{x})^2 \text{ with maximal invariant } \tau^2 \text{ on}$$

$$\theta = \{\mu, w_1, \dots, w_m, \tau, \tau_1 | -\infty < \mu < \infty, -\infty < w_i < \infty; i = 1, \dots, m;$$

$$\tau > 0, \tau_1 > 0\} \text{ where } \tau_1^2 = \sigma_I^2 + \sigma_0^2. \text{ Also } g_r(nm) \sim \tau^2 \chi_{n-1}^2(0).$$

The test procedure at stage r is

$$(H_0) : g_r(nm) < g_r(nm) < \bar{g}_r(nm) : (H_1) \quad (5.11.1)$$

where $\underline{g}_r(nm)$ and $\bar{g}_r(nm)$ are obtained from $\underline{g}_0$ and $\bar{g}_0$ of (5.4.1) by
 changing q to $n-1$ and c to mr . Also, since

$$\ln G_r(g_r) = \frac{n-1}{2} \ln \frac{1 + m\delta_0}{1 + m\delta_1} - \frac{g_r(nm)}{2s^2} \left(\frac{1}{1 + m\delta_1} - \frac{1}{1 + m\delta_0} \right) \quad (5.11.2)$$

a comparison shows that equations giving $E(r|H_0, t)$ and $E(r|H_1, t)$
 are obtained from (5.6.3)(b) and (5.6.4)(b) respectively by replacing m
 by mr .

5.12. The Two Stage Nested Design

We consider the following two cases of model (5.9.0).

(i) u_i 's are n constants and $\sum_{i=1}^n u_i = 0$. In this case, test for $H_1 : \sigma_w^2/\sigma_0^2 = \delta_1$; $i = 0, 1$ is the same as in section 5.9. The test statistic is derived by an argument similar to that section under the same sampling scheme.

(ii) w_{ij} 's are nm constants satisfying $\sum_{j=1}^m w_{ij} = 0$ ($i = 1, \dots, n$) and u_i 's constitute a random sample of n main group effects. The null and the alternative hypotheses are $H_0 : \sigma_u^2/\sigma_0^2 = \delta_0$ and $H_1 : \sigma_u^2/\sigma_0^2 = \delta_1$. Let s^2 be as in section 5.11; so $\bar{H}_1$ ($i = 1, 2$) are the same as in that section.

Procedure: First choose $n > 1$ main groups and $m > 1$ subgroups within each main group, then start with two observations from each of the nm subgroups and draw an additional observation (if required) at each subsequent stage. Proceeding as in the preceding section, we derive the same test statistic $mr \sum_{i=1}^n (x_{i..} - x_{...})^2$. For $v = r - 1$, we observe that $E(v|H_1, t)$; $i = 1, 2$; are the same as in section 5.11.

5.13. Numerical Evaluation and Comparison

Table 5.13.1

OC Function of the Sequential χ^2 -Test in the One Way Layout
Under Scheme (a) (the number ($m = 5$) of observations per
group is fixed; preliminary sample consists of 5 groups with
11 observations in each; $\delta_0 = 0$, $\delta_1 = 2$; $\alpha = 0.05 = \beta$)

τ^2/σ_0^2	OC	τ^2/σ_0^2	OC
0.5	0.9965	4	0.278
1	0.9452	4.5	0.232
1.25	0.875	5	0.196
1.5	0.803	6	0.145
1.75	0.728	7	0.111
2	0.654	8	0.0885
2.25	0.586	9	0.0723
2.5	0.524	10	0.0604
2.75	0.468	11	0.0512
3	0.419	11.5	0.0475

Table 5.13.2

Comparison of the A.S.N. of the Sequential χ^2 -test with the A.S.N. of the Sequential F-test and the Sample Size of the Fixed Sample F-test in the One-Way Layout (initial sample consists of 4 groups and 9 observations from each; $\delta_0 = 0$)

Scheme (a); Procedure I. The number (m) of observations per group is fixed.

$\delta_1 = 1$

m	$\alpha = 0.05 = \beta$			$\alpha = 0.01 = \beta$		
	Sequ-F		Sequ- χ^2	Sequ-F		Sequ- χ^2
	Fixed-F	H_0		Fixed-F	H_0	
2	76	39.8	24.64	150	65.6	45.56
3	57	34.1	25.03	111	55.4	45.46
4	52	34.1	26.17	100	54.8	47.02
5	55	35.6	27.58	95	56.6	49.21
6	54	37.5	29.1	96	59.3	51.66
7	56	39.7	30.65	105	62.4	54.23

$\delta_1 = 2$

m	$\alpha = 0.05 = \beta$			$\alpha = 0.01 = \beta$		
	Sequ-F		Sequ- χ^2	Sequ-F		Sequ- χ^2
	Fixed-F	H_0		Fixed-F	H_0	
2	38	20.9	13.09	72	33.6	23.51
3	33	20.5	14.55	57	32.4	25.83
4	32	22.2	16.11	56	34.7	28.43
5	35	24.3	17.68	55	37.6	31.07
6	36	26.6	19.22	60	40.9	33.69
7	42	29	20.73	63	44.2	36.27

Scheme (b). The number (n) of groups is fixed

n	$\delta_1 = 1$					
	$\alpha = 0.05 = \beta$		$\alpha = 0.01 = \beta$			
	Fixed-F	Sequ-F(H_0)	Sequ- $\chi^2(H_0)$	Fixed-F	Sequ-F(H_0)	Sequ- $\chi^2(H_0)$
6	60	38.4	34.06	162	90.2	86.14
7	56	35.7	30.64	140	75.2	70.3
8	56	34.5	29	120	67	61.53
9	54	33.9	27.81	108	62.1	56.38
10	50	33.8	27.05	110	59.1	52.7
11	55	34	26.62	99	57.1	50.12
12	48	34.3	26.31	96	55.9	48.16
13	52	34.8	26.12	91	55.1	46.93
14	56	35.4	26.05	98	54.7	45.92

n	$\delta_1 = 2$					
	$\alpha = 0.05 = \beta$		$\alpha = 0.01 = \beta$			
	Fixed-F	Sequ-F(H_0)	Sequ- $\chi^2(H_0)$	Fixed-F	Sequ-F(H_0)	Sequ- $\chi^2(H_0)$
6	30	21.4	17	84	47	43.07
7	28	20.4	15.32	70	39.9	35.15
8	32	20.2	14.52	64	36.2	30.76
9	27	20.3	13.95	63	34.1	28.19
10	30	20.7	13.58	60	32.9	26.37
11	33	21.2	13.47	55	32.3	25.1
12	36	21.8	13.22	60	32.1	24.19
13	39	22.5	13.16	52	32.1	23.52
14	28	23.2	13.1	56	32.4	23.01

n	$\delta_1 = 1$			
	$\alpha = 0.05 = \beta$		$\alpha = 0.01 = \beta$	
	Sequ- $\chi^2(H_0)$	Sequ- $\chi^2(H_0)$	Sequ- $\chi^2(H_0)$	Sequ- $\chi^2(H_0)$
17	26.14	44.14	13.16	22.15
18	26.25	43.86	13.2	22
19	26.39	43.55	13.25	21.89
20	26.55	43.42	13.33	21.83

CHAPTER VI

SOME EXACT RESULTS

6.1. *Introduction*

Exact properties of sequential tests of composite hypotheses have been derived in some special cases of probability models. Inferences on the location parameter of the exponential distribution, with the scale parameter unknown, are made in section 6.2. The test may be chosen to give a strength at least $(\alpha, 0)$. An analogous test for testing a parameter of Pareto distribution in the presence of nuisance shape parameter is discussed in section 6.3. Section 6.4 contains a procedure for testing the mean of the rectangular distribution with unknown variance. A description of a procedure to choose one of the three hypotheses about the mean of this distribution is in section 6.5.

6.2. *Testing Location Parameter of the Exponential Distribution*

X has probability density function $g(x, \theta_1, \theta_2)$ given by (4.9.1).

The null and the alternative hypotheses are

$$H_0 : \theta_1 = \theta_{10} \text{ vs. } H_1 : \theta_1 = \theta_{11}; \theta_{11} > \theta_{10}; \theta_2 \text{ unknown.}$$

The maximum likelihood estimator $\hat{\theta}_2$ of θ_2 given by the initial sample $\{x'_1, \dots, x'_{n_0}\}$ is $\bar{x}'_{n_0} - x'_{(1)}^{(n_0)}$ where $x_{(1)}^{(r)} = \min(x_1, \dots, x_r)$

and $\bar{x}'_{n_0}$ is the mean of this sample. The density function of $\hat{\theta}_2/\theta_2$ is

$$p(t) = n_0^{-1} t^{n_0-2} \exp(-n_0 t) / \Gamma(n_0 - 1); \quad t > 0 \quad (6.2.1)$$

Writing $z_j(\hat{\theta}_2) = \ln(g(x_j, \theta_{11}, \hat{\theta}_2) / g(x_j, \theta_{10}, \hat{\theta}_2))$ and defining $z_j(\hat{\theta}_2) = -\infty$ for $x_j < \theta_{10}$; $j = 1, 2, \dots$; we have,

$$\sum_{j=1}^n z_j(\hat{\theta}_2) = \begin{cases} -\infty & \text{for } x_{(1)}^{(n)} < \theta_{11} \\ \frac{n(\theta_{11} - \theta_{10})}{\hat{\theta}_2} & \text{for } x_{(1)}^{(n)} \geq \theta_{11} \end{cases} \quad (6.2.2)$$

For a constant $C > 0$ (to be determined later to achieve a desired strength of the test); we define $S(\hat{\theta}_2, \theta_2)$ to be a test which follows the decision rule:

accept H_0 if $x_{(1)}^{(n)} < \theta_{11}$,

accept H_1 if $n \geq C \hat{\theta}_2 / (\theta_{11} - \theta_{10})$,

otherwise continue by observing x_{n+1} (6.2.3)

Properties of the Test

Let $n(t) = [Ct\theta_2 / (\theta_{11} - \theta_{10})] + 1$ where $t = \hat{\theta}_2 / \theta_2$ and where $[q]$ denotes the largest integer less than or equal to q (> 0).

$L(\theta_1, \hat{\theta}_2, \theta_2) = \Pr\{S(\hat{\theta}_2, \theta_2) \text{ accepts } H_0 | \theta_1, \hat{\theta}_2, \theta_2\} =$
 $\Pr\{\text{at least one of the first } n(t) \text{ observations falls below } \theta_{11} | \theta_1, \hat{\theta}_2, \theta_2\}$

$$= \begin{cases} 0 & \text{if } \theta_1 \geq \theta_{11} \\ 1 - \exp\{-(\theta_{11} - \theta_1)n(t)/\theta_2\} & \text{if } \theta_1 < \theta_{11} \end{cases} \quad (6.2.4)$$

Writing $\lambda = C\theta_2 / (\theta_{11} - \theta_{10})$, we have, for $j = 1, 2, \dots$

$\Pr\{n(t) = j\} = \Pr\{j-1 \leq \lambda t < j\} = G(j) - G(j-1)$ where $G(j) =$

$$\int_0^j \left\{ \left(\frac{n_0}{\lambda} \right)^{n_0-1} t^{n_0-2} e^{-(n_0/\lambda)t} / \Gamma(n_0-1) \right\} dt. \quad \text{Let } L(\theta_1, \theta_2) =$$

$\Pr(H_0 \text{ is accepted} | \theta_1, \theta_2)$, then

$$L(\theta_1, \theta_2) = \begin{cases} 0 & \text{if } \theta_1 \geq \theta_{11} \\ \sum_{j=1}^{\infty} \Pr(n(t) = j) L(\theta_1, \hat{\theta}_2, \theta_2 | n(t)=j, \theta_1 < \theta_{11}) & \text{if } \theta_1 < \theta_{11} \end{cases} \quad (6.2.5)$$

$$\text{Thus } L(\theta_1, \theta_2 | \theta_1 < \theta_{11}) = \sum_{j=1}^{\infty} (1 - \exp\{-(\theta_{11} - \theta_1)j/\theta_2\}) (G(j) - G(j-1))$$

which, after integration and a rearrangement, gives

$$L(\theta_1, \theta_2 | \theta_1 < \theta_{11}) = 1 - \frac{(n_0/l)^{n_0-1}}{\Gamma(n_0-1)} \sum_{r=0}^{\infty} \frac{(-n_0/l)^r}{r!(n_0+r-1)} \sum_{j=1}^{\infty} \{j^{n_0+r-1} - (j-1)^{n_0+r-1}\} \exp\{-(\theta_{11} - \theta_1)j/\theta_2\} \quad (6.2.6)$$

Now for $\theta_1 < \theta_{11}$, we see that the following relations hold:

$$\begin{aligned} \sum_{j=1}^{\infty} j^s e^{(\theta_1 - \theta_{11})j/\theta_2} &= \theta_2^s D_{\theta_1}^{(s)} (e^{(\theta_{11} - \theta_1)/\theta_2} - 1)^{-1}, \\ \sum_{j=1}^{\infty} (j-1)^s e^{(\theta_1 - \theta_{11})j/\theta_2} &= \theta_2^s e^{(\theta_1 - \theta_{11})/\theta_2} D_{\theta_1}^{(s)} (e^{(\theta_{11} - \theta_1)/\theta_2} - 1)^{-1} \text{ and} \\ \text{consequently,} \\ \sum_{j=1}^{\infty} \{j^s - (j-1)^s\} e^{(\theta_1 - \theta_{11})j/\theta_2} &= \theta_2^s (1 - e^{(\theta_1 - \theta_{11})/\theta_2}) D_{\theta_1}^{(s)} (e^{(\theta_{11} - \theta_1)/\theta_2} - 1)^{-1} \end{aligned} \quad (6.2.7)$$

where $D_{\theta_1}^{(s)}(\dots) = \frac{\partial^s}{\partial \theta_1^s}(\dots)$ is the s th partial derivative (w.r.t. θ_1) of the expression within the parentheses. Thus (6.2.5) - (6.2.7) imply

$$L(\theta_1, \theta_2 | \theta_1 \geq \theta_{11}) = 0 \text{ and } L(\theta_1, \theta_2 | \theta_1 < \theta_{11}) = 1 - (1 - e^{(\theta_1 - \theta_{11})/\theta_2}) \frac{(n_0/l)^{n_0-1}}{\Gamma(n_0-1)} \sum_{r=0}^{\infty} \frac{(-n_0/l)^r}{r!(n_0+r-1)} \theta_2^{n_0+r-1} D_{\theta_1}^{(n_0+r-1)} (e^{(\theta_{11} - \theta_1)/\theta_2} - 1)^{-1} \quad (6.2.8)$$

The A.S.N. of $S(\hat{\theta}_2, \theta_2)$ is given by

$$E(n|\theta_1, \hat{\theta}_2, \theta_2) = \sum_{j=1}^{n(t)} j \Pr\{X_{(1)}^{(j-1)} \geq \theta_{11}, X_j < \theta_{11}\} +$$

$$n(t) \Pr\{X_{(1)}^{(n(t))} \geq \theta_{11}\}. \text{ Thus } E(n|\theta_1, \hat{\theta}_2, \theta_2, \theta_1 < \theta_{11}) =$$

$$n(t)e^{-((\theta_{11}-\theta_1)/\theta_2)n(t)} + (1 - e^{-(\theta_{11}-\theta_1)/\theta_2}) \sum_{j=1}^{n(t)} j e^{-((\theta_{11}-\theta_1)/\theta_2)(j-1)} =$$

$$\text{(after simplification)} \sum_{j=1}^{n(t)} e^{-((\theta_{11}-\theta_1)/\theta_2)(j-1)} =$$

$$(1 - e^{-(\theta_{11}-\theta_1)n(t)/\theta_2}) / (1 - e^{-(\theta_{11}-\theta_1)/\theta_2}), \quad E(n|\theta_1, \hat{\theta}_2, \theta_2, \theta_1 \geq \theta_{11}) =$$

$$n(t). \text{ Multiplying by the density of } n(t), \text{ summing and proceeding as in}$$

$$\text{the case of OC function, we obtain,}$$

$$E(n|\theta_1, \theta_2; \theta_1 \geq \theta_{11}) = E n(t) \quad \text{and} \quad E(n|\theta_1, \theta_2; \theta_1 < \theta_{11}) =$$

$$(1 - e^{\frac{\theta_{11}-\theta_1}{\theta_2}})^{-1} - \frac{(n_0/l)^{n_0-1}}{\Gamma(n_0-1)} \sum_{r=0}^{\infty} \frac{(-n_0/l)^r}{r!(n_0+r-1)} \theta_2^{n_0+r-1} \frac{(\theta_{11}-\theta_1)^{n_0+r-1}}{D_{\theta_1}} (e^{\frac{\theta_{11}-\theta_1}{\theta_2}} - 1)^{-1} \quad (6.2.9)$$

$$\text{where } E n(t) = \sum_{j=1}^{\infty} j \{G(j) - G(j-1)\} = \sum_{j=1}^{\infty} j (\Gamma_p(n_0-1) - \Gamma_q(n_0-1)) / \Gamma(n_0-1) \quad (6.2.10)$$

where $p = n_0 j / l$, $q = p - (n_0 / l)$ and where $\Gamma_x(\alpha)$ is an incomplete

gamma function defined by $\Gamma_x(\alpha) = \int_0^x t^{\alpha-1} e^{-t} dt$.

Choice of C to Control 'First Kind of Error' Probability

Since $\Pr\{H_0 \text{ is accepted} | \theta_1 = \theta_{11}\} = 0$; the test has power one. To ensure that probability of accepting H_1 (when $\theta_1 = \theta_{10}$) is controlled by α , say; we need to find C which satisfies $L(\theta_{10}, \theta_2) \geq 1-\alpha$. Using $n(t) > lt$ and $\theta_{11} > \theta_{10}$, we have, $-(\theta_{11}-\theta_{10})n(t)/\theta_2 < -Ct$. Thus (6.2.4) implies

$$\Pr\{S(\hat{\theta}_2, \theta_2) \text{ accepts } H_1 | \theta_{10}, \hat{\theta}_2, \theta_2\} < \exp\{-(\theta_{11}-\theta_{10})n(t)/\theta_2\} < \exp(-Ct)$$

Multiplying by the density of t , integrating and simplifying, we have,

$$1 - L(\theta_{10}, \theta_2) < (1 + (C/n_0))^{-(n_0-1)}. \text{ Thus the choice } C =$$

$n_0(\alpha^{-1/(n_0-1)} - 1)$ ensures that 'first kind of error probability' is always less than α . A lower bound on the error probability will be found below.

Bounds on the OC and A.S.N. Functions

Using the inequality $lt < n(t) \leq lt + 1$, we have, for $\theta_1 < \theta_{11}$,
 $\exp\{-(\theta_{11}-\theta_1)lt/\theta_2\} > \exp\{-(\theta_{11}-\theta_1)n(t)/\theta_2\} \geq \exp\{-(\theta_{11}-\theta_1)(lt+1)/\theta_2\}$,
 thus, $E \exp\{-(\theta_{11}-\theta_1)lt/\theta_2\} > 1 - L(\theta_1, \theta_2 | \theta_1 < \theta_{11}) \geq$
 $E \exp\{-(\theta_{11}-\theta_1)(lt+1)/\theta_2\}$, which simplifies to

$$1 - (1 + \frac{C(\theta_{11}-\theta_1)}{n_0(\theta_{11}-\theta_{10})})^{-(n_0-1)} < L(\theta_1, \theta_2 | \theta_1 < \theta_{11}) \leq$$

$$1 - e^{-(\theta_{11}-\theta_1)/\theta_2} (1 + \frac{C(\theta_{11}-\theta_1)}{n_0(\theta_{11}-\theta_{10})})^{-(n_0-1)} \quad (6.2.11)$$

The choice $C = n_0(\alpha^{-1/(n_0-1)} - 1)$ gives the bounds

$$\alpha \exp\{-(\theta_{11}-\theta_{10})/\theta_2\} \leq 1 - L(\theta_{10}, \theta_2) < \alpha \quad (6.2.12)$$

on the probability of wrong decision when H_0 is true. Using

$E t = (n_0-1)/n_0$, we similarly obtain the following bounds on the A.S.N.,

$$C\theta_2(1-n_0^{-1})(\theta_{11}-\theta_{10})^{-1} < E(n | \theta_1, \theta_2, \theta_1 \geq \theta_{11}) \leq 1 + C\theta_2(1-n_0^{-1})(\theta_{11}-\theta_{10})^{-1} \quad (6.2.13)$$

It follows that $E(n | \theta_1, \theta_2, \theta_1 \geq \theta_{11})$ will either be equal to its upper bound or to its lower bound. Also

$$\begin{aligned}
& (1 - e^{-\frac{\theta_{11}-\theta_1}{\theta_2}})^{-1} \left\{ 1 - \left(1 + \frac{C(\theta_{11}-\theta_1)}{n_0(\theta_{11}-\theta_{10})} \right)^{-(n_0-1)} \right\} < \\
& E(n | \theta_1, \theta_2, \theta_1 < \theta_{11}) \leq (1 - e^{-\frac{\theta_{11}-\theta_1}{\theta_2}})^{-1} \left\{ 1 - e^{-\frac{\theta_{11}-\theta_1}{\theta_2}} \left(1 + \frac{C(\theta_{11}-\theta_1)}{n_0(\theta_{11}-\theta_{10})} \right)^{-(n_0-1)} \right\}
\end{aligned}
\tag{6.2.14}$$

For $C = n_0(\alpha^{-1/(n_0-1)} - 1)$, we have from (6.2.14), when H_0 is true;

$$\begin{aligned}
& (1 - \alpha)(1 - \exp\{-(\theta_{11}-\theta_{10})/\theta_2\})^{-1} < E(n | \theta_{10}, \theta_2) \leq \\
& (1 - \alpha \exp\{-(\theta_{11}-\theta_{10})/\theta_2\})(1 - \exp\{-(\theta_{11}-\theta_{10})/\theta_2\})^{-1}
\end{aligned}
\tag{6.2.15}$$

6.3. Testing a Parameter of Pareto Distribution

$X_1, X_2, \dots$ are independent with a common p.d.f. $g(x, a, k)$ given by (4.8.1). The hypotheses to be discriminated are

$$H_0 : k = k_0 \text{ vs. } H_1 : k = k_1; \quad k_0 < k_1; \quad a \text{ being unknown} \tag{6.3.1}$$

The initial sample $\{x_1, \dots, x_{n_0}\}$ gives $\hat{a} = n_0 \left\{ \sum_{i=1}^{n_0} \ln x_i' / x_{(1)}^{(n_0)} \right\}^{-1}$,

the maximum likelihood estimator of a . The density function of $\hat{a}$ is

$$p(u) = a^{n_0-1} \frac{n_0-1}{n_0} e^{-n_0 a/u} \frac{n_0}{u} \Gamma(n_0-1); \quad u > 0. \text{ Using the notation of the}$$

previous section, we have, defining $Z_j(\hat{a}) = -\infty$ if $x_j < k_0$; $j = 1, 2, \dots$

$$\sum_{j=1}^n Z_j(\hat{a}) = \begin{cases} n \hat{a} \ln(k_1/k_0) & \text{for } x_{(1)}^{(n)} \geq k_1 \\ -\infty & \text{for } x_{(1)}^{(n)} < k_1 \end{cases}$$

For a constant C (to be suitably chosen later); the test $S(\hat{a}, a)$ is defined by the decision rule:

- (i) accept H_0 if $x_{(1)}^{(n)} < k_1$ (ii) accept H_1 if $n \geq C/\hat{a} \ln(k_1/k_0)$
- (iii) otherwise continue by observing x_{n+1} . (6.3.2)

Writing $\ell = C/\hat{a} \ln(k_1/k_0)$, $t = \hat{a}/a$, $n(t) = [\ell/t] + 1$, we see that,

$\Pr\{n(t) = j\} = \Pr\{j-1 \leq t/t < j\} = G(j) - G(j-1)$ where $G(j)$ has been defined in section 6.2.

Properties of the Test

$$\begin{aligned} L(k, \hat{a}, a) &= \Pr\{S(\hat{a}, a) \text{ accepts } H_0 | k, \hat{a}, a\} = 1 - \Pr\{X_{(1)}^{(n(t))} \geq k_1 | k, \hat{a}, a\}, \text{ so we easily observe that,} \\ L(k, \hat{a}, a | k \geq k_1) &= 0 \text{ and } L(k, \hat{a}, a | k < k_1) = 1 - (k/k_1)^{an(t)} \end{aligned} \quad (6.3.3)$$

Since $L(k, a | k < k_1) = \sum_{j=1}^{\infty} \{1 - (k/k_1)^{aj}\} \{G(j) - G(j-1)\} =$
 $\sum_{j=1}^{\infty} \{1 - (k/k_1)^{aj}\} \frac{(n_0/\ell)^{n_0-1}}{\Gamma(n_0-1)} \int_{j-1}^j t^{n_0-2} e^{-(n_0/\ell)t} dt$, we obtain, after simplifying,

$$\begin{aligned} L(k, a | k \geq k_1) &= 0 \text{ and } L(k, a | k < k_1) = \\ 1 - \sum_{j=1}^{\infty} (k/k_1)^{ja} (\Gamma_p(n_0-1) - \Gamma_q(n_0-1)) / \Gamma(n_0-1) \end{aligned} \quad (6.3.4)$$

where p and q are obtained from (6.2.10), ℓ being as in this section and where $\Gamma_x(\alpha)$ is as defined before.

The expected sample size of $S(\hat{a}, a)$ is $E(n | k, \hat{a}, a) =$
 $\sum_{j=1}^{n(t)} j \Pr\{X_{(1)}^{(j-1)} \geq k_1, X_j < k_1\} + n(t) \Pr\{X_{(1)}^{(n(t))} \geq k_1\}$ which gives
 $E(n | k, \hat{a}, a, k \geq k_1) = n(t)$ and $E(n | k, \hat{a}, a, k < k_1) =$
 $\sum_{j=1}^{n(t)} j (k/k_1)^{a(j-1)} \{1 - (k/k_1)^a\} + n(t) (k/k_1)^{an(t)}$. After simplifying, we have,

$$\begin{aligned} E(n | k, \hat{a}, a, k \geq k_1) &= n(t) \text{ and } E(n | k, \hat{a}, a, k < k_1) = \\ \{1 - (k/k_1)^{an(t)}\} / \{1 - (k/k_1)^a\} \end{aligned} \quad (6.3.5)$$

Multiplying by the probability function of $n(t)$ and summing,

$$E(n|k, a, k > k_1) = \sum_{j=1}^{\infty} j(\Gamma_p(n_0-1) - \Gamma_q(n_0-1))/\Gamma(n_0-1) \text{ and } E(n|k, a, k < k_1) \\ = (1 - (k/k_1)^a)^{-1} \{1 - \sum_{j=1}^{\infty} (k/k_1)^{ja} (\Gamma_p(n_0-1) - \Gamma_q(n_0-1))/\Gamma(n_0-1)\} \quad (6.3.6)$$

Choice of C to Achieve a Strength At Least $(\alpha, 0)$

$\Pr(H_0 \text{ is accepted} | k_1, a) = 0$. We need C such that $\Pr(H_1 \text{ is accepted} | k_0, a) \leq \alpha$ which implies $L(k_0, a) \geq 1 - \alpha$ where $L(k, a)$ is as in (6.3.4). Solving $L(k_0, a) = 1 - \alpha$ for C is quite complicated. Since $n(t) > (l/t)$, $k_0 < k_1$, we observe that $\Pr(\hat{S}(a, a) \text{ accepts } H_1 | k_0, \hat{a}, a) < (k_0/k_1)^{al/t} = e^{-C/t}$. Multiplying by the p.d.f. of t and integrating, we obtain,

$\Pr(H_1 \text{ is accepted} | k_0, a) = (1 + (C/n_0))^{-(n_0-1)}$. Equating the right side to α , we obtain,

$C = n_0(\alpha^{-1/n_0-1} - 1)$. A lower bound on the probability of first kind of error will be obtained below.

Bounds on OC and A.S.N. Functions

The inequality $(l/t) < n(t) \leq (l/t) + 1$ implies, for $k < k_1$, $(k/k_1)^a (k/k_1)^{al/t} \leq (k/k_1)^{an(t)} < (k/k_1)^{al/t}$; this together with (6.3.3) gives $(k/k_1)^a E(k/k_1)^{al/t} \leq 1 - L(k, a | k < k_1) < E(k/k_1)^{al/t}$. Since $E(k/k_1)^{al/t} = (1 - \frac{al}{n_0} \ln \frac{k}{k_1})^{-(n_0-1)}$, we obtain the following bound on the OC function,

$$1 - (1 - \frac{al}{n_0} \ln \frac{k}{k_1})^{-(n_0-1)} < L(k, a | k < k_1) \leq 1 - (\frac{k}{k_1})^a (1 - \frac{al}{n_0} \ln \frac{k}{k_1})^{-(n_0-1)} \quad (6.3.7)$$

If H_0 is true, the choice $C = n_0(\alpha^{-1/(n_0-1)} - 1)$ gives,

$$\alpha(k_0/k_1)^a \leq 1 - L(k_0, a) < \alpha \quad (6.3.8)$$

which provides a bound on the first kind of error probability. Since $E(l/t) = l(1 - n_0^{-1})$, (6.3.5) gives the following bounds on the A.S.N. function:

$$l(1 - n_0^{-1}) < E(n|k, a, k \geq k_1) \leq 1 + l(1 - n_0^{-1}) \quad (6.3.9)$$

which implies that for $k \geq k_1$, $E(n|k, a)$ will be equal to either its upper bound or its lower bound. Also,

$$(1 - (k/k_1)^a)^{-1} \{1 - (1 - \frac{al}{n_0} \ln \frac{k}{k_1})^{-(n_0-1)}\} < E(n|k, a, k < k_1) \leq (1 - (k/k_1)^a)^{-1} \{1 - (k/k_1)^a (1 - \frac{al}{n_0} \ln \frac{k}{k_1})^{-(n_0-1)}\} \quad (6.3.10)$$

When the null hypothesis is true, the choice $C = n_0(\alpha^{-1/(n_0-1)} - 1)$ implies,

$$(1 - \alpha)(1 - (k_0/k_1)^a)^{-1} < E(n|k_0, a) \leq (1 - \alpha(k_0/k_1)^a)(1 - (k_0/k_1)^a)^{-1} \quad (6.3.11)$$

6.4. Testing the Mean of the Rectangular Distribution

Let $\{X_i; i = 1, 2, \dots\}$ be independent with a common probability density function

$$g(x, \theta, \sigma) = \sigma^{-1} \text{ for } \theta - \sigma/2 \leq x \leq \theta + \sigma/2 \quad (6.4.1)$$

The null and the alternative hypotheses are

$$H_0 : \theta \leq \theta_0 \text{ vs. } H_1 : \theta \geq \theta_1; -\infty < \theta_0 < \theta_1 < \infty; \sigma \text{ unknown.}$$

The first stage sample of size n_0 gives $\hat{\sigma}$, the maximum likelihood estimator of σ as $\hat{\sigma} = \text{range}(x'_1, \dots, x'_{n_0})$ and the p.d.f. of $\hat{\sigma}$ is $p(u) = n_0(n_0-1)\sigma^{-n_0}(u^{n_0-2} - u^{n_0-1})$; $0 \leq u < \sigma$. Let $\tilde{\sigma} = p\hat{\sigma}$ where p is greater than 1 so that $I(p) = \Pr\{X < \theta_0 - \tilde{\sigma}/2\} + \Pr\{X > \theta_1 + \tilde{\sigma}/2\} < \Pr\{X < \theta_0 - \hat{\sigma}/2\} + \Pr\{X > \theta_1 + \hat{\sigma}/2\}$ and $I(p)$ is a strictly decreasing function of p .

In the usual notation, we have, for $\theta_1 - \theta_0 < \tilde{\sigma}$ and $j = 1, 2, \dots$

$$Z_j(\tilde{\sigma}) = \begin{cases} -\infty & \text{if } \theta_0 - \tilde{\sigma}/2 \leq x_j < \theta_1 - \tilde{\sigma}/2 \\ 0 & \text{if } \theta_1 - \tilde{\sigma}/2 \leq x_j \leq \theta_0 + \tilde{\sigma}/2 \\ \infty & \text{if } \theta_0 + \tilde{\sigma}/2 < x_j \leq \theta_1 + \tilde{\sigma}/2 \end{cases}$$

Defining $Z_j(\tilde{\sigma}) = -\infty$ if $x_j < \theta_0 - \tilde{\sigma}/2$ and $Z_j(\tilde{\sigma}) = \infty$ if $x_j > \theta_1 + \tilde{\sigma}/2$; let $S(\tilde{\sigma}, \sigma)$ be the test with the decision rule:

Accept H_0 without taking further observations if $\theta_1 \geq \theta_0 + \tilde{\sigma}$, otherwise accept or reject H_0 according as the lower or the upper inequality in $\theta_1 - \tilde{\sigma}/2 < x_n < \theta_0 + \tilde{\sigma}/2$ is first violated for $n \geq 1$ (6.4.2)
Since $\Pr\{n > \tilde{n} | \theta, \sigma, \tilde{\sigma} > \theta_1 - \theta_0\} = \Pr\{\theta_1 - \tilde{\sigma}/2 < X < \theta_0 + \tilde{\sigma}/2 | \theta, \sigma, \tilde{\sigma}\}^{\tilde{n}}$; it follows that if $\tilde{\sigma} \geq \theta_1 - \theta_0 + \sigma$, the procedure continues indefinitely with probability one for $\theta \in [\theta_1 - \frac{\tilde{\sigma}}{2} + \frac{\sigma}{2}, \theta_0 + \frac{\tilde{\sigma}}{2} - \frac{\sigma}{2}]$. Thus we assume

$$\theta_0 + (p-1)\sigma/2 < \theta_1 - (p-1)\sigma/2 \quad (6.4.3)$$

The Probability of Accepting the Null Hypothesis

In the usual notation, we have,

$$(i) \text{ for } \theta \leq \theta_0 + (p-1)\sigma/2; L(\theta, \tilde{\sigma}, \sigma | \tilde{\sigma} \in [0, \theta_1 - \theta_0]) = 1,$$

$$L(\theta, \tilde{\sigma}, \sigma | \tilde{\sigma} \in (\theta_1 - \theta_0, 2(\theta - \theta_0) + \sigma)) =$$

$$\sum_{n=1}^{\infty} \Pr\{\theta_1 - \frac{\tilde{\sigma}}{2} < X_j < \theta_0 + \frac{\tilde{\sigma}}{2}; j = 1, \dots, n-1; X_n < \theta_1 - \frac{\tilde{\sigma}}{2}\} =$$

$$\sum_{n=1}^{\infty} \left(\frac{\theta_0 - \theta_1 + \tilde{\sigma}}{\sigma}\right)^{n-1} \left(\frac{\theta_1 - \theta - \frac{\tilde{\sigma}}{2} + \frac{\sigma}{2}}{\sigma}\right) = \frac{\theta_1 - \theta - \frac{\tilde{\sigma}}{2} + \frac{\sigma}{2}}{\theta_1 - \theta_0 - \tilde{\sigma} + \sigma},$$

$$L(\theta, \tilde{\sigma}, \sigma | \tilde{\sigma} \in [2(\theta - \theta_0) + \sigma, p\sigma]) = 1.$$

$$(ii) \text{ for } \theta_0 + (p-1)\sigma/2 < \theta < \theta_1 - (p-1)\sigma/2; L(\theta, \tilde{\sigma}, \sigma | \tilde{\sigma} \in [0, \theta_1 - \theta_0]) = 1,$$

$$L(\theta, \tilde{\sigma}, \sigma | \tilde{\sigma} \in (\theta_1 - \theta_0, p\sigma)) = \frac{\theta_1 - \theta - \frac{\tilde{\sigma}}{2} + \frac{\sigma}{2}}{\theta_1 - \theta_0 - \tilde{\sigma} + \sigma}$$

$$(111) \text{ for } \theta \geq \theta_1 - (p-1)\sigma/2; L(\theta, \tilde{\sigma}, \sigma | \tilde{\sigma} \in [0, \theta_1 - \theta_0]) = 1,$$

$$L(\theta, \tilde{\sigma}, \sigma | \tilde{\sigma} \in (\theta_1 - \theta_0, 2(\theta_1 - \theta) + \sigma)) = \frac{\theta_1 - \theta - \frac{\tilde{\sigma}}{2} + \frac{\sigma}{2}}{\theta_1 - \theta_0 - \tilde{\sigma} + \sigma},$$

$$L(\theta, \tilde{\sigma}, \sigma | \tilde{\sigma} \in [2(\theta_1 - \theta) + \sigma, p\sigma]) = 0.$$

$$\text{Writing } L(\theta, \sigma) = \int_0^{p\sigma} L(\theta, \tilde{\sigma}, \sigma | \tilde{\sigma} = t) n_0(n_0-1)(p\sigma)^{-n_0} (p\sigma t)^{n_0-2} (-t)^{n_0-1} dt,$$

integrating and observing that for a positive integer n

$$\int \frac{t^n}{A-t} dt = -A^n \ln |t-A| - \sum_{r=1}^n \binom{n}{r} A^{n-r} \frac{(t-A)^r}{r}, \text{ we have, after simplify-}$$

ing;

$$(1) \text{ for } \theta \leq \theta_0 + (p-1)\sigma/2$$

$$\begin{aligned} L(\theta, \sigma) &= \lambda [(\theta_1 - \theta_0)^{n_0-1} \left(\frac{p\sigma}{n_0-1} - \frac{\theta_1 - \theta_0}{n_0} \right) - A^{n_0-2} \{ \ln(\theta_1 + \theta_0 - 2\theta) - \ln \sigma \}] \\ &+ (mp\sigma - A(m + \frac{p\sigma}{2}) + \frac{A^2}{2}) + \frac{1}{2} \sum_{r=1}^{n_0} \binom{n_0}{r} \frac{(-1)^r A^{n_0-r}}{r} \{ \sigma^r - (\theta_1 + \theta_0 - 2\theta)^r \} + \\ &+ (m + \frac{p\sigma}{2}) \sum_{r=1}^{n_0-1} \binom{n_0-1}{r} \frac{(-1)^r A^{n_0-1-r}}{r} \{ (\theta_1 + \theta_0 - 2\theta)^r - \sigma^r \} + mp\sigma \sum_{r=1}^{n_0-2} \binom{n_0-2}{r} \frac{(-1)^r A^{n_0-2-r}}{r} \cdot \\ &\{ \sigma^r - (\theta_1 + \theta_0 - 2\theta)^r \} + \{ n_0(n_0-1) \}^{-1} \{ (p\sigma)^{n_0} - n_0 p\sigma (2(\theta - \theta_0) + \sigma)^{n_0-1} + \\ &+ (n_0-1)(2(\theta - \theta_0) + \sigma)^{n_0} \} \end{aligned} \quad (6.4.4)$$

$$\text{where } \lambda = n_0(n_0-1)(p\sigma)^{-n_0}, \quad m = \theta_1 - \theta + \sigma/2 \text{ and } A = \theta_1 - \theta_0 + \sigma.$$

$$(11) \text{ for } \theta_0 + (p-1)\sigma/2 < \theta < \theta_1 - (p-1)\sigma/2$$

$$\begin{aligned}
L(\theta, \sigma) &= 2 \left[(\theta_1 - \theta_0)^{n_0-1} \left(\frac{p\sigma}{n_0-1} - \frac{\theta_1 - \theta_0}{n_0} \right) + A^{n_0-2} \{ \ln \sigma - \ln(\theta_1 - \theta_0 - (p-1)\sigma) \} \right. \\
&\quad \left\{ mp\sigma - A \left(m + \frac{p\sigma}{2} \right) + \frac{A^2}{2} \right\} + \frac{1}{2} \sum_{r=1}^{n_0} \binom{n_0}{r} \frac{(-1)^r A^{n_0-r}}{r} \{ \sigma^r - (\theta_1 - \theta_0 - (p-1)\sigma)^r \} + \\
&\quad \left(m + \frac{p\sigma}{2} \right) \sum_{r=1}^{n_0-1} \binom{n_0-1}{r} \frac{(-1)^r A^{n_0-1-r}}{r} \{ (\theta_1 - \theta_0 - (p-1)\sigma)^r - \sigma^r \} + \\
&\quad mp\sigma \sum_{r=1}^{n_0-2} \binom{n_0-2}{r} \frac{(-1)^r A^{n_0-2-r}}{r} \{ \sigma^r - (\theta_1 - \theta_0 - (p-1)\sigma)^r \} \} \quad (6.4.5)
\end{aligned}$$

(iii) for $\theta \geq \theta_1 - (p-1)\sigma/2$

$$\begin{aligned}
L(\theta, \sigma) &= 2 \left[(\theta_1 - \theta_0)^{n_0-1} \left(\frac{p\sigma}{n_0-1} - \frac{\theta_1 - \theta_0}{n_0} \right) + A^{n_0-2} \{ \ln \sigma - \ln(2\theta - \theta_1 - \theta_0) \} \right. \\
&\quad \left\{ mp\sigma - \left(m + \frac{p\sigma}{2} \right) A + \frac{A^2}{2} \right\} + \frac{1}{2} \sum_{r=1}^{n_0} \binom{n_0}{r} \frac{(-1)^r A^{n_0-r}}{r} \{ \sigma^r - (2\theta - \theta_1 - \theta_0)^r \} + \\
&\quad \left(m + \frac{p\sigma}{2} \right) \sum_{r=1}^{n_0-1} \binom{n_0-1}{r} \frac{(-1)^r A^{n_0-1-r}}{r} \{ (2\theta - \theta_1 - \theta_0)^r - \sigma^r \} + \\
&\quad mp\sigma \sum_{r=1}^{n_0-2} \binom{n_0-2}{r} \frac{(-1)^r A^{n_0-2-r}}{r} \{ \sigma^r - (2\theta - \theta_1 - \theta_0)^r \} \} \quad (6.4.6)
\end{aligned}$$

The Expected Sample Size

In the usual notation, we have,

(i) for $\theta \leq \theta_1 - (p+1)\sigma/2$; $E(n|\theta, \tilde{\sigma}, \sigma, \tilde{\sigma} \in [0, \theta_1 - \theta_0]) = 0$,

$E(n|\theta, \tilde{\sigma}, \sigma, \tilde{\sigma} \in (\theta_1 - \theta_0, p\sigma)) = 1$

(ii) for $\theta_1 - (p+1)\sigma/2 < \theta < \theta_0 + (p-1)\sigma/2$;

$E(n|\theta, \tilde{\sigma}, \sigma, \tilde{\sigma} \in [0, \theta_1 - \theta_0]) = 0$, $E(n|\theta, \tilde{\sigma}, \sigma, \tilde{\sigma} \in (\theta_1 - \theta_0, 2(\theta_1 - \theta) - \sigma))$

$= 1$, if $\tilde{\sigma} > 2(\theta_1 - \theta)\sigma$, then in the present case $\tilde{\sigma} > 2(\theta - \theta_0) + \sigma$,

i.e., $\theta_1 - \tilde{\sigma}/2 < \theta + \sigma/2 < \theta_0 + \tilde{\sigma}/2$ and so,

$$E(n|\theta, \tilde{\sigma}, \sigma, \tilde{\sigma} \in (2(\theta_1 - \theta_0) - \sigma, p\sigma)) =$$

$$\sum_{n=1}^{\infty} n [\Pr\{\theta_1 - \tilde{\sigma}/2 < X_j < \theta_0 + \tilde{\sigma}/2; j = 1, \dots, n-1; X_n \leq \theta_1 - \tilde{\sigma}/2\} +$$

$$\Pr\{\theta_1 - \tilde{\sigma}/2 < X_j < \theta_0 + \tilde{\sigma}/2; j = 1, \dots, n-1; X_n \geq \theta_0 + \tilde{\sigma}/2\}] =$$

$$= \sum_{n=1}^{\infty} n \left(\frac{\theta + \frac{\sigma}{2} - \theta_1 + \frac{\tilde{\sigma}}{2}}{\sigma} \right)^{n-1} \left(\frac{\theta_1 - \frac{\tilde{\sigma}}{2} - \theta + \frac{\sigma}{2}}{\sigma} \right) = \frac{\sigma}{\theta_1 - \frac{\tilde{\sigma}}{2} - \theta + \frac{\sigma}{2}}$$

$$(iii) \text{ for } \theta_0 + (p-1)\sigma/2 \leq \theta \leq \theta_1 - (p-1)\sigma/2;$$

$E(n|\theta, \tilde{\sigma}, \sigma, \tilde{\sigma} \in [0, \theta_1 - \theta_0]) = 0$, if $\theta_1 - \theta_0 < \tilde{\sigma}$ then since in the present case $\Pr\{\tilde{\sigma} \leq 2(\theta - \theta_0) + \sigma\} = 1$ and $\Pr\{\tilde{\sigma} \leq 2(\theta_1 - \theta) + \sigma\} = 1$, we have, $E(n|\theta, \tilde{\sigma}, \sigma, \tilde{\sigma} \in (\theta_1 - \theta_0, p\sigma)) =$

$$\sum_{n=1}^{\infty} n [\Pr\{\theta_1 - \tilde{\sigma}/2 < X_j < \theta_0 + \tilde{\sigma}/2; j = 1, \dots, n-1; X_n \leq \theta_1 - \tilde{\sigma}/2\} +$$

$$\Pr\{\theta_1 - \tilde{\sigma}/2 < X_j < \theta_0 + \tilde{\sigma}/2; j = 1, \dots, n-1; X_n \geq \theta_0 + \tilde{\sigma}/2\}] =$$

$$= \sum_{n=1}^{\infty} n \left(\frac{\theta_1 - \tilde{\sigma} + \sigma - \theta_0}{\sigma} \right) \left(\frac{\theta_0 + \tilde{\sigma} - \theta_1}{\sigma} \right)^{n-1} = \frac{\sigma}{\theta_1 - \tilde{\sigma} + \sigma - \theta_0}$$

$$(iv) \text{ for } \theta_1 - (p-1)\sigma/2 < \theta < \theta_0 + (p+1)\sigma/2; E(n|\theta, \tilde{\sigma}, \sigma, \tilde{\sigma} \in [0, \theta_1 - \theta_0])$$

$$= 0, E(n|\theta, \tilde{\sigma}, \sigma, \tilde{\sigma} \in (\theta_1 - \theta_0, 2(\theta - \theta_0) - \sigma)) = 1,$$

$$E(n|\theta, \tilde{\sigma}, \sigma, \tilde{\sigma} \in (2(\theta - \theta_0) - \sigma, p\sigma)) = \sum_{n=1}^{\infty} n \left(\frac{\theta_0 + \frac{\tilde{\sigma}}{2} - \theta + \frac{\sigma}{2}}{\sigma} \right)^{n-1} \left(\frac{\theta - \theta_0 + \frac{\sigma}{2} - \frac{\tilde{\sigma}}{2}}{\sigma} \right)$$

$$= \frac{\sigma}{\theta + \frac{\sigma}{2} - \theta_0 - \frac{\tilde{\sigma}}{2}}$$

$$(v) \text{ for } \theta \geq \theta_0 + (p+1)\sigma/2; E(n|\theta, \tilde{\sigma}, \sigma, \tilde{\sigma} \in [0, \theta_1 - \theta_0]) = 0,$$

$$E(n|\theta, \tilde{\sigma}, \sigma, \tilde{\sigma} \in (\theta_1 - \theta_0, p\sigma)) = 1.$$

Writing $E(n|\theta, \sigma) = \int E(n|\theta, \tilde{\sigma}, \sigma, \tilde{\sigma} = t) p_1(t) dt$ ($p_1(t)$ being the p.d.f. of $\tilde{\sigma}$), $B = 2(\theta_1 - \theta) + \sigma$, $C = 2(\theta - \theta_0) + \sigma$, integrating and observing by (6.4.3) that $p\sigma - B < 0$ in case (ii), $p\sigma - A < 0$ in case (iii), $p\sigma - C < 0$ in case (iv); we have, after simplifying,

(i) for $\theta \leq \theta_1 - (p+1)\sigma/2$

$$E(n|\theta, \sigma) = \ell(n_0(n_0-1))^{-1} \{ (p\sigma)^{n_0} + (\theta_1 - \theta_0)^{n_0-1} ((n_0-1)(\theta_1 - \theta_0) - n_0 p\sigma) \} \quad (6.4.7)$$

(ii) for $\theta_1 - (p+1)\sigma/2 < \theta < \theta_0 + (p-1)\sigma/2$

$$\begin{aligned} E(n|\theta, \sigma) = & \ell(n_0(n_0-1))^{-1} \{ (\theta_1 - \theta_0)^{n_0-1} \{ n_0(\theta_1 - \theta_0 - p\sigma) - (\theta_1 - \theta_0) \} + \\ & (2(\theta_1 - \theta) - \sigma)^{n_0-1} \{ n_0(p\sigma - 2(\theta_1 - \theta) + \sigma) + 2(\theta_1 - \theta) - \sigma \} \} + \\ & 2\ell\sigma [B^{n_0-2} \{ \ln 2\sigma - \ln(2(\theta_1 - \theta) - (p-1)\sigma) \} \{ p\sigma - B \} + \\ & \sum_{r=1}^{n_0-1} \binom{n_0-1}{r} \frac{(-1)^r B^{n_0-1-r}}{r} \{ (2(\theta_1 - \theta) - (p-1)\sigma)^r - (2\sigma)^r \} + \\ & p\sigma \sum_{r=1}^{n_0-2} \binom{n_0-2}{r} \frac{(-1)^r B^{n_0-2-r}}{r} \{ (2\sigma)^r - (2(\theta_1 - \theta) - (p-1)\sigma)^r \}] \end{aligned} \quad (6.4.8)$$

(iii) for $\theta_0 + (p-1)\sigma/2 \leq \theta \leq \theta_1 - (p-1)\sigma/2$

$$\begin{aligned} E(n|\theta, \sigma) = & \ell\sigma [A^{n_0-2} \{ \ln \sigma - \ln((\theta_1 - \theta_0) - (p-1)\sigma) \} \{ p\sigma - A \} + \\ & \sum_{r=1}^{n_0-1} \binom{n_0-1}{r} \frac{(-1)^r A^{n_0-1-r}}{r} \{ (\theta_1 - \theta_0 - (p-1)\sigma)^r - \sigma^r \} + \\ & p\sigma \sum_{r=1}^{n_0-2} \binom{n_0-2}{r} \frac{(-1)^r A^{n_0-2-r}}{r} \{ \sigma^r - (\theta_1 - \theta_0 - (p-1)\sigma)^r \}] \end{aligned} \quad (6.4.9)$$

(iv) for $\theta_1 - (p-1)\sigma/2 < \theta < \theta_0 + (p+1)\sigma/2$

$$\begin{aligned} E(n|\theta, \sigma) = & \ell(n_0(n_0-1))^{-1} \{ (2(\theta - \theta_0) - \sigma)^{n_0-1} \{ n_0 p\sigma - (n_0-1)(2(\theta - \theta_0) - \sigma) \} \\ & + (\theta_1 - \theta_0)^{n_0-1} \{ n_0(\theta_1 - \theta_0 - p\sigma) - (\theta_1 - \theta_0) \} \} + \\ & 2\ell\sigma \{ (\ln(2(\theta - \theta_0) - (p-1)\sigma) - \ln 2\sigma) C^{n_0-2} \{ C - p\sigma \} + \\ & \sum_{r=1}^{n_0-1} \binom{n_0-1}{r} \frac{(-1)^r C^{n_0-1-r}}{r} \{ (2(\theta - \theta_0) - (p-1)\sigma)^r - (2\sigma)^r \} + \\ & p\sigma \sum_{r=1}^{n_0-2} \binom{n_0-2}{r} \frac{(-1)^r C^{n_0-2-r}}{r} \{ (2\sigma)^r - (2(\theta - \theta_0) - (p-1)\sigma)^r \} \} \end{aligned} \quad (6.4.10)$$

(v) for $\theta \geq \theta_0 + (p+1)\sigma/2$

$$E(n|\theta, \sigma) = 2(n_0(n_0-1))^{-1} \{ (n_0)^{n_0} + (\theta_1 - \theta_0)^{n_0-1} \{ (n_0-1)(\theta_1 - \theta_0) - n_0 \sigma \} \} \quad (6.4.11)$$

6.5. Choosing One of the Three Hypotheses About the Mean of the Rectangular Distribution With Unknown Variance

The probability density function of X is as in (6.4.1).

We consider the problem of choosing one of the three mutually exclusive and exhaustive hypotheses

$$\begin{aligned} H_0 : \theta < a_0 ; \quad H_1 : a_0 \leq \theta \leq a_1 ; \quad H_2 : \theta > a_1 ; \\ -\infty < a_0 < a_1 < \infty ; \quad \sigma > 0 \end{aligned} \quad (6.5.1)$$

We choose intervals (θ_0, θ_1) and (θ_2, θ_3) satisfying

$$\begin{aligned} \text{i) } \theta_0 < a_0 < \theta_1 \leq \theta_2 < a_1 < \theta_3 \quad \text{ii) } \theta_1 - \theta_0 = \theta_3 - \theta_2 \\ \text{iii) } a_0 = (\theta_1 + \theta_0)/2, \quad a_1 = (\theta_2 + \theta_3)/2. \end{aligned}$$

Defining $\tilde{\sigma}$ as in section 6.4, let $S_1(\tilde{\sigma}, \sigma)$ be the test of $H_{\theta_0} : \theta = \theta_0$ vs. $H_{\theta_1} : \theta = \theta_1$ with decision rule (6.4.2) and let $S_2(\tilde{\sigma}, \sigma)$ be the test of $H_{\theta_2} : \theta = \theta_2$ vs. $H_{\theta_3} : \theta = \theta_3$ with decision rule obtained from (6.4.2) by replacing θ_0 by θ_2 and θ_1 by θ_3 . Let $S(\tilde{\sigma}, \sigma)$ be defined by:

$S_1(\tilde{\sigma}, \sigma)$	$S_2(\tilde{\sigma}, \sigma)$	$S(\tilde{\sigma}, \sigma)$
accepts θ_0	accepts θ_2	accepts H_0
accepts θ_1	accepts θ_2	accepts H_1
accepts θ_1	accepts θ_3	accepts H_2

It follows from the definitions of $S_1(\tilde{\sigma}, \sigma)$; $i = 1, 2$; that $S(\tilde{\sigma}, \sigma)$ is well defined since

$$\text{acceptance of both } \theta_0 \text{ and } \theta_3 \text{ is impossible} \quad (6.5.2)$$

From (6.4.3), a necessary and sufficient condition for $S_1(\bar{\sigma}, \sigma)$ ($i=1,2$) to terminate with probability one is that

$$\theta_{j-1} + (p-1)\sigma/2 < \theta_j - (p-1)\sigma/2 ; \quad j = 1, 3 \quad (6.5.3)$$

Thus we assume (6.5.3).

OC Functions

Let $L_{s_1}(\theta, \bar{\sigma}, \sigma) = \Pr\{S_1(\bar{\sigma}, \sigma) \text{ accepts } \theta_j | \theta, \bar{\sigma}, \sigma\}$; $(1,j) = (1,0)$, $(2,2)$ and $L_{s_1}(\theta, \sigma) = E L_{s_1}(\theta, \bar{\sigma}, \sigma)$; $i = 1, 2$. By the definition of $S(\bar{\sigma}, \sigma)$ and by (6.5.2), we obtain (denoting $\Pr\{H_1 \text{ is accepted} | \theta, \sigma\}$ by $L(H_1 | \theta, \sigma)$; $i = 0, 1, 2$),

$$\begin{aligned} L(H_0 | \theta, \sigma) &= L_{s_1}(\theta, \sigma); \quad L(H_1 | \theta, \sigma) = L_{s_2}(\theta, \sigma) - L_{s_1}(\theta, \sigma); \\ L(H_2 | \theta, \sigma) &= 1 - L_{s_2}(\theta, \sigma) \end{aligned} \quad (6.5.4)$$

If we denote the right sides of (6.4.4) - (6.4.6) by $L_{s_1}^{(1)}(\theta, \sigma)$; $i = 1, 2, 3$ respectively and if we obtain $L_{s_2}^{(1)}(\theta, \sigma)$ from $L_{s_1}^{(1)}(\theta, \sigma)$ ($i = 1 - 3$) by changing θ_0 and θ_1 to θ_2 and θ_3 respectively, we have (using (6.5.4)) the following formulae for the OC functions:

$$\begin{aligned} L(H_0 | \theta, \sigma) &= L_{s_1}^{(1)}(\theta, \sigma) \quad \text{if } \theta \leq \theta_0 + (p-1)\sigma/2 \\ &= L_{s_1}^{(2)}(\theta, \sigma) \quad \text{if } \theta_0 + (p-1)\sigma/2 < \theta < \theta_1 - (p-1)\sigma/2 \\ &= L_{s_1}^{(3)}(\theta, \sigma) \quad \text{if } \theta \geq \theta_1 - (p-1)\sigma/2 \end{aligned} \quad (6.5.5)$$

$$\begin{aligned} L(H_1 | \theta, \sigma) &= L_{s_2}^{(1)}(\theta, \sigma) - L_{s_1}^{(1)}(\theta, \sigma) \quad \text{if } \theta \leq \theta_0 + (p-1)\sigma/2 \\ &= L_{s_2}^{(1)}(\theta, \sigma) - L_{s_1}^{(2)}(\theta, \sigma) \quad \text{if } \theta_0 + (p-1)\sigma/2 < \theta < \theta_1 - (p-1)\sigma/2 \\ &= L_{s_2}^{(1)}(\theta, \sigma) - L_{s_1}^{(3)}(\theta, \sigma) \quad \text{if } \theta_1 - (p-1)\sigma/2 \leq \theta \leq \theta_2 + (p-1)\sigma/2 \\ &= L_{s_2}^{(2)}(\theta, \sigma) - L_{s_1}^{(3)}(\theta, \sigma) \quad \text{if } \theta_2 + (p-1)\sigma/2 < \theta < \theta_3 - (p-1)\sigma/2 \\ &= L_{s_2}^{(3)}(\theta, \sigma) - L_{s_1}^{(3)}(\theta, \sigma) \quad \text{if } \theta \geq \theta_3 - (p-1)\sigma/2 \end{aligned} \quad (6.5.6)$$

$$\begin{aligned}
L(H_2|\theta, \sigma) &= 1 - L_{s_2}^{(1)}(\theta, \sigma) \quad \text{if } \theta \leq \theta_2 + (p-1)\sigma/2 \\
&= 1 - L_{s_2}^{(2)}(\theta, \sigma) \quad \text{if } \theta_2 + (p-1)\sigma/2 < \theta < \theta_3 - (p-1)\sigma/2 \\
&= 1 - L_{s_2}^{(3)}(\theta, \sigma) \quad \text{if } \theta \geq \theta_3 - (p-1)\sigma/2 \quad (6.5.7)
\end{aligned}$$

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