	0	7	8	9	1	3	5	2	4	6
	6	1	7	8	9	2	4	3	5	0
	5	0	2	7	8	9	3	4	6	1
	4	6	1	3	7	8	9	5	0	2
0	6	5	4	9	8	7	1	2	3	
7	1	0	6	5	9	8	2	3	4	
8	7	2	1	0	6	9	3	4	5	
9	8	7	3	2	1	0	4	5	6	
1	9	8	7	4	3	2	5	6	0	
3	2	9	8	7	5	4	6	0	1	
5	4	3	9	8	7	6	0	1	2	
2	3	4	5	6	0	1	7	8	9	
4	5	6	0	1	2	3	8	9	7	
6	0	1	2	3	4	5	9	7	8	

COMBINATORIAL
MATHEMATICS
YEAR

February 1969 - June 1970

LINEAR REPRESENTATION OF A CLASS OF PROJECTIVE PLANES IN
FOUR DIMENSIONAL PROJECTIVE SPACE*

by

R.C. Bose
University of North Carolina
at Chapel Hill

and

A. Barlotti[†]
University of Perugia,
Italy

Institute of Statistics Mimeo Series No. 600.20

Department of Statistics
University of North Carolina at Chapel Hill, 27514

FEBRUARY, 1970

* This research was sponsored by the National Science Foundation under Grant No. GP-8624, and the U.S. Air Force Office of Scientific Research under Grant No. AFOSR-68-1406.

† This research was conducted while the author was visiting professor at the University of North Carolina at Chapel Hill. His research was also partially supported by C.N.R.

LINEAR REPRESENTATION OF A CLASS OF PROJECTIVE PLANES IN
FOUR DIMENSIONAL PROJECTIVE SPACE*

by

R.C. Bose
University of North Carolina
at Chapel Hill

A. Barlotti[†]
University of Perugia,
Italy

ABSTRACT

The theory of linear representations of projective planes was developed by Bruck and one of the authors (Bose) in two earlier papers [J. Algebra 1 (1964), 85-102 and 4 (1966), 117-172] can be further extended by generalizing the concept of incidence adopted there. A linear representation is obtained for a class of non-Desarguesian projective planes illustrating this concept of generalized incidence. It is shown that in the finite case, the planes represented by the new construction are derived planes in the sense defined by Ostrom [Trans. Amer. Math. Soc. 111 (1964), 1-18] and Albert [Boletin Soc. Mat. Mex., 11 (1966), 1-13] of the duals of translation planes which can be represented in 4-space by the Bose-Bruck construction. An analogous interpretation is possible for the infinite case.

* This research was sponsored by the National Science Foundation under Grant No. GP-8624, and the U.S. Air Force Office of Scientific Research under Grant No. AFOSR-68-1406.

[†] This research was conducted while the author was visiting professor at the University of North Carolina at Chapel Hill. His research was also partially supported by C.N.R.

1. INTRODUCTION

Bruck and one of the authors (Bose) in two earlier papers [2], [3], developed a theory of linear representation of projective planes, in which a projective plane S_2 (Desarguesian or not) is an isomorphic representation of S_2 in a t -dimensional (Desarguesian) projective space, S_t , in which each point or line of S_2 becomes a non-empty finite dimensional projective subspace of S_t , and incidence is given by the containing relation in S_t . In particular an explicit linear representation was obtained for all translation planes coordinatized by a right Veblen-Wedderburn system which is finite dimensional over its left operator skew-field. (This includes the class of all finite translation planes.) The present paper is a further step in the same direction. It now appears that the earlier definition of incidence based on the containing relation in S_t is too restrictive. In general the incidence between a subspace α representing a line, and a subspace β representing a point can be defined by specifying the dimension of the projective space $\alpha \cap \beta$.

A linear representation is developed for a class of projective planes Δ , illustrating this concept of generalized incidence. Points and lines of Δ , are represented by certain subspaces of a projective space S_4 of four dimensions. Σ is a fixed 3-space S_4 , S is a spread of lines in Σ (i.e., a set of lines, such that each point of Σ is contained in exactly one line of S) and δ is a fixed line of S . Ω is a fixed 3-space, intersecting Σ in a plane containing δ .

Points of Δ are of three types, (1) those represented by planes of S_4 not contained in Σ and passing through lines of S , other than δ ; (2) those represented by planes of S_4 , not contained in Σ or Ω and passing through δ ; (3) those represented by the points of the line δ .

Lines of Δ are of three types, (1) those which are represented by points of S_4 not contained in Σ or Ω ; (2) those which are represented by planes contained in Ω , and not containing δ ; (3) a single line represented by the line δ . Incidence is defined by the containing relation in S_4 , except that a point of type (1) or type (2) is defined to be incident with a line of type (2), when the planes representing them intersect in a line. It is shown in Section 3, that with this definition of incidence Δ is a projective plane when S_4 is a finite space or when S_4 is infinite and S has the property that each plane of Σ contains not more than (and therefore exactly) one line of S .

In Section 4, the relation between translation planes and Δ -planes is studied. A translation plane T , which can be coordinatized by a right Veblen-Wedderburn system which is two dimensional over its left operator skew-field may be said to be of degree two. It can be represented in a projective space S_4 in the manner described in Section 2 (being a special case of the construction given in [2]). The line at infinite in T is represented by a particular 3-space Σ of S_4 . Let δT be the dual of T . Then Σ represents a point of δT . It is shown that in the finite case the Δ -planes are the derived planes (in the sense defined by Albert [1] and Ostrom [10], [11], [12]) dual translation planes δT . Any two points of δT collinear with Σ can be embedded in a derivation set (which includes Σ). An analogous construction is possible in the infinite case. It should be noted that Σ is a point in δT , such that δT is (Σ, δ) transitive for any line δ through Σ , and is unique except for the cases when δT is a Moufang plane or is Desarguesian, when Σ is arbitrary.

In Section 5, we show that the process of dualization and derivation by which the plane Δ is obtained from T can be continued if the line

δ of the spread S , belongs to a regulus all of whose lines are in S . This gives the geometrical representation of the analogous sequences of planes obtained by Fryxell [7] and Johnson [8]. Starting from the translation plane $T = \Pi_1$, we obtain a cyclic sequence $\{\Pi_i\}$, ($i = 1, 2, 3, \dots$) of projective planes such that $\Pi_i = \Pi_{i+8}$, Π_{2i} is the dual of Π_{2i-1} and Π_{2i+1} is the derived of Π_{2i} .

2. TRANSLATION PLANES

1. Let F be a skew field (that is, an associative division ring which may or may not be commutative). Let S_4 be a four dimensional projective space based on F . The skew field F may be finite or infinite. We shall be specially interested in the first case, when F must be a Galois field $GF(q)$ of order q , where q is a prime power.

Let Σ be a fixed 3-space of S_4 . A spread S is defined as a set of lines contained in Σ , such that every point of Σ is contained in exactly one line of S . A plane Π in Σ cannot contain more than one line of S (otherwise they would intersect in a point). If Π contains a line Δ belonging to S , then every other line Δ' of S meets Π in a unique point S' , and through any point S' of Π , not in S , there passes a unique line of S . In the finite case S contains exactly q^2+1 lines, and any plane of Σ contains exactly one line of S . For further properties of spreads see references [2], [3], [4], and [13].

2. The following is a special case of a construction given in [2] for obtaining translation planes. The translation plane to be constructed will be denoted by T and its points and lines will be called T-points and T-lines, to avoid confusing them with elements of S_4 which represent them.

T-points are of two types. T-points of type (1) are represented by the points of S_4 not contained in Σ . In the finite case there are q^4 T-points of type (1). T-points of type (2) are represented by the lines of S . In the finite case there are q^2+1 T-points of type (2).

T-lines are of two types. T-lines of type (1) are represented by planes of S_4 , passing through the lines of S and not contained in Σ .

In the finite case through each line of S there pass q^2 planes not contained in Σ , so that there are $q^4 + q^2$ T-lines of type (1). There is only one line of type (2), which is represented by the fixed 3-space Σ .

To complete our description of the plane T , we must define incidence between T-points and T-lines. A T-line and a T-point are incident if and only if the element of S_4 representing the T-line contains the element of S_4 representing the T-point. It was shown in [2], that the T-points and T-lines form a translation plane, which is Desarguesian if and only if S is a regular spread. The spread S is said to be regular if for any hyperbolic quadric containing three lines $\delta_1, \delta_2, \delta_3$ of S all the generators (ruling lines) of the regulus to which $\delta_1, \delta_2, \delta_3$ belong are elements of S . In the finite case the order of the translation plane T is q^2 .

3. Δ -PLANES

1. We shall now obtain a new projective plane Δ , related to the T -plane described in Section 2. The four space S_4 , the fixed 3-space Σ , and the spread S of lines in Σ , remain as in Section 2. In the infinite case we shall additionally assume that S has the property, that every plane of Σ contains at least (and therefore exactly) one line of S . As noticed earlier this is always true in the finite case. The points and lines of Δ will be called Δ -points and Δ -lines. We shall first construct an affine plane $\alpha\Delta$ and later complete it to a projective plane Δ . The relationship between the planes T and Δ will be considered in Section 4.

Let δ be a fixed line of the spread S . Let Ω be a fixed 3-space of S_4 which passes through δ and therefore intersects Σ in a plane Π containing δ . $\alpha\Delta$ -points are all of the same type. They are represented by the planes of S_4 , which pass through a line of S other than δ , and are not contained in Σ . In the finite case there are q^2 lines of S other than δ , through each of which pass q^2 planes not contained in Σ . Hence the number of $\alpha\Delta$ -points is q^4 .

$\alpha\Delta$ -lines are two types. $\alpha\Delta$ -lines of type (1) are represented by points of S_4 not contained in Σ or Ω . In the finite case S_4 has $(q^5-1)/(q-1)$ points, Σ and Ω each have $(q^4-1)/(q-1)$ points, and Π which is contained both in Σ and Ω has $(q^3-1)/(q-1)$ points. Hence the number of points contained neither in Ω nor in Σ is q^4-q^3 . This is the number of type (1) $\alpha\Delta$ -lines. Again $\alpha\Delta$ -lines of type (2) are represented by planes contained in Ω , not passing through δ . In the finite case the number of planes contained in Ω is $(q^4-1)/(q-1)$ of which

$(q^2-1)/(q-1)$ pass through δ . Hence the number of $\alpha\Delta$ -lines of type (2) is q^3+q^2 . The total number of $\alpha\Delta$ -lines in the finite case is q^4+q^2 . Let (R) be the set of points of S_4 not contained in Σ or Ω , and (P) be the set of planes of Ω not containing δ . Then $\alpha\Delta$ -lines of type (1) are represented by points of (R), and $\alpha\Delta$ -lines of type (2) are represented by planes of (P).

To complete the description of the $\alpha\Delta$ -plane we have to define incidence between points and lines of $\alpha\Delta$. An $\alpha\Delta$ -line of type (1) is said to be incident with an $\alpha\Delta$ -point, if the point representing the former is contained in the plane representing the later. Thus if an $\alpha\Delta$ -line is represented by a point Q of (R), then the $\alpha\Delta$ -points incident with it are represented by the planes joining Q to the lines of S , other than δ . In the finite case every $\alpha\Delta$ -line of type (1), is incident with q^2 $\alpha\Delta$ -points. An $\alpha\Delta$ -line of type (2) is said to be incident with an $\alpha\Delta$ -point if the planes representing them intersect in a line. Let μ be a plane of (P), then μ intersects Π in a line ℓ other than δ . Let L be the point of intersection of ℓ and δ . If δ is a plane representing an $\alpha\Delta$ -point incident with the $\alpha\Delta$ -line represented by μ , then δ intersects μ in a line m , which intersects ℓ in a point S' . Hence δ passes through S' , and must contain the unique line δ' of S which passes through S' . Thus δ is the plane through S' , containing δ' and m . Note that S' cannot be a point of δ , since δ' is disjoint from δ . Hence to each line m lying in μ and not passing through L (i.e., not meeting δ) there corresponds a unique plane representing an $\alpha\Delta$ -point incident with the $\alpha\Delta$ -line represented by μ . It is obtained by finding the intersection S' of m and Π and taking the plane containing m and the line δ' of S which passes through S' . In the finite case there are q^2+q+1 lines in μ ,

of which $q+1$ pass through L . Hence any $\alpha\Delta$ -line of type (2) is incident with exactly q^2 $\alpha\Delta$ -points.

Let us consider the set of $\alpha\Delta$ -lines incident with a given $\alpha\Delta$ -point, represented by a plane δ passing through a line $s' \neq s$ of S , and not contained in Σ . $\alpha\Delta$ -lines of the first type incident with the $\alpha\Delta$ -point under consideration are represented by the points of (R) lying in δ . Also δ intersects Ω in a line m , not meeting s . The planes of (P) passing through m , represent $\alpha\Delta$ -lines of type (2) incident with the $\alpha\Delta$ -point represented by δ . Thus each $\alpha\Delta$ -point is incident with lines of both types. In the finite case points of (R) contained in δ , are the points not lying in s' or m . There are $q^2 - q$ such points. Also every plane passing through m belongs to (P), since no such plane can contain s . Hence in the finite case each $\alpha\Delta$ -point is incident with $q^2 + 1$ $\alpha\Delta$ -lines, $q^2 - q$ of type (1) and $q + 1$ of type (2). We now proceed to show that $\alpha\Delta$ is an affine plane.

2. Lemma (1). Two distinct $\alpha\Delta$ -points are incident with a unique $\alpha\Delta$ -line.

We have to consider two separate cases:

Case I. Let the planes δ_1 and δ_2 representing the two $\alpha\Delta$ -points, pass through the same line $s' \neq s$, of S . Since the points common to δ_1 and δ_2 , are points of s' , which is contained in Σ , there is no point of (R) common to both δ_1 and δ_2 . Hence there is no $\alpha\Delta$ -line of type (1), incident with both δ_1 and δ_2 . $\alpha\Delta$ -lines of type (2) are represented by planes of (P). Now by construction Π contains the line s of S . Hence every other line of S meets Π in a unique point. Let s' meet Π in the point S' . Now δ_1 and δ_2 meet Ω in lines m_1 and m_2 through S' . There is a unique plane $\mu = (m_1, m_2)$ of (P), which

contains m_1 and m_2 . This plane intersects δ_1 and δ_2 in lines and represents the unique $\alpha\Delta$ -line which is incident with the $\alpha\Delta$ -points represented by δ_1 and δ_2 .

Case II. Let the planes δ_1 and δ_2 representing two distinct $\alpha\Delta$ -points, pass through different lines s_1 and s_2 ($s_1 \neq s$, $s_2 \neq s$) of S . Now δ_1 and δ_2 cannot meet in a line. Suppose if possible they meet in a line u . Then u would meet Σ in a point U common to s_1 and s_2 . This is contradiction since s_1 and s_2 are disjoint. But two planes in 4-space must have at least one point in common. Let this point be Q . Clearly Q cannot belong to Σ . But it may or may not belong to Ω . Hence we have to consider two subcases:

(a) Suppose Q does not belong to Ω . Then it is an element of the set (R) , and represents an $\alpha\Delta$ -line, which is the unique $\alpha\Delta$ -line incident with the $\alpha\Delta$ -points represented by δ_1 and δ_2 .

(b) Suppose Q belongs to Ω . The lines s_1 and s_2 of S meet Π in points S_1 and S_2 not contained in s . Hence δ_1 and δ_2 intersect Ω in lines $m_1 = QS_1$ and $m_2 = QS_2$. The lines m_1 and m_2 determine a plane $\mu = (m_1, m_2)$ lying in Ω and intersecting Σ in the line S_1S_2 lying in Π and distinct from s . Hence μ is an element of the set (P) , and represents an $\alpha\Delta$ -line of type (2). It intersects δ_1 and δ_2 in m_1 and m_2 . Hence μ represents the unique $\alpha\Delta$ -line incident with the $\alpha\Delta$ -points represented by δ_1 and δ_2 . This completes the proof of Lemma (1).

3. The $\alpha\Delta$ -lines can be divided into parallel classes. Parallel classes are of two kinds. Let (P^*) be the set of planes passing through s and not lying in either Σ or Ω . In the finite case (P^*) consists of

exactly $q^2 - q$ planes, since through δ there pass $q^2 + q + 1$ planes, $q + 1$ of which lie in each of Σ and Ω , including Π which lies in both. Two $\alpha\Delta$ -lines of type (1), represented by the points Q_1 and Q_2 of (R) are defined to belong to a parallel class of the first kind, if both Q_1 and Q_2 are contained in a plane λ of the set (P^*) . This parallel class will be said to correspond to the plane λ . It is clear that $\alpha\Delta$ -lines of type (1) represented by Q_1 and Q_2 are parallel (belong to the same parallel class) if and only if the line Q_1Q_2 meets δ in a point L , in which case they belong to the parallel class corresponding to the plane λ which contains the line δ and the line LQ_1Q_2 . Every $\alpha\Delta$ -line of type (1), belongs to a unique parallel class of the first kind. If Q is a point of (R) , representing an $\alpha\Delta$ -line, then the unique parallel class to which the $\alpha\Delta$ -line belongs is represented by the plane λ of (P^*) , containing Q and δ . $\alpha\Delta$ -lines of type (1) can thus be partitioned into parallel classes. In the finite case the $q^4 - q^3$ $\alpha\Delta$ -lines of type (1), are partitioned into $q^2 - q$ parallel classes each containing q^2 members.

Lemma (2). Two $\alpha\Delta$ -lines of type (1) are simultaneously incident with one or zero $\alpha\Delta$ -points according as they are non-parallel (belong to different parallel classes) or are parallel.

Consider two $\alpha\Delta$ -lines of type (1), represented by the points Q_1 and Q_2 of (R) . If δ is a plane representing an $\alpha\Delta$ -point simultaneously incident with both the $\alpha\Delta$ -lines, then δ passes through the line Q_1Q_2 . Let Q_1Q_2 intersect Σ in S' . Then δ must be the plane containing Q_1Q_2 and the line δ' of S which passes through S' . If S' is not on δ , then the $\alpha\Delta$ -lines represented by Q_1 and Q_2 are non-parallel and the $\alpha\Delta$ -point represented by δ is uniquely determined. However if S'

is on δ , i.e., the $\alpha\Delta$ -lines represented by Q_1 and Q_2 are parallel, then the only line of S through S' is δ . By definition planes through δ do not represent any $\alpha\Delta$ -point. Hence the two $\alpha\Delta$ -lines are not simultaneously incident with any $\alpha\Delta$ -point. On the other hand they belong to the unique parallel class of the first kind corresponding to the plane λ of (P^*) containing Q_1Q_2 and δ .

Let (σ) denote the set of points on δ . In the finite case (σ) consists of exactly $q+1$ points. Two $\alpha\Delta$ -lines of type (2), represented by the planes μ_1 and μ_2 of Ω , not containing δ , are defined to belong to a parallel class of the second kind if μ_1 and μ_2 meet δ in the same point L . This parallel class will be said to correspond to the point L of (σ) . Every $\alpha\Delta$ -line of type (2) belongs to a unique parallel class of the second kind. If μ is a plane of (P) representing an $\alpha\Delta$ -line, then the unique parallel class to which the $\alpha\Delta$ -line corresponds is represented by the point L of (σ) , in which μ intersects δ . $\alpha\Delta$ -lines of type (2) can thus be partitioned into parallel classes. In the finite case the q^3+q^2 $\alpha\Delta$ -lines of type (2), are partitioned into $q+1$ parallel classes, each containing q^2 members.

Lemma (3). Two $\alpha\Delta$ -lines of type (2) are simultaneously incident with one or zero $\alpha\Delta$ -points according as they are non-parallel (belong to different parallel classes) or are parallel.

Consider two $\alpha\Delta$ -lines of type (2) represented by the planes μ_1 and μ_2 of (P) . By definition μ_1 and μ_2 cannot contain δ . Let m be the line of intersection of μ_1 and μ_2 . Suppose first that m does not lie in Π . Then m meets Π in a point S' . A plane δ representing

an $\alpha\Delta$ -point incident with both the $\alpha\Delta$ -lines represented by μ_1 and μ_2 must contain m and the line δ' of S which passes through S' . If S' is not on δ , i.e., the $\alpha\Delta$ -lines represented by μ_1 and μ_2 are non-parallel, then δ is uniquely determined. If however S' is on δ , i.e., the $\alpha\Delta$ -lines represented by μ_1 and μ_2 are parallel, then the only line of S through S' is δ . By definition planes through δ do not represent any $\alpha\Delta$ -point. Hence the two $\alpha\Delta$ -lines are not simultaneously incident with any $\alpha\Delta$ -point. On the other hand they belong to the unique parallel class of the second kind corresponding to the point S' of (σ) . Next suppose that m , the line of intersection of μ_1 and μ_2 lies in Π . Then m is different from δ , and meets it in a point L . In this case the two $\alpha\Delta$ -lines represented by μ_1 and μ_2 , belong to the unique parallel class of the second kind corresponding to the point L of (σ) . We shall show that there is no $\alpha\Delta$ -point simultaneously incident with these $\alpha\Delta$ -lines. If possible suppose such an $\alpha\Delta$ -point exists and is represented by the plane δ . Then the line of intersection of δ and Ω must lie in both μ_1 and μ_2 and therefore coincides with m . Hence δ intersects Σ in m . Therefore m belongs to S , which contradicts the fact that δ is the only line of S , which lies in Π .

Finally two $\alpha\Delta$ -lines of different types are defined to be non-parallel. The appropriateness of this is apparent from the following Lemma.

Lemma (4). Two $\alpha\Delta$ -lines one of which is of type (1), and the other of type (2), are simultaneously incident with a unique $\alpha\Delta$ -point.

Consider an $\alpha\Delta$ -line of type (1) represented by a point Q of (R) , and an $\alpha\Delta$ -line of type (2) represented by a plane μ of (P) . Let ϕ be

the 3-space containing both Q and μ . Since Q is not in Ω or Σ , Φ is distinct from Ω or Σ . Hence Φ does not contain δ , for Ω is the only 3-space containing δ and μ . Hence Φ intersects Σ in a plane Π^* not containing δ . Then Π^* contains a unique line ℓ of δ , which is distinct from δ . If δ is a plane representing an $\alpha\Delta$ -point incident with the $\alpha\Delta$ -lines represented by μ and Q , then δ must contain Q and must intersect μ in a line. Hence δ lies in Φ . Also δ must pass through a line of S . Since ℓ is the only line of S in Φ , δ is the uniquely determined plane containing ℓ and Q .

Lemmas (2), (3) and (4) taken together show that two $\alpha\Delta$ -lines are incident with no $\alpha\Delta$ -point if they are parallel and with a unique $\alpha\Delta$ -point if they are non-parallel.

4. Lemma (5). Given an $\alpha\Delta$ -line and an $\alpha\Delta$ -point not incident with it, there exists a unique $\alpha\Delta$ -line parallel to the given $\alpha\Delta$ -line, and incident with the given $\alpha\Delta$ -point.

Let the given $\alpha\Delta$ -point be represented by the plane δ , passing through a line δ' of S ($\delta \neq \delta'$). By definition δ is not contained in Σ . We have two cases to consider.

Case I. Let the given $\alpha\Delta$ -line be of type (1), represented by the point Q of (R) . Then Q is not contained in δ . Let λ be the plane joining δ and Q . Lines of the parallel class to which the given $\alpha\Delta$ -line belongs are represented by the points of λ . Clearly λ intersects δ in a unique point Q' which represents the required $\alpha\Delta$ -line.

Case II. Let the given $\alpha\Delta$ -line be of type (2), represented by a plane μ of (P) . Then δ does not intersect μ in a line. Let L be the

point of intersection of μ and δ . Then the line of intersection m of δ and Ω does not lie in μ , and does not pass through L . The required $\alpha\Delta$ -line is represented by the plane containing L and m .

Lemma (6). There exist three $\alpha\Delta$ -points not incident with the same $\alpha\Delta$ -line.

Let $\delta_1, \delta_2, \delta_3$ be three lines of S , $\delta_i \neq \delta$ ($i = 1, 2, 3$). Let Q_1 be a point of (R) , and let δ_2, δ_3 be the planes joining Q_1 to δ_2 and δ_3 respectively. They represent $\alpha\Delta$ -points. Choose a point Q_2 of (R) on δ_3 , other than Q_1 . Then the plane δ_3 joining Q_2 and δ_1 represents an $\alpha\Delta$ -point, and meets δ_2 in a point Q_3 different from Q_1 . Then the $\alpha\Delta$ -points represented by $\delta_1, \delta_2, \delta_3$ are not incident with the same $\alpha\Delta$ -line. In fact the $\alpha\Delta$ -points represented by δ_j and δ_k are incident with the unique line Q_i , where i, j, k are the indices 1, 2, 3 in some order or other. This completes the proof that $\alpha\Delta$ is an affine plane.

5. The completion of the affine plane $\alpha\Delta$ to the projective plane Δ is now straight forward. $\alpha\Delta$ -points may be called Δ -points of type (1). $\alpha\Delta$ -lines of types (1) and (2) may be called Δ -lines of types (1) and (2) respectively. We now define new Δ -points corresponding to parallel classes. Thus Δ -points of type (2) are represented by planes of (P^*) , and Δ -points of type (3) are represented by points of (σ) , i.e., points of δ . Finally we have to define a new Δ -line of type (3) incident with all Δ -points of types (2) and (3), which may be represented by δ . Hence the completed Δ plane may be described as follows. Points and lines are of three types:

Points of type (1) are represented by planes, passing through the lines of S other than δ , and not contained in Σ .

Points of type (2) are represented by planes passing through δ , and not contained in Ω or Σ .

Points of type (3) are represented by points of δ .

Lines of type (1) are represented by points not contained in Σ or Ω .

Lines of type (2) are represented by planes contained in Ω , and not containing δ .

There is a single line of type (3) represented by δ .

A Δ -line of type (2) is incident with a Δ -point of type (1) or (2) if and only if the planes representing them intersect in a line. In every other case a Δ -line is incident with a Δ -point if and only if the element of S_4 representing the Δ -line is contained in or contains the element of S_4 representing the Δ -point. Thus the Δ -line of type (3) represented by δ is incident with every Δ -point of type (2) or (3) and non-incident with any Δ -point of type (1). A Δ -point of type (2) represented by the plane λ , is incident with a Δ -line of type (1) if the point Q representing it is contained in λ . A Δ -point of type (2) is not incident with any Δ -line of type (2). A Δ -point of type (3) represented by a point L is incident with a Δ -line of type (2) represented by the plane μ , if μ passes through L . A Δ -point of type (3) is not incident with any Δ -line of type (1). The incidence between Δ -points of type (1), and Δ -lines of type (1) or (2) is carried over from the affine plane $\alpha\Delta$.

In the finite case the order of Δ is q^2 . The number of points of types (1), (2) and (3) respectively is q^4 , $q^2 - q$ and $q + 1$. The number of lines of types (1), (2), (3) respectively is $q^4 - q^3$, $q^3 + q^2$ and 1.

4. DERIVATION OF Δ -PLANES FROM DUAL TRANSLATION PLANES

1. In this section we shall consider the relation between translation planes described in Section 2, and the Δ -planes obtained in Section 3.

Let us start with the translation plane T . We can dualize it and obtain a dual translation plane δT by taking the T -lines for δT -points, and the T -points for δT -lines.

Thus we have a 4-space S_4 , together with a fixed 3-space Σ of S_4 . S is a spread of lines in Σ . Then δT lines are of two types. δT lines of type (1) are represented by points of S_4 not contained in Σ . δT lines of type (2) are represented by lines of S . δT points are of two types. δT -points of type (1) are represented by planes of S_4 not contained in Σ , and passing through the lines of S . There is a single δT -point of type (2), represented by the 3-space Σ . A δT -line and a δT -point are incident if and only if the element of S_4 representing the δT -line is contained in the element of S_4 representing the δT -point.

δT is a dual translation plane. We can make it affine, by choosing a line at infinity. Let s be a line of the spread S . Then s represents a δT -line. The δT -points incident with s are represented by the 3-space Σ , and by the planes of S_4 , passing through s and not contained in Σ . We will choose the δT line represented by s to be the line at infinity in δT . The δT -points not incident with the line at infinity, may be called $\alpha\delta T$ -points, and the δT -lines other than the line at infinity may be called $\alpha\delta T$ -lines. The incidence between $\alpha\delta T$ -points and $\alpha\delta T$ -lines is carried over from δT . Thus the elements of the affine dual translation plane $\alpha\delta T$ are as follows:

$\alpha\delta T$ -points are all of one type. They are represented by the planes of S_4 , passing through the lines of the spread S other than δ . In the finite case their number is q^4 , since through each of the q^2 lines of S , other than δ , there pass q^2 planes of S_4 , not contained in Σ . $\alpha\delta T$ -lines are of two types. $\alpha\delta T$ -lines of type (1) are represented by points of S_4 not contained in Σ . $\alpha\delta T$ lines of type (2) are represented by lines of S other than δ . In the finite case there are q^4 $\alpha\delta T$ -lines of type (1) and q^2 $\alpha\delta T$ -lines of type (2).

The lines of the affine plane $\alpha\delta T$ can be divided into parallel pencils in the usual manner. All $\alpha\delta T$ -lines passing through a given δT -point on the line at infinity in δT belong to the same parallel pencil. The δT -point at infinity with which all $\alpha\delta T$ lines of a parallel pencil are incident will be called the vertex of the parallel pencil. In the finite case the number of parallel pencils is q^2+1 , each parallel pencil containing q^2 $\alpha\delta T$ -lines.

Let ρ_1, ρ_2 be two planes of S_4 , passing through δ and not contained in Σ . Then ρ_1 and ρ_2 represent two δT -points of the line at infinity in δT . Let Ω be the 3-space of S_4 containing the planes ρ_1 and ρ_2 , and let Π be the plane of intersection of Σ and Ω . We shall identify Π and Ω with the spaces denoted by the same symbols as in Section 3. Let (D) denote the set of planes of Ω other than Π , which pass through δ . In the finite case (D) consists of q planes.

The set of δT -points at infinity represented by the planes of (D), and the 3-space Σ , will be called the derivation set. In the finite case the derivation set consists of $q+1$ δT -points at infinity.

Each δT -point of the derivation set is the vertex of a parallel pencil of the affine plane $\alpha\delta T$. We shall now construct a new affine plane from

$\alpha\delta T$ in the following manner: The points of the new affine plane will be the same as the $\alpha\delta T$ -points. The $\alpha\delta T$ -lines belonging to the parallel pencils whose vertices are δT -points of the derivation set will be deleted. In what follows we shall refer to these as the deleted lines, and the pencils formed by them as the deleted pencils. The deleted $\alpha\delta T$ -lines will be replaced by new lines, which are certain (affine) Baer subplanes of $\alpha\delta T$. Before proceeding to consider these, we shall make a (1,1) correspondence between $\alpha\delta T$ -points and the lines of Ω not intersecting δ .

2. Any $\alpha\delta T$ -point is represented by a plane δ passing through a line δ' of S (other than δ), and not contained in Σ . Then δ cannot intersect δ , otherwise δ would be contained in Σ . Hence δ intersects Ω in a unique line m , not intersecting δ . We shall say that this line m corresponds to the $\alpha\delta T$ -point represented by δ . This correspondence is (1,1). If m is any line of Ω not intersecting δ , then m meets Π in a point S' . Through S' passes a unique line δ' of S . Then the $\alpha\delta T$ -point corresponding to m , is represented by the plane δ containing δ' and m .

3. Let μ be a plane of the set (P) of Section 3, i.e., μ is a plane of Ω , not passing through δ . We shall show that μ can be regarded as representing an affine subplane of $\alpha\delta T$. Consider the set (B_μ) of $\alpha\delta T$ -points corresponding to the lines of μ not intersecting δ . This we define to be the set of $\alpha\delta T$ -points belonging to the subplane represented by μ . To each line m in μ , there is a unique $\alpha\delta T$ -point of (B_μ) represented by a plane δ , passing through some line of S (other than δ) and intersecting μ in m . In the finite case (B_μ) contains q^2 $\alpha\delta T$ -

points corresponding to the q^2 lines of μ , which do not pass through L , the point of intersection of μ and δ . We shall now define certain subsets of (B_μ) , called sublines, which will be the 'lines' of our subplane.

A subline of (B_μ) is defined to be a subset of $\alpha\delta T$ -points common to (B_μ) and the set of $\alpha\delta T$ -points incident with an $\alpha\delta T$ -line, provided that the subset contains at least two elements. Since any two $\alpha\delta T$ -points belonging to (B_μ) are incident with a unique $\alpha\delta T$ -line, they determine a unique subline of (B_μ) . We recall that ℓ is the line of intersection of μ and Π , and ℓ meets δ in L . Let m_1 and m_2 be two lines of μ . We have to consider two cases:

Case I. Let m_1 and m_2 meet in point Q not on ℓ . If δ_1 and δ_2 are the planes representing the $\alpha\delta T$ -points corresponding to m_1 and m_2 , then they contain the lines m_1 and m_2 , and therefore the point Q . Hence the unique $\alpha\delta T$ -line incident with the $\alpha\delta T$ -points represented by δ_1 and δ_2 , is represented by the point Q . Denote by (Δ_Q) the set of $\alpha\delta T$ -points incident with the $\alpha\delta T$ -line of type (1) represented by the point Q . Then the subline determined by the $\alpha\delta T$ -points corresponding to m_1 and m_2 is the set $(\ell_Q) = (B_\mu) \cap (\Delta_Q)$. Any two $\alpha\delta T$ -points of (B_μ) whose corresponding lines pass through Q determine the same subline. Hence to each point Q of μ , not on ℓ , there corresponds a unique subline of (B_μ) , which is a subset of the $\alpha\delta T$ -points incident with the $\alpha\delta T$ -line represented by Q . These sublines will be said to be of the first type. In the finite case, there are q^2 sublines of the first type, since this is the number of points on μ , not belonging to ℓ .

Case II. Let m_1 and m_2 meet at a point S' on ℓ ($S' \neq L$). Let δ' be the line of S ($\delta' \neq \delta$), passing through S' . Then the $\alpha\delta T$ -point

corresponding to m_i ($i = 1, 2$) is represented by the plane δ_i containing s' and m_i . Hence the unique $\alpha\delta T$ -line incident with the $\alpha\delta T$ -points represented by δ_1 and δ_2 is represented by s' . Denote by (Δ_s) the set of $\alpha\delta T$ -points incident with the $\alpha\delta T$ -line of type (2) represented by s' , then the subline determined by the $\alpha\delta T$ -points corresponding to m_1 and m_2 is the set $(\ell_S) = (B_\mu) \cap (\Delta_s)$. Note there is a (1,1) correspondence between S' and s' . Any two $\alpha\delta T$ -points whose corresponding lines pass through S' determine the same subline. Hence to each point S' of μ on ℓ ($S' \neq L$), there corresponds a unique subline of the subplane represented by μ , which is a subset of the $\alpha\delta T$ -line represented by s' , the line of S passing through S' . These sublines will be said to be of the second type. In the finite case, their number is q . Taking the two cases together, we see that to each point of μ (other than L , the point of intersection of μ and s), there corresponds a unique subline.

The set of points (B_μ) , together with the set of sublines of (B_μ) , constitutes a subplane of the affine plane $\alpha\delta T$. It is clear that the $\alpha\delta T$ -line of which (ℓ_Q) or (ℓ_S) is a subset is incident in $\alpha\delta T$ with an $\alpha\delta T$ -point of (B_μ) , if and only if the subline (ℓ_Q) or (ℓ_S) contains this $\alpha\delta T$ -point. Hence incidence in the subplane can be naturally defined by the containing contained relation, and is carried over from $\alpha\delta T$. The line m of μ (m not passing through L) may be called the image of the $\alpha\delta T$ -point of (B_μ) to which it corresponds, and the point Q not on ℓ , or a point S' on ℓ ($S' \neq L$) may be called the image of the subline (ℓ_Q) or (ℓ_S) . Then a subline of (B_μ) is incident with an $\alpha\delta T$ -point of (B_μ) , if and only if the point which is the image of the subline is contained in the line which is the image of the $\alpha\delta T$ -point.

4. Two sublines of (B_μ) may be defined to be parallel if they have no element in common. We shall show that the parallelism in $\alpha\Delta T$ is carried over to the subplane represented by μ . There are three cases to be considered.

Case I. Let Q_1 and Q_2 be two points of μ , not on ℓ . Then Q_1 and Q_2 are images of sublines (ℓ_{Q_1}) and (ℓ_{Q_2}) of the first type. If the line Q_1Q_2 , does not intersect s , then $m = Q_1Q_2$ is the image of a unique $\alpha\delta T$ -point of (B_μ) common to (ℓ_{Q_1}) and (ℓ_{Q_2}) . If, however, Q_1Q_2 meets s , at a point L , then (ℓ_{Q_1}) and (ℓ_{Q_2}) have no common $\alpha\Delta T$ -point and are parallel. However, in this case, the plane ρ containing Q_1Q_2 and s represents a δT -point of the derivation set, so that the $\alpha\delta T$ -lines represented by Q_1 and Q_2 belong to the parallel pencil whose vertex is represented by ρ .

Case II. Let Q be a point of μ , not on ℓ and S' be a point of μ , on ℓ ($S' \neq L$). Then Q is the image of a subline (ℓ_Q) of the first type and S' is the image of a subline $(\ell_{S'})$ of the second type. They are not parallel, and contain the unique $\alpha\delta T$ -point of (B_μ) , whose image is the line $S'Q$.

Case III. Let S'_1 and S'_2 be two points of μ both on ℓ , but different from L . Then S'_1 and S'_2 are images of sublines $(\ell_{S'_1})$ and $(\ell_{S'_2})$. These sublines do not have any common $\alpha\Delta T$ -point, and are parallel, since the line of μ joining S'_1 and S'_2 is ℓ , which is not the image of an $\alpha\Delta T$ -point. However, in this case, the 3-space Σ represents a δT -point of the derivation set. Consider the δT lines represented by the lines s'_1 and s'_2 of S , passing through S'_1 and S'_2 . These are the δT -lines of which $(\ell_{S'_1})$ and $(\ell_{S'_2})$ are subsets. They belong to the parallel pencil whose vertex is represented by Σ .

We have now shown that any two sublines of (B_μ) which are parallel are subsets of points incident with $\alpha\delta T$ -lines which are themselves parallel being incident with a δT -point of the derivation set. Hence the affine subplane (B_μ) represented by μ can be completed to a projective plane (B_μ^0) by adjoining the δT -points of the derivation set, and a new subline consisting of the points of the derivation set. This subline (the subline at infinity in (B_μ)) can be appropriately denoted by (ℓ_δ) and is a subset of the δT -points incident with the δT -line δ , which was chosen as the line at infinity in δT , to obtain $\alpha\delta T$.

It is readily verified that (B_μ^0) is a Baer subplane of δT , i.e., each δT -line is incident with at least one δT point of (B_μ^0) , and each δT -point is incident with a δT -line intersecting (B_μ^0) in a subline. In the finite case, this follows directly from the fact that the order of δT is q^2 and the order of (B_μ^0) is q . In the infinite case, a geometrical argument is necessary.

5. We are now ready to obtain a new affine plane from the affine plane $\alpha\delta T$. This will turn out to be the same as the affine plane $\alpha\Delta$ of Section 3. Let the points of $\alpha\Delta$ be the same as the points of $\alpha\delta T$. Thus $\alpha\Delta$ -points are represented by planes of S_4 passing through lines of S other than δ , and which are not contained in Σ . We, however, delete all lines of $\alpha\Delta T$, which belong to any parallel pencil whose vertex is a δT -point of the derivation set. Thus the deleted lines are represented by (i) points contained in some plane ρ of Ω passing through δ and not contained in δ , i.e., all points of Ω not in Π ; (ii) lines of S other than δ . In the finite case, there are $q^3 + q^2$ deleted $\alpha\delta T$ -lines, since there are q^3 points in Ω other than points of Π , and there

are q^2 lines of S , other than s . The lines of $\alpha\delta T$ which are not deleted are also lines of $\alpha\Delta$ and are called lines of type (1), incidence between undeleted $\alpha\delta T$ -lines and $\alpha\delta T$ -points being carried over to $\alpha\Delta$. The deleted $\alpha\delta T$ -lines are replaced by the subplanes represented by the planes of Ω not passing through s , which are now called $\alpha\Delta$ -lines of type (2). An $\alpha\Delta$ -point is incident with a line of type (2) represented by μ if the corresponding $\alpha\delta T$ -point belongs to the set (B_μ) . It is now readily seen that we have obtained the same representation of points and lines, and the same definition for incidence for $\alpha\Delta$ as was given in Section 3.

In the finite case, the procedure by which we have obtained $\alpha\Delta$ from $\alpha\delta T$ has been called derivation by Albert [1] and Ostrom [10], [12], [5, p.223]. They have shown that if a finite projective plane of order q^2 is made affine by choosing a line at infinity s , and if in s we can find a set (called the derivation set) of $q+1$ points $\Sigma, \rho_1, \rho_2, \dots, \rho_q$ such that there exists a set of affine Baer subplanes (whose projective completion contains all the points of the derivation set), and which have the further property that any two affine points collinear with a point of the derivation set are contained in exactly one Baer subplane of the set then we can obtain a new affine plane, by deleting the q^3+q^2 affine lines passing through the points of the derivation set and replacing them by the Baer subplanes. The original plane can in this case be called derivable and the new plane is called the derived of this plane. The subplanes defined by us are Baer subplanes of $\alpha\delta T$. We have already shown that the projective completion of any affine Baer subplane represented by a plane μ of Ω , contains the δT -points of the derivation set defined by us. Thus to show that the affine plane $\alpha\Delta$ is the derived of the dual translation plane δT (with respect to the given derivation set) we have only to show that

given any two $\alpha\delta T$ -points collinear in δT , with a δT -point of the derivation set, they are contained in a unique Baer subplane. This means that we have to show that any two $\alpha\delta T$ -points belonging to a deleted $\alpha\Delta T$ -line are contained in a unique Baer subplane of our set.

Case I. Let the deleted $\alpha\Delta T$ -line be of the type (1). Then it is represented by a point Q of Ω (where Q is not a point of Δ). Any two $\alpha\Delta T$ -points incident with this $\alpha\delta T$ -line are represented by planes δ_1 and δ_2 (not contained in Σ) passing through different lines of S (other than Δ) and intersecting in Q . We have proved in Section 3, Lemma (1), Case II (b), that there is exactly one $\alpha\Delta$ -line of type (2) incident with the $\alpha\Delta$ -points represented by δ_1, δ_2 . This is precisely equivalent to saying that any two $\alpha\delta T$ -points incident with an $\alpha\delta T$ -line of type (1) represented by a point Q of Ω , are contained in a unique Baer subplane of the set considered.

Case II. Let the deleted $\alpha\delta T$ line be of type (2), and be represented by a line Δ' of S ($\Delta' \neq \Delta$). Any two $\alpha\delta T$ -points incident with this $\alpha\delta T$ -line are represented by planes δ_1 and δ_2 passing through Δ (and not contained in Σ). We have proved in Section 3, Lemma (1), Case I, that there is exactly one $\alpha\Delta$ -line of type (2), incident with the $\alpha\Delta$ -points represented by δ_1, δ_2 . This is precisely equivalent to saying that any two $\alpha\Delta T$ -points incident with an $\alpha\Delta T$ -line of type (2), are contained in a unique Baer subplane of the set considered.

Thus in the finite case, the proof that $\alpha\Delta$ is an affine plane could have been accomplished by proving that the sets (B_μ) represented affine Baer subplanes in $\alpha\Delta T$, together with an appeal to the theorem of Albert and Ostrom, coupled with a rephrasing of Lemma (1) of Section 3. We, however, chose to give a direct proof in Section 3, in order to include the

infinite case. It is clear that even in the infinite case the relation between $\alpha\delta T$ and $\alpha\Delta$ is quite analogous to that in the finite case.

6. We conclude this section with some final remarks. It should be noted that the translation plane T which forms the starting point of our investigation is restricted to those translation planes which can be represented in a 4-dimensional projective space by the Bose-Bruck construction given in [2]. These are exactly the planes for which there exists a coordinating right Veblen-Wedderburn system, which has dimension 2 over some skew-field F contained in the left-operator skew-field of R . These planes have been extensively studied by Bruck in [4]. We shall call them translation planes of degree 2. Note that there is no restriction on the spread S in Σ to which the translation plane T is associated. Thus the spread may be subregular in the sense considered by Bruck in [4], but it need not be. For example, it may be the spread associated with the names of Segre, Tits, Lüneburg and Suzuki -- which is far from being subregular [4], [9], [14]. Again it may be a Foulser type spread [6], [11]. The dual of a translation plane of degree 2 may be called a dual translation plane of degree 2.

In the translation plane T , there is a special line Σ , such that T is (Σ, δ) transitive for any point δ of T on Σ . (The line Σ is uniquely determined except when T is a Moufang plane or is Desarguesian, when Σ is arbitrary). If T is of degree 2, then in the Bose-Bruck representation of T , the T -line Σ , is represented by the 3-space Σ containing the spread S , through the lines of which pass planes which represent T -lines. In the dual plane δT , Σ becomes a special point such that δT is (Σ, δ) transitive for any line δ containing Σ . In our representation of δT , the δT -point Σ is represented by the 3-space

Σ , and the line δ represents a particular line of δT incident with the δT -point Σ . The planes ρ_1 and ρ_2 through δ , determining Ω represent two arbitrary δT -points collinear with the δT -point Σ . We have thus proved the following:

Theorem 1. Every finite dual translation plane δT of degree 2 is derivable. If Σ is a point of δT such that δT is (Σ, δ) transitive for any line δ incident with Σ , then Σ together with any two points ρ_1 and ρ_2 of δT collinear with it, can be embedded in a derivation set (Σ is a unique point, except when δT is a Desarguesian plane).

A procedure analogous to derivation is still applicable when δT is an infinite dual translation plane of degree 2. Let Σ be a point of δT , such that δT is (Σ, δ) transitive for any line δ incident with Σ . Given any two points ρ_1 and ρ_2 collinear with Σ , we can find a set of points (ρ) , collinear with Σ and including Σ , ρ_1 and ρ_2 , such that a new affine plane $\alpha\Delta$ can be obtained by deleting the lines of the parallel pencils with vertices belonging to (ρ) , and replacing them by new lines corresponding to certain affine Baer subplanes. $\alpha\Delta$ then can be completed to a projective plane (Σ is a unique point except when δT is a Moufang plane or is Desarguesian).

The choice of ρ_1 and ρ_2 will influence the structure of the derived plane. Thus it should be possible to derive non-isomorphic planes from the same dual affine plane of degree 2.

5. SEQUENCES OF PLANES OBTAINABLE FROM TRANSLATION PLANES OF DEGREE 2, BY DUALIZATION AND DERIVATION

1. We shall now show that the process of dualization and derivation by which we obtained the plane Δ from T can be continued if the line δ of the spread S , belongs to a regulus all of whose lines are in S . This gives the geometrical representation of the analogous sequences of planes obtained by Fryxell [7] and Johnson [8]. Although our procedure is applicable in the infinite case, we shall confine ourselves only to the finite case, in the first instance.

2. Let us start with the Δ -plane constructed in Section 3. We can dualize it and obtain a new plane $\delta\Delta$ such that the lines of Δ are the points of $\delta\Delta$ and the points of Δ are the lines of $\delta\Delta$. We shall show that $\delta\Delta$ is derivable if there exists in the spread S , a regulus R such that δ belongs to R . R consists of $q+1$ lines which are one set of ruling lines of a hyperbolic quadric. Let the lines of R other than δ be denoted by δ_i ($i = 1, 2, \dots, q$). Let R^* be the regulus conjugate to R , the lines of R^* being the second set of ruling lines of the quadric under consideration. Since Π passes through δ , it is a tangent plane to the quadric, and contains exactly one line of R^* say t , meeting δ at the point O . Let the other lines of R^* be denoted by t_j ($j = 1, 2, \dots, q$). We shall denote the intersection of δ_i and t by S_i , the intersection of t_j and δ by T_j , and the intersection of δ_i and t_j by P_{ij} ($i, j = 1, 2, \dots, q$).

We shall choose the $\delta\Delta$ line represented by O , as the line at infinity in $\delta\Delta$. Then the points at infinity in $\delta\Delta$ are represented by

(i) the q^2 planes through O contained in Ω but not containing s and (ii) the line s . Let μ_i ($i = 1, 2, \dots, q$) be the planes of Ω , other than Π , passing through the line t . We shall choose as our derivation set the $\delta\Delta$ -points at infinity represented by the line s and the planes $\mu_1, \mu_2, \dots, \mu_q$.

We can convert $\delta\Delta$ into an affine plane $\alpha\delta\Delta$ by deleting the $\delta\Delta$ -line represented by O , and the $\delta\Delta$ -points incident with it. Then $\alpha\delta\Delta$ -points are of two types. Points of type (1) are represented by points of the set (R) , i.e., points of S_4 not contained in Σ or Ω , their number being $q^4 - q^3$. Points of type (2) are represented by planes of Ω , not passing through O , q^3 in number. Again $\alpha\delta\Delta$ -lines are of three types. Lines of type (1) are represented by planes through lines of S other than s , and not contained in Σ , their number being q^4 . Lines of type (2) are represented by planes through s not contained in Ω or Σ , $q^2 - q$ in number. Lines of type (3) are represented by the q points of s other than O .

Now we can divide the $\alpha\delta\Delta$ -lines into parallel pencils in the usual manner. $\alpha\delta\Delta$ -lines belonging to parallel pencils whose vertices are $\delta\Delta$ -points of the derivation set will be deleted and replaced by certain (affine) Baer subplanes of $\alpha\delta\Delta$. Thus there are $q+1$ deleted pencils each consisting of q^2 $\alpha\delta\Delta$ -lines.

3. In what follows, we shall consider three different types of (affine) Baer subplanes, which replace the deleted lines. We shall first define these subplanes and prove the appropriateness of the designation subplane later.

(a) Let δ^* be a plane through t_j not contained in Σ . Then δ^* is not contained in Ω , since it contains only the point T_j of Π . Hence δ^* intersects Ω in a line m through T_j , where m is not contained in Π . Then δ^* represents a Baer subplane of type (a). The set of q^2 $\alpha\delta\Delta$ -points belonging to this subplane may be denoted by (B_{δ^*}) . (B_{δ^*}) consists of the $q^2 - q$ $\alpha\delta\Delta$ -points represented by the points of δ^* not contained in the lines t_j or m and (ii) the q $\alpha\delta\Delta$ -points represented by planes of Ω passing through m but not through O . Through t_j there pass q^2 planes not contained in Σ . Since $1 \leq j \leq q$, there are q^3 subplanes of type (a).

(b) Let λ^* be a plane through t , not contained in Σ or Ω . Then λ^* represents a Baer subplane of type (b). The set of $\alpha\delta\Delta$ -points belonging to this may be denoted by (B_{λ^*}) . (B_{λ^*}) consists of the q^2 $\alpha\delta\Delta$ -points represented by points of λ^* not belonging to t . Since through t there pass $q^2 - q$ planes not belonging to Σ or Ω , there are $q^2 - q$ subplanes of type (b).

(c) Let S_i be a point of t , other than O . Then S_i represents a Baer subplane of type (c). The set of $\alpha\delta\Delta$ -points belonging to it may be denoted by (B_{S_i}) . (B_{S_i}) consists of the q^2 $\alpha\delta\Delta$ -points represented by planes of Ω passing through S_i but not through O . There are q subplanes of type (c).

Let (B_α) be the set of points belonging to a subplane of type (a), (b) or (c) of $\alpha\delta\Delta$ represented by the element α of S_4 . Also let $(\delta\Delta_\nu)$ be the set of points incident with the $\alpha\delta\Delta$ -line of type (1), (2), or (3), represented by the element ν of S_4 . Then the subset $(B_\alpha) \cap (\delta\Delta_\nu)$ is defined to be the subline of (B_α) corresponding to the $\alpha\delta\Delta$ -line represented by ν , provided that the subset contains at least two $\alpha\delta\Delta$ -points.

Any two $\alpha\delta\Delta$ -points of (B_α) are incident with a unique $\alpha\delta\Delta$ -line, represented by an element v of S_4 , and hence determine the unique subline $(B_\alpha) \cap (\delta\Delta_v)$.

Two sublines of (B_α) will be defined to be parallel if they have no point in common. Consider two sublines $(B_\alpha) \cap (\delta\Delta_{v_1})$ and $(B_\alpha) \cap (\delta\Delta_{v_2})$ corresponding to the $\alpha\delta\Delta$ -lines represented by the elements v_1 and v_2 of S_4 . We shall show that the sublines are parallel if the corresponding $\alpha\delta\Delta$ -lines are parallel. Otherwise, they intersect in the unique $\alpha\delta T$ -point incident with the corresponding $\alpha\delta T$ -lines. This will show that (B_α) together with the sublines is an affine subplane of $\alpha\delta\Delta$. Since $\alpha\delta\Delta$ is of order q^2 and (B_α) is of order q , (B_α) must be a Baer subplane.

4. In the notation of paragraph 3, let δ^* be a plane through t_j (not contained in Σ) representing a subplane of type (a). We shall study the sublines of (B_{δ^*}) . Let v be a plane representing an $\alpha\delta\Delta$ -line of type (1), which is incident with at least two $\alpha\delta\Delta$ -points of (B_{δ^*}) . Then v must intersect δ^* in a line ℓ . Recall that δ^* meets Ω in the line m , where m and t_j intersect at the point T_j of δ . Then ℓ meets m at the point M , and t_j at the point P_{ij} , ($M \neq T_j$, $P_{ij} \neq T_j$). Through P_{ij} there passes a unique line s_i of S , which meets t in S_i . Thus v must be the plane joining s_i and M , and must intersect Ω in the line $d = MS_i$. Let γ be the plane T_jMS_i containing m and S_i . The $\alpha\delta\Delta$ -point represented by γ belongs to both (B_{δ^*}) and $(\delta\Delta_v)$. The subline $(\ell) = (B_{\delta^*}) \cap (\delta\Delta_v)$ consists of the $\alpha\delta\Delta$ -point represented by γ , and the $q-1$ $\alpha\delta\Delta$ -points represented by points of ℓ , other than P_{ij} and M . This subline may be said to be of type (al), and may be appropriately represented by ℓ . (B_{δ^*}) has q^2 sublines of this type, represented by lines of δ^* , not passing through T_j .

Again let λ be a plane representing an $\alpha\delta\Delta$ -line of type (2), which is incident with at least two $\alpha\delta\Delta$ -points of $(B_{\delta*})$. Then λ must meet δ^* in a line k . Since λ passes through δ , k must meet δ in the point T_j which is the only point common to δ^* and δ . The subline $(k) = (B_{\delta*}) \cap (\delta\Delta_\lambda)$ consists of the q $\alpha\delta\Delta$ -points represented by the q points of k , other than T_j . This subline may be said to be of the type (a2), and may appropriately be represented by the line k . $(B_{\delta*})$ has $q-1$ sublines of this type represented by lines of δ^* passing through T_j , other than t_j or m .

Finally, the only $\alpha\delta\Delta$ -line of type (3), which is incident with at least two $\alpha\delta\Delta$ -points of $(B_{\delta*})$ is the $\alpha\delta\Delta$ -line represented by T_j . The subline $(m) = (B_{\delta*}) \cap (\delta\Delta_{T_j})$ consists of the q $\alpha\delta\Delta$ -points represented by planes of Ω through m , not passing through O . Thus there is only one subline of $(B_{\delta*})$ of type (a3). It is represented by the point T_j .

It is clear that if two sublines have an $\alpha\delta\Delta$ -point in common, then this must be the unique $\alpha\delta\Delta$ -point incident with both the corresponding $\alpha\delta\Delta$ -lines. If two sublines of $(B_{\delta*})$ both of type (a1), or one of type (a1) and the other of type (a2), meet at a point A of δ^* , not on t_j or m , then clearly A represents the $\alpha\delta\Delta$ -point common to the two sublines.

Consider two sublines of $(B_{\delta*})$ of type (a1), represented by lines ℓ and ℓ' meeting at the point P_{ij} of t_j . Let ℓ and ℓ' meet m in M and M' . Let δ_i be the line of S passing through P_{ij} and meeting t at the point S_i . Let δ and δ' be the planes representing $\alpha\delta\Delta$ -lines corresponding to ℓ and ℓ' . Then the $\alpha\delta\Delta$ -point represented by the plane γ containing m and S_i belongs to both the sublines.

Next consider two sublines of $(B_{\delta*})$ of type (a1), represented by the lines ℓ and ℓ_0 and meeting m at the point M . Let ℓ and ℓ_0 meet

t_j at the points P_{ij} and P_{kj} . Let s_i and s_k be the line of S , passing through P_{ij} and P_{kj} respectively and meeting t at S_i and S_k respectively. Then the $\alpha\delta\Delta$ -points of type (2) belonging to ℓ and ℓ_0 are represented by the planes γ and γ' joining m with S_i and S_k respectively. Thus the sublines represented by ℓ and ℓ_0 have no $\alpha\delta\Delta$ -point in common, and are therefore parallel. The corresponding $\alpha\delta\Delta$ -lines are represented by the planes δ and δ_0 joining M to s_i and s_k respectively. The $\alpha\delta\Delta$ -lines are parallel, and meet the line at infinity in the $\delta\Delta$ -point represented by the plane $S_i S_k M$ (which is an element of the derivation set).

Consider now the intersection of the subline represented by ℓ (defined as before) and the subline of type (a3) represented by the point T_j . The plane $\gamma = MS_i T_j$ joining m and S_i , represents an $\alpha\delta\Delta$ -point common to both the sublines.

Finally it is readily seen that two sublines both of type (a2) or one of type (a2) and the other of type (a3) are parallel. The corresponding $\alpha\delta\Delta$ -lines are also parallel meeting the line at infinity in the $\delta\Delta$ -point represented by s (which is an element of the derivation set).

This completes the proof that $(B_{\delta*})$ together with its sublines is an affine plane of order q , with the same incidence and parallelism as $\alpha\delta\Delta$. Hence it is a (affine) Baer subplane of $\alpha\delta\Delta$.

In the same way, we can prove that the sets $(B_{\lambda*})$ and (B_{S_i}) defined in paragraph 3, together with their sublines are (affine) Baer subplanes of $\alpha\delta\Delta$. Details of the proof will be omitted.

5. In the previous two paragraphs, we have defined a set of $q^3 + q^2$ affine Baer subplanes of $\alpha\delta\Delta$, q^3 of type (a), $q^2 - q$ of type (b) and q of

type (c). We shall now show:

Lemma (7). Any two points of a deleted line (i.e., an $\alpha\delta\Delta$ -line which meets the line at infinity in a point of the derivation set) are contained in a unique Baer subplane of the above set.

The $\delta\Delta$ -points at infinity belonging to the derivation set are represented by the line δ , and the q planes of Ω passing through t . Let μ be one of these planes. An $\alpha\delta\Delta$ -line, the point at infinity on which is represented by μ , must be of type (1). It is represented by a plane δ meeting μ in a line d . Let d and t intersect in S_i and let δ_i be the line of S passing through S_i . Thus δ is the plane containing d and δ_i .

(i) Consider two $\alpha\delta\Delta$ -points of type (1), incident with the $\alpha\delta\Delta$ -line represented by δ . Let them be represented by the points P and P' of δ . Suppose first that the line $\ell = PP'$ meets δ_i at the point P_{ij} , other than S_i . Through P_{ij} there passes a ruling line t_j belonging to R^* , and meeting δ at the point T_j . Then the plane δ^* containing t_j and ℓ represents the unique Baer subplane (B_{δ^*}) which contains the given $\alpha\delta\Delta$ -points. It is of type (a). Next suppose that the line $\ell = PP'$ passes through S_i . Then the plane λ^* containing ℓ and t represents the unique Baer subplane (B_{λ^*}) containing the given $\alpha\delta\Delta$ -points. It is of type (b).

(ii) Consider two $\alpha\delta\Delta$ -points, one of type (1) and the other of type (2) incident with the $\alpha\delta\Delta$ -line represented by the plane δ . Let them be represented by the point P of δ , and the plane γ of Ω . Since d is the line of intersection of δ and Ω , d is contained in γ . Hence γ

meets Π in a line $S_i T_j$ where T_j is a point of s , other than 0. Through T_j there passes a line t_j of R^* meeting s_i in P_{ij} . Let the line joining P and P_{ij} meet d in M . Let m be the line $T_j M$. Then δ^* the plane containing t_j and m represents the unique Baer subplane (B_{δ^*}) which contains the given $\alpha\delta\Delta$ -points. It is of type (a).

(iii) Next consider two $\alpha\delta\Delta$ -points both of type (2) incident with the $\alpha\delta\Delta$ -line represented by the plane δ . Let them be represented by the planes γ and γ' , which must both pass through the line d , and meet Π in the lines $S_i T_j$ and $S_i T_k$ where $T_j \neq 0, T_k \neq 0$. Then S_i represents the unique Baer subplane (B_{S_i}) , which contains the given $\alpha\delta\Delta$ -points. It is of type (c).

Let us now consider $\alpha\delta\Delta$ -lines for which the point at infinity is represented by s . Such a line may be of type (2) or type (3).

(iv) Let λ be a plane representing an $\alpha\delta\Delta$ -line of type (2). Then λ contains the line s . Let P_1 and P_2 be two points of λ . We have to consider two separate cases. Let us first suppose that the line $P_1 P_2$ meets s at a point T_j other than 0. Let t_j be the line of R^* passing through T_j . Let δ^* be the plane containing the lines t_j and $P_1 P_2$. Then δ^* represents the unique Baer subplane (B_{δ^*}) which contains the $\alpha\delta\Delta$ -points represented by P_1 and P_2 . It is of type (a). Next suppose that the line $P_1 P_2$ meets the line s at the point 0. Recall that t passes through 0. Let λ^* be the plane containing the lines t and $P_1 P_2$. Then (B_{λ^*}) is the unique Baer subplane which contains the $\alpha\delta\Delta$ -points represented by P_1 and P_2 . It is of type (b).

(v) Let T_j be a point representing an $\alpha\delta\Delta$ -line of type (3). Then T_j is a point of s , other than 0. Let γ_1 and γ_2 be two planes of Ω representing $\alpha\delta\Delta$ -points of type (2) incident with the $\alpha\delta\Delta$ -line represented

by T_j . Then m the line of intersection of γ_1 and γ_2 passes through T_j . Also let t_j be the line of R^* passing through T_j . Let δ^* be the plane containing t_j and m . Then δ^* represents the unique Baer subplane (B_{δ^*}) containing the $\alpha\delta\Delta$ -points represented by γ_1 and γ_2 . It is of type (a). This completes the proof of Lemma (7).

b. We can now use the theorem of Albert and Ostrom already referred to in Section 4, paragraph 5, to derive a new affine plane from $\alpha\delta\Delta$, which can then be completed to a projective plane. Before proceeding to do this, we shall make two observations necessary for elucidating the structure of the new plane.

Let $S_0 = S - R$, be the set of lines obtained by deleting the lines belonging to the regulus R from the lines of the spread S . Let $S^* = S_0 \cup R^*$. Then S^* is a spread since the lines of R and R^* contain exactly the same points. The lines $s, s_1, \dots, s_q$ of S merely get replaced by $t, t_1, \dots, t_q$ to give S^* .

Note that for an affine Baer subplane of type (a) or (b), represented by the plane σ^* or λ^* , an $\alpha\delta\Delta$ -point of type (1), represented by P is contained in (B_{σ^*}) or (B_{λ^*}) if and only if P is a point of the plane σ^* or γ^* . Again an $\alpha\delta\Delta$ -point of type (2) represented by the plane γ is contained in (B_{σ^*}) or (B_{λ^*}) if and only if γ intersects the plane σ^* or λ^* in a line.

Similarly for an affine Baer subplane of type (c) represented by the point of S_1 , the set (B_{S_1}) contains $\alpha\delta\Delta$ -points of type (2) only. An $\alpha\delta\Delta$ -point of type (2) represented by the plane γ is contained in the (B_{S_1}) if and only if the point S_1 is contained in the plane γ .

We now apply the procedure of Albert and Ostrom to the affine plane $\alpha\delta\Delta$. The points of $\alpha\delta\Delta$ are retained but the q^3+q^2 lines of $\alpha\delta\Delta$ which belong to parallel pencils, having one of the $\delta\Delta$ -points of the derivation set vertex, are deleted and replaced by the q^3+q^2 affine Baer subplanes obtained in paragraphs 4 and 5. It follows from Lemma (7) and Ostrom's theorem, that we obtain a new affine plane which we may appropriately denote by $\alpha\delta\Delta^*$ since it bears exactly the same relation to the spread S^* as $\alpha\delta\Delta$ bears to S . The points of $\alpha\delta\Delta^*$ are the same as the points of $\alpha\delta\Delta$, and the lines are of three types: Lines of type (1) are represented by planes through lines of S^* , other than t , and not contained in Σ , their number being q^4 . Lines of type (2) are represented by planes through t not contained in Ω or Σ , q^2-q in number. Lines of type (3) are represented by the q points of t other than 0 . Incidence is always given by the containing contained relation except between $\alpha\delta\Delta^*$ -points of type (2), and $\alpha\delta\Delta^*$ -lines of type (2) or (3), when incidence means that the representative planes intersect in a line.

We can now complete $\alpha\delta\Delta^*$ to a projective plane $\delta\Delta^*$. The line at infinity in $\delta\Delta^*$ is obtained from the line at infinity in $\delta\Delta$ by replacing the $q+1$ $\delta\Delta$ -points of the derivation set by new $\delta\Delta^*$ -points represented by the line t , and the q planes of Ω which contain s , but not t . If we denote by (D^*) the set of new $\delta\Delta^*$ -points, and incidence is defined as before, i.e., by the containing contained relation except in the case when a $\delta\Delta^*$ -point and a $\delta\Delta^*$ -line are both represented by planes, when incidence means that the representative planes intersect in a line, then it is readily verified that the element of S_4 representing any affine Baer subplane of our set is incident with exactly one element of (D^*) . The affine Baer subplane can therefore be completed to a projective line, by adding this

element of (D^*) . Any undeleted line of $\alpha\delta\Delta$ is a line of $\alpha\delta\Delta^*$ also. It was obtained in the first place by deleting a point at infinity from a corresponding $\delta\Delta$ line. This is now again added to it. In this way, we get a projective plane $\delta\Delta^*$, which is the derived of $\delta\Delta$ and bears exactly the same relation to the spread S^* as the plane $\delta\Delta$ does to S .

7. We can now dualize $\delta\Delta^*$ to obtain a plane Δ^* , where the lines of the one are points of the other and vice versa.

Since derivation is an involuntary process (i.e., if Π_1 is a derived of Π_2 , then Π_2 is a derived of Π_1) and since Δ^* bears the same relation to the spread S^* as Δ to S , we can by reversing the procedure of Section 4, derive from Δ^* a dual translation plane δT^* . Dualizing δT^* , we obtain the translation plane T^* . Finally, we can derive T from T^* by using the 'switching' process of Bruck and Bose [2], which is equivalent to derivation in which the line at infinity in T^* is represented by Σ , the points at infinity by the lines of S^* , and the derivation set by the lines of R^* . We thus get a cycle of eight planes, $T, \delta T, \Delta, \delta\Delta, \delta\Delta^*, \Delta^*, \delta T^*, T^*$ the relation between which can be schematically represented as

$$\begin{array}{ccccccc}
 & \delta & & D & & \delta & \\
 T & \rightarrow & \delta T & \rightarrow & \Delta & \rightarrow & \delta\Delta \\
 & & & & & & \\
 D \uparrow & & & & & & D \downarrow \\
 & \delta & & D & & \delta & \\
 T^* & \leftarrow & \delta T^* & \leftarrow & \Delta^* & \leftarrow & \delta\Delta^*
 \end{array}$$

where $\overset{\delta}{\rightarrow}$ stands for dualization, and $\overset{D}{\rightarrow}$ for derivation. Of course the direction of the arrows could have been reversed.

Finally we note that even in the infinite case we can obtain $\delta\Delta^*$ from $\delta\Delta$ by a procedure analogous to derivation, under the assumption that the spread S , contains a regulus R , and the line δ is chosen to belong to R . We can thus enunciate the following theorem (compare with Fryxell [7] and Johnson [8]).

Theorem 2. Given a translation plane $T = \Pi_1$, which can be linearly represented in 4-dimensional projective space by the Bose-Bruck construction of Section 2, and if the spread S defined there contains a regulus R then we can obtain a cyclic sequence $\{\Pi_i\}$, $i = 1, 2, 3, \dots$ of projective planes, such that $\Pi_i = \Pi_{i+8}$, Π_{2i} is the dual of Π_{2i-1} , and Π_{2i+1} is the derived of Π_{2i} , provided that in the infinite case we further assume that each plane of Σ contains at least one line of S .

ACKNOWLEDGMENT

The authors acknowledge with pleasure the helpful discussions with Mrs. J.C.D.S. Yaqub of Ohio State University.

REFERENCES

1. Albert, A.A. "The finite planes of Ostrom". *Boletín de la Sociedad Matemática Mexicana*, 11 (1966), 1-13.
2. Bruck, R.H. and Bose, R.C. "The construction of translation planes from projective spaces". *J. Algebra*, 1 (1964), 85-102.
3. Bruck, R.H. and Bose, R.C. "Linear representations of projective planes in projective spaces". *J. Algebra*, 4 (1966), 117-172.
4. Bruck, R.H. "Construction problems of finite projective planes". *Combinatorial Mathematics and its Applications. University of North Carolina Monograph Series in Probability and Statistics*, Chapter 27 (1969), 416-514.
5. Dembowski, P. *Finite Geometries*. Springer Verlag, New York Inc. 1968.
6. Foulser, D. "Solvable flag transitive groups". *Math. Z.*, 86 (1964), 191-204.
7. Fryxell, R. "Sequences of planes constructed from nearfield planes of square order". Ph.D. Thesis, Washington State University (1964).
8. Johnson, N.L. "Derivable chains of planes". *Notices, Am. Math. Soc.*, 16 (1969), 153.
9. Lüneburg, H. "Die Suzukigruppen und ihre Geometrien." *Lecture notes in Mathematics*, 10. Springer Verlag, Berlin-Göttingen-Heidelberg-New York, 1965.
10. Ostrom, T.G. "Semi-translation planes". *Trans. Amer. Math. Soc.*, 111 (1964), 1-18.
11. Ostrom, T.G. "Translation planes of order p^2 ". *Combinatorial Mathematics and its Applications. University of North Carolina Monograph Series in Probability and Statistics*, Chapter 26 (1969), 416-425.
12. Ostrom, T.G. "Vector spaces and construction of finite projective planes". *Arch. Math.*, 19 (1968), 1-25.
13. Segre, B. "Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane". *Ann. Mat. Pura Appl.*, 64 (1964), 1-76.
14. Tits, J. "Ovoides et groupes de Suzuki". *Arch. Math.*, 13 (1962), 187-198.