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MULTITYPE BRANCHING PROCESSES IN RANDOM ENVIRONMENTS

by

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ABSTRACT

Consider a population of particles and k fixed particle classifications labeled type 1, type 2, ..., type k . Suppose that to each classification there corresponds exactly one k -variate p.g.f. and that each particle of the population is classified by one and only one type. Further, assume that after each unit of time, each particle, independently of the other particles, splits or disintegrates into particles of several types, in accordance with the k -variate p.g.f. which corresponds to the parent particle's type. Given these hypotheses, the probability of the population ever becoming extinct is known. This process, in fact, is the well-known multitype Galton-Watson process.

In this work, we remove the restrictive assumption that particles of the same type always divide in accordance with the same k -variate p.g.f. Instead, we assume that at each unit of time, Nature chooses a k -vector of k -variate p.g.f.s from a class of k -vectors of k -variate p.g.f.s, independently of the population, past and present, and the previously selected k -vectors, which is then assigned to the present population. Each particle of the present population then splits or disintegrates, independently of the others, in accordance with the k -variate p.g.f. assigned to its

classification. We call this process a multitype branching process in a random environment (MBPRE).

When there is only one particle classification, necessary and sufficient conditions for almost certain extinction of the MBPRE are known [Smith and Wilkinson (Ann. Math. Statist. 40 814-827) and Smith (Ann. Math. Statist. 39 2136-2140)]. In this work, we give some necessary and some sufficient conditions for almost certain extinction of the MBPRE when there are at least two particle classifications. To obtain these results, we use Jensen's inequality, the dual process suggested by Smith and Wilkinson (Ann. Math. Statist. 40 814-827), and some results on products of random matrices. Since our theorems require the user to evaluate limits of sequences of products of random matrices, we include many corollaries which involve simpler, though less general, conditions. Further, when Nature has only a finite number of choices and when the class of moment matrices of the k -vectors of k -variate p.g.f.s enjoys a special property, we give conditions which are nearly necessary and sufficient for almost certain extinction of the MBPRE.

NOTATION AND TERMS

Notation

$[n]$	the <u>nth</u> reference listed in the References
BPRE	the abbreviation for branching process in a random environment (p. 2)
MBPRE	the abbreviation for multitype branching process in a random environment (Definition 2.1, p. 7)
$\underline{1}$	the k -vector $(1, 1, 1, \dots, 1)$
$\underline{e}_j$	the k -vector whose <u>jth</u> coordinate is one and whose other coordinates are zero
$j=a(b)a+Nb$	notation for $j = a, a+b, a+2b, \dots, a+Nb$ where $a, b \geq 0$ and N is a positive integer
$j=a(b)-$	notation for $j = a, a+b, a+2b, \dots$ where $a, b \geq 0$
T	the set of all k -vectors whose coordinates are nonnegative integers
ζ_n	an environmental random variable (Assumption A1, p. 5)
Θ	the environmental space (Assumption A1, p. 5)
m.p.g.f.	the abbreviation for multivariate probability generating function(s)
$\phi(\theta, \underline{z})$	the k -vector of m.p.g.f. associated with the environmental state θ (Assumption A2, p. 5)
$\underline{z}(n)$	the size of the <u>nth</u> generation of the MBPRE (Definition 2.1, p. 7)

- $(P(\underline{a}, \underline{b}))$ the one step transition matrix of the MBPRE
(Defintion 2.1, p. 7)
- $\xi(\theta)$ the matrix of moments of the k -vector of m.p.g.f.
 $\phi(\theta, \underline{g})$ associated with the environmental state θ
(p. 10)
- ξ the nonrandom matrix of moments of the MBPRE;
 $= E\{\xi(\zeta_0)\}$ (p. 11)
- $G(n, \underline{g} | \underline{a})$ the p.g.f. of $\underline{Z}(n)$ given that $\underline{Z}(0) = \underline{a}$
(Definition 3.1, p. 11)
- $\underline{u} > \underline{v}$ each component of the vector $\underline{u} - \underline{v}$ is positive
- $\underline{u} \geq \underline{v}$ each component of the vector $\underline{u} - \underline{v}$ is nonnegative
- $\underline{u} = \underline{v}$ each component of the vector $\underline{u} - \underline{v}$ is zero
- $|\underline{u}|$ the sum of the absolute values of the coordinates
of the vector $\underline{u}$
- $|\underline{A}|$ the sum of the absolute values of the entries of
the matrix $\underline{A}$
- $||\underline{A}||$ $\max_i \sum_j |A(i, j)|$, if $\underline{A}$ is a square matrix
- $\lambda(\underline{A})$ the largest positive eigenvalue of the primitive
matrix $\underline{A}$ (Theorem 4.1, p. 15)
- $q(\underline{a})$ the extinction probability of the MBPRE given that
 $\underline{Z}(0) = \underline{a}$ (Definition 11.1, p. 56)
- $\underline{q}$ the k -vector $(q(\underline{e}_1), q(\underline{e}_2), \dots, q(\underline{e}_k))$
- $\underline{X}(n)$ a random vector of the dual process
(Definition 7.1, p. 24)
- $\log^+ x$ $\max(0, \log x)$, if x is a positive number
- $M(C, N)$ a special class of matrices
(Definition 8.1, p. 27)

r	a critical parameter associated with products of random matrices (Sections 8, 9, and 10)
$\tilde{Y}$	the $\tilde{X}(n)$ process converges in distribution to this random k -vector (Remark 11.1, p. 57)
$F(\tilde{x})$	the ergodic distribution function (Definition 11.2, p. 57)

Term	Definition	Page
dual process	7.1	24
ergodic distribution	11.2	57
extinction probability	11.1	56
impartial MBPRE	4.3	16
insecure MBPRE	5.2	18
primitive matrix	4.2	15
singular MBPRE	9.1	42
transient state	5.1	16

CHAPTER I
INTRODUCTION AND PRELIMINARY RESULTS

1. Introduction

Consider a population of particles and k fixed particle classifications labeled type 1, type 2, ..., type k . Suppose that to each classification there corresponds exactly one k -variate probability generating function and that each particle of the population is classified by one and only one type. Further, assume that after each unit of time, each particle, independently of the other particles, splits or disintegrates into particles of several types, in accordance with the k -variate probability generating function which corresponds to the parent particle's type. Given these hypotheses, the probability of the population ever becoming extinct is known. This process, in fact, is the well-known multitype Galton-Watson process described by Harris [8, p. 34].

If $k=1$, the process described is the classical Galton-Watson process in which each particle is of the same type and each particle splits or disintegrates in accordance with the same given probability generating function. Recently, this model has been generalized by

Wilkinson ([11], [12]) and Smith ([10], [11]). In his thesis, Wilkinson removed the restrictive assumption that the particles always divide in accordance with the same probability generating function. Instead, he proposed that at each unit of time, Nature be allowed to choose a probability generating function from a class of probability generating functions, independently of the population, past and present, and the previously selected probability generating functions, which would then be assigned to the present population. Each particle of the present population would then split, independently of the others, in accordance with this probability generating function. This process, called a branching process in a random environment (BPRE), is, of course, much more applicable than the Galton-Watson process. Moreover, Wilkinson's search for necessary and sufficient conditions for almost certain extinction seems to be not only elegant, but also fairly complete. Further, the conditions are almost as simple to verify as those for the Galton-Watson process. The search for necessary and sufficient conditions for almost certain extinction of the BPRE has been completed now by Smith [10].

It is the purpose of this work to consider the multitype branching process in a random environment (MBPRE); that is, the BPRE process for the case $k \geq 2$. As Wilkinson did with the BPRE process, we endeavored to find necessary and sufficient conditions for almost sure extinction of the population. However, it appears that the determination of

the probability of extinction of the MBPRE is not a simple extension of the techniques used by Wilkinson. Where he was able to consider products of nonnegative random variables, the MBPRE yielded products of nonnegative random matrices. Where he was able to evaluate limits of sequences of products of random variables by applying the strong law of large numbers, we could only find existence theorems for the limits of sequences of products of random matrices which were of little immediate practical value. Where Wilkinson used linear approximations to the probability generating functions of the BPRE to excellent advantage, we were forced to use higher order approximations to the k -variate probability generating functions of the MBPRE which in some cases gave excellent results, but in other cases gave no results. Many of these difficulties are, in fact, stimulating research problems in themselves. Thus, we shall only be able to give some conditions for almost sure extinction and nonextinction of the MBPRE.

Let us begin by defining the process mathematically. We shall then discuss the probability generating functions of the process, the moments of the process, and the transience of its states. Then in Chapter II, we shall again define the dual process (Smith and Wilkinson [11]) and exhibit some conditions for extinction and nonextinction of the process. By including many corollaries and some results of Furstenberg and Kesten [5], we will endeavor to make the rather complex conditions more applicable.

Finally, in Chapter III, we shall introduce the ergodic distribution which is associated with the process and which gives us virtually all the information we require about extinction of the MBPRE. As we proceed, we shall observe, through the many remarks, many of the strengths and weaknesses of the various approaches and methods.

2. Definition of the process

Notation. We shall denote vectors and matrices by lightface letters underscored with a wavy line. Unless specifically noted, we will assume that all vectors (matrices) are of order k ($k \times k$), where k is the fixed positive integer, greater than one, which corresponds to the number of distinct types. A vector premultiplying (postmultiplying) a matrix will be considered a row (column) vector; otherwise we do not distinguish between row and column vectors. Further, if r is any real number, let $\underline{r} = (r, r, \dots, r)$ and if j is any integer, $1 \leq j \leq k$, let $\underline{e}_j$ denote the vector whose j th component is 1 and whose other components are zero. Also, we shall denote the j th component of the vector $\underline{a}(n)$ by $a(j, n)$ for $j = 1, 2, \dots, k$ and the entry in the i th row and j th column of the matrix $\underline{A}(n)$ by $A(i, j, n)$ for $i, j = 1, 2, \dots, k$.

We shall denote the underlying probability space by $(\Omega, \mathcal{A}, P)$, where Ω is some abstract set, $\mathcal{A}$ is a sigma-field of subsets of Ω , and P is a probability measure defined on

A. If m is any positive integer, let B_m denote the Borel sets of the real Euclidean space R^m . Further, if f is defined on a measurable space (E, F) and is a mapping into the m -dimensional real Euclidean space (R^m, B_m) , then f is F -measurable if the inverse image of every set of B_m is a member of F .

For simplicity, let $j = a, a+b, a+2b, \dots, a+Nb$, for some positive integers b and N , be denoted by $j = a(b)a+Nb$; if $N = \infty$, denote by $j = a(b)-$. Finally, let T denote the set of all k -dimensional vectors whose components are nonnegative integers.

Assumption A1. Let $\{\zeta_n, n=0(1)-\}$ be an infinite sequence of independent and identically distributed "environmental" random variables defined on (Ω, A, P) and taking values in Θ , a Borel set of the Euclidean space $R^{m(\Theta)}$ for some positive integer $m(\Theta)$.

Assumption A2. Suppose that associated with each point $\theta \in \Theta$ there is a vector of multivariate probability generating functions (m.p.g.f.),

$$\phi(\theta, \underline{s}) = [\phi(1, \theta, \underline{s}), \phi(2, \theta, \underline{s}), \dots, \phi(k, \theta, \underline{s})],$$

where $\underline{s} = (s(1), s(2), \dots, s(k)), \underline{s} \in [0, 1]^k$, and

$$(2.1) \quad \phi(j, \theta, \underline{s}) = \sum_{a \in T} p(j, \theta, a) \prod_{i=1}^k s(i)^{a(i)},$$

for $j=1(1)k$.

Assumption A3. Suppose that for each fixed $\underline{s} \in [0,1]^k$, $\underline{\phi}(\theta, \underline{s})$ is a mapping from Θ into $[0,1]^k$ which is Borel measurable.

Remark 2.1. If Θ has at most countably many points, then Assumption A3 is trivially satisfied.

Remark 2.2. We may immediately conclude from Assumption A3 that for fixed $\underline{t}(1), \underline{t}(2), \dots, \underline{t}(k) \in [0,1]^k$ $[\phi(j, \theta, \underline{t}(i)), i, j = 1(1)k]$, a mapping from Θ into $[0,1]^{k^2}$, is Borel measurable. Thus, for fixed $\underline{t}(1), \underline{t}(2), \dots, \underline{t}(k) \in [0,1]^k$, Assumption A1 implies that

$$\{[\phi(j, \zeta_n, \underline{t}(i)), i, j = 1(1)k], n=0(1)-\}$$

is an infinite sequence of independent and identically distributed random arrays defined on $(\Omega, \mathcal{A}, P)$. More specifically, for fixed $\underline{s} \in [0,1]^k$, $\{\underline{\phi}(\zeta_n, \underline{s}), n=0(1)-\}$ is an infinite sequence of independent and identically distributed random vectors defined on $(\Omega, \mathcal{A}, P)$.

Remark 2.3. For fixed $\underline{s} \in [0,1]^k$ $\underline{\phi}(\theta_1, \underline{\phi}(\theta_2, \dots, \underline{\phi}(\theta_n, \underline{s}) \dots))$ is a mapping from Θ^n into $[0,1]^k$ which is Borel measurable for $n=1(1)-$. To demonstrate this, let j be any integer such that $1 \leq j \leq k$ and define $f(\theta, \underline{t})$ on $\Theta \times [0,1]^k$ by $f(\theta, \underline{t}) = \phi(j, \theta, \underline{t})$. Clearly $f(\theta, \underline{t})$ is a mapping into $[0,1]$. For fixed $\theta \in \Theta$, Assumption A2 implies that $f(\theta, \underline{t})$ is a mapping from $[0,1]^k$ into $[0,1]$ which is continuous. For fixed $\underline{t} \in [0,1]^k$,

Assumption A3 is that $f(\theta, \underline{t})$ is a mapping from Θ into $[0,1]$ which is Borel measurable. Thus, since each section f_{θ} is continuous and each section $f^{\underline{t}}$ is Borel measurable, $f(\theta, \underline{t})$ is Borel measurable on $\Theta \times [0,1]^k$. Hence, $\phi(\theta, \underline{t})$ is also Borel measurable. Now fix $\underline{s} \in [0,1]^k$ and consider the vector $(\theta_1, \phi(\theta_2, \underline{s}))$ defined on Θ^2 and taking values in $\Theta \times [0,1]^k$. Clearly this vector is Borel measurable and therefore $\phi(\theta_1, \phi(\theta_2, \underline{s}))$ is Borel measurable. A simple induction argument now completes the proof.

Remark 2.4. For fixed $\underline{s} \in [0,1]^k$, $\phi(\zeta_0, \phi(\zeta_1, \dots, \phi(\zeta_n, \underline{s}) \dots))$ and $\phi(\zeta_n, \phi(\zeta_{n-1}, \dots, \phi(\zeta_0, \underline{s}) \dots))$ are random vectors defined on $(\Omega, \mathcal{A}, P)$ for $n=0(1)-$.

In the next few sections the importance of these remarks will be self-evident. Without these results, the measurability of many random variables would be suspect.

Definition 2.1. The multitype branching process in a random environment (MBPRE) is defined to be a temporally homogeneous vector Markov chain $\{\underline{Z}(n), n=0(1)-\}$ whose states are vectors in T . We shall always assume that $\underline{Z}(0)$ is nonrandom and we shall interpret the ith component of $\underline{Z}(n)$, $Z(i,n)$, as the number of type i particles in the nth generation.

The transition law for the process is as follows. For $\underline{a}, \underline{b} \in T$, $\underline{s} \in [0,1]^k$ and $n=0(1)-$, define the matrix $(P(\underline{a}, \underline{b}))$ by

$$(2.2) \quad P(\underline{a}, \underline{b}) = \text{coefficient of } \prod_{i=1}^k s(i)^{b(i)} \text{ in}$$

$$E\{\prod_{j=1}^k \phi(j, \zeta_n, \underline{s})^{a(j)}\}.$$

Remark 2.2 implies that $P(\underline{a}, \underline{b})$ is independent of the choice of n . Plainly $P(\underline{a}, \underline{b}) \geq 0$ for $\underline{a}, \underline{b} \in T$ and since

$$(2.3) \quad \sum_{\underline{b} \in T} P(\underline{a}, \underline{b}) \prod_{i=1}^k s(i)^{b(i)} = E\{\prod_{i=1}^k \phi(i, \zeta_n, \underline{s})^{a(i)}\},$$

it follows that $\sum_{\underline{b} \in T} P(\underline{a}, \underline{b}) = 1$ for each $\underline{a} \in T$. We can now define the vector Markov chain $\{\underline{Z}(n), n=0(1)-\}$ by choosing initial probabilities

$$\Pr\{\underline{Z}(0) = \underline{a}\} = \begin{cases} 1, & \text{if } \underline{a} = \underline{c}, \\ 0, & \text{if } \underline{a} \neq \underline{c}, \end{cases}$$

for some $\underline{c} \in T$,

and defining

$$\Pr\{\underline{Z}(i) = \underline{a}(i), i=0(1)n\} = \Pr\{\underline{Z}(0) = \underline{a}(0)\} \prod_{i=1}^n P(\underline{a}(i-1), \underline{a}(i))$$

where $\underline{a}(i) \in T, i=0(1)n$. If $\Pr\{\underline{Z}(n) = \underline{a}\}$ is positive, then $P(\underline{a}, \underline{b})$ is the transition probability $\Pr\{\underline{Z}(n+1) = \underline{b} | \underline{Z}(n) = \underline{a}\}$.

While all our results can be derived using Definition 2.1 it may be extremely helpful if we pause here to give a more concrete interpretation to the MBPRE.

Consider a population of particles. Suppose that, at any time, there are at most k distinct types of particles, labeled $1, 2, \dots, k$, that may be represented in the population. Suppose, also that after one unit of time each particle of the population either disintegrates or splits

randomly, generating particles of several types. The initial population of particles, called the zeroth generation, generates particles which comprise the first generation, the first generation generates particles which comprise the second generation, and so on. Since we are primarily interested in the size of the n th generation ($n=0(1)-$), let $\underline{z}(n) = (Z(1,n), Z(2,n), \dots, Z(k,n))$ denote the respective number of particles of each type in the n th generation; that is, $Z(j,n)$ is the number of type j particles in the n th generation. By assumption, $\underline{z}(0) = \underline{a}$ for some fixed $\underline{a} \in T$.

Now assume that Nature chooses an environmental state, $\zeta_n = \zeta$, independently of the past environmental states and the population, past and present, which uniquely determines the stochastic manner in which the particles of the n th generation will generate particles of the $(n+1)$ st generation. One may think of ζ as being a particular combination of environmental factors such as heat, pressure, or light which will affect the splitting or disintegration of the particles. Mathematically, this effect is completely defined by some vector of m.p.g.f., $\phi(\zeta, \underline{s})$. If $\phi(\zeta, \underline{s})$ is defined as in Assumption A2, we may interpret the environmental effect ζ as follows: (1) if $\underline{b} \in T$, then $p(j, \zeta, \underline{b})$ is the probability that one type j particle will generate $b(i)$ type i particles, $i=1(1)k$, and (2) each particle of the n th generation will split independently of the other particles present, but in

accordance with $\phi(\zeta, \underline{s})$.

Thus, we assume that the environment passes through a sequence of states governed by the infinite sequence $\{\zeta_n, n=0(1)-\}$ of independent and identically distributed random variables. Then, given $\zeta_n = \zeta$, each particle of the n th generation splits or disintegrates independently of the other particles of the n th generation, but in accordance with the specified vector of m.p.g.f. $\phi(\zeta, \underline{s})$.

Remark 2.5. Observe that if Θ contains only one point, then the MBPRE is precisely the standard multitype Galton-Watson process.

Let us now consider certain moments associated with the MBPRE. Since for arbitrary $\theta \in \Theta$ and i any integer such that $1 \leq i \leq k$, $\phi(i, \theta, \underline{s})$ is a m.p.g.f., as $t \rightarrow 0+$

$$(2.4) \quad [1 - \phi(i, \theta, \underline{1} - t\mathbf{e}_j)]/t, \quad j=1(1)k,$$

increases to a nonnegative (possibly infinite) limit, say $\xi(i, j, \theta)$, which is the mean number of type j particles produced by one type i particle in the environmental state θ .

Notation. For $\theta \in \Theta$, let $\xi(\theta)$ be the matrix defined by $(\xi(i, j, \theta))$. Clearly $\xi(\theta)$ is of order $k \times k$.

Assumption A4. Suppose that for each $\theta \in \Theta$, $\xi(\theta)$ is finite; that is $\xi(i, j, \theta)$ is finite for $i, j = 1(1)k$.

Remark 2.6. We can now prove that $\{\xi(\zeta_n), n=0(1)-\}$

is an infinite sequence of independent and identically distributed nonnegative random matrices defined on (Ω, A, P) . First, observe that Assumption A4 and Assumption A3 applied to (2.4) implies that $\xi(\theta)$, a mapping from θ into the nonnegative $k \times k$ matrices, is Borel measurable. Assumption A1 then implies that $\{\xi(\zeta_n), n=0(1)-\}$ is an infinite sequence of independent random matrices on (Ω, A, P) . Clearly the terms of this sequence are also identically distributed.

Notation. Let ξ denote the square matrix defined by $\xi = E\{\xi(\zeta_n)\} = (E\{\xi(i, j, \zeta_n)\})$. Plainly, ξ is independent of n and well-defined, since the entries of $\xi(\zeta_n)$ are nonnegative for $n=0(1)-$. Observe that the entries of ξ are nonnegative. Some may even be infinite.

3. Generating functions and moments of $Z(n)$

Definition 3.1. For $n=0(1)-$ and $\underline{a} \in T$, let $G(n, \underline{s} | \underline{a})$ be the multivariate probability generating function of the random vector $Z(n)$, given that $Z(0) = \underline{a}$.

Theorem 3.1. For $n=1(1)-$, $\underline{s} \in [0, 1]^k$, and $\underline{a} \in T$

$$(3.1) \quad G(n, \underline{s} | \underline{a}) = E\{G(n-1, \phi(\zeta_{n-1}, \underline{s}) | \underline{a})\}.$$

In accordance with Definition 2.1, $G(0, \underline{s} | \underline{a}) = \prod_{j=1}^k s(j)^{a(j)}$.

Proof. Let n be any integer greater than one, choose $\underline{a} \in T$, and fix $\underline{s} \in [0, 1]^k$. From equation (2.3), we see that

$$\begin{aligned}
G(n, \underline{s} | \underline{a}) &= E\{\prod_{j=1}^k s(j)^{Z(j,n)} \mid Z(0) = \underline{a}\} \\
&= \sum_{\underline{b} \in T} \Pr\{Z(n-1) = \underline{b} \mid Z(0) = \underline{a}\} E\{\prod_{j=1}^k s(j)^{Z(j,n)} \mid Z(n-1) = \underline{b}\} \\
&= \sum_{\underline{b} \in T} \Pr\{Z(n-1) = \underline{b} \mid Z(0) = \underline{a}\} E\{\prod_{j=1}^k \phi(j, \zeta_{n-1}, \underline{s})^{b(j)}\}.
\end{aligned}$$

By the dominated convergence theorem, we obtain (3.1) for $n=2(1)-$, $\underline{s} \in [0,1]^k$ and $\underline{a} \in T$. To show (3.1) for $n = 0, 1$, we appeal directly to Definition 2.1.

Remark 3.1. By repeated application of Theorem 3.1, we obtain the representation

$$\begin{aligned}
(3.2) \quad G(n, \underline{s} | \underline{a}) &= \\
&E\{\prod_{j=1}^k \phi(j, \zeta_0, \phi(\zeta_1, \phi(\zeta_2, \dots, \phi(\zeta_{n-1}, \underline{s}) \dots))^{a(j)}\}
\end{aligned}$$

for $n=1(1)-$, $\underline{a} \in T$ and $\underline{s} \in [0,1]^k$.

Remark 3.2. Since the random variables $\zeta_0, \zeta_1, \dots, \zeta_{n-1}$ are independent and identically distributed, we may renumber them without affecting the value of the right hand side of (3.2). Thus, we also have the representation

$$\begin{aligned}
(3.3) \quad G(n, \underline{s} | \underline{a}) &= \\
&E\{\prod_{j=1}^k \phi(j, \zeta_{n-1}, \phi(\zeta_{n-2}, \phi(\zeta_{n-3}, \dots, \phi(\zeta_0, \underline{s}) \dots))^{a(j)}\}
\end{aligned}$$

for $n=1(1)-$, $\underline{a} \in T$ and $\underline{s} \in [0,1]^k$.

Remark 3.3. Let N be any fixed positive integer. Then $\{Z(nN), n=0(1)-\}$ is also a temporally homogeneous vector Markov chain whose states are vectors in T . Using (3.2), we observe that if $\underline{a}, \underline{b} \in T$ and $\Pr\{Z(nN) = \underline{a}\}$ is positive, then

$\Pr\{\tilde{Z}(nN+N) = \tilde{b} | \tilde{Z}(nN) = \tilde{a}\} = \text{coefficient of } \prod_{i=1}^k s(i)^{b(i)} \text{ in}$

$$E\{\prod_{j=1}^k \phi(j, \zeta_0, \phi(\zeta_1, \phi(\zeta_2, \dots, \phi(\zeta_{N-1}, s) \dots))^{a(j)}\}$$

for $n=0(1)-$.

Now Assumption A1 implies that

$\{(\zeta_{nN}, \dots, \zeta_{nN+N-1}), n=0(1)-\}$ is an infinite sequence of independent and identically distributed "environmental" random vectors defined on (Ω, A, P) and taking values in Θ^N , a Borel set of the Euclidean space $R^{Nm(\Theta)}$. Assumption A2 implies that for each point $(\theta_1, \theta_2, \dots, \theta_N) \in \Theta^N$,

$$(3.4) \quad \phi(\theta_1, \phi(\theta_2, \phi(\theta_3, \dots, \phi(\theta_N, s) \dots))$$

is a vector of m.p.g.f. defined on $[0,1]^k$. Also, Remark 2.3 implies that for each fixed $s \in [0,1]^k$, (3.4) is a mapping from Θ^N into $[0,1]^k$ which is Borel measurable. Thus, a close review of Definition 2.1 reveals that we may also consider the temporally homogeneous vector Markov chain $\{\tilde{Z}(nN), n=0(1)-\}$ as a MBPRE.

Theorem 3.2. $E\{\tilde{Z}(n+m) | \tilde{Z}(n)\} = \tilde{Z}(n) \xi^m,$

with probability one, for $n, m = 0(1)-$.

Proof. Since the theorem is clearly true if $m=0$, we begin by setting $m=1$. Suppose also that $\tilde{Z}(n) = \tilde{a}$ for some $\tilde{a} \in T$. Then, by Definition 2.1, we observe that the m.p.g.f. of $\tilde{Z}(n+1)$, given $\tilde{Z}(n) = \tilde{a}$, is given by

$$G(1, s | \tilde{a}) = E\{\prod_{j=1}^k \phi(j, \zeta_0, s)^{a(j)}\}.$$

Since $\prod_{j=1}^k \phi(j, \theta, \underline{s})^{a(j)}$ is a m.p.g.f. for each $\theta \in \Theta$, it follows, using the monotone convergence theorem, that we may evaluate the left hand partials of $G(1, \underline{s} | \underline{a})$ with respect to $s(i)$ at $\underline{s} = \underline{1}$, $i = 1(1)k$; in particular,

$$\begin{aligned} \lim_{t \rightarrow 0+} [1 - G(1, \underline{1} - t \underline{e}_i | \underline{a})] / t &= E\{\sum_{j=1}^k a(j) \xi(j, i, \zeta_0)\} \\ &= \sum_{j=1}^k a(j) \xi(j, i), \quad i = 1(1)k. \end{aligned}$$

Therefore, $E\{\underline{Z}(n+1) | \underline{Z}(n) = \underline{a}\} = \underline{a} \underline{\xi}$ and thus, with probability one,

$$(3.5) \quad E\{\underline{Z}(n+1) | \underline{Z}(n)\} = \underline{Z}(n) \underline{\xi}, \quad n = 0(1)-.$$

Now using (3.5), we have that, with probability one,

$$\begin{aligned} E\{\underline{Z}(n+m) | \underline{Z}(n)\} &= E\{E\{\underline{Z}(n+m) | \underline{Z}(n+m-1), \dots, \underline{Z}(n)\} | \underline{Z}(n)\} \\ &= E\{E\{\underline{Z}(n+m) | \underline{Z}(n+m-1)\} | \underline{Z}(n)\} \\ &= E\{\underline{Z}(n+m-1) \underline{\xi} | \underline{Z}(n)\} \\ &= \dots = \underline{Z}(n) \underline{\xi}^m, \end{aligned}$$

for $n = 0(1)-$ and $m = 2(1)-$. This completes the proof.

Corollary 3.2.1. $E\{\underline{Z}(n) | \underline{Z}(0) = \underline{e}_j\} = \underline{e}_j \underline{\xi}^n$
for $j = 1(1)k$ and $n = 0(1)-$.

4. Positivity properties

Before considering the transience of the nonzero states, let us pause to define and state some important properties of matrices. Since these properties will be used frequently, the reader is urged to pay particular attention to the definitions and conventions.

Definition 4.1. We shall call a vector or a matrix positive, nonnegative, or zero if all its components have these properties. If $\underline{u}$ and $\underline{v}$ are vectors or matrices, then $\underline{u} > \underline{v}$ ($\underline{u} \geq \underline{v}$) means $\underline{u} - \underline{v}$ is positive (nonnegative).

Notation. Let absolute value signs enclosing a vector or a matrix denote the sum of the absolute values of the elements of the vector or the matrix, respectively. For a square matrix $\underline{A}$, with real entries, denote the norm of $\underline{A}$ by $||\underline{A}|| = \max_i \sum_j |A(i,j)|$.

Definition 4.2. A nonnegative square matrix $\underline{A}$ is said to be primitive if there is a positive integer N such that $\underline{A}^N$ is positive.

Theorem 4.1. Let $\underline{A}$ be a nonnegative matrix of order m which is primitive. Then $\underline{A}$ has a positive characteristic root λ that is simple and greater in absolute value than any other characteristic root; λ corresponds to positive left and right eigenvectors $\underline{u}$ and $\underline{v}$ which are the only nonnegative eigenvectors. Moreover, we have

$$\underline{A}^n = \lambda^n \underline{A}_1 + \underline{A}_2^n, \quad n=1(1)-,$$

where $\underline{A}_1 = (u(i)v(j))$, with the normalization

$\sum_{i=1}^m u(i)v(i) = 1$. Hence $\underline{A}_1 \underline{A}_1 = \underline{A}_1$. Furthermore,

(a) $\underline{A}_1 \underline{A}_2 = \underline{A}_2 \underline{A}_1 = \underline{0}$; (b) $|\underline{A}_2^n| = O(\delta^n)$ for some δ , $0 < \delta < \lambda$;

(c) for each r , $1 \leq r \leq m$, we have $\min_{1 \leq i \leq r} \sum_{j=1}^r A(i,j) \leq \lambda$;

(d) if j is a positive integer, then λ^j corresponds to $\underline{A}^j$ just as λ does to $\underline{A}$.

For a proof of this theorem the reader is referred to Gantmacher [6, p. 65] and Karlin [9, p. 475].

Notation. If $\tilde{A}$ is a nonnegative matrix of order m which is primitive, let $\lambda(\tilde{A})$ be the largest positive characteristic root of $\tilde{A}$.

Definition 4.3. The MBPRE is said to be impartial if the nonnegative square matrix $\tilde{M}$ is primitive, where $\tilde{M}$ is defined by

$$M(i,j) = \begin{cases} 0, & \text{if } \xi(i,j) = 0, \\ 1, & \text{if } \xi(i,j) \text{ is positive or infinite,} \end{cases}$$

and ξ is the matrix defined by $\xi = E\{\xi(z_0)\}$.

Remark 4.1. If the square matrix ξ is finite, then the MBPRE is impartial if and only if there is a positive integer N such that ξ^N is positive.

Observe that if the MBPRE is impartial, there is a positive probability that there will be at least one type j particle, $1 \leq j \leq k$, generated in the next N generations. Hence, we expect to find type j particles, $1 \leq j \leq k$, in the population frequently; we do not expect the population to consist only of type i_1 , type i_2 , ..., type i_h particles, $h < k$, from some generation on.

5. Transience of nonzero states

Definition 5.1. The state $a \in T$ is said to be transient

if $\Pr\{Z(n) = a \text{ for some } n=1(1)-|Z(0) = a\} < 1$, or equivalently if $\Pr\{Z(n) = a \text{ infinitely often } |Z(0) = a\} < 1$.

Remark 5.1. From Definition 2.1 we observe that $P(0,0) = 1$. Thus, with probability one, if $Z(n) = 0$, then $Z(m) = 0$ for $m=n(1)-$; that is, 0 is an absorbing state.

Now if $a \in T-\{0\}$ and $\Pr\{Z(n) = 0 | Z(0) = a\}$ is positive, then by Definition 5.1 and Remark 5.1, we infer that a is transient. Further, since $\Pr\{Z(n) = 0 | Z(0) = a\}$ is nondecreasing as n increases, if $a \in T-\{0\}$ and $\lim_{n \rightarrow \infty} \Pr\{Z(n) = 0 | Z(0) = a\}$ is positive, then, again, we can infer that a is transient.

Since we are interested, however, in evaluating

$$(5.1) \quad \lim_{n \rightarrow \infty} \Pr\{Z(n) = 0 | Z(0) = a\}$$

for $a \in T-\{0\}$, it is doubtful that we will always know in advance whether or not (5.1) is positive. In fact, the proof of the well-known result for the almost certain extinction of the multitype Galton-Watson process presupposes that the nonzero states of T are transient (Harris [8, p. 38]). Our approach, however, to evaluating (5.1) will not require us to study the transience of the nonzero states first. But, to illustrate the methods of the standard type of proof, we shall give a crude condition under which transience of the nonzero states of T occurs. Later in Section 6 we shall use this condition to prove a generalization of the well-known theorem for the almost

sure extinction of the multitype Galton-Watson process.

Definition 5.2. The MBPRE is said to be insecure if $\Pr\{Z(n) = 0 | Z(0) = 1\} = G(n, 0 | 1) > 0$ for some positive integer n .

Theorem 5.1. Suppose the MBPRE is insecure. If
 $a \in T - \{0\}$, then a is a transient state.

Proof. Choose $a \in T - \{0\}$ and suppose that $\Pr\{Z(n) = 0 | Z(0) = a\} = 0$ for $n=1(1)-$. Then $G(n, 0 | a) = 0$ for $n=1(1)-$ and we have, using Remark 3.2, that

$$E\{\prod_{j=1}^k \phi(j, \zeta_{n-1}, \phi(\zeta_{n-2}, \dots, \phi(\zeta_0, 0) \dots)^{a(j)}\} = 0$$

for $n=1(1)-$. Thus, with probability one,

$$\prod_{j=1}^k \phi(j, \zeta_{n-1}, \phi(\zeta_{n-2}, \dots, \phi(\zeta_0, 0) \dots)^{a(j)} = 0$$

for $n=1(1)-$ and therefore, with probability one,

$$\prod_{j=1}^k \phi(j, \zeta_{n-1}, \phi(\zeta_{n-2}, \dots, \phi(\zeta_0, 0) \dots) = 0$$

for $n=1(1)-$. It now follows that $G(n, 0 | 1) = 0$ for $n=1(1)-$. But this violates the hypothesis that the MBPRE is insecure. Thus there is a positive integer n such that $\Pr\{Z(n) = 0 | Z(0) = a\}$ is positive. From our previous comments, we now conclude that a is transient.

Theorem 5.2. Suppose the MBPRE is insecure. If
 N is an arbitrary positive integer, then

$$(5.2) \quad \Pr\{0 < |\underline{z}(n)| < N\} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

and

$$(5.3) \quad \Pr\{0 < |\underline{z}(n)| < N \text{ infinitely often}\} = 0.$$

Proof. This result follows from Theorem 5.1 and Feller [4, Vol. I, Chapter 15].

Theorem 5.3. Suppose the MBPRE is insecure. For each $\underline{a} \in T - \{0\}$, there exists a number $u(\underline{a}) \in [0,1]$ such that for every $\underline{s} \in [0,1]^k$, $G(n, \underline{s} | \underline{a}) \rightarrow u(\underline{a})$, as $n \rightarrow \infty$.

Proof. Choose $\underline{s} \in [0,1]^k$ and $\underline{a} \in T - \{0\}$. If N is a positive integer, then

$$\begin{aligned} G(n, \underline{s} | \underline{a}) &= G(n, 0 | \underline{a}) + \sum_{0 < |\underline{b}| < N} \Pr\{\underline{z}(n) = \underline{b} | \underline{z}(0) = \underline{a}\} \prod_{j=1}^k s(j)^{b(j)} \\ &\quad + \sum_{|\underline{b}| \geq N} \Pr\{\underline{z}(n) = \underline{b} | \underline{z}(0) = \underline{a}\} \prod_{j=1}^k s(j)^{b(j)}. \end{aligned}$$

Thus,

$$\begin{aligned} G(n, 0 | \underline{a}) &\leq G(n, \underline{s} | \underline{a}) \leq G(n, 0 | \underline{a}) + \Pr\{0 < |\underline{z}(n)| < N\} \\ &\quad + \{\max_{1 \leq i \leq k} s(i)\}^N. \end{aligned}$$

Since $G(n, 0 | \underline{a})$ is a bounded sequence which is nondecreasing, as n increases, there exists a number, say $u(\underline{a})$, such that

$$u(\underline{a}) = \lim_{n \rightarrow \infty} G(n, 0 | \underline{a}).$$

Clearly $u(\underline{a}) \in [0,1]$ and applying Theorem 5.2, we obtain

$$\begin{aligned}
 u(\underline{a}) &\leq \liminf_{n \rightarrow \infty} G(n, \underline{s} | \underline{a}) \leq \limsup_{n \rightarrow \infty} G(n, \underline{s} | \underline{a}) \\
 &\leq u(\underline{a}) + \{\max_{1 \leq i \leq k} s(i)\}^N.
 \end{aligned}$$

Since N is arbitrary and $\underline{s} \in [0, 1]^k$, the theorem is proved.

CHAPTER II
EXTINCTION PROBABILITIES FOR POPULATIONS
CONSISTING INITIALLY OF ONE PARTICLE

6. Extinction probabilities

In this chapter we will consider the probability of extinction of the MBPRE given that $|Z(0)| = 1$. We will begin by introducing extinction probabilities and by proving a result for the almost sure extinction of the process which is similar to the well-known theorem for the multitype Galton-Watson process. Then, since the proofs of the theorems in the later sections require some knowledge of the dual process, we will define and study this process. Also, we will review some results on products of random matrices by Furstenberg and Kesten [5]. Following these preliminaries, we will then give the best conditions we could obtain for extinction and nonextinction of the MBPRE given that $|Z(0)| = 1$. Further, to make our results more applicable for practical work, we will give some corollaries whose conditions are easier to verify than the conditions of the main results.

We begin by defining extinction probabilities.

Definition 6.1. Let $q(\underline{e}_i)$ be the extinction probability of the MBPRE given that $\underline{Z}(0) = \underline{e}_i$, $i=1(1)k$; that is

$$q(\underline{e}_i) = \Pr\{\underline{Z}(n) = \underline{0} \text{ for some } n=1(1)\infty \mid \underline{Z}(0) = \underline{e}_i\}$$

for $i=1(1)k$. Let $\underline{q}$ denote the vector $(q(\underline{e}_1), q(\underline{e}_2), \dots, q(\underline{e}_k))$.

Remark 6.1. Since $\underline{0}$ is an absorbing state, the extinction probability may be calculated as follows:

$$\begin{aligned} q(\underline{e}_i) &= \Pr\{\underline{Z}(n) = \underline{0} \text{ for some } n=1(1)\infty \mid \underline{Z}(0) = \underline{e}_i\} \\ &= \Pr\{\bigcup_{m=1}^{\infty} [\underline{Z}(m) = \underline{0}] \mid \underline{Z}(0) = \underline{e}_i\} \\ &= \lim_{n \rightarrow \infty} \Pr\{\bigcup_{m=1}^n [\underline{Z}(m) = \underline{0}] \mid \underline{Z}(0) = \underline{e}_i\} \\ &= \lim_{n \rightarrow \infty} \Pr\{\underline{Z}(n) = \underline{0} \mid \underline{Z}(0) = \underline{e}_i\} \\ &= \lim_{n \rightarrow \infty} G(n, \underline{0} \mid \underline{e}_i), \end{aligned} \quad i=1(1)k.$$

From these calculations and the fact that $\underline{0}$ is absorbing, it is clear that the extinction probability $q(\underline{e}_i)$ is precisely the probability that the MBPRE ever becomes extinct, given that $\underline{Z}(0) = \underline{e}_i$.

Theorem 6.1. Suppose the MBPRE is impartial. If for some integer i , $1 \leq i \leq k$, $q(\underline{e}_i) = 1$, then $\underline{q} = \underline{1}$.

Proof. Assume $q(\underline{e}_i) = 1$ for some integer i , $1 \leq i \leq k$. Since the MBPRE is impartial, there exists a positive integer N such that $\underline{M}^N$ is positive (see Definition 4.3). For $n > N$,

$$G(n, \underline{0} \mid \underline{e}_i) = \sum_{\underline{a} \in T} \Pr\{\underline{Z}(N) = \underline{a} \mid \underline{Z}(0) = \underline{e}_i\} G(n-N, \underline{0} \mid \underline{a}).$$

Now, if $\underline{a} \in T$ and $a(j) > 0$ for some integer j , $1 \leq j \leq k$, then

Remark 3.2 implies that $G(n-N, 0 | \underline{a}) \leq G(n-N, 0 | \underline{e}_j)$ for $n > N$.

Thus for $j=1(1)k$ and $n > N$,

$$G(n, 0 | \underline{e}_i) \leq \Pr\{Z(j, N) = 0 | Z(0) = \underline{e}_i\} + \Pr\{Z(j, N) > 0 | Z(0) = \underline{e}_i\} G(n-N, 0 | \underline{e}_j).$$

Taking limits, as $n \rightarrow \infty$, we obtain for $j=1(1)k$,

$$1 \leq \Pr\{Z(j, N) = 0 | Z(0) = \underline{e}_i\} + \Pr\{Z(j, N) > 0 | Z(0) = \underline{e}_i\} q(\underline{e}_j).$$

Since $\underline{M}^N$ is positive, $\Pr\{Z(j, N) > 0 | Z(0) = \underline{e}_i\}$ is positive for $j=1(1)k$. Thus, $q(\underline{e}_j) = 1$ for $j=1(1)k$. Equivalently, $\underline{q} = \underline{1}$.

Before defining the dual process, let us prove a result for the almost sure extinction of the MBPRE. Observe that the method of proof is essentially the same method used in proving a similar result for the multitype Galton-Watson process (Harris [8, p. 41]).

Theorem 6.2. Suppose the MBPRE is insecure, impartial, and has finite moment matrix $\underline{\xi}$. If $\lambda(\underline{\xi}) \leq 1$, then $\underline{q} = \underline{1}$.

Proof. Suppose the MBPRE is insecure, impartial, and has finite moment matrix $\underline{\xi}$. Further, suppose $Z(0) = \underline{e}_i$ for some integer i , $1 \leq i \leq k$. Now since the process is insecure, (5.3) of Theorem 5.2 implies that

$$\Pr\{|Z(n)| \rightarrow 0\} + \Pr\{|Z(n)| \rightarrow \infty\} = 1.$$

Also, Remark 4.1 implies that $\underline{\xi}$ is primitive and hence $\lambda(\underline{\xi})$

is well-defined. Thus, combining Corollary 3.2.1, Theorem 4.1, and the assumption that $\lambda(\xi) \leq 1$, we can infer that $E\{|Z(n)|\}$ is a bounded function of n . It follows that $\Pr\{|Z(n)| \rightarrow \infty\} = 0$ and therefore that $\Pr\{|Z(n)| \rightarrow 0\} = 1$. Thus, $q(e_i) = 1$ for $i=1(1)k$.

Remark 6.2. We note that the transience of the nonzero states is used to obtain this result; in particular, we have used Theorem 5.2.

Further, if Θ contains only one point, then Theorem 6.2 is similar to the standard result for almost certain extinction of the multitype Galton-Watson process.

7. The dual process

Definition 7.1. Consider the discrete parameter vector Markov process $\{\tilde{X}(n), n=0(1)-\}$ taking values in $[0,1]^k$ and defined as follows: for arbitrary, but fixed $\underline{s} \in [0,1]^k$,

$$\begin{aligned}\tilde{X}(0) &= \underline{s}, \\ \tilde{X}(n+1) &= \phi(\zeta_n, \tilde{X}(n)), \quad n=0(1)-.\end{aligned}$$

By Remark 2.4, $\{\tilde{X}(n), n=0(1)-\}$ is indeed a family of well-defined random vectors. The stochastic process $\{\tilde{X}(n), n=0(1)-\}$ will be called the dual process associated with the MBPRE.

Remark 7.1. Note that the dual process is not unique.

In fact, it would be better to say that $\{\tilde{X}(n), n=0(1)-\}$ is the dual process associated with the MBPRE and the vector $\underline{s} \in [0,1]^k$. For convenience however, we will call each process the dual process.

Remark 7.2. A direct consequence of Definition 7.1 is that

$$(7.1) \quad \tilde{X}(n+1) = \phi(\zeta_n, \phi(\zeta_{n-1}, \dots, \phi(\zeta_0, \tilde{X}(0)) \dots), \quad n=0(1)-.$$

Since for any nonnegative integer n , $\underline{s} \in [0,1]^k$, and $(\theta_0, \theta_1, \dots, \theta_n) \in \Theta^{n+1}$, $\phi(\theta_n, \phi(\theta_{n-1}, \dots, \phi(\theta_0, \underline{s})) \dots)$ is a vector of m.p.g.f., it follows, with probability one, that

$$(7.2) \quad \tilde{X}(n) | \tilde{X}(0) = \underline{s} \in [0,1]^k \geq \tilde{X}(n) | \tilde{X}(0) = \underline{0}, \quad n=0(1)-.$$

Remark 7.3. For the expected location of the dual process after n steps, we obtain straight forwardly that

$$E\{\tilde{X}(n) | \tilde{X}(0) = \underline{s}\} = E\{\phi(\zeta_{n-1}, \phi(\zeta_{n-2}, \dots, \phi(\zeta_0, \underline{s})) \dots)\},$$

whenever $\underline{s} \in [0,1]^k$. Combining this with (3.3) of Remark 3.2, we have

$$(7.3) \quad E\{X(j,n) | \tilde{X}(0) = \underline{s}\} = G(n, \underline{s} | e_j)$$

for $\underline{s} \in [0,1]^k$, $j=1(1)k$, and $n=0(1)-$.

8. Products of random matrices

At this point we introduce some results on products of random matrices. We will begin by stating some very elegant results by Furstenberg and Kesten [5] on the norms of

products of random matrices. Since we will be mainly interested in the row sums of products of random matrices however, we will then introduce a class of matrices in which the results of Furstenberg and Kesten hold for the row sums of products of random matrices. After some corollaries and remarks, we take up the main question of conditions for extinction and nonextinction of the MBPRE.

Notation. If x is any positive number, let $\log^+ x = \max(0, \log x)$.

Theorem 8.1. Suppose $\{M_n, n=1(1)-\}$ is an infinite sequence of independent and identically distributed $h \times h$ random matrices. If $E\{\log^+ ||M_1||\} < \infty$, then

$$(8.1) \quad r = \lim_{n \rightarrow \infty} n^{-1} E\{\log ||M_1 M_2 \cdots M_n||\}$$

exists and $-\infty \leq r < \infty$. Further,

$$(8.2) \quad r = \lim_{n \rightarrow \infty} n^{-1} \log ||M_1 M_2 \cdots M_n||,$$

almost surely.

Proof. This result follows from the work done by Furstenberg and Kesten [5, pp. 459-462] and Theorem 1.2 of Doob [3, p. 460]. A very clear and concise proof of part of this result is also given by Grenander [7, p. 161].

Remark 8.1. Observe that if A and B are $h \times h$ matrices with real components, then $||AB|| \leq ||A|| \cdot ||B||$.

Corollary 8.1.1. Suppose $\{\underline{M}_n, n=1(1)-\}$ is an infinite sequence of independent and identically distributed $h \times h$ random matrices. If $E\{\log^+ ||\underline{M}_1||\} < \infty$, then r , as defined in (8.1), exists and $-\infty \leq r < \infty$. Further,

$$(8.3) \quad r \leq n^{-1} E\{\log ||\underline{M}_1 \underline{M}_2 \cdots \underline{M}_n||\}, \quad n=1(1)-.$$

Proof. Let m and n be any positive integers. Using Remark 8.1, we obtain

$$\log ||\underline{M}_1 \underline{M}_2 \cdots \underline{M}_{mn}|| \leq \sum_{i=0}^{m-1} \log ||\underline{M}_{ni+1} \cdots \underline{M}_{ni+n}||,$$

almost surely. Thus, since $\{\underline{M}_n, n=1(1)-\}$ is an infinite sequence of independent and identically distributed random arrays,

$$E\{\log ||\underline{M}_1 \underline{M}_2 \cdots \underline{M}_{mn}||\} \leq m E\{\log ||\underline{M}_1 \underline{M}_2 \cdots \underline{M}_n||\}.$$

Using (8.1), we can now verify (8.3).

Since, in later sections, we will be more concerned with the row sums, instead of the norms, of products of random matrices, let us introduce a special class of matrices $M(C, N)$. We will show that if $\{\underline{M}_n, n=1(1)-\}$ takes only values in $M(C, N)$ and the hypotheses of Theorem 8.1 are satisfied, then Theorem 8.1 will remain valid if we replace the norm in (8.2) by any particular row sum.

Definition 8.1. For $C \geq 1$ and N a positive integer, let $M(C, N)$ be a class of $h \times h$ nonnegative, finite matrices such that if $\underline{A}_i \in M(C, N)$ for $i=1(1)N$ and $\underline{A} = \underline{A}_1 \underline{A}_2 \cdots \underline{A}_N$,

then

- a) $\tilde{A}$ is positive
 and b) $1 \leq [\max_{i,j} A(i,j)] / [\min_{i,j} A(i,j)] \leq C < \infty$.

Example 8.1. The following four matrices form a class of the type $M(3,2)$.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Observe that once we verify condition (a) of Definition 8.1, we automatically know there is a $C \geq 1$ satisfying condition (b) of Definition 8.1 since there are only a finite number of 2-fold products. Of course, if we require a value for C , we must make further calculations. Thus, to demonstrate that any finite class of $h \times h$ nonnegative, finite matrices is of the form $M(C,N)$, we need only verify condition (a) of Definition 8.1.

Lemma 8.1. If $\tilde{A}_i \in M(C,N)$ for $i=1(1)n$, where $n \geq N$, and $\tilde{A} = \tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_n$, then $\tilde{A}$ is positive and

$$(8.4) \quad C^{-1} \|\tilde{A}\| \leq \underline{e}_j \tilde{A} \underline{e}_j \leq \|\tilde{A}\|, \quad j=1(1)h.$$

Proof. If $\tilde{B} \in M(C,N)$, then $\tilde{B}^N$ is a positive matrix. This implies that each row and each column of $\tilde{B}$ contains nonzero elements. Since, by Definition 8.1, $\tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_N$ is positive, $\tilde{A}$ must be a positive matrix also.

Since the second inequality of (8.4) is obvious, we

we need only verify the first inequality of (8.4). Indeed, if $D = A_1 A_2 \cdots A_N$ and $B = A_{N+1} A_{N+2} \cdots A_n$, then

$$\begin{aligned} e_{i \sim \sim} A_1 / e_{j \sim \sim} A_1 &= (\sum_{m=1}^h D(i, m) e_{m \sim \sim} B_1) / (\sum_{m=1}^h D(j, m) e_{m \sim \sim} B_1) \\ &\leq (\max_{r, s} D(r, s)) (\sum_{m=1}^h e_{m \sim \sim} B_1) / (\min_{r, s} D(r, s)) (\sum_{m=1}^h e_{m \sim \sim} B_1) \end{aligned}$$

for $i, j = 1(1)h$. Since A is positive, $\sum_{m=1}^h e_{m \sim \sim} B_1$ is positive and therefore

$$e_{i \sim \sim} A_1 / e_{j \sim \sim} A_1 \leq C, \quad i, j = 1(1)h.$$

Hence, $C^{-1} e_{i \sim \sim} A_1 \leq e_{j \sim \sim} A_1$ for $i, j = 1(1)h$ and moreover,

$$C^{-1} ||A|| \leq e_{j \sim \sim} A_1, \quad j=1(1)h.$$

Corollary 8.1.2. Suppose $\{M_n, n=1(1)-\}$ is an infinite sequence of independent and identically distributed $h \times h$ random matrices which takes only values in some space $M(C, N)$.
If $E\{\log^+ ||M_1||\} < \infty$, then

$$(8.5) \quad r = \lim_{n \rightarrow \infty} n^{-1} E\{\log ||M_1 M_2 \cdots M_n||\}$$

exists and $-\infty \leq r < \infty$. Further,

$$(8.6) \quad r = \lim_{n \rightarrow \infty} n^{-1} \log(e_{j \sim \sim} M_1 M_2 \cdots M_n 1),$$

almost surely, for $j=1(1)h$.

Proof. This result is a direct consequence of Theorem 8.1 and Lemma 8.1.

Remark 8.2. Although these results seem to be strong and informative, they have two drawbacks. First we will be

more interested in row sums than norms of products of random matrices when we consider extinction of the MBPRE. Thus, if we could replace the norm in (8.2) of Theorem 8.1 by a particular row sum and if we could replace the hypothesis of Theorem 8.1 that $E\{\log^+ ||\tilde{M}_1||\} < \infty$ by $E\{\log^+ e_{j\tilde{M}_1}\} < \infty$ for some integer j , then Theorem 8.1 would be more useful to us. Now, however, to use the above results effectively, we will have to add conditions to our theorems on extinction of the MBPRE which will limit their generality. Secondly, and probably most important, r , as defined in (8.1), appears to be exceedingly difficult to calculate. This is most unfortunate, since we shall soon see that whether $r < 0$ or $r > 0$ determines, under suitable conditions, whether or not the probability of extinction of the MBPRE is 1. Certainly, Corollary 8.1.1 does provide an upper bound for r which in some cases will allow us to conclude that $r < 0$, but we have no similar result from which we will be able to conclude that $r > 0$. Therefore, if we are to make the results of this section more applicable to the MBPRE, we must find a simpler method for evaluating r , and we must consider the relation between the row sums and the norms of products of random arrays more carefully.

9. A condition for almost certain extinction

We will now state and prove the strongest result we

have been able to obtain for almost certain extinction of the MBPRE, given that $|\tilde{z}(0)| = 1$. To make this result easier to use, we will then give some corollaries which involve simpler, albeit less general, conditions. After these corollaries, we will combine the results on products of random matrices with Theorem 9.1 to obtain a less general, but more informative result for almost certain extinction of the MBPRE. From this result we will then obtain a corollary which is possibly the most useful corollary we shall give. We will conclude this section with some remarks on extensions of our results and state an important conjecture.

Theorem 9.1. Suppose the MBPRE is impartial. Suppose, also, that for some integer j_0 , $1 \leq j_0 \leq k$, and some number r , $-\infty \leq r < 0$,

$$(9.1) \quad r \geq \limsup_{n \rightarrow \infty} (n+1)^{-1} \log [e_{j_0} \tilde{\xi}(\zeta_0) \tilde{\xi}(\zeta_1) \cdots \tilde{\xi}(\zeta_n) \tilde{1}],$$

almost surely. Then $\underline{g} = \tilde{1}$.

Proof. Assume that for some integer j_0 , $1 \leq j_0 \leq k$, and some number r , $-\infty \leq r < 0$, (9.1) is true almost surely.

Now, for fixed $\underline{s} \in [0,1]^k$, Assumptions A2 and A4 and Jensen's inequality imply that for $i=1(1)k$ and $n=0(1)-$,

$$(9.2) \quad \phi(i, \zeta_n, \underline{s}) \geq \prod_{j=1}^k s(j)^{\xi(i,j,\zeta_n)},$$

almost surely. For notational convenience, let $\psi(\zeta_n, \underline{s})$ be defined for $\underline{s} \in [0,1]^k$ by

$$\psi(i, \zeta_n, \underline{s}) = \prod_{j=1}^k s(j)^{\xi(i, j, \zeta_n)}$$

for $i=1(1)k$ and $n=0(1)-$.

Consider now the infinite sequence of random vectors $\{\underline{Y}(n), n=0(1)-\}$ which takes values in $(0,1]^k$, almost surely, and is defined by

$$\begin{aligned} (9.3) \quad \underline{Y}(0) &= t\underline{1}, \text{ for some arbitrary, but fixed } t \in (0,1), \\ \underline{Y}(n+1) &= \underline{\psi}(\zeta_n, \underline{Y}(n)), \quad n=0(1)-. \end{aligned}$$

If $\underline{Y}(n) = [Y(1,n), \dots, Y(k,n)]$ for $n=0(1)-$, observe that

$$(9.4) \quad Y(j, n+1) = \exp\{e_j \xi(\zeta_n) \xi(\zeta_{n-1}) \cdots \xi(\zeta_0) \underline{1} \cdot \log t\}$$

for $j=1(1)k$ and $n=0(1)-$, and that, indeed $Y(j,n)$ is a random variable. Furthermore, combining Assumption A2, (9.2) and (9.3), we obtain

$$\begin{aligned} (9.5) \quad E\{\underline{Y}(n) | \underline{Y}(0) = t\underline{1}\} &= E\{\underline{\psi}(\zeta_{n-1}, \underline{\psi}(\zeta_{n-2}, \underline{\psi}(\zeta_{n-3}, \dots, \underline{\psi}(\zeta_0, t\underline{1}) \dots))\} \\ &\leq E\{\underline{\phi}(\zeta_{n-1}, \underline{\psi}(\zeta_{n-2}, \underline{\psi}(\zeta_{n-3}, \dots, \underline{\psi}(\zeta_0, t\underline{1}) \dots))\} \\ &\leq E\{\underline{\phi}(\zeta_{n-1}, \underline{\phi}(\zeta_{n-2}, \underline{\psi}(\zeta_{n-3}, \dots, \underline{\psi}(\zeta_0, t\underline{1}) \dots))\} \\ &\leq \dots \\ &\leq E\{\underline{\phi}(\zeta_{n-1}, \underline{\phi}(\zeta_{n-2}, \underline{\phi}(\zeta_{n-3}, \dots, \underline{\phi}(\zeta_0, t\underline{1}) \dots))\} \\ &= E\{\underline{X}(n) | \underline{X}(0) = t\underline{1}\}, \quad n=0(1)-. \end{aligned}$$

Let us now choose ε so that $t < 1-\varepsilon < 1$ and write, for convenience,

$$\beta(j, n) = e_j \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) \underline{1}$$

for $j=1(1)k$ and $n=0(1)-$. Then there exists $\delta > 0$ such that $\log\{\log(1-\varepsilon)\}/\log t = -\delta$. Also, using Remark 2.6, we observe that $e_j \xi(\zeta_n) \xi(\zeta_{n-1}) \cdots \xi(\zeta_0) \frac{1}{t}$ and $\beta(j,n)$ are identically distributed for $j=1(1)k$ and $n=0(1)-$. Thus,

$$(9.6) \quad \Pr\{Y(j_0, n+1) < 1-\varepsilon\} = \Pr\{\beta(j_0, n) > [\log(1-\varepsilon)]/\log t\} \\ = \Pr\{(n+1)^{-1} \log \beta(j_0, n) > -\delta/(n+1)\}$$

for $n=0(1)-$. Thus, for sufficiently large n , since $r < 0$,

$$(9.7) \quad \Pr\{Y(j_0, n+1) < 1-\varepsilon\} \leq \Pr\{(n+1)^{-1} \log \beta(j_0, n) > r/2\}.$$

But, by assumption, $\limsup_{n \rightarrow \infty} (n+1)^{-1} \log \beta(j_0, n) \leq r < r/2$, almost surely; so we conclude

$$\Pr\{Y(j_0, n+1) < 1-\varepsilon\} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Therefore, $Y(j_0, n) \rightarrow 1$, in probability, as $n \rightarrow \infty$. Since $Y(j_0, n)$ is bounded by 1, the dominated convergence theorem implies that $E\{Y(j_0, n) | \mathcal{Y}(0)\} \rightarrow 1$, as $n \rightarrow \infty$. Combining this with (9.5) and (7.3), we have that $G(n, t | e_{j_0}) \rightarrow 1$, as $n \rightarrow \infty$. Since $t < 1$ and

$$G(n, t | e_{j_0}) \leq G(n, 0 | e_{j_0}) + t[1 - G(n, 0 | e_{j_0})]$$

for $n=0(1)-$, it follows that $G(n, 0 | e_{j_0}) \rightarrow 1$, as $n \rightarrow \infty$. Thus $q(e_{j_0}) = 1$ and by Theorem 6.1, $q = 1$.

Remark 9.1. The application of Jensen's inequality in the proof of Theorem 9.1 should be noted. While studying Smith and Wilkinson's [11, p. 820] method of linear

approximation, we were tempted to use hyperplanes to bound the m.p.g.f. $\phi(i, \zeta_n, \underline{s})$ in (9.2). Although this approach might have led to a sharper result (i.e., $r \leq 0$), we found that the necessary approximations were difficult to handle, since they could not be placed in closed form.

If, on the other hand, we apply the Jensen's inequality approach to the BPPE and observe that the right hand side of condition (9.1) is a generalization of the BPPE condition for almost certain extinction, we obtain the BPPE result for $r < 0$. Smith and Wilkinson have a sharper result, however, which is apparently due to their use of linear approximations in the case $r = 0$.

Remark 9.2. Suppose the MBPRE is insecure, impartial, and has finite moment matrix ξ . If $\lambda(\xi) < 1$, then Theorem 6.2 implies that $\underline{q} = \underline{1}$. This result is also a consequence of Theorem 9.1. To demonstrate this, let $\lambda = \lambda(\xi)$ and choose λ_1 such that $\lambda < \lambda_1 < 1$. If $r = \log \lambda_1$, then $r < 0$. Moreover, using the Markov lemma, we obtain

$$\begin{aligned} \Pr\{(n+1)^{-1} \log[\underline{e}_1 \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) \underline{1}] > \log \lambda_1\} \\ &= \Pr\{\underline{e}_1 \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) \underline{1} > \lambda_1^{n+1}\} \\ &\leq \underline{e}_1 \xi^{n+1} \underline{1} \lambda_1^{-n-1}, \quad n=0(1)-. \end{aligned}$$

Since ξ is primitive, Theorem 4.1 implies that there exists positive integers K and L such that

$$|\xi^{n+1}| \leq K\lambda^{n+1}, \quad n \geq L.$$

Thus,

$$\begin{aligned} \Pr\{(n+1)^{-1} \log[e_1 \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) 1] > \log \lambda_1\} \\ \leq K\lambda^{n+1} \lambda_1^{-n-1} = K(\lambda/\lambda_1)^{n+1}, \quad n \geq L. \end{aligned}$$

Since $\sum_{n=L}^{\infty} K(\lambda/\lambda_1)^{n+1}$ is finite, the Borel-Cantelli lemma implies that there exists a positive random variable, finite almost surely, say $N(\omega)$, such that if $n \geq N(\omega)$,

$$(n+1)^{-1} \log[e_1 \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) 1] \leq \log \lambda_1.$$

Hence condition (9.1) holds, almost surely, for $j_0 = 1$ and $r = \log \lambda_1 < 0$. Theorem 9.1 can now be applied to prove that $q = 1$.

Note that this result is weaker in one respect and stronger in another, than Theorem 6.2. On the one hand, we have made no use of the fact that the MBPRE is insecure. On the other hand, we have proved this result only for $\lambda(\xi) < 1$ and not for $\lambda(\xi) \leq 1$.

Corollary 9.1.1. Suppose the MBPRE is impartial and has finite moment matrix ξ . If $\lambda(\xi) < 1$, then $q = 1$.

Corollary 9.1.2. Suppose the MBPRE is impartial. Further, suppose $\Delta(\zeta_n) = \max\{e_j \xi(\zeta_n) 1, j=1(1)k\}$ for $n=0(1)-$. Then, $\{\Delta(\zeta_n), n=0(1)-\}$ is an infinite sequence of independent and identically distributed random variables. If $E\{\log^+ \Delta(\zeta_0)\} < \infty$ and $E\{\log \Delta(\zeta_0)\} < 0$, then $q = 1$.

Proof. If $\Delta(\zeta_n) = \max\{\xi_j(\zeta_n), j=1(1)k\}$ for $n=0(1)-$, then Remark 2.6 implies that $\{\Delta(\zeta_n), n=0(1)-\}$ is an infinite sequence of independent and identically distributed random variables. Moreover, $\{\log \Delta(\zeta_n), n=0(1)-\}$ is an infinite sequence of independent and identically distributed (possibly extended) random variables. Suppose $E\{\log^+ \Delta(\zeta_0)\} < \infty$ and $E\{\log \Delta(\zeta_0)\} < 0$. Then, with probability one,

$$e_1 \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) \leq \prod_{i=0}^n \Delta(\zeta_i), \quad n=0(1)-.$$

It follows, using the strong law of large numbers, that

$$\limsup_{n \rightarrow \infty} (n+1)^{-1} \log[e_1 \xi(\zeta_0) \cdots \xi(\zeta_n)] \leq E\{\log \Delta(\zeta_0)\},$$

almost surely. Since this is condition (9.1), Theorem 9.1 implies that $q = 1$.

Corollary 9.1.3. Suppose the MBPRE is impartial.
Further, suppose there is an infinite sequence of
independent and identically distributed nonnegative random
matrices $\{\xi(\zeta_n), n=0(1)-\}$ such that $\xi(\zeta_n) \leq \delta(\zeta_n)$, with
probability one, for $n=0(1)-$, and that for some integer
 $j_0, 1 \leq j_0 \leq k$, and some number $r, -\infty \leq r < 0$,

$$r \geq \limsup_{n \rightarrow \infty} (n+1)^{-1} \log[e_{j_0} \delta(\zeta_0) \delta(\zeta_1) \cdots \delta(\zeta_n)],$$

almost surely. Then $q = 1$.

Remark 9.3. Since $\{\xi(\zeta_n), n=0(1)-\}$ is an infinite sequence of independent and identically distributed random

matrices, it follows that $\Pr\{|\xi(\zeta_n)| = 0\}$ is independent of n . Note that, if $\Pr\{|\xi(\zeta_0)| = 0\}$ is positive, then there would be a positive probability of extinction at each generation. In this case, $q = 1$.

Corollary 9.1.4. Suppose the MBPRE is impartial.
Suppose, also, that for some positive vector
 $\underline{v} = (v(1), \dots, v(k))$, there is an infinite sequence of
positive random variables $\{\lambda(n), n=0(1)-\}$ such that
 $\xi(\zeta_n) \underline{v} = \lambda(n) \underline{v}$, almost surely, for $n=0(1)-$. If
 $E\{\log^+ \lambda(0)\} < \infty$ and $E\{\log \lambda(0)\} < 0$, then $q = 1$.

Proof. Since $\lambda(n) = \sum_{j=1}^k \xi(1, j, \zeta_n) v(j)/v(1)$, almost surely, for $n=0(1)-$, it follows, from Remark 2.6, that $\{\lambda(n), n=0(1)-\}$ is an infinite sequence of independent and identically distributed random variables. Observe, that if $E\{\log^+ \lambda(0)\}$ is finite, then $\{\log \lambda(n), n=0(1)-\}$ is an infinite sequence of independent and identically distributed random variables whose first moments are well-defined. Assume now that $E\{\log^+ \lambda(0)\} < \infty$ and $E\{\log \lambda(0)\} < 0$.

If we let $u = \min\{v(i), i=1(1)k\} > 0$, then, with probability one,

$$\begin{aligned} v(1) \prod_{j=0}^n \lambda(j) &= e_1 \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) \underline{v} \\ &\geq u e_1 \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) \underline{1}, \quad n=0(1)-. \end{aligned}$$

It follows, with probability one, that

$$\sum_{j=0}^n \log \lambda(j) + \log(v(1)/u) \geq \log[e_1 \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) 1]$$

for $n=0(1)-$. Taking limits, as $n \rightarrow \infty$, and applying the strong law of large numbers, we obtain, with probability one,

$$E\{\log \lambda(0)\} \geq \limsup_{n \rightarrow \infty} (n+1)^{-1} \log[e_1 \xi(\zeta_0) \cdots \xi(\zeta_n) 1].$$

Theorem 9.1 now implies that $q = 1$.

Remark 9.4. Corollary 9.1.4 is particularly useful if for each environmental state θ , the mean number of particles generated by any particle is the same for all types. In this case, $v = 1$.

Lemma 9.1. Let M denote a space of primitive $h \times h$ nonnegative matrices which commute under multiplication. Then there is a positive vector v such that $Mv = \lambda(M)v$ for each $M \in M$.

Proof. Choose $A, B \in M$. Let $\lambda(A)$ and $\lambda(B)$ denote the corresponding largest positive characteristic root of A and B . Denote by $v(A)$ and $v(B)$ the right, positive, normalized eigenvector associated with $\lambda(A)$ and $\lambda(B)$, respectively. Then $Av(A) = \lambda(A)v(A)$ and $Bv(B) = \lambda(B)v(B)$. Recall that the right eigenspaces associated with $\lambda(A)$ and $\lambda(B)$ are one-dimensional and that $v(A)$ and $v(B)$ are the only positive, normalized right eigenvectors associated with A and B , respectively. Thus, if $Aw = cw$, where w is a positive vector and c is a

nonzero constant, then $\underline{w}$ is a multiple of $\underline{v}(\underline{A})$. Of course, a similar result holds for $\underline{B}$.

Now since

$$\underline{B} \underline{A} \underline{v}(\underline{B}) = \underline{A} \underline{B} \underline{v}(\underline{B}) = \underline{A} \lambda(\underline{B}) \underline{v}(\underline{B}) = \lambda(\underline{B}) \underline{A} \underline{v}(\underline{B})$$

and since $\underline{A} \underline{v}(\underline{B})$ is positive, it follows that $\underline{A} \underline{v}(\underline{B}) = c \underline{v}(\underline{B})$ for some nonzero constant c . Since $\underline{v}(\underline{B})$ is positive, $\underline{v}(\underline{A}) = \underline{v}(\underline{B})$. Thus, since $\underline{A} \in M$ is arbitrary, there exists a positive vector $\underline{v}$ such that $\underline{v}(\underline{A}) = \underline{v}$ for every $\underline{A} \in M$. The result now follows.

Corollary 9.1.5. Suppose the MBPRE is impartial and suppose that $\{\xi(\zeta_n), n=0(1)-\}$ takes values in a space of primitive $k \times k$ matrices, which are commutative under multiplication. Then there exists a positive vector $\underline{v}$ and an infinite sequence of positive random variables $\{\lambda(n), n=0(1)-\}$ such that, almost surely, $\xi(\zeta_n) \underline{v} = \lambda(n) \underline{v}$ for $n=0(1)-$. If, in addition, $E\{\log^+ \lambda(0)\} < \infty$ and $E\{\log \lambda(0)\} < 0$, then $\underline{q} = \underline{1}$.

Proof. This result follows directly from Lemma 9.1 and Corollary 9.1.4.

Clearly, the primary difficulty in applying Theorem 9.1 is the verification of condition (9.1). However, if we are willing to assume that $E\{\log^+ ||\xi(\zeta_0)||\} < \infty$, then we can gain greater insight into condition (9.1) by applying the results on products of random matrices to the MBPRE. Hence,

let us assume that $E\{\log^+ ||\xi(\zeta_0)||\} < \infty$.

We can now present a result which is extremely informative, but unfortunately not as sharp as Theorem 9.1. This drawback is small, however, in view of the very applicable corollary which is a direct consequence of this theorem.

Theorem 9.2. Suppose the MBPRE is impartial and
 $E\{\log^+ ||\xi(\zeta_0)||\} < \infty$. Then

$$(9.8) \quad r = \lim_{n \rightarrow \infty} (n+1)^{-1} E\{\log ||\xi(\zeta_0)\xi(\zeta_1) \cdots \xi(\zeta_n)||\},$$

exists and $-\infty \leq r < \infty$. Further,

$$(9.9) \quad r = \lim_{n \rightarrow \infty} (n+1)^{-1} \log ||\xi(\zeta_0)\xi(\zeta_1) \cdots \xi(\zeta_n)||,$$

almost surely. If $r < 0$, then $q = \frac{1}{\sim}$.

Proof. Combining the results of Theorem 8.1 with Remark 2.6, we obtain (9.8) and (9.9). Also, since

$$e_{\frac{1}{\sim}} \xi(\zeta_0)\xi(\zeta_1) \cdots \xi(\zeta_n) \frac{1}{\sim} \leq ||\xi(\zeta_0)\xi(\zeta_1) \cdots \xi(\zeta_n)||,$$

almost surely, for $n=0(1)-$, it follows that

$$\limsup_{n \rightarrow \infty} (n+1)^{-1} \log[e_{\frac{1}{\sim}} \xi(\zeta_0)\xi(\zeta_1) \cdots \xi(\zeta_n) \frac{1}{\sim}] \leq r,$$

almost surely. Thus, if $r < 0$, then Theorem 9.1 implies that $q = \frac{1}{\sim}$.

Corollary 9.2.1. Suppose the MBPRE is impartial and
 $E\{\log^+ ||\xi(\zeta_0)||\} < \infty$. Suppose also, that there is a

positive integer N such that

$$(9.10) \quad E\{\log ||\xi(\zeta_0)\xi(\zeta_1)\cdots\xi(\zeta_N)||\} < 0.$$

Then $\underline{q} = \underline{1}$.

Proof. This is a direct consequence of Corollary 8.1.1 and Theorem 9.2.

Remark 9.5. Observe that (9.8) gives an expression for r . If we could evaluate this limit in (9.8) to find r exactly however, we could probably, just as easily, evaluate the limit superior in condition (9.1) or the limit in condition (9.9).

On the other hand, Corollary 9.2.1 does provide us with conditions which might be easily verified in a real situation. If we were extremely fortunate and could verify (9.10) for some positive integer N , then Corollary 9.2.1 would be the most important result stated so far. However, if we could not find an appropriate N to satisfy (9.10) and we could not infer that $\{E \log[||\xi(\zeta_0)\cdots\xi(\zeta_n)||], n=0(1)-\}$ was bounded below by some positive number, then we could not conclude that $r < 0$ or $r > 0$. In this case, the above results would have little significance.

We now conclude this section on almost certain extinction of the MBPRE, given that $|Z(0)| = 1$, with a conjecture. First, however, we need to define a singular MBPRE.

Definition 9.1. The MBPRE is said to be singular if the multivariate probability generating functions $G(1, \underline{s} | e_j)$, $j=1(1)k$, are all linear in $s(i)$, $i=1(1)k$, with no constant terms; that is, each particle generates exactly one particle.

Remark 9.6. If the MBPRE is singular, then $\xi(\zeta_n) \underline{1} = \underline{1}$, almost surely, for $n=0(1)-$ and

$$\limsup_{n \rightarrow \infty} (n+1)^{-1} \log[e_j \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) \underline{1}] = 0,$$

almost surely, for $j=1(1)k$. It also follows that $q = 0$.

As noted in Remark 9.1, Smith and Wilkinson were able to prove, for the BPPE, a result similar to Theorem 9.1 for $r \leq 0$ and not just $r < 0$. They did, however, use a result by Chung and Fuchs [2] which appears to be crucial in proving the result for $r = 0$. We believe that "similar" results are needed for the case $r = 0$ of the MBPRE and that, in view of Remark 9.6, the following conjecture is true.

Conjecture 9.1. Suppose the MBPRE is impartial and nonsingular. Suppose, also, that for some integer j_0 , $1 \leq j_0 \leq k$, and some r , $-\infty \leq r \leq 0$,

$$r \geq \limsup_{n \rightarrow \infty} (n+1)^{-1} \log[e_{j_0} \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) \underline{1}],$$

almost surely. Then $q = \underline{1}$.

10. A condition for nonextinction

In this section we shall state and prove a theorem which gives conditions under which there is a positive probability of survival of the MBPRE, whenever $|\underline{z}(0)| = 1$. Before presenting this result, however, we must prove a technical lemma. After doing this and after presenting our main result, we will comment on the various hypotheses. Then, by applying the results on products of random matrices, we will give an equivalent theorem which is more informative, but equally difficult, we believe, to apply. As in Section 9, we will then state some corollaries which involve simpler, but less general, conditions. We will conclude with a comment on necessary and sufficient conditions for almost certain extinction of the MBPRE, given that $|\underline{z}(0)| = 1$.

Lemma 10.1. Let $W(1), W(2), \dots, W(h)$ be nonnegative, integral valued random variables and suppose $E|W(i)| < \infty$ for $i=1(1)h$. Denote the finite mean $E W(i)$ by $m(i)$ for $i=1(1)h$. Then for every $\epsilon, 0 < \epsilon < 1$, there exists a real number $t(\epsilon) \in [0,1)$ such that

$$(10.1) \quad E\{\prod_{j=1}^h s(j)^{W(j)}\} \leq \prod_{j=1}^h s(j)^{\epsilon m(j)},$$

whenever $\underline{s} \in [t(\epsilon), 1]^h$.

Proof. Assume $h=1$. If $m(1) = 0$, then $W(1) = 0$, almost surely, and hence (10.1) will be satisfied for every

ε , $0 < \varepsilon < 1$, if $t(\varepsilon) = 0$. Suppose then that $m(1) > 0$ and that for some ε , $0 < \varepsilon < 1$, there does not exist $t(\varepsilon) \in [0, 1)$ such that (10.1) is satisfied. Then there exists an increasing sequence $t_1, t_2, t_3, \dots$ of real numbers such that

$$i) \quad t_n \in (0, 1) \quad \text{for } n=1(1)-,$$

$$ii) \quad t_n \rightarrow 1, \quad \text{as } n \rightarrow \infty,$$

$$\text{and} \quad iii) \quad t_n^{\varepsilon m(1)} < E\{t_n^{W(1)}\}, \quad n=1(1)-.$$

Thus, using (iii), we obtain for $n=1(1)-$

$$(1 - t_n^{\varepsilon m(1)}) / (1 - t_n) > (1 - E\{t_n^{W(1)}\}) / (1 - t_n),$$

which implies, as $n \rightarrow \infty$, that $\varepsilon m(1) \geq m(1)$. Since $m(1)$ is positive and $0 < \varepsilon < 1$, this is a contradiction. Hence, there exists $t(\varepsilon) \in [0, 1)$ which satisfies (10.1) for $h=1$.

Assume now that h is an integer greater than one. By repeated application of the Schwarz inequality, we have

$$\begin{aligned} E\{\prod_{j=1}^h s(j)^{W(j)}\} \\ \leq [\prod_{j=1}^{h-1} (E\{s(j)^{2^j W(j)}\})^{2^{-j}}] (E\{s(h)^{2^{h-1} W(h)}\})^{2^{1-h}}, \end{aligned}$$

for $s \in [0, 1]^h$. Choose ε such that $0 < \varepsilon < 1$. Since (10.1) holds for $h=1$, there exist real numbers $t(i, \varepsilon) \in [0, 1)$ for $i=1(1)h$ such that $E\{s(i)^{W(i)}\} \leq s(i)^{\varepsilon m(i)}$, whenever $t(i, \varepsilon) \leq s(i) \leq 1$, for $i=1(1)h$. Combining these results, we obtain inequality (10.1), whenever $t(i, \varepsilon) \leq s(i)^{2^i} \leq 1$ for $i=1(1)h-1$ and $t(h, \varepsilon) \leq s(h)^{2^{h-1}} \leq 1$. Thus, if we define $t(\varepsilon)$ by

$$t(\varepsilon) = \max\{t(i, \varepsilon)^{2^{-i}}, i=1(1)h\},$$

then $t(\varepsilon) \in [0,1)$ and (10.1) is satisfied whenever $\underline{s} \in [t(\varepsilon), 1]^h$.

Theorem 10.1. Suppose the space Θ of the MBPRE contains only a finite number of points. Further, suppose that $\{\xi(\zeta_n), n=0(1)-\}$ takes only values in some space $M(C,N)$ and that for some integer j_0 , $1 \leq j_0 \leq k$, and some real number $r > 0$,

$$(10.2) \quad r \leq \liminf_{n \rightarrow \infty} (n+1)^{-1} \log [e_{j_0} \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) 1],$$

with probability one. Then $q < 1$.

Proof. We consider two cases.

Case 1. ($N = 1$) We begin by choosing ε such that $\varepsilon = \exp(-r/2)$. Clearly $0 < \varepsilon < 1$. Applying Lemma 10.1, we have for every $\theta \in \Theta$ there exists a real number $t(\theta)$ in $(0,1)$ such that

$$(10.3) \quad \phi(j, \theta, \underline{s}) \leq \prod_{i=1}^k s(i)^{\varepsilon \xi(j, i, \theta)}$$

for $j=1(1)k$, whenever $\underline{s} \in [t(\theta), 1]^k$. Since Θ contains only a finite number of points, if $t = \max\{t(\theta), \theta \in \Theta\}$, then $t \in (0,1)$ and (10.3) is satisfied for $j=1(1)k$ and $\theta \in \Theta$, whenever $\underline{s} \in [t, 1]^k$.

Now for any $\underline{s} \in [t, 1]^k$ there is a unique nonnegative vector $\underline{w}$ such that $\underline{w} \in [0, 1]^k$ and $\underline{s} = (t^{w(1)}, t^{w(2)}, \dots, t^{w(k)})$. For $j=1(1)k$, $\theta \in \Theta$, $\underline{s} \in [t, 1]^k$ and $\underline{s} = (t^{w(1)}, \dots, t^{w(k)})$, let us define $\psi(j, \theta, \underline{s})$ by

$$\begin{aligned}
 (10.4) \quad \psi(j, \theta, \underline{s}) &= \psi(j, \theta, (t^{w(1)}, t^{w(2)}, \dots, t^{w(k)})) \\
 &= \begin{cases} \exp\{(\epsilon \sum_{i=1}^k w(i) \xi(j, i, \theta)) \log t\}, & \text{if } m \leq 1, \\ \exp\{(\epsilon m^{-1} \sum_{i=1}^k w(i) \xi(j, i, \theta)) \log t\}, & \text{if } m > 1, \end{cases}
 \end{aligned}$$

where

$$m = \max\{\epsilon \sum_{i=1}^k w(i) \xi(j, i, \theta), j=1(1)k\}.$$

Observe that $\psi(j, \theta, \underline{s}) \in [t, 1]$ and that by (10.3)

$$(10.5) \quad \phi(j, \theta, \underline{s}) \leq \psi(j, \theta, \underline{s})$$

for $j=1(1)k$, $\theta \in \Theta$, and $\underline{s} \in [t, 1]^k$.

Consider now the infinite sequence of random vectors $\{\underline{Y}(n) = [Y(1, n), \dots, Y(k, n)], n=0(1)-\}$ which takes values in $[t, 1]^k$ and is defined by

$$\begin{aligned}
 \underline{Y}(0) &= t\underline{1}, \\
 \underline{Y}(n+1) &= \underline{\psi}(\underline{\zeta}_n, \underline{Y}(n)), \quad n=0(1)-.
 \end{aligned}$$

It follows that, for $j=1(1)k$,

$$Y(j, 1) = \exp\{(\epsilon e_j \xi(\zeta_0) \underline{1} / \max[1, \|\epsilon \xi(\zeta_0)\|]) \log t\}$$

and $Y(j, 2) =$

$$t^{\epsilon^2 e_j \xi(\zeta_1) \xi(\zeta_0) \underline{1} / \max[1, \|\epsilon \xi(\zeta_0)\|, \|\epsilon^2 \xi(\zeta_1) \xi(\zeta_0)\|]}.$$

For convenience, let us write

$$\rho(j,n) = \varepsilon^{n+1} e_j \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) \frac{1}{\varepsilon}$$

$$\pi(j,n) = \varepsilon^{n+1} e_j \xi(\zeta_n) \xi(\zeta_{n-1}) \cdots \xi(\zeta_0) \frac{1}{\varepsilon}$$

$$\Delta(j,n) = \max[1; ||\varepsilon^{i+1} \xi(\zeta_i) \xi(\zeta_{i-1}) \cdots \xi(\zeta_0)||, \quad i=0(1)n]$$

for $n=0(1)-$ and $j=1(1)k$. Then, in general, we have

$$Y(j, n+1) = \exp\{[\pi(j,n)/\Delta(j,n)] \log t\}$$

for $n=0(1)-$ and $j=1(1)k$. Clearly $Y(j,n)$ is a random variable, and by Assumption A2 and (10.5) we obtain

$$\begin{aligned} (10.6) \quad E\{\underline{Y}(n) | \underline{Y}(0) = t \frac{1}{\varepsilon}\} \\ &= E\{\psi(\zeta_{n-1}, \psi(\zeta_{n-2}, \psi(\zeta_{n-3}, \dots, \psi(\zeta_0, t \frac{1}{\varepsilon}) \dots))\} \\ &\geq E\{\phi(\zeta_{n-1}, \psi(\zeta_{n-2}, \psi(\zeta_{n-3}, \dots, \psi(\zeta_0, t \frac{1}{\varepsilon}) \dots))\} \\ &\geq E\{\phi(\zeta_{n-1}, \phi(\zeta_{n-2}, \psi(\zeta_{n-3}, \dots, \psi(\zeta_0, t \frac{1}{\varepsilon}) \dots))\} \\ &\geq \dots \\ &\geq E\{\phi(\zeta_{n-1}, \phi(\zeta_{n-2}, \phi(\zeta_{n-3}, \dots, \phi(\zeta_0, t \frac{1}{\varepsilon}) \dots))\} \\ &= E\{\underline{X}(n) | \underline{X}(0) = t \frac{1}{\varepsilon}\}, \quad n=0(1)-. \end{aligned}$$

Observe now that (10.2) implies that

$$\liminf_{n \rightarrow \infty} (n+1)^{-1} \log \rho(j_0, n) \geq r/2,$$

almost surely. Hence for small positive δ , $0 < \delta < 1$, there exists a positive integer n_0 such that

$$\Pr\{\log \rho(j_0, n) \geq 0, \quad n=n_0(1)-\} \geq 1 - \delta/2.$$

Further, there exists a sufficiently large positive x_0 such that

$$\Pr\{\log \rho(j_0, n) \geq -x_0, \quad n=0(1)n_0\} \geq 1 - \delta/2.$$

This implies that

$$(10.7) \quad \Pr\{\log \rho(j_0, n) \geq -x_0, \quad n=0(1)-\} \geq 1 - \delta.$$

To conclude this proof for $N = 1$, let $\alpha = C^{-1} \exp(-x_0)$.

Then for $n=0(1)-$,

$$(10.8) \quad \Pr\{Y(j_0, n+1) \leq t^\alpha\} = \Pr\{\pi(j_0, n)/\Delta(j_0, n) \geq \alpha\} = \\ \Pr\{\min[\pi(j_0, n); \pi(j_0, n)/|\varepsilon^{i+1} \xi(\zeta_i) \cdots \xi(\zeta_0)|], \quad i=0(1)n] \geq \alpha\}.$$

Note that if $A = A_1 A_2 \cdots A_u$ and $B = B_1 B_2 \cdots B_v$ where $A_1, \dots, A_u, B_1, \dots, B_v \in M(C, 1)$, then by (8.4)

$$(\varepsilon_j A B \varepsilon) / |B| = (\sum_{i=1}^k A(j, i) \varepsilon_i B \varepsilon) / |B| \\ \geq C^{-1} \varepsilon_j A \varepsilon, \quad j=1(1)k.$$

Combining this fact with the fact that $\alpha C < 1$, we obtain,

for $n=0(1)-$,

$$(10.9) \quad \Pr\{Y(j_0, n+1) \leq t^\alpha\} \\ \geq \Pr\{\min[1; \varepsilon^{n+1-i} \varepsilon_{j_0} \xi(\zeta_n) \xi(\zeta_{n-1}) \cdots \xi(\zeta_i) \varepsilon, \quad i=0(1)n] \geq \alpha C\} \\ = \Pr\{\min[\varepsilon^{i+1} \varepsilon_{j_0} \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_i) \varepsilon, \quad i=0(1)n] \geq \alpha C\}.$$

This last equality follows by renumbering the independent and identically distributed random matrices; in particular, we substitute $\xi(\zeta_m)$ for $\xi(\zeta_{n-m})$, $m=0(1)n$. It now follows, using (10.7), that

$$\begin{aligned}
(10.10) \quad & \Pr\{Y(j_0, n+1) \leq t^\alpha\} \\
& \geq \Pr\{\log[\varepsilon_{j_0}^{i+1} \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_i)] \geq -x_0, i=0(1)n\} \\
& \geq 1 - \delta, \quad n=0(1)-.
\end{aligned}$$

Since $0 < \alpha < 1$ and $t \in (0, 1)$, we conclude that

$$E\{Y(j_0, n+1) | \tilde{Y}(0)\} \leq 1 - (1-t^\alpha)(1-\delta) < 1, \quad n=0(1)-.$$

Hence, $\limsup_{n \rightarrow \infty} E\{Y(j_0, n) | \tilde{Y}(0) = t\} < 1$. Further, by (10.6), $\limsup_{n \rightarrow \infty} E\{X(j_0, n) | \tilde{X}(0) = t\} < 1$. It follows by (7.2) of Remark 7.2 that $\limsup_{n \rightarrow \infty} E\{X(j_0, n) | \tilde{X}(0) = 0\} < 1$. Thus, combining (7.3) and Remark 6.1, we can infer that $q(\varepsilon_{j_0}) < 1$.

Now since $\{\xi(\zeta_n), n=0(1)-\}$ takes only values in some space $M(C, 1)$, Definition 8.1 implies the MBPRE is impartial. Thus, Theorem 6.1 implies that $q < 1$.

Case 2. ($N > 1$) In Remark 3.3 we verified that the process $\{\tilde{Z}(nN), n=0(1)-\}$ is a MBPRE. It follows immediately from the hypotheses of this theorem that θ^N contains only a finite number of points, that

$$\{\xi(\zeta_{nN}) \xi(\zeta_{nN+1}) \cdots \xi(\zeta_{nN+N-1}), n=0(1)-\}$$

takes only values in some space $M(C, 1)$, and that

$$r \leq \liminf_{n \rightarrow \infty} (n+1)^{-1} \log[\varepsilon_{j_0} \tau_0 \tau_1 \cdots \tau_n],$$

almost surely, where $\tau_i = \xi(\zeta_{iN}) \xi(\zeta_{iN+1}) \cdots \xi(\zeta_{iN+N-1})$ for $i=0(1)-$. Thus, we may apply the results of Case 1 to the

process $\{\tilde{Z}(nN), n=0(1)-\}$. In particular, we have
 $\lim_{n \rightarrow \infty} G(nN, 0 | \tilde{e}_j) < 1$ for $j=1(1)k$. Remark 6.1 now
 implies that $\tilde{q} < 1$.

Remark 10.1. It might be helpful to give some comments
 on the hypotheses of Theorem 10.1.

To begin, by assuming that $\{\tilde{\xi}(\zeta_n), n=0(1)-\}$ takes
 only values in some space $M(C, N)$, we not only guarantee
 that the MBPRE is impartial, but we also make it possible
 to continue the proof from (10.8) to (10.9). Of course,
 a more judicious choice of m in the definition of $\psi(j, \theta, \tilde{s})$
 (see (10.4)) could possibly have eliminated the use of the
 norm. However, we need to be sure that $\psi(j, \theta, \tilde{s})$ always
 takes values in $[t, 1]$ so that the $\tilde{Y}(n)$ process is well-
 defined and so that we may verify inequality (10.6). If we
 had let $m = \varepsilon \sum_{i=1}^k w(i) \xi(j, i, \theta)$, say, then the same effect
 would have been obtained and the norm would have been
 removed. But, then the formulation of $Y(j, n)$ would have
 become so complex that we would have been unable to evaluate
 $\Pr\{Y(j_0, n) \leq t^\alpha\}$. Thus, we assumed that $\{\tilde{\xi}(\zeta_n), n=0(1)-\}$
 takes only values in some space $M(C, N)$.

Observe also, that if $\{\tilde{\xi}(\zeta_n), n=0(1)-\}$ takes only
 values in some space $M(C, 1)$, then we only need the
 hypothesis that θ contains only a finite number of points
 to show that there exists some $t \in (0, 1)$ so that (10.3) is
 satisfied whenever $j \in \{1, 2, \dots, k\}$, $\theta \in \theta$, and $\tilde{s} \in [t, 1]^k$. Thus,
 in a particular case, if $N = 1$ and we know that there is

some $t \in (0,1)$ for which (10.3) is true for appropriate j , θ , and ξ , then it is irrelevant whether Θ consists of finitely many points or infinitely many points.

Finally, it appears that if we wish to prove a result similar to Theorem 10.1 for which Θ may be any set, we must find an additional condition as was the case in the BPRE (Smith and Wilkinson [11, p. 820] and Smith [10]). At present, we cannot hypothesize what this condition might be.

Before applying the results on products of random matrices to Theorem 10.1, let us pause to consider further consequences of Theorem 10.1 with respect to the multitype Galton-Watson process.

Remark 10.2. Suppose the MBPRE is impartial and the space Θ contains exactly one point, say θ . By Assumption A4 and Remark 4.1, $\xi(\theta) = \xi$ is primitive. Thus, if $\lambda(\xi) > 1$, the standard result for extinction of the multitype Galton-Watson process (Harris [8, p. 41]) implies that $q < 1$.

This result is also a consequence of Theorem 10.1. Indeed, since $\xi(\theta)$ is primitive, $\xi(\theta)$ forms, by itself, a class of matrices of the form $M(C,N)$. Further, if $\lambda(\xi) > 1$, then Theorem 4.1 implies that

$$\lim_{n \rightarrow \infty} (n+1)^{-1} \log [e_1 \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) 1] = \log \lambda(\xi) > 0,$$

with probability one. Applying Theorem 10.1, we have, again, $q < 1$.

The primary difficulty in applying Theorem 10.1 is the verification of (10.2). Recall, however, that in Section 9, by assuming that $E\{\log^+ ||\xi(\zeta_0)||\} < \infty$, we gained considerable insight into condition (9.1) when we applied the results on products of random matrices. In this section, we will not need to assume that $E\{\log^+ ||\xi(\zeta_0)||\}$ is finite since Assumption A4 and our hypothesis that Θ contains only a finite number of points already imply this condition. Hence, we can apply the theorems of Section 8 to Theorem 10.1 to obtain a better result. The one noticeable defect, unfortunately, will be the absence of an applicable corollary similar to Corollary 9.2.1.

Theorem 10.2. Suppose the space Θ of the MBPRE contains only a finite number of points. Further, suppose that $\{\xi(\zeta_n), n=0(1)-\}$ takes only values in some space $M(C,N)$. Then

$$r = \lim_{n \rightarrow \infty} (n+1)^{-1} E\{\log ||\xi(\zeta_0)\xi(\zeta_1)\cdots\xi(\zeta_n)||\}$$

exists and $-\infty \leq r < \infty$. Further,

$$r = \lim_{n \rightarrow \infty} (n+1)^{-1} \log ||\xi(\zeta_0)\xi(\zeta_1)\cdots\xi(\zeta_n)||$$

and

$$r = \lim_{n \rightarrow \infty} (n+1)^{-1} \log [e_j \xi(\zeta_0)\xi(\zeta_1)\cdots\xi(\zeta_n) 1],$$

almost surely, for $j=1(1)k$. If $r > 0$, then $q \sim 1$.

Proof. This result follows directly from the above comment, Corollary 8.1.2 and Theorem 10.1.

We now state some corollaries which parallel those in Section 9; their proofs are too similar to merit repetition.

Corollary 10.1.1. Suppose the space Θ of the MBPRE contains only a finite number of points. Further, suppose that $\{\xi(\zeta_n), n=0(1)-\}$ takes only values in some space $M(C,N)$ and that $\Delta(\zeta_n) = \min\{e_{\tilde{j}} \xi(\zeta_n) \tilde{1}, j=1(1)k\}$ for $n=0(1)-$. Then, $\{\Delta(\zeta_n), n=0(1)-\}$ is an infinite sequence of independent and identically distributed random variables. If $E\{\log^+ \Delta(\zeta_0)\} < \infty$ and $E\{\log \Delta(\zeta_0)\} > 0$, then $q < \tilde{1}$.

Corollary 10.1.2. Suppose the space Θ of the MBPRE contains only a finite number of points. Further, suppose that $\{\xi(\zeta_n), n=0(1)-\}$ takes only values in some space $M(C,N)$ and that there is an infinite sequence of independent and identically distributed nonnegative random matrices $\{\delta(\zeta_n), n=0(1)-\}$ such that $\xi(\zeta_n) \geq \delta(\zeta_n)$, almost surely, for $n=0(1)-$. If for some integer $j_0, 1 \leq j_0 \leq k$, and some real number $r > 0$,

$$(10.11) \quad r \leq \liminf_{n \rightarrow \infty} (n+1)^{-1} \log[e_{\tilde{j}_0} \delta(\zeta_0) \cdots \delta(\zeta_n) \tilde{1}],$$

almost surely, then $q < \tilde{1}$.

Remark 10.3. One particularly interesting choice for the infinite sequence $\{\delta(\zeta_n), n=0(1)-\}$ is defined by

$$\delta(i, j, \zeta_n) = \begin{cases} \xi(i, i, \zeta_n), & \text{if } i=j, \\ 0, & \text{if } i \neq j, \end{cases}$$

for $i, j = 1(1)k$ and $n=0(1)-$. In this case, the strong law of large numbers and Remark 2.6 imply that the right hand side of (10.11) is equal to $E\{\log \xi(j_0, j_0, \zeta_0)\}$, almost surely, if $E|\log \xi(j_0, j_0, \zeta_0)| < \infty$. Thus, if for some $j_0, 1 \leq j_0 \leq k$, $E\{\log \xi(j_0, j_0, \zeta_0)\} > 0$, then $\underline{q} < \underline{1}$, whenever the other two hypotheses are satisfied.

Corollary 10.1.3. Suppose the space Θ of the MBPRE contains only a finite number of points. Further, suppose that $\{\xi(\zeta_n), n=0(1)-\}$ takes only values in some space $M(C, N)$ and that for some positive vector $\underline{v}$ there is an infinite sequence of positive random variables $\{\lambda(n), n=0(1)-\}$ such that $\xi(\zeta_n) \underline{v} = \lambda(n) \underline{v}$, almost surely, for $n=0(1)-$. If $E\{\log^+ \lambda(0)\} < \infty$ and $E\{\log \lambda(0)\} > 0$, then $\underline{q} < \underline{1}$.

Corollary 10.1.4. Suppose the space Θ of the MBPRE contains only a finite number of points. Further suppose that $\{\xi(\zeta_n), n=0(1)-\}$ takes only values in some space $M(C, N)$ and that in this space multiplication is commutative. Then there exists a positive vector $\underline{v}$ and an infinite sequence of positive random variables $\{\lambda(n), n=0(1)-\}$ such that, with probability one, $\xi(\zeta_n) \underline{v} = \lambda(n) \underline{v}$, for $n=0(1)-$. If $E\{\log^+ \lambda(0)\} < \infty$ and $E\{\log \lambda(0)\} > 0$, then $\underline{q} < \underline{1}$.

We conclude this section with a remark about the existence of necessary and sufficient conditions for almost certain extinction of the MBPRE, given that $|\underline{z}(0)| = 1$. The following result, which is a corollary

to Theorems 9.2 and 10.2, will form the basis for our remarks.

Theorem 10.3. Suppose the space Θ of the MBPRE contains only a finite number of points. Further, suppose that $\{\xi(\zeta_n), n=0(1)-\}$ takes only values in some space $M(C,N)$. Then

$$(10.12) \quad r = \lim_{n \rightarrow \infty} (n+1)^{-1} E\{\log ||\xi(\zeta_0)\xi(\zeta_1)\cdots\xi(\zeta_n)||\}$$

exists and $-\infty \leq r < \infty$. If $r < 0$, then $q = 1$. If $r > 0$, then $q < 1$.

Remark 10.4. Suppose the MBPRE is nonsingular and that the two hypotheses of Theorem 10.3 are satisfied. Further, assume that Conjecture 9.1 is valid. Then, if r is defined by (10.12), we can conclude, by combining Theorem 10.3 and Conjecture 9.1, that $r \leq 0$ is a necessary and sufficient condition for $q = 1$. Equivalently, $r \leq 0$ if and only if the probability of the MBPRE ever becoming extinct, given that $|Z(0)| = 1$, is one.

CHAPTER III
EXTINCTION PROBABILITIES FOR POPULATIONS
WITH ANY GIVEN INITIAL POPULATION

11. The ergodic distribution

In Chapter II we considered conditions for almost certain extinction and nonextinction of the MBPRE with an initial population of only one particle. Let us extend our discussion to the MBPRE with a fixed, but arbitrary, initial population. We shall begin by extending our earlier results on extinction probabilities and on the moments of the dual process. We will then be able to demonstrate the existence of a multivariate distribution function each moment of which is precisely the probability of extinction of the MBPRE for some given initial population. After this, we will show that the results of Sections 9 and 10 can be easily extended to include the MBPRE with any given initial population.

Let us begin by extending our definition of extinction probabilities.

Definition 11.1. For $a \in T$, let $q(a)$ denote the extinction probability of the MBPRE given that the initial

population $Z(0) = \underline{a}$; that is, define $q(\underline{a})$ by

$$q(\underline{a}) = \Pr\{\underline{Z}(n) = \underline{0} \text{ for some } n=1(1)- \mid \underline{Z}(0) = \underline{a}\}.$$

Remark 11.1. Observe that Definition 11.1 is consistent with Definition 6.1 and that by applying the same reasoning as was used in Remark 6.1, we can conclude that

$$(11.1) \quad q(\underline{a}) = \lim_{n \rightarrow \infty} G(n, \underline{0} \mid \underline{a}), \quad \underline{a} \in T.$$

Moreover, combining (11.1) with our results on the various representations of $G(n, \underline{s} \mid \underline{a})$ (see Remark 3.2) and our results on the dual process (see Remark 7.2), we immediately obtain that

$$q(\underline{a}) = \lim_{n \rightarrow \infty} E\{\prod_{j=1}^k X(j, n)^{a(j)} \mid \underline{X}(0) = \underline{0}\}$$

whenever $\underline{a} \in T$. Now, since the dual process takes only values in the unit k -cube, Example (d) of Feller [4, Vol. 2, p. 244] implies that $\underline{X}(n)$ converges in distribution to some random vector $\underline{V}$ as $n \rightarrow \infty$. Further, $\underline{V}$ takes only values in $[0, 1]^k$ and

$$q(\underline{a}) = E\{\prod_{j=1}^k V(j)^{a(j)}\}, \quad \underline{a} \in T.$$

Definition 11.2. For $\underline{x} \in [0, 1]^k$, let $F(\underline{x}) = \Pr\{\underline{V} \leq \underline{x}\}$. We shall refer to $F(\underline{x})$ as the ergodic distribution function associated with the dual process $\{\underline{X}(n), n=0(1)-\}$ (and $\underline{X}(0) = \underline{0}$).

Theorem 11.1. To any MBPRE, $\{\underline{Z}(n), n=0(1)-\}$, there corresponds a dual Markov process, $\{\underline{X}(n), n=0(1)-\}$, which

is defined on the unit k-cube and which possesses a limiting ergodic distribution F. The moments of F are precisely the extinction probabilities $q(\underline{a})$, $\underline{a} \in T$; that is,

$$(11.2) \quad q(\underline{a}) = E\{\prod_{j=1}^k v(j)^{a(j)}\} = \int \prod_{j=1}^k x(j)^{a(j)} dF(\underline{x}).$$

Theorem 11.2. Suppose the MBPRE is impartial. Assume also, that for some integer j_0 , $1 \leq j_0 \leq k$, and some number r , $-\infty \leq r < 0$,

$$r \geq \limsup_{n \rightarrow \infty} (n+1)^{-1} \log [e_{j_0} \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) \underline{1}],$$

almost surely. Then the ergodic distribution F is degenerate and concentrates all its mass at $\underline{1}$. Also $q(\underline{a}) = 1$ for every $\underline{a} \in T$.

Proof. From Theorem 9.1 we can infer that $q(\underline{e}_i) = 1$ for $i=1(1)k$. Applying (11.2) of Theorem 11.1, we see immediately that $\underline{y} = \underline{1}$, with probability one, and moreover that $q(\underline{a}) = 1$ for every $\underline{a} \in T$.

Theorem 11.3. Suppose the space Θ of the MBPRE contains only a finite number of points. Further, suppose that $\{\xi(\zeta_n), n=0(1)\infty\}$ takes only values in some space $M(C,N)$ and that for some integer j_0 , $1 \leq j_0 \leq k$, and some real number $r > 0$,

$$r \leq \liminf_{n \rightarrow \infty} (n+1)^{-1} \log [e_{j_0} \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) \underline{1}],$$

almost surely. Then $F(\underline{1}-) = 1$, $q(\underline{a}) < 1$ for $\underline{a} \in T - \{0\}$, and $q(\underline{a}) \rightarrow 0$ as $|\underline{a}| \rightarrow \infty$.

Proof. As in the proof of Theorem 10.1, we consider two cases.

Case 1. Suppose our hypotheses are valid and that $N=1$. Then the initial arguments employed in the proof of Theorem 10.1 imply that there exists a number $t \in (0,1)$ such that (10.3) is true whenever $s \in [t,1]^k$, $\theta \in \Theta$, and $1 \leq j \leq k$. Further, from (10.10) we have that for each $\delta > 0$ there is a sufficiently large x_0 such that

$$(11.3) \quad \Pr\{Y(j_0, n+1) \leq t^{C^{-1}\exp(-x_0)} \mid \underline{Y}(0) = \underline{t}\} \geq 1 - \delta$$

for $n=0(1)-$. From the definition of the $\underline{Y}(n)$ process (see (10.5) - (10.6)) and Remark 7.2 we have that

$$\underline{Y}(n) \mid \underline{Y}(0) = \underline{s} \geq \underline{X}(n) \mid \underline{X}(0) = \underline{s} \geq \underline{X}(n) \mid \underline{X}(0) = \underline{0},$$

almost surely, whenever $s \in [0,1]^k$ and n is a nonnegative integer. Combining this result with (11.3), we obtain that

$$\Pr\{X(j_0, n+1) \leq t^{C^{-1}\exp(-x_0)} \mid \underline{X}(0) = \underline{0}\} \geq 1 - \delta$$

for $n=0(1)-$. Since $\underline{X}(n)$ converges in distribution to $\underline{V}$,

$$\Pr\{V(j_0) \leq t^{C^{-1}\exp(-x_0)}\} \geq 1 - \delta$$

and, in particular, $V(j_0) < 1$, with probability one.

Now Theorem 10.2 implies that

$$\lim_{n \rightarrow \infty} (n+1)^{-1} \log[e_j \xi(\zeta_0) \xi(\zeta_1) \cdots \xi(\zeta_n) \underline{1}] = r,$$

almost surely, for every j , $1 \leq j \leq k$. Thus, in the above remarks, we can let j_0 be any integer, $1 \leq j_0 \leq k$. So, $V(j) < 1$,

with probability one, for $j=1(1)k$. Moreover, by applying (11.2) it follows immediately that $q(\underline{a}) < 1$ for $\underline{a} \in T - \{0\}$ and that $q(\underline{a}) \rightarrow 0$ as $|\underline{a}| \rightarrow \infty$.

Case 2. Suppose now that our hypotheses are satisfied and that $N > 1$. Then Remark 3.3 implies that the process $\{Z(nN), n=0(1)-\}$ is also a MBPRE. For convenience, let us denote the extinction probabilities associated with the $Z(nN)$ process by $q_N(\underline{a})$ for $\underline{a} \in T$, the dual process associated with the $Z(nN)$ process by $X_N(n)$, and the ergodic distribution associated with the $X_N(n)$ process by F_N . If we apply the arguments of Case 2 of the proof of Theorem 10.1, we can conclude immediately that $F_N(1-) = 1$, $q_N(\underline{a}) < 1$ for $\underline{a} \in T - \{0\}$, and $q_N(\underline{a}) \rightarrow 0$ as $|\underline{a}| \rightarrow \infty$. A brief review of the definition of $q_N(\underline{a})$ (see Definition 11.1 and Remark 11.1) however, reveals that $q_N(\underline{a}) = q(\underline{a})$ for $\underline{a} \in T$. Moreover, although the $X_N(n)$ process and the $X(n)$ process are not, in general, equivalent, the respective moments of the ergodic distributions, F_N and F , are equal. Since the support of both distributions is the compact $[0,1]^k$, F_N and F are completely determined by their respective moments and, we can conclude that $F_N = F$. Thus $F(1-) = 1$, $q(\underline{a}) < 1$ for $\underline{a} \in T - \{0\}$, and $q(\underline{a}) \rightarrow 0$ as $|\underline{a}| \rightarrow \infty$.

It should now be clear that all the results presented in Sections 9 and 10 can be somewhat generalized. In particular, instead of concluding that $q = 1$ or $q < 1$,

we may now infer, under the same hypotheses, that $q(\underline{a}) = 1$ for every $\underline{a} \in T$ or that $q(\underline{a}) < 1$ for every $\underline{a} \in T - \{0\}$. Further, we may continue to apply all our corollaries, whenever possible, to obtain these generalized results for the MBPRE.

We conclude this work by stating a theorem which, together with Conjecture 9.1, suggests that $r \leq 0$ may be a necessary and sufficient condition for almost certain extinction of the MBPRE, under certain hypotheses. The veracity of this statement depends, of course, on the probability of extinction of the MBPRE when $r = 0$. Unfortunately, we have been unable to draw any significant conclusions about the probability of extinction of the MBPRE when $r = 0$.

Theorem 11.4. Suppose the space Θ of the MBPRE contains only a finite number of points. Further, suppose that $\{\xi(\zeta_n), n=0(1)-\}$ takes only values in some space $M(C,N)$. Then

$$r = \lim_{n \rightarrow \infty} (n+1)^{-1} E\{\log ||\xi(\zeta_0)\xi(\zeta_1) \cdots \xi(\zeta_n)||\}$$

exists and $-\infty \leq r < \infty$. If $r < 0$, then $q(\underline{a}) = 1$ for every $\underline{a} \in T$. If $r > 0$, then $q(\underline{a}) < 1$ for every $\underline{a} \in T - \{0\}$.

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