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NEW CONDITIONS FOR EXACTNESS OF A
SIMPLE PENALTY FUNCTION

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Stephen Howe

Department of Statistics
University of North Carolina at Chapel Hill
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ABSTRACT

We consider penalty function methods for finding the maximum of a function f over the set

$$S_0 = \{x \in \mathbb{R}^n: g_i(x) \leq 0 \text{ for } i=1, \dots, m \text{ and } h_j(x)=0 \text{ for } j=1, \dots, p\}.$$

New conditions, extending earlier work done by Pietrzykowski, are presented under which the penalty function

$$P(x, \mu) = \mu f(x) - \sum_{i=1}^m U(g_i(x)) - \sum_{j=1}^p |h_j(x)|$$

is locally exact. The relationships among the new conditions, Pietrzykowski's conditions, and Kuhn-Tucker constraint qualifications are explored.

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Stephen Howe*

1. INTRODUCTION. In 1969 Pietrzykowski [1], [2] presented conditions under which a local maximum x_0 of the function f on the set

$$S_0 = \{x \in \mathbb{R}^n: g_i(x) \leq 0, i=1, \dots, m \text{ and } h_j(x)=0, j=1, \dots, p\}$$

is also an unconstrained isolated (strong) local maximum, for μ sufficiently small, of the penalty function

$$p(x, \mu) = \mu f(x) - \sum_{i=1}^m U(g_i(x)) - \sum_{j=1}^p |h_j(x)|$$

where $U: \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$U(\alpha) = \begin{cases} \alpha & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha \leq 0 \end{cases}$$

and where $f, g_1, \dots, g_m, h_1, \dots, h_p$ are continuously differentiable real-valued functions on $\mathbb{R}^n$. Specifically, Pietrzykowski's condition is that x_0 be an isolated local maximum of f on S_0 and that the set of "active gradients"

$$\{\nabla g_i(x_0) \text{ each } i \in I, \text{ and } \nabla h_j(x_0) \text{ each } j=1, \dots, p\}$$

be linearly independent, where $I = \{i: g_i(x_0) = 0\}$. It is worth noting that the linear independence of the gradients, as specified, constitutes a constraint qualification which validates the Kuhn-Tucker necessary optimality

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criterion at x_0 . This suggests that any constraint qualification which validates the Kuhn-Tucker criterion might also guarantee the above result at an isolated local maximum of f on S_0 .

In this note a new condition for the above result is presented. The new condition, unlike that of Pietrzykowski, depends upon the objective function f , and furthermore it neither implies nor is implied by Pietrzykowski's condition. It is also shown that the weak constraint qualification [3] does not guarantee the result at an isolated local maximum of f on S_0 , and hence that the Kuhn-Tucker constraint qualifications are not sufficient for this result in general.

2. RESULT. Let $f, g_1, \dots, g_m, h_1, \dots, h_p, S_0, x_0, I$ and $p(x, \mu)$ be defined as above.

THEOREM. Suppose the following condition is satisfied.

G: For every nonzero $y \in \mathbb{R}^n$ such that $\nabla g_i(x_0)^T y \leq 0$ each $i \in I$ and $\nabla h_j(x_0)^T y = 0$ each $j = 1, \dots, p$ it follows that $\nabla f(x_0)^T y < 0$.

Then there is a $\mu_0 > 0$ such that if $0 < \mu \leq \mu_0$ then $p(x, \mu)$ has an unconstrained isolated local maximum at x_0 .

Remark: It can be shown that G is a sufficient, but not necessary, condition that

x_0 be an isolated local maximum of f on S_0 .

Proof: Since $g_1, \dots, g_m$ are continuous, there is a $\delta > 0$ such that $g_i(x) < 0$ for every $x \in B(x_0, \delta)$ and $i \in I$. Here $B(x_0, \delta)$ is the open ball of radius δ about x_0 . Now suppose, to get a contradiction, that there is a

sequence $\{\mu_n\}$ of positive scalars such that $\mu_n \rightarrow 0$ as $n \rightarrow \infty$ and where $p(x, \mu_n)$ does not have an unconstrained isolated local maximum at x_0 for each n . Then for each n there is a point x_n such that $0 < \|x_n - x_0\| < \min\{\delta, \mu_n\}$ but for which $p(x_n, \mu_n) \geq p(x_0, \mu_n)$. Denote $y_n = x_n - x_0$ each n , noting that $y_n \rightarrow 0$ as $n \rightarrow \infty$. Consider the sequence $\{y_n / \|y_n\|\}$ of vectors in the compact set $\{z \in \mathbb{R}^n : \|z\| = 1\}$. This sequence must have a subsequential limit $y_0 \in \mathbb{R}^n$ such that $\|y_0\| = 1$. Assume without loss of generality that $y_n / \|y_n\| \rightarrow y_0$ as $n \rightarrow \infty$.

Now suppose that for some $r \in I$, $\nabla g_r(x_0)^T y_0 > 0$. Then for each n

$$p(x_n, \mu_n) - p(x_0, \mu_n) \leq \mu_n [f(x_n) - f(x_0)] - U(g_r(x_n)).$$

Note that

$$\begin{aligned} \frac{1}{\|y_n\|} g_r(x_n) &= \frac{1}{\|y_n\|} [g_r(x_n) - g_r(x_0)] \\ &= \nabla g_r(x_0)^T \frac{y_n}{\|y_n\|} + \frac{o(\|y_n\|)}{\|y_n\|} \\ &\rightarrow \nabla g_r(x_0)^T y_0 > 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Clearly for n sufficiently large $g_r(x_n) > 0$ so that $U(g_r(x_n)) = g_r(x_n)$, whence for such n

$$\begin{aligned} \frac{1}{\|y_n\|} [p(x_n, \mu_n) - p(x_0, \mu_n)] &\leq \frac{\mu_n}{\|y_n\|} [f(x_n) - f(x_0)] - \frac{1}{\|y_n\|} g_r(x_n) \\ &= \mu_n \nabla f(x_0)^T \frac{y_n}{\|y_n\|} - \frac{1}{\|y_n\|} g_r(x_n) + \frac{o(\|y_n\|)}{\|y_n\|} \end{aligned}$$

but the latter converges to $-\nabla g_r(x_0)^T y_0 < 0$ as $n \rightarrow \infty$, implying that for large n $p(x_n, \mu_n) < p(x_0, \mu_n)$ which is a contradiction. Thus it follows that $\nabla g_1(x_0)^T y_0 \leq 0$ for each $i \in I$. Similarly, it can be shown that $\nabla h_j(x_0)^T y_0 = 0$ for $j = 1, \dots, p$, and hence from the assumption that G holds it follows that $\nabla f(x_0)^T y_0 < 0$. But then

$$\begin{aligned} \frac{1}{\|y_n\|} [f(x_n) - f(x_0)] &= \nabla f(x_0)^T \frac{y_n}{\|y_n\|} + \frac{o(\|y_n\|)}{\|y_n\|} \\ &\rightarrow \nabla f(x_0)^T y_0 < 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

implying that for n sufficiently large $f(x_n) < f(x_0)$ and hence $p(x_n, \mu_n) < p(x_0, \mu_n)$. This final contradiction establishes the theorem. $\square$

3. EXAMPLES.

Example 1. Let $f, g_1, g_2: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x$, $g_1(x) = x$, $g_2(x) = -x$. Both constraints are active at $x_0 = 0$, which is the only feasible point and hence the constrained maximum. Since there is no nonzero $y \in \mathbb{R}$ such that $yg'_i(0) \leq 0$ for $i = 1, 2$, G holds trivially, but the gradients $g'_1(0) = 1$ and $g'_2(0) = -1$ are not linearly independent. Therefore, the new condition G does not imply Pietrzykowski's condition.

Example 2. Let $f, g: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^3$, $g(x) = x$. The constraint is active at the constrained maximum $x_0 = 0$. Note that $g'(0) = 1$, and that for $y = -1$ $yg'(0) = -1 \leq 0$ but $yf'(0) = 0$. Therefore, Pietrzykowski's condition does not imply G .

Example 3. Let $f, g_1, g_2: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned} f(x, y) &= y + x^4 \\ g_1(x, y) &= y \\ g_2(x, y) &= y^3 + x^6 \end{aligned}$$

(see Figure 1 below). It can be shown that the weak constraint qualification [3] is satisfied at the isolated local maximum $(0, 0)$. It is not true, however, that for μ sufficiently small the point $(0, 0)$ is a local maximum of

$p((x,y),\mu)$. To see this, define $h: \mathbb{R} \rightarrow \mathbb{R}$ by $h(x) = p((x,0),\mu) = \mu x^4 - x^6$ for any $\mu > 0$, noting that h has a local *minimum* at $x = 0$ for any $\mu > 0$. Hence the condition that x_0 be an isolated local constrained maximum at which the weak constraint qualification holds does not guarantee that the penalty function will be exact.

Figure 1

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