THE USE OF NON-PARAMETRIC METHODS IN THE STATISTICAL
ANALYSIS OF A COMPLEX SPLIT PLOT EXPERIMENT

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Summary

In many instances, statisticians are confronted with data which arise from a complex experimental design and which do not necessarily appear to satisfy one or more of the traditional assumptions of the analysis of variance; namely those of independence, homogeneity of variance, and normality. Moreover, the presence of interaction in the sense of lack of additivity of factor effects may cause difficulty. Such situations appropriately call for a non-parametric approach. Indeed many useful and powerful procedures are discussed in the literature (for example, see Hodges and Lehmann [1962], Lehmann [1963], Chatterjee and Sen [1966] and Puri and Sen [1966]). However, because these methods are not widely known among practitioners and because their application appears to involve difficult computations, they are seldom applied. Instead, the analyst first attempts to adjust the data by performing transformations or discarding certain extreme observations etc.; then he performs an analysis of variance. Often, this may lead to a perfectly satisfactory analysis. On the other hand, in some cases the results may be questionable.

The purpose of this paper is to reveal that non-parametric methods do actually provide a realistic alternative to the analysis of such data. The approach to be followed is similar to that of Koch and Sen [1968] with respect to the mixed model. The procedures to be discussed are based on ranks (although some other types of scoring procedure could be used).

As such, they may be presumed to be valid under fairly general conditions and to have good power properties. However, most important from the analyst's point of view, they involve reasonably simple calculations. In this paper, emphasis will be placed on describing the steps of algorithms for the efficient computation of the various test statistics. Computer programs for some of these algorithms have already been written (see Koch and Sen [1968]) and others are being prepared.

0. A complex split-plot experiment

Let us now consider the data from an experiment undertaken at the Department of Pathology, Duke University Medical Center, Durham, North Carolina and reproduced with the kind permission of Dr. Nathan Kaufman and Dr. J. V. Klavins. The design is as follows: 34 animals are randomly placed in one of two groups - an experimental one receiving ethionine in their diets and a pair fed control group (i.e., a control animal is given the same amount of food as the experimental animal with which it is paired). After a period of time, the liver of each animal is extracted and divided into three parts. At this point the matched pairs of sets of liver thirds are divided at random into two blocks. In the first block, each liver third is treated with radioactive iron in a solution that is of low, medium, or high acidity (i.e., pH in the range of either 7.0-7.7, 4.5-5.5, or 2.0-3.0) at 37°C; while in the second block, each liver third is so treated at 25°C. The data consist of the amounts of iron absorbed by the variously treated liver thirds and are as follows:

Table I

block	pair	Ethionine			Control		
		Low	Medium	High	Low	Medium	High
I	1	2.23	2.59	4.50	1.34	1.40	3.87
	2	1.14	1.54	3.92	0.84	1.51	2.81
	3	2.63	3.68	10.33	0.68	2.49	8.42
	4	1.00	1.96	8.23	0.69	1.74	3.82
	5	1.35	2.94	2.07	2.08	1.59	2.42
	6	2.01	1.61	4.90	1.16	1.36	2.85
	7	1.64	1.23	6.84	0.96	3.00	4.15
	8	1.13	6.96	6.42	0.74	4.81	5.64
	Mean	1.64	2.81	5.90	1.06	2.24	4.25
II	9	3.00	6.77	2.95	1.56	4.71	2.64
	10	4.78	4.97	3.42	2.30	1.60	2.48
	11	0.71	1.46	6.85	1.01	0.67	3.66
	12	1.01	0.96	0.94	1.61	0.71	0.68
	13	1.70	5.59	3.72	1.06	5.21	3.20
	14	1.31	9.56	6.00	1.35	5.12	3.77
	15	2.13	1.08	3.13	1.40	0.95	3.94
	16	2.42	1.58	2.74	1.18	1.56	2.62
	17	1.32	8.09	3.91	1.55	1.68	2.40
	Mean	2.04	4.45	3.74	1.45	2.47	2.82
	Mean	1.85	3.68	4.76	1.27	2.36	3.48

As can be seen, there are a number of extreme valued observations which are much larger than the others. These should not necessarily be discarded because they convey some information about the magnitude of the effects of the factors on some animals. However, if they are retained, an analysis of variance may become contaminated and thus difficult to interpret.

In what follows, a number of appropriate test procedures (both parametric and non-parametric) are given for the various hypotheses of interest:

1. H_1 : equality of temperature effects
2. H_2 : equality of diet effects
3. H_3 : no diet x temperature interaction
4. H_4 : equality of pH effects
5. H_5 : no pH x temperature interaction
6. H_6 : no pH x diet interaction
7. H_7 : no pH x diet x temperature interaction

In this way, their relative performances can be seen and compared. No attempt is made to investigate the validity of the underlying assumptions of the various procedures. However, some parametric tests for such hypotheses are given in Anderson [1958, Chapter 10] while non-parametric ones are discussed by Sen [1967].

1. Testing the equality of temperature effects

A parametric test for this hypothesis may be obtained by totaling all the measurements in each pair and then forming the associated F-ratio for the between blocks mean square to the within blocks mean square. This statistic accounts for the pair effects being random. For the given data

$$F(1, 15) = \frac{0.61}{(95.71/6)} = 0.10$$

which indicates differential temperature effects are small. This test must be interpreted carefully because it presumes (perhaps unrealistically) that temperature effects do not interact with diet effects and/or pH effects (i.e., H_3 , H_5 , H_7 must hold). If this is not the case, then the significance of block effects needs to be examined either within each diet x pH condition or by means of certain multivariate procedures (see Anderson [1958] and Cole and Grizzle [1966]).

Non-parametric procedures for testing this hypothesis may be based on the multivariate version of the well-known Kruskal-Wallis [1952] test, the theory of which is discussed by Chatterjee and Sen [1966] and Puri and Sen [1966]. First of all, we form the matrix giving the ranks of the pairs within each diet x pH combination. This is given below.

Table II

block	pair	Ethionine			Control			Total
		Low	Medium	High	Low	Medium	High	
I	1	13	9	10	10	5	13	60
	2	5	5	9	4	6	7	36
	3	15	11	17	1	12	17	73
	4	2	8	16	2	11	12	51
	5	8	10	2	16	8	3	47
	6	11	7	11	8	4	8	49
	7	9	3	14	5	13	15	59
	8	4	15	13	3	15	16	66
	mean	8.38	8.50	11.50	6.12	9.25	11.38	55.12
II	9	16	14	4	14	14	6	68
	10	17	12	6	17	9	4	65
	11	1	4	15	6	1	10	37
	12	3	1	1	15	2	1	23
	13	10	13	7	7	17	9	63
	14	6	17	12	11	16	11	73
	15	12	2	5	12	3	14	48
	16	14	6	3	9	7	5	44
	17	7	16	8	13	10	2	56
	mean	9.56	9.44	6.78	11.56	8.78	6.89	53.00

In forming ranks, any ties are handled by the mid-rank method. To compute the test statistic L we proceed as follows:

1. Compute the mean vector for each block (the components of these vectors correspond to the various diet x pH combinations).
2. Compute the sums of cross-products of deviations matrix S for the combined blocks array (in the case of ranks, deviations are measured from $(\frac{N+1}{2})$ where N is the number of pairs (= 17 here).
3. Compute the mean deviation vector $\underline{u}_i$ for each block by subtracting the combined blocks mean vector from each of the mean vectors obtained in (1). Here, this is achieved by subtracting $(\frac{N+1}{2})$ from each component of each mean vector.
4. Compute $L = (N-1) \sum_i n_i \underline{u}_i' S^{-1} \underline{u}_i$ where n_i is the number of pairs in the i -th block. In large samples, L has approximately the χ^2 -distribution with $p(b-1)$ degrees of freedom where b is the number of blocks and p is the number of measurements on each unit (pair) within each block. Here, it is the number of diet x pH combinations (i.e., $p=6$). For the given data, we have

$$\underline{u}_1' = [-0.62 \quad -0.50 \quad 2.50 \quad -2.88 \quad 0.25 \quad 2.38], \quad n_1 = 8$$

$$\underline{u}_2' = [0.56 \quad 0.44 \quad -2.22 \quad 2.56 \quad -0.22 \quad -2.11], \quad n_2 = 9$$

$$\frac{S}{(N-1)} = \begin{bmatrix} 25.50 & 4.81 & -7.25 & 7.88 & 3.69 & 0.12 \\ 4.81 & 25.50 & 2.94 & 2.44 & 18.81 & -0.25 \\ -7.25 & 2.94 & 25.50 & -20.50 & 6.12 & 19.00 \\ 7.88 & 2.44 & -20.50 & 25.50 & -5.62 & -18.31 \\ 3.69 & 18.81 & 6.12 & -5.62 & 25.50 & 7.19 \\ 0.12 & -0.25 & 19.00 & -18.31 & 7.19 & 25.50 \end{bmatrix}$$

$$L = [8(0.3453) + 9(0.2729)] = 5.22, \text{ D.F.} = 6.$$

Hence, there appears to be no significant differential temperature effects.

In addition, L can be computed for each diet x pH combination (D.F. = 1),

each pH (D.F. = 2), or each diet (D.F. = 3). The results for this are

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	Low	Medium	High	Joint
Ethionine	0.23	0.15	3.70	4.12
Control	4.90	0.04	3.34	5.15
Joint	4.94	0.64	4.05	5.22

Only for the low condition of the control group are the differential temperature effects statistically significant at the $\alpha = .05$ level.

A further modification can be introduced by exploiting the reasonable assumption that the measurements on the two animals within the same pair are independent. One then views the sets of scores for each diet as a combined set ($n_1 = 16$, $n_2 = 18$). Then L is computed as before but modified by being multiplied by $2(N-1)/(2N-1)$. The computations give

low	medium	high	joint
3.63	0.02	7.04	7.86

where D.F. = 1 for each pH and D.F. = 3 for the joint test.

Alternatively, L can be computed for the scores obtained by summing the ranks associated with each pair. In this case

$$L = \{8(55.12 - 54.00)^2 + 9(53.00 - 54.00)^2\} / \{195.125\}$$

$$= 0.098$$

with D.F. = 1. This test does not specifically require that the joint (diet x pH) effects and the temperature effects be additive. Instead, the no interaction condition here presumes that the general ranking of the pairs is similar within the respective diet x pH combinations.

Finally, the computation of L can be based on the ranks of the totals of the measurements associated with each pair; i.e., the same information used to determine F . In this instance, the statistic is essentially that of Kruskal Wallis [1952].

$$L = \{8(.125)^2 + 9(.111)^2\} / 25.50 \approx 0.01 \quad \text{D.F.}=1$$

This procedure represents the non-parametric alternative which is analogous to the analysis of variance test.

2. Testing the equality of diet effects

As demonstrated in the previous section, there are a number of procedures by which the significance of differential diet effects may be judged. The standard analysis of variance test statistic is produced by just totaling the measurements for each animal in each pair (i.e., summing over pH levels) and then forming the F-ratio for the between diets mean square to the "diet x pairs in block" mean square. The error term here is adjusted for the pair effects and the diet effects within each separate block. Finally, this statistic accounts for the pair effects being random. For the given data

$$F(1, 15) = \frac{28.55}{(15.19/15)} = 28.50$$

which indicates that differential diet effects are highly significant ($\alpha = .01$). This test must be interpreted carefully because it presumes (perhaps unrealistically) that diet effects do not interact with either temperature effects and/or pH effects (i.e., H_3 , H_6 , H_7 must hold). If this is not the case, then an alternative analysis should be used; for example, certain multivariate procedures (see Cole and Grizzle [1966]).

One set of non-parametric procedures for testing this hypothesis may be based on the multivariate version of the well-known Friedman [1937] test, the theory of which is discussed by Gerig [1968]. First of all, we form the matrix giving the ranks of diets within each pair x pH combination. This is given below.

Table III

block	pair	Ethionine			Control		
		Low	Medium	High	Low	Medium	High
I	1	2	2	2	1	1	1
	2	2	2	2	1	1	1
	3	2	2	2	1	1	1
	4	2	2	2	1	1	1
	5	1	2	1	2	1	2
	6	2	2	2	1	1	1
	7	2	1	2	1	2	1
	8	2	2	2	1	1	1
	mean	1.88	1.88	1.88	1.12	1.12	1.12
II	9	2	2	2	1	1	1
	10	2	2	2	1	1	1
	11	1	2	2	2	1	1
	12	1	2	2	2	1	1
	13	2	2	2	1	1	1
	14	1	2	2	2	1	1
	15	2	2	1	1	1	2
	16	2	2	2	1	1	1
	17	1	2	2	2	1	1
	mean	1.56	2.00	1.89	1.44	1.00	1.11
Grand mean		1.71	1.94	1.88	1.29	1.06	1.12

To compute the test statistic Q , we proceed as follows:

1. Compute the mean vector (averaging over pairs) for each of the diets (the components of these vectors correspond to the various pH's).

2. Compute the sums of cross-products of deviations from $(\frac{d+1}{2})$ matrix R (the summations are over pairs and diets) where d is the number of diets (=2 here).

3. Compute the mean deviation vector $\underline{x}_j$ for each diet by subtracting $(\frac{d+1}{2})$ from each component of each mean vector.

4. Compute $Q = N^2(d-1) \sum_j \underline{x}_j' R^{-1} \underline{x}_j$ where N is the number of pairs.

In large samples, Q has approximately the χ^2 -distribution with $h(d-1)$ degrees of freedom where " h " is the number of pH levels (i.e., $h=3$).

For the given data we have

$$\underline{x}_e' = (0.21 \quad 0.44 \quad 0.38)$$

$$\underline{x}_c' = (-0.21 \quad -0.44 \quad -0.38)$$

$$R = \frac{1}{2} \begin{bmatrix} 17 & 5 & 7 \\ 5 & 17 & 11 \\ 7 & 11 & 17 \end{bmatrix}$$

$$Q = (17)^2(1)(2)(\frac{1}{2})[0.21 \quad 0.44 \quad 0.38] \begin{bmatrix} 17 & 5 & 7 \\ 5 & 17 & 11 \\ 7 & 11 & 17 \end{bmatrix}^{-1} \begin{bmatrix} 0.21 \\ 0.44 \\ 0.38 \end{bmatrix}$$

$$= 14.46, \text{ D.F. } = 3$$

Hence, there tends to be a significant ($\alpha = .05$) difference between the diet effects. In addition, Q can be computed for each temperature x pH combination (D.F. = 1), each pH (D.F. = 1), or each temperature (D.F. = 3). The results for this are

	Low	Medium	High	Joint
37°C Group	4.50	4.50	4.50	6.00*
25°C Group	0.11	9.00	5.44	9.00
Combined	2.88	13.24	9.94	14.46

Although approximating χ^2 -distributions may not be realistic for all of the above statistics(because of the small sample sizes), it is still apparent that diet effects are significantly different. For exact significance levels, one would need to refer to the appropriate permutation distribution.

Another test can be produced by computing Q for the scores obtained by summing the ranks associated with each diet x pair combination (i.e., summing over pH levels). The only difference in the calculation is that $(\frac{d+1}{2})$ is replaced by $h(\frac{d+1}{2})$. In this case

$$Q = (2)(17)^2(1)(1.03)^2/(48.5) = 12.63$$

with D.F. = 1. This test presumes no diet x pH interaction in the sense that the general ranking of the responses to the different diets is similar

* Because the low and high pH levels have the same scores, the D.F. here becomes reduced to D.F. = 2.

for the various pH levels.

Finally, a test analogous to the parametric one can be produced by basing the computation of Q on the relative ranks of the totals for each animal within each pair (i.e., the sums over pH). The resulting statistic is essentially that of Friedman [1937].

$$Q = 2(17)^2 (.44)^2 / (8.5) = 13.24; \text{ D.F.} = 1.$$

From the preceding discussion, it is apparent that this class of non-parametric tests are almost as sensitive at detecting the significance of differential diet effects as the analysis of variance test.

An alternative non-parametric approach may be used if it is reasonable to assume that pair effects are additive within each specific pH level. The test statistic represents a multivariate version of the procedure discussed in Sen [1968]. First of all, we form the matrix giving the deviations of each measurement from the corresponding pair $\times$ pH mean (i.e., mean over diets). For each pH level, ranks are then assigned to the members of this set of deviations. The obtained matrix of ranks is as follows:

Table IV

block	pair	Ethionine			Control			Ethionine	Control
		Low	Medium	High	Low	Medium	High	Total	Total
I	1	30	26.5	23	5	8.5	12	79.5	25.5
	2	20.5	19	27	14.5	16	8	66.5	38.5
	3	33	26.5	29	2	8.5	6	88.5	16.5
	4	22	21	34	13	14	1	77	28
	5	7.5	28	14	27.5	7	21	49.5	55.5
	6	29	22.5	30	6	12.5	5	81.5	23.5
	7	26	6	32	9	29	3	64	41
	8	23	31	24	12	4	11	78	27
	mean	23.88	22.56	26.62	11.12	12.44	8.38	73.06	31.94
II	9	32	30	20	3	5	15	82	23
	10	34	32	26	1	3	9	92	13
	11	14.5	25	33	20.5	10	2	72.5	32.5
	12	11	22.5	19	24	12.5	16	52.5	52.5
	13	25	24	22	10	11	13	71	34
	14	17	33	31	18	2	4	81	24
	15	27.5	20	10	7.5	15	25	57.5	47.5
	16	31	18	18	4	17	17	67	38
	17	16	34	28	19	1	7	78	27
	mean	23.11	26.50	23.00	11.89	8.50	12.00	72.61	32.39
	mean	23.48	24.65	24.71	11.52	10.35	10.29	72.84	32.16

To compute the test statistic $\tilde{Q}$, we proceed as follows:

1. Compute the mean vectors (averaging over pairs) for each diet (the components of these vectors correspond to the pH levels)
2. Compute the mean deviation vector $\tilde{\underline{x}}_j$ for each diet by subtracting $(\frac{Nd+1}{2})$ from each component of each mean vector
3. Compute the sums of cross products of deviations from corresponding within pair mean vectors matrix $\tilde{R}$ (the summation being over pairs and diets).

4. Compute $\tilde{Q} = N^2(d-1) \sum_j \tilde{\underline{x}}_j \tilde{R}^{-1} \tilde{\underline{x}}_j$. In large samples $\tilde{Q}$ has approximately the χ^2 -distribution with $h(d-1)$ degrees of freedom

For the given data, we have

$$\tilde{Q} = 12.93$$

with D.F. = 3.

In addition, $\tilde{Q}$ can be computed separately for each pH level. The results for this are

low	medium	high	joint
6.30	9.03	9.17	12.93

each with D.F. = 1.

Alternatively, the computations can be based on the total scores for each diet obtained by summing over pH levels. In this case, we have

$$\tilde{Q} = 12.80; \quad \text{D.F.} = 1.$$

Other tests are possible. In particular, $\tilde{Q}$ could be obtained for the data obtained by summing the measurements for each animal within each pair. Also, the various types of $\tilde{Q}$ could be computed within each temperature group (actually the test outlined here which combines the temperature groups presumes that there is no temperature x diet interaction). Finally, if the measurements on the different liver fractions of each animal are independent (or symmetrically dependent), then tests like those of section 4 are applicable.

3. Testing the interaction between temperature and diet effects

The usual parametric test for this hypothesis is based on the totals (summing over pH levels) for each animal and is obtained by forming the F-ratio for the diet x temperature mean square to the "diet x pairs in block" mean square (the error term here is the same as the one used for the F-test in section 2). For the given data

$$F(1, 15) = \frac{0.33}{(15.19/15)} = 0.33$$

Hence, there does not appear to be any temperature x diet interaction. However, one should note that the application of this test requires the (perhaps unrealistic) assumption that neither diet effects nor temperature effects interact with pH effects (i.e., H_5 , H_6 , H_7 must hold). When these conditions do not apply, then an alternative multivariate analysis should be considered (see Cole and Grizzle [1966]).

A non-parametric approach to this hypothesis may be based on a multivariate Kruskal-Wallis procedure similar to the one outlined in section 3. First of all, we obtain the differences between the ethionine measurements and the corresponding control measurements within each pair x pH combination. We then form the matrix giving the ranks of the pairs with respect to these differential diet effects within each pH level. This is given below.

Table V

block	pair	EL-CL	EM-CM	EH-CH	Total Score
I	1	13	10.5	7	30.5
	2	6	3	10	19
	3	16	10.5	12	38.5
	4	7	5	17	29
	5	1	12	2	15
	6	12	6.5	13	31.5
	7	10	1	15	26
	8	8	14	8	30
	mean	9.12	7.81	10.50	27.44
II	9	15	13	5	33
	10	17	15	9	41
	11	3	9	16	28
	12	2	6.5	4	12.5
	13	9	8	6	23
	14	5	16	14	35
	15	11	4	1	16
	16	14	2	3	19
	17	4	17	11	32
	mean	8.89	10.06	7.67	26.61

The test statistic $\tilde{L}$ is computed by applying steps (1-4) of section 1 to the above matrix. For the given data, we have

$$\tilde{u}'_1 = [0.12 \quad -1.19 \quad 1.50]$$

$$\tilde{u}'_2 = [-0.11 \quad 1.06 \quad -1.33]$$

$$\tilde{S}/(N-1) = \begin{bmatrix} 25.50 & -1.59 & -2.38 \\ -1.59 & 25.44 & 0.75 \\ -2.38 & 0.75 & 25.50 \end{bmatrix}$$

$$\tilde{L} = [8(0.1494) + 9(0.1180)] = 2.26; \text{ D.F. } = 3.$$

Hence, there appears to be no significant temperature x diet interaction.

In addition, $\tilde{L}$ can be computed for each pH level (D.F. = 1). The results are

Low	Medium	High	Joint
0.01	0.84	1.33	2.26

If the computations of $\tilde{L}$ are based on the total scores obtained by summing the ranks associated with each pair, then

$$\begin{aligned}\tilde{L} &= \{8(27.44 - 27.00)^2 + 9(26.61 - 27.00)^2\} / \{70.00\} \\ &= 0.04\end{aligned}$$

with D.F. = 1. The use of this test presumes that the general rankings of the pairs with respect to differential diet effects tend to be similar within the various pH levels.

Finally, the computation of $\tilde{L}$ can be based on the ranks of the differences between the ethionine totals and the control totals associated with each pair, i.e., the same information (and hence requiring the same assumptions) as used to determine F. The resulting statistic is

$$\begin{aligned}\tilde{L} &= \{8(0.375)^2 + 9(0.333)^2\} / (25.50) \\ &= 0.08\end{aligned}$$

with D.F. = 1.

In concluding this section, we should remark that the computations have been considerably simplified because there are only $d = 2$ diets. However, there do exist similar procedures which are not overly complicated for the case $d > 2$. The principals behind the use of these will be illustrated in section 5 in connection with tests of no temperature x pH interaction.

4. Testing the equality of pH effects

The analysis of variance test of this hypothesis is similar in structure to that described in section 4 for the equality of diet effects; the principal difference being that it is based on the totals (obtained by summing over diets) associated with each pair x pH combination. The computed F-statistic is the ratio of the between pH levels mean square to the "pH x pairs in block" mean square. It should be noted that the error term is adjusted for the pair effects and the pH effects within each separate block; it also accounts for the pair effects being random. For the given data

$$F(2, 30) = \frac{(112.59/2)}{(136.03/30)} = 12.43$$

which indicates that differential pH effects are highly significant ($\alpha = .01$). However, one should be careful in interpreting the results of this test because it presumes that pH effects do not interact with either diet effects or temperature effects (i.e., H_5 , H_6 , H_9 must hold). This remark is noteworthy here since inspection of the means in Table I are suggestive of some pH x temperature interaction. An alternative parametric procedure would be to compute F separately for each temperature or each temperature x diet combination. Also, since the sets of measurements associated with the different pH levels are all on the same animals, certain multivariate tests might be useful (see Cole and Grizzle [1966]).

Certain non-parametric procedures for testing this hypothesis may be based on the matrix giving the ranks of the different measurements (corresponding to pH levels) within each animal (i.e., diet x pair combination). This is given below.

Table VI

block	pair	Ethionine			Control			Total Score		
		Low	Medium	High	Low	Medium	High	Low	Medium	High
I	1	1	2	3	1	2	3	2	4	6
	2	1	2	3	1	2	3	2	4	6
	3	1	2	3	1	2	3	2	4	6
	4	1	2	3	1	2	3	2	4	6
	5	1	3	2	2	1	3	3	4	5
	6	2	1	3	1	2	3	3	3	6
	7	2	1	3	1	2	3	3	3	6
	8	1	3	2	1	2	3	2	5	5
	mean	1.25	2.00	2.75	1.12	1.88	3.00	2.37	3.88	5.75
II	9	2	3	1	1	3	2	3	6	3
	10	2	3	1	2	1	3	4	4	4
	11	1	2	3	2	1	3	3	3	6
	12	3	2	1	3	2	1	6	4	2
	13	1	3	2	1	3	2	2	6	4
	14	1	3	2	1	3	2	2	6	4
	15	2	1	3	2	1	3	4	2	6
	16	2	1	3	1	2	3	3	3	6
	17	1	3	2	1	2	3	2	5	5
	mean	1.67	2.33	2.00	1.56	2.00	2.44	3.23	4.33	4.44
	grand mean	1.47	2.18	2.35	1.35	1.94	2.71	2.82	4.12	5.06

Two types of tests may be based on the above array. The first one which we shall consider is discussed in Koch and Sen [1968] (under the heading of Case I). It is appropriate for situations where the measurements on the different liver fractions (i.e., the different pH levels) for the same animal may not be independent or symmetrically dependent (i.e., equally correlated and homoscedastic). The computations of the test statistic W are as follows:

1. Compute the vector of sums $\underline{T}$ (totaling over pairs and diets) associated with the different pH levels
2. Compute the sums of cross-products of deviations from $\left(\frac{h+1}{2}\right)$ matrix V (where h is the number of pH levels, = 3 here)
3. Specify an $(h-1) \times h$ matrix C of contrasts; e.g.,

$$C = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

4. Compute $W = \underline{T}' C' [C V C']^{-1} C \underline{T}$. In large samples, W has approximately the χ^2 -distribution with $(h-1)$ degrees of freedom.

For the given data, we have

$$\underline{T}' = [48 \quad 70 \quad 86]$$

$$V = \begin{bmatrix} 24 & -8 & -16 \\ -8 & 18 & -10 \\ -16 & -10 & 26 \end{bmatrix}$$

$$W = [-22 \quad -38] \begin{bmatrix} 58 & 38 \\ 38 & 82 \end{bmatrix}^{-1} \begin{bmatrix} -22 \\ -38 \end{bmatrix} = 18.09$$

D.F. = 2. Hence, there tend to be significant differences among the pH effects. Because of the possibility of interaction between pH effects and the other factors, it is worthwhile to compute W separately for each temperature x diet combination (D.F. = 2), each temperature (D.F. = 2), and each diet (D.F. = 2). The results for this are

	Ethionine	Control	Combined Group
37°C Group	7.20	8.00	14.92
25°C Group	2.08	3.56	4.52
Combined Group	7.50	11.29	18.09

Although the approximating χ^2 -distributions may not be realistic for all of the above statistics because of the small sample sizes (being 8 or 9 for the cell entries) it is evident that the significant differences among pH effects arise almost entirely from the 37°C temperature group.

Another test can be obtained by computing W for the scores obtained by summing the ranks associated with each pH x pair combination (i.e., summing over diets). The only difference in the calculations being that $(\frac{h+1}{2})$ is replaced by $d(\frac{h+1}{2})$ in determining V .

In this case, we have

37°C Group	W = 7.68; D.F. = 2
25°C Group	W = 2.61; D.F. = 2
Combined group	W = 9.91; D.F. = 2

As mentioned in previous sections, the use of this type of test presumes no diet x pH interaction in the sense that the general ranking of the responses to the pH levels for the animals in the same pair are similar.

Finally, a test based on the same information as the parametric one can be produced by computing W for the relative ranks of the totals for each pH x pair combination (i.e., the sums over diet). This gives

37°C Group	W = 7.54; D.F. = 2
25°C Group	W = 1.31; D.F. = 2
Combined group	W = 7.54; D.F. = 2

The previously described tests are applicable under fairly general conditions. However, if the measurements on the different liver fractions of the same animal may be reasonably assumed to be symmetrically dependent, then it is appropriate to apply the traditional univariate Friedman test to the sets of ranks in Table VI. The resulting statistics (each with D.F. = 2) are as follows

	Ethionine	Control	Combined Group
37°C Group	9.00	14.25	22.88
25°C Group	2.00	3.56	4.11
Combined Group	7.41	15.64	21.41

Similarly, if this test procedure is applied to the total rank scores, we have

37°C Group	14.08; D.F. = 2
25°C Group	3.72; D.F. = 2
Combined Group	13.46; D.F. = 2

while if it is applied to the ranks of the totals for each pH x pair combination, we have

37°C Group	12.25; D.F. = 2
25°C Group	1.55; D.F. = 2
Combined Group	9.88; D.F. = 2

The interpretation of the results of this series of tests is similar to that given for the preceding ones based on W.

To conclude this section, we shall consider a third series of non-parametric tests which are appropriate for situations where the measurements on the different liver fractions are not symmetrically dependent. On the other hand, their use does necessarily presume that pair effects are purely additive. For a discussion of their other properties, the reader is referred to Koch and Sen [1968] (under the heading of Case III). The computations of the test statistic W^* are as follows:

1. Within each temperature x diet combination
 - a. compute the $\binom{h}{2}$ differences associated with the possible pairs of pH levels; e.g., L-M, L-H, M-H.
 - b. Let signed ranks be associated with each of the values associated with the variables (i.e., differences) computed in (a) — ties are handled by the mid-rank method and zero is assigned to zero values.
 - c. Within each pair, form scores for each pH level by adding the sum of the signed ranks associated with differences in which it

is the minuend to the negative of the sum of those associated with differences in which it is the subtrahend.

These scores are given in the following table

Table VII

Ethionine		Rank			Scores		
block	pair	(1) L-M	(2) L-H	(3) M-H	(1)+(2) Low	(3)-(1) Medium	-(2)-(3) High
I	1	- 1	- 2	- 3	- 3	- 2	5
	2	-2.5	- 3	- 4	-5.5	-1.5	7
	3	-6	- 8	- 8	-14	- 2	16
	4	- 5	- 7	- 7	-12	- 2	14
	5	- 7	- 1	2	- 8	9	- 1
	6	2.5	- 4	- 5	-1.5	-7.5	9
	7	4	- 5	- 6	- 1	-10	11
	8	- 8	- 6	1	-14	9	5
	mean	-2.88	-4.50	-3.75	-7.38	-0.88	8.25
II	9	- 6	1	7	- 5	13	- 8
	10	- 2	5	3	3	5	- 8
	11	- 3	- 9	- 9	-12	- 6	18
	12	1	2	1	3	0	- 3
	13	- 7	- 6	4	-13	11	2
	14	- 9	- 8	6	-17	15	2
	15	5	- 4	- 5	1	-10	9
	16	4	- 3	- 2	1	- 6	5
	17	- 8	- 7	8	-15	16	- 1
	mean	-2.78	-3.22	1.44	-6.00	4.22	1.78
	mean	-2.82	-3.82	-1.00	-6.65	1.82	4.82

Control		Ranks			Scores			Total scores over Treatments		
block	pair	(1) L-M	(2) L-H	(3) M-H	(1)+(2) Low	(3)-(1) Medium	-(2)-(3) High	Low	Medium	High
I	1	-1	-4	-7	-5	-6	11	-8	-8	16
	2	-4	-3	-4	-7	0	7	-12.5	-1.5	14
	3	-6	-8	-8	-14	-2	16	-28	-4	32
	4	-5	-5	-6	-10	-1	11	-22	-3	25
	5	3	-1	-1.5	2	-4.5	2.5	-6	4.5	1.5
	6	-2	-2	-5	-4	-3	7	-5.5	-10.5	16
	7	-7	-6	-3	-13	4	9	-14	-6	20
	8	-8	-7	-1.5	-15	6.5	8.5	-29	15.5	13.5
	mean	-3.75	-4.50	-4.50	-8.25	-0.75	9.00	-15.63	-1.63	17.25
II	9	-7	-4	7	-11	14	-3	-16	27	-11
	10	5	-1	-3	4	-8	4	7	-3	-4
	11	2	-9	-8.5	-7	-10.5	17.5	-19	-16.5	35.5
	12	6	3	1	9	-5	-4	12	-5	-7
	13	-9	-6	6	-15	15	0	-28	26	2
	14	-8	-7	5	-15	13	2	-32	28	4
	15	4	-8	-8.5	-4	-12.5	16.5	-3	-22.5	25.5
	16	-3	-5	-4	-8	-1	9	-7	-7	14
	17	-1	-2	-2	-3	-1	4	-18	15	3
	mean	-1.22	-4.33	-0.78	-5.56	0.44	5.11	-11.56	4.67	6.89
	mean	-2.41	-4.41	-2.53	-6.82	-0.12	6.94	-13.47	1.71	11.76

2. Compute the vector of sums $\underline{T}^*$ (totaling over pairs within a temperature x diet combination) associated with the different pH levels

3. Compute the sums of cross-products of deviations from zero matrix V^*

4. Compute $W^* = \underline{T}^{*'} C' [CV^*C']^{-1} C \underline{T}^*$ where C is as before. In large samples, W^* has approximately the χ^2 -distribution with $(h-1)$ degrees of freedom

For the given data, the results are

	Ethionine	Control	Combined Group
37°C Group	10.16	21.63	12.96
25°C Group	5.61	5.75	7.20
Combined Group	9.07	10.88	19.59

In each of the above, D.F. = 2. The computations for the overall test are

$$W^* = \begin{bmatrix} -229 & 29 & 200 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \left\{ \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3104.5 & -1337.5 & -1767.0 \\ -1337.5 & 2376.0 & -1039.0 \\ -1767.0 & -1039.0 & 2806.0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -229 \\ 29 \\ 200 \end{bmatrix}$$

If this test procedure is applied to the total scores (obtained by summing over diets), we have

37°C Group	18.70
25°C Group	6.10
Combined Group	17.27

Both of these results lead to conclusions consistent with those associated with the previously described tests in this section. In addition, W^* can be computed from scores assigned to the totals of the measurements associated with each pH x pair combination. However, this will not be done here.

Finally, one should note that there are a number of other tests which might be useful when applicable. For example, if both pair effects and temperature effects are additive, then W^* can be based on signed ranks associated with the combined group of pairs (i.e., all 17 pairs taken together instead of in one separate group of 8 and another of 9). In addition, if both the conditions of additive pair effects and symmetric dependency of the measurements on the same animal are satisfied, then the test of Sen [1968] may be applied in a fashion similar to that indicated in the latter part of Section 2. The results of this type of analysis are given below:

Table VIIIA

block	pair	Ethionine			Control			Total Score		
		Low	Medium	High	Low	Medium	High	Low	Medium	High
I	1	10	14	17	7	8	21	17	22	38
	2	9	13	19	6	13	19	15	26	38
	3	2	5	24	1	5	24	3	10	48
	4	3	6	23	4	12	22	7	18	45
	5	12	16	15	14	10	16	26	26	31
	6	11	8	20	9	11	17	20	19	37
	7	7	4	22	3	15	20	10	19	42
	8	1	21	18	2	18	23	3	39	41
	mean	6.88	10.87	19.75	5.75	11.50	20.25	12.63	22.37	40.00
II	9	7	24	6	3	24	10.5	10	48	16.5
	10	19	21	9	17	9	18	36	30	27
	11	3	5	26	6	5	26	9	10	52
	12	15	13	12	21	13	12	36	26	24
	13	4	23	16	1	27	16	5	50	32
	14	1	27	18	2	23	19	3	50	37
	15	14	8	22	7	4	25	21	12	47
	16	17	10	20	8	14	22	25	24	42
	17	2	25	11	10.5	15	20	12.5	40	31
	mean	9.11	17.33	15.56	8.39	14.89	18.72	17.50	32.22	34.28

Table VIIIB

$\chi^2(2)$	Ethionine	Control	Total
Group I	10.12	12.21	11.95
Group II	3.86	5.54	4.83

For a further discussion of the application of this test, the reader is referred to Koch and Sen [1968] (Case IV).

5. Testing the interaction between temperature and pH effects

The usual parametric procedure for this hypothesis is similar in structure to that described in section 3 for the temperature x diet interaction; the principal difference being that it is based on the totals (obtained by summing over diets) associated with each pair x pH combination. The test statistic is obtained by forming the F-ratio of the temperature x pH mean square to the "pH x pairs in block" mean square (the error term here is the same as the one used for the F-test in section 4). For the given data,

$$F(2, 30) = \frac{(35.34/2)}{(136.03/30)} = 3.90$$

Hence, there tends to be a statistically significant ($\alpha = .05$) temperature x pH interaction. As mentioned in previous sections, care should be given to the interpretation of the results of this test because they are hinged to the assumption that neither temperature effects nor pH effects interact with diet effects (i.e., H_3 , H_6 , H_7 must hold). When these conditions do not apply, then it may be worthwhile to compute F separately for each diet or use some multivariate procedure (see Cole and Grizzle [1966]).

A non-parametric procedure for this hypothesis may be based on a statistic which is a synthesis of the multivariate Kruskal-Wallis test of section 3 and the W^* test of section 4. Its theoretical properties follow from its resemblance to the former and hence are essentially discussed in Chatterjee and Sen [1966] and Puri and Sen [1966]. The computations of the test statistic L^* are as follows:

1. Within each diet level

- a. Compute the $\binom{h}{2}$ differences associated with the possible pairs of pH levels; e.g., L-M, L-H, M-H.

b. Form the matrix giving the ranks of the pairs (with respect to the combined set of blocks) for each of these differential pH effect variables.

c. Within each pair, form scores for each pH level by adding the sum of the ranks associated with differences in which it is the minuend to the sum of the deviations of ranks from $(N+1)$ associated with differences in which it is the subtrahend

These scores are given in the following table

Table IX

Ethionine		Ranks			Scores		
		(1)	(2)	(3)	(1)+(2)	(3)+18-(1)	56-(2)-(3)
block	pair	L-M	L-H	M-H	Low	Medium	High
I	1	11	10	8	21	15	18
	2	10	8	6	18	14	22
	3	7	1	1	8	12	34
	4	8	2	2	10	12	32
	5	6	13	12	19	24	11
	6	14	7	5	21	9	24
	7	15	5	3	20	6	28
	8	3	4	11	7	26	21
	mean	9.25	6.25	6.00	15.50	14.75	23.75
II	9	5	15	16	20	29	5
	10	12	17	13	29	19	6
	11	9	3	4	12	13	29
	12	13	16	10	29	15	10
	13	4	11	14	15	28	11
	14	1	6	15	7	32	15
	15	17	12	7	29	8	17
	16	16	14	9	30	11	13
	17	2	9	17	11	33	10
	mean	8.78	11.44	11.67	20.22	20.89	12.89

Control	block	pair	Ranks			Scores			Total Scores		
			(1) L-M	(2) L-H	(3) M-H	(1)+(2) Low	(3)+18-(1) Medium	36-(2)-(3) High	Ethionine + Control		
I		1	12	7	4	19	10	25	40	25	43
		2	8	10	7	18	17	19	36	31	41
		3	6	1	1	7	13	34	15	25	68
		4	7	4	5	11	16	27	21	28	59
		5	15	15	11.5	30	14.5	9.5	49	38.5	20.5
		6	10	11	6	21	14	19	42	23	43
		7	5	3	8	8	21	25	28	27	53
		8	2	2	11.5	4	27.5	22.5	11	53.5	43.5
		mean	8.12	6.62	6.75	14.75	16.62	22.63	30.25	31.37	46.38
II		9	4	13	17	17	31	6	37	60	11
		10	16	16	10	32	12	10	61	31	16
		11	13	5	2.5	18	7.5	28.5	30	20.5	57.5
		12	17	17	14	34	15	5	63	30	15
		13	1	9	16	10	33	11	25	61	22
		14	3	8	15	11	30	13	18	62	28
		15	14	6	2.5	20	6.5	27.5	49	14.5	44.5
		16	9	12	9	21	18	15	51	29	28
		17	11	14	13	25	20	9	36	53	19
		mean	9.78	11.11	11.00	20.89	19.22	13.89	41.11	40.11	26.78

2. Compute $\underline{u}_i^*$ and S^* by applying the same steps as given for $\underline{u}_i$ and S in section 1 except deviations are measured from $(k-1)(\frac{N+1}{2})$ instead of $(\frac{N+1}{2})$.

3. Compute $L^* = (N-1) \sum_i n_i \underline{u}_i^*{}' C' [CS^*C']^{-1} C \underline{u}_i^*$ where C is the matrix of contrasts defined in section 6. In large samples, L^* has approximately the χ^2 -distribution with $(b-1)(k-1)$ degrees of freedom

For the given data, the results are

Ethionine	Control
6.11	4.17

each with D.F. = 2. The above tests are appropriate only for the specified diet groups. To obtain an overall test, there are several possibilities. One of these involves considering the data for the two diets jointly and accounts for the possibility that the measurements on the two animals

within the same pair may be correlated. The computations are similar to those indicated above except $\underline{u}_i^*$ has six components instead of 3, S^* is 6×6 instead of 3×3 , and

$$C = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

The value of L^* here is $L^* = 6.28$; D.F. = 4. On the other hand, one can view the sets of scores for each diet as a combined set. Compute $\underline{u}_i^*$ and S^* as in (2) but with respect to this combined set. Then L^* is given by

$$\begin{aligned} L^* &= 2(N-1) \sum_i 2n_i \underline{u}_i^* 'C' [CS^*C']^{-1} \underline{C u}_i^* \\ &= 10.14 \end{aligned}$$

with D.F. = $(b-1)(h-1) = 2$.

Another alternative is to base the computation of L^* on the total scores (obtained by summing over diets) associated with each pair x pH combination. The only difference in the calculations is that the deviations are measured from $d(h-1)(\frac{N+1}{2})$. For the given data, the results of this test are

$$L^* = 5.38; \text{ D.F. } = 2$$

The only restriction on this test is that it presumes that the general rankings of the pairs with respect to differential pH effects tend to be similar within the various diets.

Finally, a test analogous to the parametric one can be based on the totals of the measurements (obtained by summing over diets) associated with each pair x pH combination. The computation of L^* for this set of marginal totals is the same as indicated in steps 1-3. The resulting statistic is

$$L^* = 5.84; \text{ D.F. } = 2.$$

As was the case with F, the use of this test presumes that there is no temperature x diet interaction and no diet x pH interaction.

In conclusion, it can be seen from the preceding results that the non-parametric tests are almost as sensitive at detecting a temperature x pH interaction as the analysis of variance.

6. Testing the interaction between diet and pH effects

The analysis of variance test of this hypothesis is based on the F-statistic associated with the ratio of the diet x pH mean square to the "diet x pH x pairs in block" mean square. The error term here has been adjusted for all the principal effects. For the given data

$$F(2, 30) = \frac{(2.82/2)}{(30.07/30)} = 1.41$$

which indicates that there does not appear to be a significant diet x pH interaction. However, this result should be interpreted carefully because it presumes that neither diet effects nor pH effects interact with temperature effects (the latter of these has already been indicated as unrealistic).

One type of non-parametric approach to this hypothesis makes use of statistics like W and the univariate Friedman test. The first step in the computations is to obtain the differences between the ethionine measurement and the corresponding control measurement within each pair x pH combination. We then form the matrix giving the ranks of these differences (as they are associated with pH levels) within each pair. This is given below

Table X

block	pair	EL-CL	EM-CM	EH-CH
I	1	2	3	1
	2	2	1	3
	3	3	1	2
	4	2	1	3
	5	1	3	2
	6	2	1	3
	7	2	1	3
	8	1	3	2
	mean	1.88	1.75	2.37
II	9	2	3	1
	10	2	3	1
	11	1	2	3
	12	1	2	3
	13	3	1	2
	14	1	3	2
	15	3	2	1
	16	3	1	2
	17	1	3	2
	mean	1.89	2.22	1.89
Grand mean		1.88	2.00	2.12

The test statistic $\tilde{W}$ is computed from this array by the same steps outlined in section 4 for obtaining W . For the given data, the results are

37°C Group	$\tilde{W} = 2.13$; D.F. = 2
25°C Group	$\tilde{W} = 0.69$; D.F. = 2
Combined Group	$\tilde{W} = 0.62$; D.F. = 2

In large samples, $\tilde{W}$ has approximately the χ^2 -distribution with (for this case of $d=2$ diets) D.F. = $(h-1)$.

Alternatively, if it is reasonable to assume that the measurements on the different liver fractions of the same animal are symmetrically dependent, then Friedman's test statistic may be applied to the sets of ranks in Table IX. In this case, we have

37°C Group	$\tilde{Q} = 1.75$; D.F. = 2
25°C Group	$\tilde{Q} = 0.67$; D.F. = 2
Combined Group	$\tilde{Q} = 0.47$; D.F. = 2

As in section 5, the computations here have been considerably simplified because there are only $d=2$ diets. However, similar procedures can be used for $d > 2$. Essentially they involve forming rank matrices like the one in Table IX for each pair of diets and then obtaining scores for each diet x pH combination by procedures similar to those indicated in the computational steps (l.c.) associated with W^* and L^* . From this matrix of scores, appropriate quadratic forms with approximately the χ^2 -distribution with $(d-1)(h-1)$ degrees of freedom can be formed.

Finally, a statistic similar to W^* can be used when it is reasonable to assume that pair effects are purely additive. It is formed as follows:

1. Within each pair of each block (temperature group)

a. Compute the $\binom{h}{2}\binom{d}{2}$ contrasts associated with the differential pH effect on differential diet effects; e.g., (EL-CL)-(EM-CM), (EL-CL)-(EH-CH), (EM-CM)-(EH-CH).

b. Let signed ranks be associated with each of the values associated with these contrasts

c. Within each pair, form scores for each diet x pH combination by adding the sum of the signed ranks associated with contrasts in which its coefficient is +1 to the negative of the sum of the signed ranks associated with contrasts in which its coefficient is -1.

These scores are partially given in the following table (when $d=2$, the scores for the control diet are the negatives of those for ethionine)

Table XI

block	pair	(1) (EL-CL) -(EM-CM)	(2) (EL-CL) -(EH-CH)	(3) (EM-CM) -(EH-CH)	Scores		
					(1)+(2) EL	(3)-(1) EM	-(2)-(3) EH
I	1	-3	2	1	-1	4	-3
	2	2	-5	-3	-3	-5	8
	3	5	1	-2	6	-7	1
	4	1	-8	-7	-7	-8	15
	5	-7	-3	5	-10	12	-2
	6	4	-6	-6	-2	-10	12
	7	8	-7	-8	1	-16	15
	8	-6	-4	4	-10	10	0
	mean	0.50	-3.75	-2.00	-3.25	-2.50	5.75
II	9	-3	4	5	1	8	-9
	10	-5	5.5	8	0.5	13	-13.5
	11	-6	-9	-7	-15	-1	16
	12	-4	-2	-1	-6	3	3
	13	1	1	-3	2	-4	2
	14	-8	-8	6	-16	14	2
	15	2	5.5	4	7.5	2	-9.5
	16	7	3	-2	10	-9	-1
	17	-9	-7	9	-16	18	-2
	mean	-2.78	-0.78	2.11	-3.56	4.89	-1.33
	mean	-1.24	-2.18	0.18	-3.41	1.41	2.00

2. Compute the vector of sums $\tilde{T}^*$ (totaling over pairs); the components of which correspond to the different diet x pH combinations.

3. Compute the sums of cross-products of deviations from zero matrix $\tilde{V}^*$

4. Compute $\tilde{W}^* = \tilde{T}^* \tilde{C}' [\tilde{C} \tilde{V}^* \tilde{C}']^{-1} \tilde{C} \tilde{T}^*$ where $\tilde{C}$ is a matrix of contrasts which here has the form

$$\tilde{C} = \begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{bmatrix} .$$

In large samples, $\tilde{W}^*$ has approximately the χ^2 -distribution with $(d-1)(h-1)$ degrees of freedom.

For the given data, the results are

37°C Group	$\tilde{W}^* = 4.38$; D.F. = 2
25°C Group	$\tilde{W}^* = 2.83$; D.F. = 2
Combined Group	$\tilde{W}^* = 3.84$; D.F. = 2

If certain other assumptions were reasonable (e.g., additivity of block effects, symmetric dependency of the observation vectors within the same pair, etc.), some additional modifications could be added to the computation of $\tilde{W}^*$. The exact nature of these will not be discussed any further beyond the fact that their development follows from the same principles that underly the construction of all the tests described in this paper.

7. Testing the temperature x diet x pH interaction

The usual parametric test here is the F-ratio of the temperature x diet x pH mean square to the "diet x pH x pair in block" mean square. For the given data

$$F(2, 30) = \frac{(5.01/2)}{(30.07/30)} = 2.50$$

which indicates that any such interaction is not statistically significant.

A non-parametric test of this hypothesis may be based on a synthesis of the computations given for $\tilde{W}^*$ and L.

1. Within each pair

a. Compute the $\binom{h}{2} \binom{d}{2}$ contrasts associated with the differential pH effects on differential diet effects

b. Form the matrix giving the ranks of the pairs (with respect to the combined set of blocks) for each of these contrasts

c. Within each pair, form scores for each diet x pH combination by adding the sum of the ranks associated with contrasts in which its coefficient is +1 to the sum of the deviations from $(N+1)$ of ranks associated with contrasts in which its coefficient is -1.

These scores are partially given in the following table (when $d=2$, the scores for the control diet and the ethionine diet sum to $(h-1)(N+1)$).

Table XII

		(1) (EL-CL) -(EM-CM)	(2) (EL-CL) -(EH-CH)	(3) (EM-CM) -(EH-CH)	(1)+(2) EL	(3)+18-(1) EM	36-(2)-(3) EH
I	1	9	13	10	22	19	13
	2	12	8	5	20	11	23
	3	15	11	6	26	9	19
	4	10	1	2	11	10	33
	5	3	10	13	13	28	13
	6	13.5	6	4	19.5	8.5	26
	7	17	4	1	21	2	31
	8	4	9	12	13	26	15
	mean	10.44	7.75	6.62	18.19	14.19	21.62
II	9	8	15	14	23	24	7
	10	6	16.5	16	22.6	28	3.5
	11	5	2	3	7	16	31
	12	7	7	9	14	20	20
	13	11	12	7	23	14	17
	14	2	3	15	5	31	18
	15	13.5	16.5	11	30	15.5	8.5
	16	16	14	8	30	10	14
+	17	1	5	17	6	34	14
	mean	7.72	10.11	11.11	17.83	21.39	14.78

2. Compute $\tilde{u}_1^*$ and $\tilde{S}^*$ by applying the same steps as given for $\underline{u}_1$ and S in section 3 except deviations are measured from $(d-1)(h-1)(\frac{N+1}{2})$ instead of $(\frac{N+1}{2})$.

3. Compute $\tilde{L}^* = (N-1) \sum_i n_i \tilde{u}_i^* \tilde{C}' [\tilde{C} \tilde{S}^* \tilde{C}']^{-1} \tilde{C} \tilde{u}_1^*$ where $\tilde{C}$ is the matrix of contrasts defined in section 6. In large samples $\tilde{L}^*$ has approximately the χ^2 -distribution with $(b-1)(d-1)(h-1)$ degrees of freedom.

For the given data, $\tilde{L}^* = 3.30$ with D.F. = 2. Hence, no significant temperature x diet x pH interaction is detected. Again modifications could be introduced to this test which might exploit certain types of symmetric dependency; but these will not be discussed here.

In conclusion, it is evident that there is a wide variety of sensitive rank test procedures which may be applied in the analysis of complex

experiments. Their various forms are applicable under assumptions much more general than those associated with analysis of variance. Finally, when properly systematized, the associated computations are not overly difficult to perform.

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