**A NORMAL SCALE MIXTURE REPRESENTATION
OF THE
LOGISTIC DISTRIBUTION**

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ABSTRACT

In this paper it is shown that the logistic distribution can be represented as a scale mixture of the standard normal distribution where the mixing density is related to the Kolmogorov-Smirnov distribution. Two derivations of the theorem are presented that give rise to two different representations of the Kolmogorov-Smirnov distribution. The induced identity is of independent interest and is not widely published nor easily derived directly.

Key words and phrases. Kolmogorov-Smirnov distribution; logistic distribution; normal scale mixture.

1. INTRODUCTION

Recently I encountered a problem in which it was advantageous to approximate the logistic distribution, $F(t) = 1/(1 + e^{-t})$, with discrete mixtures having the form

$$F_k(t) = \sum_{i=1}^k p_{k,i} \Phi(ts_{k,i}), \quad (k = 1, 2, \dots), \quad (1.1)$$

where Φ is the standard normal cumulative distribution function. These approximations provide a solution to a problem posed by Cox (1970, p. 110) and are useful in statistical models involving convolutions of a normal distribution with F , such as logistic regression measurement error models and random effects logistic regression models; see, for example, Carroll, Spiegelman, Bailey, Lan and Abbott (1984). Monahan and Stefanski (1989) discuss applications in detail and provide tables of $\{p_{k,i}, s_{k,i}\}_{i=1}^k$ that minimize

$$D_k^* = \sup_t |F(t) - F_k(t)|, \quad (1.2)$$

for $k = 1, 2, \dots$. The approximation for $k = 3$ is very good and improves significantly with increasing k .

In this paper I prove the following theorem, providing explanation for the quality of the approximations and mathematical justification for the class of approximants in (1.1).

THEOREM. *Let F and Φ denote the standard logistic and normal cumulative distribution functions respectively. Then*

$$F(t) = \int_0^\infty \Phi(t/\sigma) q(\sigma) d\sigma, \quad (1.3)$$

where $q(\sigma) = (d/d\sigma)L(\sigma/2)$ and L is the Kolmogorov-Smirnov distribution,

$$L(\sigma) = 1 - 2 \sum_{n=1}^{\infty} (-1)^{(n+1)} \exp(-2n^2 \sigma^2). \quad (1.4)$$

In Section 2, two derivations of (1.3) are presented that give rise to two different representations of the Kolmogorov-Smirnov distribution. The induced identity is of independent interest and is not widely published nor easily derived directly.

2. THE LOGISTIC DISTRIBUTION AS A GAUSSIAN SCALE MIXTURE

Consider the integral equation

$$f(t) = \int_0^\infty \sigma^{-1} \phi(t/\sigma) q(\sigma) d\sigma, \quad (2.1)$$

when $f(t) = e^{-t}/(1 + e^{-t})^2$ and ϕ is the standard normal density. A change-of-variables $v = 1/2\sigma^2$ in the integral and evaluation at $t = \sqrt{s}$, $s > 0$, shows that

$$f(\sqrt{s}) = \int_0^\infty e^{-sv} h(v) dv, \quad (2.2)$$

where

$$h(v) = q(1/\sqrt{2v})/\sqrt{8\pi v^2}. \quad (2.3)$$

It follows from (2.2) that $f(\sqrt{s})$ is the Laplace transform of h .

Note that for $t > 0$, $F(t) = (1 + e^{-t})^{-1} = \sum_{n=0}^\infty (-1)^n \exp(-nt)$. Upon differentiation and appeal to symmetry it follows that

$$f(t) = \sum_{n=1}^\infty (-1)^{(n+1)} n \exp\{-n | t | \}, \quad (2.4)$$

and thus

$$f(\sqrt{s}) = \sum_{n=1}^\infty (-1)^{(n+1)} n \exp\{-n\sqrt{s}\}. \quad (2.5)$$

Since $e^{-a\sqrt{s}}$ is the Laplace transform of $h_a(t) = a \exp\{-a^2/4t\}/\sqrt{4\pi t^3}$, it follows from (2.5) that

$$h(t) = \sum_{n=1}^\infty (-1)^{(n+1)} \frac{n^2}{2\sqrt{\pi t^3}} \exp\left\{-\frac{n^2}{4t}\right\},$$

and from (2.3) that

$$q(\sigma) = 2\sigma \sum_{n=1}^{\infty} (-1)^{(n+1)} n^2 \exp \left\{ -\frac{n^2 \sigma^2}{2} \right\}. \quad (2.6)$$

Note that in (2.6), $q(\sigma) = (d/d\sigma)L(\sigma/2)$ where L is given in (1.4). $\square$

An alternative method of solving (2.1) leads to an interesting identity that is not easily derived directly. In terms of moment generating functions (2.1) becomes

$$\frac{t\pi}{\sin(t\pi)} = \int_0^{\infty} \exp \left(\frac{t^2 \sigma^2}{2} \right) q(\sigma) d\sigma.$$

The change-of-variables $v = \sigma^2/2$, evaluation at $t = i\sqrt{s}$, and a geometric series expansion of $1/\{1 - \exp(-2\pi\sqrt{s})\}$ results in the identities

$$\int_0^{\infty} e^{-sv} g(v) dv = \frac{2\pi\sqrt{s} \exp(-\pi\sqrt{s})}{1 - \exp(-2\pi\sqrt{s})} = 2\pi \sum_{n=0}^{\infty} \sqrt{s} \exp\{-\pi(2n+1)\sqrt{s}\}, \quad (2.7)$$

where

$$g(v) = q(\sqrt{2v})/\sqrt{2v}. \quad (2.8)$$

Since $\sqrt{s} \exp(-a\sqrt{s})$ is the Laplace transform of

$$-\left(\frac{d}{da}\right) h_a(t) = -\left(\frac{d}{da}\right) \left[\frac{a \exp\{-a^2/4t\}}{\sqrt{4\pi t^3}} \right],$$

it follows from (2.7) that

$$g(t) = 2\pi \sum_{n=0}^{\infty} \frac{\pi^2(2n+1)^2 - 2t}{4\sqrt{\pi t^5}} \exp \left\{ -\frac{\pi^2(2n+1)^2}{4t} \right\}.$$

Using (2.8) and integrating q term-by-term shows that $q(\sigma) = (d/d\sigma)L^*(\sigma/2)$ where

$$L^*(\sigma) = \frac{\sqrt{2\pi}}{\sigma} \sum_{n=0}^{\infty} \exp \left\{ -\frac{\pi^2(2n+1)^2}{8\sigma^2} \right\}. \quad (2.9)$$

Of course L and L^* must be equal thus showing that the right-hand sides of (1.4) and (2.9) are equal. This identity is not easily established directly; see for example, Feller (1948), Smirnov (1948) and Monahan (1989). The alternative representation (2.9) is useful for computing $L(\sigma)$ for small σ , Monahan (1989).

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