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For testing the hypothesis that two (symmetric) distributions differ only in locations, a Kolmogorov-Smirnov type test based on the aligned observations is considered and its properties studied.

1. Introduction. Let $X_1, \dots, X_m$ be m independent and identically distributed random variables (i.i.d.r.v.) with a continuous distribution function (d.f.) F , defined on the real line $R = (-\infty, \infty)$. Also, let $Y_1, \dots, Y_n$ be n i.i.d.r.v. with a d.f. G , defined on R . It is assumed that

$$(1.1) \quad F(x) = F_0(x - \theta_1) \text{ and } G(x) = G_0(x - \theta_2), \quad x \in R,$$

where $\tilde{\theta} = (\theta_1, \theta_2)$ is an unknown vector of the location parameters, and, we want to test for

$$(1.2) \quad H_0: F_0 = G_0 \text{ against } H_1: F_0 \neq G_0,$$

treating $\tilde{\theta}$ as a nuisance parameter. If $\theta_1 = \theta_2$, then an omnibus goodness of fit test for (1.2) is based on the classical Kolmogorov-Smirnov statistics

$$(1.3) \quad K_{mn}^+ = \sup\{ F_m(x) - G_n(x) : x \in R \},$$

$$(1.4) \quad K_{mn} = \sup\{ |F_m(x) - G_n(x)| : x \in R \},$$

where F_m and G_n are the two sample (empirical) d.f. This test is not suitable when θ_1 and θ_2 are not equal. One possibility to overcome this problem is to estimate $\tilde{\theta}$ by some convenient estimator $\hat{\tilde{\theta}}$ and to base the test on the aligned observations $X_i - \hat{\theta}_1$, $i=1, \dots, m$ and $Y_j - \hat{\theta}_2$, $j=1, \dots, n$. Unfortunately, the resulting test may not be distribution-free, even asymptotically. In this note, we consider

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a variant form of the Kolmogorov-Smirnov statistics where the alignment procedure works out well when the underlying d.f.'s are symmetric. Along with the preliminary notions, this proposed test is considered in Section 2 and its properties are studied in the concluding section.

2. The proposed test. Let $\hat{\theta}_{1,m}$ and $\hat{\theta}_{2,n}$ be two arbitrary translation-invariant estimators of θ_1 and θ_2 , respectively, such that

$$(2.1) \quad m^{1/2} |\hat{\theta}_{1,m} - \theta_1| = o_p(1) \quad \text{and} \quad n^{1/2} |\hat{\theta}_{2,n} - \theta_2| = o_p(1).$$

Consider then the aligned observations

$$(2.2) \quad \hat{X}_{mi} = X_i - \hat{\theta}_{1,m}, \quad i=1, \dots, m; \quad \hat{Y}_{nj} = Y_j - \hat{\theta}_{2,n}, \quad j=1, \dots, n,$$

and let

$$(2.3) \quad \hat{F}_m(x) = m^{-1} \sum_{i=1}^m I(\hat{X}_{mi} \leq x) \quad \text{and} \quad \hat{G}_n(x) = n^{-1} \sum_{j=1}^n I(\hat{Y}_{nj} \leq x), \quad x \in \mathbb{R}$$

be the two empirical d.f.'s based on the aligned observations. Further, let

$$(2.4) \quad \hat{F}_m^*(x) = \hat{F}_m(x) - \hat{F}_m(-x-) \quad \text{and} \quad \hat{G}_n^*(x) = \hat{G}_n(x) - \hat{G}_n(-x-), \quad x \geq 0.$$

Our proposed tests are based on the statistics

$$(2.5) \quad \hat{K}_{mn}^{*+} = \sup\{ \hat{F}_m^*(x) - \hat{G}_n^*(x) : x \geq 0 \},$$

$$(2.6) \quad \hat{K}_{mn}^* = \sup\{ |\hat{F}_m^*(x) - \hat{G}_n^*(x)| : x \geq 0 \}.$$

We intend to show that the tests based on the statistics in (2.5)-(2.6) are asymptotically distribution-free and have the same properties as the ones based on (1.3)-(1.4), when F_0 is symmetric about 0.

3. Properties of the test. Because of the translation-invariance of the estimators $\hat{\theta}_{1,m}$ and $\hat{\theta}_{2,n}$, without any loss of generality, we may take $\theta_1 = \theta_2 = 0$ and note that the residuals in (2.2) are translation invariant too. Further, we assume that there exists a positive λ_0 , such that on letting $N = m+n$,

$$(3.1) \quad 0 < \lambda_0 \leq \lambda_N = m/N \leq 1 - \lambda_0 < 1, \quad \text{for all } N.$$

Also, we assume that the d.f. F_0 (and G_0) are symmetric and have uniformly continuous probability density functions (p.d.f.) f_0 (and g_0) almost everywhere (a.e.). Finally, replacing $\hat{\theta}_{1,m}$ and $\hat{\theta}_{2,n}$ by θ_1 and θ_2 , respectively, in (2.2), we denote the corresponding empirical d.f.s in (2.4) by F_m^* and G_n^* , respectively.

Then, we have the following

Theorem 1. Under (2.1), (3.1) and the assumed regularity conditions on F_0 and G_0 ,

$$(3.2) \quad \sup\{ m^{\frac{1}{2}} |\hat{F}_m^*(x) - F_m^*(x)| : x \geq 0 \} = o_p(1),$$

$$(3.2) \quad \sup\{ n^{\frac{1}{2}} |\hat{G}_n^*(x) - G_n^*(x)| : x \geq 0 \} = o_p(1), \text{ as } N \rightarrow \infty.$$

Proof. We shall only prove (3.2), and (3.3) follows precisely on the same line.

Let $t_m = m^{\frac{1}{2}}(\hat{\theta}_{1,m} - \hat{\theta}_1)$. Then, by (2.2) and (2.3), for every $x \geq 0$,

$$(3.4) \quad \begin{aligned} \hat{F}_m^*(x) &= F_m(x + m^{-\frac{1}{2}} t_m) - F_m(-x + m^{-\frac{1}{2}} t_m) \\ &= F_m^*(x) + \{F_m(x + m^{-\frac{1}{2}} t_m) - F_m(x) - F_m(-x + m^{-\frac{1}{2}} t_m) + F_m(-x)\}, \end{aligned}$$

so that

$$(3.5) \quad \begin{aligned} m^{\frac{1}{2}}\{\hat{F}_m^*(x) - F_m^*(x)\} &= m^{\frac{1}{2}}\{F_0(x + m^{-\frac{1}{2}} t_m) - F_0(x) - F_0(-x + m^{-\frac{1}{2}} t_m) + F_0(-x)\} \\ &\quad + m^{\frac{1}{2}}\{F_m(x + m^{-\frac{1}{2}} t_m) - F_0(x + m^{-\frac{1}{2}} t_m) - F_m(x) + F_0(x)\} \\ &\quad - m^{\frac{1}{2}}\{F_m(-x + m^{-\frac{1}{2}} t_m) - F_0(-x + m^{-\frac{1}{2}} t_m) - F_m(-x) + F_0(-x)\}. \end{aligned}$$

Now, the first term on the right hand side of (3.5) is equal to $t_m\{f_0(x + am^{-\frac{1}{2}} t_m) - f_0(-x + bm^{-\frac{1}{2}} t_m)\}$, where $0 < a, b < 1$, and using the symmetry and uniform continuity of the pdf f_0 , we conclude that under (2.1) (insuring the boundedness of t_m , in probability), this converges to 0, in probability, as $m \rightarrow \infty$, uniformly in $x \geq 0$.

On the other hand, if we make use of the weak convergence of the empirical process $m^{\frac{1}{2}}\{F_m - F_0\}$ to a Brownian bridge, then, by the tightness part of this weak convergence, (2.1) and the uniform continuity of f_0 , each of the other two terms on the right hand side of (3.5) converges to 0, in probability, as $m \rightarrow \infty$; the conclusion remains true under the sup-norm metric when we make use of the modulus of continuity for the empirical process $m^{\frac{1}{2}}(F_m - F_0)$. Hence, the proof of (3.2) is complete.

By virtue of Theorem 1, we conclude that under the hypothesis of Theorem 1,

$$(3.6) \quad \sup\{ N^{\frac{1}{2}} |\hat{F}_m^*(x) - \hat{G}_n^*(x) - F_m^*(x) + G_n^*(x)| : x \geq 0 \} \xrightarrow{P} 0, \text{ as } N \rightarrow \infty.$$

Note that F_m^* is the empirical d.f. of the $X_i - \theta_1$ whose true d.f. is $F_0^*(x) = F_0(x) - F_0(-x)$, $x \geq 0$. Similarly, G_n^* is the empirical d.f. corresponding to the true d.f. $G_0^*(x) = G_0(x) - G_0(-x)$, $x \geq 0$. Hence, if we define

$$(3.7) \quad K_{mn}^{*+} = \sup\{F_m^*(x) - G_n^*(x) : x \geq 0\} \text{ and } K_{mn}^* = \sup\{|F_m^*(x) - G_n^*(x)| : x \geq 0\},$$

then, these are respectively the one and two-sided Kolmogorov-Smirnov statistics for the two samples with observations $|X_i - \theta_1|$, $i=1, \dots, m$ and $|Y_j - \theta_2|$, $j=1, \dots, n$.

From (2.5), (2.6), (3.6) and (3.7), we conclude that under the hypothesis of Theorem 1, as $N \rightarrow \infty$,

$$(3.8) \quad N^{\frac{1}{2}} |\hat{K}_{mn}^{*+} - K_{mn}^{*+}| \xrightarrow{P} 0 \text{ and } N^{\frac{1}{2}} |\hat{K}_{mn}^* - K_{mn}^*| \xrightarrow{P} 0.$$

This exhibits the asymptotic equivalence of the proposed tests and the ones based on the statistics in (3.7). In particular, under $F_0 = G_0$, F_0^* and G_0^* are also the same, and hence, we have for every $d \geq 0$,

$$(3.9) \quad \lim_{N \rightarrow \infty} P_{H_0} \{ (mn/N)^{\frac{1}{2}} K_{mn}^{*+} \geq d \} = \exp(-2d^2),$$

$$(3.10) \quad \lim_{N \rightarrow \infty} P_{H_0} \{ (mn/N)^{\frac{1}{2}} K_{mn}^* \geq d \} = 2 \sum_{r=1}^{\infty} (-1)^{r-1} \exp(-2r^2 d^2),$$

see for example, Hájek and Šidák (1967, p.190). Thus, under $H_0: F_0 = G_0$ and the regularity conditions of Theorem 1, for $(mn/N)^{\frac{1}{2}} \hat{K}_{mn}^{*+}$ and $(mn/N)^{\frac{1}{2}} \hat{K}_{mn}^*$, the limiting distributions in (3.9) and (3.10) apply. Also, for contiguous alternatives, the power properties of K_{mn}^{*+} and K_{mn}^* are shared by the proposed tests.

In this context, we may use the sample medians for $\hat{\theta}_{1,m}$ and $\hat{\theta}_{2,n}$ and these satisfy (2.1) whenever $f_0(0)$ and $g_0(0)$ are positive. Other robust estimates of location may also be used, and, this may avoid the assumption of a finite second moment which is usually needed with the sample mean. It is also possible to improve (3.8) by using the Bahadur-Kiefer representation for the sample quantiles in (3.5) and the Komlos-Major-Tusnady strong approximation for the reduced empirical processes. However, for our specific purpose, these are not really needed.

REFERENCE

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