ON THE THEORY OF ELLIPTICALLY CONTOURED DISTRIBUTIONS

by

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Abstract

The theory of elliptically contoured distributions is presented in an unrestricted setting (without reference to moment restrictions or assumptions of absolute continuity). These distributions are defined parametrically through their characteristic functions, and then studied primarily through the use of stochastic representations which naturally follow from the seminal work of Schoenberg on spherically symmetric distributions, appearing in 1938. It is shown that the conditional distributions of elliptically contoured distributions are elliptically contoured, and the conditional distributions are precisely identified. In addition, a number of the properties of normal distributions (which constitute a type of elliptically contoured distributions) are shown, in fact, to characterize normality. A by-product of the research is a new (and useful) characterization of certain classes of characteristic functions appearing in Schoenberg's work.

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1. Introduction.

The elliptically contoured distributions on $\mathbb{R}^n$ (n-dimensional Euclidean space) are defined as follows. If X is an n-dimensional random (row) vector and, for some $\mu \in \mathbb{R}$ and some $n \times n$ nonnegative definite matrix Σ, the characteristic function $\phi_{X-\mu}(t)$ of $X - \mu$ is a function of the quadratic form $t \Sigma t'$, $\phi_{X-\mu}(t) = \phi(t\Sigma t')$, we say that X has an elliptically contoured distribution with parameters μ, Σ and ϕ, and we write $X \sim EC_n(\mu,\Sigma,\phi)$. When $\phi(u) = \exp(-u/2)$, $EC_n(\mu,\Sigma,\phi)$ is the normal distribution $N_n(\mu,\Sigma)$; and when $n = 1$, the class of elliptically contoured distributions coincides with the class of one-dimensional symmetric distributions. The location and scale parameters μ and Σ can be any vector in $\mathbb{R}^n$ and any $n \times n$ nonnegative definite matrix, while the class of admissible functions ϕ will be discussed below.

It is clear that for any $c > 0$, $EC_n(\mu,\Sigma,\phi(\cdot)) = EC_n(\mu,c\Sigma,\phi(c^{-1}\cdot))$. It is shown in Theorem 1 that this is the only redundancy in the parametric representation of a nondegenerate elliptically contoured distribution. The scale parameter Σ is proportional to the covariance matrix of X when the latter exists. Excluding the degenerate case $\Sigma = 0$, all of the components of X have finite second moments if and only if ϕ has a finite right-hand derivative $\phi'(0)$ at zero, and in this case the covariance matrix is $-2\phi'(0)\Sigma$. (See Theorem 4.) When the components of X have finite first moments, it is clear that the location parameter μ is the mean of X.

Several properties of elliptically contoured distributions have been obtained by Kelker (1970) and by Das Gupta et al. (1972) when Σ is invertible and a density exists. Here, we consider the general case of elliptically contoured distributions and, by making use of a convenient stochastic representation which follows from the results of Schoenberg (1938), we show that their conditional distributions are also elliptically contoured and permit an intuitively appealing stochastic representation. From this work follow several new characterizations of the normal distributions. Section 2 deals with the theory of stochastic representations; Section 3 is concerned with the subject of conditional distributions; Section 4 discusses densities and the circumstances of their existence; and Section 5 is concerned with characterizations of normality.

In addition to the references already cited, there is an interesting discussion of a subclass of the elliptically contoured distributions by Dempster (1969).

2. Stochastic representations.

Our approach to defining elliptically contoured distributions by means of parametric triplets (μ,Σ,ϕ), besides providing greater generality than other approaches which require X to be absolutely continuous, has the advantage that the class of distributions is closed under linear transformations of X with ϕ preserved under such transformations and with μ and Σ transformed in the same way as a mean vector and covariance matrix. Nevertheless, many of the properties of these distributions are more easily studied and described by means of stochastic representations of the form

(1) $$X \stackrel{d}{=} \mu + RU^{(\ell)}A ,$$

where $U^{(\ell)}$ is a random vector of dimension ℓ which is uniformly distributed on the unit sphere in $\mathbb{R}^{\ell}$ ($\ell \geq 1$), where R, independent of $U^{(\ell)}$, is a nonnegative random variable, and where μ and A are a nonstochastic vector and matrix of appropriate dimensions.

Denoting the characteristic function of $U^{(\ell)}$ by $\Omega_{\ell}(||t||^2)$, $t \in \mathbb{R}^{\ell}$, and the distribution function of R by F, it easily follows that a random vector $X \in \mathbb{R}^n$ which is representable as in (1) is distributed as $EC_n(\mu,\Sigma,\phi)$, where $\Sigma = A'A$ and

$$\phi(u) = \int_{[0,\infty)} \Omega_{\ell}(r^2u)dF(r) , \qquad u \geq 0 . \tag{2}$$

Thus *every random vector* X *which is representable as in* (1) *is elliptically contoured.*

Let Φ_{ℓ}, $\ell \geq 1$, denote the class of functions $\phi : [0,\infty) \to \mathbb{R}$ which are expressible as in (2) for some distribution function F on $[0,\infty)$; and let Φ_{∞} denote the same kind of class when $\Omega_{\ell}(r^2u)$ is replaced by $\exp(-r^2u/2)$. Schoenberg (1938) showed that for each $\ell \geq 1$, $\phi(||t||^2)$, $t \in \mathbb{R}^{\ell}$, is a characteristic function if and only if $\phi \in \Phi_{\ell}$, and that $\Phi_{\ell} \downarrow \Phi_{\infty}$ as $\ell \to \infty$.

Let $X \sim EC_n(\mu,\Sigma,\phi)$ and the rank $r(\Sigma)$ of Σ be $k \geq 1$. Further let $\Sigma = A'A$ be a "rank factorization" of Σ, i.e., a factorization such that A is $k \times n$, necessarily of rank k. Then $Y = (X-\mu)A^-$, where A^- is a generalized inverse of A, has characteristic function

$$\phi_Y(s) = \phi(sA^{-'}\Sigma A^- s') = \phi(sA^{-'}A'AA^- s') = \phi(||s||^2), \qquad s \in \mathbb{R}^k ,$$

since AA^- equals the $k \times k$ identity matrix I_k, due to the fact that A is of full rank k. (Cf., Rao and Mitra (1971), page 23.) Thus

$Y \sim EC_k(0,I_k,\phi)$ and $\phi(||s||^2)$, $s\in\mathbb{R}^k$, is a characteristic function on $\mathbb{R}^k$. Consequently $\phi\in\Phi_k$, and it assumes the form (2) for $\ell = k$ and some distribution function F on $[0,\infty)$. Thus if R is independent of $U^{(k)}$ and has the distribution function F, then $\mu + RU^{(k)}A \sim EC_n(\mu,\Sigma,\phi)$, and therefore

$$X \overset{d}{=} \mu + RU^{(k)}A \ . \tag{3}$$

In summary, $X \sim EC_n(\mu,\Sigma,\phi)$ *with* $r(\Sigma) = k$ *if and only if* X *is representable as in* (3), *where* R *is a nonnegative random variable which is independent of* $U^{(k)}$ *and* $\Sigma = A'A$ *is a rank factorization of* Σ $(A{:}k\times n, r(A)=k)$. *The function* ϕ *and the distribution function* F *are related through* (2) *with* $\ell = k$. (*When* Σ *is the zero matrix,* $k = 0$ *and* $X = \mu$ *a.s.*) Also, it follows from (3) that *the class of admissible* ϕ *in the parametric representation* $EC_n(\mu,\Sigma,\phi)$, *when* $r(\Sigma) = k$, *is* Φ_k.

We shall refer to the representation given in (3) as a *canonical representation* of X. (It is not unique.) It is to be distinguished from the more general representation in (1) which might hold for $\ell > k = r(\Sigma)$. Observe, for instance, that if $\mu + RU^{(k)}A$ is a canonical representation of X and $Y = XB + C$ is a linear transformation of X, then the representation $(\mu B + C) + RU^{(k)}(AB)$ of Y is canonical if $r(AB) = k$, and noncanonical if $r(AB) < k$. For every index $\ell \geq k$ for which $\phi\in\Phi_\ell$, there is a representation of the type shown in (1). (See Corollary 2 below.) The distribution of R in (1) depends on ℓ, which is apparent from (2). There are precise relationships between the corresponding parts of any two representations of an elliptically contoured random vector; these are described in Theorem 3 below.

When Σ is of full rank n, then $X \sim EC_n(\mu,\Sigma,\phi)$ has a canonical representation taking the form

$$X \overset{d}{=} \mu + RU^{(n)}\Sigma^{\frac{1}{2}} . \tag{4}$$

The distribution function F appearing in (2) for $\ell = k = r(\Sigma)$ will be called the *canonical distribution function associated with* X. Its special significance is made clear in the following lemma.

Lemma 1. *If* F *is the canonical distribution function associated with* $X \sim EC_n(\mu,\Sigma,\phi)$ *and* $k = r(\Sigma) \geq 1$, *then the quadratic form*

$$Q(X) = (X-\mu)\Sigma^{-}(X-\mu)' \tag{5}$$

where Σ^{-} *is any generalized inverse of* Σ, *has the distribution function* $F(\sqrt{\cdot}\,)$.

Proof. From the canonical representation (3), one obtains $(X-\mu)\Sigma^{-}(X-\mu)' \overset{d}{=} R^2U^{(k)}A(A'A)^{-}A'U^{(k)\prime}$. According to Rao (1973, page 26, (vii)), $A(A'A)^{-}A'$ does not depend upon the particular generalized inverse chosen. Choosing the Moore-Penrose generalized inverse Σ^{+}, for which $(A'A)^{+} = A^{+}A^{+\prime}$ (cf., Boullion and Odell (1971), page 8), we have $A(A'A)^{+}A' = AA^{+}A^{+\prime}A' = I_kI_k' = I_k$. Thus $Q(X) \overset{d}{=} R^2$, and the theorem follows. □

Remarks. The distribution function F and the quadratic form Q(X) are both defined with respect to the parameters (μ,Σ,ϕ) used to describe the distribution of X. As Theorem 1 below points out, other parameterizations are possible. If $k = r(\Sigma) = 0$, then $X = \mu$ a.s. and $Q(X) = 0$ a.s.

In the remainder of this section, we develop the theory germane to stochastic representations of elliptically contoured random vectors.

Theorem 1. *If* $X \sim EC_n(\mu,\Sigma,\phi)$ *and* $X \sim EC_n(\mu_o,\Sigma_o,\phi_o)$, *then*

(6) $$\mu_o = \mu \ .$$

Moreover, if X *is nondegenerate, then there exists a* $c > 0$ *such that*

(7) $$\Sigma_o = c\Sigma$$

and

(8) $$\phi_o(\cdot) = \phi(c^{-1}\cdot) \ .$$

Proof. By examining characteristic functions, one can easily see that $X - \mu \overset{d}{=} -(X-\mu) = \mu - \mu_o - (X-\mu_o) \overset{d}{=} \mu - \mu_o + (X-\mu_o) = X - (2\mu_o-\mu)$, from which (6) follows. Write $\Sigma = (\sigma_{ij})$, $\Sigma_o = (\sigma^o_{ij})$, $\mu = (\mu_1,\ldots,\mu_n)$. If $X = (X_1,\ldots,X_n)$ is not degenerate, then one of its components X_j is not degenerate, and the characteristic function of $X_j - \mu_j$ is given by

$$\phi(\sigma_{jj}u^2) = \phi_o(\sigma^o_{jj}u^2), \qquad u\in\mathbb{R} ,$$

with $\sigma_{jj}, \sigma^o_{jj} > 0$, which establishes (8) with $c = \sigma^o_{jj}/\sigma_{jj}$. The hypotheses and (8) imply that the characteristic function of $X - \mu$ satisfies

(9) $$\phi(t\Sigma t') = \phi_o(t\Sigma_o t') = \phi(c^{-1}t\Sigma_o t) , \qquad t\in\mathbb{R}^n .$$

Now, if (7) is not true then for some $t_o\in\mathbb{R}^n$

$$(a^2=) \qquad t_o\Sigma t_o' \neq c^{-1}t_o\Sigma_o t_o' \qquad (=b^2) \ .$$

Substituting ut_o $(u\in\mathbb{R})$ for t in (9) leads to

(10) $$\phi(a^2u^2) = \phi(b^2u^2), \qquad u\in\mathbb{R} .$$

Then for $d = b^2/a^2<1$ if $b^2 < a^2$, or for $d = a^2/b^2<1$ if $a^2 < b^2$, it follows recursively from (10) that

$$\phi(u^2) = \phi(d^n u^2), \qquad u\in\mathbb{R} , \qquad n = 1,2,\ldots,$$

which in the limit, as $n \to \infty$, yields $\phi(u^2) = 1$, $u\in\mathbb{R}$. This contradicts the nondegeneracy of X, and hence (7) follows. □

Thus the distribution of an elliptically contoured random vector X does not uniquely determine the parameters (Σ,ϕ) in $EC_n(\mu,\Sigma,\phi)$. This redundancy has its analogues in (1) and (3), where it is apparent that if R is multiplied by a positive constant and A is multiplied by the constant's reciprocal, then the distribution of X is unchanged. Except for this indeterminancy in scale, the distributions of R in (1) and (3) are determined by the distribution of X when X is not degenerate. This is a consequence of Corollary 2 below.

Theorem 2. *A function* $\phi:[0,\infty) \to \mathbb{R}$ *belongs to* Φ_ℓ *if and only if* ϕ *is continuous and*

(11) $$\int_0^\infty \phi(2sv)g_\ell(v)dv , \qquad s \geq 0 ,$$

is the Laplace transform of a nonnegative random variable, where g_ℓ *denotes the chi-squared density with* ℓ *degrees of freedom* $(1\leq\ell<\infty)$.

Proof. The normal random vector $X \sim N_\ell(0,I_\ell)$ is elliptically contoured $EC_\ell(0,I_\ell,\phi)$ where $\phi(u) = \exp(-u/2)$. This $\phi\in\Phi_\ell$, and (2) yields the identity

$$\exp(-u/2) = \int_0^\infty \Omega_\ell(uv)g_\ell(v)dv \;, \qquad u \geq 0 \;,$$

where g_ℓ is the density of the quadratic form $Q(X) = XX'$ (see Lemma 1), or, what is more convenient for our purposes:

$$(12) \qquad \exp(-sr^2) = \int_0^\infty \Omega_\ell(2sr^2v)g_\ell(v)dv \;, \qquad s \geq 0,\ r\in\mathbb{R} \;.$$

Now suppose ϕ is an arbitrary member of Φ_ℓ and F is the distribution function on $[0,\infty)$ associated with ϕ through (2). It follows immediately from (2) and (12) that (11) is the Laplace transform of R^2 where R has the distribution function F. Conversely, suppose (11) is the Laplace transform of a nonnegative random variable S and ϕ is continuous on $[0,\infty)$. Let F be the distribution function of $S^{\frac{1}{2}}$, and let $\phi_o\epsilon\Phi_\ell$ be the function defined in (2). We shall show that $\phi_o = \phi$. From the first part of the proof we have

$$\int_0^\infty \phi_o(2sv)g_\ell(v)dv = \int_0^\infty \phi(2sv)g_\ell(v)dv \;, \qquad s \geq 0 \;,$$

which yields

$$\int_0^\infty h(u)\cdot u^{\frac{\ell}{2}-1} e^{-u/4s}du = 0 \;, \qquad s > 0 \;,$$

where $h = \phi_o - \phi$, and $u = 2sv$ describes a change of variables. The latter is the Laplace transform of $h(u)u^{\frac{\ell}{2}-1}$ in the variable $1/4s$, and hence $h(u) = 0$ a.e. on $[0,\infty)$. Since ϕ is continuous by assumption, and ϕ_o is continuous in as much as $\phi_o(u^2)$ is a characteristic function, it follows that $\phi_o = \phi$, and hence $\phi\epsilon\Phi_\ell$. $\square$

<u>Remarks.</u>

1. It is apparent in the proof of Theorem 2 that *for each fixed* ℓ

$(1 \le \ell < \infty)$, *there is a one-to-one correspondence between functions* $\phi \in \Phi_\ell$ *and distribution functions* F: Equation (2) defines ϕ in terms of F, and (11) describes the Laplace transform of the distribution function $F(\sqrt{\cdot}\,)$ in terms of ϕ.

2. According to Lemma 1, (11) is the Laplace transform of the quadratic form $Q(X) = (X-\mu)\Sigma^{-}(X-\mu)'$ when X is not degenerate and $\ell = k = r(\Sigma)$. It follows, in particular, that *an elliptically contoured random vector* X *has a nondegenerate normal distribution if and only if* Q(X) *is a positive multiple of a chi-squared random variable with* $k = r(\Sigma)$ *degrees of freedom.* (The multiple is one if Σ is the covariance matrix of X. This corresponds to $\phi(u) = \exp(-u/2)$, $u \ge 0$.)

3. Theorem 2, together with Bernstein's theorem (cf., Feller (1971), page 439), can be used effectively to show when a candidate ϕ belongs to the class Φ_ℓ. For instance, it is easy to check that

$\phi(\cdot) = (1-\alpha\cdot)\exp(-\cdot/2) \in \Phi_\ell$ if and only if $0 \le \alpha\ell \le 1 \qquad (1 \le \ell < \infty)$.

<u>Corollary 2</u>. *Suppose* $X \sim EC_n(\mu,\Sigma,\phi)$ *and* X *is nondegenerate. Then* X *is representable as in* (1) *if and only if* $\ell \ge r(\Sigma)$ *and* $\phi \in \Phi_\ell$; *equivalently, if and only if* $\ell \ge r(\Sigma)$ *and* (11) *is the Laplace transform of a nonnegative random variable. If* X *is representable as in* (1), *then* A'A *is a positive multiple of* Σ. *Moreover, if* A *is scaled so as to make* $A'A = \Sigma$, *then the square of the random variable* R *appearing in* (1) *must have the Laplace transform given in* (11).

<u>Proof</u>. We have already shown that (1) leads to $X \sim EC_n(\mu,\Sigma_o,\phi_o)$ where $\Sigma_o = A'A$ and ϕ_o is the function defined in (2). Since, by assumption, $X \sim EC_n(\mu,\Sigma,\phi)$, it follows from Theorem 1 that $\phi \in \Phi_\ell$ and

$\ell \geq r(A) \geq r(\Sigma_o) = r(\Sigma)$. Conversely, if $\phi \epsilon \Phi_\ell$ and R is a random variable, independent of $U^{(\ell)}$, whose distribution function is the F appearing in (2), then $RU^{(\ell)} \sim EC_\ell(0, I_\ell, \phi)$ (a restatement of (2)), and hence $\mu + RU^{(\ell)}A \sim EC_n(\mu, \Sigma, \phi)$, where A is $\ell \times n$ and chosen so as to make $A'A = \Sigma$. That A can be so chosen is easily established when $\ell \geq r(\Sigma)$. The remaining part of the first sentence, following the semi-colon, in the statement of the corollary then becomes an immediate application of Theorem 2. Now suppose X is representable as in (1), so that $X \sim EC_n(\mu, \Sigma_o, \phi_o)$ with $\Sigma_o = A'A$. Then A'A is a positive multiple of Σ, and it is apparent that A can be rescaled so as to make $A'A = \Sigma$ and $\phi_o = \phi$ (cf., Theorem 1). Then it follows from the first remark following the proof of Theorem 2, that the square of the random variable R appearing in (1) has the Laplace transform given in (11). □

Thus *if a nondegenerate elliptically contoured random vector* X *has two representations* $X \stackrel{d}{=} \mu + RU^{(\ell)}A$ *and* $X \stackrel{d}{=} \mu_o + R_oU^{(\ell)}A_o$ *and* $A_o = A$ *(or* $A_o'A_o = A'A$), *then* $\mu_o = \mu$ *and* $R_o \stackrel{d}{=} R$. (If $A_o'A_o = cA'A$, then $R_o \stackrel{d}{=} c^{-\frac{1}{2}} R$.)

<u>Lemma 2</u>. *Suppose* $X \stackrel{d}{=} RU^{(n)} \sim EC_n(0, I_n, \phi)$ *and* $P(X=0) = 0$. *Then*

$$||X|| \stackrel{d}{=} R\ , \qquad X/||X|| \stackrel{d}{=} U^{(n)}\ ,$$

and they are independent.

<u>Proof</u>. Since the mapping $x \to (||x||, x/||x||)$ is Borel measurable on $\mathbb{R}^n - \{0\}$, it follows from the representation $X \stackrel{d}{=} RU^{(n)}$ that $(||X||, X/||X||) \stackrel{d}{=} (R, U^{(n)})$, which proves the lemma. □

Write $U^{(n)} = (U_1^{(n)}, U_2^{(n)})$ where $U_1^{(n)}$ is m-dimensional $(1 \leq m < n)$.

<u>Lemma 3.</u> $(U_1^{(n)}, U_2^{(n)}) \overset{d}{=} (R_{nm}U^{(m)}, (1-R_{nm}^2)^{\frac{1}{2}} U^{(n-m)})$

where R_{nm} (≥ 0), $U^{(m)}$ *and* $U^{(n-m)}$ *are independent, and*

$$R_{nm}^2 \sim \mathrm{Beta}(\tfrac{m}{2}, \tfrac{n-m}{2}),$$

i.e., R_{nm} *has the density function*

$$\frac{2\Gamma(\frac{n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n-m}{2})} r^{m-1}(1-r^2)^{\frac{n-m}{2}-1}, \qquad 0 < r < 1 .$$

<u>Proof.</u> Let $X = (X_1, X_2) \sim N_n(0, I_n)$, where the dimension of X_1 is m. Since X_1 and X_2 are independent, it follows from Lemma 2 that

$$X_1/||X_1|| , \qquad X_2/||X_2|| , \qquad ||X_1|| , \qquad ||X_2||$$

are jointly independent and

$$X_1/||X_1|| \overset{d}{=} U^{(m)} , \quad X_2/||X_2|| \overset{d}{=} U^{(n-m)} .$$

In what follows, we set $U^{(m)}$ and $U^{(n-m)}$ equal to $X_1/||X_1||$ and $X_2/||X_2||$, respectively, and define R_{nm} as $||X_1||/||X||$ $(=||X_1||/(||X_1||^2+||X_2||^2)^{\frac{1}{2}})$. Clearly R_{nm}, $U^{(m)}$ and $U^{(n-m)}$ are independent, and R_{nm} has the required distribution since $||X_1||^2$ and $||X_2||^2$ are independent chi-squared variables with degrees of freedom m and n - m respectively. Finally, applying Lemma 2 again, we obtain

$$(U_1^{(n)}, U_2^{(n)}) = U^{(n)} \overset{d}{=} \frac{X}{||X||} = (X_1/||X||,\ X_2/||X||)$$

$$= (R_{nm}U^{(m)},\ (1-R_{nm}^2)^{\frac{1}{2}} U^{(n-m)}) . \qquad \square$$

<u>Remark</u>. If $\phi\epsilon\Phi_N$ $(N<\infty)$, then for each n, $1 \le n \le N$, $\phi\epsilon\Phi_n$ and, according to (2),

$$\phi(u) = \int_{[0,\infty)} \Omega_n(r^2u)dF_n(r) \ , \qquad u \ge 0 \ ,$$

for some unique distribution function F_n on $[0,\infty)$. F_m and F_n $(1\le m<n\le N)$ are related in the following way: If R_m and R_n have distribution functions F_m and F_n, then

$$R_m \overset{d}{=} R_n R_{nm} \tag{13}$$

where R_{nm}, distributed as in Lemma 3, is independent of R_n. Consequently (cf., Lemma 4 below), R_m has an atom of size $P(R_n=0)$ at zero if this probability is positive, and it is absolutely continuous on $(0,\infty)$ with the density

$$f_m(s) = \frac{2s^{m-1}\Gamma(\frac{n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n-m}{2})} \int_s^\infty r^{-(n-2)}(r^2-s^2)^{\frac{n-m}{2}-1} dF_n(r) \ , \qquad 0 < s < \infty \ . \tag{14}$$

Thus F_m takes the form

$$F_m(\rho) = F_n(0) + \int_0^\rho \int_s^\infty \frac{2\Gamma(\frac{n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n-m}{2})} s^{m-1} r^{-(n-2)}(r^2-s^2)^{\frac{n-m}{2}-1} dF_n(r)ds$$

$$= F_n(0) + \int_1^\infty \frac{2\Gamma(\frac{n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n-m}{2})} t^{-(n-2)}(t^2-1)^{\frac{n-m}{2}-1} dF_n(\rho t) \ , \qquad \rho \ge 0 \ .$$

To show (13), let $X = (X_1,X_2) \overset{d}{=} R_n U^{(n)} \sim EC_n(0,I_n,\phi)$, where X_1 is of dimension m. Then $X_1 \overset{d}{=} R_m U^{(m)} \sim EC_m(0,I_m,\phi)$. But, on the other hand, we have from Lemma 3,

$$X_1 \overset{d}{=} R_n U_1^{(n)} \overset{d}{=} R_n R_{nm} U^{(m)} ,$$

where R_n, R_{nm} and $U^{(m)}$ are independent. Thus X_1 has two canonical representations, $R_m U^{(m)}$ and $R_n R_{nm} U^{(m)}$, and (13) follows from the remark following Corollary 2. □

On the basis of this remark and the foregoing discussion, we are able to easily justify the following theorem and its corollaries.

Theorem 3. *Suppose the nondegenerate elliptically contoured random vector* X *has two representations* $X \overset{d}{=} \mu + RU^{(\ell)}A$ *and* $X \overset{d}{=} \mu_o + R_o U^{(\ell_o)} A_o$, *where* $\ell \geq \ell_o$. *Then*

(i) $\mu_o = \mu$,

(ii) $A'A = cA_o'A_o$ *for some* $c > 0$,

(iii) $c^{\frac{1}{2}} RR_{\ell\ell_o} \overset{d}{=} R_o$, *where* $R_{\ell\ell_o}$ *is independent of* R *and* $R_{\ell\ell_o} \sim \text{Beta}(\frac{\ell_o}{2}, \frac{\ell-\ell_o}{2})$. (*Set* $R_{\ell\ell_o} \equiv 1$ *if* $\ell_o = \ell$.)

Corollary 3a. *If the elliptically contoured random vector* X *has the representation* (1) *and* X *is not degenerate, then* X *has a canonical representation* $\mu + RR_{\ell k}U^{(k)}A_o$ *where* $k = r(A'A)$, $A_o'A_o$ *is a rank factorization of* $A'A$, *the stochastic entities* R, $R_{\ell k}$ *and* $U^{(k)}$ *are independent, and* $R_{\ell k}^2 \sim \text{Beta}(\frac{k}{2}, \frac{\ell-k}{2})$. ($R_{\ell k} \equiv 1$ *if* $\ell = k$.)

Corollary 3b. *If* $X \sim EC_n(\mu, \Sigma, \phi)$ *has the representation* (1) *with* $A'A = \Sigma$ *and* $k = r(\Sigma) \geq 1$, *then the quadratic form* $Q(X) = (X-\mu)\Sigma^-(X-\mu)' \overset{d}{=} R^2R_{\ell k}^2$, *where* Σ^- *is any generalized inverse of* Σ, R *and* $R_{\ell k}$ *are independent, and* $R_{\ell k}^2 \sim \text{Beta}(\frac{k}{2}, \frac{\ell-k}{2})$ ($R_{\ell k} = 1$ *if* $\ell = k$.)

Suppose $X \sim EC_n(\mu,\Sigma,\phi)$ is nondegenerate and, for convenience, R and A in (3) are scaled (see the remark following Theorem 1) so that $A'A = \Sigma$ and R has the distribution F appearing in (2) when $\ell = k = r(\Sigma)$. Since, from (3),

$$(X-\mu)'(X-\mu) \overset{d}{=} R^2 \cdot A'U^{(k)'}U^{(k)}A \ ,$$

it follows that the covariance matrix of X, denoted Σ_o, exists if and only if $ER^2 < \infty$. In this case $EX = \mu$ and

$$\begin{aligned} \Sigma_o &= ER^2 \cdot A' \operatorname{diag}(\tfrac{1}{k},\ldots,\tfrac{1}{k})A \\ &= k^{-1}ER^2 \cdot \Sigma = c\Sigma \qquad \text{(say)}. \end{aligned} \tag{15}$$

Because the distribution function of R (i.e., F) cannot be readily expressed in terms of ϕ, it is desirable to relate the existence of Σ_o and the constant c to ϕ directly.

<u>Theorem 4</u>. *Let* $X \sim EC_n(\mu,\Sigma,\phi)$ *with* X *nondegenerate. The covariance matrix* Σ_o *of* X *exists if and only if the right-hand derivative of* $\phi(u)$ *at* $u = 0$, *denoted* $\phi'(0)$, *exists and is finite. When it exists and is finite,*

$$\Sigma_o = -2\phi'(0)\Sigma \ .$$

<u>Proof</u>. We continue with the setting and facts established in the paragraph preceding the theorem. Clearly

$$\Sigma_o \text{ exists} \iff ER^2 < \infty \iff E(RU_1^{(k)})^2 < \infty \ ,$$

where $U_1^{(k)}$ denotes the first component of $U^{(k)}$. Since $RU^{(k)}$ has the characteristic function $\phi(||t||^2)$, $t \in \mathbb{R}^k$, $RU_1^{(k)}$ has the characteristic function

$$\phi_{RU_1^{(k)}}(u) = \phi(u^2) \ , \qquad u \in \mathbb{R} \ .$$

If Σ_o exists, then $\phi_{RU_1^{(k)}}$ is twice differentiable and

$$k^{-1}ER^2 = E(RU_1^{(k)})^2 = -\phi''_{RU_1^{(k)}}(0) = -\lim_{h\to 0} \frac{\phi(h^2) - 2\phi(0) + \phi((-h)^2)}{h^2} = -2\phi'(0) \ .$$

Thus the existence of Σ_o implies the existence and finiteness of $\phi'(0)$, and, moreover, $-2\phi'(0) = k^{-1}ER^2$, so that (cf., (15))

$$\Sigma_o = -2\phi'(0)\Sigma \ .$$

Conversely if $\phi'(0)$ exists and is finite then for $h\neq 0$,

$$\frac{1-\phi(h^2)}{h^2} = \frac{-\phi_{RU_1^{(k)}}(h) + 2\phi_{RU_1^{(k)}}(0) - \phi_{RU_1^{(k)}}(-h)}{2h^2}$$

$$= \int_{-\infty}^{\infty} \frac{1-\cos hx}{h^2}\, dH(x) \ ,$$

where H is the distribution function of $RU_1^{(k)}$. Then by Fatou's Lemma,

$$E(RU_1^{(k)})^2 = \int_{-\infty}^{\infty} x^2 dH(x) = 2\int_{-\infty}^{\infty} \lim_{h\to 0} \frac{1-\cos hx}{h^2}\, dH(x)$$

$$\leq 2 \lim_{h\to 0} \int_{-\infty}^{\infty} \frac{1-\cos hx}{h^2}\, dH(x) = -2\phi'(0) \ .$$

Thus $E(RU_1^{(k)})^2 < \infty$ and, hence, Σ_o exists. $\square$

When the covariance matrix Σ_o of $X \sim EC_n(\mu,\Sigma,\phi)$ exists, choosing Σ to be Σ_o has special appeal. This occurs (automatically) if and only if ϕ is chosen so as to make $-2\phi'(0) = 1$. This happens, for instance, in the case of normality when, among the possibilities $\{\exp(-cu),\ c>0\}$, one chooses $\phi(u) = \exp(-u/2),\ u \geq 0$.

3. Conditional distributions.

In this section, it is shown that if two random vectors have a joint elliptically contoured distribution, then the conditional distribution of one given the other is also elliptically contoured. The location and scale parameters of the conditional distribution do not depend upon the third parameter ϕ of the joint distribution and, consequently, the formulas which apply in the normal case apply in this more general setting as well. The situation with the third parameter of the conditional distribution is much more complicated, unfortunately, and needs further discussion. The case in which the scale parameter Σ of the joint distribution is an identity matrix is considered first in Theorem 5, and the general case is handled in Corollary 5. The statement of each includes parametric and stochastic descriptions of all distributions, the latter being made in terms of canonical representations.

The following lemma, whose proof is straightforward and therefore omitted, is needed in the proof of Theorem 5.

Lemma 4. *Suppose* R *is a nonnegative random variable with (right continuous) distribution function* F, *and* S *is an independent random variable which is nonnegative and absolutely continuous with density* g. *Then the product* $T = RS$ *has an atom of size* $F(0)$ *at zero if* $F(0) > 0$, *and* *it is absolutely continuous on* $(0,\infty)$ *with density* h *given by*

$$h(t) = \int_{(0,\infty)} r^{-1} g(t/r)\,dF(r) \ .$$

Moreover, a regular version of the conditional distribution of R *given* $T = t$ *is expressible as*

$$(16) \qquad P(R\le\rho \mid T=t) \begin{cases} = 0 \text{ for } \rho < 0 \ ; \\ = 1 \text{ for } t = 0, \text{ or } t > 0 \text{ with } h(t) = 0,\ \rho \ge 0 \ ; \\ = (h(t))^{-1}\int_{(0,\rho]} r^{-1} g(t/r)\,dF(r) \text{ for } t > 0 \text{ with } \\ \qquad h(t) \ne 0,\ \rho \ge 0 \ . \end{cases}$$

<u>Theorem 5</u>. *Let* $X = RU^{(n)} \sim EC_n(0,I_n,\phi)$ *and* $X = (X_1,X_2)$, *where* X_1 *is m-dimensional,* $1 \le m < n$. *Then a regular conditional distribution of* X_1 *given* $X_2 = x_2$ *is given by*

$$(17) \qquad (X_1 \mid X_2=x_2) \sim EC_m(0,I_m,\phi_{||x_2||^2})$$

with canonical representation

$$(18) \qquad (X_1 \mid X_2=x_2) \overset{d}{=} R_{||x_2||^2} U^{(m)} \ ,$$

where for each $a \ge 0$, R_{a^2} *and* $U^{(m)}$ *are independent, the distribution of* R_{a^2} *is given by* (21) *(below),*

$$(19) \qquad R_{||x_2||^2} \overset{d}{=} ((R^2-||x_2||^2)^{\frac{1}{2}} \mid X_2=x_2) \ ,$$

and the function ϕ_{a^2} *is given by* (2) *with* $\ell = m$ *and* F *replaced by the distribution function of* R_{a^2}.

Proof. From Lemma 3,

$$(X_1,X_2) = R(U_1^{(n)},U_2^{(n)}) \stackrel{d}{=} R(R_{nm}U^{(m)},(1-R_{nm}^2)^{\frac{1}{2}}\, U^{(n-m)}),$$

where R, R_{nm}, $U^{(m)}$ and $U^{(n-m)}$ are independent. Observe that $RR_{nm} = (R^2-||x_2||^2)^{\frac{1}{2}}$ when $R(1-R_{nm}^2)^{\frac{1}{2}}\, U^{(n-m)} = x_2$. Thus

$$\begin{aligned}(X_1|X_2=x_2) &\stackrel{d}{=} (RR_{nm}U^{(m)}|R(1-R_{nm}^2)^{\frac{1}{2}}\, U^{(n-m)}=x_2)\\
&\stackrel{d}{=} ((R^2-||x_2||^2)^{\frac{1}{2}}\, U^{(m)}|R(1-R_{nm}^2)^{\frac{1}{2}}\, U^{(n-m)}=x_2)\\
&\stackrel{d}{=} ((R^2-||x_2||^2)^{\frac{1}{2}}|R(1-R_{nm}^2)^{\frac{1}{2}}\, U^{(n-m)}=x_2)U^{(m)}\\
&\stackrel{d}{=} R_{||x_2||^2}U^{(m)} ,\end{aligned}$$

which is the canonical representation described in (18), where $R_{||x_2||^2}$ is distributed as expressed in (19). That $R_{||x_2||^2}$ depends upon x_2 only through the value of $||x_2||^2$, as the notation indicates, is obvious when $x_2 = 0$. When $x_2 \neq 0$, this follows from

$$\begin{aligned}((R^2-||x_2||)^{\frac{1}{2}}|X_2=x_2) &\stackrel{d}{=} ((R^2-||x_2||^2)^{\frac{1}{2}}|R(1-R_{nm}^2)^{\frac{1}{2}}\, U^{(n-m)}=x_2)\\
&\stackrel{d}{=} ((R^2-||x_2||^2)^{\frac{1}{2}}|R^2(1-R_{nm}^2)=||x_2||^2,U^{(n-m)}=x_2/||x_2||) \qquad (20)\\
&\stackrel{d}{=} ((R^2-||x_2||^2)^{\frac{1}{2}}|R^2(1-R_{nm}^2)=||x_2||^2) ,\end{aligned}$$

since $U^{(n-m)}$ is independent of R and R_{nm}. Clearly (17) follows from (18) with $\phi_{||x_2||^2}$ defined in the manner described. Finally from (19), (20), and Lemma 4 with the density of $(1-R_{nm}^2)^{\frac{1}{2}}$ given by (cf., Lemma 3)

$$g(t) = \frac{\Gamma(\frac{m}{2})\Gamma(\frac{n-m}{2})}{\Gamma(\frac{n}{2})} t^{n-m-1}(1-t^2)^{\frac{m}{2}-1}, \qquad 0 < t < 1 ,$$

(zero elsewhere) and the distribution function of R given by F, one obtains

$$R_{a^2} = 0 \quad \text{a.s. when } a = 0 \text{ or } F(a) = 1 ,$$

$$(21) \qquad P(R_{a^2} \leq \rho) = \frac{\int_{(a,\sqrt{\rho^2+a^2}]} (r^2-a^2)^{\frac{m}{2}-1} r^{-(n-2)} dF(r)}{\int_{(a,\infty)} (r^2-a^2)^{\frac{m}{2}-1} r^{-(n-2)} dF(r)} ,$$

$$\rho \geq 0,\ a > 0 \text{ and } F(a) < 1 . \qquad \square$$

<u>Remarks</u>.

1. There is no simple way (that we know of) to explicitly express ϕ_{a^2} in terms of ϕ (together with m, n and a) when $a > 0$. The relationship arises implicitly as follows: (i) ϕ determines the Laplace transform of R^2 (see the first remark following Theorem 2) which implicitly determines the distribution function F; (ii) F determines F_{a^2}, the distribution function of R_{a^2}, by means of (21); (iii) F_{a^2} determines ϕ_{a^2} by means of (2) (letting $\ell = m$ and replacing F by F_{a^2}). One complication is that the values of F on [0,a) influence all of the values of ϕ (apparent from (2) with $\ell = n$), while these same values of F have no effect on F_{a^2} (see (21))

and, therefore, no effect on ϕ_{a^2}. Thus the mapping $\phi \to \phi_{a^2}$ is many-to-one in a very complicated manner. In contrast, the description of $(X_1|X_2=x_2)$ through (19) and the canonical representation given in (18) is quite explicit and straightforward, as far as it goes.

2. While (21) shows that the distribution function F_{a^2} (of R_{a^2}) is determined by F (together with m, n and a), in the converse direction we have the following which is germane to Theorem 7 below: Denoting the denominator in (21) by C_{a^2}, we obtain from (20):

$$(22)\quad 1 - F_{a^2}(r) = C_{a^2}\int_{(\sqrt{r^2-a^2},\infty)} (\rho^2+a^2)^{\frac{n}{2}-1}\rho^{-(m-2)}dF_{a^2}(\rho)\ , \qquad r \geq a\ ,$$

which is valid for all $a > 0$. Thus, when $a > 0$, F_{a^2} determines F on the interval $[a,\infty)$ up to the unknown multiplicative factor $C_{a^2} \geq 0$, and, of course, it contains no information about the values of F on $[0,a)$. If $F(r)$ is known for some $r \geq a$ and $F(r) < 1$, then F_{a^2} determines F uniquely on $[a,\infty)$ by means of (22).

<u>Corollary 5</u>. *Let* $X = \mu + RU^{(k)}A \sim EC_n(\mu,\Sigma,\phi)$ *with* $A'A = \Sigma$ *and* $r(A) = r(\Sigma) = k \geq 1$. *Further, let*

$$X = (X_1,X_2)\ , \qquad \mu = (\mu_1,\mu_2)\ , \qquad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\ ,$$

where the dimensions of X_1, μ_1 *and* Σ_{11} *are, respectively,* m, m *and* $m \times m$, *and assume* $k_2 = r(\Sigma_{22}) \geq 1$ *and* $k_1 = k - k_2 \geq 1$. *Finally let* S *denote the row space of* Σ_{22}. *Then a regular conditional distribution of* X, *given* $X_2 = x_2$, *is given by*

(23a) $\quad (X_1|X_2=x_2) \sim EC_m(\mu_{x_2},\Sigma^*,\phi_{q(x_2)})$ *for* $x_2\epsilon\mu_2 + S$,

(23b) $\quad (X_1|X_2=x_2) \overset{d}{=} \mu_1$ *for* $x_2\not\epsilon\mu_2 + S$,

with a canonical representation

$$(X_1|X_2=x_2) \overset{d}{=} \mu_{x_2} + R_{q(x_2)}U^{(k_1)}A^* \quad \textit{for } x_2\epsilon\mu_2 + S ,$$

where, with Σ_{22}^- *denoting any generalized inverse of* Σ_{22},

$$\mu_{x_2} = \mu_1 + (x_2-\mu_2)\Sigma_{22}^-\Sigma_{21} , \qquad \Sigma^* = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^-\Sigma_{21} ,$$

$$q(x_2) = (x_2-\mu_2)\Sigma_{22}^-(x_2-\mu_2)' ,$$

and A^* *is a* $k_1 \times m$ *matrix of full rank* k_1 *satisfying* $A^{*\prime}A^* = \Sigma^*$. *Moreover, for each* $a \geq 0$, R_{a^2} *is independent of* $U^{(k_1)}$ *and its distribution is given by* (21) *with* $n = k$ *and* $m = k_1$;

$$R_{q(x_2)} \overset{d}{=} ((R^2-q(x_2))^{\frac{1}{2}}|X_2=x_2) \quad \textit{for } x_2\epsilon\mu_2 + S$$

and the function ϕ_{a^2} *is given by* (2) *with* $\ell = k_1$ *and* F *replaced by the distribution function of* R_{a^2}.

Remarks.

1. The description of $(X_1|X_2=x_2)$ given in (23a) is of primary interest since $X_2\epsilon\mu_2 + S$ a.s. (apparent from the proof below).

2. The excluded cases $k_2 = 0$ and $k_1 = 0$ are trivial and largely uninteresting: If $k_2 = 0$, then $X_2 = \mu_2$ a.s., and the conditional distribution of X_1 given X_2 is that of X_1. If $k_1 = 0$,

then $X_1 = \mu_{X_2} = \mu_1 + (X_2-\mu_2)\Sigma_{22}^{-}\Sigma_{21}$ a.s., and the conditional distribution of X_1 given X_2 is degenerate a.s.

Proof of corollary. Write $RU^{(k)} = (Z_1,Z_2)$, where Z_1 is k_1-dimensional, and let $\Sigma_{22} = A_2'A_2$ be a rank factorization of Σ_{22}, so that A_2 is $k_2 \times (n-m)$ and $r(A_2) = k_2$. It is easily checked through their characteristic functions that

$$(X_1,X_2) \stackrel{d}{=} (\mu_1+Z_1A^*+Z_2A_2\Sigma^{-}\Sigma_{21},\mu_2+Z_2A_2) \ .$$

Thus

$$\begin{aligned}(X_1|X_2=x_2) &\stackrel{d}{=} (\mu_1+Z_1A^*+Z_2A_2\Sigma_{22}^{-}\Sigma_{21}|Z_2A_2=x_2-\mu_2)\\ &\stackrel{d}{=} (\mu_1+(x_2-\mu_2)\Sigma_{22}^{-}\Sigma_{21}) + (Z_1|Z_2A_2=x_2-\mu_2)A^*\\ &= \mu_{x_2} + (Z_1|Z_2A_2=x_2-\mu_2)A^*\end{aligned}$$

Since $A_2'A_2 = \Sigma_{22}$, the row space of A_2 is also S, and it follows that the equation $z_2A_2 = x_2 - \mu_2$ admits a solution for z_2 if and only if $x_2 \epsilon \mu_2 + S$. Moreover, when a solution exists, it follows from Theorems 2.2.1 and 2.3.1 of Rao and Mitra (1971), pages 23-24, that it is unique and assumes the form $z_2 = (x_2-\mu_2)A_2^{-}$, where A_2^{-} is any generalized inverse of A_2. Thus

$$(X_1|X_2=x_2) \stackrel{d}{=} \mu_{x_2} + (Z_1|Z_2=(x_2-\mu_2)A_2^{-})A^* \ , \qquad x_2 \epsilon \mu + S \ .$$

But by Theorem 5,

$$(Z_1|Z_2=(x_2-\mu_2)A_2^-) \overset{d}{=} R_{||(x_2-\mu_2)A^-||^2} U^{(k_1)} ,$$

where for each $a \geq 0$, R_{a^2} is independent of $U^{(k_1)}$, and its distribution is given by (21) with $n = k$, $m = k_1$. Also for $x_2 \epsilon \mu_2 + S$, $x_2 - \mu_2 = z_2 A_2$ for some z_2, and hence

$$||(x_2-\mu_2)A_2^-||^2 = z_2A_2A_2^-A_2^{-'}A_2^{'}z_2^{'} = z_2A_2\Sigma_{22}^-A_2^{'}z^{'} = q(x_2) ,$$

since $A_2^-A_2^{-'}$ is a generalized inverse of Σ_{22}, and the value of the product $A_2\Sigma^-A_2^{'}$ is independent of which generalized inverse Σ^- one chooses. (Cf., Rao (1973), page 26, (vii).) Thus the claims for $x_2 \epsilon \mu_2 + S$ are true. Since $P(X_2 \epsilon \mu_2 + S) = P(Z_2A_2 \epsilon S) = 1$, the conditional distribution of X_1 given $X_2 = x_2$ may be defined arbitrarily on the complement of S. In particular, definition (23b) leads to a regular version of the conditional distribution. □

4. Densities.

Here, we discuss the densities of elliptically contoured distributions with an emphasis on their relationship to the random variable R appearing in their canonical representations. Much of this material appears elsewhere, such as in Kelker (1970), with a somewhat different emphasis. It is included here because we shall have occasion to refer to it, and, in part, for the sake of completeness.

If $X \sim EC_n(0, I_n, \phi)$ is absolutely continuous, its density is invariant under all orthogonal transformations and thus is expressible in terms of a function $g_n:[0,\infty) \to [0,\infty)$ as

$$(24) \qquad g_n(x_1^2+\ldots+x_n^2) \ , \qquad x = (x_1,\ldots,x_n) \in \mathbb{R}^n \ .$$

Moreover X has a canonical representation of the form $R_n U^{(n)}$, where R_n is also absolutely continuous, and its density f_n is expressible as

$$(25) \qquad f_n(r) = s_n r^{n-1} g_n(r^2) \ , \qquad r \geq 0 \ ,$$

where s_n denotes the surface area of the unit ball in $\mathbb{R}^n$ ($s_1=1$). Thus g_n must satisfy the integrability condition $\int_0^\infty r^{n-1} g_n(r^2)dr < \infty$. (The special case $g_n(u) = (2\pi)^{-n/2} \exp(-u/2)$ corresponds to normality and, as we have noted for this case, f_n is a chi-density with n degrees of freedom.) Conversely, if R is absolutely continuous, then so is X and their densities are related through (25).

If an absolutely continuous random vector $X = (X_1,\ldots,X_n)$ has a density of the form shown in (24), then f_n, defined by (25), is a density of a nonnegative random variable R, and $X \sim EC_n(0,I_n,\phi)$, where ϕ is defined by (2) with $\ell = n$ and $dF(x) = f_n(x)dx$.

Let $Y \sim EC_m(0,I_m,\phi)$ denote an arbitrary marginal of X of dimension m ($1 \leq m < n$). Y has the canonical representation $R_n R_{nm} U^{(m)}$, where R_n, R_{mn} and $U^{(m)}$ are independent, and $R_{nm} \sim \text{Beta}(\frac{m}{2},\frac{n-m}{2})$ (see (13) and the adjacent discussion). Consequently, Y is absolutely continuous if and only if $P(R_n=0) = 0$. When this occurs, the density f_m of R_m is given in (14), where F_n denotes the distribution function of R_n, and thus the density of Y takes the form

$$(26) \qquad g_m(||y||^2) = s_n^{-1} s_{(n-m)} \int_{||y||^2}^\infty r^{-(n-2)} (r^2-||y||^2)^{\frac{n-m}{2}-1} dF_n(r) \ ,$$

$$y \in \mathbb{R}^m \ .$$

When, moreover, R_n is absolutely continuous, this simplifies to

(27) $$g_m(||y||^2) = s_{(n-m)} \int_0^\infty g_n(||y||^2+r^2)r^{(n-m)-1}dr .$$

More generally, if $X \sim EC_n(\mu,\Sigma,\phi)$ with Σ of full rank n, then $(X-\mu)\Sigma^{-\frac{1}{2}} \sim EC_n(0,I_n,\phi)$. Thus the density of X exists and assumes the form

(28) $$|\Sigma|^{-\frac{1}{2}} g_n((x-\mu)\Sigma^{-1}(x-\mu))$$

if and only if R, appearing in the canonical representation $\mu + RU^{(n)}\Sigma^{\frac{1}{2}}$, is absolutely continuous with the density f_n defined in (25). Since $R^2 \overset{d}{=} Q(X) = (X-\mu)\Sigma^{-1}(X-\mu)'$, it is seen that X is absolutely continuous if and only if its quadratic form is, and that when they are absolutely continuous, the quadratic form Q(X) has a density of the form

$$f_Q(q) = \frac{s_n}{2} q^{\frac{n}{2}-1} g_n(q) , \qquad q \geq 0 .$$

The occurrence of absolute continuity for Y, when Y is a marginal of $X \sim EC_n(\mu,\Sigma,\phi)$, is as one would expect, and it follows that, in terms of the notation of Corollary 5, the conditional density of X_1 given $X_2 = x_2$ is given by

$$\frac{g_n([x_1-\mu_1-(x_2-\mu_2)\Sigma_{22}^{-1}\Sigma_{21}]\Sigma^{*-1}[x_1-\mu_1-(x_2-\mu_2)\Sigma_{22}^{-1}\Sigma_{21}]' + (x_2-\mu_2)\Sigma_{22}^{-1}(x_2-\mu_2)')}{s_m|\Sigma^*|^{\frac{1}{2}} \int_0^\infty g_n(r^2+(x_2-\mu_2)\Sigma_{22}^{-1}(x_2-\mu_2)')r^{m-1}dr}$$

when Σ is of rank n and X is absolutely continuous.

Finally, suppose $X \sim EC_n(\mu,\Sigma,\phi)$ with $r(\Sigma) = n$ and $\phi\epsilon\Phi_\infty$. By the definition of Φ_∞, there exists a distribution function F_∞ such that

$$\phi(u) = \int_{[0,\infty)} \exp(-ur^2/2)dF_\infty(r) \ , \qquad u \geq 0 \ .$$

Since $\phi(t\Sigma t')$, $t\epsilon\mathbb{R}^n$, is the characteristic function of $X - \mu$, X is absolutely continuous and has the density

$$\begin{aligned} &|\Sigma|^{-\frac{1}{2}} g_n((x-\mu)\Sigma^{-1}(x-\mu)') \\ (29) \qquad & \\ &= |\Sigma|^{-\frac{1}{2}} \int_{(0,\infty)} (2\pi r^2)^{-n/2} \exp\{-(x-\mu)\Sigma^{-1}(x-\mu)'/2r^2\}dF_\infty(r) \end{aligned}$$

if and only if $F_\infty(0) = 0$. Otherwise X is absolutely continuous away from the origin with the density given in (29), and it is atomic at the origin with atom $F_\infty(0)$.

5. Characterizations of normality.

In this section we focus attention on several properties of the normal distributions which do not extend to other elliptically contoured distributions. We have already discussed one of these in the second remark following Theorem 2, appearing in Section 2.

(a) When $X = (X_1,X_2)$ is normally distributed, then, of course, the conditional distribution of X_1 given X_2 is normally distributed and the function $\phi_{q(x_2)}$ in Corollary 5 assumes the form $\phi_{q(x_2)}(u) = \exp(-cu/2)$, where $c \geq 0$ is independent of x_2. The failure of $\phi_{q(x_2)}$ to depend upon the value of $q(x_2)$ characterizes normality:

Theorem 6. *Assume the rank values* k_1 *and* k_2 *appearing in Corollary* 5 *are strictly positive. Then* $\phi_{q(X_2)}(u)$ *is degenerate for each* $u \geq 0$ *if and only if* X *is normally distributed.*

Proof. In view of the foregoing discussion, it is only necessary to show the "only if" part. By Corollary 2, for all $t = (t_1, t_2) \in \mathbb{R}^n$,

$$\phi(t\Sigma t') = E \exp[it(X-\mu)'] = E \exp[it_1(X_1-\mu_1)' + it_2(X_2-\mu_2)']$$

$$= E\{\exp[it_2(X_2-\mu_2)']E(\exp[it_1(X_1-\mu_1)'] \mid X_2)\}$$

$$= E\{\exp[it_2(X_2-\mu_2)' + it_1(\mu_{X_2}-\mu_1)']\phi_{q(X_2)}(t_1\Sigma^* t_1)\} .$$

Hence, putting, for each $u \geq 0$, $\phi_{q(X_2)}(u) = \psi(u)$ a.s., it follows that

$$\phi(t\Sigma t') = \psi(t_1\Sigma^* t_1')E \exp[i(t_2+t_1\Sigma_{12}\Sigma_{22}^-)(X_2-\mu_2)']$$

$$= \psi(t_1\Sigma^* t_1)\phi\{(t_2+t_1\Sigma_{12}\Sigma_{22}^-)\Sigma_{22}(t_2+t_1\Sigma_{12}\Sigma_{22}^-)'\} .$$

And since

$$t\Sigma t' = t_1\Sigma^* t_1' + (t_2+t_1\Sigma_{12}\Sigma_{22}^-)\Sigma_{22}(t_2+t_1\Sigma_{12}\Sigma_{22}^-)' ,$$

it follows that $\phi(u+v) = \psi(u)\phi(v)$, $u,v \geq 0$. (Here, one makes use of the assumption $k_1, k_2 \geq 1$.) Setting $v = 0$, yields $\phi = \psi$, and thus $\phi(u+v) = \phi(u)\phi(v)$, $u,v \geq 0$. Since ϕ is continuous with $\phi(0) = 1$ and $|\phi(u)| \leq 1$, it follows that $\phi(u) = \exp(-cu/2)$ for some $c \geq 0$, and thus $X \sim N_n(\mu, c\Sigma)$. □

(b) It is apparent from (21) that the normality of $(X_1|X_2=x_2)$ does not entail the normality of X, for the distribution function $F_{||x_2||^2}$ can be that of a chi-variable without F being that of a chi-variable, or a multiple of a chi-variable. (See the remarks following Theorem 2.) Nevertheless, the following is true:

Theorem 7. *Assume* $X = (X_1, X_2)$ *has a nondegenerate elliptically contoured distribution and the rank values* k_1 *and* k_2*, appearing in Corollary* 5, *are both positive. Then* X *is normally distributed if and only if, with probability one, the conditional distribution of* X_1 *given* X_2 *is nondegenerate normal.*

Proof. The "only if" part states a well-known fact, and so we will only show the "if" part. We shall refer freely to the notation and results appearing in the statement and proof of Corollary 5. For $x_2 \epsilon \mu_2 + S$, $q(x_2) = ||(x_2-\mu_2)A_2^-||^2 = ||z_2||^2$, and, consequently,

$$q(X_2) \overset{d}{=} ||Z_2||^2 , \tag{30}$$

where Z_2 denotes the vector consisting of the last k_2 components of $RU^{(k)}$. Thus $P(q(X_2)=0) = P(R=0) \leq P((X_1|X_2)$ is degenerate$) = 0$, and hence $q(X_2) > 0$ a.s. Also, since $(X_1|X_2)$ is nondegenerate normal a.s., the function $\phi_{q(X_2)}(\cdot)$ must assume the form

$$\phi_{q(X_2)}(u) = \exp(-c(q(X_2))u/2) , \qquad u \geq 0 \qquad \text{a.s.}$$

for some function $c:(0,\infty) \to (0,\infty)$. If it can be shown that $c(q(X_2))$ is a degenerate random variable, then the normality of X follows from

Theorem 6. Let A be the set of all $a > 0$ such that

(31) $$\phi_{a^2}(u) = \exp(-c(a^2)u/2)\ , \qquad u \geq 0\ .$$

Note $P(q(X_2)^{\frac{1}{2}} \epsilon A) = 1$. Now (31) implies that the distribution function F_{a^2} of R_{a^2}, $a \epsilon A$, is that of a chi-squared variable with k_1 degrees of freedom times $c(a^2)^{\frac{1}{2}}$. Combining this with (22), when $n = k$ and $m = k_1$, we have for $a \epsilon A$

$$1 - F(r) = \text{Const.} \int_r^\infty S^{k-1} \exp(-S^2/2c(a^2))dS\ , \qquad r \geq a$$

i.e.

(32) $$F'(r) = \text{Const.}\ r^{k-1} e^{-r^2/2c(a^2)}\ , \qquad r \geq a\ .$$

It is obvious from (32) that $c(a^2)$ is a constant for all $a \epsilon A$, and the degeneracy of $c(q(X_2))$ follows. □

Remarks.

1. An alternative proof of the "if" part of Theorem 7 without using Theorem 6 can be given as follows. (32) implies

(33) $$F'(r) = \text{const.}\ r^{k-1} e^{-r^2/2c}\ , \qquad r > a_o$$

where c is some positive constant and a_o is the infimum of A. Since, on account of (30), $P(0 < q(X_2)^{\frac{1}{2}} < \varepsilon) > 0$ for every $\varepsilon > 0$, we have $A \cap (0,\varepsilon) \neq \emptyset$ for every $\varepsilon > 0$ and, therefore, $a_o = 0$. Now it follows from (33) and the fact $F(0) = P(R=0) = 0$, as noted in the previous proof, that

F is the distribution of a chi-squared variable with k degrees of freedom times $c^{\frac{1}{2}}$. Thus X must have a normal distribution. (See the second remark following Theorem 2.)

2. The reasoning in the previous remark can be used under weaker assumptions. X is normal if X_1 given $X_2 = x_2$ is nondegenerate normal for all $x_2 \epsilon \mu_2 + S$ for which $|x_2 - \mu_2| < \varepsilon$ for some $\varepsilon > 0$.

3. The restriction in the statement of Theorem 7 that the conditional distribution of X_1 given X_2 be nondegenerate is necessary: Suppose R in (3) has a mass F(0) at zero with $0 < F(0) < 1$ and a density of the form (33) on $(0,\infty)$. Then the conditional distribution of X_1 given X_2 is normal with probability one, but X is *not* normally distributed. (Of course, if $F(0) = 1$, then X is normally distributed, but degenerate.)

4. Theorem 7 was obtained by Kelker (1970) under the assumptions that Σ is nonsingular and X has a density.

(c) We shall refer to the function $g_n(\cdot)$, appearing in (28) as the *functional form* of the density of $EC_n(\mu,\Sigma,\phi)$ whenever it exists. Naturally its specification is arbitrary on a Lebesgue null subset of $[0,\infty)$. In addition, it is apparent from Theorem 1 that for any $c > 0$, $c^{-\frac{1}{2}} g(c\cdot)$ is also a functional form for a different parameterization of $EC_n(\mu,\Sigma,\phi)$.

<u>Theorem 8</u>. $EC_n(\mu,\Sigma,\phi)$ *with* $r(\Sigma) \geq 2$ *is a normal distribution if and only if two marginal densities of different dimensions exist and have functional forms which agree up to a positive multiple.*

<u>Proof</u>. The "only if" part is obvious, so we shall only show the "if" part. Suppose $X \sim EC_n(\mu,\Sigma,\phi)$ has marginals of dimensions p and p + q with functional forms g_p and g_{p+q}, and

$$(34) \qquad g_{p+q}(u) = \text{Const. } g_p(u) \ , \qquad u \geq 0 \ .$$

(Here and below, "Const." stands, generically, for a positive constant, not always the same.) Without loss of generality we assume that $r(\Sigma) = n$ and, in fact, that $\mu = 0$ and $\Sigma = I_n$. (Otherwise, we could consider $Y = (X-\mu)A^- \sim EC_k(0,I_k,\phi)$, where $k = r(\Sigma)$ and $A'A$ is a rank factorization of Σ.) Then

$$g_p(x_1^2+\ldots+x_p^2) = \int_{\mathbb{R}^q} g_{p+q}(x_1^2+\ldots+x_{p+q}^2)\,dx_{p+1}\ldots dx_{p+q}$$

$$= \text{Const. } \int_{\mathbb{R}^q} g_p(x_1^2+\ldots+x_{p+q}^2)\,dx_{p+1}\ldots dx_{p+q}$$

which implies

$$g_p(u) = \text{Const. } \int_{\mathbb{R}^q} g_p(u+z_1^2+\ldots+z_q^2)\,dz_1\ldots dz_q \ , \qquad u \geq 0 \ .$$

It follows that

$$g_{p+q}(x_1^2+\ldots+x_{p+q}^2) = \text{Const. } \int_{\mathbb{R}^q} g_p(x_1^2+\ldots+x_{p+2q}^2)\,dx_{p+q+1}\ldots dx_{p+2q} \ .$$

Thus a multiple of $g_p(x_1^2+\ldots+x_{p+2q}^2)$ is a density of an elliptically contoured random vector $Z \sim E_{p+2q}(0,I_{p+2q},\psi)$ (see the third paragraph of Section 4), and Z has the $(p+q)$-dimensional marginal density $g_{p+q}(x_1^2+\ldots+x_{p+q}^2)$. Consequently, $\phi = \psi \epsilon \Phi_{p+2q}$. Similarly it follows that $\phi \epsilon \Phi_{p+jq}$ for all $j = 1,2,\ldots$. Hence $\phi \epsilon \Phi_\infty$ and there exists a distribution function F_∞ on $[0,\infty)$ such that (cf., (29))

$$g_j(u) = \int_{(0,\infty)} (2\pi r^2)^{-j/2} \exp(-u/2r^2)\,dF_\infty(r) \ , \qquad u \geq 0 \ , \qquad j = p,p+q \ .$$

By the uniqueness of Laplace transforms and (34), we obtain

$$r^{-p} dF_\infty(r) = \text{Const. } r^{-(p+q)} dF_\infty(r) ,$$

from which it follows that F_∞ is degenerate at some point $\sigma > 0$. Thus $g_p(u) = \exp[-u/(2\sigma^2)]$ and X is normally distributed. (Observe that in (34) the constant must be unity.) □

(d) Because of the one-to-one correspondence between functions ϕ and distribution functions F described in the remark following Theorem 2, it is easily seen from Theorem 6, that, *under the assumptions of Theorem* 6, *the conditional distribution of* $R_{q(X_2)}$ (defined in Corollary 5) *given* X_2 *is independent of* X_2 *if and only if* X *is normally distributed*. Thus if X is normally distributed, then every conditional moment of positive order of $R_{q(X_2)}$ given X_2 is nonstochastic (degenerate). This invites the following characterization of normality:

Corollary 8a. *Suppose* $X = (X_1, X_2) \sim EC_n(\mu, \Sigma, \phi)$ *and the ranks* k_1 *and* k_2 *appearing in Corollary* 5 *are strictly positive. Then, for any fixed positive integer* p, $E((R_{q(X_2)})^p | X_2)$ *is finite and degenerate* (*i.e., independent of* X_2) *if and only if* X *is normally distributed.*

Proof. Just the "only if" part requires proof. From the proof of Corollary 5, it is apparent the assumption "$E((R_{q(X_2)})^p | X_2)$ is finite and degenerate" is equivalent to "$E(||Z_1||^p | Z_2)$ is finite and degenerate", where $Z = (Z_1, Z_2) \sim EC_k(0, I_k, \phi)$ with Z_1 and Z_2 of dimensions

k_1 and k_2 respectively. Moreover, if Z is normally distributed, so is X. Thus, without loss of generality, we may assume $X = (X_1, X_2) \sim EC_n(0, I_n, \phi)$, where X_1 is of dimension m ($1 \le m < n$), as in Theorem 5, and show that the normality of X is implied by the finiteness and degeneracy of $E(||X_1||^p | X_2)$. The latter permits two cases: $P(X=0) = 1$ and $P(X=0) = 0$. The first case is trivial: X is degenerate normal. In the second case, $P(||X_2||=0) = 0$, and according to (18) and (21),

$$
\begin{aligned}
(||X_1||^p | X_2) &= \frac{\int_{(||X_2||,\infty)} (r^2 - ||X_2||^2)^{\frac{m+p}{2}-1} r^{-(n-2)} dF(r)}{\int_{(||X_2||,\infty)} (r^2 - ||X_2||^2)^{\frac{m}{2}-1} r^{-(n-2)} dF(r)} && \text{when } F(||X_2||) < 1, \\
&= 0 && \text{when } F(||X_2||) = 1,
\end{aligned}
$$

where F is the distribution function of R in the canonical representation $X \overset{d}{=} RU^{(n)}$. Thus

$$
\int_{(||X_2||,\infty)} (r^2 - ||X_2||^2)^{\frac{m+p}{2}-1} r^{-(n-2)} dF(r)
$$

$$
= \text{Const.} \int_{(||X_2||,\infty)} (r^2 - ||X_2||^2)^{\frac{m}{2}-1} r^{-(n-2)} dF(r) \quad \text{a.s.}
$$

which implies

$$
\int_{(u,\infty)} (r^2 - u^2)^{\frac{m+p}{2}-1} r^{-(n-2)} dF(r) = \text{Const.} \int_{(u,\infty)} (r^2 - u^2)^{\frac{m}{2}-1} r^{-(n-2)} dF(r)
$$

for almost every $u > 0$ (Leb.), since $||X_2||$ is absolutely continuous on $(0,\infty)$ with (according to (14)) a positive density at every $u > 0$ for which $F(u) < 1$. It follows that

$$(35)\quad \int_{(u,\infty)} (r^2-u^2)^{\frac{m+p}{2}-1} r^{-(n+p-2)} dF_{n+p}(dr)$$

$$= \text{Const.} \int_{(u,\infty)} (r^2-u^2)^{\frac{m}{2}-1} r^{-(n+p-2)} dF_{n+p}(r)$$

for almost every $u > 0$, where $F_{n+p}(\rho) = \int_{(0,\rho]} r^p dF(r) / \int_{(0,\infty)} r^p dF(r)$ defines the distribution function F_{n+p}. (That $\int_{(0,\infty)} r^p dF(r) > 0$ is because $P(X=0) = 0$; that it is finite is because $E||X_1||^p = E[E(||X_1||^p|X_2)] = E(||X_1||^p|X_2)$ a.s., and hence is finite.) It is apparent from (26) that the left and right integrals of (35) are functional forms for marginal densities of $Z \sim EC_{n+p}(0, I_{n+p}, \psi)$ of dimensions $n - m$ and $n - m + p$ respectively, where $\psi(u)$ is defined by the right side of (2) with $\ell = n + p$ and $F = F_{n+p}$. Hence by Theorem 8, Z is normally distributed. Thus, F_{n+p} is the distribution function of a multiple of a chi-variable with n+p degrees of freedom. This implies F_n is the distribution function of a multiple of a chi-variable with n degrees of freedom, and, in turn, implies X is normally distributed. $\square$

<u>Remarks</u>. We suspect that the corollary would hold for any real $p > 0$, not necessarily an integer. The size and nature of the collection of functions $h: [0,\infty) \to \mathbb{R}$ which guarantee that "$E(h(||X_1||)|X_2)$ is finite and degenerate" $\Rightarrow$ "X is normal" is unknown.

(e) For $X = (X_1, X_2) \sim N_n(\mu, \Sigma)$, it is well-known that the conditional central moments of all orders of X_1 given X_2 are finite and independent of X_2. This fact characterizes normality:

Corollary 8b. *Suppose* $X = (X_1, X_2) \sim EC_n(\mu, \Sigma, \phi)$, *the ranks* k_1 *and* k_2 *appearing in Corollary* 5 *are strictly positive, and* p *is a positive integer. Then a* p-*th order conditional central moment of* X_1 *given* X_2 *is finite and degenerate (i.e., independent of* X_2*) if and only if* X *is normally distributed.*

Proof. Just the "only if" part requires proof. From Corollary 5 we have the canonical representation

$$(X_1 | X_2 = x_2) \stackrel{d}{=} \mu_{x_2} + R_{q(x_2)} U^{(k_1)} A^* ,$$

valid for all x_2 in a set $\mu_2 + S$ for which $P(X_2 \epsilon \mu_2 + S) = 1$. Putting $X_1 = (X^1, \ldots, X^m)$, $\mu_{x_2} = (\mu^1, \ldots, \mu^m)$ and $A^* = (a_1, \ldots, a_m)$, where a_i is the i-th column of A, we have for $p_1 + \ldots + p_m = p$,

$$((X^1 - \mu^1)^{p_1} \ldots (X^m - \mu^m)^{p_m} | X_2 = x_2) \stackrel{d}{=} (R_{q(x_2)})^p (U^{(k_1)} a_1)^{p_1} \ldots (U^{(k_1)} a_m)^{p_m} .$$

Since $R_{q(x_2)}$ and $U^{(k_1)}$ are independent, the normality of X is immediate from Corollary 8a. □

(f) The following, while not a characterization of normality, is an interesting extension of Theorem 6.

Theorem 9. *Let* $X = (X_1, X_2) \sim EC_n(\mu, \Sigma, \phi)$ *and assume the ranks* k_1 *and* k_2 *appearing in Corollary* 5 *are strictly positive. Then*

(36) $$\phi(u) = \int_{[0,\infty)} \exp(-ur^2/2)\,dG(r)\ , \qquad u \geq 0\ ,$$

for some distribution function G *on* $[0,\infty)$ *if and only if*

(37) $$\phi_{q(X_2)}(u) = \int_{[0,\infty)} \exp(-ur^2/2)\,dG_{q(X_2)}(r)\ , \qquad u \geq 0\ , \qquad a.s.$$

where G_{a^2} *is a distribution function on* $[0,\infty)$ *given by*

(38) $$G_0(\rho) = 1 \qquad \rho \geq 0\ ,$$

$$G_{a^2}(\rho) = \frac{\int_{(0,\rho]} r^{-k_2} \exp[-a^2/(2r^2)]\,dG(r)}{\int_{(0,\infty)} r^{-k_2} \exp[-a^2/(2r^2)]\,dG(r)} \qquad a > 0\ .$$

Proof. By Lemma 5 below the distribution function G is related to the distribution function F of R, appearing in the canonical representation $X \stackrel{d}{=} \mu + RU^{(k)}A$ with $A'A = \Sigma$, through the relationships

(39) $$F(0) = G(0)$$

$$f(r) = \frac{r^{k-1}}{2^{\frac{k}{2}-1}\,\Gamma(\frac{k}{2})} \int_{(0,\infty)} \rho^{-k} \exp[-r^2/(2\rho^2)]\,dG(\rho)\ , \qquad r > 0\ ,$$

where f is the density of F on $(0,\infty)$ on which it is absolutely continuous. If F_{a^2} denotes the distribution function of R_{a^2} appearing in Corollary 5, then it follows from (21) that

$$F_0(0) = 1\ ,$$

$$f_{a^2}(r) = \frac{r^{k_1-1} \int_{(0,\infty)} \rho^{-k} \exp[-(r^2+a^2)/(2\rho^2)]dG(\rho)}{2^{\frac{k_1}{2}-1} \Gamma(k_1/2) \int_{(0,\infty)} \rho^{-k_2} \exp[-a^2/(2\rho^2)]dG(\rho)} , \quad a > 0, r \geq 0 ,$$

where f_{a^2} denotes the density of F_{a^2} (which is absolutely continuous when $a > 0$). Again by Lemma 5, the latter is equivalent to (37) and (38). □

Lemma 5. *Suppose* $X \sim EC_n(\mu,\Sigma,\phi)$ *has the canonical representation* $X \overset{d}{=} \mu + RU^{(k)}A$ *with* $A'A = \Sigma$. *Then* ϕ *is given by* (36) *if and only if* G *is related to the distribution function* F *of* R *as described in* (39).

Proof. Now $RU^{(k)} \sim EC_k(0,I_k,\phi)$. If ϕ is given by (36), then $RU^{(k)} \overset{d}{=} YZ$ where Y and Z are independent, Y has distribution function G, and $Z \sim N_k(0,I_k)$. Setting $W = ||Z||$, we have $R \overset{d}{=} YW$ which implies

$$F(r) = G(0) + \int_{(0,\infty)} f_W(r/\rho)dG(\rho) , \qquad r > 0 ,$$

where f_W is the density of W, and the result follows from

$$f_W(r) = \frac{r^{k-1} \exp(-r^2/2)}{2^{\frac{k}{2}-1} \Gamma(\frac{k}{2})} , \qquad r > 0 .$$

□

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It is shown that the conditional distributions of elliptically contoured distributions are elliptically contoured, and the conditional distributions are precisely identified. In addition, a number of the properties of normal distributions (which constitute a type of elliptically contoured distributions) are shown, in fact, to characterize normality. A by-product of the research is a new characterization of certain classes of characteristic functions appearing in Schoenberg's work.