Stochastic vs. Deterministic Models for Systems with Delays

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Abstract: We consider population models with nodal delays which result in dynamical systems with delays. For small population models the appropriate models are discrete stochastic systems with delays. We consider these delay systems and present new theoretical and computational results for such systems. In particular, in this note we summarize results on the effects of different types of delays (a fixed delay and a random delay) on the dynamics of stochastic system as well as their relationship with each other in the context of a just-in-time network model. In addition, we numerically explore the corresponding deterministic approximations for the stochastic systems with these two different types of delays.

Keywords: Stochastic Markov chains, ordinary differential equations, delay equations

1. INTRODUCTION

Continuous time Markov Chain models are widely used to model physical and biological processes (e.g., see [1,3]). These models are typically used when dealing with dynamic systems involving low species count. However, because simulations are quite expensive for large population stochastic systems, in such models one often wishes to know whether or not the stochastic system can be approximated by a deterministic one when the population size is sufficiently large. Theory established by Kurtz (e.g. [11–15]) gives a way to construct a deterministic system to approximate density dependent continuous time Markov Chains as the population size grows large (this result is often called *Kurtz’s limit theorem*). In general, deterministic systems are much easier to analyze compared to stochastic systems. Techniques, such as parameter estimation methods, are well developed for deterministic systems, whereas parameter estimation is much more difficult in a stochastic framework. For example, in [16], the authors developed a method for estimating parameters in dynamic stochastic (Markov Chain) models based on Kurtz’s limit theory coupled with inverse problem methods developed for deterministic dynamical systems and illustrated these ideas in the context of disease dynamics. The methodology relies on finding an approximate large-population behavior of an appropriate scaled stochastic system. The approach detailed there leads to a deterministic approximation obtained as solutions of rate equations (ordinary differential equations) in terms of the large sample size average over sample paths or trajectories (limits of pure jump Markov

processes). Using the resulting deterministic model the authors discussed how to select parameter subset combinations that can be estimated using an ordinary-least-squares (OLS) or generalized-least-squares (GLS) inverse problem formulation with a given data set. The authors illustrated the ideas with a stochastic model for the transmission of vancomycin-resistant enterococcus (VRE) in hospitals and VRE surveillance data from an oncology unit. The dynamics of the VRE colonization of patients in a hospital unit are modeled as a continuous time Markov Chain (MC) with discrete state space embedded in R^3 . More importantly the transition parameters in such models are precisely the coefficients in the large population mean dynamics. This leads to rather obvious parameter estimation techniques for continuous time Markov Chain models.

Delays occur and are important in many physical and biological processes, but especially in modeling of just-in-time networks with transport delays. For example, recent studies show that delayed-induced stochastic oscillations can occur in certain elements of gene regulation networks [10]. In addition, delays are of practical importance in general supply networks for investigation of a wide range of perturbations (either accidental or deliberate) such as a node being rendered inoperable due to bad weather, and technical difficulties in communication system. Hence, continuous time Markov chain models with delays incorporated (simply referred to stochastic models with delays in this note) have enjoyed considerable research attention in the past decade, especially the efforts on the development of algorithms to simulate such systems (e.g., [2,8,10,17]). However, it appears that there is only minimal effort on the convergence of stochastic solutions to deterministic systems with delays as the sample size goes to infinity (that is, the analogy of the Kurtz’s limit theorem). We found two works in this spirit given in [9] and [19]. Specifically, Bortolussi and Hillston [9] extended the Kurtz’s limit the-

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orem to the case where fixed delays are incorporated into a density dependent continuous time Markov chain. Schlicht and Winkler [19] showed that if all the transition rates are linear, then the mean solution of stochastic system with random delays can be described by deterministic differential equations with distributed delays. However, to our knowledge, there is still no theoretical results on the convergence of the solution to a general stochastic system with random delays; that is, there is not yet an analog of the Kurtz's limit theorem for a general stochastic system with random delays. In this note we discuss these issues in the context of a pork production supply network. This model was originally developed in [4], where four nodes of production are considered: sows, nurseries, finishers, and slaughterhouses. The movement of pigs from one node to the next is assumed to occur only in the forward direction. That is, from sows to nurseries, from nurseries to finishers and from finishers to slaughterhouses

2. THE PORK PRODUCTION NETWORK MODEL WITH A FIXED DELAY

The assumption made on the original pork production network model derived and studied in [4] was that the transition from one node to the next is made instantaneously. It was made clear in [4] that this is a simplifying assumption, and that incorporating delays would give a more realistic model due to the possible long distance between the nodes or bad weather or some other disruptions/interruptions. Presented here is a first attempt to account for delays in the following way. Assume that all transitions occur instantaneously except for the arrival of pigs transitioning from Node 1 to 2. That is, the pigs leave Node 1 immediately, but the time of arrival at Node 2 is delayed. We only consider a delay in one of the transitions for simplicity, but depending on the physical proximity of the nodes, it may be a reasonable assumption to have delays in not all of the transition.

$$\begin{aligned}
X_1(t) &= X_1(0) - Y_1 \left(\int_0^t \lambda_1(\mathbf{X}(s)) ds \right) \\
&\quad + Y_4 \left(\int_0^t \lambda_4(\mathbf{X}(s)) ds \right) \\
X_2(t) &= X_2(0) - Y_2 \left(\int_0^t \lambda_2(\mathbf{X}(s)) ds \right) \\
&\quad + Y_1 \left(\int_0^t \lambda_1(\mathbf{X}(s - \tau)) ds \right) \\
X_i(t) &= X_i(0) - Y_i \left(\int_0^t \lambda_i(\mathbf{X}(s)) ds \right) \\
&\quad + Y_{i-1} \left(\int_0^t \lambda_{i-1}(\mathbf{X}(s)) ds \right) \quad i = 3, 4.
\end{aligned} \tag{1}$$

Here $\{Y_j(t), t \geq 0\}$, $j = 1, 2, 3, 4$ are independent standard Poisson processes, and transition rates $\lambda_j(\mathbf{x}) = k_j x_j (L_{j+1} - x_{j+1})_+$, $j = 1, 2, 3$ and transition rate $\lambda_4(\mathbf{x}) = k_4 \min(x_4, S_m)$, where $(z)_+ = \max(z, 0)$, k_i and L_i respectively denote the service rate and capacity constraint at Node i , and S_m is the maximal exit constraint at Node 4.

2.1 The Stochastic Model With A Fixed Delay

As a first consideration, we take the delayed time of arrival at node 2, τ , to be a fixed value. If we assume that the process starts from $t = 0$ (that is, $\mathbf{X}(t) = 0$ for $t < 0$), then this results in a stochastic model with a fixed delay given by system (1). Note that λ_1 only depends on the state. Hence, the assumption of $\mathbf{X}(t) = 0$ for $t < 0$ leads to $\lambda_1(\mathbf{X}(t)) = 0$ for $t < 0$. The interpretation of stochastic model (1) with a fixed delay is as follows. When any of the transitions λ_2 , λ_3 or λ_4 fires at time t , the system is updated accordingly at time t . When the transition λ_1 fires at time t , one unit is subtracted from the first node. Since the completion of the transition is delayed, at time $t + \tau$ the unit is added onto Node 2.

2.2 The Corresponding Deterministic System For The Stochastic Model With A Fixed Delay

In [9], Bortolussi and Hillston extended the Kurtz's limit theorem to a scenario with fixed delays incorporated into a density dependent continuous time Markov chain, where the convergence is in the sense of convergence in probability. We will use our pork production model (1) to illustrate this theorem (referred to as BH limit theorem). An approximating deterministic system can be constructed based on the BH limit theorem for a scaled stochastic system with a fixed delay. Let $\mathbf{C}(t) = \mathbf{X}(t)/N$ with $\mathbf{X}(t)$ described by (1). This approximating deterministic system is given by

$$\begin{aligned}
\dot{c}_1(t) &= -\kappa_1 c_1(t)(l_2 - c_2(t))_+ + \kappa_4 \min(c_4(t), s_m) \\
\dot{c}_2(t) &= -\kappa_2 c_2(t)(l_3 - c_3(t))_+ \\
&\quad + \kappa_1 c_1(t - \tau)(l_2 - c_2(t - \tau))_+ \\
\dot{c}_3(t) &= -\kappa_3 c_3(t)(l_4 - c_4(t))_+ + \kappa_2 c_2(t)(l_3 - c_3(t))_+ \\
\dot{c}_4(t) &= -\kappa_4 \min(c_4(t), s_m) + \kappa_3 c_3(t)(l_4 - c_4(t))_+,
\end{aligned} \tag{2}$$

where $(z)_+ = \max(z, 0)$, $\kappa_4 = k_4$, $s_m = S_m/N$ and $\kappa_i = N k_i$, $i = 1, 2, 3$. We note that this approximating deterministic system is not a system of ordinary differential equations, but rather a system of delay (ordinary) differential equations with a fixed delay. The delay differential equation is a direct result of the delay term present in (1). Since there is a delay term, the system is dependent on the previous states, for this reason it is necessary to have some past history functions as initial conditions. It should be noted that past history functions should not be chosen in an arbitrary fashion as they should capture the limit dynamics of the scaled stochastic system with a fixed delay.

Now we illustrate how to construct the initial conditions for the delay differential equation (2). Noted that in the interval $[0, \tau]$ the delay term has no affect ($X(t) = 0$, $t < 0$!), thus we can ignore the delay term in this interval. This yields a stochastic system with no delays, the concentration of which can be approximated by a system of ODE's. This gives the deterministic system (as in [4])

$$\begin{aligned}
\dot{c}_1(t) &= -\kappa_1 c_1(t)(l_2 - c_2(t))_+ + \kappa_4 \min(c_4(t), s_m) \\
\dot{c}_2(t) &= -\kappa_2 c_2(t)(l_3 - c_3(t))_+ \\
\dot{c}_3(t) &= -\kappa_3 c_3(t)(l_4 - c_4(t))_+ + \kappa_2 c_2(t)(l_3 - c_3(t))_+ \\
\dot{c}_4(t) &= -\kappa_4 \min(c_4(t), s_m) + \kappa_3 c_3(t)(l_4 - c_4(t))_+ \\
\mathbf{c}(0) &= \mathbf{c}_0
\end{aligned} \tag{3}$$

for $t \in [0, \tau]$, and let $\Phi(t)$ denote the solution to (3). Thus we have that $\mathbf{C}(t)$ converges to $\Phi(t)$ as $N \rightarrow \infty$ on the interval $[0, \tau]$.

In the interval $[\tau, t_f]$, where t_f is the final time, the delay has an affect, so we approximate with the DDE system (2), and the solution $\Phi(t)$ to the ODE system (3) on the interval $[0, \tau]$ serves as the initial function. Explicitly the system can be written as

$$\begin{aligned} \dot{c}_1(t) &= -\kappa_1 c_1(t)(l_2 - c_2(t))_+ + \kappa_4 \min(c_4(t), s_m) \\ \dot{c}_2(t) &= -\kappa_2 c_2(t)(l_3 - c_3(t))_+ \\ &\quad + \kappa_1 c_1(t - \tau)(l_2 - c_2(t - \tau))_+ \\ \dot{c}_3(t) &= -\kappa_3 c_3(t)(l_4 - c_4(t))_+ + \kappa_2 c_2(t)(l_3 - c_3(t))_+ \\ \dot{c}_4(t) &= -\kappa_4 \min(c_4(t), s_m) + \kappa_3 c_3(t)(l_4 - c_4(t))_+ \\ \mathbf{c}(s) &= \Phi(s), \quad s \in [0, \tau]. \end{aligned} \quad (4)$$

The BH limit theorem indicates that $\mathbf{C}(t)$ converges in probability to the solution of (4) as $N \rightarrow \infty$.

2.3 Comparison Of The Stochastic Model With A Fixed Delay And Its Corresponding Deterministic System

In this section we compare the results of the stochastic system (1) with a fixed delay to its corresponding deterministic system (in terms of number of pigs, i.e., $N\mathbf{c}(t)$ with $\mathbf{c}(t)$ being the solution to (4)). The stochastic system with a fixed delay (1) was simulated using an algorithm given in detail in [6], and the deterministic system (4) was solved numerically using a linear spline approximation method (e.g., see [5,7] for details).

All parameter values and initial conditions are taken from [6]. The value of the delay was set to be $\tau = 5$. Two sample sizes were considered, $N = 100$ and $N = 10,000$, for the stochastic system. In Figures 1 and 2 the deterministic approximation ($N\mathbf{c}(t)$ with $\mathbf{c}(t)$ being the solution to (4)) is compared to five typical sample paths of solution to stochastic system (1) as well as to the mean solution for the stochastic system (1), where the mean solution was calculated by averaging 10,000 sample paths. It is clear that the trajectories of the stochastic simulations follow the solution of its corresponding deterministic system, and the variance among sample paths of the stochastic solution decreases as the sample size increases. It is also seen that as the sample size increases the mean solution of the stochastic system become closer to that of its corresponding deterministic system. Convergence rates in Nodes 3 and 4 are even more rapid as is depicted in [6].

3. THE PORK PRODUCTION NETWORK MODEL WITH A RANDOM DELAY

Now we return to the issue of how to implement random delays into the original pork production model. In the previous section we assumed that the delay was fixed. The interpretation of this is that every transition from Node 1 to Node 2 was delayed by the same amount of time. Now, we want to consider the amount of delayed time varying at each transition. The motivation for doing so is that in practice we would expect that the amount of time it takes to travel from Node 1 to Node 2 will vary based on a number of conditions, e.g. weather, traffic, road construction, etc. In this case it may not be a reasonable assumption that every transition is delayed by the same

Fig. 1. Node 1 obtained by the stochastic system with a fixed delay (S) and by the corresponding deterministic system (D), with (a) and (b) $N = 100$, and (c) and (d) $N = 10,000$. (a) and (c) five typical sample paths of the solution to stochastic system with a fixed delay vs. the solution of its corresponding deterministic system, and (b) and (d) the mean solution of the stochastic system with a fixed delay vs. the solution of its corresponding deterministic system.

Fig. 2. Node 2 obtained by the stochastic system with a fixed delay (S) and by the corresponding deterministic system (D), with (a) and (b) $N = 100$, and (c) and (d) $N = 10,000$. (a) and (c) five typical sample paths of the solution to stochastic system with a fixed delay vs. the solution of its corresponding deterministic system, and (b) and (d) the mean solution of the stochastic system with a fixed delay vs. the solution of its corresponding deterministic system.

amount of time, but rather may vary for each transition that occurs. One way to implement this variation of delay times is to consider the delay to be a random variable that

will be sampled for every transition. The resulting system is called a stochastic model with a random delay. For this model, we still assume here, for ease in exposition, that all transitions occur instantaneously except for the arrival of pigs transitioning from Node 1 to 2, and the process starts from $t = 0$, that is, $\mathbf{X}(t) = 0$ for $t < 0$.

3.1 The Corresponding Deterministic System For The Stochastic Model With A Random Delay

In [19], Schlicht and Winkler showed that if all the transition rates are linear, then the mean solution of the stochastic system with random delays can be described by a system of deterministic differential equations with a distributed delay, where the delay kernel is the probability density function of the given distribution for the random delay. Even though the transition rates in our pork production model are nonlinear, we still would like to explore whether or not such deterministic system can be used as a possible corresponding deterministic system for our stochastic system with random delay and explore the relationship between them.

Let $G(t)$ be the probability density function of the random delay. Then the corresponding deterministic system for our stochastic system with delay is given by

$$\begin{aligned} \dot{c}_1(t) &= -\kappa_1 c_1(t)(l_2 - c_2(t))_+ + \kappa_4 \min(c_4(t), s_m) \\ \dot{c}_2(t) &= -\kappa_2 c_2(t)(l_3 - c_3(t))_+ \\ &\quad + \int_{-\infty}^t G(t-s) \kappa_1 c_1(s)(l_2 - c_2(s))_+ ds \\ \dot{c}_3(t) &= -\kappa_3 c_3(t)(l_4 - c_4(t))_+ + \kappa_2 c_2(t)(l_3 - c_3(t))_+ \\ \dot{c}_4(t) &= -\kappa_4 \min(c_4(t), s_m) + \kappa_3 c_3(t)(l_4 - c_4(t))_+ \\ c_i(0) &= c_{i0}, \quad i = 1, 2, 3, 4, \\ c_i(s) &= 0, \quad s < 0, \quad i = 1, 2, 3, 4, \end{aligned} \quad (5)$$

We remark that numerically solving a system of the form (5) may prove to be difficult due to the distributed delay term. However, if we make additional assumptions on the delay kernel, we can transform a system with a distributed delay into a system of ODE's. Specifically, if we assume that the delay kernel has the form

$$G(u; \alpha, n) = \frac{\alpha^n u^{n-1} e^{-\alpha u}}{(n-1)!} \quad (6)$$

with $\alpha > 0$ and n being a positive integer number, that is, G is the probability density function of a Gamma distributed random variable with mean being n/α and the variance being n/α^2 , then by way of the linear chain trick (e.g., see [18] and the references therein) we can transform the system (5) into a system of ODE's. For example, for the case $n = 1$, if we let

$$c_5(t) = \int_{-\infty}^t \alpha e^{-\alpha(t-\theta)} \kappa_1 c_1(\theta)(l_2 - c_2(\theta))_+ d\theta,$$

then this substitution yields the following system of ODE's

$$\begin{aligned} \dot{c}_1(t) &= -\kappa_1 c_1(t)(l_2 - c_2(t))_+ + \kappa_4 \min(c_4(t), s_m) \\ \dot{c}_2(t) &= -\kappa_2 c_2(t)(l_3 - c_3(t))_+ + c_5(t) \\ \dot{c}_3(t) &= -\kappa_3 c_3(t)(l_4 - c_4(t))_+ + \kappa_2 c_2(t)(l_3 - c_3(t))_+ \\ \dot{c}_4(t) &= -\kappa_4 \min(c_4(t), s_m) + \kappa_3 c_3(t)(l_4 - c_4(t))_+ \\ \dot{c}_5(t) &= \alpha \kappa_1 c_1(t)(l_2 - c_2(t))_+ - \alpha c_5(t) \\ c_i(0) &= c_{i0}, \quad i = 1, 2, 3, 4, \\ c_5(0) &= \int_{-\infty}^0 \alpha e^{\alpha\theta} \kappa_1 c_1(\theta)(l_2 - c_2(\theta))_+ d\theta = 0, \end{aligned} \quad (7)$$

which is equivalent to (5).

The advantage of introducing the linear chain trick is two fold. The resulting system of ODE's is much easier to solve compared to the system (5) where there is a distributed delay. In addition, we can use the Kurtz's limit theorem to construct a corresponding stochastic system which converges to the resulting system of ODE's.

3.2 Comparison Of The Stochastic Model With A Random Delay And Its Corresponding Deterministic System

All parameter values and initial conditions remain as in [6], i.e., the same as in [4]. For the probability density function $G(u; \alpha, n)$, n was taken to be 1 and α to be 0.2, which implies that the mean value of the random delay is 5 and its variance is 25.0. Sample sizes of $N = 100$ and $N = 10,000$ were considered for the stochastic system with a random delay. As before, the stochastic system was simulated for 10,000 trials, and the mean solution was computed.

Figures 3 through 6 compare the solution of deterministic system for the nodes (in terms of number of pigs, i.e., $N\mathbf{c}(t)$ with $\mathbf{c}(t)$ being the solution to (7)) and the results of the stochastic system with a random delay. From these figures we observe that the trajectories of the stochastic simulations follow the solution of deterministic system, and the variation of sample paths of solution to stochastic system with a random delay decreases as the sample size increases. It is also seen that as the sample size increases the mean solution of stochastic system with a random delay become closer to the solution of deterministic system. Hence, not only are the sample paths of the solution to stochastic system with a random delay showing less variation for larger sample sizes, but the expected value of the solution is better approximated by the solution of deterministic system for large sample sizes. Thus, the deterministic system (7) (or deterministic differential equation with a distributed delay (5)) could be used to serve as a reasonable corresponding deterministic system for this particular stochastic system with a random delay (with the given parameter values and initial conditions).

4. CONCLUSION

In this note we extended the stochastic pork production model in [4] to incorporate delay to account for the phenomenon that movement from one node to the next is often not instantaneous in practice due to the physical distance and/or some unexpected disruptions/interruptions. We considered two different types of delays, a fixed delay and a random delay, and numerically explored the corresponding deterministic approximations for these two

Fig. 3. Node 1 obtained by the stochastic system with a random delay (S) and by the corresponding deterministic system (D), with (a) and (b) $N = 100$, and (c) and (d) $N = 10,000$. (a) and (c) five typical sample paths of the solution to stochastic system with a random delay vs. the solution of its corresponding deterministic system, and (b) and (d) the mean solution of the stochastic system with a random delay vs. the solution of its corresponding deterministic system.

Fig. 4. Node 2 obtained by the stochastic system with a random delay (S) and by the corresponding deterministic system (D), with (a) and (b) $N = 100$, and (c) and (d) $N = 10,000$. (a) and (c) five typical sample paths of the solution to stochastic system with a random delay vs. the solution of its corresponding deterministic system, and (b) and (d) the mean solution of the stochastic system with a random delay vs. the solution of its corresponding deterministic system.

resulting stochastic models. Numerical results show that when the sample size is sufficiently large the stochastic model with a fixed delay can be well approximated by a

Fig. 5. Node 3 obtained by the stochastic system with a random delay (S) and by the corresponding deterministic system (D), with (a) and (b) $N = 100$, and (c) and (d) $N = 10,000$. (a) and (c) five typical sample paths of the solution to stochastic system with a random delay vs. the solution of its corresponding deterministic system, and (b) and (d) the mean solution of the stochastic system with a random delay vs. the solution of its corresponding deterministic system.

Fig. 6. Node 4 obtained by the stochastic system with a random delay (S) and by the corresponding deterministic system (D), with (a) and (b) $N = 100$, and (c) and (d) $N = 10,000$. (a) and (c) five typical sample paths of the solution to stochastic system with a random delay vs. the solution of its corresponding deterministic system, and (b) and (d) the mean solution of the stochastic system with a random delay vs. the solution of its corresponding deterministic system.

system of deterministic differential equations with a fixed delay. This confirms with the recent theoretical results presented in [9]. We also numerically showed that the

mean solution of the stochastic model with a Gamma distributed random delay can be well approximated by the solution of a system of deterministic differential equations with a Gamma distributed delay when the sample size is sufficiently large. Hence, the system of deterministic differential equations with a Gamma distributed delay can be used as a possible corresponding deterministic one for this particular stochastic model with a Gamma distributed random delay (with the given parameter values and initial conditions).

In additional efforts [6], we have compared the stochastic model with a Gamma distributed random delay to the stochastic system constructed based on the Kurtz's limit theorem from a system of deterministic differential equations with a Gamma distributed delay. Even though the same system of deterministic differential equations with a Gamma distributed delay can be used as the corresponding deterministic ones for these two stochastic systems, it was found that with the same sample size the histogram plots of the state solutions to the constructed stochastic system appear more flattened than the corresponding ones obtained for the stochastic model with a random delay. However, there is more agreement between the histograms of these two stochastic systems as the variance of the random delay decreases. We also found that with the same variance for the random delay the histogram plots for the stochastic model with a random delay are symmetric for all the sample size investigated, while those for the constructed stochastic system are asymmetric when the sample size is small, but becomes more symmetric as the sample size increases.

Finally in [6] we compared the histogram plots of the state solutions to the stochastic model with a fixed delay to those obtained for the stochastic model with a random delay, where the value of the fixed delay is chosen as the mean value of the random delay. Numerical results show that for those states affected by the delay most their histogram plots obtained for the stochastic system with a fixed delay have similar unimodal shapes and dispersion as the corresponding ones for the stochastic system with a random delay, but their mode values become smaller (i.e., shifted more to the left side as compared to the corresponding ones obtained for the stochastic system with a random delay) as the sample size increases. We also found that when the variance of the random delay is sufficiently small, the histograms of state solutions to the stochastic model with a fixed delay agree well with the corresponding ones obtained for the stochastic model with a random delay regardless of the sample size. Detailed descriptions of all these results can be found in [6].

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