ABSTRACT

BATTEN RAY, CHELSIE ANN. Complemented Leibniz Algebras. (Under the direction of Dr. Ernest Stitzinger.)

Leibniz algebras were introduced by Jean Loday in 1993 as a noncommutative generalization of Lie algebras. The theory of Leibniz algebras has been studied for twenty-one years, and during this time many results from the theory of Lie algebras have been extended. In the present work, we make contributions to the theory of Complemented Algebras and introduce Leibniz algebra generalizations, which have Lie and Group theory counterparts.

Complemented Leibniz Algebras

by
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DEDICATION

In honor of Peggy Grady Batten.

In memory of Frances Pearce, Fred and Peggy Grady.

BIOGRAPHY

Chelsie Ann Batten Ray was born September 26, 1988 in Wilson, North Carolina to Anthony and Peggy Batten. She grew up in Kenly, North Carolina with siblings, Hunter and Madison, and attended North Johnston High School. After graduating in 2006, she attended Meredith College, where she earned a B.S. in mathematics and a minor in chemical physics in 2010. She then went to North Carolina State University, where she served as a Graduate Teaching Assistant, while working toward master's and doctorate degrees in mathematics.

Chelsie's first memory of the joy of mathematics goes back to high school calculus when the beauty of the subject was just beginning to be revealed. The pursuit of studying math came as no surprise to most of Chelsie's family. Her mother also earned a Ph.D. in mathematics from North Carolina State University when Chelsie was a young child. A plethora of exceptional teachers helped encourage and nurture the pathway to her eventual career in mathematics.

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I greatly appreciate all of my friends and family members that have tolerated, supported, encouraged and listened to me along my journey to completing my Ph.D. I am sure you are all anxious for me to discuss something other than my hectic weeks of work.

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Chapter 1

Introduction

1.1 History of Leibniz Algebras

Leibniz algebras were introduced by Jean Loday in [12] as a noncommutative generalization of Lie algebras. Previously, they had been studied by A.M. Bloch who called them D-algebras [7]. He gave the algebras the name of D-algebras due to their connection with derivations. It was much later when Loday independently rediscovered Leibniz algebras. While studying the properties of the homology of Lie algebras, Loday discovered that the antisymmetry property was not needed to prove the derived property defined on chains. This discovery motivated the introduction to the notion of Leibniz algebras. He called the algebras Leibniz since it was Leibniz who proved the 'Leibniz rule" for differentiation of functions.

With this being the case, a natural problem arises for Leibniz algebras in order to obtain analogues of results from the theory of Lie algebras. The theory of Leibniz algebras has been studied for 21 years, and during this time many results from the theory of Lie algebras have been extended. There are many cases where it has been shown that certain results from Lie algebras can not be extended to Leibniz. The issues are rooted in the lack of antisymmetry. In this work, our principal aim is to focus on complemented Leibniz algebras extended from Lie algebra. Throughout this work, unless indicated otherwise, all algebras are assumed to be finite dimensional over a field of characteristic zero.

1.2 Preliminaries

We begin with the basic definitions and properties of Leibniz algebras in this section. Let L be an algebra over a field F. We denote the left multiplication by L_a for $a \in L$, where $L_a(x) = [a, x]$ for all $x \in L$. Similarly, the right multiplication is denoted by R_a for $a \in L$, where $R_a(x) = [x, a]$ for all $x \in L$.

Definition 1.2.1. A (left) Leibniz algebra is an algebra L which satisfies the Leibniz identity

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]]$$

for all $a, b, c \in A$.

This definition is equivalent to saying L_a is a derivation of L. Many authors chose to work with right Leibniz algebras. In this case, R_a is a derivation of L. The right Leibniz algebra results can be proven for left Leibniz algebras. The same is true of left Leibniz algebra results. In this work, we will work with only left Leibniz algebras and will simply call them Leibniz algebras.

Example 1.2.2. Let L be a 2-dimensional algebra with the following multiplications.

$$[x, x] = 0, [x, y] = 0, [y, x] = x, [y, y] = x$$

L is a left Leibniz algebra, but it is not a right Leibniz algebra, since $[[y,y],y] \neq [y,[y,y]] + [[y,y],y]$ which implies $0 = [x,y] \neq [y,x] + [x,y] = x + 0 = x$.

Further we note that any Lie algebra is a Leibniz algebra. Also, if a Leibniz algebra L additionally satisfies $[a,a] = a^2 = 0$ for $a \in L$, then the Leibniz identity will be the same as the Jacobi identity $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$. Hence Leibniz algebra L is a Lie algebra.

Definition 1.2.3. Let I be a subspace of a Leibniz algebra L. Then I is a *subalgebra* if $[I,I] \subseteq I$. I is a *left* (resp. right) *ideal* if $[L,I] \subseteq I$ (resp. $[I,L] \subseteq I$). I is an *ideal* of L if it is both a left and right ideal.

As is the case in Lie algebras, the sum and intersection of two ideals of a Leibniz algebra is an ideal; however the product of two ideals need not be an ideal.

Example 1.2.4. Let $A = span\{x,a,b,c,d\}$ with multiplications $[x,b] = c, [b,a] = d, [x,a] = a = -[a,x], [x,d] = d, [c,x] = d, [d,x] = -d$, and the rest are zero.

Let $I = span\{a,c,d\}$ and $J = span\{b,c,d\}$.

Then I and J are ideals, but $[I,J] = span\{c\}$ which is not an ideal.

Let L be a Leibniz algebra. Then the series of ideals $L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \dots$ where $L^{(1)} = [L,L]$, $L^{(i+1)} = [L^{(i)}, L^{(i)}]$ is called the derived series of L.

Definition 1.2.5. A Leibniz algebra L is *solvable* if $L^{(m)} = 0$ for some integer $m \geq 0$.

As is in the case of Lie algebras, the sum and intersection of two solvable ideals of a Leibniz algebra is solve. Hence any Leibniz algebra L contains a unique maximal solvable ideal $rad(L)$ called the *radical of* L which contains all solvable ideals.

For a Leibniz algebra L the series of ideals $L \supseteq L^1 \supseteq L^2 \supseteq \dots$ where $L^2 = [L, L]$ and $L^{i+1} = [L, L^i]$ is called the lower central series of L.

Definition 1.2.6. A Leibniz algebra A is *nilpotent* of class c if $A^{c+1} = 0$ but $A^c \neq 0$.

As in Lie algebras, the sum and intersection of two solvable (resp. nilpotent) ideals of a Leibniz algebra are solvable (resp. nilpotent).

1.3 Cyclic Leibniz Algebras

An algebra L is called Leibniz if it sastifies the identity $x(yz) = (xy)z + y(xz)$. Thus left multiplication is a derivation. If the algebra also satisfies $xy = -yx$, then it is a Lie algebra. Results on low-dimensional classification, nilpotency, solvability, derivations, and semisimplicity of these algebras are found throughout this section.

Cyclic Leibniz algebras, those generated by a single element, are especially basic in structure. In this case L has a basis $\{a, a^2, \dots, a^n\}$ and $aa^n = \alpha_1 a + \alpha_2 a^2 + \dots + \alpha_n a^n$. The Leibniz identity on a, a^2, and a,

$$0 = [a, [a^n, a]] = [[a, a^n], a] + [a^n, [a, a]] = [\alpha_1 a + \dots \alpha_n a^n, a] = \alpha_1 [a, a]$$

we see that $\alpha_1 = 0$. A particularly important ideal of any Leibniz algebra L is the span of the squares of the elements of L, called $Leib(L)$. Since $a^j L = 0$ for all $j > 1$, $Leib(L)$ is abelian and it is the smallest ideal of L for which the quotient is Lie. Thus for cyclic Leibniz algebras the left multiplication by a, characterized by the above polynomial, determines the algebra. The matrix for this multiplication is the companion matrix for the polynomial, hence the polynomial is both the characteristic and minimum polynomial for the matrix. Consequences of this are summarized in [5]. In particular, the number of maximal subalgebras, Engel subalgebras, and Fitting null components of left multiplications are greatly restricted and easily computed. Thus Frattini subalgebras, Cartan subalgebras, and minimal Engel subalgebras [1] are easily found. Throughout this section all algebras are taken over the field of complex numbers.

Let $L = span\{a_1, a_2, a_3\}$ be a Leibniz algebra. Then $aa^3 = \alpha a^2 + \beta a^3$ for some $\alpha, \beta \in \mathbf{C}$. In fact, choosing any α and any β yields a Leibniz algebra [5]. However, differing choices for α and β are not guaranteed to yield non-isomorphic Leibniz algebras. In [8], the 3-dimensional cyclic Leibniz algebras were classified up to isomorphism.

First, the case when $\alpha = \beta = 0$ is the nilpotent cyclic Leibniz algebra of which there is only one up to isomorphism. In the case where $\alpha = 0$ and $\beta \neq 0$, let $x = \frac{1}{\beta}a$. Then $xx^3 = x^3$. For the case when $\alpha \neq 0$, let $x = \frac{1}{\sqrt{\alpha}}a$. It follows that $xx^3 = x^2 + \frac{\beta}{\sqrt{\alpha}}x^3$. Thus any 3-dimensional non-nilpotent cyclic Leibniz algebra with $\alpha \neq 0$ has a generator x which either has multiplication $xx^3 = x^2 + \gamma x^3$ for some $\gamma \in \mathbf{C}$, or multiplication $xx^3 = x^3$. The following theorem, given in [8], completes the classification of 3-dimensional cyclic Leibniz algebras up to isomorphism.

Theorem 1.3.1. *Let $\mathbb{L}$ be a 3-dimensional cyclic Leibniz algebra. Then $\mathbb{L}$ has a generator a with one and only one of the following multiplications:*

(i) $aa^3 = 0$

(ii) $aa^3 = a^3$

(iii) $aa^3 = a^2 + \gamma a^3$ *for precisely one* $\gamma \in \mathbf{C}$.

Let L be a cyclic Leibniz algebra generated by a, and let T be the matrix for L_a

with respect to the basis $\{a, a^2, \ldots, a^n\}$. Then T is the companion matrix for $p(x) = x^n - \alpha_n x^{n-1} - \cdots - \alpha_2 x = p_1(x)^{n_1} \cdots p_x(x)^{n_s}$, where the p_j are the distinct irreducible factors of $p(x)$. Let $r_j(x) = px/p_j(x)$ for $1 \leq j \leq s$. The maximal subalgebras of L are the null spaces of each $r_j(T)$, and the Frattini subalgebra $F(L)$ is the null space of $q(T)$ where $q(x) = p_1(x)^{n_1-1} \cdots p_s(x)^{n_s-1}$ [5].

As an example of these results, we directly compute the maximal subalgebras and the Frattini subalgebra for each 3-dimensional algebra from Theorem 1.3.1. The results are summarized below. Note that if L is an n-dimensional algebra, it has no more than n maximal subalgebras (since $p(x)$ has no more than n distinct irreducible factors). In this example, the greatest number of maximal subalgebras is three.

Example 1.3.2. For 3-dimensional cyclic Leibniz algebras we find the following:

- For the type (i) 3-dimensional cyclic Leibniz algebra, the maximal subalgebra is L^2 and the Frattini subalgeba is L^2.

- For the type (ii) 3-dimensional cyclic Leibniz algebra, the maximal subalgebras are L^2 and $span\{a - a^2, a - a^3\}$, and the Frattini subalgebra is $span\{a^2 - a^3\}$.

- For the type (iii) 3-dimensional cyclic Leibniz algebra with $\gamma = 2i$, the maximal subalgebras are L^2 and $span\{ia - a^2, a + a^3\}$, and the Frattini subalgebra is $span\{ia^2 - a^3\}$.

- For the type (iii) 3-dimensional cyclic Leibniz algebra with $\gamma = -2i$, the maximal subalgebras are L^2 and $span\{ia + a^2, a - a^3\}$, and the Frattini subalgebra is $span\{ia^2 + a^3\}$.

- For the type (iii) 3-dimensional cyclic Leibniz algebra with all other $\gamma \in \mathbf{C}$, the maximal subalgebras are L^2, $span\{r_1a^2 - a^3\}, a + r_2a^2$, $span\{r_2a^2 - a^3, a + r_1a^2\}$, and the Frattini subalgebra is 0.

Recall that if L is a cyclic Leibniz algebra, then $L^2 = Leib(L)$ is abelian. A Leibniz algebra with L^2 nilpotent is elementary if and only if its Frattini subalgebra is trivial [4]. Thus the only algebras in Table 1 which are elementary are those of type (iii) with $\gamma \neq \pm 2i$.

1.4 Overview

In [19], Towers gave a characterization of complemented Lie algebras over any field of characteristic zero. This was accomplished by considering the results known about elementary and ϕ-free Lie algebras and further extending them to incorporate this idea of complemented Lie algebras. His main result was that complemented lie algebras and ϕ-free Lie algebras are the same over any field of characteristic zero. We were able to show an equivalent theorem in Leibniz algebras.

In Chapter 3, we continue to look at the theory of complemented Leibniz algebras based off of the work done by Towers in [20]. The focus is strictly on solvable complemented Leibniz algebras. Similar to Towers, we break this chapter up into two sections. The first section involves complements of subalgebra intervals. We begin by defining a complemented subalgebra interval. At first the results are similar to those found in group theory for the same topic, but later the theory diverges from that of groups. We use our study of these complements of subalgebra intervals to introduce the concept of prefrattini subalgebras of L. Before introducing the definition of prefrattini subalegbra, it was necessary to investigate the idea of a Frattini chief factor and when it is complemented. This leads to the definition of a U-Frattini chief factor, which in turn allows us to develop enough material to define a U-prefrattini subalgebra. We proved a lemma that gives us the ability to calculate the dimension of our U-prefrattini subalgebra which lead to a series of lemmas that gave way to an important result. We find that the set of U-prefrattini subalgebras of L are exactly the same as the minimal elements in the set of subalgebras that are complemented in an interval.

Further, in Chapter 4, we employ what we developed involving prefrattini subalgebras in the previous chapter to study complemented solvable Leibniz algebras following Towers' work from [21]. The ultimate goal set before us in this section is to give a characterization of solvable complemented Leibniz algebras following Towers' lead in Lie theory. The chapter is divided up into three sections. The first pulls together all developed ideas involving prefrattini subalgebras of solvable Leibniz algebras. Building on these ideas about prefrattini subalgebras, we move into the second section devoted to the prefrattini residual. From here we are able to prove two classification theorems, one for completely solvable Leibniz algebras and another for solvable Leibniz algebras. Finally we consider decomposition results for complemented Leibniz algebras in the third section.

During the development of this work, we came across several topics that were related to our study but was not directly involved. These results are shared in Chapter 5. The section on Cartan subalgebras supports the work completed in the decomposition results in Chapter 4.

Chapter 2

Complemented Leibniz Algebras

2.1 Introduction

Assume L is a finite-dimensional Leibniz algebra over a field F. For the main results in this section, assume F is a field of characteristic zero. We will call L *complemented* if its subalgebra lattice is complemented; that is, if, given any subalgebra A of L there is a subalgebra B of L such that $A \cap B = 0$ and $\langle A, B \rangle = L$. The *Frattini subalgebra*, $F(L)$, of L is the intersection of the maximal subalgebras of L; the *Frattini ideal*, $\phi(L)$, is the largest ideal of L contained in $F(L)$. We say that L is $\phi-$*free* if $\phi(L) = 0$, and that L is *elementary* if every subalgebra of L (including L itself) is $\phi-$free.

Unlike in groups, $F(L)$ is not always an ideal. This is the case in both Lie and Leibniz algebras. The only known example where the Frattini subalgebra of a Lie algebra is not an ideal is as follows:

Example 2.1.1. For the 3-dimensional algebra $L = \langle x, y, z \rangle$ over the field of 2 elements with multiplication $[x, y] = z, [y, z] = x, [z, x] = y$, $F(L) = \langle x + y + z \rangle$, which is not an ideal. In this example, $\Phi(L) = 0$.

We will now consider an example of determining the Frattini subalgebra and ideal of a cyclic Leibniz algebra.

Example 2.1.2. Let A be a 2-dimensional cyclic Leibniz algebra with multiplication $aa = a^2$ and $aa^2 = a^2$. We find there are two maximal subalgebras: $\langle a - a^2 \rangle$ and $\langle a^2 \rangle$. This implies $F(A) = 0$ and $\phi(A) = 0$.

2.2 Complemented Leibniz Algebras

Theorem 2.2.1. *Let L be a Leibniz algebra over a field of characteristic zero.*

1. *If L is complemented, then L is $\phi-$free.*
2. *If L is elementary, then L is complemented.*
3. *Suppose L is solvable. Then L is elementary if and only if L is complemented if and only if L is $\phi-$free.*

Proof. 1. Assume $\phi(L) \neq 0$. Let $H \subseteq L$. Then $L = \langle \phi(L), H \rangle$. Let H be contained in the maximal subalgebra, M. This implies $L = \langle \phi(L), M \rangle$. But $\phi(L) \subseteq M$. Thus, $M = L$. Contradiction.

2. Let H be a subalgebra of L. Then $\phi(L) = 0$ implies that there exists a proper subalgebra K such that $L = \langle H, K \rangle$. If $H \cap K \neq 0$, then there exists a proper subalgebra S such that $S \subset K$ such that $K = \langle H \cap K, K_1 \rangle$. This implies $L = \langle H, H \cap K, K_1 \rangle = \langle H, K_1 \rangle$ and $dim(K_1) < dim(K)$. If $H \cap K_1 \neq 0$, then there

exists a proper subalgebra K_2 of K_1 such that $K_1 = \langle H \cap K_1, K_2 \rangle$. This implies $K = \langle H, H \cap K_1, K_2 \rangle = \langle H, K_2 \rangle$ and $dim(K_2) < dim(K_1)$. Continue and we get K_t such that $L = \langle H, K_t \rangle$ and $H \cap K_t = 0$.

3. Assume L is solvable. The assume that L is ϕ-free. Since L^2 is nilpotent and $\phi(L) = 0$, L is elementary by Theorem 3.4 of [4]. Therefore, when L^2 is nilpotent, we know elementary iff complemented iff $\phi(L) = 0$.

□

Lemma 2.2.2. *Let L be a complemented Leibniz algebra (over any field F). Then every factor algebra of L is complemented.*

Proof. Assume K is an ideal in L. Assume H is a complemented subalgebra in L that contains K. Let H/K be a subalgebra of L/K. Then $L = \langle H, J \rangle$ and $H \cap J = 0$, where H and J are subalgebras of L, since L is a complemented Leibniz algebra. Consider $H \cap (K + J) = K + (H \cap J) = K$. This implies $L = \langle H, K + J \rangle$ and $H \cap (K + J) = K$. Thus, $L/K = \langle H/K, (K + J)/K \rangle$ and $H/K \cap (K + J)/K = 0$. □

The following three lemmas were originally proved in [17] for algebras in general over any field.

Lemma 2.2.3. *If C is a subalgebra of L and B is an ideal of L contained in $F(C)$, then B is contained in $F(L)$.*

Lemma 2.2.4. *Let B be an ideal of an algebra L and let U be a subalgebra of L which is minimal with respect to the property that $L = B + U$. Then $B \cap U \subseteq \phi(U)$.*

Proof. Suppose that $B \cap U \nsubseteq \phi(U)$. Then, since $B \cap U$ is an ideal of U, $B \cap U \nsubseteq F(U)$. It follows that there is a maximal subalgebra M of U such that $B \cap U \nsubseteq M$. Clearly, $B \cap U + M = U$, so $L = B + (B \cap U + M) = B + M$, which contradicts the minimality of U. Thus, $B \cap U \subseteq \phi(U)$. □

Lemma 2.2.5. *Let B be an abelian ideal of an algebra such that $B \cap \phi(L) = 0$. Then there exists a subalgebra C of L such that $L = B + C$.*

Proof. Choose C to be a subalgebra of L which is minimal with respect to the property that $L = B + C$. Then by Lemma 2.2.4, $B \cap C \subseteq \phi(C)$. Now $B \cap C$ is an ideal of C and $(B \cap C)B + B(B \cap C) \subseteq B^2 = 0$, so $B \cap C$ is an ideal of L. Hence, by Lemma 2.2.3, $B \cap C \subseteq \phi(L) \cap B = 0$ and $L = B + C$. □

Lemma 2.2.6. *Let A be a minimal abelian ideal of L. Then L is complemented if and only if there is a subalgebra B of L such that $L = A + B$ where $A \cap B = 0$ and B is complemented.*

Proof. Suppose that L is complemented. Then $\phi(L) = 0$ by Theorem 2.2.1. This implies that there exists a maximal subalgebra M, $A \nsubseteq M$ such that $L = A + M$ and $A \cap M = 0$. Since $M \cong L/A$, M is complemented by Lemma 2.2.2.

Conversely, assume $L = A + B$ and $A \cap B = 0$, and B is a complemented subalgebra of L. Let U be a subalgebra of L. Since B complemented, there exists a subalgebra x of B such that $\langle B \cap (U + A), x\rangle = B$ and $(U + A) \cap x = B \cap (U + A) \cap X = 0$. Hence $\langle U + A, X + A\rangle = L$ and $(U + A) \cap (X + A) = 0$. Let $V = X + A$. Then $L = \langle U, V\rangle = \langle U + A, V\rangle$ and $(U + A) \cap V = A$. Suppose that $A \subseteq U$. Take $W = V \cap B$.

Then $V = V \cap L = V \cap (A + B) = A + (V \cap B)$ and $L = \langle U, V \rangle = \langle U, A + (V \cap B) \rangle = \langle U, W \rangle$ and $U \cap W = U \cap V \cap B = A \cap B = 0$. Hence W is a complement of U.

Suppose $A \cap U = 0$. Then $\langle U, V \rangle = L$ and $V \cap U = V \cap U \cap (U + A) = A \cap U = 0$, since $V \cap (U + A) = A$. So, V is a complement of U.

Suppose $A \cap U \neq 0$ (and $A \nsubseteq U$). Put $W = V \cap B$. Assume $A \nsubseteq \langle U, W \rangle$. Let $\langle U, W \rangle \subseteq M$, where M is a maximal subalgebra of L. This implies $\langle U, W, A \rangle = \langle U, V \cap B, A \rangle = \langle U, V \cap (A + B) \rangle = \langle U, A, V \rangle = L$, so $A \nsubseteq M$. Thus, $L = A + M$ and $A \cap M = 0$. This implies $U \cap A \subset M \cap A = 0$. Contradiction. Hence, $A \subseteq \langle U, W \rangle$. Thus, $\langle U, W \rangle = \langle U, W + A \rangle = \langle U, V \rangle = L$ and $U \cap W = U \cap V \cap B \subseteq A \cap B = 0$. Hence W is a complement of U. $\square$

Theorem 2.2.7. *Let L be a semisimple Leibniz algebra over a field F of characteristic zero, and let x be any element of L which has non-trivial projections on each of the simple ideal of L. Then there is a $y \in L$ such that x and y generate L.*

Proof. Let L be a semisimple Leibniz algebra over a field of characteristic zero. This implies that L is a Lie algebra. Thus, by Theorem 5 of [19], there is a $y \in L$ such that x and y generate L. $\square$

Corollary 2.2.8. *Let L be a semisimple Leibniz algebra over a field of characteristic zero. Then L is complemented.*

Proof. Let L be a semisimple Leibniz algebra over a field of characteristic zero. This implies that L is a Lie algebra. Hence, by Corollary 6 in [19], L is complemented. $\square$

Theorem 2.2.9. *If a field F is of characteristic zero, then the class of complemented Leibniz algebras is equivalent to the class of $\phi-$free Leibniz algebras.*

Proof. Assume L is a minimal counterexample. L is not semisimple. Let A be a minimal abelian ideal of L. Since L is $\phi-$free, $L = A + M$ where M is a subalgebra of L and $A \cap M = 0$. Clearly M must be complemented. This means, by Lemma 2.2.6, L is complemented. □

Chapter 3

Complements of Intervals and Prefrattini Subalgebras of Solvable Leibniz Algebras

3.1 Introduction

Throughout this section, L will denote a solvable Leibniz algebra over a field F of characteristic zero. For a subalgebra U of L we denote by $[U : L]$ the set of all subalgebras S of L with $U \subseteq S \subseteq L$. We will call this set an *interval*. We say that $[U : L]$ is *complemented* if, for any $S \in [U : L]$ there is a $T \in [U : L]$ such that $S \cap T = U$ and $\langle S, T\rangle = L$. We set out to study the set:

$$\Omega(U, L) = \{S \in [U : L] \mid [S : L] \text{ is complemented}\}.$$

Example 3.1.1. Let L be the Leibniz algebra with basis $\{x, y_1, y_2, y_3\}$ with multiplications $[x, y_1] = y_2$, $[x, y_2] = y_3$, and all other 0. Let $U = \{y_2, y_3\}$. The general subalgebra of L in $[U : L]$ is $\{\alpha x + \beta y_1, y_2, y_3\}$.

There is a difference of two dimensions between U and L. We need to consider only the proper subalgebras. Consider the following:

Let $S = \{\alpha x + \beta y_1, U\}$. To find a complement of S we must determine a subalgebra T such that $S \cap T = U$ and $\langle S, T\rangle = L$. Now consider all possibilities for α and β. If

$$\alpha = 0, T = \{x, y_2, y_3\}$$

$$\beta = 0, T = \{y_1, y_2, y_3\}$$

$$\alpha = 0, \beta = 0, T = L$$

$$\alpha \neq 0, \beta \neq 0, \text{any linearly independent vector containing U will be a complement of S}$$

In any case, the criterion for S to be complemented is satisfied. Hence every possible subalgebra, S, between U and L is complemented in $[U : L]$.

Thus, $[U : L]$ is complemented when $U = \{y_2, y_3\}$.

Example 3.1.2. Let L be the Leibniz algebra with basis $\{x, y_1, y_2, y_3\}$ with multiplications $[x, y_1] = y_2$, $[x, y_2] = y_3$, and all other 0. Let $U = \{y_2\}$.

(i) Let $S = \{y_1, y_2\}$. Note that S is a subalgebra between U and L. To find a complement, T, of S it must contain U and $\langle S, T\rangle$ must equal L. Thus, T must contain y_2, as well as x and y_3. Hence $T = \{x, y_2, y_3\}$. Then $S \cap T = U$ and $\langle S, T\rangle = L$. Thus S complemented in $[U : L]$.

(ii) Now let $S = \{y_1, y_2, y_3\}$. When we consider what T must contain in order to complement S, we know that again it must contain x and y_2 by nature of containing U and $\langle S, T\rangle$ needing to equal L. If T contains x and y_2, it must also contain y_3 due to the multiplications and requirement that T be a subalgebra. Hence $T = \{x, y_2, y_3\}$. Then $S \cap T \neq U$. Thus, S is not complemented in $[U : L]$.

Therefore, $[U : L]$ is not complemented when $U = \{y_2\}$ because there is a subalgebra in $[U : L]$ that is not complemented.

Now let's consider the set $\Omega(U, L) = \{S \in [U : L] | [S : L] \text{ is complemented}\}$.

Example 3.1.3. Using a previous example where $U = \{y_2\}$. We will consider various choices of S and then determine whether or not $[S : L]$ is complemented. Provided that it is, we will have found a subalgebra contained in $\Omega(U, L)$.

(i) Consider $S = \{y_1, y_2\}$. Recall that S is complemented in $[U : L]$. Let $S_1 \in [S : L]$. Let $S_1 = \{y_1, y_2, y_3\}$. For T to complement S, it must contain a vector x which appears nontrivially and it must contain y_1 and y_2. Due to the multiplications and necessity for T to be a subalgebra, T must also contain y_3. Thus, $T = \{x, y_1, y_2, y_3\} = L$. Then $S_1 \cap T \neq S$. Hence $[S : L]$ is not complemented for $S = \{y_1, y_2\}$.

    Therefore S is not contained in $\Omega(U, L)$.

(ii) As in example 3.1.1, consider $S = \{y_2, y_3\} \in [U : L]$. It was previously determined that the only proper subalgebra between $S$and L must be $\{\alpha x + \beta y_1, y_2, y_3\}$. Furthermore, it was determine that for every choice of α and β an appropriate complement could be found.

Therefore, we can conclude that $S = \{y_2, y_3\}$ is contained in $\Omega(U, L)$.

Now we want to consider an S with lower dimension.

(iii) Consider $S = \{y_2\}$. Let $S_1 = \{y_1, y_2, y_3\}$ and $T = \{x, y_2, y_3\}$ (see example ii for explanation of T). Then $S_1 \cap T \neq S$. Thus, $[S : L]$ is not complemented and hence $S = \{y_2\}$ is not in $\Omega(U, L)$.

Therefore, when $U = \{y_2, y_3\} = \Omega(U, L)_{min} = F(U, L)$.

We denote by $[U : L]_{max}$ the set of maximal subalgebras in $[U : L]$; that is, the set of maximal subalgebras of L containing U. Note that if $L = A + B$ where A and B are subalgebras of L and $A \cap B = 0$, then we will write $L = A \oplus B$.

3.2 Complements of subalgebra intervals

Lemma 3.2.1. *If $S \in \Omega(U, L)$, $S \neq L$ then $S = \bigcap\{M | M \in [S : L]_{max}\}$.*

Proof. Let $T = \bigcap\{M | M \in [S : L]_{max}\}$. Then $[S : L]$ is complemented, since $S \in \Omega(U, L)$, by definition, and so T has a complement $C \in [S : L]$.

If $C \neq L$, then $C \subseteq M$ for some $M \in [S : L]_{max}$. Then $\langle T, C\rangle = M$, which contradicts that C is a complements of T in $[S : L]$. Hence $C = L$ and $S = T \cap C = T \cap L = T$, as required. $\square$

Example 3.2.2. As in example 5.3.ii, let $U = \{y_2\}$ and $S = \{y_2, y_3\}$. Previously it was determined that $S \in \Omega(U, L)$. Clearly, $S \neq L$. Now consider $[S : L]_{max}$. The maximal subalgebras between S and L are $M_1 = \{y_1, y_2, y_3\}$ and $M_2 = \{x, y_2, y_3\}$. Note that $M_1 \cap M_2 = \{y_2, y_3\} = S$.

Lemma 3.2.3. *If I is an ideal of L and $S \in \Omega(U, L)$, then $S + I \in \Omega(U, L)$.*

Proof. Let $B \in [S + I : L] \subseteq [S : L]$. Since $S \in \Omega(U, L)$, B has a complement $D \in [S : L]$; this implies $B \cap D = S$ and $\langle B, D\rangle = L$. Let $C = D + I$. Then $\langle B, C\rangle = L$ and $B \cap C = B \cap (D + I) = B \cap D + I = S + I$. Thus, C is a complement for B in $[S + I : L]$ and $S + I \in \Omega(U, L)$. □

Example 3.2.4. Consider ideals of L, where L is the Leibniz algebra with basis

$\{x, y_1, y_2, y_3\}$ with multiplications $[x, y_1] = y_2, [x, y_2] = y_3$ and all others 0. The only ideals are $\{y_2, y_3\}$, and $\{y_2, y_3\}$. As in example 5.3.ii, let $U = \{y_2\}$ and $S = \{y_2, y_3\}$. Previously it was determined that $S \in \Omega(U, L)$. Regardless of the choice from the possibilities of ideals, $S + I = \{y_2, y_3\} \subseteq \Omega(U, L)$.

Lemma 3.2.5. *Let A be a minimal ideal of L and let M be a complement of A in L containing U. Then $\Omega(U, M) = \{S \in \Omega(U, L)|S \subseteq M\}$. In particular, $\Omega(U, M)_{min} = \{S \in \Omega(U, L)_{min}|S \subseteq M\}$.*

Proof. Since L is solvable, M is a maximal subalgebra of L and $L = A \oplus M$. Suppose that $S \in \Omega(U, L)$ with $S \subseteq M$. Then by the previous lemma, $S + A \in \Omega(U, L)$. The interval $[S : M]$ is lattice isomorphic to $[S + I : L]$ and so is complemented. Hence $S \in \Omega(U, M)$.

Conversely, let $S \in \Omega(U, M)$. Then $[S : M]$ is complemented. (WTS: $S \in \Omega(U, L)$ where $[S : L]$ is complemented.) Let $B \in [S : L]$. Then $B \cap M \in [S : M]$, so there exists a subalgebra $D \in [S : M]$ such that $\langle B \cap M, D\rangle = M$ and $B \cap D = B \cap M \cap D = S$.

If $B \nsubseteq M$, then M is a proper subalgebra of $\langle B, D\rangle$. But M is a maximal subalgebra of L, so $\langle B, D\rangle = L$ and D is a complement of B in $[S : L]$. Hence $[S : L]$ is complemented.

If $B \subseteq M$, put $C = D + A$. Then $L = A \oplus M \subseteq \langle B, A\rangle \subseteq \langle B, D + A\rangle = \langle B, C\rangle$, so $\langle B, D+A\rangle = L$. Also, $B \cap C = B \cap (D+A) = B \cap M \cap (D+A) = B \cap (D+M \cap A) = B \cap D = S$, yielding that C is a complement of B in $[S : L]$ and $[S : L]$ is complemented. □

Example 3.2.6. Let $U = \{y_2\}$. Let $A = \{y_3\}$, a minimal ideal in L. Then $M = L$, is a complement to A since M must contain U, as well as x and y_1. Thus $\Omega(U, M) = \Omega(U, L)$, so as seen in previous examples, $\Omega(U, M) = \Omega(U, L) = \{y_2, y_3\}$.

Lemma 3.2.7. *Let A be a minimal ideal of L and let $S \in \Omega(U, L)_{min}$ with $A \nsubseteq S$. Then there is an $M \in [S : L]_{max}$ such that $A \nsubseteq M$.*

Proof. The proof follows from Lemma 3.2.1. □

Example 3.2.8. Let $L = L_1 \oplus L_2$ where $L_1 = \langle a, a^2\rangle$ and $L_2 = \langle b, b^2, b^3\rangle$ with multiplications $aa = a^2, aa^2 = a^2, bb = b^2, bb^2 = b^3, bb^3 = b^3$ and all other multiplications are 0. Let $U = \{b^2\}$. If A is the minimal ideal of L, then $A = \{a^2\}$. If $S \in \Omega(U, L)_{min}$, then $S = \{b^2, b^3\}$. Note that $A \nsubseteq S$.

Consider $[S : L]_{max}$. Then $M = L_2 \oplus \{a - a^2\} \in [S : L]_{max}$. Note that $A \nsubseteq M$. Hence we have an example of Lemma 3.2.7.

Example 3.2.9. Let $L = \langle a, a^2, a^3\rangle$ with multiplications $aa = a^2$, $aa^2 = a^3$, $aa^3 = a^3$. Then let $U = \{a^2 - a^3\}$. Note that if A is the minimal ideal of L that $A = \{a^3\}$. Now consider the set $\Omega(U, L)_{min}$. If $S \in \Omega(U, L)_{min}$, then $S = \{a - a^2, a - a^3\}$. Note that $A \nsubseteq S$. S is also equal to $[S : L]_{max}$. Hence $M = S$ and $A \nsubseteq M$.

Lemma 3.2.10. *Let A be a minimal ideal of L. Then the following are equivalent:*

(i.) $A \nsubseteq S$ *for some* $S \in \Omega(U, L)_{min}$*;*

(ii.) $A \nsubseteq M$ *for some* $M \in [U : L]_{max}$*; and*

(iii.) for every $S \in \Omega(U, L)_{min}$ *there is a complement of* A *in* L *containing* S*.*

Proof. $(i \to ii)$ This follows from Lemma 3.2.7.

$(ii \to iii)$ Assume $A \nsubseteq M$ for some $M \in [U : L]_{max}$. Then $L = A \oplus M$. Let $S \in \Omega(U, L)_{min}$. Suppose $A \subseteq S$. Then $S = A \oplus M \cap S$ and $M \cap S = S/A$ so the interval $[S : L]$ is lattice isomorphic to $[M \cap S : M]$. It follows that $M \cap S \in \Omega(U, M)$. But Lemma 3.2.5 gives that $M \cap S \in \Omega(U, L)$ which contradicts the minimality of S. Hence $A \nsubseteq S$ and Lemma 3.2.7 gives a complement of A containing S.

$(iii \to i)$ Trivial. □

Lemma 3.2.11. *If A is an ideal of L and $S \in \Omega(U, L)_{min}$ then $S + A \in \Omega(U + A, L)_{min}$.*

Proof. We want to show $(S + A)/A \in \Omega((U + A)/A, L/A)_{min}$ and so we can suppose A is a minimal ideal of L. The result is clear is $A \subseteq S$, since then $U + A \subseteq S$ (because $U \subseteq S$ too). So, assume $A \nsubseteq S$. Then there is a complement M of A in L, by Lemma 3.2.10, and $L = A \oplus M$. Moreover, $S + A \in \Omega(U + A, L)$. Choose $C \in \Omega(U + A, L)_{min}$ such that $C \subseteq S + A$. Then $U \subseteq M \cap C \subseteq S \subseteq M$ and the interval $[M \cap C : M]$ is lattice isomorphic to $[C : L]$. It follows that $M \cap C \in \Omega(U, M)$ and so $M \cap C \in \Omega(U, L)$, by Lemma 3.2.5. But $S \in \Omega(U, L)_{min}$, which yields that $M \cap C = S$; hence, $C = S + A$. □

We say that L is *completely solvable* if L^2 is nilpotent. If $char(F) = 0$, then all Leibniz algebras are completely solvable because L^2 is always nilpotent.

Theorem 3.2.12. *Let L be completely solvable and let U be a subalgebra of L. Then $\Omega(U, L)_{min} = \{F(U, L)\}$. In particular, if $U = 0$ then $\Omega(U, L)_{min} = \{F(L)\}$.*

Proof. Put $B = \Omega(U, L)_{min}$, $C = F(U, L)$. Then $F(U, L) \subseteq B$ and so $C \subseteq B$, by Lemma 3.2.1. We use induction on the dimension of L. Suppose first that there is a minimal ideal A of L with $A \subseteq C$. Then $B/A \in \Omega((U + A)/A, L/A)_{min}$, by Lemma 3.2.11, and so $B/A = F((U + A)/A, L/A)$, by the inductive hypothesis. Then it is clear that $B = C$.

Now suppose that no such minimal ideal exists. Then L is ϕ-free and so L is complemented by Theorem 2.2.1. Thus there is a subalgebra V such that $\langle C, V\rangle = L$ and $C \cap V = 0$. It follows that $\langle C, U + V\rangle = L$ and $C \cap (U + V) = U + C \cap V = U$, hence $C \in [U : L]$ and $[C : L]$ is complemented. Thus, $C \in \Omega(U, L)$ and the minimality of B yields that $B = C$. □

Example 3.2.13. Let L be a Leibniz algebra with basis $\{x, y_1, y_2, y_3\}$ with multiplications $[x, y_1] = y_2$, $[x, y_2] = y_3$ with all others 0. Recall that the Frattini subalgebra of a Leibniz algebra is the intersection of maximal subalgebras. Consider the following:

(i) Let $U = \{0\}$. Then $F(U, L) = \{y_2, y_3\}$ since the maximal subalgebras are $\{x, y_2, y_3\}$ and $\{y_1, y_2, y_3\}$. As seen in previous examples in this section, $\Omega(U, L)_{min} = \{y_2, y_3\}$. Hence $F(L) = \Omega(U, L)_{min}$ when $U = \{0\}$.

(ii) Let $U = \{y_2, y_3\}$. Then $F(U, L) = F(L)$ and $\Omega(U, L)_{min} = \{y_2, y_3\}$, as in the previous example. Hence $F(U, L) = \Omega(U, L)_{min}$ when $U = \{y_2, y_3\}$.

(iii) Let $U = \{y_1, y_2, y_3\}$. Since the set

$$\Omega(U,L)_{min} = \{S \in [U:L] | [S:L] \text{is complemented}\}_{min},$$

we find in this case when U contains three elements that $\Omega(U,L)_{min} = \emptyset$. Also, $F(U,L) = \emptyset$, because there are no subalgebras, hence no maximal subalgebras, between U and L. Thus, the theorem remains true. A similar argument can be held for $U = \{x, y_2, y_3\}$.

Example 3.2.14. Let L be the cyclic Leibniz algebra $L = \langle a, a^2 \rangle$ with multiplications $aa^2 = a^2$, $aa = a^2$ and all others 0. Note that the maximal subalgebras of L are $\langle a - a^2 \rangle$ and $\langle a^2 \rangle$. The intersection of these subalgebras is 0. Thus, $F(L) = \{0\}$. Consider the following:

(i) Let $U = \{0\}$. In this case, $F(U,L) = F(L) = \{0\}$. Consider $\Omega(U,L)_{min}$. Since nothing is complemented other than $\{0\}$ when $U = \{0\}$, $\Omega(U,L)_{min} = \{0\}$. Thus, $F(L) = \Omega(U,L)_{min}$ for $U = \{0\}$.

(ii) Let $U = \{a^2\}$. Now if we consider $F(U,L)$, we get only one maximal subalgebra that contains U and hence $F(U,L) = \{a^2\}$. Checking to see if $\{a^2\}$ is complemented in L, we find that it is complemented by all of L. Thus, $\{a^2\} \in \Omega(U,L)$. Since $\{a^2\}$ is the only 1-dimensional complemented subalgebra from L when $U = \{a^2\}$, we can determine that $\Omega(U,L)_{min} = \{a^2\}$. Thus, $F(U,L) = \Omega(U,L)_{min}$ for $U = \{a^2\}$.

3.3 U-prefrattini subalgebras

Let $0 = A_0 \subset A_1 \subset \cdots \subset A_n = L$ be a *fixed chief series* for L. We say A_i/A_{i-1} is a *Frattini chief factor* if $A_i/A_{i-1} \subseteq F(L/A_{i-1})$; it is *complemented* if there is a maximal subalgebra M of L such that $L = A_i + M$ and $A_i \cap M = A_{i-1}$. When L is solvable it is easy to see that a chief factor is Frattini if and only if it is not complemented. This can be generalized in the following way.

The factor algebra A_i/A_{i-1} is called a *U-frattini chief factor* if $A_i \subseteq F(U+A_{i-1}, L)$ or if $U+A_{i-1} = L$. If A_i/A_{i-1} is not a U-Frattini chief factor there is an $M \in [U+A_{i-1} : L]_{max}$ for which $A_i \nsubseteq M$; that is, M is a complement of the chief factor A_i/A_{i-1}.

Lemma 3.3.1. *Let A_1, A_2 be distinct minimal ideals of the solvable Lie algebra L. Then there is a bijection $\theta : \{A_1, (A_1+A_2)/A_1\} \to \{A_2, (A_1+A_2)/A_2\}$ such that corresponding chief factors have the same dimension and U-Frattini chief factors correspond to another.*

Proof. Assume $U \neq L$. Put $A = A_1 \oplus A_2$. Suppose first that A_1 is a U-Frattini chief factor. Then $A_1 \subseteq F(U, L)$. Thus, $A \subseteq F(U + A_2, L)$ and A/A_2 is a U-Frattini chief factor.

If A/A_1 is also a U-Frattini chief factor, then $A \subseteq F(U + A_1, L)$. This implies $A \subseteq F(U, L)$ and all four factors are U-Frattini.

In this case, we can choose θ so that $\theta(A_1) = A/A_2$ and $\theta(A/A_1) = A_2$.

If A/A_1 is not a U-Frattini chief factor, then nor is A_2, by the same argument as above, and so the same choice of *theta* is sufficient; likewise if none of the factors are U-Frattini chief factors.

The remaining case is where A_1 and A_2 are not U-Frattini chief factors but A/A_2 is.

Then $A_1 \nsubseteq F(U, L)$, $A_2 \nsubseteq F(U, L)$ and either $A \subseteq F(U + A_2, L)$ or $U + A_2 = L$. Thus there exists $M \in [U : L]_{max}$ such that $A_1 \nsubseteq M$, giving $L = A_1 \oplus M$.

Put $A_3 = M \cap A$. Then $A_3 \oplus A_1 = M \cap A \oplus A_1 = (M + A_1) \cap A = A$ and so $A_3 \cong A/A_1 \cong A_2$. If $A_3 = A_2$, then $U + A_2 \subseteq M$ which gives $A \subseteq M$ - a contradiction.

Hence $A_3 \neq A_2$, $A = A_3 \oplus A_2$ and $A_3 \cong A/A_2 \cong A_1$. It follows that all chief factors have the same dimension.

If $U + A_1 = L$, then A/A_1 is a U-Frattini chief factor, so we can choose θ such that $\theta(A_1) = A_2$ and $\theta(A/A_1) = A/A_2$.

If $U + A_1 \neq L$, let $N \in [U + A_1, L]_{max}$. If $A_2 \nsubseteq N$ then $L = A_2 \oplus N$ and $N \cap A = A_1$. But $A \subseteq F(U + A_2, L)$ implies that $A \subseteq F(U, L) + A_2$, when $A + A_2 + F(U, L) \cap A$. It follows that $F(U, L) \cap A \subseteq N \cap A = A_1$, giving $F(U, L) \cap A = A_1$. But now $A_1 \subseteq F(U, L)$ - a contradiction. Therefore, we have $A_2 \subseteq N$ and so $A \subseteq N$. Thus, $A \subseteq F(U + A_1, L)$; that is, A/A_i is a U-Frattini chief factor. In this case, we can again choose θ such that $\theta(A_1) = A_2$ and $\theta(A/A_1) = A/A_2$. $\square$

Theorem 3.3.2. *Let*

$$0 < A_1 < A_2 < \cdots < A_n = L$$

$$0 < B_1 < B_2 < \cdots < B_n = L$$

be chief series for the solvable Leibniz algebra L. Then there is a bijection between the chief factors of these two series such that corresponding factors have the same dimension and such that the U-Frattini chief factors in the two series correspond.

Proof. These two series have the same length by Lemma 3.3.1. We induct on n. This is clear for $n = 1$. Let $n > 1$ and suppose the result holds for all solvable Leibniz algebras with chief series of length less than or equal to $n - 1$.

If $A_1 = B_1$, then applying the inductive hypothesis to L/A_1 gives a suitable bijection between factors above A_1, and then we can map A_1 to B_1, and we have the same result.

So suppose A_1 and B_1 are distinct and put $A = A_1 \oplus B_1$. Then A/A_1 and A/B_1 are chief factors of L and there are chief series of the form $0 < A_1 < A < C_3 < \cdots < C_n = L$ and $0 < B_1 < A < C_3 < \cdots < C_n = L$.

Define an equivalence relation on the chief series of L by saying that two such series are equivalent if there is a bijection between their chief factors satisfying the requirements of the theorem.

Since the series $0 < A_1 < A_2 < \cdots < A_n = L$ and $0 < A_1 < A < C_3 < \cdots < C_n = L$ have a minimal ideal in common, they are equivalent. Similarly, series $0 < B_1 < B_2 < \cdots < B_n = L$ and $0 < B_1 < A < C_3 < \cdots < C_n = L$ are equivalent. Moreover, since series $0 < A_1 < A < C_3 < \cdots < C_n = L$ and $0 < B_1 < A < C_3 < \cdots < C_n = L$ coincide above A they are also equivalent by Lemma 3.3.1. Hence series $0 < A_1 < A_2 < \cdots < A_n = L$ and $0 < B_1 < B_2 < \cdots < B_n = L$ are equivalent. □

Define the set I by $i \in I$ if and only if A_i/A_{i-1} is not a U-Frattini chief factor of L. For each $i \in I$ say $\mathcal{M}_i = \{M \in [U + A_{i-1} : L]_{max} | A_i \not\subseteq M\}$. Then B is a *U-prefrattini subalgebra* of L if $B = \bigcap_{i \in I} M_i$ for some $M_i \in \mathcal{M}_i$. If $U = \{0\}$, we will refer to B simply as a *prefrattini subalgebra* of L.

The subalgebra B *avoids* A_i/A_{i-1} if $B \cap A_i = B \cap A_{i-1}$; likewise, B covers A_i/A_{i-1} if $B + A_i = B + A_{i-1}$. Then we have the following important property of U-prefrattini subalgebras of L.

Lemma 3.3.3. *If B is a U-prefrattini subalgebra of L then it covers all U-Frattini chief factors of L in $0 < A_1 < A_2 < \cdots < A_n = L$ and avoids the rest.*

Proof. Let B be a U-prefrattini subalgebra of L and let A_i/A_{i-1} be a chief factor of L. If it is U-Frattini chief factor then either $A_i \subseteq F(U + A_{i-1}, L)$ or else $U + A_{i-1} = L$. In the first case, every maximal subalgebra of L that contains $U + A_{i-1}$ also contains A_i, and so $A_i \subseteq B$. In either case, B covers A_i/A_{i-1}. If it is not U-Frattini chief factor we have $B \subseteq M_i$ where $L = A_i + M_i$ and $A_I \cap M_i = A_{i-1}$. Hence $B \cap A_i = B \cap M_i \cap A_i = B \cap A_{i-1} \subseteq B \cap A_{i-1}$, and so B avoids A_i/A_{i-1}. □

The next four results show how U-prefrattini subalgebras relate to the previous section.

Lemma 3.3.4. *Let B be a U-prefrattini subalgebra of L. Then*

$$dim(B) = \sum_{i \notin \mathcal{I}} (dim A_i - dim A_{i-1});$$

in particular, all U-prefrattini subalgebras of L have the same dimension.

Proof. We use induction on $dim(L)$. The result is clear if L is abelian, so suppose it holds for Leibniz algebras of smaller dimension then L. It is easy to check that $(B + A_1)/A_1$ is a $(U + A_1/A_1)$-prefrattini subalgebra of L/A_1 and so

$$dim(\frac{B + A_1}{A_1}) = \sum_{i \in \mathcal{I}, i \neq 1} (dim(A_i - dim(A_{i-1})$$

by the inductive hypothesis.

If A_1/A_0 is a U-Frattini chief factor of L, the B covers A_1/A_0, whence $B = B + A_1$ and

$$dimB = dimA_1 + dim(\frac{B + A_1}{A_1}) = \sum_{i\in\mathcal{I}}(dimA_i - dimA_{i-1}).$$

If A_1/A_0 is not a U-Frattini chief factor of L, then B avoids A_1/A_0 whence $B \cap A_1 = 0$ and

$$dimB = dim(\frac{B + A_1}{A_1}) = \sum_{i\in\mathcal{I}}(dimA_i - dimA_{i-1}).$$

□

Let $\Pi(U, L)$ be the set of U-prefrattini subalgebras of L.

Lemma 3.3.5. *$\Pi(U, L) \subseteq \Omega(U, L)$.*

Proof. Use induction on $dimL$. The result is clear if L is abelian, so suppose it holds for Leibniz algebras of dimension less than that of L. Let $B \in \Pi(U, L)$. Then

$$\frac{B + A_1}{A_1} \in \Pi(\frac{U + A_1}{A_1}, \frac{L}{A_1}) \subseteq \Omega(\frac{U + A_1}{A_1}, \frac{L}{A_1})$$

whence $B + A_1 \in \Omega(U, L)$. If $A_1 \subseteq B$ we have $B \in \Omega(U, L)$. So suppose that $A_1 \nsubseteq B$. Then B does not cover A_1/A_0, so A_1/A_0 is not a U-Frattini chief factor of L.

It follows that $1 \in \mathcal{I}$, and so there is a maximal subalgebra M of L with $B \subseteq M$ and $A_1 \nsubseteq M$. But now $L = A_1 \oplus M$ and the intervals $[B + A_1 : L]$ and $[B : M]$ are lattice isomorphic, which yields that $[B : M]$ is complemented.

By Lemma 3.2.5, $B \in \Omega(U, L)$ again. □

Lemma 3.3.6. *$\Omega(U, L)_{min} \subseteq \Pi(U, L)$*

Proof. Let $B \in \Omega(U, L)_{min}$ and let A_i/A_{i-1} be a chief factor of L. By Lemma 3.2.11, $(\frac{B+A_{i-1}}{A_{i-1}}) \in \Omega(\frac{U+A_{i-1}}{A_{i-1}}, \frac{L}{A_{i-1}})_{min}$. Applying Lemma 3.2.10 to the minimal ideal A_i/A_{i-1} of L/A_{i-1}. If A_i/A_{i-1} is a U-Frattini chief factor then it doesn't have a complement in L/A_{i-1} and Lemma 3.2.10 gives that $A_i \subseteq B + A_{i-1}$, whence $A_i + B = A_{i-1} + B$ and B covers A_i/A_{i-1}.

If A_i/A_{i-1} is not U-Frattini chief factor then it has a complement M_i/A_{i-1} in L/A_{i-1} and Lemma 3.2.10 gives that it has such a complement containing $(B + A_{i-1})/A_{i-1}$; that is $L = M_i + A_i$, $M_i \cap A_i = A_{i-1}$, and $B + A_{i-1} \subseteq M_i$. But now $B \cap A_i \subseteq B \cap A_i + A_{i-1} = (B + A_{i-1}) \cap A_i \subseteq M_i \cap A_i = A_{i-1}$. It follows that $B \cap A_i \subseteq B \cap A_i = B \cap A_{i-1}$ and B avoids A_i/A_{i-1}. Clearly, $M_i \in \mathcal{M}_i$ and $B \subseteq C = \cap_{i \in \mathcal{I}} M_i \in \Pi(U, L)$. But B covers or avoids the same chief factors of $0 < A_1 < A_2 < \cdots < A_n = L$ as C, so the proof of Lemma 3.3.4 shows dim $B =$ dim C. It follows that $B = C \in \Pi(U, L)$. □

Combining the previous three lemmas we get the following result.

Theorem 3.3.7. $\Omega(U, L)_{min} = \Pi(U, L)$.

This result shows that the definition of U-prefrattini subalgebras does not depend on the choice of chief series.

Corollary 3.3.8. *If A is an ideal of L and $S \in \Pi(U, L)$ then $(S + A)/A \in \Pi((U + A)/A, L/A)$.*

Proof. This follows from Theorem 3.3.7 and Lemma 3.2.11. □

Corollary 3.3.9. *For every solvable Leibniz algebra L, $F(U, L) = \bigcap_{B \in \Pi(U,L)} B$.*

Proof. Put $P = \bigcap_{B \in \Pi(U,L)} B$. Then $F(U,L) \subseteq P$, by Theorem 3.3.7 and Lemma 3.2.1. Let $M \in [U : L]_{max}$. There is an I such that $A_{i-1} \subseteq M$ but $A_i \nsubseteq M$ $(1 \leq i \leq n)$. Then A_i/A_{i-1} is not a U-Frattini chief factor of L, so $i \in \mathcal{I}$ and $M \in \mathcal{M}_i$. Thus there exists $B \in \Pi(U,L)$ such that $B \subseteq M$, whence $P \subseteq M$. Hence $P \subseteq F(U,L)$. □

Corollary 3.3.10. *Let L be completely solvable and let U be a subalgebra of L. Then $\Pi(U,L) = \{F(U,L)\}$. In particular, $\Pi(0,L) = \{F(L)\}$.*

Proof. This follows from Theorem 3.3.7 and Theorem 3.2.12. □

3.3.1 Examples

Example 3.3.11. Let L be the cyclic Leibniz algebra $L = \langle a, a^2 \rangle$ with multiplications $aa^2 = a^2$, $aa = a^2$ and all others 0. Note that the maximal subalgebras of L are $\langle a - a^2 \rangle$ and $\langle a^2 \rangle$. The intersection of these subalgebras is contains only 0. Thus, $F(L) = \{0\}$. A fixed chief series for L is $0 \subseteq \{a^2\} \subseteq L$. We will consider $0 = A_0$, $\{a^2\} = A_1$, and $L = A_2$.

Consider $F(A_2/A_1)$. We find that $F(A_2/A_1) = \{a^2\}$. To determine if A_2/A_1 is a Frattini chief factor we consider whether or not $A_2/A_1 \subseteq F(L/A_1)$. In this case we find that $\{a^2\} \subseteq \{a^2\}$. Hence A_2/A_1 is a Frattini chief factor.

Consider $F(A_1/A_0)$. Here we find $F(A_1/A_0) = \{a^2\}$. Further we see $A_1/A_0 \subseteq F(L/A_0)$, since $\{a^2\} \nsubseteq \{0\}$. Thus we determine that A_1/A_0 is not a Frattini chief factor. This leads us to considering whether or not the factor algebra is complemented. To be complemented we must find a maximal subalgebra M such that $L = A_1 + M$ and $A_1 \cap M = A_0$. If we take $M = \{a - a^2\}$, we find that A_1/A_0 is complemented. This result is consistent with the fact that we previously determined if L is solvable, a chief Factor is Frattini if and only if it is not complemented.

For chief factors that are not Frattini, they still could be U-frattini chief factors. Hence we next want to determine whether or not A_1/A_0 is a U-frattini chief factor. We get to make our own selection for U so long as it is a subalgebra of L. To determine whether or not a chief factor is U-frattini, we consider a factor algebra A_i/A_{i-1} and check to see if $A_i \subseteq F(U + A_{i-1}, L)$ or if $U + A_{i-1} = L$. If so, then the chief factor is U-frattini.

For A_1/A_0, we want to consider a U such that $A_1 = F(U + A_0, L)$ or $U + A_0 = L$. In this case, we will use $U = \{a^2\}$. We check to see if $A_1 \subseteq F(U + A_0, L)$. Here $\{a^2\} \subseteq F(L) = \{a^2\}$. Hence A_1/A_0 is a U-frattini chief factor when $U = \{a^2\}$.

Now we will consider if A_2/A_1 is a U-Frattini chief factor when $U = \{a^2\}$. We find that $A_2 \nsubseteq F(U + A_1, L)$ and that $U + A_1 \neq L$. Hence A_2/A_1 is not a U-Frattini chief factor when $U = \{a^2\}$.

We defined the set $\mathscr{I}$ by $i \in \mathscr{I}$ if and only if A_i/A_{i-1} is not a U-frattini chief factor of L. Here $\mathscr{I} = \{2\}$ because A_2/A_1 is not a U-frattini chief factor but A_1/A_0 is.

From here we consider the set

$$\mathscr{M}_2 = \{M \in [U + A_1, L]_{max} : A_2 \nsubseteq M\}.$$

$$\mathscr{M} = \{\{a^2\}\}$$

.

From here we define $B = \bigcap_{i \in \mathscr{I}} M_i$ for some $M_i \in \mathscr{M}_i$ to be a U-prefrattini subalgebra. For this example, $B = \{a^2\}$.

We now turn to Lemma 3.3.4 to check the dimension of B. Lemma 3.3.4 guarantees that the dimension of our prefrattini subalgebra will be equal to $\sum_{i \notin \mathscr{I}} (dim A_i - A_{i-1})$.

$$dimB = \sum_{i \notin \mathscr{I}} (dimA_i - A_{i-1})$$

$$1 = (dim(A_1) - dim(A_0))$$

$$1 = (1 - 0)$$

Now we can consider if our prefrattini subalgebra avoids A_i/A_{i-1} meaning $B \cap A_i = B \cap A_{i-1}$ or if B covers A_i/A_{i-1} meaning $B + A_i = B + A_{i-1}$. We first consider if $B \cap A_2 = B \cap A_1$. This statement is true. Next we consider $B \cap A_1 = B \cap A_0$. This statement indicates $\{a^2\} = \{0\}$, which is false. Hence we conclude that B avoids A_2/A_1 but does not avoid A_1/A_0. Now we consider whether or not B covers A_1/A_0. We check $B + A_1 = B + A_0$. This is true. Hence B covers A_1/A_0.

Since B covers A_1/A_0, we know by Lemma 3.3.3 that B must be a U-frattini chief factor and since B avoids A_2/A_1 we know it is not a U-frattini chief factor. This agrees with all that was previously discussed in this example.

Finally we consider the set $\Pi(U, L)$ which is the set of U-prefrattini subalgebras. For this example, $\Pi(U, L) = \{a^2\}$ for $U = \{a^2\}$.

Finally, we check to see whether our example validates the important theorem of this section: $\Omega(U, L)_{min} = \Pi(U, L)$ and even more specifically, we check the corollary that states that when L is completely solvable $\Pi(U, L) = \{F(U, L)\}$. Recall that solvable Leibniz algebras are always completely solvable. In each case, the statements are true. Recall from Example 3.2.14 (ii) that $\Omega(U, L)_{min} = \{a^2\}$. Hence we have connected our concept of the set of prefrattini subalgebras in this chapter to our concept of the set of subalgebras that are complemented in an interval.

Example 3.3.12. We will now consider a 3-dimensional cyclic Leibniz algebra of type (ii) as mentioned in the Cyclic Leibniz Algebra section of the Introduction chapter. There it was determined that a cyclic Leibniz algebra, $L = \{a, a^2, a^3\}$ with multiplications $aa = a^2$, $aa^2 = a^3$, and $aa^3 = a^3$, has maximal subalgebras L^2 and $\{a - a^2, a - a^3\}$. This leads to the fact that its Frattini subalgebra is $F(L) = \{a^2 - a^3\}$.

We want to consider the following fixed chief series: $0 \subseteq F(L) \subseteq \{a^2, a^3\} \subseteq L$

First we want to check to see if there are any Frattini chief factors. If a chief factor of this solvable Leibniz algebra is Frattini, we have established that it is not complemented. If the chief factor is not Frattini, we want to investigate whether or not it is complemented.

Consider $F(A_3/A_2)$. For A_3/A_2 to be a Frattini chief factor $A_3/A_2 \subseteq F(L/A_2)$; however, $A_3/A_2 \not\subseteq F(L/A_2)$. Hence A_3/A_2 is not a Frattini chief factor.

Consider $F(A_2/A_1)$. In this case, $A_2/A_1 \subseteq F(L/A_1)$. Hence A_2/A_1 is a Frattini chief factor.

Consider $F(A_1/A_0)$. Here we find $A_1/A_0 \subseteq F(L/A_0)$. Hence A_1/A_0 is a Frattini chief factor.

Next we will determine whether the chief factor A_3/A_2 is complemented. To do this we are looking to see if there is a maximal subalgebra, $M_1 = \{a^2, a^3\}$ or $M_2 = \{a-a^2, a-a^3\}$, of L such that $L = A_i + M$ and $A_i \cap M = A_{i-1}$.

Since $A_3/A_2 = \{a\}$, we consider M_1.

$$L = A_3 + M_2 \,\text{and}\, A_3 \cap M_2 = A_2$$

Hence A_3/A_2 is a complemented chief factor.

For chief factors that are not Frattini, they still could be U-frattini chief factors. Hence we next want to determine whether or not A_3/A_2 is a U-frattini chief factor. We get to make our own selection for U so long as it is a subalgebra of L. To determine whether or not a chief factor is U-frattini, we must consider a factor algebra A_i/A_{i-1} to see if $A_i \subseteq F(U + A_{i-1}, L)$ or if $U + A_{i-1} = L$. If so, then the chief factor is U-frattini.

We will choose $U = \{a^2, a^3\}$.For A_3/A_2, we want to consider $A_3 = F(U + A_2, L)$ or $U + A_2 = L$. Neither of these are possible for our chosen U. Hence there must be an $M \in [U + A_{i-1} : L]_{max}$ for which $A_i \nsubseteq M$; meaning that M is a complement of the chief factor A_i/A_{i-1}. In this case, $M = \{a - a^2, a - a^3\}$. This holds for any choice of U. Note that A_2/A_1 and A_1/A_0 are both U-Frattini factors under our chosen U.

We defined the set $\mathscr{I}$ by $i \in \mathscr{I}$ if and only if A_i/A_{i-1} is not a U-frattini chief factor of L. Here $\mathscr{I} = \{3\}$ because A_3/A_2 is not a U-frattini chief factor but A_2/A_1 and A_1/A_0 are.

From here we consider the set

$$\mathscr{M}_3 = \{M \in [U + A_2, L]_{max} : A_3 \nsubseteq M\}.$$

$$\mathscr{M} = \{a^2, a^3\}.$$

From here we define $B = \bigcap_{i \in \mathscr{I}} M_i$ for some $M_i \in \mathscr{M}_i$ to be a U-prefrattini subalgebra. For this example, there is only one U-prefrattini subalgebra and it is $B = \{a^2, a^3\}$.

We now turn to Lemma 3.3.4 to check the dimension of B. Lemma 3.3.4 guarantees that the dimension of our prefrattini subalgebra will be equal to $\sum_{i \notin \mathscr{I}} (dimA_i - A_{i-1})$.

$$dimB = \sum_{i \notin \mathscr{I}} (dimA_i - A_{i-1})$$

$$2 = (dim(A_2) - dim(A_1)) + (dim(A_1) - dim(A_0))$$

$$2 = (2 - 1) + (1 - 0)$$

Now we can consider if our prefrattini subalgebra avoids A_i/A_{i-1} meaning $B \cap A_i = B \cap A_{i-1}$ or if B covers A_i/A_{i-1} meaning $B + A_i = B + A_{i-1}$. We first consider if $B \cap A_2 = B \cap A_1$. This statement is true. Next we consider $B \cap A_1 = B \cap A_0$. This statement is also true. Hence we conclude that B covers A_2/A_1 and A_1/A_0. If we consider the statement $B + A_3 = B + A_2$, we will find that it is false. Since B does not cover A_3/A_2, we will want to consider whether or not it avoids it. We check $B \cap A_3 = B \cap A_2$. This is true. Hence B avoids A_3/A_2.

Since B covers A_2/A_1 and A_1/A_0, we know by Lemma 3.3.3 that they must be a U-frattini chief factors and since B avoids A_3/A_2 we know it is not a U-frattini chief factor. This agrees with all that was previously discussed in this example.

Finally we consider the set $\Pi(U, L)$ which is the set of U-prefrattini subalgebras. For this example, $\Pi(U, L) = \{a^2, a^3\}$ for $U = \{a^2, a^3\}$.

Finally, we check to see whether our example validates the important theorem of this section: $\Omega(U, L)_{min} = \Pi(U, L)$ and even more specifically, we check the corollary that states that when L is completely solvable $\Pi(U, L) = \{F(U, L)\}$. Recall that solvable Leibniz algebras are always completely solvable. In each case, the statements are true. Hence we have connected our concept of the set of prefrattini subalgebras in this chapter to our concept of the set of subalgebras that are complemented in an interval.

Example 3.3.13. We will now consider a 4-dimensional nilpotent Leibniz algebra. This algebra, $L = \{x, y_1, y_2, y_3\}$ with multiplications $xy_1 = y_2$, $xy_2 = y_3$, and $xy_3 = 0$ and all other multiplication zero and has maximal subalgebras $\{x, y_2, y_3\}$ and $\{y_1, y_2, y_3\}$. This leads to the fact that its Frattini subalgebra is $F(L) = \{y_2, y_3\}$.

We want to consider the following fixed chief series:

$$0 \subseteq \{y_3\} \subseteq \{y_2, y_3\} \subseteq \{x, y_2, y_3\} \subseteq L$$

First we want to check to see if there are any Frattini chief factors. If a chief factor of this solvable Leibniz algebra is Frattini, we have established that it is not complemented.

To determine whether a chief factor is U-Frattini we must select and U and then check to see for chief factor A_i/A_{i-1} if $A_i \subseteq F(U + A_{i-1}, L)$. We will select $U = \{y_2, y_3\}$. Going through this process leads us to find that A_4/A_3 and A_3/A_2 are U-Frattini chief factors and A_2/A_1 and A_1/A_0 are not U-Frattini chief factors.

We defined the set $\mathscr{I}$ by $i \in \mathscr{I}$ if and only if A_i/A_{i-1} is not a U-frattini chief factor of L. Here $\mathscr{I} = \{3, 4\}$ because A_4/A_3 and A_3/A_2 is not a U-frattini chief factor but A_2/A_1 and A_1/A_0 are.

From here we consider the set

$$\mathscr{M}_3 = \{M \in [U + A_2, L]_{max} : A_3 \nsubseteq M\}.$$

$$\mathscr{M} = \{y_1, y_2, y_3\}.$$

We also must consider the set

$$\mathscr{M}_4 = \{M \in [U + A_3, L]_{max} : A_4 \nsubseteq M\}.$$

$$\mathscr{M} = \{x, y_2, y_3\}.$$

From here we define $B = \bigcap_{i \in \mathscr{I}} M_i$ for some $M_i \in \mathscr{M}_i$ to be a U-prefrattini subalgebra. For this example, there is only one U-prefrattini subalgebra and it is $B = \{y_2, y_3\}$.

We now turn to Lemma 3.3.4 to check the dimension of B. Lemma 3.3.4 guarantees that the dimension of our prefrattini subalgebra will be equal to $\sum_{i \notin \mathscr{I}} (dimA_i - A_{i-1})$.

$$dimB = \sum_{i \notin \mathscr{I}} (dimA_i - A_{i-1})$$

$$2 = (dim(A_2) - dim(A_1)) + (dim(A_1) - dim(A_0))$$

$$2 = (2 - 1) + (1 - 0)$$

Now we can consider if our prefrattini subalgebra avoids A_i/A_{i-1} meaning $B \cap A_i = B \cap A_{i-1}$ or if B covers A_i/A_{i-1} meaning $B + A_i = B + A_{i-1}$. We first consider if $B \cap A_3 = B \cap A_2$. This statement is true. Next we consider $B \cap A_3 = B \cap A_4$. This statement is also true. Hence we conclude that B avoids A_4/A_3 and A_3/A_2. If we consider the statement $B \cap A_1 = B \cap A_2$, we will find that it is false. Since B does not avoid A_2/A_1, we will want to consider whether or not it covers it. We check $B + A_2 = B + A_1$. This is true. Hence B covers A_2/A_1. A similar argument can be made for A_1/A_0 and we can determine that B covers A_1/A_0 also.

Since B covers A_2/A_1 and A_1/A_0, we know by Lemma 3.3.3 that they must be a U-frattini chief factors and since B avoids A_3/A_2 and A_4/A_3, we know they are not a U-frattini chief factors. This agrees with all that was previously discussed in this example.

Finally we consider the set $\Pi(U, L)$ which is the set of U-prefrattini subalgebras. For this example, $\Pi(U, L) = \{y_2, y_3\}$ for $U = \{y_2, y_3\}$.

Finally, we check to see whether our example validates the important theorem of this section: $\Omega(U, L)_{min} = \Pi(U, L)$ and even more specifically, we check the corollary that states that when L is completely solvable $\Pi(U, L) = \{F(U, L)\}$. Recall that solvable Leibniz algebras are always completely solvable. Also, recall from Example 3.2.13 that $\Omega(U, L)_{min} = \{y_2, y_3\}$. In each case, the statements are true. Hence we have connected our concept of the set of prefrattini subalgebras in this chapter to our concept of the set of subalgebras that are complemented in an interval.

Chapter 4

Solvable Complemented Leibniz Algebras

4.1 Introduction

Throughout this chapter, L will denote a solvable Leibniz algebra over a field F of characteristic zero. We define the *nilpotent residual*, L^∞, of L to be the smallest ideal of L such that L/L^∞ is nilpotent. Clearly this is the intersection of the terms of the lower central series for L. The *derived series* for L is the sequence of ideals $L^{(i)}$ of L defined by $L^{(0)} = L, L^{(i+1)} = [L^{(i)}, L^{(i)}]$ for $i \geq 0$; we will also write L^2 for $L^{(1)}$. If $L^{(n)} = 0$ but $L^{(n-1)} \neq 0$ we say that L has *derived length n*. We say that L is *completely solvable* if L^2 is nilpotent. Algebra direct sums will be denoted by $\oplus$, whereas vector space direct sums will be denoted by $\dot{+}$.

The *Frattini subalgebra* of L, $\phi(L)$, is the intersection of the maximal subalgebras of L. When L is solvable this is always an ideal of L. This is true since we are in a field of characteristic zero. For a subalgebra U of Lwe denote by $[U : L]$ the set of all subalgebras S of L with $U \subseteq S \subseteq L$. We say that $[U : L]$ is *complemented* if, for any $S \in [U : L]$ there is a $T \in [U : L]$ such that $S \cap T = U$ and $\langle S, T \rangle = L$. We denote by $[U : L]_{max}$ the set of maximal subalgebras in $[U : L]$, that is, the set of maximal subalgebras of L containing U.

Let $0 = A_0 \subset A_1 \subset \cdots \subset A_n = L$ be a chief series for L. We say that A_i/A_{i-1} is a *Frattini* chief factor is $A_i/A_{i-1} \subseteq \phi(L/A_{i-1})$; it is *complemented* if there is a maximal subalgebra M of L such that $L = A_i + M$ and $A_i \cap M = A_{i-1}$. When L is solvable it is easy to see that a chief factor is Frattini if and only if it is not complemented.

We define the set $\mathscr{I}$ by $i \in \mathscr{I}$ if and only if A_i/A_{i-1} is not a Frattini chief factor of L. For each $i \in \mathscr{I}$ put $\mathscr{M}_i = \{M \in [A_{i-1}, L]_{max} : A_i \nsubseteq M\}$. Then B is a *prefrattini subalgebra* of L if $B = \bigcap_{i \in \mathscr{I}} M_i$ for some $M_i \in \mathscr{M}_i$. It was shown in the previous section that the definition of prefrattini subalgebras does not depend on the choice of chief series.

The subalgebra B *avoids* $A_i/Ai - 1$ if $B \cap A_i = B \cap A_{i-1}$; likewise, B *covers* A_i/A_{i-1} if $B + A_i = B + A_{i-1}$. Let $\Pi(L)$ be the set of prefrattini subalgebras of L.

4.2 Prefrattini Subalgebras

Theorem 4.2.1. *Let L be a solvable Leibniz algebra over a field F.*

(i) If B is a prefrattini subalgebra of L, then it covers all Frattini chief factors of L and avoids the rest.

(ii) If B is a prefrattini subalgebra of L, then $\dim B = \sum_{i \notin \mathscr{I}} (\dim A_i - \dim A_{i-1})$; in particular, all prefrattini subalgebras of L have the same dimension.

(iii) *If A is an ideal of L and $S \in \Pi(L)$, then $(S + A)/A \in \Pi(L/A)$.*

(iv) $F(L) = \cap_{B \in \Pi(L)} B$.

(v) *If L is completely solvable, then* $\Pi(L) = \{F(L)\}$.

(vi) *L is complemented if and only if* $\Pi(L) = \{0\}$.

If L^2 is not nilpotent, then $\Pi(L)$ can contain more than one element (see previous section).

4.3 The Prefrattini Residual

Now we will use the ideas of the previous section to again examine *complemented* Leibniz algebras, as previously studied in section 2.

Theorem 4.3.1. *Let L be a solvable Leibniz algebra over any field F. Then the following are equivalent:*

(i) *L is complemented.*

(ii) *The prefrattini subalgebras of L are all trivial.*

(iii) *L and all of its epimorphic images are ϕ-free.*

(iv) *L splits over all of its ideals.*

Proof. $(i) \Rightarrow (ii)$: If L is complemented, then $\Pi(L) = \{0\}$, by Theorem 4.2.1 (vi).

$(ii) \Rightarrow (iii)$: Assume $\Pi(L) = \{0\}$, let L/B be any epimorphic image of L, and suppose that $F(L/B) \neq 0$. Then there is a Frattini cheif factor of L, C/B, contained in $F(L/B)$.

But now any prefrattini subalgebra has dimiension great than or equal to $\dim(C/B)$, by Theorem 4.2.1 (ii), which is a contradiction. Thus, (iii) is established.

$(iii) \Rightarrow (iv)$: Suppose that L and all of its epimorphic images are ϕ-free. We use induction on the $\dim L$. If $\dim L = 1$, then the result is clear. Assume it holds for a Leibniz algebra of dimension less than $\dim L$, and let B be a non-trivial ideal of L. If B is a minimal ideal of L, then the result follows from Lemma 4.3.2 found below. If B is not a minimal ideal, let A be a minimal ideal contained in B. Then L/A splits over B/A by the inductive hypothesis. Hence there is a subalgebra C of L with $A \subset C$ such that $L = B + C$ and $B \cap C = A$. Moreover, there is a subalgebra M of L such that $L = A \dot{+} M$, by Lemma 4.3.2. But now $C = A \dot{+} (M \cap C)$, thus, $L = B \dot{+} (M \cap C)$ and (iv) is established.

$(iv) \Rightarrow (i)$: Suppose that L splits over all of its ideals. We use induction on the $\dim L$ again. Th result is clear if $\dim L = 1$. Assume it holds for a Leibniz algebra of dimension less than $\dim L$. Then $L = A \oplus M$ for some subalgebra M of L. Hence $M \cong L/A$ splits over all its ideals and so is complemented by the inductive hypothesis. It follows from Lemma 2.2.6 that L is complemented. $\square$

Lemma 4.3.2. *Let B be a zero ideal ($B^2 = 0$) of an algebra L such that $B \cap \Phi(L) = 0$. Then there is a subalgebra C of L such that $L = B \dot{+} C$.*

Proof. Let C be a subalgebra of A that is minimal with respect to the property that $L = B + C$. Then, by [[4], Lemma 3.6], $B \cap C \subseteq \phi(C)$. Now $B \cap C$ is an ideal of C and $(B \cap C)B + B(B \cap C) \subseteq B^2 = 0$, so $B \cap C$ is an ideal of A. Hence using, [[4], Lemma 3.7], $B \cap C \subseteq \phi(L) \cap B = 0$ and $L = B \dot{+} C$. $\square$

We say that L is *elementary* if $\phi(B) = 0$ for every subalgebra B of L. Let A be a vector space of finite dimension and let B be an abelian completely reducible subalgebra of gl(A). A construction of solvable elementary Lie algebras is given in [18]. Let A be a vector space and let B be an abelian Lie subalgebra of gl(A) which act completely reducibly on A. Forming the semidirect sum of A and B one obtains an elementary, solvable, almost algebraic Lie algebra L. We can use this construction to obtain elementary, solvable, Leibniz algebras. Since $A = Asoc(L)$, A is the direct sum of minimal ideals of L. Let $A = A_1 + A_2$, where each summand is an ideal in L. Define the Leibniz algebra L^* to be the vector space $A + B$ with the same multiplication as in L except that $A_2B = 0$. This algebra has $\Phi(L^*) \subseteq (L^*)^2 \subseteq A$ and $\Phi(L^*) \subseteq B$, hence $\Phi(L^*) = 0$ and therefore, L^* is elementary. Following [18], such a algebras will be called elementary Leibniz algebras of type I.

Theorem 4.3.3. *Let L be a completely solvable Leibniz algebra. The the following are equivalent:*

1. *L is complemented;*
2. *$F(L) = 0$;*
3. *L is elementary;*
4. *$L \cong A \oplus E$, where A is an abelian Leibniz algebra and E is an elementary Leibniz algebra of type I.*

Proof. The equivalence of (i), (ii) and (iii) is shown in Theorem 2.2.1. The equivalence of (iv) follows from [[4], Corollary 3.9]. □

Lemma 4.3.4. *Let L be a solvable Leibniz algebra, let B, C be ideals of L with $B \cap C = 0$, and suppose that L/B and L/C are complemented. Then L splits over B and over C.*

Proof. We show that L splits over C. Since L/B is complemented there is a subalgebra U of L with $B \subseteq U$ such that $L = (B + C) + U = C + U$ and $(B + C) \cap U = B$. Hence $C \cap U \subseteq C \cap (B + C) \cap U = C \cap B = 0$. □

A class $\mathscr{H}$ of finite-dimensional solvable Leibniz algebras is called a *homomorph* if it contains, along with an algebra L, all epimorphic images of L. A homomorph $\mathscr{H}$ is called a *formation* if $L/M, L/N \in \mathscr{H}$, where M, N are ideals of L, implies that $L/M \cap N \in \mathscr{H}$. If $\mathscr{H}$ is a formation, then for every solvable Leibniz algebra L there is a smallest ideal R such that $L/R \in \mathscr{H}$; this is called the $\mathscr{H} - \textit{residual}$ of L. We denote the class of solvable complemented Leibniz algebras by $\mathscr{C}$.

Theorem 4.3.5. *$\mathscr{C}$ is a formation.*

Proof. Note that $\mathscr{C}$ is a homomorph, by Lemma 2.2.2. Let B, C be distinct ideals of L with $L/B, L/C \in \mathscr{C}$. We want to show that $L/B \cap C \in \mathscr{C}$. Without loss of generality, assume that $B \cap C = 0$. Let $0 \subset B_k \subset \cdots \subset B_1 = B$ be part of a chief series for L. We will induct on k. If $k = 1$, then B is a minimal idea of L and the result follows from Lemma 4.3.4 and Lemma 2.2.6. So suppose it holds whenever $k < n$ and that we have $k = n$. Then B/B_n, $(C + B_n)/B_n$ are distinct ideals of L/B_n and the corresponding factor algebras are isomorphic to L/B and $(L/C)/((C + B_n)/C)$ respectively. These are both complemented (by Lemma 2.2.2 in the second case). It follows from the inductive hypothesis that L/B_n is complemented. But now L is complemented by Lemma 4.3.4 and Lemma 2.2.6, and the result follows. □

We define the *prefrattini residual* of a solvable Leibniz algebra L to be

$$\pi(L) = \bigcap\{B|B \text{ is an ideal of } L \text{ and } L/B \in \mathscr{C}\}.$$

Clearly $\pi(L)$ is the smallest ideal of L such that $L/\pi(L) \in \mathscr{C}$. It is also the ideal closure of the prefrattini subalgebras of L, by Theorem 4.3.1.

We define that *nilpotent series* for L inductively by $N_0 = 0, N_{i+1}/N_i = N(L/N_i(L))$ for $i > 0$, where $N(L)$ denotes the nilradical of L, the maximal nilpotent ideal of L. Now we obtain the following characterization of solvable complemented Leibniz algebras.

Theorem 4.3.6. *The solvable Leibniz algebra L is complemented if and only if $F(L/N_i(L)) = 0$ for all $i \geq 0$.*

Proof. Suppose L is complemented. Then $L/N_i(L)$ is complemented, by Lemma 2.2.2 and so $F(L/N_i(L)) = 0$ by Theorem 4.3.1.

Suppose conversely that $F(L/N_i(L)) = 0$ for all $i \geq 0$. We induct on the $\dim L$. The result is clear for $\dim L = 1$. Assume the result holds for all solvable Leibniz algebras of dimension less than that of L. The $L/N(L)$ is complemented, by the induction hypothesis. Further, we have that $L = N(L) \dot{+} B$ for some subalgebra B of L and $N(L) = AsocL$, by [[4], Theorem 2.4]. Let $AsocL = A_1 \oplus \cdots \oplus A_n$. If $n = 1$, then L splits over A_1 and L/A_1 is complemented by Lemma 2.2.6. So assume that $n > 1$ and let $C_i = A_1 \oplus \cdots \oplus \hat{A}_i \oplus \cdots \oplus A_n$, where $\hat{A}_i$ means that A_i is missing from the direct sum. Then L/C_i splits over $AsocL/C_i$ and $(L/C_i)/(AsocL/C_i) \cong L/N(L)$ is complemented, so L/C_i is complemented, by Lemma 2.2.6 again. It follows from Theorem 4.3.5 that $L \cong L/\cap_{i=1}^n C_i$ is complemented. □

4.4 Decomposition Results for Complemented Algebras

A Leibniz algebra L is called an *A-algebra* if every nilpotent subalgebra of L is abelian. In this section we have a few basic structure theorems. To begin we see that L splits over the terms in its derived series. To establish this result the following lemma is useful.

Lemma 4.4.1. *Let L be a metabelian Leibniz algebra that splits over L^2. Then $L^2 = L^\infty$.*

Proof. Since L splits over L^2, we have that $L = L^2 \dot{+} C$, where C is abelian. Then $L^2 = [L^2, C] = [L^2, L] = L^3$. This implies $L^2 = L^\infty$. □

Theorem 4.4.2. *Let L be a solvable complemented Leibniz algebra. Then L splits over each term in its derived series. Moreover, the Cartan subalgebras of $L^{(i)}/L^{(i+2)}$ are precisely the subalgebras that are complements to $L^{(i+1)}/L^{(i+2)}$ for $i \geq 0$.*

Proof. The first assertion comes directly from Theorem 4.3.1 iv. The second assertion follows from Lemma 4.4.1 and the result proven in the following section on Cartan subalgebras. □

Now we obtain the following characterization of solvable complemented Leibniz algebras.

Corollary 4.4.3. *Let L be a solvable Leibniz algebra of derived length $n+1$. Then L is complemented if and only if the following hold:*

(i) $L = A_n \dot{+} A_{n-1} \dot{+} \cdots \dot{+} A_0$ *where A_i is an abelian subalgebra of L for each $0 \leq i \leq n$;*

(ii) $L^{(i)} = A_n \dotplus A_{n-1} \dotplus \dots A_i$ *for each* $0 \leq i \leq n$*;*

(iii) $L^{(i)}/L^{(i+1)}$ *is completely reducible as an* $(L/L^{(i+1)})$*-module for each* $1 \leq i \leq n$*.*

Proof. Assume L is complemented. By Theorem 4.4.2, there exists a subalgebra B_n of L such that $L = L^n \dotplus B_n$. Put $A_n = L^{(n)}$. Similarly, $B_n \cong L/L^{(n)}$ is complemented by Lemma 3 in [19], so $B_n = A_{n-1} \dotplus B_{n-1}$, where $A_{n-1} = (B_n)^{(n-1)}$. Continuing in this way, we get our result in (i). By induction we can obtain (ii) straightforwardly. Lastly, $L^{(i)}/L^{(i+1)} \subseteq N(L/L^{(i+1)} = Asc(L/L^{(i+1)})$, by Theorem 4.3.1 (iii) and Theorem 7.4 in [4], which gives (iii).

Conversely, assume that (i), (ii), and (iii) hold. Then by repeated use of Lemma 2.2.6 L is complemented. $\square$

Our next goal is to show the relationship between ideals of L and the decomposition in Corollary 4.4.3. To do this we need the following lemmas.

Lemma 4.4.4. *Let* L *be a Leibniz algebra. Then* $Z(L) \cap L^2 \subseteq \phi(L)$*.*

Proof. Let M be any maximal subalgebra of L. If $Z(L) \nsubseteq M$, then $L = M + Z(L)$ and $L^2 \subseteq M$. $\square$

Chapter 5

A Few Extentions

5.1 Cartan Subalgebras

To get the desired result from Theorem 4.4.2 above, we needed the following lemmas and theorem. First we give the definition and an example showing a difference between Cartan subalgebras for Lie algebras verses Leibniz algebras.

Definition 5.1.1. A Cartan subalgebra of a Leibniz algebra L is a nilpotent subalgebra such that $C = N_A(C)$.

Unlike Lie algebras, the sets $N_L^l(C)$ and $N_L^r(C)$ do not coincide because of the lack of antisymmetry in Leibniz algebras.

Example 5.1.2. Let L be a Leibniz algebra given by the following multiplications:

$$[z,x] = x, \qquad [z,y] = -y, \qquad [y,z] = y, \qquad [z,z] = x,$$

where $\{x, y, z\}$ is the basis of the algebra L and omitted products are equal to zero. Then

$C = \{x - z\}$ is a Cartan subalgebra, but $N_L^r(C) = \{x, z\}$.

Lemma 5.1.3. *Let H be a Cartan subalgebra of L and let A be an ideal of L. Then $(H + A)/A$ is a Cartan subalgebra of L/A.*

Proof. $(H+A)/A$ is nilpotent since $(H+A)/A \cong H/(H \cap A)$. Suppose $x+A \in N_{L/A}((H+A)/A)$. Then $(x + A)(H + A) \subseteq H + A$. This implies $x(H + A) \subseteq H + A$ and $x \in N_L(H + A) = H + A$. Thus, $N_{L/A}((H + A)/A) = (H + A)/A$. Hence $(H + A)/A$ is a Cartan subalgebra of L/A. □

Lemma 5.1.4. *Let A be and ideal of L, $A \subseteq U \subseteq L$. Suppose U/A is a Cartan subalgebra of L/A and suppose H is a Cartan subalgebra of U. Then H is a Cartan subalgebra.*

Proof. Since H is a Cartan subalgebra of U, H is nilpotent. Suppose $x \in N_L(H)$. Then $x + A \in N_{L/A}((H + A)/A)$. But $(H + A)/A$ is a Cartan subalgebra of U/A and U/A is nilpotent. Therefore, $H + A = U$ and $x + A \in N_{L/A}(U/A) = U/A$, since U/A is a Cartan subalgebra U/A. Therefore, $x \in U$ and so $x \in N_U(H) = H$. Hence H is a Cartan subalgebra. □

Theorem 5.1.5. *Let L be a solvable Leibniz algebra. Then there exists a Cartan subalgebra of L.*

Proof. Let L be a minimal counterexample and let A be a minimal ideal of L. Then there exists a Cartan subalgebra U/A of L/A.

If $U \subset L$, then there exists a Cartan subalgebra H of U. By Lemma 5.1.4, H is a Cartan subalgebra of L contrary to the choice of L. Therefore, $U = L$ and L/A is nilpotent.

Let M be a maximal subalgebra of L. If $A \subseteq M$, then M/A is a maximal subalgebra of the nilpotent algebra L/A, and it follows that M is an ideal of L. If every maximal subalgebra of L is an ideal, then L is nilpotent. Therefore, there exists a maximal subalgebra H of L which is not an ideal. For this H, we have $N_L(H) = H$ and $A \nsubseteq H$.

Since A is an abelian ideal and $L = H + A$, it follows that $H \cap A$ is an ideal of L. Since A is minimal, $H \cap A = 0$. Therefore, $H \cong L/A$ which is nilpotent, and H is a Cartan subalgebra of L. □

5.2 The Jacobson Radical

The Jacobson radical $J(L)$, is the intersection of maximal ideals of L. This concept was considered in [9] and [13] when L is a Lie algebra. If L is nilpotent, then $J(L) = \Phi(L)$, since then all maximal subalgebras are ideals [1]. Clearly $J(L) \subseteq L^2$, since if x is not in L^2, then we can find a complementary subspace, M, of x in L that contains L^2 and, since $L^2 \subseteq M$, M is a maximal ideal of L and x is not in M.

If L is a linear Lie algebra, let $R = Rad(L)$, and let $Rad(L^*)$ be the radical of the associative envelope,L^* of L. Then, by Corollary 2 p. 45 of [11], $L \cap Rad(L^*) =$ all nilpotent elements of R and $[R, L] \subseteq Rad(L^*)$.

Theorem 5.2.1. *Let L be a Leibniz algebra and $R = Rad(L)$ be the radical of L. Then $LR + RL \subseteq N = Nil(L)$.*

Proof. Let $\mathscr{L}(L) = L_x : x \in N$. The map $\pi : L \to \mathscr{L}(L)$ is a homomorphism and $\mathscr{L}(L)$ is a Lie algebra under commutation. Also $\pi : R \to R(\mathscr{L}(L))$, the radical of $\mathscr{L}(L)$. By the result in the first paragraph, [$\mathscr{L}$(L),R($\mathscr{L}$(L))]⊆R($\mathscr{L}$(L)*). Hence there exists an n such that $[L_{s_n},L_{t_n}]...[L_{s_2},L_{t_2}][L_{s_1},L_{t_1}]=0$, where $s_i \in L$, t_i R or $s_i \in R$ and $t_i \in L$. Hence $L_{s_n t_n}...L_{s_1 t_1} = 0$. Hence $s_n t_n(...(s_1 t_1(x))...) = 0$ for all x∈L. Putting $x = s_0 t_0$, we have $s_n t_n(...(s_1 t_1(s_0 t_0))...) = 0$. Therefore$(LR+RL)^{n+1}=0$. Since LR+ RL is an ideal in L, LR+RL ⊆ N. □

Proposition 5.2.2. *If L is of characteristic 0, then $L^2 \cap R = LR+RL \subseteq N$.*

Proof. L=R+S as in the Levi decomposition for Leibniz algebras in [2].

Then $L^2 = S^2 + LR + RL$, and $L^2 \cap R = S^2 \cap R + (LR+RL) \cap R = LR+RL$ since $S \cap R = 0$ and $LR+RL \subseteq R$. □

Lemma 5.2.3. *If L is solvable, then $J(L)=L^2$.*

Proof. If M is a maximal ideal of L, then L/M is abelian and, in fact, one-dimensional since L is solvable. Hence $L^2 \subseteq M$ for all M and $L^2 \subseteq J(L)$. Since $J(L) \subseteq L^2$ always holds, $L^2=J(L)$. □

Proposition 5.2.4. *Let L be a Leibniz algebra over a field of characteristic 0. Let R=R(L). Then J(L)=LR+RL.*

Proof. Let S be a Levi factor of L. Then S is Lie and $S=S_1 \bigoplus ... \bigoplus S_t$ where each S_i is simple. Let M_i be the sum of R and all of the S_j except S_i. Then M_i is a maximal ideal of L. Then $J(L) \subseteq \bigcap M_i = R$ and $J(L) \subseteq R \cap L^2$.

If M is a maximal ideal of L, then L/M is abelian or simple. In the first case, $L^2 \subseteq M$, and in the second case, $R \subseteq M$. The intersection of all of the first type of M contains L^2, and all of the second type contains R. Therefore, $R \cap L^2 \subseteq J(L)$ and the result follows. □

Corollary 5.2.5. *Let L be a Leibniz algebra over a field of characteristic 0. Then J(L) is nilpotent.*

Proof. J(L)=LR+RL $\subseteq$ Nil(L) by Propositions 5.2.2 and 5.2.4. □

Corollary 5.2.6. Φ*(L)* $\subseteq$ *J(L) when L has characteristic 0.*

Proof. L=R+S as in the Levi decomposition and S is Lie. Hence Φ(S)=0. Thus Φ(L) $\subseteq$ R. Since Φ(L) $\subseteq L^2$, it follows that $\Phi(L) \subseteq R \cap L^2$=LR+RL=J(L) using Propositions 5.2.2 and 5.2.4. □

Proposition 5.2.7. *Let B be a nilpotent ideal in a Leibniz algebra, L. Then J(B)* $\subseteq$ *J(L).*

Proof. Since B is nilpotent, J(B)=B^2, which is an ideal in L. Suppose that x $\in$ J(B), x $\notin$ J(L). Let M be a maximal ideal of L such that x is not in M. Then L=J(B)+M and B=J(B)+(M$\cap$B). M$\cap$B is a proper ideal of B that supplements J(B), a contradiction. □

5.3 Leibniz Algebras With A Unique Maximal Ideal

Lie algebras with a unique maximal ideal were classified in [9] when the field of scalars has characteristic 0. We extend this result to Leibniz algebras.

Lemma 5.3.1. *Suppose that* $L=N+<x>$ *where* $<x>$ *is a one dimensional vector space,* $x^2 \neq 0$*, and N is an abelian ideal. Then xL is an ideal in L.*

Proof. Since xL $\subseteq$ N, N(xL)=(xL)N=0. (xL)x=(x(x+N))x=(x^2+xN)x=(xN)x=x(Nx)-N(x^2) $\subseteq$ xL since $x^2 \in$ N and N is abelian. Furthermore x(xL) $\subseteq$ xL. Hence the result holds. □

Lemma 5.3.2. *Suppose that $L=N+<x>$ where $<x>$ is a one dimensional vector space, $x^2 \neq 0$ and N is an abelian ideal. If $xL+Lx=N$, then $xL=N$.*

Proof. If $xL \neq N$, then xL is contained in an ideal M of L , $M \subseteq N$, and N/M is a minimal ideal of L/M. Note that $x^2 \in M$. We may take M=0. Then xN=0, which yields Nx=0 by [1]. Then Lx=(N+x)x =0. Therefore xL+Lx=0, a contradiction. □

We now show the following:

Theorem 5.3.3. *Let L be a Leibniz algebra over a field of characteristic 0. L has a unique maximal ideal if and only if one of the following holds.*

1) L is nilpotent and cyclic, or

2) $L=Nil(L)+S$ where S is simple and $N/N^2=S(N/N^2)+(N/N^2)S$, or

3) $L=Nil(L)+< x >$ where $< x >$ is a one dimensional vector space, $x^2 \notin N^2$, and $N/(N^2)=x(L/N^2)$, or

4) $L=Nil(L)+< x >$ where $x^2 \in N^2$ and $N/(N^2)=x(N/N^2)+(N/N^2)x$.

Proof. Let L have only one maximal ideal which is then J(L)=J. Then L/J is simple or one-dimensional. Since $J \subseteq Nil(L)=N$, if L/J is simple, then J=N and L=J+S=N+S. To show the second part of (2), assume that $N^2=0$ and let R=Rad(L). Then N=R and J=LR+RL=SN+NS.

Suppose that dim(L/J)=1. Since $J \subseteq N \subseteq L$, either J=N or N=L. If N=L, then L is nilpotent and $J=\Phi(L)=L^2$, so L is cyclic generated by some a and L is nilpotent and (1) holds.

Suppose that N=J.

Then L=N+$< x >$ where $< x >$ is a one-dimensional subspace. By Theorem 3.1 of [5], L/N^2 is not nilpotent, hence Nil(L/N^2) =N/N^2 =J/N^2=J(L/N^2)=x(L/N^2)+(L/N^2)x. Suppose that $x^2 \in N^2$. Then Nil(L/N^2) = x(N/N^2)+(N/N^2)x, which is as in condition (4). Then suppose that $x^2 \notin N^2$. Hence, Nil(L/N^2) = N/N^2 = x(L/N^2)+(L/N^2)x = x(L/N^2) by Lemma 5.3.2, as in (3).

We now show that each of the algebras in 1-4 have a unique maximal ideal. For the algebras in (3), we have the following lemma.

Lemma 5.3.4. *Suppose that L is Leibniz, and L=Nil(L)+$< x >$ where $x^2 \neq 0$. Suppose that xL=N and N^2=0 where N=Nil(L). Then Nil(L)=J(L). The algebras in (3) have a unique maximal ideal.*

Proof. Let J=J(L). Since xL=N, N=L^2. Let T be the subalgebra generated by x. Since dim(Ker $L_x|_L$)=1 and $0 \neq$ Ker $L_x|_T \subseteq$ Ker $L_x|_L$, it follows that Ker $L_x|_T$= Ker $L_x|_L$. Since dim(Ker $L_x|_T$)=1, the Fitting null component of $L_x|_T$ is the union $T_0 \subseteq T_1 \subseteq ...T_k$ where dim(T_j)=j and x$T_j \subseteq T_{j-1}$. The same holds for the Fitting null component of L_x on L where the invariant subspaces end with L_m. Clearly T_i=L_i for i $\leq$ k. We claim that m=k. There is a T_i that is not contained in N since the Fitting one component of L_x on T is contained in N. Then αx+n $\in T_i$, $\alpha \neq 0$ n $\in$ N. If $i \neq$ k, then there exists βx+p such that x(βx+p)=αx+n, which is impossible since x(βx+p) $\in$ N=L^2. Thus $T_{k-1} \subseteq$ N while T_k is not contained in N. Since T_k=L_k is not contained in N, the chain of L_i's must stop at L_k since only the final term in the string is not in N. Hence the Fitting null component for L_x acting on L is the same as L_x acting on T and the Fitting null component of L_x on N and T^2 are the same, both equal to $T_{k-1} \subseteq T^2$.

Let J=J(L). Since xL=N , N=L^2. Let N=N_0+N_1 be the Fitting decomposition of

L_x on N. Let K be a maximal ideal of L and suppose that K is not contained in L^2. There exists y $\in$ K such that y=αx+n_0+n_1 where $\alpha \neq 0$, $n_0 \in N_0$ and $n_1 \in N_1$. For any t$\in N_1$, there exists s $\in N_1$ such that xs=t. Then ys=(αx +n_0+n_1)s=αt since $N^2 = 0$. Thus αt $\in$ K and $N_1 \subseteq K$. Also $n_0 \in N_0 = T_{k-1} \subseteq T^2$ by the last paragraph. Hence $n_0 = \alpha_2 x^2 + ... + \alpha_t x^t$ and yx=(αx+$\alpha_2 x^2 + ... + \alpha_t x^t$ +n_1)x =αx^2 +n_1x $\in$ K. Since n_1x $\in$K, $\alpha x^2 \in$ K. Therefore $x^2 \in$ K and $T^2 \subseteq$ K. Since $N_0 \subseteq T^2$ and $N_1 \subseteq$ K, N $\subseteq$ K. Since J(L)=N has codimension 1, J(L)=N=K for all maximal ideals K of L. Hence algebras as in (3) have a unique maximal ideal. □

We now consider the algebras in (1), (2), and (4). Suppose that L is as in (2). We may assume that N^2=0 since N/N^2=Nil(L/N^2). Thus J=LR+RL=SN+NS=Nil(L) and J(L) is the only maximal ideal of L. If L is as in (4), then assume that N^2=0 since N/N^2=Nil(L/N^2). Then J(L)=LR+RL=xN+Nx. Since dim(L/N)=1, N is the only maximal ideal of L.

If L is as in (1), then L^2=J(L) since L is nilpotent, and J(L) is the unique maximal ideal. □

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