ABSTRACT

SUTHERLAND, AMANDA KAY. Generalization of the Cartan and Iwasawa Decompositions to $\mathrm{SL}_2(k)$. (Under the direction of Loek Helminck.)

The Cartan and Iwasawa decompositions of real reductive groups were developed over 100 years ago and have been extensively studied. They play a fundamental role in the representation theory of the groups and their corresponding symmetric spaces. These decompositions are defined by an involution with a compact fixed-point group, called a Cartan involution. Removing the requirement of having a Cartan involution or being defined over the real numbers causes this decomposition to break down. For an arbitrary involution, one can consider similar decompositions over other fields. We offer a generalization of the Cartan and Iwasawa decompositions for the algebraic group $G = \mathrm{SL}_2(k)$ over an arbitrary field k and a general involution. Additionally, we give a detailed analysis of the structure of the symmetric and extend symmetric spaces over any field and defined by a general involution.

Generalization of the Cartan and Iwasawa Decompositions to $\mathrm{SL}_2(k)$

by
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DEDICATION

For my Mom and Dad.

BIOGRAPHY

Mandi was born in Lexington, Kentucky in 1987 to Glenn and Liz Sutherland. She graduated from West Jessamine High School in Nicholasville, Kentucky in 2006. She graduated from the University of Louisville with a Bachelor's of Science in Mathematics in 2010. During her college career, she was an active sister in the Chi Omega fraternity and remains to this day an avid University of Louisville athletics fan.

Upon graduation, she moved to Raleigh, North Carolina to pursue her graduate career at North Carolina State University. She received her Master's of Science in Pure Mathematics in 2012 and her Doctor of Philosophy in Pure Mathematics in 2015. During her graduate career, she enjoyed her spare time long distance running and teaching yoga and cycling classes for University Recreation.

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TABLE OF CONTENTS

LIST OF TABLES

Chapter 1

Introduction

Decompositions of Lie groups allows us to study how they are constructed from subgroups and subsets within themselves. Popular Lie group decompositions include the Cartan, Iwasawa, Bruhat [Bor91], Langlands [Kna96], and Jordan-Chevalley [Hel78, Hum72]. These decompositions of real reductive Lie groups are used as tools in representation theory and in the structure of their corresponding real reductive symmetric spaces.

Originally developed in the late 1800's by Cartan and Killing, the Cartan decomposition is integral in the structure and representation theory of Lie groups and algebras. The Cartan decomposition of a real reductive Lie group G factors the group into $G = HQ$, where H is a maximal compact subgroup and Q is the symmetric space with respect to a Cartan involution. The Cartan decomposition is a generalization of the polar decomposition or singular value decomposition of matrices [Kna96].

Constructed by Kenkichi Iwasawa in the mid 1900's, the Iwasawa decomposition is a generalization of the Gram-Schmidt orthogonalization process. The Iwasawa decomposition of a real reductive Lie group G factors the group into analytical subgroups, $G = HP$, where H is a maximal compact subgroup and P is a minimal parabolic $\mathbb{R}$-subgroup. This decomposition results from combining the Cartan decomposition of a semisimple Lie algebra with the root space decomposition of its complexification [Hel78].

The classical Cartan and Iwasawa decompositions are constructed for real groups using an involution with a compact fixed-point group, called a Cartan involution. In this thesis, we work to generalize these notions to the algebraic group $\mathrm{SL}_2(k)$ defined over any field k with an arbitrary involution of the group. In particular, we consider the algebraically closed fields, finite fields, and the real, rational, and $\mathfrak{p}$-adic numbers.

Additionally, we define and discuss the factors of these groups with respect to an arbitrary involution of the group.

In Chapter 2, we review the necessary background material to set up our results. In [HW93], a generalization of the notion of a Cartan involution is given and the Cartan and Iwasawa decompositions are generalized to groups with such an involution, the details of which are included in this chapter. The assumptions made in [HW93] are more restrictive than in this thesis. We will also discuss the isomorphy classes of involutions of an algebraic group. These will be important for out results because the factors of these decompositions are defined by an involution of the group.

In Chapter 3, we introduce our initial combined generalization of the Cartan and Iwasawa decompositions. For $G = \mathrm{SL}_2(k)$, we generalize the Cartan and Iwasawa decompositions of G as $G = H\widetilde{Q}U$ for any field k of characteristic not two with an arbitrary involution θ. Here, H is the fixed-point group of θ, $\widetilde{Q} = \{g \mid \theta(g) = g^{-1}\}$ is the extended symmetric space, and U is a unipotent subgroup of G. We also discuss the results for fields of characteristic two.

After we have set up the notions of a generalized Cartan involution and decomposition, we compare these new ideas with the traditional. In Chapter 4, we delve into a detailed analysis of the symmetric and extended symmetric spaces of $G = \mathrm{SL}_2(k)$. We look at their relationship with respect to each other and determine cases in which they consist of semisimple elements.

After discussing the details of the factors H, $\widetilde{Q}$ and U, we can begin to simplify our result from Chapter 2. In Chapter 5 we begin to refine the decomposition. We study several specific cases for when we can simply the decomposition $G = H\widetilde{Q}U$. In particular, we find that for G paired with a Cartan involution, $G = HU$. We also deduce that for G with a non-Cartan involution, $G = HWU$, where H and U are defined as above and W is the Weyl group of a maximal torus.

In Chapter 6, the focus moves to analyzing the extended symmetric space on its own. In many cases, the extended symmetric spaces has a finite number of connected components, the symmetric space being the connected component containing the identity. We characterize the quotient space $\widetilde{Q}/Q$ and determine the coset representatives by determining the intersection of the symmetric space with a maximal torus of the group.

In Chapter 7, we begin to extend earlier results to the larger group, $\mathrm{SL}_n(k)$, $n \geq 3$. We discuss tools and challenges to generalizing the result from Chapter 2 to larger dimensions.

At the conclusion of this thesis, one should have an understanding of the construction of the Cartan and Iwasawa decompositions and their limitations. One should be able to apply this generalization to $G = \mathrm{SL}_2(k)$ and compare the properties of the traditional decompositions to this generalization. Lastly, one will recognize the structure of the extended symmetric space and determine its connected components in specific cases.

Chapter 2

Background

In this chapter, we review the relevant background material used in the research for this thesis. This includes the derivation of the Cartan and Iwasawa decompositions [Hel78], a characterization of the involutions of $\mathrm{SL}_2(k)$ [HW02] and $\mathrm{SL}_n(k)$ [HWD06], the structure of the symmetric space of $\mathrm{SL}_2(\mathbb{F}_p)$ [BHK$^+$], the notion of a generalized Cartan involution and a Cartan decomposition for groups with such an involution [HW93], H_k-conjugacy classes of tori in $\mathrm{SL}_2(k)$ [BH09, Nor13], and the polar decomposition of matrices [Ser02].

2.1   Symmetric spaces

Our main resources for the preliminaries concerning symmetric spaces and their generalizations comes from [Hel78]. The study of symmetric spaces began over 100 years ago and initially only concerned groups defined over the real numbers. In application, symmetric spaces over the real numbers are seen in representation theory [Car29], differential geometry, and mathematical physics. More recently, work including [HW93], have begun to generalize this notion, defining symmetric spaces of algebraic groups over arbitrary fields of characteristic not two, called symmetric k-varieties. Among other applications, these are used in representation theory [vdB88], algebraic geometry [DCP83, DCP85], automorphic functions, integrable systems [Dix94], number theory, perverse sheaves [Lus90, Gro92], harmonic analysis, and singularity theory [LV83, HS90].

To construct a symmetric space, consider the vector space $V = \mathbb{R}^n$. By choosing a basis for V, we get $\mathrm{GL}(V) \simeq \mathrm{GL}_n(\mathbb{R})$.

Definition 2.1.1. Let $M_n(k)$ represent the set of $n \times n$ matrices with entries in k, $\mathrm{GL}_n(k)$

4

be the *general linear group* (2.1.1), and $\mathrm{SL}_n(k)$ the *special linear group* (2.1.2).

$$\mathrm{GL}_(k) = \{g \in M_n(k) \mid \det(g) \neq 0\} = \{\text{invertible matrices}\} \tag{2.1.1}$$

$$\mathrm{SL}_(k) = \{g \in \mathrm{GL}_n(k) \mid \det(g) = 1\} \tag{2.1.2}$$

Because V is a Euclidean vector space it has an inner product with which we can define the symmetric bilinear form $B(x,y) = x^T M y$, for some symmetric $n \times n$ matrix M.

Definition 2.1.2. The *adjoint* of $A \in \mathrm{GL}(V)$ with respect to the bilinear form B is A' such that $B(A(x), y) = B(x, A'(y))$.

Note that $B(A(x), y) = B(x, A'(y))$ implies the following.

$$(Ax)^T M y = x^T A^T M y = x^T M A' y \tag{2.1.3}$$

Because (2.1.3) must hold for all $x, y \in V$, we may solve for the adjoint of A as $A' = M^{-1} A^T M$.

This leads to our initial definition of symmetric space.

Definition 2.1.3. The (Riemannian) *symmetric space* of $\mathrm{GL}(V)$ is

$$Q = \{AA' \mid A \in \mathrm{GL}(V)\}$$

where A' is the adjoint of $A \in \mathrm{GL}(V)$.

Consider the following subgroup H of $\mathrm{GL}(V)$ (2.1.4).

$$H = \{A \in \mathrm{GL}(V) \mid B(A(x), A(y)) = B(x,y)\} \tag{2.1.4}$$

If we let θ be the automorphism of order two defined as $\theta(A) := (A')^{-1}$, then H is the fixed-point group of θ by a chain of equivalent statements (2.1.5).

$$B(A(x), A(y)) = B(x, A'A(y) = B(x) \Leftrightarrow A'A = \mathrm{Id} \Leftrightarrow \theta(A) = A \tag{2.1.5}$$

Furthermore, $H = O(V, B)$, the group of orthogonal reflections. We will show that $g \in \mathrm{GL}(V)$ can be written uniquely as an orthogonal matrix and a positive definite symmetric matrix because we have a diffeomorphism from $H \times Q$ to G.

Initially, the notion of a symmetric space was constructed over the real numbers. There are several ways to define the symmetric space over an arbitrary field. We will extend this idea to any field by defining a symmetric space of a group G with respect to an involution of G.

Definition 2.1.4. An automorphism θ of a group G is an *involution* if we have the following:

1. $\theta \neq id$

2. $\theta^2 = id$

Here, id is the identity automorphism.

We define the *generalized symmetric space* of G as the image of the map $\tau : G \to G$ defined by $g \mapsto g\theta(g)^{-1}$, for some involution θ. We have $\text{Im}(G) = Q = \{g\theta(g)^{-1} \mid g \in G\}$.

Definition 2.1.5. For an involution θ of G, the *fixed-point group* H (2.1.6) is the subgroup of elements fixed under the action of θ.

$$H = \{g \in G \mid \theta(g) = g\} \tag{2.1.6}$$

Because the kernel of τ is the fixed-point group H, we have $Q \simeq G/H$ by the first isomorphism theorem. By definition, $G, H,$ and Q are defined over the algebraic closure of k, so we take G_k and H_k to be the k-rational points of G and H and $Q_k = \{g\theta(g)^{-1} \mid g \in G_k\}$.

Definition 2.1.6. The *extended symmetric space* $\widetilde{Q}$ of G (2.1.7) is the set of elements in G mapped to their inverse under the involution θ.

$$\widetilde{Q} = \{g \in G \mid \theta(g) = g^{-1}\} \tag{2.1.7}$$

Lemma 2.1.7. *The symmetric space is contained in the extended symmetric space.*

Proof. For $g\theta(g)^{-1} \in Q$, $\theta(g\theta(g)^{-1}) = (g\theta(g)^{-1})^{-1}$. (4.1.1)

$$\theta(g\theta(g)^{-1}) = \theta(g)\theta(\theta(g)^{-1}) = \theta(g)g^{-1} = (g\theta(g)^{-1})^{-1} \tag{2.1.8}$$

Hence $g\theta(g)^{-1} \in \widetilde{Q}$. $\qquad\square$

Remark 2.1.8. If the group G is defined over a field k with a topology, then Q is connected as it is the image of the continuous mapping τ on G. The symmetric space contains the identity, thus $\widetilde{Q}^\circ = Q$. In particular, $\widetilde{Q}^\circ = Q$ when k is algebraically closed, the real or $\mathfrak{p}$-adic numbers.

Example 2.1.9. If $M = \mathrm{Id}$, then $A' = A^T$ and $Q = \{AA^T\}$. Here, Q is the positive definite symmetric matrices. The extended symmetric space is the set of symmetric matrices, $\widetilde{Q^\theta} = \{A \in \mathrm{GL}(V) \mid A = A^T\}$. Furthermore, $Q = \widetilde{Q}^\circ$, the connected component of the extended symmetric space containing the identity. The other connected components depend on the number of negative eigenvalues of $A \in \widetilde{Q}$.

Riemannian symmetric spaces were first developed using involutions with a compact fixed-point group H over the real numbers. Real reductive symmetric spaces are the homogeneous spaces $G_\mathbb{R}/H_\mathbb{R}$, where $G_\mathbb{R}$ is a reductive Harish Chandra Lie group and $H_\mathbb{R}$ is the fixed-point group of an involution of $G_\mathbb{R}$ [HC84]. The idea of symmetric spaces has been generalized in several ways. First, by dropping the condition that H is compact, we have the study of *affine symmetric spaces* [Hel78, Ber57].

Example 2.1.10. Let $G = \mathrm{SL}_2(\mathbb{R})$ and θ the automorphism of G defined by (2.1.9) for all $g \in G$.

$$\theta(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.1.9}$$

Note that θ is an involution because $\theta^2(g) = \mathrm{Id}\, g\, \mathrm{Id} = g$, thus we can define the symmetric space of g with respect to θ (2.1.10).

$$Q = \left\{ \begin{pmatrix} a^2 - b^2 & -(ac - bd) \\ ac - bd & -c^2 + d^2 \end{pmatrix} \,\Bigg|\, ad - bc = 1 \right\} \tag{2.1.10}$$

Note that the fixed-point group H (2.1.11) in this case in not compact.

$$H = \left\{ \begin{pmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{pmatrix} \,\Bigg|\, x \in \mathbb{R} \right\} \tag{2.1.11}$$

From here, mathematicians generalized the notion to algebraically closed fields [Spr85, Ric82, Hel88, Vus74]. These are called *symmetric varieties* and are defined as the subvariety $Q = \{g\theta(g)^{-1} \mid g \in G\}$ of an algebraic group G with involution θ. As in the real case, $Q \simeq G/H$, and both are referred to as a symmetric variety.

Symmetric spaces were then generalized to algebraic groups defined over an arbitrary field, called *symmetric k-varieties* [HW93]. Introduced in the late 1900's, symmetric k-varieties generalize real reductive symmetric spaces to similar spaces over other fields. A symmetric k-variety of the algebraic group G defined over k with involution θ is $Q_k = \{g\theta(g)^{-1} \mid g \in G_k\}$. Again, $Q_k \simeq G_k/H_k$, and both are referred to as a symmetric k-variety.

For a reductive algebraic group G, we use the following generalization.

Definition 2.1.11. For G with involution θ and fixed-point group H, the *generalized symmetric space* is $Q = \{g\theta(g)^{-1} \mid g \in G\} \simeq G/H$.

Example 2.1.12. Let G be an arbitrary group G. Then G may be viewed as a symmetric space itself of the group $G \times G$. Consider the automorphism $\theta : G \times G \to G \times G$ defined by $(x, y) \mapsto (y, x)$. Then clearly θ has order two and thus is an involution. The fixed-point group of θ is $H = \{(x, y) \in G \times G \mid (y, x) = (x, y)\} = \{(x, x) \mid x \in G\}$. The symmetric space is $Q = \{(x, y) \cdot \theta(x, y)^{-1} \mid (x, y) \in G \times G\} = \{(xy^{-1}, yx^{-1}) \mid x, y \in G\} = \{(x, x^{-1}) \mid x \in G\} \simeq G$.

More recently, mathematicians have begun to expand this idea to automorphisms τ of G of order greater than two. While the symmetric space is defined similarly, $Q = \{g\tau(g)^{-1} \mid g \in G\}$, most of the properties regarding the symmetric space, fixed-point group, and decompositions no longer hold.

2.2 Polar decomposition of matrices

The Cartan decomposition is a generalization of the polar decomposition of matrices. The background material in this section comes for [Ser02]. The polar decomposition of a matrix is analogous to writing a complex number $x \in \mathbb{C}^*$ as $z = re^{i\theta}$, where r is a positive real number and $e^{i\theta}$ lies on the unit circle.

Notation 2.2.1. In this section, we will consider the fields $k = \mathbb{R}$ or $k = \mathbb{C}$. Let $\mathrm{M}_n(k)$ be the set of $n \times n$ matrices with entries in k. When $k = \mathbb{C}$, HPD_n and U_n will be the Hermitian positive definite matrices and unitary matrices, respectively. When $k = \mathbb{C}$, SPD_n and O_n will be the symmetric positive definite matrices and orthogonal matrices, respectively. We will consider the polar decomposition of matrices in $\mathrm{GL}_n(k)$, the general linear group.

The polar decomposition of $\mathrm{GL}_n(k)$ is given as follows.

Theorem 2.2.2 (Polar Decomposition)**.** *For* $A \in \mathrm{GL}_n(\mathbb{C})$, *there exists* $Q \in \mathrm{HPD}_n$ *and* $H \in \mathrm{U}_n$ *such that* $A = QH$. *For* $A \in \mathrm{GL}_n(\mathbb{R})$, *there exists* $Q \in \mathrm{SPD}_n$ *and* $H \in \mathrm{O}_n$ *such that* $A = QH$.

Theorem 2.2.3. *For* $Q \in \mathrm{HPD}_n$, *there exists a unique* $Q_0 \in \mathrm{HPD}_n$ *such that* $Q_0^2 = Q$. *If* $Q \in \mathrm{SPD}_n$, *then so is* Q_0. *The element* Q_0 *is called the* square root of Q *and is denoted* $Q_0 = \sqrt{Q}$.

Idea of Proof of 2.2.2. For $A \in \mathrm{GL}_n(\mathbb{C})$, $AA^* \in \mathrm{HPD}_n$. By Theorem 2.2.3, define $Q :=$ $\sqrt{AA^*} \in \mathrm{HPD}_n$ and define $H := Q^{-1}A \in \mathrm{U}_n$ to get $A = QH$. $\qquad\square$

Remark 2.2.4. The polar decomposition of $\mathrm{GL}_n(\mathbb{C})$ may also be described as $\mathrm{GL}_n(\mathbb{C}) = \mathrm{U}_n \times \mathrm{HPD}_n$, and similarly $\mathrm{GL}_n(\mathbb{R}) = \mathrm{O}_n \times \mathrm{SPD}_n$. Although $A = HQ \in \mathrm{GL}_n(k)$ can also be written as $A = Q_1 H_1$, H_1 is not necessarily the same as H, and the same for Q_1 and Q.

We may extend this polar decomposition to subgroups of $\mathrm{GL}_n(k)$.

Proposition 2.2.5. *Let* G *be a subgroup of* $\mathrm{GL}_n(\mathbb{C})$. *If* G *is stable under the map* $A \mapsto A^*$ *and for all* $Q \in \mathrm{HPD}_n$, *we have* $\sqrt{Q} \in \mathrm{HPD}_n$, *then* $G = (G \cap \mathrm{U}_n) \times (G \cap \mathrm{HPD}_n)$.
Let G *be a subgroup of* $\mathrm{GL}_n(\mathbb{R})$. *If* G *is stable under the map* $A \mapsto A^T$ *and for all* $Q \in \mathrm{SPD}_n$, *we have* $\sqrt{Q} \in \mathrm{SPD}_n$, *then* $G = (G \cap \mathrm{O}_n) \times (G \cap \mathrm{SPD}_n)$.

2.3 Traditional Cartan decomposition

The background provided on the construction of the Cartan decomposition comes primarily from [Hel78]. We begin with a description of the decomposition on the Lie algebra and then lift to the group level. In this section, let G be a real, connected, semisimple Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of G.

Definition 2.3.1. A group involution of G is a *Cartan involution* if it has a compact fixed-point group.

Remark 2.3.2. The involution $\theta(A) = (A')^{-1}$ for $A \in \mathrm{GL}_(V)$ from Section 2.1 is a Cartan involution.

By abuse of notation, we let $\theta := d\theta$ be the involution on the algebra level as well. For example, for $\theta(A) = (A')^{-1}$ for all $A \in \mathrm{GL}(V)$, we use $d\theta(x) = \theta(x) = -x'$ for all $x \in \mathfrak{g}$. It will be clear from context which θ we are using.

9

Given a Cartan involution θ of $\mathfrak{g}$, we will consider the +1 and −1 eigenspaces. The +1 eigenspace is $\mathfrak{h} = \mathrm{Lie}(H) = \{x \mid \theta(x) = x\}$, where H is the fixed point group of G. The −1 eigenspace is $\mathfrak{q} = \mathrm{Lie}(Q) = \{x \mid \theta(x) = -x\}$, where Q is the symmetric space of G. Then the eigenspace decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$.

Remark 2.3.3. The Lie algebras of the symmetric and extended symmetric spaces are equivalent, $\mathrm{Lie}(Q) = \mathrm{Lie}(\widetilde{Q^\theta})$. The topology disappears because the Lie algebra is the tangent space at a point and the eigenspaces are isomorphic. Hence, we do not define the notion of extended symmetric space on the algebra level.

Lemma 2.3.4. *For a compact Lie group we have the following.*

1. $\exp(\mathfrak{h}) = H^\circ$

2. $\exp(\mathfrak{q}) = Q$

This leads to the traditional Cartan decomposition of a Lie group.

Theorem 2.3.5 (Cartan Decomposition)**.** *Let G be a Lie group with Cartan involution θ then the map $(H \times Q) \to G$ is a diffeomorphism onto, defined by $(h, x) \mapsto h \cdot \exp(x)$ for $h \in H, x \in \mathfrak{q}$.*

Lemma 2.3.6. *For a compact connected Lie group G, we have the following.*

1. $G = G^{ss}$, *where G^{ss} is the set of semisimple elements in G.*

2. *All the eigenvalues of $\mathrm{Lie}(G)$ are imaginary.*

Remark 2.3.7. Note that for $g \in G$, g is uniquely written as $g = h \cdot \exp(x)$. The map is unique because $H \cap Q = \varnothing$ since the eigenvalues of Q must be real while eigenvalues of H are imaginary.

Example 2.3.8. Let $G = \mathrm{SL}_n(\mathbb{R})$ and θ the involution of G defined by $\theta(g) = (g^T)^{-1}$ for all $g \in G$. The fixed-point group H is SO_n (2.3.1), which is compact.

$$H = \{g \in G \mid \theta(g) = (g^T)^{-1}\} = \{g \in G \mid g^T = g^{-1}\} = \mathrm{SO}_n \qquad (2.3.1)$$

The symmetric space Q (2.3.2) is the set of positive-definite symmetric matrices.

$$Q = \{gg^T \mid g \in G\} \qquad (2.3.2)$$

By the Cartan decomposition, for $g \in G$, there exists unique orthogonal $h \in \mathrm{SO}_n$ and positive-definite symmetric $q \in Q$ such that $g = hq$.

Example 2.3.9. To highlight to necessity that the fixed-point group H is compact, consider the case from Example 2.1.10 where $G = \mathrm{SL}_2(\mathbb{R})$ with the involution $\theta(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}$. As demonstrated, H (2.3.3) is non-compact and we can determine the extended symmetric space $\widetilde{Q}$ (2.3.4) which contains the symmetric space Q.

$$H = \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} \;\middle|\; x^2 - y^2 = 1 \right\} \tag{2.3.3}$$

$$\widetilde{Q} = \left\{ \begin{pmatrix} a & b \\ -b & c \end{pmatrix} \;\middle|\; ac + b^2 = 1 \right\} \tag{2.3.4}$$

Let $g = \begin{pmatrix} 1 & 2(\sqrt{5}-3)^{-1} \\ \frac{1}{2}(\sqrt{5}-3) & 2 \end{pmatrix} \in G_{\mathbb{R}}$. We will show that g is not in the larger set $H\widetilde{Q}$, thus is not in HQ. If $g \in H\widetilde{Q}$, then there exists $h^{-1} = \begin{pmatrix} x & y \\ y & x \end{pmatrix} \in H$ such that $h^{-1}g \in \widetilde{Q}$. Solving h^{-1} such that $\theta(h^{-1}g) = (h^{-1}g)^{-1}$ (2.3.5) implies $x = y$, hence $h^{-1} \notin H$, therefore g cannot be in $H\widetilde{Q}$ and thus $g \notin HQ$.

$$\theta(h^{-1}g) = \begin{pmatrix} \frac{2x}{\sqrt{5}-3} + 2x & y + x\left(\frac{\sqrt{5}-3}{2}\right) \\ \frac{2x}{\sqrt{5}-3} + 2y & x + y\left(\frac{\sqrt{5}-3}{2}\right) \end{pmatrix} = \begin{pmatrix} \frac{2y}{\sqrt{5}-3} + 2x & -2y - \frac{2x}{\sqrt{5}-3} \\ -\left(\frac{\sqrt{5}-3}{2}\right)x - y & x + y\left(\frac{\sqrt{5}-3}{2}\right) \end{pmatrix} = (h^{-1}g)^{-1} \quad (2.3.5)$$

The Cartan decomposition is derived from the study of real groups. Real groups are convenient because of the relationship between the group and its Lie algebra. We must have that the exponential map is defined and its image closed. Thus far, we have only considered the real group $\mathrm{GL}(V)$. The following theorem allows us to apply the Cartan decomposition to any real Lie group.

Theorem 2.3.10 (Ado's Theorem). *Any Lie group is isomorphic to a subgroup of* $\mathrm{GL}(V)$.

Once we have a subgroup $G \subset \mathrm{GL}(V)$ with Lie algebra $\mathfrak{g}$, we can align it appropriately

with the Cartan decomposition (2.3.6) by either conjugating the group or the involution.

$$G = (G \cap H).(\exp(\mathfrak{g} \cap \mathfrak{q})) \qquad (2.3.6)$$

Recall that an automorphism θ of order two of a real reductive Lie group with a maximal compact fixed-point group is a Cartan involution. The involution θ of $\mathrm{GL}(V)$ defined by $\theta(A) = A'$ is a Cartan involution, which induces a Cartan Decomposition. Conversely, given any maximal compact subgroup H, there exists a (Cartan) involution θ such that H is the fixed-point group of θ.

Theorem 2.3.11 (Cartan). *Any reductive real Lie group has a unique (up to conjugation) Cartan involution.*

Given a real Lie group G, we will construct its Cartan involution using its Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is a real Lie algebra, then $g \otimes_{\mathbb{R}} \mathbb{C} = \bar{\mathfrak{g}}$ is a complex Lie algebra with real subspace $\mathfrak{g}$. Moreover, $\bar{\mathfrak{g}}$ is two copies of $\mathfrak{g}$, $\bar{\mathfrak{g}} = \mathfrak{g} \oplus i\mathfrak{g}$.

Definition 2.3.12. A map $\sigma : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$ is a *conjugation* if we have the following.

1. $\sigma^2 = id$

2. $\sigma(x + iy) = x - iy$, for all $x + iy \in \mathfrak{g}$

3. $\sigma(\lambda x)$ for all $x \in \mathfrak{g}$, $\lambda \in \mathbb{C}$

i.e. σ is $\mathbb{R}$-linear, but not $\mathbb{C}$-linear.

For $\mathfrak{g} \subset \bar{\mathfrak{g}}$, there is a conjugation σ such that $\mathfrak{g} = \{x \in \bar{g} \mid \sigma(x) = x\}$.

Definition 2.3.13. A Lie algebra $\mathfrak{g}$ is *compact* if the Killing form is negative definite on $\mathfrak{g}$.

For an alternative definition, we say $\mathfrak{g}$ is compact if $\mathrm{ad}(x)$ has imaginary eigenvalues for all $x \in \mathfrak{g}$. Hence, $\exp(\mathfrak{g})$ lies on the unit circle and is compact.

Theorem 2.3.14. *If $\bar{\mathfrak{g}}$ is a complex reductive Lie algebra, then $\bar{\mathfrak{g}}$ has a unique (up to isomorphism) compact subalgebra $\mathfrak{g}$, called a compact real form.*

Theorem 2.3.15. *Let $\mathfrak{g} \subset \bar{\mathfrak{g}}$ be a real Lie algebra with conjugation σ, then there exists compact real form $\mathfrak{u}$ of $\bar{\mathfrak{g}}$ with conjugation τ such that $\sigma\tau = \tau\sigma$. Moreover, $\theta = \sigma\tau$ is a Cartan involution.*

Lemma 2.3.16. *Let $\theta = \sigma\tau = \tau\sigma$, then θ is an involution of $\bar{\mathfrak{g}}$ which keeps $\mathfrak{g}$ invariant.*

We call θ the Cartan involution of $\mathfrak{g}$ and will show it induces a Cartan decomposition of $\mathfrak{g}$. Consider the $+1$ (2.3.7) and -1 (2.3.8) eigenspaces of θ.

$$\{x \in \mathfrak{g} \mid \theta(x) = x\} = \{x \in \mathfrak{g} \mid \tau(x) = x\} = \mathfrak{g} \cap \mathfrak{u} \tag{2.3.7}$$

$$\{x \in \mathfrak{g} \mid \theta(x) = -x\} = \{x \in \mathfrak{g} \mid \tau(x) = -x\} = \mathfrak{g} \cap i\mathfrak{u} \tag{2.3.8}$$

Because $\bar{\mathfrak{g}} = \mathfrak{u} \oplus i\mathfrak{u}$, we have $\mathfrak{g} = (\mathfrak{g} \cap \mathfrak{u}) \oplus (\mathfrak{g} \cap i\mathfrak{u})$. Hence, $\mathfrak{u}$ a compact real form implies the Killing form κ is negative definite on $\mathfrak{u} \times \mathfrak{u}$ and positive definite on $i\mathfrak{u} \times i\mathfrak{u}$. The Killing form is negative definite on $\mathfrak{g} \cap \mathfrak{u}$ and therefore $\mathfrak{g} \cap \mathfrak{u}$ is a maximal compact Lie algebra. From here, we can exponentiate to get the Cartan decomposition on the group.

Theorem 2.3.17. *All compact real forms of $\bar{\mathfrak{g}}$ are conjugate.*

Corollary 2.3.18. *The Cartan involution of $\mathfrak{g}$ is unique up to isomorphism.*

Let G be a semisimple, connected Lie group with Cartan involution θ. Because of the compactness of the fixed-point group H, G has a nice structure. We have the Cartan decomposition, $\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, where $\mathfrak{h}$ and $\mathfrak{q}$ are as defined above. Because the Killing form is negative definite on $\mathfrak{h}$ and positive definite on $\mathfrak{q}$, both $\mathfrak{h}$ and $\mathfrak{q}$ consist of semisimple elements.

Theorem 2.3.19. *The symmetric space of G with the Cartan involution, $\exp(\mathfrak{q}) = Q = \{g\theta(g)^{-1} \mid g \in G\}$, consists of semisimple elements.*

Example 2.3.20. Let $G = \mathrm{SL}_2(\mathbb{R})$ and θ be the involution $\theta(g) = (g^T)^{-1}$ for all $g \in G$. Then the symmetric space (2.3.9) is the set of positive definite symmetric matrices, hence is semisimple.

$$Q = \{g((g^T)^{-1})^{-1} \mid g \in G\} = \{gg^T \mid g \in G\} = \left\{ \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix} \middle| ad - bc = 1 \right\} \tag{2.3.9}$$

2.4 Parabolics acting on symmetric spaces

The orbits of parabolic subgroups acting on symmetric spaces have been characterized in a number of ways. One can consider the P-orbits acting on the symmetric variety G/H

by twisted conjugation, as H-orbits acting of G/P by conjugation, or the $(P \times H)$-orbits on G, where P is a parabolic subgroup and H is the fixed-point group of an involution. These actions are studied in detail in [BH00]. For an arbitrary field, as in the case we are interested in, [HW93] has characterized these orbits.

Notation 2.4.1. Let G be a connected, reductive algebraic group defined over a field k or characteristic not two. For a subgroup A of G, let A_k be the k-rational points of A.

Definition 2.4.2. A subgroup $B \subset G$ is a *Borel subgroup* if it is a connected maximal solvable subgroup. A subgroup $P \subset G$ is a *parabolic subgroup* if it contains a Borel subgroup. A subgroup $P \subset G$ is a *parabolic k-subgroup* if it is a parabolic subgroup defined over k.

Definition 2.4.3. A *torus* is a semisimple abelian subgroup. A subgroup is *k-split* if it can be diagonalized over the base field k. For an involution θ, a subgroup A is *θ-split* if $\theta(a) = a^{-1}$ for all $a \in A$. A subgroup is *(θ, k)-split* if it is both k-split and θ-split. A subgroup is *θ-stable* if it is invariant under θ. A subgroup is *k-anisotropic* if it does not contain a k-split torus.

Remark 2.4.4. The extended symmetric space $\widetilde{Q}$ is θ-split by definition.

Theorem 2.4.5 (Richardson [Ric82])**.** *All maximal θ-split tori are conjugate under the fixed-point group of θ.*

Remark 2.4.6. Theorem 2.4.5 does not hold for non-algebraically closed fields. Over other fields, there may be more than one conjugacy class. For example, consider $G = \mathrm{SL}_2(\mathbb{Q})$ with the Cartan involution θ defined by $\theta(g) = (g^T)^{-1}$ for all $g \in G$. There are infinitely many $H_{\mathbb{Q}}$-conjugacy classes of maximal θ-split tori.

Example 2.4.7. Let $G = \mathrm{GL}_n(\mathbb{R})$ with the involution θ defined by $\theta(g) = (g^T)^{-1}$ for all $g \in G$. Then the fixed-point group of θ is $H = \{g \in G \mid \theta(g) = g\} = \mathrm{O}_n$, the symmetric space is $Q = \{g \in G \mid g^T = g, g \text{ is positive definite}\}$, and the extended symmetric space is the set of symmetric matrices. Let A be a maximal θ-split torus, then $A \in \widetilde{Q}$. We can diagonalize A via O_n. Therefore, all maximal θ-split tori are O_n-conjugate to the diagonal matrices.

2.4.1 Iwasawa decomposition

Our main resources for the Iwasawa decomposition are Knapp and Helgason [Kna96, Hel78]. The Iwasawa decomposition is another global decomposition, the advantage of

this decomposition is that the factors are analytical subgroups.

Theorem 2.4.8. *Let G be defined over $k = \mathbb{R}$ with a Cartan involution θ and $P \subset G$ a minimal parabolic $\mathbb{R}$-subgroup. Then $P_{\mathbb{R}}$ has a unique open orbit in $G_{\mathbb{R}}/H_{\mathbb{R}} = Q_{\mathbb{R}}$.*

Theorem 2.4.9 (Iwasawa Decomposition)**.** *Let G be a real semisimple Lie group and θ a Cartan involution of G. Let H be the fixed-point group of θ and P a minimal parabolic subgroup. Then θ induces the Iwasawa decomposition $G = HP$.*

Because any group can be written as a semisimple (or reductive) part and a unipotent part, we can write $P_{\mathbb{R}}$ as $P_{\mathbb{R}} = Z_{G_{\mathbb{R}}}(A_{\mathbb{R}})U$, where $U = (R_u(P))_{\mathbb{R}}$ is the unipotent radical of P and A is a maximal $\mathbb{R}$-split torus. We can assume $A_{\mathbb{R}} \subset Q_{\mathbb{R}}$, thus $Z_G(A_{\mathbb{R}}) = Z_{H_{\mathbb{R}}}(A_{\mathbb{R}})A_{\mathbb{R}}$. This implies $P_{\mathbb{R}} = Z_{H_{\mathbb{R}}}(A_{\mathbb{R}})U_{\mathbb{R}}$. Finally, we arrive at the Iwasawa decomposition of $G_{\mathbb{R}}$, $G_{\mathbb{R}} = H_{\mathbb{R}}A_{\mathbb{R}}U_{\mathbb{R}}$.

The Iwasawa decomposition of a Lie algebra arose as a combination its Cartan decomposition and the complexification of its root space decomposition. This decomposition is a generalization of the Gram-Schmidt orthogonalization process. The factors, while similar to those of the Cartan decomposition, are closed subgroups.

To construct the Iwasawa decomposition of a Lie group G, we first look at the decomposition on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, and then lift it through the exponential map to G. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the Cartan decomposition of G and θ the Cartan involution of $\mathfrak{g}$ (and respectively G) as above. Let $\mathfrak{a}$ be a maximal torus contained in $\mathfrak{q}$, which exists because $\mathfrak{q}$ is finite. For a root λ of $\mathfrak{g}$, we have the restricted root space $g_\lambda = \{X \in \mathfrak{g} \mid (\mathrm{ad}\, Y)X = \lambda(Y)X \text{ for all } Y \in \mathfrak{a}\}$.

Proposition 2.4.10. *The restricted root space decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \Phi} \mathfrak{g}_\lambda$, where Φ is the set of roots such that $\mathfrak{g}_\lambda \neq 0$.*

Let $\mathfrak{n} = \bigoplus_{\lambda \in \Phi^+} g_\lambda$, where Φ^+ is the set of positive roots of $\mathfrak{g}$. Then $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$, which leads to the Iwasawa decomposition of $\mathfrak{g}$.

Theorem 2.4.11 (Iwasawa decomposition of a Lie algebra)**.** *For a Lie algebra $\mathfrak{g}$ with a Cartan decomposition, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{h}, \mathfrak{a}$, and $\mathfrak{n}$ are as above. Furthermore, $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable Lie subalgebra of $\mathfrak{g}$.*

From Theorem 2.4.11, we can lift to the Iwasawa decomposition on the group level.

Theorem 2.4.12 (Iwasawa decomposition)**.** *Let G be a semisimple Lie group with the algebra-level Iwasawa decomposition $\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{n}$. If H, A, U are the analytic subgroups of G such that $\mathrm{Lie}(H) = \mathfrak{h}, \mathrm{Lie}(A) = \mathfrak{a}$ and $\mathrm{Lie}(U) = \mathfrak{n}$, then $G = HAU$.*

Remark 2.4.13. Here, H is the fixed-point group of the Cartan involution θ, A is a maximal (θ, k)-split torus, and U is a unipotent subgroup.

Example 2.4.14. Let $G = \mathrm{SL}_n(\mathbb{R})$ with the Cartan involution θ defined by $\theta(g) = (g^T)^{-1}$ for all $g \in G$. The fixed-point group H is the special orthogonal group $\mathrm{SO}_n(\mathbb{R})$. Because G is a $\mathbb{R}$-split group, we may take the minimal parabolic subgroup to be the Borel subgroup consisting of upper triangular matrices. The unipotent radical of P, U, is the upper triangular matrices with ones on the diagonal. Finally, let the maximal torus A be the subgroup of diagonal matrices in G. By the Iwasawa decomposition, for $g \in G$, there exists orthogonal $h \in H$, diagonal $a \in A$, and unipotent $u \in U$ such that $g = hau$.

Using the action of the parabolic subgroups on the symmetric space, we can form the connection between the Cartan and Iwasawa decompositions. First, we give an alternate Cartan decomposition of G defined over $\mathbb{R}$.

Let G be defined over $k = \mathbb{R}$ and θ a Cartan involution of G, i.e. the fixed-point group H is compact. By the Cartan decomposition of G, we have $G = H_{\mathbb{R}} Q_{\mathbb{R}} = Q_{\mathbb{R}} H_{\mathbb{R}}$.

Lemma 2.4.15. *All maximal $(\theta, \mathbb{R})$-split tori are $H_{\mathbb{R}}^{\circ}$-conjugate.*

Let A_1 and A_2 be $(\theta, \mathbb{R})$-split tori in G. From [BT72, BT65], all maximal k-split tori are G_k-conjugate. Hence there exists $g \in G_{\mathbb{R}}$, such that $gA_1g^{-1} = A_2$. Because $g = qh$, for some $q \in Q_{\mathbb{R}}$ and $h \in H_{\mathbb{R}}$, and hA_1h^{-1} is maximal $(\theta, \mathbb{R})$-split, we may assume $g = q$. Hence, $qA_1q^{-1} = A_2$ where $q = \exp(x)$ for some $x \in \mathrm{Lie}(Q) = \mathfrak{q}$. On the Lie algebra if $x, y \in \mathfrak{q}$, then $[x, y] \in \mathfrak{h}$. Thus for $A_1 = \exp(\mathfrak{a}_1)$, we have $qA_1q^{-1} = \exp(x)\exp(\mathfrak{a}_1)\exp(x)^{-1} = \exp([x, \mathfrak{a}_1])$. However, $[x, \mathfrak{a}_1] \subset \mathfrak{h}$ which can not be the case, hence $q \in Q$ must be the identity. All elements of Q are $\mathbb{R}$-split semisimple, so they are contained in a torus, therefore $Q = \{hAh^{-1} \mid h \in H\}$. This implies $G = HQ = H \cdot HAH = HAH$.

2.5 Parabolics acting on generalized symmetric spaces

For a general field k, the orbits of parabolics on generalized symmetric spaces are not as simple as the case in Section 2.4, where $k = \mathbb{R}$ and θ is a Cartan involution. In the

section let G be an algebraic group defined over an arbitrary k and θ a generalized Cartan involution, where the fixed-point group is k-anisotropic rather than compact. Our primary resources in this section will be [HW93] and [BH09]. The fundamental tool for studying the action of minimal parabolic k-subgroups on an algebraic group defined over k is the Bruhat decomposition.

Definition 2.5.1. For a torus $T \subset G$, the *Weyl group* of T in G is $W_G(T) = N_G(T)/Z_G(T)$, where $N_G(T) = \{g \in G \mid gtg^{-1} = t \; \forall \; t \in T\}$ is the normalizer of T and $Z_G(T) = \{g \in G \mid gtg^{-1} \in T\}$ is the centralizer of T. For a subset $A \subset G$, $W_A(T) = A \cap W_G(T)$.

Theorem 2.5.2 (Bruhat Decomposition)**.** *Let G be defined over a field k with minimal parabolic subgroup P and A a maximal k-split torus contained in P. Then G_k decomposes as the disjoint union of double cosets of P_k parameterized by $W_G(A)$ (2.5.1).*

$$G_k = \biguplus_{\omega \in W_G(A)} P_k \omega P_k \tag{2.5.1}$$

Remark 2.5.3. For a k-split group G, such as $\mathrm{GL}_n(k)$, $P = B$ is a Borel subgroup and $A = T$ is a maximal torus. For G_k, the Bruhat decomposition is $G_k = \biguplus_{\omega \in W_G(T)} = B_k \omega B_k$.

Example 2.5.4. Let $G = \mathrm{SL}_2(k)$ with the Cartan involution θ defined by $\theta(g) = (g^T)^{-1}$, T the subgroup of diagonal matrices (2.5.2), and $B \supset T$ the Borel subgroup of upper triangular matrices (2.5.3). Then the Weyl group in G of T is $W(T)$ (2.5.4) and we can determine the Bruhat decomposition of G_k (2.5.5).

$$T_k = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| \; a \in k^* \right\} \tag{2.5.2}$$

$$B_k = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \middle| \; x \in k, y \in k^* \right\} \tag{2.5.3}$$

$$W(T) = \left\{ \mathrm{Id}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \right\} \tag{2.5.4}$$

$$G_k = B_k \uplus B_k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B_k \tag{2.5.5}$$

To analyze the structure of the symmetric spaces, we must know the action of H_k on $G_k/H_k = Q_k$ and the action of the parabolic k-subgroups P_k, on G_k/H_k.

Proposition 2.5.5. *Let $\{A_i \mid i \in I\}$ be representatives of the H_k-conjugacy classes of θ-stable maximal k-split tori in G. Then $H_k \backslash G_k / P_k \simeq \bigcup_{i \in I} W_{H_k}(A_i) / W_{G_k}(A_i)$.*

Example 2.5.6. By Theorem 2.4.8, $|P_\mathbb{R}/G_\mathbb{R}\backslash H_\mathbb{R}| = 1$. The unique orbit contains the identity, thus $G_\mathbb{R} = P_\mathbb{R} H_\mathbb{R} = H_\mathbb{R} P_\mathbb{R}$.

Proposition 2.5.7. *Let P be a minimal parabolic θ-split parabolic k-subgroup of G, A a θ-stable maximal k-split torus of P, then the following are equivalent.*

1. $g \in G_k \cap H_k P_k$

2. $g \in \{x \in G_k \mid x^{-1}\theta(x) \in N_{Z_G(A^-)}(A)\}P_k$, *where A^- are the θ-split elements in A*

3. $g \in G_k$ *and $gP_k g^{-1}$ is a θ-split parabolic k-subgroup of G_k*

Example 2.5.8. Usually $|H_k \backslash G_k / P_k|$ is infinite and minimal parabolic subgroups are not H_k-conjugate. Let $G = \mathrm{SL}_2(\mathbb{Q})$ with the Cartan involution θ defined by $\theta(g) = (g^T)^{-1}$ for all $g \in G$, B the Borel subgroup of upper triangular matrices, and A the subgroup of diagonal matrices. Here, we will consider "twisted orbits". For $g, x \in G$, we define the twisted action of g on x as $g * x := gx\theta(g)^{-1}$. There are only a finite number of twisted orbits in $\widetilde{Q}$ and each orbit is closed.

We define $a_1, a_2 \in A_\mathbb{Q}$ (2.5.6).

$$a_1 = \begin{pmatrix} x_1^2 + y_1^2 & 0 \\ 0 & (x_1^2 + y_1^2)^{-1} \end{pmatrix} \quad a_2 = \begin{pmatrix} x_2^2 + y_2^2 & 0 \\ 0 & (x_2^2 + y_2^2)^{-1} \end{pmatrix} \in A \qquad (2.5.6)$$

Then a_1 and a_2 are in the same twisted $B_\mathbb{Q}$ orbit if and only if $(x_1^2 + y_1^2)^{-1}(x_2^2 + y_2^2) \in (\mathbb{Q}^*)^2$. If follows that $|H_\mathbb{Q} \backslash G_\mathbb{Q} / B_\mathbb{Q}| = \left| \mathbb{Q}^*/(\mathbb{Q}^*)^2 \right| = \infty$.

Furthermore, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $g\theta(g)^{-1} \in A = N_{Z_G(A^-)}(A)$ if and only if $ab + cd = 0$.

Choose $u, v, t \in \mathbb{Q}$ such that $uv(1+t)^2 = 1$ and $(1+t)^2 \notin (\mathbb{Q}^*)^2$. If we set $g = \begin{pmatrix} vt & v \\ -u & ut \end{pmatrix}$,

then $g\theta(g)^{-1} = \begin{pmatrix} v^2(1+t^2) & 0 \\ 0 & u^2(1+t^2) \end{pmatrix}$. By the above proposition, gBg^{-1} is a θ-split maximal parabolic $\mathbb{Q}$-subgroup of G. Then $g \notin H_\mathbb{Q}$ because $(1+t^2)$ is not a square. Hence the Borel (minimal parabolic) $\mathbb{Q}$-subgroups are not $H_\mathbb{Q}$-conjugate to each other.

2.6 Generalization of the Cartan decomposition

The traditional Cartan decomposition is defined for a Lie group G over $k = \mathbb{C}$ or $\mathbb{R}$. An involution θ with a compact fixed-point group H is a Cartan involution. For the symmetric space $Q = \{g\theta(g)^{-1} \mid g \in G\}$, the Cartan decomposition of G is $G = HQ$. For involutions of G with a non-compact fixed-point group H or for G an algebraic group defined over $k \neq \mathbb{R}$ or $\mathbb{C}$, we no longer have this decomposition.

In [HW93], the notions of a generalized Cartan involution and a generalized Cartan decomposition for an algebraic group are defined. Specifically, conditions are given in which the Cartan decomposition implies the Iwasawa decomposition and vice versa.

Notation 2.6.1. In this section, let G be a connected reductive algebraic group defined over a field k with involution θ and H the fixed-point group of θ. Let τ denote the map on G defined by $\tau(g) = g\theta(g)^{-1}$ for $g \in G$. For a group A, let A_k be the k-rational points of A. Let P be a minimal parabolic k-subgroup, $U = R_u(P)$ the unipotent radical of P, and A a θ-stable maximal k-split torus contained in P.

Proposition 2.6.2. *If H_k is k-anisotropic, then the following are equivalent.*

1. $(Z_G(A)H)_k = A_k H_k$

2. $G_k = U_k A_k H_k = U_k A_k H_k^\circ$

Remark 2.6.3. Proposition 2.6.2 is a generalization of the Iwasawa decomposition to algebraic groups.

Corollary 2.6.4. *If H_k is k-anisotropic and $G_k = U_k A_k H_k$, then we have the following.*

1. *A is a maximal θ-split torus and therefore A is a maximal (θ, k)-split torus.*

2. *All maximal (θ, k)-split tori are H_k-conjugate.*

3. *$N_{G_k}(A) = A_k N_{H_k}(A)$ and therefore $W_{G_k}(A) = W_{H_k}(A)$.*

Proposition 2.6.5. *If H_k is k-anisotropic and $G_k = H_k A_k H_k$, then the following are equivalent.*

1. *$Q = \{g\theta(g)^{-1} \mid g \in G_k\}$ consists of k-split semisimple elements.*

2. *$G_k = U_k A_k H_k$*

3. If $(k^*)^2 = (k^*)^4$, then $G_k = QH_k$.

Remark 2.6.6. For a general field k, $G_k = H_k A_k H_k$ does not imply $G_k = H_k Q$.

The following result provides the conditions in which both versions or the Cartan decomposition and the Iwasawa decomposition are equivalent, which leads to the definition of a generalized Cartan involution.

Proposition 2.6.7. *If H_k is k-anisotropic and $(k^*)^2 = (k^*)^4$, then the following are equivalent.*

1. $G_k = H_k A_k H_k$

2. $G_k = U_k A_k H_k$ and Q consists of k-split semisimple elements.

3. $G_k = QH_k$ and Q consists of k-split semisimple elements.

Definition 2.6.8. For an involution θ of G, we call θ a *generalized Cartan involution* of G if H is k-anisotropic, $(k^*)^2 = (k^*)^4$, and G_k satisfies the conditions in Proposition 2.6.7.

Remark 2.6.9. By abuse of terminology, we refer to a generalized Cartan involution as simply a Cartan involution. When $k \neq \mathbb{R}$, it is assumed we mean generalized Cartan involution.

2.7 Generalizing the concept of Cartan involution

Recall from Section 2.1, if θ is a Cartan involution of a real Lie group, then the symmetric space consists of semisimple elements. When we generalize to an algebraic group with a Cartan involution, we have similar properties. In [HW93], this idea is studied extensively, we summarize the results.

Proposition 2.7.1. *Let G be a connected reductive algebraic group defined over a field k and θ an involution of G. For the fixed-point group H and minimal parabolic k-subgroup P, $G_k = (HP)_k$.*

Proposition 2.7.2. *Let G be a connected reductive algebraic group defined over a field k with characteristic zero and θ an involution of G. Then we have the following.*

1. The symmetric space Q_k of G_k consists of semi-simple elements.

2. The symmetric space $\mathfrak{q}$ of $\mathfrak{g}$ consists of semi-simple elements.

2.8 Isomorphy classes of involutions of $\mathrm{SL}_2(k)$

The symmetric spaces of a connected reductive algebraic group $\mathcal{G}$ are defined by the automorphisms of order two, called involutions, of $\mathcal{G}$. To work with the symmetric spaces of $G = \mathrm{SL}_2(k)$ we must know what the involutions are and, more importantly, the k-isomorphy classes of these automorphisms. A full classification of these involutions is offered in[HW02].

Notation 2.8.1. In this section, let k be a field of characteristic not two and $\bar{k}$ the algebraic closure of k. Let $\mathcal{G}$ be a generic group, $G = \mathrm{SL}_2(\bar{k})$, and G_k the k-rational points of G.

Definition 2.8.2. A group automorphism ϕ of $\mathcal{G}$ is an *inner automorphism* of $\mathcal{G}$ if there exists $g \in \mathcal{G}$ such that $\phi(X) = gXg^{-1}$ for all $X \in \mathcal{G}$.

Notation 2.8.3. For $B \in \mathrm{GL}_2(k)$, let $\mathrm{Inn}(B)$ denote the inner automorphism of $\mathrm{GL}_2(k)$ defined by $\mathrm{Inn}(B)(X) = BXB^{-1}$; $\mathrm{Inn}(B)$ is an automorphism of G. Let $\mathrm{Aut}(G)$ denote the automorphisms of G and $\mathrm{Aut}(G, G_k)$ the group of automorphisms of G which keep G_k invariant.

Definition 2.8.4. Two automorphisms $\phi, \theta \in \mathrm{Aut}(G)$ are *isomorphic* if there exists an automorphism $\chi \in \mathrm{Aut}(G)$ such that $\chi \phi \chi^{-1} = \theta$; denoted $\phi \simeq \theta$. Two automorphisms $\phi, \theta \in \mathrm{Aut}(G, G_k)$ are *k-isomorphic* if there exists an automorphism $\chi \in \mathrm{Aut}(G, G_k)$ such that $\chi \phi \chi^{-1} = \theta$. This is denoted $\phi \underset{k}{\simeq} \theta$ or simply $\phi \simeq \theta$ when the field k is clear from context.

From [Bor91], we have the following.

Lemma 2.8.5. *For $\phi \in \mathrm{Aut}(G)$, there exists $B \in \mathrm{GL}_2(\bar{k})$ such that $\phi \simeq \mathrm{Inn}(B)$.*

Remark 2.8.6. For a non-algebraically closed field k, if $\phi \in \mathrm{Aut}(G, G_k)$, there exists $B \in \mathrm{GL}_2(\bar{k})$ such that $\mathrm{Inn}(B)\big|_{G_k} = \phi$.

Definition 2.8.7. The *order* of an automorphism ϕ is the smallest positive integer n such that $\phi^n = id$. An automorphism ϕ of order two is an *involution*.

Lemma 2.8.8. *For $A \in G_k, \mathrm{Inn}(A) \in \mathrm{Aut}(G, G_k)$ if and only if $A = pB$ for some $p \in \bar{k}$ and $B \in G_k$.*

This allows us to recognize which automorphisms of G will be automorphisms of G_k. From here, we must determine which of these automorphisms are of order two.

Lemma 2.8.9. *Let $\phi \in \mathrm{Aut}(G, G_k)$ be an involution, then there exists $A = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \in$ $\mathrm{GL}_2(k)$ such that $\mathrm{Inn}(A)\big|_{G_k} = \phi$.*

Because we can multiply A by any constant and not affect the action of the automorphism, we can determine the representatives of the k-isomorphy classes.

Corollary 2.8.10. *The k-isomorphy classes of involutions of G_k can be represented by $\mathrm{Inn}(B)$, where $B = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \in \mathrm{GL}_2(k)$.*

Lemma 2.8.11. *Let $\mathrm{Inn}(B_1), \mathrm{Inn}(B_2) \in \mathrm{Aut}(G_k)$ (2.8.1).*

$$B_1 = \begin{pmatrix} 0 & 1 \\ b_1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 \\ b_2 & 0 \end{pmatrix} \tag{2.8.1}$$

Then $\mathrm{Inn}(B_1) \underset{k}{\simeq} \mathrm{Inn}(B_2)$ if and only if $\dfrac{b_1}{b_2}$ is a square in k.

Notation 2.8.12. For a field k, let k^* denote the product group of non-zero elements in k and $(k^*)^2$ the normal subgroup of squares in k^* defined by $(k^*)^2 = \{a^2 \mid a \in k^*\}$. The quotient group $k^*/(k^*)^2$ represents the set of square classes in k.

Corollary 2.8.13. *The number of isomorphy classes of involutions of G_k is $|k^*/(k^*)^2|$.*

Notation 2.8.14. Let m be a representative of the square class of $\overline{m}$ in k. We will use θ_m to denote the involution $\mathrm{Inn}(M)$ of G_k (2.8.2).

$$M = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix} \tag{2.8.2}$$

Remark 2.8.15. For all involutions $\theta \in \mathrm{Aut}(G, G_k)$, we can assume $\theta \underset{k}{\simeq} \theta_m$ for some $m \in k^*$, the representative of the square class $\overline{m}$. For the class of squares, we use θ_1.

Example 2.8.16 (Real numbers). Let $k = \mathbb{R}$. The square classes of $\mathbb{R}$ are $\mathbb{R}^*/(\mathbb{R}^*)^2 = \{1, -1\}$. Up to isomorphy, there are two involutions of $G_{\mathbb{R}}$; specifically θ_1 and θ_{-1}.

The involution θ_{-1} is equivalent to θ in $G_{\mathbb{R}}$, where θ is defined by $\theta(g) = (g^T)^{-1}$ for all $g \in G_{\mathbb{R}}$.

$$\theta(g) = \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}^T\right)^{-1} = \begin{pmatrix} w & -z \\ -y & x \end{pmatrix}$$

$$\theta_{-1}(g) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} w & -z \\ -y & x \end{pmatrix}$$

Example 2.8.17 (Finite fields)**.** For $k = \mathbb{F}_q$ for some $q = p^r$, p an odd prime, $\mathbb{F}_q^*/(\mathbb{F}_q^*)^2 \simeq \mathbb{Z}_2$. There are two square classes which can be represented by $\{1, N_p\}$, where N_p is the "smallest non-square" in $\mathbb{F}_q$. Therefore θ_1 and θ_{N_p} represent the two k-isomorphy classes of involutions of $G_{\mathbb{F}_q}$.

Example 2.8.18 (𝔭-adic numbers)**.** For $k = \mathbb{Q}_p$, for some odd prime p, there are four square classes which can be represented by $\{1, p, N_p, pN_p\}$, where N_p is the "smallest non-square" not equal to p in $\mathbb{Q}_p$. Therefore $\theta_1, \theta_p, \theta_{N_p}$, and θ_{pN_p} represent the four k-isomorphy classes of involutions of $G_{\mathbb{Q}_p}$.

Example 2.8.19 (Rational numbers)**.** For $k = \mathbb{Q}$, there are an infinitely many square classes. Thus, there are an infinite number of $\mathbb{Q}$-isomorphy classes of involutions of G_k, all represented by θ_m, for some $\overline{m} \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$.

2.9 Isomorphy classes of involutions of $\mathrm{SL}_n(k)$

Similar to the $G = \mathrm{SL}_2(k)$ case, the k-isomorphy classes of involutions of $\mathrm{SL}_n(k)$, $n > 2$, have also been characterized. When $n > 2$, not all involutions are inner automorphisms and therefore the characterization is not as simple as when $n = 2$. In [HWD06], the isomorphy classes of $\mathrm{SL}_n(k)$ are given for fields k with characteristic not two. Furthermore, they discuss in which cases we have a generalized Cartan involution and when the symmetric space consists of semisimple elements.

Notation 2.9.1. Throughout this section, let $G = \mathrm{SL}_n(\bar{k})$ and G_k the k-rational points of G. We take k to be a field of characteristic not two, and $\bar{k}$ to be the algebraic closure of k. Let $V = k^n$ be a finite-dimensional vector space defined over k. We will use the same definitions of isomorphy and k-isomorphy classes of involutions as in Section 2.8.

Given a bilinear form, we can induce an involution. Let $\beta : V \to V$ be a bilinear form on V and $\mathcal{B} = \{e_1, \ldots, e_n\}$ an ordered basis for V. Then the matrix $M_{\mathcal{B}}$ is the matrix of β with respect to $\mathcal{B}$, $M_{\mathcal{B}} = (\beta(e_i, e_j))$. We can define the bilinear form β as $\beta(x, y) = x_{\mathcal{B}}^T M_{\mathcal{B}} y_{\mathcal{B}}$, for all $x, y \in V$ where $x_{\mathcal{B}}$ and $y_{\mathcal{B}}$ are the coordinate vectors of x and y with respect to the ordered basis $\mathcal{B}$.

Theorem 2.9.2. *Two matrices M_1 and M_2 represent the same bilinear form if and only if there exists $Q \in \mathrm{GL}_n(k)$ such that $M_2 = Q^T M_1 Q$.*

Notation 2.9.3. If there exists $Q \in \mathrm{GL}_n(k)$ such that $M_2 = Q^T M_1 Q$, we write $M_2 \simeq M_1$ and say M_1 and M_2 are congruent.

Recall that the adjoint of a matrix $A \in \mathrm{GL}_n(k)$ is the matrix A' such that $\beta(Ax, y) = \beta(x, A'y)$. Furthermore, the adjoint can be determined as $A' = M^{-1} A^T M$, where M is the matrix representative of the bilinear form β. We can then use this adjoint to construct involutions of $\mathrm{GL}_n(k)$.

Proposition 2.9.4. *If the bilinear form β is symmetric or skew-symmetric, then the automorphism θ_M, defined by $\theta_M(A) = M^{-1}(A^T)^{-1}M$ for all $A \in G$ is an involution of $\mathrm{GL}_n(k)$.*

Theorem 2.9.5. *Let M_1 and M_2 be the matrices of bilinear forms β_1 and β_2, respectively. If $M_1 \simeq M_2$, then $\theta_{M_1} \simeq \theta_{M_2}$.*

Theorem 2.9.6. *Let M_1 and M_2 be matrix representatives of symmetric or skew-symmetric bilinear forms. If $\theta_{M_1} \simeq \theta_{M_2}$, then $M_2 = \alpha Q^T M_1 Q$ for some $Q \in \mathrm{GL}_n(k)$ and $\alpha \in k$.*

Notation 2.9.7. If there exists $Q \in \mathrm{GL}_n(k)$ and $\alpha \in k$ such that $M_2 = \alpha Q^T M_1 Q$, we write $M_2 \simeq^s M_1$ and say M_1 and M_2 are semi-congruent.

This allows us the classify the k-isomorphy classes of involutions of $\mathrm{GL}_n(k)$ of the form θ_M.

Theorem 2.9.8. *Let M_1 and M_2 be matrix representatives of symmetric or skew-symmetric bilinear forms as defined above. Then $M_1 \simeq^s M_2$ over k if and only if $\theta_{M_1} \underset{k}{\simeq} \theta_{M_2}$.*

From here, we can restrict to the smaller group $G = \mathrm{SL}_n(k)$.

Lemma 2.9.9. *An automorphism θ_M induced from a bilinear form with matrix representative M is an involution of $\mathrm{GL}_n(k)$ if and only if θ_M is an involution of G.*

Theorem 2.9.10. *Two involution θ_1 and θ_2 are isomorphic over $\mathrm{GL}_n(k)$ if and only if θ_1 and θ_2 are isomorphic over G.*

We now look at the two types of involutions, those which are outer automorphisms and those which are inner automorphisms. Recall an automorphism $\theta \in \mathrm{Aut}(G)$ is inner if there exists $A \in G$ such that $\theta(g) = \mathrm{Inn}(A)(g) = AgA^{-1}$ for all $g \in G$, denoted $\theta \in \mathrm{Inn}(G)$. Otherwise, we say it is an outer automorphism.

Lemma 2.9.11. *If k is algebraically closed, then $|\mathrm{Aut}(G)/\mathrm{Inn}(G)| = 2$.*

Lemma 2.9.12. *All outer automorphisms of G are isomorphic to $\mathrm{Inn}(M)\phi$, where ϕ is the fixed outer automorphism defined by $\phi(g) = (g^T)^{-1}$.*

To construct the involutions of G which come from outer automorphisms, we will use $\theta_M(A) = \mathrm{Inn}(M)(\phi(A)) = M^{-1}(A^T)^{-1}M$, where M is the matrix representative of a bilinear form β as defined above. All involutions of G which come from outer automorphisms are of the form θ_M, for some M.

Theorem 2.9.13. *If θ_1 and θ_2 are involutions and outer automorphisms of G, then they come from symmetric or skew-symmetric bilinear forms represented by M_1 and M_2 respectively. Furthermore, $\theta_1 \simeq \theta_2$ if and only if $\mathrm{Inn}(M_1)\phi \simeq \mathrm{Inn}(M_2)\phi$ if and only if $M_1 \simeq^s M_2$.*

Theorem 2.9.14. *Symmetric matrices are congruent to diagonal matrices whose entries are representatives of the $k^*/(k^*)^2$. Skew-symmetric matrices are congruent to J_{2m}, where $n = 2m$ (4.2.19).*

$$J_{2m} = \begin{pmatrix} 0 & I_{m \times m} \\ -I_{m \times m} & 0 \end{pmatrix} \tag{2.9.1}$$

We can now classify the isomorphy classes of involutions of G which are outer automorphisms. We will do so for certain fields and for each case, we will consider n even and n odd separately.

Example 2.9.15 ($k = \overline{k}$)**.** Assume k is algebraically closed. If n is odd, then there is one isomorphy class of outer involutions which can be represented by the standard involution ϕ. If n is even, then there are two isomorphy classes of involutions which are outer automorphisms. These can be represented by ϕ and $\mathrm{Inn}(J_{2m})\phi$.

Example 2.9.16 ($k = \mathbb{R}$)**.** Assume $k = \mathbb{R}$ and define $I_{n-i,i}$ (2.9.2).

$$I_{n-i,i} = \begin{pmatrix} I_{n-i \times n-1} & 0 \\ 0 & -I_{i \times i} \end{pmatrix} \tag{2.9.2}$$

If n is odd, then there are $\frac{n+1}{2}$ isomorphy classes of involutions which are outer automorphisms of G. These are represented by ϕ and $\mathrm{Inn}(I_{n-i,i})\phi$ for $i = 1, \ldots, \frac{n-1}{2}$. If n is even, there are $\frac{n}{2} + 2$ isomorphy classes of involutions which are outer automorphisms. These are represented by $\phi, \mathrm{Inn}(J_{2m})\phi$, and $\mathrm{Inn}(I_{n-i,i})\phi$ for $i = 1, \ldots, \frac{n}{2}$.

Example 2.9.17 ($k = \mathbb{F}_p, p \neq 2$)**.** Assume $k = \mathbb{F}_p$ and define $M_{n,x,y,z}$ (2.9.3).

$$M_{n,x,y,z} = \begin{pmatrix} I_{n-3 \times n-3} & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{pmatrix} \tag{2.9.3}$$

If n is odd, there are two isomorphy classes of involutions which are outer automorphism of G, represented by ϕ and $\mathrm{Inn}(M_{n,1,1,N_p})\phi$, where N_p is the smallest non-square in $\mathbb{F}_p$. If n is even, we have the same two classes as the odd case and additionally $\mathrm{Inn}(J_{2m})\phi$ for a total of three isomorphy classes.

We now discuss the isomorphy classes of involutions which are inner automorphisms of G. We borrow the same results from Section 2.8 concerning inner automorphisms of G.

Lemma 2.9.18. *Suppose θ is an inner automorphism and involution of G and define $L_{n,x}$ (2.9.4).*

$$L_{n,x} = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ x & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & x & 0 \end{pmatrix} \tag{2.9.4}$$

There exists $Y \in \mathrm{GL}_n(k)$ such that $\theta = \mathrm{Inn}_Y$, where $Y \simeq cI_{n-1,i}$ for some $i = 0, 1, \ldots, n$ and $c \in k^*$ or $Y \simeq L_{\frac{n}{2},p}$ for some $p \in k^*/(k^*)^2$.

The following list of lemmas allows us to determine which involutions are isomorphic to each other.

Lemma 2.9.19.

1. $I_{n-i,i} \simeq cI_{n-j,j}$ for some $c \in k$ if and only if $c = 1$ and $i = j$ or $c = -1$ and $j = n - 1$.

2. For $p, q \in k^*/(k^*)^2$, $L_{\frac{n}{2},p} \simeq cL_{\frac{n}{2},q}$ if and only if $\frac{p}{q} \in (k^*)^2$.

3. $\mathrm{Inn}(Y_1) \simeq \mathrm{Inn}(Y_2)$ if and only if $Y_1 \simeq cY_2$ for some $c \in \overline{k}$.

4. $L_{n,1} \simeq I_{n \times n}$

Theorem 2.9.20. *Suppose θ is an involution of G which is an inner automorphism. Then up to isomorphism, $\theta = \mathrm{Inn}(Y)$ where $Y = I_{n-i,i} \in \mathrm{GL}_n(k)$ where $i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ or $Y = L_{\frac{n}{2},p} \in \mathrm{GL}_n(k)$ where $p \in k^*/(k^*)^2, p \not\equiv \overline{1}$.*

Corollary 2.9.21. *The number of involutions of G which are inner automorphisms is $|k^*/(k^*)^2| + \frac{n}{2} - 1$ if n is even and $\frac{n-1}{2}$ if n is odd.*

Example 2.9.22 ($k = \overline{k}$). If k is algebraically closed, then there are $\lfloor \frac{n}{2} \rfloor$ isomorphy classes of involutions of G which are inner automorphisms. These isomorphy classes are represented by $\theta = \mathrm{Inn}(Y)$ where $Y = I_{n-i,i}$, for $i = 1, 2, \ldots, \frac{n}{2}$ if n is even or $i = 1, 2, \ldots, \frac{n-1}{2}$ if n is odd.

Example 2.9.23 ($k = \mathbb{R}$). Assume $k = \mathbb{R}$. If n is odd, then there are $\frac{n-1}{2}$ isomorphy classes of involutions of G which are inner automorphisms. These are represented by $\theta = \mathrm{Inn}(Y)$, where $Y = I_{n-i,i}$, for $i = 1, 2, \ldots, \frac{n-1}{2}$. If n is even, then there are $\frac{n}{2} + 1$ isomorphy classes. These are represented by $\theta = \mathrm{Inn}(Y)$, where $Y = L_{n,-1}$ or $Y = I_{n-i,i}$, for $i = 1, 2, \ldots, \frac{n}{2}$.

Example 2.9.24 ($\mathbb{F}_p, p \neq 2$). Assume $k = \mathbb{F}_p, p \neq 2$. If n is odd, then there are $\frac{n-1}{2}$ isomorphy classes of involutions of G which are inner automorphisms. These are represented by $\theta = \mathrm{Inn}(Y)$, where $Y = I_{n-i,i}$, for $i = 1, 2, \ldots, \frac{n-1}{2}$. If n is even, then there are $\frac{n}{2} + 1$ isomorphy classes. These are represented by $\theta = \mathrm{Inn}(Y)$, where $Y = L_{n,N_p}$ or $Y = I_{n-i,i}$, for $i = 1, 2, \ldots, \frac{n}{2}$.

Example 2.9.25 ($k = \mathbb{Q}_p, p \neq 2$). Assume $k = \mathbb{Q}_p$. If n is odd, then there are $\frac{n-1}{2}$ isomorphy classes of involutions of G which are inner automorphisms. These are represented by $\theta = \mathrm{Inn}(Y)$, where $Y = I_{n-i,i}$, for $i = 1, 2, \ldots, \frac{n-1}{2}$. If n is even, there are $\frac{n}{2} + 3$ isomorphy classes. These are represented by $\theta = \mathrm{Inn}(Y)$, where $y = I_{n-i,i}$, for $i = 1, 2, \ldots, \frac{n}{2}$ or $Y = L_{n,\alpha}$, for $\alpha \in \{p, N_p, pN_p\}$.

Example 2.9.26 $(k = \mathbb{Q})$**.** If $k = \mathbb{Q}$, then there are infinitely many isomorphy classes of involutions of G. The isomorphy classes of involutions which are inner automorphisms are represented by $\theta = \text{Inn}(Y)$, where $Y = I_{n-i,i}$ for $i = 1, 2, \ldots, \frac{n}{2}$, or $Y = L_{n,\alpha}$ for $\alpha \notin \overline{1}$. Similarly, there are an infinite number of isomorphy classes of involutions of G which are outer automorphisms.

As far as the fixed-point groups of these involutions of $\text{SL}_n(k)$, it turns out there very few, if any in some cases, have k-anisotropic fixed-point groups. In [HWD06] the cases when $k = \mathbb{R}$ and $k = \mathbb{Q}_p$ are considered.

When $k = \mathbb{R}$, the only involution with a compact fixed-point group is ϕ, defined by $\phi(g) = (g^T)^{-1}$ for all $g \in G$. When $k = \mathbb{Q}_p, p \neq 2$, the following cases have k-anisotropic fixed-point groups.

1. $n = 3$: $\text{Inn}(M_{3,1,1,p})\phi$ and $\text{Inn}(M_{3,1,1,pN_p})\phi$

2. $n = 4$: $\text{Inn}(M_{4,1,pp})\phi$, if $p \equiv 1 \mod 4$

3. $n > 4$: none

2.10 H_k-conjugacy classes of tori in $\text{SL}_2(k)$

The tori in $\text{SL}_2(k)$ play a role in the structure of its symmetric spaces. In [BH09, Nor13], these tori are analyzed in detail. This section offers a summary of the key results used in the results of this thesis.

Notation 2.10.1. In this section, let k be a field of characteristic not two, $G = \text{SL}_2(\overline{k})$, G_k the k-rational points of G, and θ an involution of G_k. We continue with the same definitions from Section 2.4. Let H be the fixed-point group of θ and $Q_k = \{g\theta(g)^{-1} \mid g \in G_k\}$ the symmetric space of G_k.

Lemma 2.10.2. *In a symmetric space Q, all maximal θ-split tori are maximal k-split.*

Example 2.10.3. Let G be defined over $k = \mathbb{R}$ and θ the involution defined by $\theta(g) = (g^T)^{-1}$ for all $g \in G$. The symmetric space is the set of positive definite symmetric matrices, $Q_{\mathbb{R}} = \{gg^T \mid g \in G_{\mathbb{R}}\}$. The torus T of diagonal matrices is maximal in Q. Additionally, all elements in $T_{\mathbb{R}}$ (2.10.1) are $(\theta, \mathbb{R})$-split.

$$T_{\mathbb{R}} = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \middle| x \in \mathbb{R}^+ \right\} \tag{2.10.1}$$

Theorem 2.10.4. *If Q_k is k-split and consists of semisimple elements, then we have the following.*

1. *All θ-split tori of G are k-split.*

2. *All maximal θ-split tori in G are maximal (θ, k)-split.*

3. *For $q \in Q_k$, there exists a maximal (θ, k)-split torus of G containing q.*

We will focus on the H_k-conjugacy class of tori in G_k.

2.10.1 (θ, k)-split tori in G_k

The H_k-conjugacy classes of (θ, k)-split tori of G_k are characterized in [BH09].

Notation 2.10.5. Let T be a maximal torus. In G_k, we may assume T is the diagonal matrices because all maximal tori are conjugate. Let θ_m be the involution $\mathrm{Inn}(M)$, where $M = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ and m is the representative of the square class $\overline{m} \in k^*/(m^*)^2$.

Theorem 2.10.6. *For G_k with the involution θ_m, let $U = \{\overline{q} \in k^*/(k^*)^2 \mid x_1^2 - m^{-1}x_2^2 = q^{-1}$ has a solution in $k\}$. The number of H_k-conjugacy class of (θ, k)-split tori is $|U/\{1, -m\}|$.*

In the case that there is one H_k-conjugacy class, we may assume the (θ, k)-split torus is T. If there is more than one H_k-conjugacy class, they can be determined as follows.

Theorem 2.10.7. *Let U be defined as in Theorem 2.10.6. For $y \in U$, there exists $r, s \in k$ such that $r^2 - m^{-1}s^2 = y^{-1}$. Let $g = \begin{pmatrix} r & sym^{-1} \\ s & ry \end{pmatrix}$, then the representatives of the H_k-conjugacy classes of maximal (θ, k)-split tori in G_k are $\{T_y = g^{-1}Tg \mid y \in U\}$.*

Example 2.10.8 ($k = \bar{k}$). For G_k defined over an algebraically closed field $\bar{k}$, up to isomorphy there is one involution, θ_1. By Theorem 2.10.6, there is one H-conjugacy class of $(\theta, \bar{k})$-split tori.

Example 2.10.9 ($k = \mathbb{R}$). For G_k defined over $k = \mathbb{R}$, up to isomorphy there are two involutions, θ_1 and θ_{-1}. In either case, there is one $H_{\mathbb{R}}$ conjugacy class of $(\theta, \mathbb{R})$-split tori.

Example 2.10.10 ($k = \mathbb{Q}$). For G_k defined over $k = \mathbb{Q}$, there are an infinite number of involutions. For each involution θ_m, there are an infinite number of $H_{\mathbb{Q}}$-conjugacy classes of $(\theta, \mathbb{Q})$-split tori.

Example 2.10.11 ($k = \mathbb{F}_p, n \neq 2$)**.** For G_k defined over $k = \mathbb{F}_p$, up to isomorphy there are two involutions, θ_1 and θ_{N_p}. The number of $H_{\mathbb{F}_p}$-conjugacy classes of $(\theta, \mathbb{F}_p)$-split tori is given in Table 2.1. We consider the cases where $p \equiv 1 \mod 4$ and $p \equiv 3 \mod 4$ separately. The major difference is $-1 \in (\mathbb{F}_p^*)^2$ if and only if $p \equiv 1 \mod 4$.

Table 2.1: $H_{\mathbb{F}_p}$-conjugacy classes of $(\theta, \mathbb{F}_p)$-split tori in $G_{\mathbb{F}_p}$

Involution	p	$H_{\mathbb{F}_p}$-conj. classes
θ_1	$p \equiv 1 \mod 4$	2
θ_1	$p \equiv 3 \mod 4$	1
θ_{N_p}	$p \equiv 1 \mod 4$	1
θ_{N_p}	$p \equiv 3 \mod 4$	2

Example 2.10.12 ($k = \mathbb{Q}_p, p \neq 2$)**.** For G_k defined over $k = \mathbb{Q}_p$, up to isomorphy there are four involutions, θ_1, θ_p, θ_{N_p}, and θ_{pN_p}. The number of $H_{\mathbb{Q}_p}$-conjugacy classes of $(\theta, \mathbb{Q}_p)$-split tori is given in Table 2.2. For the same reason as when $k = \mathbb{F}_p$, we consider the cases where $p \equiv 1 \mod 4$ and $p \equiv 3 \mod 4$ separately.

Table 2.2: $H_{\mathbb{Q}_p}$-conjugacy classes of $(\theta, \mathbb{Q}_p)$-split tori in $G_{\mathbb{Q}_p}$

Involution	p	$H_{\mathbb{Q}_p}$-conj. classes
θ_1	$p \equiv 1 \mod 4$	4
θ_1	$p \equiv 3 \mod 4$	2
θ_p	-	1
θ_{N_p}	$p \equiv 1 \mod 4$	1
θ_{N_p}	$p \equiv 3 \mod 4$	2
θ_{pN_p}	-	1

Example 2.10.13. From [Beu08] we have the following example of an instance when G_k has more than once H_k-conjugacy class of (θ, k)-split tori. Let G be defined over $k = \mathbb{Q}_5$ and $\theta = \mathrm{Inn}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the involution of G. We will let $N_p = 3$, then $U = \{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{15}\}$. We

know $T_{\mathbb{Q}_5}$ is a maximal $(\theta, \mathbb{Q}_5)$-split torus. For $y \in U$, we must find $k, l \in \mathbb{Q}_5$ such that $y^{-1} = k^2 - l^2$. We then conjugate $t \in T$ with $A_i = \begin{pmatrix} k & kl \\ l & ky \end{pmatrix}$ to get representatives of the $H_{\mathbb{Q}_5}$-conjugacy classes. Solving $y^{-1} = k^2 - l^2$ for each $y \in U$, we get the matrices A_i, for $i = 1, \ldots, 4$ (2.10.2).

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & \frac{1}{3} \\ 1 & \frac{2}{3} \end{pmatrix}, \quad A_3 = \begin{pmatrix} 3 & \frac{2}{5} \\ 2 & \frac{3}{5} \end{pmatrix}, \quad A_4 = \begin{pmatrix} 4 & \frac{1}{15} \\ 1 & \frac{4}{15} \end{pmatrix} \tag{2.10.2}$$

Conjugating $t \in T$ by A_i for $i = 1 \ldots 4$, we get the $H_{\mathbb{Q}_5}$-conjugacy classes of $(\theta, \mathbb{Q}_5)$-split tori, T_1, T_2, T_3, T_4 (2.10.3)-(**??**).

$$T_1 = \{A_1 t A_1^{-1} \mid t \in T\} = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \; \middle| \; x \in \mathbb{Q}_5^* \right\} \tag{2.10.3}$$

$$T_2 = \{A_2 t A_2^{-1} \mid t \in T\} = \left\{ \begin{pmatrix} \frac{1}{3(4x-x^{-1})} & \frac{2}{3(x^{-1}-x)} \\ \frac{-2}{3(x^{-1}-x)} & \frac{1}{3(4x^{-1}-x)} \end{pmatrix} \; \middle| \; x \in \mathbb{Q}_5^* \right\} \tag{2.10.4}$$

$$T_3 = \{A_3 t A_3^{-1} \mid t \in T\} = \left\{ \begin{pmatrix} \frac{1}{5(9x-4x^{-1})} & \frac{6}{5(x^{-1}-x)} \\ \frac{-6}{5(x^{-1}-x)} & \frac{1}{5(9x^{-1}-4x)} \end{pmatrix} \; \middle| \; x \in \mathbb{Q}_5^* \right\} \tag{2.10.5}$$

$$T_4 = \{A_4 t A_4^{-1} \mid t \in T\} = \left\{ \begin{pmatrix} \frac{1}{15(16x-x^{-1})} & \frac{4}{15(x^{-1}-x)} \\ \frac{-4}{15(x^{-1}-x)} & \frac{1}{15(16x^{-1}-x)} \end{pmatrix} \; \middle| \; x \in \mathbb{Q}_5^* \right\} \tag{2.10.6}$$

2.10.2 θ-split, k-anisotropic tori in G_k

The H_k-conjugacy classes of θ-split, k-anisotropic toral subalgebras of $\mathfrak{sl}_2(k)$ are studied in [Nor13]. We use these results and lift the toral subalgebras to the group level through the exponential map to obtain the H_k-conjugacy classes of θ-split, k-anisotropic tori in G_k.

Definition 2.10.14. On the algebra level, a *toral subalgebra* is an abelian, semisimple subalgebra. A toral subalgebra $\mathfrak{a}$ is *θ-split* if $\theta(a) = -a$ for all $a \in \mathfrak{a}$. A toral subalgebra is *k-anisotropic* if it can not be diagonalized over the base field k.

The maximal θ-split, k-anisotropic toral subalgebras in $\mathfrak{sl}_2(k)$ take on two forms. Using the terminology from [Nor13], we will refer to these forms as Type II and Type III. Type I are the (θ, k)-split toral subalgebras.

31

For the involution θ_m of $\mathfrak{sl}_2(k)$ we have the toral subalgebras of Type II (2.10.7).

$$\mathfrak{a}_2 = \left\{ \begin{pmatrix} 0 & x \\ -mx & 0 \end{pmatrix} \,\middle|\, x \in k \right\} \tag{2.10.7}$$

For $a \in \mathfrak{a}_2$, we can lift to the group level using $\mathrm{Exp}(ta)$, for some parameter t to get the θ-split, k-anisotropic Type II tori in G_k (2.10.8).

$$A_2 = \left\{ \begin{pmatrix} x & y \\ -my & x \end{pmatrix} \,\middle|\, x^2 + my^2 = 1, \ x, y \in k \right\} \tag{2.10.8}$$

Again, we are concerned with the H_k-conjugacy classes of the θ-split, k-anisotropic tori in G_k.

Lemma 2.10.15. *Let G_k be defined over k with the involution θ_m. There is exactly one H_k-conjugacy class of maximal θ-split, k-anisotropic tori of Type II.*

Example 2.10.16 $(k = \bar{k})$**.** For G defined over an algebraically closed field $\bar{k}$, there are no H-conjugacy classes of θ-split, $\bar{k}$-anisotropic Type II tori in G, because all tori can be diagonalized over the base field.

Example 2.10.17 $(k = \mathbb{R})$**.** For G_k defined over $k = \mathbb{R}$, there are two isomorphy classes of involutions, which can be represented by θ_1 and θ_{-1}. In the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, the toral subalgebras of Type II are generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for θ_1 and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for θ_{-1}. When $m = 1$, the eigenvalues of the generator are $\pm i$, hence the toral subalgebra is $\mathbb{R}$-anisotropic There is one $H_\mathbb{R}$-conjugacy class of θ_1-split, $\mathbb{R}$-anisotropic Type II tori in $G_\mathbb{R}$. When $m = -1$, the eigenvalues of the generator are ± 1, hence the toral subalgebra is $\mathbb{R}$-split and there are no $H_\mathbb{R}$-conjugacy class of θ_{-1}-split, $\mathbb{R}$-anisotropic Type II tori in $G_\mathbb{R}$.

Example 2.10.18 $(k = \mathbb{F}_p, \ p \neq 2)$**.** For G_k defined over $k = \mathbb{F}_p$ with the involution θ, the number of $H_{\mathbb{F}_p}$-conjugacy classes of θ-split, $\mathbb{F}_p$-anisotropic tori of Type II are given in Table 2.3.

Table 2.3: $H_{\mathbb{F}_p}$-conjugacy classes of θ-split, $\mathbb{F}_p$-anisotropic Type II tori in $G_{\mathbb{F}_p}$

Involution	p	$H_{\mathbb{F}_p}$-conj. classes
θ_1	$p \equiv 1 \mod 4$	0
θ_1	$p \equiv 3 \mod 4$	1
θ_{N_p}	$p \equiv 1 \mod 4$	1
θ_{N_p}	$p \equiv 3 \mod 4$	0

Example 2.10.19 ($k = \mathbb{Q}_p$, $p \neq 2$). For G_k defined over $k = \mathbb{Q}_p$ with the involution θ, the number of $H_{\mathbb{Q}_p}$-conjugacy classes of θ-split, $\mathbb{Q}_p$-anisotropic tori of Type II are given in Table 2.4.

Table 2.4: $H_{\mathbb{Q}_p}$-conjugacy classes of θ-split, $\mathbb{Q}_p$-anisotropic Type II tori in $G_{\mathbb{Q}_p}$

Involution	p	$H_{\mathbb{Q}_p}$-conj. classes
θ_1	$p \equiv 1 \mod 4$	0
θ_1	$p \equiv 3 \mod 4$	3
θ_p	-	3
θ_{N_p}	$p \equiv 1 \mod 4$	3
θ_{N_p}	$p \equiv 3 \mod 4$	0
θ_{pN_p}	-	3

Example 2.10.20 ($k = \mathbb{Q}$). For G_k defined over $k = \mathbb{Q}$ with the involution $\theta = \theta_m$, $-m \notin (\mathbb{Q}^*)^2$, there is exactly one $H_{\mathbb{Q}}$-conjugacy class of θ-split, $\mathbb{Q}$-anisotropic Type II tori.

For the involution θ_m of $\mathfrak{sl}_2(k)$, we have the toral subalgebras of Type III (2.10.9).

$$\mathfrak{a}_3 = \left\{ \begin{pmatrix} x & \gamma x \\ -m\gamma x & -x \end{pmatrix} \,\middle|\, \gamma \text{ fixed s.t. } 1 - m\gamma^2 \notin (k^*)^2 \right\} \tag{2.10.9}$$

For $a \in \mathfrak{a}_3$, we can lift to the group level using $\mathrm{Exp}(ta)$, for some parameter t to get the θ-split, k-anisotropic Type III tori in G_k (2.10.10).

$$A_3 = \left\{ \begin{pmatrix} x+y & \gamma y \\ -m\gamma y & x-y \end{pmatrix} \,\middle|\, \gamma \text{ fixed s.t. } 1 - m\gamma^2 \notin (k^*)^2; x^2 - (1 - m\gamma^2)y^2 = 1 \right\} \tag{2.10.10}$$

For Type III tori, this classification has not yet been completed, although [Nor13] offers an upper bound when the exact number of H_k-conjugacy classes is not known.

Example 2.10.21 $(k = \mathbb{Q})$. Let G_k be defined over $k = \mathbb{Q}$ with the involution θ_m. There are an infinite number of conjugacy classes of maximal θ_m-split, $\mathbb{Q}$-anisotropic Type III tori.

Example 2.10.22 $(k = \mathbb{F}_p$ or $\mathbb{Q}_p,\ p \neq 2)$. For G_k defined over $k = \mathbb{F}_p$ or $k = \mathbb{Q}_p$, [Nor13] offers an upper bound for the H_k-conjugacy classes. For G_k defined over $k = \mathbb{F}_p$, there is either one or two $H_{\mathbb{F}_p}$-conjugacy classes of maximal θ-split, $\mathbb{F}_p$-anisotropic tori of Type III. For G_k defined over $k = \mathbb{Q}_p$, there is at most 12 $H_{\mathbb{Q}_p}$-conjugacy classes of maximal θ-split, $\mathbb{Q}_p$-anisotropic tori of Type III.

Chapter 3

Generalizations to $\mathrm{SL}_2(k)$

We begin to generalize the Cartan and Iwasawa decompositions to the algebraic group $\mathrm{SL}_2(k)$, where k is an arbitrary field of characteristic not two and θ is any involution.

3.1 Factors of the decomposition

Notation 3.1.1. Throughout this chapter, let $G = \mathrm{SL}_2(k)$, where k is a field of characteristic not two and $\bar{k}$ is the algebraic closure of k. We will denote the k-rational point of G by G_k. In general, the k-rational points of a set A will be denoted A_k. Recall the isomorphy classes of G from Section 2.8. Using the same notation, we let $\theta_m = \mathrm{Inn}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$, for $\overline{m} \in k^*/(k^*)^2$, be an involution of G.

We define the fixed-point group, symmetric and extended symmetric spaces with respect to different involutions, thus we add the following notation. The H^θ is the fixed-point group of θ, Q^θ is the symmetric space with respect to θ, and $\widetilde{Q^\theta}$ is the extended symmetric space with respect to θ.

For G_k with an arbitrary involution θ_m, we can determine the fixed-point group H^{θ_m} (3.1.1), symmetric space Q^{θ_m} (3.1.2), and extended symmetric space $\widetilde{Q^{\theta_m}}$ (3.1.3) of G_k.

$$H_k^{\theta_m} = \left\{ \begin{pmatrix} a & b \\ mb & a \end{pmatrix} \,\middle|\, a, b \in k, \ a^2 - mb^2 = 1 \right\} \tag{3.1.1}$$

$$Q_k^{\theta_m} = \left\{ \begin{pmatrix} a^2 - mb^2 & -\frac{1}{m}(ac - bd) \\ m(ac - bd) & -\frac{1}{m}c^2 + d^2 \end{pmatrix} \,\middle|\, a, b, c, d \in k, \ ad - bc = 1 \right\} \tag{3.1.2}$$

$$\widetilde{Q_k^{\theta_m}} = \left\{ \begin{pmatrix} a & b \\ -mb & c \end{pmatrix} \,\middle|\, a, b, c \in k, \ ac + mb^2 = 1 \right\} \tag{3.1.3}$$

Example 3.1.2. Consider G defined over $k = \mathbb{R}$. The square classes of $\mathbb{R}$ are represented by $\{1, -1\}$. The fixed-point group of θ_{-1} is the special orthogonal group $\mathrm{SO}(2)$ which is compact, thus θ_{-1} is a Cartan involution of G. The symmetric space with respect to θ_{-1} is the set of positive-definite symmetric matrices, while the extended symmetric space is the set of all symmetric matrices.

For the involution θ_1 of G, the fixed-point group is the indefinite special orthogonal group $\mathrm{SO}(1, 1)$ (3.1.4). The symmetric space is Q^{θ_1} (3.1.5) and the extended symmetric space is $\widetilde{Q^{\theta_1}}$ (3.1.6).

$$H_{\mathbb{R}}^{\theta_1} = \mathrm{SO}(1, 1) = \left\{ \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} \tag{3.1.4}$$

$$Q_{\mathbb{R}}^{\theta_1} = \left\{ \begin{pmatrix} a^2 - b^2 & -ac + bd \\ ac - bd & -c^2 + d^2 \end{pmatrix} \,\middle|\, a, b, c, d \in k, \ ad - bc = 1 \right\} \tag{3.1.5}$$

$$\widetilde{Q_{\mathbb{R}}^{\theta_1}} = \left\{ \begin{pmatrix} x & y \\ -y & z \end{pmatrix} \,\middle|\, x, y, z \in k, \ xz + y^2 = 1 \right\} \tag{3.1.6}$$

3.2 Generalizing the Cartan and Iwasawa decompositions

As previously discussed, the Cartan and Iwasawa decompositions are defined for real semisimple Lie groups when paired with a Cartan involution. In general, for any field k with an arbitrary involution θ, the set $H_k^\theta Q_k^\theta$ is contained in, but not equal to, G_k. In fact, even for G defined over $k = \mathbb{R}$ with the non-Cartan involution θ_1, the Cartan decomposition fails, as shown in Example 2.3.9. When we switch to another field, such as a finite field, it is also clear that the Cartan decomposition fails, even with the generalized Cartan involution. In [BH09], we get the fixed-point group of H^{θ_m} is k-anisotropic if and only if $m \notin \bar{1}$. In other words, θ_m is a generalized Cartan involution if and only if m is not a square in the field.

Example 3.2.1. Let G be defined over $k = \mathbb{F}_7$ and $\theta = \theta_{-1}$ the involution of G. Because

$-1 \notin (\mathbb{F}_7^*)^2$, θ serves as a Cartan involution. Let $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. If $g \in H_{\mathbb{F}_7}^{\theta} Q_{\mathbb{F}_7}^{\theta} \subset H_{\mathbb{F}_7}^{\theta} \widetilde{Q_{\mathbb{F}_7}^{\theta}}$, then

there exists $h^{-1} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in H_{\mathbb{F}_7}^{\theta}$ such that $h^{-1}g \in \widetilde{Q_{\mathbb{F}_7}^{\theta}}$ (3.2.1).

$$h^{-1}g = \begin{pmatrix} a & a+b \\ -b & a-b \end{pmatrix} \tag{3.2.1}$$

For $h^{-1}g$ to be symmetric, $h^{-1} = \begin{pmatrix} 5b & b \\ 6b & 5b \end{pmatrix}$. For $\det(h^{-1}) = 5b^2 = 1$, it must be that $3 \in$ $(\mathbb{F}_7^*)^2$, which is not true. Hence, $G_{\mathbb{F}_7} \notin H_{\mathbb{F}_7}^{\theta} \widetilde{Q_{\mathbb{F}_7}^{\theta}}$ and therefore the Cartan decomposition $G_{\mathbb{F}_7} = H_{\mathbb{F}_7}^{\theta} Q_{\mathbb{F}_7}^{\theta}$ does not hold.

To account for the missing elements, we introduce the unipotent subgroup U of G consisting of upper triangular matrices with ones in the diagonal (3.2.2).

$$U_k = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \;\middle|\; \alpha \in k \right\} \tag{3.2.2}$$

With the addition of the new subgroup, we have the following result.

Theorem 3.2.2. *For G defined over with the involution θ, fixed-point group H^{θ}, extended symmetric space $\widetilde{Q^{\theta}}$ and unipotent subgroup U, $G_k = H_k^{\theta} \widetilde{Q_k^{\theta}} U_k$.*

This decomposition simultaneously serves as a generalization of both the Cartan and Iwasawa decompositions. It contains the fixed-point group and symmetric space similar to the Cartan decomposition. Additionally, because the maximal k-split torus is contained in $\widetilde{Q^{\theta}}$ and we have a unipotent subgroup, it generalizes the Iwasawa decomposition. The proof of Theorem 3.2.2 relies on the Bruhat decomposition of G. Recall from Example 2.5.4, the Bruhat decomposition of G_k. We establish the necessary results to prove Theorem 3.2.2 here.

Theorem 3.2.3 (Bruhat Decomposition of $\mathrm{SL}_2(k)$). *Let G be defined over k, $B \subset G$ a Borel subgroup, T a maximal torus, and $W(T)$ the Weyl group of G. Then we have the Bruhat decomposition of G_k (3.2.3)*

$$G_k = \biguplus_{\omega \in W(T)} B_k \omega B_k = B_k \uplus B_k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B_k \tag{3.2.3}$$

37

Let the maximal torus T be the subgroup of diagonal matrices in G (3.2.4) and the Borel subgroup $B \supset T$ be the upper triangular matrices (3.2.5).

$$T_k = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \ \middle| \ \alpha \in k^* \right\} \tag{3.2.4}$$

$$B_k = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \ \middle| \ x \in k^*, y \in k \right\} \tag{3.2.5}$$

For the Borel subgroup B, we can write $B = TU$, where T is the k-split maximal torus and U is the unipotent radical of B.

Lemma 3.2.4. *Let G be defined over k with the involution θ and T a k-split maximal torus. Then T is invariant under θ and is maximal (θ, k)-split.*

Proof. Let $\theta = \theta_m$ be the involution of G and $t = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in T$. Then T is θ-split and $\theta(t) \in T$ (3.2.6).

$$\theta(t) = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 0 & m^{-1} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} = t^{-1} \in T \tag{3.2.6}$$

$\square$

Lemma 3.2.5. *Let G be defined over k and θ an involution of G. If $H_k^\theta = \{\pm \mathrm{Id}\}$, then $k \simeq \mathbb{F}_3$.*

Proof. From [HW93], the fixed-point group of an involution is always reductive. Thus for G, the fixed-point group is a torus. Since $H \simeq \overline{k^*}$, if $|H| = 2$, then $k \simeq \mathbb{F}_3$.

$\square$

Remark 3.2.6. For G defined over $k = \mathbb{F}_3$, $H_k^{\theta_m} = \{\pm \mathrm{Id}\}$ only for $m \in (\mathbb{F}_3^*)^2 = \{1\}$.

Proof of Theorem 3.2.2. Let $\theta = \theta_m$ be the involution of G .Because $H_k^\theta, \widetilde{Q_k^\theta}$, and U_k are contained in G_k, $H_k^\theta \widetilde{Q_k^\theta} U_k \subset G_k$ is clear. We will show the reverse containment, $G \subset H_k^\theta \widetilde{Q_k^\theta} U_k$, using an equivalent statement (3.2.7), replacing G_k with its Bruhat decomposition.

$$B_k \bigcup B_k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B_k \subset H_k^\theta \widetilde{Q_k^\theta} U_k \tag{3.2.7}$$

First, consider $g \in B_k$. Then there exists $u \in U_k$ such that $u^{-1}g = t \in T_k$. By Lemma 3.2.4, $u^{-1}g = t$ is θ-split, hence $u^{-1}g \in \widetilde{Q_k^\theta}$. Therefore, $g \in \widetilde{Q_k^\theta} U_k \subset H_k^\theta \widetilde{Q_k^\theta} U_k$.

Second, consider $g \in B_k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B_k$. Then for some $a, b \in k^*$ and $\alpha, \beta \in k$, we can rewrite g (3.2.8).

$$g = \begin{pmatrix} a & \alpha \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b & \beta \\ 0 & b^{-1} \end{pmatrix} \tag{3.2.8}$$

If $\alpha \neq 0$, let $u = \begin{pmatrix} 1 & \frac{ma^2 - b^2 - m\alpha\beta ab}{m\alpha ab^2} \\ 0 & 1 \end{pmatrix}$. Then $gu \in \widetilde{Q_k^\theta}$ (3.2.9) and therefore $g \in H_k^\theta \widetilde{Q_k^\theta} U_k$.

$$\theta(gu) = \begin{pmatrix} -\frac{a^2 m - b^2}{a^2 bm\alpha} & -\frac{b}{ma} \\ \frac{b}{a} & -\alpha b \end{pmatrix} = (gu)^{-1} \tag{3.2.9}$$

If $\alpha = 0$, let $h = \begin{pmatrix} x & y \\ my & x \end{pmatrix} \in H_k^\theta \smallsetminus \{\pm \mathrm{Id}\}$, and $u = \begin{pmatrix} 1 & \frac{mxa^2 - xb^2 - my\beta b}{mb^2 y} \\ 0 & 1 \end{pmatrix}$. Then $hgu \in \widetilde{Q_k^\theta}$ (3.2.10) and therefore $g \in H_k^\theta \widetilde{Q_k^\theta} U_k$.

$$\theta(hgu) = \begin{pmatrix} -\frac{yb}{a} & \frac{xb}{am} \\ -\frac{xb}{a} & \frac{a^2 m^2 y^2 - ma^2 x^2 + b^2 x^2}{abmy} \end{pmatrix} = (hgu)^{-1} \tag{3.2.10}$$

Now assume $H_k^\theta = \{\pm \mathrm{Id}\}$, then by Lemma 3.2.5 we may assume $k = \mathbb{F}_3$ and $\theta = \theta_1$. By direct calculation of $G_{\mathbb{F}_3}$, we get that most of $G_{\mathbb{F}_3}$ is already in $H_{\mathbb{F}_3}^\theta, \widetilde{Q_{\mathbb{F}_3}^\theta}$, or $U_{\mathbb{F}_3}$. We can calculate the factorization directly of the remaining elements (3.2.11).

$$\mathrm{SL}_2(\mathbb{F}_3) \smallsetminus (H_{\mathbb{F}_3}^\theta \cup \widetilde{Q_{\mathbb{F}_3}^\theta} \cup U_{\mathbb{F}_3}) = \left\{ \pm \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}, \pm \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \pm \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\} \tag{3.2.11}$$

$$\pm \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix} = \pm \mathrm{Id} \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\pm \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \pm \mathrm{Id} \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\pm \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} = \pm \mathrm{Id} \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\pm \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \pm \operatorname{Id} \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Therefore, the decomposition $G_k = H_k^\theta \widetilde{Q_k^\theta} U_k$ for an arbitrary involution θ and any field k of characteristic not 2.

$\square$

Remark 3.2.7. For a general field k and involution θ, $G_k \neq H_k^\theta Q_k^\theta U_k$ and thus expanding to the extended symmetric space is necessary. Further on, we give cases in which the symmetric space will suffice.

For G defined over a field k with the involution θ, we can consider the $(H_k^\theta \times U_k)$-orbits on G_k defined by $(h, u) \bullet g := hgu$ for $h \in H_k^\theta, u \in U_k$ and $g \in G_k$. From [HW93], we have the following result which allows us to choose orbit representatives in $\widetilde{Q_k^\theta}$.

Proposition 3.2.8. *If $g \in \widetilde{Q_k^\theta}$, then there exists $u \in U_k$ such that $ug\theta(u)^{-1} \in N_G(T)$.*

Similarly, we can define the twisted U_k-orbits on $\widetilde{Q_k^\theta}$ using the θ-twisted action $u * q := u^{-1}q\theta(u)$ for $u \in U_k$ and $q \in Q_k^\theta$. We have $Q_k^\theta \simeq G_k/H_k^\theta$, thus the U_k-orbits of Q_k^θ are in bijective correspondence with the $(H_k^\theta \times U_k)$-orbits of G_k if and only if $H_k^\theta Q_k^\theta U_k = H_k^\theta \widetilde{Q_k^\theta} U_k$.

Example 3.2.9. Let G be defined over $k = \mathbb{Q}$ and $\theta = \theta_{-1}$ the involution of G. For convenience, we will use $Q_\mathbb{Q}^\theta = \{g^{-1}\theta(g) \mid g \in G_\mathbb{Q}\}$. Let the $(H_\mathbb{Q}^\theta \times U_\mathbb{Q})$-orbits on $G_\mathbb{Q}$ and twisted $U_\mathbb{Q}$-orbits on $Q_\mathbb{Q}^\theta$ be as above. Consider the map from $U_\mathbb{Q} * Q_\mathbb{Q}^\theta$ to $(H_\mathbb{Q}^\theta \times U_\mathbb{Q}) \bullet G_\mathbb{Q}$, where $q = g^{-1}\theta(g)$ (3.2.12).

$$U_\mathbb{Q} * q \mapsto (H_\mathbb{Q}^\theta \times U_\mathbb{Q}) \bullet g \tag{3.2.12}$$

For $q \in Q_\mathbb{Q}^\theta$, assume there exists $g_1, g_2 \in G_\mathbb{Q}$ such that $q = g_1^{-1}\theta(g_1) = g_2^{-1}\theta(g_2)$. Then $g_1 = hg_2$ for some $h \in H_\mathbb{Q}^\theta$ by the following implications (3.2.13).

$$g_1^{-1}\theta(g_1) = g_2^{-1}\theta(g_2) \Rightarrow \theta(g_1 g_2^{-1} = g_1 g_2^{-1} \Rightarrow g_1 g_2^{-1} \in H_\mathbb{Q}^\theta \tag{3.2.13}$$

The map is well-defined because it is independent of coset representative and surjective by definition of $Q_\mathbb{Q}^\theta$. We may also reverse the map (3.2.14).

$$(H_\mathbb{Q}^\theta \times U_\mathbb{Q}) \bullet g \mapsto g^{-1}\theta(g) \tag{3.2.14}$$

By Theorem 3.2.2, let $g = hqu$, where $h \in H_\mathbb{Q}^\theta, q \in \widetilde{Q_\mathbb{Q}^\theta}$ and $u \in U_\mathbb{Q}$. Then $(H_\mathbb{Q}^\theta \times U_\mathbb{Q}) \bullet g$ maps to $U_\mathbb{Q} * q_0$, for some $q_0 = q^{-2} \in Q_\mathbb{Q}^\theta$ (3.2.15).

$$(g^{-1}\theta(g) = (hqu)^{-1}\theta(hqu) = u^{-1}q^{-1}h^{-1}\theta(h)\theta(q)\theta(u) = u^{-1}q^{-1}\theta(q)\theta(u) = u * q_0 \quad (3.2.15)$$

Over $k = \mathbb{Q}$, this map is not surjective because not all elements of $Q_\mathbb{Q}^\theta$ can be written as q^{-2} for some $q \in \widetilde{Q_\mathbb{Q}^\theta}$. From Proposition 3.2.8, the $U_\mathbb{Q}$-orbits on $Q_\mathbb{Q}^\theta$ can always be represented by an element from the normalizer in G of a θ-stable, maximal $\mathbb{Q}$-split torus T, $N_G(T)$. In this case, $N_G(T) \cap Q_\mathbb{Q}^\theta$ is the set of diagonal elements in $G_\mathbb{Q}$. Furthermore, the action of $(H_\mathbb{Q}^\theta \times U_\mathbb{Q})$ on $\lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \in N_G(T) \cap Q_\mathbb{Q}^\theta$ cannot map to another element $\mu = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix} \in N_G(T) \cap Q_\mathbb{Q}^\theta$, if $\mu \neq \pm\lambda$.

$$(H_\mathbb{Q}^\theta \times U_\mathbb{Q}) \bullet \lambda = h\lambda u = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a\lambda_1 & \alpha a\lambda_1 + b\lambda_1^{-1} \\ -b\lambda_1 & -b\lambda_1\alpha + a\lambda_1^{-1} \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix}$$
$$(3.2.16)$$

Solving (3.2.16), we get $h = \pm\operatorname{Id}$ and $u = \operatorname{Id}$. If we let $q^{-1} = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \in \widetilde{Q_\mathbb{Q}^\theta}$, then the only $U_\mathbb{Q}$-orbits on $Q_\mathbb{Q}^\theta$ which correspond to the $(H_\mathbb{Q}^\theta \times U_\mathbb{Q})$-orbits on $G_\mathbb{Q}$ are the ones whose representative in $N_G(T)$ are of the form $q_0 = \begin{pmatrix} x^2 + y^2 & 0 \\ 0 & (x^2 + y^2)^{-1} \end{pmatrix}$. Let $g = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \in G_\mathbb{Q}$ such that $\lambda_1 > 0$ and λ_1 is not the sum of two squares. Then $(H_\mathbb{Q}^\theta \times U_\mathbb{Q}) \bullet g$ cannot be obtained as a $U_\mathbb{Q}$-orbit on $Q_\mathbb{Q}^\theta$. Hence, $G_\mathbb{Q} \neq H_\mathbb{Q}^\theta \widetilde{Q_\mathbb{Q}^\theta} U_\mathbb{Q}$.

3.3 Results for $\mathrm{GL}_2(k)$

The results for $\mathrm{GL}_2(k)$ are much simpler. Ensuring that the factors in Theorem 3.2.2 have a determinant of one makes the decomposition over $\mathrm{SL}_2(k)$ more involved. Note that the characterization of isomorphy classes of involutions of $\mathrm{SL}_2(k)$ in [HW93] also serves as a characterization over $\mathrm{GL}_2(k)$.

Theorem 3.3.1. *Let $G = \mathrm{GL}_2(\bar{k})$ be defined over a field k of characteristic not two and*

θ an involution of G. Then $G_k = H_k^\theta \widetilde{Q_k^\theta}$.

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_k$ and $\theta = \theta_m$ be an involution of G. Then $hg \in \widetilde{Q_k^\theta}$ for $h = \begin{pmatrix} -\frac{m(a+d)}{bm+c} & 1 \\ m & -\frac{m(a+d)}{bm+c} \end{pmatrix} \in H_k^\theta$ (3.3.1).

$$hg = \begin{pmatrix} c - \frac{ma(a+d)}{bm+c} & \frac{cd-mab}{bm+c} \\ -m\left(\frac{cd-mab}{bm+c}\right) & bm - \frac{md(a+d)}{bm+c} \end{pmatrix} \in \widetilde{Q_k^\theta} \tag{3.3.1}$$

Hence, $g \in H_k^\theta \widetilde{Q_k^\theta}$. Note that if $bm + c = 0$, then $g \in \widetilde{Q_k^\theta}$. $\qquad \square$

Example 3.3.2 $(k = \mathbb{R})$**.** Let G be defined over $k = \mathbb{R}$. In this case, there are two involutions, $\theta = \theta_{-1}$ and $\theta = \theta_1$. In both cases, we can simplify the decomposition to $G_\mathbb{R} = H_\mathbb{R}^\theta Q_\mathbb{R}^\theta$. For $\theta = \theta_{-1}$, $H_\mathbb{R}^\theta$ is compact, and therefore θ is a Cartan involution and the traditional Cartan decomposition holds. For $\theta = \theta_1$, we will show in section 4.1 that $Q_\mathbb{R}^\theta = \widetilde{Q_\mathbb{R}^\theta}$.

Example 3.3.3 $(k = \mathbb{F}_p, p \neq 2)$**.** Let G be defined over $k = \mathbb{F}_p$. In section 4.1, we will show $Q_{\mathbb{F}_p}^\theta = \widetilde{Q_{\mathbb{Q}_p}^\theta}$ for any involution θ of G. Thus, we can simply the decomposition to $G_{\mathbb{F}_p} = H_{\mathbb{F}_p}^\theta Q_{\mathbb{F}_p}^\theta$.

3.4 Results for fields of characteristic two

Dealing with fields of characteristic two presents a new challenge. In [Sch13], the isomorphy classes of involutions of $\mathrm{SL}_n(k)$ are classified for such fields. We use these results to verify Theorem 3.2.2 for fields of characteristic two.

Note that in a field k with characteristic 2, we have $x = -x$ for all x in k. The finite field $\mathbb{F}_2 = \{0, 1\}$ has characteristic two and is the smallest of the finite fields. While most fields of characteristic two are isomorphic to $\mathbb{F}_2$, these are not the fields of characteristic two. We will also consider algebraically closed fields. The algebraic closure of a finite field is defined as $\overline{\mathbb{F}_p} = \bigcup_{n \in \mathbb{N}} F_{p^n}$.

Notation 3.4.1. In this section, let $G = \mathrm{SL}_2(\overline{k})$, k a field of characteristic two, and G_k the k-rational points of G.

Theorem 3.4.2. *If k is a finite field or algebraically closed, then there is one isomorphism class of involutions of G_k.*

Notation 3.4.3. For $k = \mathbb{F}_{2^r}$ or $k = \bar{k}$, we will represent this isomorphy class of involutions by $\theta_0 = \mathrm{Inn}(N)$, where $N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Remark 3.4.4. The fixed-point group of θ_0 is the unipotent subgroup H^{θ_0} (3.4.1) and the extended symmetric space of θ_0 is $\widetilde{Q^\theta}$ (3.4.2).

$$H_k^{\theta_0} = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a^2 = 1 \right\} \tag{3.4.1}$$

$$\widetilde{Q_k^{\theta_0}} = \left\{ \begin{pmatrix} x & y \\ z & x+z \end{pmatrix} \;\middle|\; x^2 + xz + yz = 1 \right\} \tag{3.4.2}$$

Because the fixed-point group of θ_0 is unipotent, we will not need to include the unipotent subgroup U from Theorem 3.2.2.

Theorem 3.4.5. *Let G be defined over an algebraically closed field or finite field k and $\theta = \theta_0$ the involution of G. For the fixed-point group H_k^θ, extended symmetric space $\widetilde{Q_k^\theta}$, and Weyl group of the maximal k-split torus $W(T)$, we can factor the group as a generalized Cartan decomposition (3.4.3).*

$$G_k = \bigcup_{\omega \in W(T)} H_k^\theta \omega \widetilde{Q_k^\theta} \tag{3.4.3}$$

Proof. Let $W(T) = \left\{ \mathrm{Id}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_k$ with $c \in k^*$, let $h = \begin{pmatrix} 1 & \frac{a+c+d}{c} \\ 0 & 1 \end{pmatrix} \in H_k^\theta$ and $q = \begin{pmatrix} c+d & \frac{1+cd+d^2}{c} \\ c & d \end{pmatrix} \in \widetilde{Q_k^\theta}$, then $g = hq \in H_k^\theta \widetilde{Q_k^\theta}$. For $g = \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} \in G_k$, let $h = \begin{pmatrix} 1 & a^2 + ab \\ 0 & 1 \end{pmatrix} \in H_k^\theta$ and $q = \begin{pmatrix} 0 & \frac{1}{a} \\ a & a \end{pmatrix} \in \widetilde{Q_k^\theta}$, then $g = h\omega q \in H_k^\theta \omega \widetilde{Q^\theta}$, where $\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\square$

Theorem 3.4.6. *If k is an infinite field which is not algebraically closed, then there is an infinite number of isomorphy classes of involutions of G_k.*

Notation 3.4.7. For an infinite field k such that $k \neq \bar{k}$, we will represent the isomorphy classes of involutions by $\theta_m = \mathrm{Inn}(M)$, where $M = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$.

43

Remark 3.4.8. We have the fixed-point group (3.4.4) and extended symmetric space (3.4.5) of G defined over k with the involution θ_m.

$$H_k^\theta = \left\{ \begin{pmatrix} a & b \\ mb & a \end{pmatrix} \middle| \ a, b \in k, \ a^2 + mb^2 = 1 \right\} \tag{3.4.4}$$

$$\widetilde{Q_k^\theta} = \left\{ \begin{pmatrix} a & b \\ mb & c \end{pmatrix} \middle| \ a, b, c \in k, \ ac + mb^2 = 1 \right\} \tag{3.4.5}$$

Theorem 3.4.9. *Let G be defined over a field k with characteristic 2 and $\theta = \theta_m$ the involution of G. For the fixed-point group H^θ, extended symmetric space $\widetilde{Q^\theta}$, and the unipotent subgroup $U_k = \left\{ \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \middle| u \in k \right\}$, then $G_k = H_k^\theta \widetilde{Q_k^\theta} U_k$.*

Proof. Let $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in G_k$. If $z \neq 0$, for $u = \begin{pmatrix} 1 & \frac{x+w}{z} \\ 0 & 1 \end{pmatrix} \in U_k$ and $h = \begin{pmatrix} 0 & b \\ mb & 0 \end{pmatrix} \in H_k^\theta$, we have $\theta(hgu) = (hgu)^{-1}$ (3.4.6). Hence, $hgu \in \widetilde{Q_k^\theta}$ and $g \in H_k^\theta \widetilde{Q_k^\theta} U_k$.

$$\theta(hgu) = \begin{pmatrix} bm(1 + x^2) & bx \\ mbx & bz \end{pmatrix} = (hgu)^{-1} \tag{3.4.6}$$

If $z = 0$, the $g = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \in G_k$. For $u = \begin{pmatrix} 1 & \frac{y}{x} \\ 0 & 1 \end{pmatrix} \in U_k$, we have $\theta(gu) = (gu)^{-1}$ (3.4.7).

$$\theta(gu) = \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} = (gu)^{-1} \tag{3.4.7}$$

Hence, $gu \in \widetilde{Q_k^\theta}$ and $g \in H_k^\theta \widetilde{Q_k^\theta} U_k$. The reverse containment is clear. $\square$

Chapter 4

Structure of the Symmetric Spaces

To understand the symmetric and extended symmetric spaces of G, we will analyze the relationship between them and then the semisimplicity of their elements.

4.1 Relationship between the symmetric and extended symmetric spaces

Notation 4.1.1. In this section, let $G = \mathrm{SL}_2(\overline{k})$ be defined over the field k, $\overline{k}$ the algebraic closure of k, and G_k the k-rational points of G. Let θ_m denote the involution $\mathrm{Inn}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$, for $\overline{m} \in k^*/(k^*)^2$. Let $Q_k^\theta = \{g\theta(g)^{-1} \mid g \in G_k\}$ be the symmetric space of G_k and $\widetilde{Q_k^\theta} = \{g \in G_k \mid \theta(g) = g^{-1}\}$ the extended symmetric space of G_k.

As their names would suggest, there exists a strong relationship between the symmetric and extended symmetric spaces.

Lemma 4.1.2. *For a group $\mathcal{G}$ with involution θ, the symmetric space is contained within the extended symmetric space. i.e. $Q^\theta \subset \widetilde{Q^\theta}$.*

Proof. For $g\theta(g)^{-1} \in Q_k^\theta$, $\theta(g\theta(g)^{-1}) = (g\theta(g)^{-1})^{-1}$ (4.1.1).

$$\theta(g\theta(g)^{-1}) = \theta(g)\theta^2(g)^{-1} = \theta(g)g^{-1} = (g\theta(g)^{-1})^{-1} \tag{4.1.1}$$

Hence, $g\theta(g)^{-1} \in \widetilde{Q_k^\theta}$.

$\square$

In general, this containment is proper and the symmetric and extended symmetric spaces are not equivalent. We will determine the cases in which we get equality.

Example 4.1.3. Let G be defined over $k = \mathbb{R}$ with the Cartan involution θ_{-1}. The symmetric space consists of the positive-definite symmetric matrices which is contained in, but not equal to, the extended symmetric space which consists of all symmetric matrices.

Theorem 4.1.4. *Let G be defined over $k = \bar{k}$ and $\theta = \theta_1$ the involution of G. Then the extended symmetric space is equivalent to the symmetric space.*

Proof. Let $q = \begin{pmatrix} a & b \\ -b & c \end{pmatrix} \in \widetilde{Q^\theta}$. For q to be in the symmetric space, we need $g \in G$ such that $g\theta(g)^{-1} = q$. Depending on the value of c, choose $g \in G$ according to Table 4.1. Choosing the appropriate g will yield $g\theta(g)^{-1} = q \in Q^\theta$. The reverse containment follows from Lemma 4.1.2.

Table 4.1: $g \in G$ such that $q = g\theta(g)^{-1} \in Q^\theta$ for $k = \bar{k}$

c	b	$g \in G$
$c \neq 0$	-	$\begin{pmatrix} \frac{1}{\sqrt{c}} & \frac{b}{\sqrt{c}} \\ 0 & \sqrt{c} \end{pmatrix}$
$c = 0$	b=1	$\begin{pmatrix} 0 & \sqrt{-a} \\ \frac{\sqrt{-a}}{a} & -\frac{b\sqrt{-a}}{a} \end{pmatrix}$
$c = 0$	$b = -1$	$\begin{pmatrix} 1 & -1 \\ \frac{1-c}{2} & \frac{c+1}{2} \end{pmatrix}$

$\square$

Theorem 4.1.5. *Let G be defined over $k = \mathbb{R}$ and $\theta = \theta_1$ the involution of G. Then the extended symmetric space is equivalent to the symmetric space.*

Proof. Let $q = \begin{pmatrix} a & b \\ -b & c \end{pmatrix} \in \widetilde{Q^\theta_\mathbb{R}}$. As in the proof of Theorem 4.1.4, we need $g \in G_\mathbb{R}$ such that $g\theta(g)^{-1} = q$. Depending on the value of a, choose $g \in G_\mathbb{R}$ according to Table 4.2. Choosing the appropriate g will yield $g\theta(g)^{-1} = q \in Q^\theta$. The reverse containment follows from Lemma 4.1.2.

$\square$

Table 4.2: $g \in G_{\mathbb{R}}$ such that $q = g\theta(g)^{-1} \in Q_{\mathbb{R}}^\theta$ for $k = \mathbb{R}$

a	b	$g \in G_{\mathbb{R}}$
$a > 0$	-	$\begin{pmatrix} \sqrt{a} & 0 \\ -\frac{b}{\sqrt{a}} & \frac{1}{\sqrt{a}} \end{pmatrix}$
$a < 0$	-	$\begin{pmatrix} 0 & \sqrt{-a} \\ \frac{\sqrt{-a}}{a} & -\frac{b\sqrt{-a}}{a} \end{pmatrix}$
$a = 0$	$b = 1$	$\begin{pmatrix} 1 & 1 \\ \frac{c-1}{2} & \frac{c+1}{2} \end{pmatrix}$
$a = 0$	$b = -1$	$\begin{pmatrix} 1 & -1 \\ \frac{1-c}{2} & \frac{c+1}{2} \end{pmatrix}$

In [BHK$^+$], the structure of the symmetric spaces of G_k and $\mathrm{GL}_2(k)$ defined over $k = \mathbb{F}_q$ are analyzed and they provide the following result.

Theorem 4.1.6. *Let G be defined over $k = \mathbb{F}_q$, with characteristic of k not 2, and θ an involution of G. The symmetric space is equivalent to the extended symmetric space.*

4.2 Results for the $\mathfrak{p}$-adic fields

We want to determine the relationship between the symmetric and extended symmetric space when G is defined over $k = \mathbb{Q}_p$ for some prime $p \neq 2$. Working over the $\mathfrak{p}$-adic fields creates a number of challenges. First, there are four isomorphy classes of involutions because $|\mathbb{Q}_p/(\mathbb{Q}_p^*)^2| = 4$, namely $\mathbb{Q}_p/(\mathbb{Q}_p^*)^2 = \{\overline{1}, \overline{p}, \overline{N}_p, \overline{p}N_p\}$, where N_p is the "smallest non-square" in $\mathbb{Q}_p$. Second, the cases when $p \equiv 1 \mod$ and $p \equiv 3 \mod 4$ are handled separately. This is because -1 is not a square in $k = \mathbb{Q}_p$ if $p \equiv 3 \mod 4$ but -1 is a square if $p \equiv 1 \mod 4$. For example, $\sqrt{-1} = 2$ in $k = \mathbb{Q}_5$. When $p \equiv 3 \mod 4$, we let $N_p = -1$.

4.2.1 Hilbert's Symbol

A useful tool when studying G over the $\mathfrak{p}$-adic fields is Hilbert's symbol [Cas78].

Definition 4.2.1. For $a, b \in \mathbb{Q}_p$, *Hilbert's symbol* is defined as

$$(a, b)_p = \begin{cases} 1 & ax^2 + by^2 - z^2 \text{ is isotropic} \\ -1 & \text{otherwise} \end{cases}$$

Remark 4.2.2. The polynomial $ax^2 + bx^2 - z^2$ is isotropic over $\mathbb{Q}_p$ if there exists non-trivial (x, y, z) in $\mathbb{Q}_p^3$ such that $ax^2 + bx^2 - z^2 = 0$.

Proposition 4.2.3 (Properties of the Hilbert Symbol). *For all $a, b, x, y \in \mathbb{Q}_p$, $p \neq 2$, we have the following.*

1. $(a, b)_p = (b, a)_p$

2. $(a, b)_p = 1$ *if $a \in (\mathbb{Q}_p^*)^2$*

3. $(a, -a)_p = 1$

4. $(ax^2, by^2)_p = (a, b)_p$

5. *For $a = a_1 p^n$, $b = b_1 p^m$, with a_1 and b_1 units, $(a, b)_p = (-1|p)^{nm}(a_1|p)^m(b_1|p)^n$, where $(a|p)$ is the Legendre symbol.*

Definition 4.2.4. The *Legendre symbol* $(a|p) = 1$ if a is a square in $\mathbb{F}_p$ and $(a|p) = -1$ if a is not a square in $\mathbb{F}_p$.

We first consider G defined over $k = \mathbb{Q}_p$ with the involution $\theta = \theta_1$. For $q = \begin{pmatrix} a & b \\ -b & c \end{pmatrix} \in \widetilde{Q_{\mathbb{Q}_p}^\theta}$, we must show there exists $g \in G_{\mathbb{Q}_p}$ such that $q = g\theta(g)^{-1}$. To find such $g \in G_{\mathbb{Q}_p}$, we solve for $x, y, z, w \in \mathbb{Q}_p$ such that the following equations hold simultaneously.

$$xw - yz = 1 \tag{4.2.1}$$

$$x^2 - z^2 = a \tag{4.2.2}$$

$$w^2 - z^2 = c \tag{4.2.3}$$

$$yw - xz = b \tag{4.2.4}$$

Using Hilbert's symbol, we obtain solutions to (4.2.2) and (4.2.3) in $\mathbb{Q}_p$. The Hilbert symbol shows that (4.2.2) has a solution because the equivalent equation (4.2.5) below is isotropic over $k = \mathbb{Q}_p$.

$$x^2 - z^2 - y^2 a = 0 \Rightarrow \frac{1}{a}x^2 + \left(-\frac{1}{a}\right)z^2 - y^2 = 0 \tag{4.2.5}$$

By Proposition 4.2.3, $(\frac{1}{a}, -\frac{1}{a})_p = 1$. Therefore (4.2.5) has a non-trivial solution (x, z, y) in $\mathbb{Q}_p^3$. We then scale our solution such that $y = 1$. Similar calculations show (4.2.4) has a solution.

For a simultaneous solution to (4.2.1)-(4.2.4), let

$$x = \frac{-b + \alpha w}{\beta}, y = \alpha, z = \beta$$

where

$$\alpha = \frac{wb \pm \sqrt{w^2 - c}}{c}, \beta = \pm\sqrt{w^2 - c}.$$

We must verify there exists $\beta \in \mathbb{Q}_p$ for all $c \in \mathbb{Q}_p$ such that $\beta^2 = w^2 - c$. Using the Hilbert symbol, this equation corresponds to $(1, -1)_p = 1$ and therefore $\beta \in \mathbb{Q}_p$ exists. Depending on the value of c, choose $g \in G_{\mathbb{Q}_p}$ according to Table 4.3. Choosing the appropriate g will yield $g\theta(g)^{-1} = q \in Q_{\mathbb{Q}_p}^\theta$. Because the reverse containment is clear by Lemma 4.1.2, we have the following result.

Table 4.3: $g \in G_{\mathbb{Q}_p}$ such that $q = g\theta(g)^{-1} \in Q_{\mathbb{Q}_p}^\theta$ for $k = \mathbb{Q}_p$

c	b	$g \in G_{\mathbb{Q}_p}$
$c \neq 0$	-	$\begin{pmatrix} \frac{-bc+w^2b+w\sqrt{w^2-c}}{c\sqrt{w^2-c}} & \frac{wb+\sqrt{w^2-c}}{c} \\ \sqrt{w^2-c} & w \end{pmatrix}$
$c = 0$	b=1	$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 & -1 \end{pmatrix}$
$c = 0$	$b = -1$	$\begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

Theorem 4.2.5. *Let G be defined over $k = \mathbb{Q}_p$ with $p \neq 2$, and $\theta = \theta_1$ the involution of G. The extended symmetric space is equivalent to the symmetric space.*

We can show by example that for the involution θ_m, $m \notin (\mathbb{Q}_p^*)^2$, the symmetric and extended symmetric spaces are not equivalent.

Example 4.2.6. Let $p = 3$ and consider the involution $\theta = \theta_3$. The extended symmetric

space is $\widetilde{Q^\theta_{\mathbb{Q}_3}}$ (4.2.6).

$$\widetilde{Q^\theta_{\mathbb{Q}_3}} = \left\{ \begin{pmatrix} a & b \\ -3b & c \end{pmatrix} \middle| a, b, c \in \mathbb{Q}_3 \text{ and } ac + 3b^2 = 1 \right\} \tag{4.2.6}$$

Let $q = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{pmatrix} \in \widetilde{Q^\theta_{\mathbb{Q}_3}}$. Then $g = g\theta(g)^{-1}$ and $\det(g) = 1$ implies g is g_1 or g_2, where $\alpha = \sqrt{3d^2 - 9}$ (4.2.7).

$$g_1 = \begin{pmatrix} \frac{\sqrt{3}}{3} & 0 \\ 0 & \sqrt{3} \end{pmatrix}, \quad g_2 = \begin{pmatrix} \frac{1}{3}d & \pm\frac{1}{9}\alpha \\ \pm\alpha & d \end{pmatrix} \tag{4.2.7}$$

Because $3 \notin (\mathbb{Q}_3^*)^2$, $g_1 \notin G_{\mathbb{Q}_3}$. For g_2, $\alpha = \sqrt{3d^2 - 9} = \sqrt{3}\sqrt{d^2 - 3} \in \mathbb{Q}_3$ if only if $d^2 - 3 = 3$, which implies $d = \sqrt{6}$. Because 6 is in the square class $pN_p = -3$ and is not a square, $d \notin \mathbb{Q}_3$. Thus there is no $g \in G_{\mathbb{Q}_3}$ such that $q = g\theta(g)^{-1}$ and the symmetric space and extended symmetric space are not equivalent for the involution $\theta = \theta_p$ when $p \equiv 3 \mod 4$.

We will do the same calculations for the other cases. For each, we will get two options for g, and for similar reasons, neither g will be in $G_{\mathbb{Q}_p}$.

Example 4.2.7. Let $p = 5$ and consider the involution $\theta = \theta_5$. The extended symmetric space is $\widetilde{Q^\theta_{\mathbb{Q}_p}}$ (4.2.8).

$$\widetilde{Q^\theta_{\mathbb{Q}_5}} = \left\{ \begin{pmatrix} a & b \\ -5b & c \end{pmatrix} \middle| a, b, c \in \mathbb{Q}_p \text{ and } ac + 5b^2 = 1 \right\} \tag{4.2.8}$$

Let $q = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \in \widetilde{Q^\theta_{\mathbb{Q}_5}}$. For g such that $q = g\theta(g)^{-1}$ and $\det(g) = 1$, g must be of the form g_1 or g_2, where $\alpha = \sqrt{5d^2 - 10}$ (4.2.9).

$$g_1 = \begin{pmatrix} \frac{1}{2}\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \quad g_2 = \begin{pmatrix} \frac{1}{2}d & \pm\frac{1}{10}\alpha \\ \pm\alpha & d \end{pmatrix} \tag{4.2.9}$$

Using calculations similar to Example 4.2.6, we see that g_1 and g_2 are not in $G_{\mathbb{Q}_5}$. Therefore, $\widetilde{Q^\theta_{\mathbb{Q}_5}}$ is not contained in $Q^\theta_{\mathbb{Q}_5}$ when $\theta = \theta_p$ and $p \equiv 1 \mod 4$.

Example 4.2.8. Let $p = 3$ and consider the involution $\theta = \theta_{N_p}$. Because $p \equiv 3 \mod 4$, we will use $\theta = \theta_{-1}$. Let $q = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{pmatrix} \in \widetilde{Q^\theta_{\mathbb{Q}_3}}$. For g such that $q = g\theta(g)^{-1}$ and $\det(g) = 1$, g must

be of the form g_1 or g_2, where $\alpha = \sqrt{-d^2 + 3}$ (4.2.10).

$$g_1 = \begin{pmatrix} \frac{1}{3}\sqrt{3} & 0 \\ 0 & \sqrt{3} \end{pmatrix}, \quad g_2 = \begin{pmatrix} \frac{1}{3}d & \pm\frac{1}{3}\alpha \\ \pm\alpha & d \end{pmatrix} \qquad (4.2.10)$$

We can show g_1 and g_2 are not defined over $\mathbb{Q}_5$ using the same method as in Example 4.2.6. Therefore, $\widetilde{Q^\theta_{\mathbb{Q}_3}}$ is not contained in $Q^\theta_{\mathbb{Q}_3}$ when $\theta = \theta_{N_p}$ and $p \equiv 3 \mod 4$.

Example 4.2.9. Let $p = 5$ and consider the involution $\theta = \theta_{N_p}$. Choose $N_p = 2$, then $\theta = \theta_2$. Let $q = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 5 \end{pmatrix} \in \widetilde{Q^\theta_{\mathbb{Q}_5}}$. For g such that $q = g\theta(g)^{-1}$ and $\det(g) = 1$, g must be of the form g_1 or g_2, where $\alpha = \sqrt{2d^2 - 10}$ (4.2.11).

$$g_1 = \begin{pmatrix} \frac{1}{5}\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix}, \quad g_2 = \begin{pmatrix} \frac{1}{5}d & \pm\frac{1}{10}\alpha \\ \pm\alpha & d \end{pmatrix} \qquad (4.2.11)$$

Again, both g_1 and g_2 are not defined over $\mathbb{Q}_5$. Hence, $\widetilde{Q^\theta_{\mathbb{Q}_5}}$ is not contained in $Q^\theta_{\mathbb{Q}_5}$ if $p \equiv 1 \mod 4$ and $\theta = \theta_{N_p}$.

Example 4.2.10. Let $p = 3$ and consider the involution $\theta = \theta_{pN_p}$. Then $pN_p = -3$, so the involution is $\theta = \theta_{-3}$. Let $q = \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 3 \end{pmatrix} \in \widetilde{Q^\theta_{\mathbb{Q}_3}}$. For g such that $q = g\theta(g)^{-1}$ and $\det(g) = 1$, g must be of the form g_1 or g_2, where $\alpha = \sqrt{-3d^2 + 9}$ (4.2.12).

$$g_1 = \begin{pmatrix} -\frac{1}{3}\sqrt{-3} & 0 \\ 0 & \sqrt{-3} \end{pmatrix}, \quad g_2 = \begin{pmatrix} -\frac{1}{3}d & \pm\frac{1}{9}\alpha \\ \pm\alpha & d \end{pmatrix} \qquad (4.2.12)$$

Again, g is not defined over $\mathbb{Q}_3$. Therefore $\widetilde{Q^\theta_{\mathbb{Q}_3}}$ is not contained in $Q^\theta_{\mathbb{Q}_3}$ for $\theta = \theta_{pN_p}$ and $p \equiv 3 \mod 4$.

Example 4.2.11. Let $p = 5$ and consider the involution $\theta = \theta_{pN_p}$. Let $N_p = 2$, then $\theta = \theta_{10}$. Let $q = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$. For g such that $q = g\theta(g)^{-1}$ and $\det(g) = 1$, g must be of the form g_1 or g_2 where $\alpha = \sqrt{10d^2 - 20}$ (4.2.13).

$$g_1 = \begin{pmatrix} \frac{1}{2}\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \quad g_2 = \begin{pmatrix} \frac{1}{2}d & \pm\frac{1}{20}\alpha \\ \pm\alpha & d \end{pmatrix} \qquad (4.2.13)$$

Clearly g_1 is not defined over $\mathbb{Q}_5$ by definition of N_p. For g_2 we need to consider when α is in $\mathbb{Q}_5$. Because $\alpha = \sqrt{10d^2 - 20} = \sqrt{5}\sqrt{2d^2 - 4}$, α will be in $\mathbb{Q}_5$ if $2d^2 - 4 = 5$. Solving for d we get $d = \frac{3}{2}\sqrt{2}$ which is not defined over $\mathbb{Q}_5$. Therefore, there is no $g \in G_{\mathbb{Q}_5}$ such that $q = g\theta(g)^{-1}$ and $\widetilde{Q^\theta_{\mathbb{Q}_5}}$ is not contained in $Q^\theta_{\mathbb{Q}_5}$ when $\theta = \theta_{pN_p}$ and $p \equiv 1 \mod 4$.

These examples serve to show that for G defined over $k = \mathbb{Q}_p$ with the involution θ_m, $\widetilde{Q^\theta_{\mathbb{Q}_p}} = Q^\theta_{\mathbb{Q}_p}$ only when $m \in \bar{1}$.

Theorem 4.2.12 (Strong Hasse Principle)**.** *Let f be a regular quadratic form over $\mathbb{Q}$. Then f is isotropic over $\mathbb{Q}$ if and only if f is isotropic over $\mathbb{Q}_p$ for all p, including $p = \infty$.*

By Theorem 4.1.5, equations (4.2.1)-(4.2.4) have a simultaneous solution over $k = \mathbb{R} = \mathbb{Q}_\infty$. Applying Theorem 4.2.12 to Theorems 4.1.5 and 4.2.5, equations (4.2.1)-(4.2.4) must also have a solution over $k = \mathbb{Q}$, yielding the following result.

Theorem 4.2.13. *Let G be defined over $k = \mathbb{Q}$ and $\theta = \theta_1$ the involution of G. Then the extended symmetric space is equivalent to the symmetric space.*

Corollary 4.2.14. *For G defined over k and $\theta = \theta_1$ the involution of G, the extended symmetric space and symmetric space are equivalent over the following fields of characteristic not two.*

1. k algebraically closed

2. $k = \mathbb{R}$

3. $k = \mathbb{F}_q$

4. $k = \mathbb{Q}_p$

5. $k = \mathbb{Q}$

Corollary 4.2.15. *For G defined over a field listed in Corollary 4.2.14 with the involution $\theta = \theta_1$, or G defined over $k = \mathbb{F}_q$ with any involution θ, the decomposition in Theorem 3.2.2 can be simplified to $G_k = H^\theta_k Q^\theta_k U_k$.*

4.2.2 Semisimplicity of the symmetric and extended symmetric spaces

We will determine when the symmetric and similarly the extended symmetric spaces consist of semisimple elements. From [HW93] and [BH09], respectively, we have the following results.

Theorem 4.2.16. *Let k be a field with characteristic zero. If H_k^θ is k-anisotropic, then the symmetric space consists of semisimple elements.*

Theorem 4.2.17. *Let G be defined over a field k and $\theta = \theta_m$ the involution of G. The fixed-point group of θ_m is k-anisotropic if and only if $m \notin \bar{1}$.*

Combining the previous two results, we have the following corollary.

Corollary 4.2.18. *Let G be defined over a field k with characteristic zero and $\theta = \theta_m$ the involution of G. If $m \notin \bar{1}$ then the symmetric space consists of semisimple elements.*

Example 4.2.19. For G defined over a field k with the involution θ_m, the corresponding symmetric space consists of semisimple elements in the following cases.

1. $k = \mathbb{R}$ and $m = -1$

2. $k = \mathbb{Q}_p$ and $m = p$, N_p, or pN_p

3. $k = \mathbb{Q}$ and $m \notin \bar{1}$

Remark 4.2.20. To complete Theorem 4.2.16 and Corollary 4.2.18 for fields of characteristic not 2, we need to determine if the symmetric space contains only semisimple elements for the involution θ_m, when $m \neq 1$, over fields with prime characteristic p.

Theorem 4.2.21. *Let G be defined over a field k and $\theta = \theta_m$ the involution of G. If $m \notin \bar{1}$, then the extended symmetric space consists of semisimple elements.*

Proof. Let $q = \begin{pmatrix} a & b \\ -mb & c \end{pmatrix} \in \widetilde{Q_k^\theta}$. To determine if q is semisimple, we analyze its eigenvalues (4.2.14). If q has two distinct eigenvalues, then q is semisimple. The cases of concern are when q has one eigenvalue with multiplicity 2.

$$\text{eigenvalues of } q = \left\{ \frac{1}{2} \left(a + c \pm \sqrt{c^2 - 2ac + a^2 - 4mb^2} \right) \right\} \tag{4.2.14}$$

Because $\det(q) = 1$, $(a + c)^2 = 4$ is a necessary and sufficient condition for q to have one eigenvalue. This implies, q has one eigenvalue if and only if $a + c = \pm 2$.

Assume $a + c = 2$, then $q = \begin{pmatrix} a & b \\ -mb & 2 - a \end{pmatrix}$ and $\det(q) = 1$ implies $y = \pm\frac{x-1}{\sqrt{m}}$. By assumption, $\sqrt{m} \notin k$, which implies $a = 1$ and $b = 0$, i.e. $q = \mathrm{Id}$. If you assume $a + c = -2$, similar calculations yield $q = -\mathrm{Id}$. Hence, q has two distinct eigenvalues and is diagonalizable or $q = \pm\mathrm{Id}$ which is already diagonal. $\qquad\square$

Corollary 4.2.22. *Let G be defined over k and $\theta = \theta_m$ the involution of G. If $m \notin \bar{1}$, then the symmetric space consists of semisimple elements.*

Proof. By Lemma 4.1.2 and Theorem 4.2.21, the elements of the symmetric space must be semisimple. $\qquad\square$

Remark 4.2.23. While combining Theorems 4.2.16 and 4.2.17 proves this result for fields with characteristic zero, our result and proof holds for any field with characteristic not 2.

As shown in the following examples, if $m \in \bar{1}$ then the symmetric space, and hence extended symmetric space, contain unipotent elements.

Example 4.2.24. Let G be defined over $k = \mathbb{R}$ and consider the involution $\theta = \theta_1$. For $g = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -1 & 1 \end{pmatrix} \in G_{\mathbb{R}}$, $q = g\theta(g)^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \in Q_{\mathbb{R}}^{\theta}$. The Jordan form of q is $J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which cannot be diagonalized and therefore q is not semisimple. Hence, not all elements of the symmetric space or extended symmetric space are semisimple.

Example 4.2.25. Let G be defined over $k = \mathbb{F}_7$ and consider the involution $\theta = \theta_1$. For $g = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix} \in G_{\mathbb{F}_7}$, $q = g\theta(g)^{-1} = \begin{pmatrix} 4 & 3 \\ 4 & 5 \end{pmatrix} \in Q_{\mathbb{Q}_7}^{\theta}$. The Jordan form of q is $J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ which cannot be diagonalized and therefore q is not semisimple. Hence not elements of the symmetric space (which is equivalent to the extended symmetric space) are semisimple.

Lemma 4.2.26. *Let G be defined over a field k and $\theta = \theta_1$ the involution of G. There exists elements in the symmetric space which are not semisimple.*

Proof. We will construct an element in the symmetric space with a unipotent factor. Let $g = \begin{pmatrix} x + 2 & x + 1 \\ -(x + 1) & -x \end{pmatrix} \in G_k$ for some $x \in k \smallsetminus \{-1\}$. Then $q = g\theta(g)^{-1} \in Q^{\theta}$ has a Jordan

decomposition with a unipotent factor (4.2.15), thus q is not semisimple.

$$q = g\theta(g)^{-1} = \begin{pmatrix} 3 + 2x & 2 + 2x \\ -(2 + 2x) & -(2x + 1) \end{pmatrix} = S^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} S \qquad (4.2.15)$$

$\square$

Corollary 4.2.27. *Let G be defined over k and $\theta = \theta_1$ the involution of G. There exists elements in the extended symmetric space which are not semisimple.*

Proof. This follows from Lemmas 4.1.2 and 4.2.26. $\square$

Corollary 4.2.28. *Let G be defined over k and $\theta = \theta_m$ the involution of G. The symmetric space and extended symmetric space consist of semisimple elements if and only if $m \notin \bar{1}$.*

Lemma 4.2.29. *Let G be defined over k, θ an involution of G, and A a (θ, k)-split torus of G. Then the image of A_k under conjugation by H_k^θ is contained in the extended symmetric space.*

Proof. Let $a \in A_k$ and $h \in H_k^\theta$. Then $hah^{-1} \in \widetilde{Q^\theta}$ (4.2.16).

$$\theta(hah^{-1}) = \theta(h)\theta(a)\theta(h^{-1}) = ha^{-1}h^{-1} = (hah^{-1})^{-1} \qquad (4.2.16)$$

$\square$

Theorem 4.2.30. *Let G be defined over k and $\theta = \theta_m$ the involution of G. If $m \notin \bar{1}$ then the extended symmetric space decomposes as the disjoint union of the H_k^θ-orbits of the maximal k-split tori $\{A_i \mid i \in I\}$ (4.2.17).*

$$\widetilde{Q_k^\theta} = \biguplus_{i \in I} H_k^\theta \cdot (A_i)_k \qquad (4.2.17)$$

Proof. Let $\theta = \theta_m$ with $m \notin \bar{1}$. By Lemma 4.2.16, $H_k \cdot (A_i)_k \subset \widetilde{Q_k^\theta}$ for all $\{A_i \mid i \in I\}$. For $q \in \widetilde{Q_k^\theta}$, q is θ-split and semisimple by Corollary 4.2.28. Thus q must be contained in the H_k^θ-conjugacy class of some k-split torus $(A_i)_k$. $\square$

Corollary 4.2.31. *Let G be defined over k and $\theta = \theta_m$ the involution of G. If $m \notin \bar{1}$, then G_k decomposes as $G_k = \bigcup_{i \in I} H_k^\theta (A_i)_k H_k^\theta U_k$, where $\{A_i \mid i \in I\}$ are the H_k^θ-conjugacy classes of maximal k-split tori.*

Notation 4.2.32. Let $(Q^\theta)^{ss}$ and $(\widetilde{Q^\theta})^{ss}$ denote the subset of semisimple elements in the symmetric space and extended symmetric space, respectively.

Lemma 4.2.33. *Let G be defined over k and $\theta = \theta_1$ the involution of G. Then G_k decomposes as $G_k = H_k^\theta (\widetilde{Q_k^\theta})^{ss} U_k$.*

Proof. By Theorem 3.2.2, it suffices to show $\widetilde{Q_k^\theta} \smallsetminus \widetilde{Q_k^\theta}^{ss} \in H_k^\theta(\widetilde{Q_k^\theta})^{ss} U_k$. Using the construction of q as in Theorem 4.2.21, let $q = \begin{pmatrix} x & x-1 \\ 1-x & 2-x \end{pmatrix} \in \widetilde{Q_k^\theta} \smallsetminus \widetilde{Q_k^\theta}^{ss}$, $x \neq 1$. Take $h = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in H_k^\theta \smallsetminus \{\pm \mathrm{Id}\}$ and $u = \begin{pmatrix} 1 & \frac{2b(-ax-bx+b)}{2b^2x-b^2+x^2} \\ 0 & 1 \end{pmatrix} \in U_k$. For u to be defined, choose a, b such that $x \neq -b^2 \pm ba$. Then $\theta(hqu) = (hqu)^{-1}$ (4.2.18), hence $hqu \in \widetilde{Q^\theta}$.

$$\theta(hgu) = \begin{pmatrix} \frac{2a^2bx^2 - 2ab^2x^2 - 2a^2bx + ab^2x - ax^3 - b^3x + bx^3 + 2ax^2 + b^3 - bx^2}{2b^2x - b^2 + x^2} & -ax + bx + a \\ -\frac{2a^2bx^2 - 2ab^2x^2 + 3ab^2x - ax^3 - b^3x + bx^3 - ab^2 + ax^2 - 2bx^2}{2b^2x - b^2 + x^2} & ax - bx + b \end{pmatrix} = (hqu)^{-1} \quad (4.2.18)$$

Furthermore, the Jordan decomposition of hqu is $hqu = S^{-1}JS$ (4.2.19). Therefore hqu is semisimple.

$$J = \begin{pmatrix} f_1(a,b,x) & 0 \\ 0 & f_2(a,b,x) \end{pmatrix} \quad (4.2.19)$$

$\square$

We can now simplify our main result, Theorem 3.2.2.

Corollary 4.2.34. *Let G be defined over k and $\theta = \theta_m$ an involution of G. Then G_k decomposes as $G_k = H_k^\theta (\widetilde{Q_k^\theta})^{ss} U_k$.*

Proof. If $m = 1$, use Lemma 4.2.33. If $m \neq 1$, then $\widetilde{Q_k^\theta} = (\widetilde{Q_k^\theta})^{ss}$. $\square$

Chapter 5

Refining the Decomposition

5.1 Pairwise intersections of H_k^θ, $\widetilde{Q_k^\theta}$, and U_k

We will begin refining the decomposition by determining the pairwise intersections of the fixed-point group, the extended symmetric space, and the unipotent subgroup. Doing so will allow us to determine when we can simplify the decomposition $G_k = H_k^\theta \widetilde{Q_k^\theta} U_k$.

Notation 5.1.1. In this chapter, let $G = \mathrm{SL}_2(\overline{k})$, G_k the k-rational points of G, and $\theta = \theta_m$ the involution of G. Let H^θ be the fixed-point group, $\widetilde{Q^\theta}$ the extended symmetric space, and U the unipotent subgroup of upper triangular matrices with ones on the diagonal. Let Id denote the 2×2 identity matrix.

Proposition 5.1.2.

$$H_k^\theta \widetilde{Q^\theta} \bigcap U_k = \left\{ \begin{pmatrix} 1 & \frac{2b}{a} \\ 0 & 1 \end{pmatrix} \middle| \ a \in k^*, \ b \in k, \ a^2 - mb^2 = 1 \right\}$$

Proof. Let $X = \begin{pmatrix} ax - mby & ay + bz \\ m(bx - ay) & mby + az \end{pmatrix} \in H_k^\theta \widetilde{Q_k^\theta}$ for some $a^2 - mb^2 = 1$ and $xz + my^2 = 1$. Then $X \in U_k$ implies there exists $u \in U_k$ such that $X = u$ (5.1.1).

$$\begin{pmatrix} ax - mby & ay + bz \\ m(bx - ay) & mby + az \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \tag{5.1.1}$$

Solving (5.1.1), we obtain $\alpha = \frac{2b}{a}$. $\qquad\square$

Example 5.1.3. For G defined over $k = \mathbb{F}_3$ with an arbitrary involution θ, we have $H^\theta_{\mathbb{F}_3} \widetilde{Q^\theta_{\mathbb{F}_3}} \cap U_{\mathbb{F}_3} = \{\pm \mathrm{Id}\}$. The only solutions when $m = 1$ which give us $a^2 - b^2 = 1$ are $\{a = \pm 1, b = 0\}$. The only solutions when $m = 2$ which give us $a^2 + b^2 = 1$ and $\{a = \pm 1, b = 0\}$ and $\{a = 0, b = \pm 1\}$.

Example 5.1.4. For G defined over $k = \mathbb{F}_5$ with the involution $\theta = \theta_1$, we have the intersection $H^\theta_{\mathbb{F}_5} \widetilde{Q^\theta_{\mathbb{F}_5}} \cap U_{\mathbb{F}_5} = \{\pm \mathrm{Id}\}$. For the same group with the involution $\theta = \theta_2$, we have the intersection $H^\theta_{\mathbb{F}_5} \widetilde{Q^\theta_{\mathbb{F}_5}} \cap U_{\mathbb{F}_5} = \left\{ \pm \mathrm{Id}, \pm \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\}$.

Example 5.1.5. For G defined over $k = \mathbb{F}_7$ with the involution $\theta = \theta_1$, we have the intersection $H^\theta_{\mathbb{F}_7} \widetilde{Q^\theta_{\mathbb{F}_7}} \cap U_{\mathbb{F}_7} = \left\{ \mathrm{Id}, \begin{pmatrix} 1 & \pm 3 \\ 0 & 1 \end{pmatrix} \right\}$. For the same group with the other involution, $\theta = \theta_3$, we have the intersection $H^\theta_{\mathbb{F}_7} \widetilde{Q^\theta_{\mathbb{F}_7}} \cap U_{\mathbb{F}_7} = \left\{ \mathrm{Id}, \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix} \right\}$.

Example 5.1.6. For G defined over $k = \mathbb{R}$ with an arbitrary involution θ, the intersection $H^\theta_{\mathbb{R}} \widetilde{Q^\theta_{\mathbb{R}}} \cap U_{\mathbb{R}}$ is infinite.

Remark 5.1.7. The order of Proposition 5.1.2 is equivalent to the order of H^θ_k minus the elements of H^θ_k with zeroes on the diagonal (5.1.2).

$$|H^\theta_k \widetilde{Q^\theta_k} \cap U_k| = |H^\theta_k| - \left| \left\{ b \in k^\star \ \middle| \ b = \pm \frac{1}{\sqrt{-m}} \right\} \right| \tag{5.1.2}$$

Proposition 5.1.8.
$$H^\theta_k \cap \widetilde{Q^\theta} = U_k \cap H^\theta_k = U_k \cap \widetilde{Q^\theta} = \pm \mathrm{Id}$$

Proof. This is clear by the definitions of $H^\theta_k, \widetilde{Q^\theta_k}$, and U_k.

$\square$

Proposition 5.1.9.

$$H^\theta_k \cap \widetilde{Q^\theta_k} U_k = \left\{ \begin{pmatrix} a & b \\ mb & a \end{pmatrix} \ \middle| \ a \in k^\star, \ b \in k, \ a^2 - mb^2 = 1 \right\}$$

Proof. Let $X = \begin{pmatrix} x & \alpha x + y \\ -my & -my\alpha + z \end{pmatrix} \in \widetilde{Q^\theta_k} U_k$, for some $xz + my^2 = 1$. Then $X \in H^\theta_k$ implies there exists $h \in H^\theta_k$ such that $X = h$ (5.1.3).

58

$$\begin{pmatrix} x & \alpha x + y \\ -my & -my\alpha + z \end{pmatrix} = \begin{pmatrix} a & b \\ mb & a \end{pmatrix} \tag{5.1.3}$$

Solving (5.1.3), we obtain $x = a$, $y = -b$ and $\alpha = \frac{2b}{a}$.

<div align="right">□</div>

Remark 5.1.10. The size of Proposition 5.1.9 is the order of H_k^θ minus the elements in H_k^θ with zeroes on the diagonal (5.1.4). Furthermore, the size of Proposition 5.1.9 is equivalent to the size of Proposition 5.1.2.

$$|(H_k^\theta \cap \widetilde{Q_k^\theta U_k})| = |H_k^\theta \widetilde{Q_k^\theta} \cap U_k| = |H_k^\theta| - \left|\left\{ b \in k^* \,\middle|\, b = \pm\frac{1}{\sqrt{-m}} \right\}\right| \tag{5.1.4}$$

This intersection is almost equivalent to H_k^θ.

$$H_k^\theta \smallsetminus (H_k^\theta \cap \widetilde{Q_k^\theta U_k}) = \left\{ \begin{pmatrix} 0 & b \\ mb & 0 \end{pmatrix} \,\middle|\, b \in k, \ -mb^2 = 1 \right\}$$

Lemma 5.1.11. *Let G be defined over k and $\theta = \theta_m$ the involution of G. The fixed-point group of θ_m is contained in $\widetilde{Q_k^\theta U_k}$ if and only if $-m \notin \bar{1}$.*

Proof. This proof follows from a chain of equivalent statements.

$$H_k^\theta \subset \widetilde{Q^\theta U_k} \qquad\qquad \Leftrightarrow$$

$$H_k^\theta \smallsetminus (H_k^\theta \cap \widetilde{Q_k^\theta U_k}) = \left\{ \begin{pmatrix} 0 & b \\ mb & 0 \end{pmatrix} \,\middle|\, b \in k, \ -mb^2 = 1 \right\} = \varnothing \qquad\qquad \Leftrightarrow$$

$$b = \pm\frac{1}{\sqrt{-m}} \notin k$$

<div align="right">□</div>

Example 5.1.12. When $H_k^\theta \subset \widetilde{Q_k^\theta U_k}$, we do not necessarily have $G_k = \widetilde{Q_k^\theta U_k}$. Let G be defined over $k = \mathbb{R}$ and $\theta = \theta_1$ the involution of G. By Lemma 5.1.11, $H_\mathbb{R} \subset \widetilde{Q_\mathbb{R}^\theta U_\mathbb{R}}$. Consider $g = \begin{pmatrix} 0 & \frac{1}{2} \\ -2 & 0 \end{pmatrix} \in G_\mathbb{R}$, then $gu \notin \widetilde{Q_\mathbb{R}^\theta}$ for any $u = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in U_\mathbb{R}$ (5.1.5). Therefore $G_\mathbb{R} \neq \widetilde{Q_\mathbb{R}^\theta U_\mathbb{R}}$.

$$\theta(gu) = \begin{pmatrix} -2\alpha & -2 \\ \frac{1}{2} & 0 \end{pmatrix} \neq \begin{pmatrix} -2\alpha & -\frac{1}{2} \\ 2 & 0 \end{pmatrix} = (gu)^{-1} \tag{5.1.5}$$

<div align="center">59</div>

5.1.1 Generalization of the Iwasawa decomposition

The following results lead to a simplified generalization of the Iwasawa decomposition for certain involutions and fields.

Proposition 5.1.13.

$$H_k^\theta U_k \cap \widetilde{Q_k^\theta} = \left\{ \begin{pmatrix} x & y \\ -my & z \end{pmatrix} \middle| x \in k^*, \ y, z \in k, \ xz + my^2 = 1 \right\}$$

Proof. Let $X = \begin{pmatrix} a & \alpha a + b \\ mb & mb\alpha + a \end{pmatrix} \in H_k^\theta U_k$, for some $a^2 - mb^2 = 1$. Then $X \in \widetilde{Q_k^\theta}$ implies there exists $q \in \widetilde{Q_k^\theta}$ such that $X = q$ (5.1.6).

$$\begin{pmatrix} a & \alpha a + b \\ mb & mb\alpha + a \end{pmatrix} = \begin{pmatrix} x & y \\ -my & z \end{pmatrix} \tag{5.1.6}$$

Solving (5.1.6), we obtain $a = x$, $b = -y$, and $\alpha = \frac{2y}{x}$.

$\square$

Remark 5.1.14. Similar to Remark 5.1.10, Proposition 5.1.13 is almost equivalent to the extended symmetric space (5.1.7).

$$\widetilde{Q_k^\theta} \smallsetminus (H_k^\theta U_k \cap \widetilde{Q_k^\theta}) = \left\{ \begin{pmatrix} 0 & y \\ -my & z \end{pmatrix} \middle| y \in k, \ my^2 = 1 \right\} \tag{5.1.7}$$

Lemma 5.1.15. *Let G be defined over k and $\theta = \theta_m$ the involution of G. Then the extended symmetric space is contained in $H_k^\theta U_k$ if and only if $m \notin \overline{1}$.*

Proof. This proof follows from a chain of equivalent statements.

$$\widetilde{Q_k^\theta} \subset H_k^\theta U_k \qquad\qquad \Leftrightarrow$$

$$\widetilde{Q_k^\theta} \smallsetminus (H_k^\theta U_k \cap \widetilde{Q_k^\theta}) = \left\{ \begin{pmatrix} 0 & y \\ -my & z \end{pmatrix} \middle| y \in k, \ my^2 = 1 \right\} = \varnothing \qquad \Leftrightarrow$$

$$y = \pm \frac{1}{\sqrt{m}} \notin k$$

$\square$

60

Theorem 5.1.16. *Let G be defined over k and $\theta = \theta_m$ the involution of G. If $m \notin \bar{1}$, then $G_k = H_k^\theta U_k$.*

Proof. Let $g \in G_k$ and $\theta = \theta_m$ the involution of G with $m \notin \bar{1}$. By Theorem 3.2.2 write $g = hqu$ for some $h \in H_k^\theta, q \in \widetilde{Q_k^\theta}$ and $u \in U_k$. By Lemma 5.1.15, write $q = h_1 u_1$ for some $h_1 \in H_k^\theta$ and $u_1 \in U_k$. Thus, $g = hh_1u_1u \in H_k^\theta U_k$. The reverse containment in clear. $\square$

Theorem 5.1.17. *Let G be defined over k and $\theta = \theta_1$ the involution of G. Then $G_k = \bigcup_{\omega \in W(T)} H_k^\theta \omega U_k$, where $W(T)$ is the Weyl group of a maximal k-split torus T.*

Proof. Let $W(T) = \left\{ \mathrm{Id}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ and $g \in G_k$. If $g \in H_k^\theta U_k \cap \widetilde{Q_k^\theta}$, then $g \in H_k^\theta U_k$. If $g \notin H_k^\theta U_k \cap \widetilde{Q_k^\theta}$, write $g = hqu$ as in Theorem 3.2.2, where $q \in \widetilde{Q_k^\theta} \setminus (H_k^\theta U_k \cap \widetilde{Q_k^\theta})$. By Remark 5.1.14, let $q = \begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix}$ without loss of generality and $u_1 = \begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix}$, then $g \in H_k^\theta \omega U_k$ for $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in W(T)$ (5.1.8).

$$g = hqu = h \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_1 u \in H_k^\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U_k \tag{5.1.8}$$

$\square$

Remark 5.1.18. Recall that θ_m is a generalized Cartan involution if and only if $m \notin 1$ by Theorem 4.2.17. Therefore, this generalization of the Iwasawa decomposition aligns nicely with the traditional version.

5.2 Commutativity of H_k^θ, $\widetilde{Q_k^\theta}$, and U_k

We will discuss whether or not the order of the factors in $G_k = H_k^\theta \widetilde{Q_k^\theta} U_k$ effect the decomposition.

Lemma 5.2.1. *Let G be defined over k and θ an involution of G. Then $H_k^\theta \widetilde{Q_k^\theta} = \widetilde{Q_k^\theta} H_k^\theta$.*

Proof. Let $g \in \widetilde{Q_k^\theta} H_k^\theta$, then $g = q_1 h_1$ for some $q_1 \in \widetilde{Q_k^\theta}$ and $h_1 \in H_k^\theta$. Using Lemma 4.2.16, $g \in H_k^\theta \widetilde{Q_k^\theta}$ (5.2.1).

$$g = q_1 h_1 = h_1(h_1^{-1} q_1 h_1) \in H_k^\theta \widetilde{Q_k^\theta} \tag{5.2.1}$$

Similarly, let $g = h_1 q_1 \in H_k^\theta \widetilde{Q_k^\theta}$, then $g \in \widetilde{Q_k^\theta} H_k^\theta$ (5.2.2).

$$g = h_1 q_1 = (h_1 q_1 h_1^{-1}) h_1 \in \widetilde{Q_k^\theta} H_k^\theta \tag{5.2.2}$$

$\square$

Remark 5.2.2. In general, $U_k H_k^\theta \neq H_k^\theta U_k$ and $U_k \widetilde{Q_k^\theta} \neq \widetilde{Q_k^\theta} U_k$.

Example 5.2.3. Let G be defined over $k = \mathbb{R}$ with the involution $\theta = \theta_1$. For $h = \begin{pmatrix} \sqrt{10} & 3 \\ 3 & \sqrt{10} \end{pmatrix} \in H_\mathbb{R}^\theta$ and $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in U_\mathbb{R}$, $hu = \begin{pmatrix} \sqrt{10} & \sqrt{10} + 3 \\ 3 & \sqrt{10} + 3 \end{pmatrix} \in H_\mathbb{R}^\theta U_\mathbb{R}$. If $hu \in U_\mathbb{R} H_\mathbb{R}^\theta$, then there exists $a, b, \alpha \in \mathbb{R}$ such that $a^2 - b^2 = 1$ and (5.2.3) has a solution.

$$hu = \begin{pmatrix} \sqrt{10} & \sqrt{10} + 3 \\ 3 & \sqrt{10} + 3 \end{pmatrix} = \begin{pmatrix} \alpha b + a & \alpha a + b \\ b & a \end{pmatrix} = u_1 h_1 \tag{5.2.3}$$

This implies $b = 3$ and $a = \sqrt{10} + 3$, but then $a^2 + b^2 \neq 1$, hence there is no solution.

Example 5.2.4. Let G be defined over $k = \mathbb{R}$ with the involution $\theta = \theta_1$. For $\begin{pmatrix} \frac{1}{2} & 2 \\ -2 & 6 \end{pmatrix} \in \widetilde{Q_\mathbb{R}^\theta}$ and $u = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \in U_\mathbb{R}$, $qu = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -2 & 0 \end{pmatrix} \in \widetilde{Q_\mathbb{R}^\theta} U_\mathbb{R}$. If $qu \in U_\mathbb{R} \widetilde{Q_\mathbb{R}^\theta}$, then there exists $u_1 = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ such that $u_1 qu \in \widetilde{Q_\mathbb{R}^\theta}$.

$$u_1 qu = \begin{pmatrix} \frac{1}{2} - 2\alpha & \frac{1}{2} \\ -2 & 0 \end{pmatrix}$$

Clearly, no matter the choice of α, $u_1 qu \notin \widetilde{Q_\mathbb{R}^\theta}$.

Lemma 5.2.5. *Let G be defined over k and θ an involution of G, then $G_k = U_k H_k^\theta \widetilde{Q_k^\theta}$. and $G_k = \widetilde{Q_k^\theta} U_k H_k^\theta$.*

The proof of Lemma 5.2.5 follows the same technique as the proof of Theorem 3.2.2, using the Bruhat decomposition. One can construct $u_1, u_2 \in U_k$ and $h_1, h_2 \in H_k^\theta$ such that $h_1 u_1 g \in \widetilde{Q_k^\theta}$ and $g h_2 u_2 \in \widetilde{Q_k^\theta}$.

Corollary 5.2.6. *Let G be defined over k and θ an involution of G. The following decompositions of G_k are equivalent.*

1. $G_k = H_k^\theta \widetilde{Q_k^\theta} U_k$

2. $G_k = H_k^\theta U_k \widetilde{Q_k^\theta}$

3. $G_k = \widetilde{Q_k^\theta} H_k^\theta U_k$

4. $G_k = \widetilde{Q_k^\theta} U_k H_k^\theta$

5. $G_k = U_k H_k^\theta \widetilde{Q_k^\theta}$

6. $G_k = U_k \widetilde{Q_k^\theta} H_k^\theta$

This corollary combines Lemmas 5.2.1, 5.2.5, and Theorem 3.2.2.

Chapter 6

Components of the Extended Symmetric Space

We want to characterize the connected components of the extended symmetric space for $G = \mathrm{SL}_2(k)$ defined over an arbitrary field with any involution. Let $\widetilde{Q^\theta}$ and Q^θ be the extended symmetric and symmetric spaces of G with respect to the involution θ. We will do this in several steps. First, we will determine $A_k \cap Q_k^\theta$ in order to find the coset representatives of $A_k/(A_k \cap Q_k^\theta)$. Here, A is a maximal (θ, k)-split torus. For a review of the tori in $\mathrm{SL}_2(k)$, see Section 2.10. Second, we will determine if $\bigcup_{i \in I} a_i Q_k^\theta = \widetilde{Q_k^\theta}$, where $A_k/(A_k \cap Q_k^\theta) = \{a_i \mid i \in I\}$, in which case we can determine $\widetilde{Q_k^\theta}/Q_k^\theta$. By abuse of notation, $\widetilde{Q_k^\theta}/Q_k^\theta$ will represent the set $\{a_i \mid i \in I\}$.

6.1 Extended symmetric space and (θ, k)-split tori

Notation 6.1.1. In this section, let $G_k = \mathrm{SL}_2(k)$ be defined over a field k with $\mathrm{char}(k) \neq 2$. Let θ_m denote the involution of G isomorphic to $\mathrm{Inn}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$, $\widetilde{Q^\theta}$ the extended symmetric space (6.1.1), and Q^θ the symmetric space of G (6.1.2). Let A be the maximal (θ, k)-split torus of diagonal matrices in G.

$$\widetilde{Q_k^\theta} = \left\{ \begin{pmatrix} x & y \\ -my & z \end{pmatrix} \;\middle|\; x, y, z \in k, \ xz + my^2 = 1 \right\} \tag{6.1.1}$$

$$Q_k^\theta = \{g\theta(g)^{-1} \mid g \in G_k\} \tag{6.1.2}$$

Example 6.1.2. Let $G = \mathrm{SL}_2(\mathbb{R})$ and θ the Cartan involution of θ defined by $\theta(g) = (g^T)^{-1}$ for all $g \in G$. The extended symmetric space has two connected components. First is the connected component containing the identity, $(\widetilde{Q_\mathbb{R}^\theta})^\circ = Q_\mathbb{R}^\theta$, which is the set of positive definite-symmetric matrices. The other connected component is the set of matrices with two negative eigenvalues. Note that no matrices in G can have one positive and one negative eigenvalue because $\det(g) = 1$ for all $g \in G$.

Lemma 6.1.3. *If $\widetilde{Q^\theta} = Q^\theta$, then $A_k/(A_k \cap Q_k^\theta) = \{\mathrm{Id}\}$.*

Proof. Because A is a θ-split torus, $A_k \subset \widetilde{Q_k^\theta} = Q_k^\theta$. Hence $A_k \cap Q_k^\theta = A_k$. $\qquad\square$

Lemma 6.1.4. $(-\mathrm{Id})Q_k^\theta \subset \widetilde{Q_k^\theta}$

Proof. Let $-g\theta(g)^{-1} \in (-\mathrm{Id})Q_k^\theta$. Then $\theta(-g\theta(g)^{-1}) = (-g\theta(g)^{-1})^{-1}$ (6.1.3).

$$\theta(-g\theta(g)^{-1}) = -\theta(g)g^{-1} = -(g\theta(g)^{-1} \tag{6.1.3}$$

Hence, $-g\theta(g)^{-1} \in \widetilde{Q_k^\theta}$. $\qquad\square$

6.1.1 $k = \mathbb{R}$

Let G be defined over $k = \mathbb{R}$. With the involution $\theta = \theta_1$, the extended symmetric and symmetric spaces are equivalent, hence $A_\mathbb{R}/(A_\mathbb{R} \cap Q_\mathbb{R}^\theta) = \{\mathrm{Id}\}$.

For the involution $\theta = \theta_{-1}$, we know the extended symmetric and symmetric spaces are not equivalent. For $a = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in A_\mathbb{R}$, we have $a \in Q_\mathbb{R}^\theta$ if and only if $x > 0$. If $x > 0$, then $a = g\theta(g)^{-1} \in Q_\mathbb{R}^\theta$ for $g = \begin{pmatrix} \sqrt{x} & 0 \\ 0 & \frac{1}{\sqrt{x}} \end{pmatrix}$.

The diagonal elements of matrices in $Q_\mathbb{R}^\theta$ are sums of squares. Hence, if $x < 0$, then $a \notin Q_\mathbb{R}^\theta$. For $-\mathrm{Id} \in A_\mathbb{R}$ and $a \in A_\mathbb{R}$ such that $a \notin Q_\mathbb{R}^\theta$, we have $(-\mathrm{Id})a \in Q_\mathbb{R}^\theta$. Therefore, $A_\mathbb{R}/(A_\mathbb{R} \cap Q_\mathbb{R}^\theta) = \{\pm\mathrm{Id}\}$ and we have the following result.

Theorem 6.1.5. *Let G be defined over $k = \mathbb{R}$ with the involution $\theta = \theta_{-1}$, then $\widetilde{Q_\mathbb{R}^\theta}/Q_\mathbb{R}^\theta = \{\pm\mathrm{Id}\}$. Furthermore, $\widetilde{Q_\mathbb{R}^\theta} = Q_\mathbb{R}^\theta \cup (-\mathrm{Id})Q_\mathbb{R}^\theta$.*

Proof. Let $q \in \begin{pmatrix} x & y \\ y & z \end{pmatrix} \in \widetilde{Q_{\mathbb{R}}^{\theta}}$. If $z > 0$, then $q = g\theta(g)^{-1} \in Q_{\mathbb{R}}^{\theta}$ for $g = \begin{pmatrix} \frac{1}{\sqrt{z}} & \frac{y}{\sqrt{z}} \\ 0 & \sqrt{z} \end{pmatrix} \in G_{\mathbb{R}}$. If $z <$

0, then $(-\mathrm{Id})q = g\theta(g)^{-1} \in Q_{\mathbb{R}}^{\theta}$ for $g = \begin{pmatrix} \frac{1}{\sqrt{-z}} & \frac{y}{\sqrt{-z}} \\ 0 & \sqrt{-z} \end{pmatrix} \in G_{\mathbb{R}}$. Note that $z = 0$ is not an option

because $\det(q) = -y^2 = 1$ does not have a solution in $\mathbb{R}$. Thus, $\widetilde{Q_{\mathbb{R}}^{\theta}} \subset (Q_{\mathbb{R}}^{\theta} \cup (-\mathrm{Id})Q_{\mathbb{R}}^{\theta})$

and the reverse containment is clear from Lemmas 4.1.2 and 6.1.4. The union is disjoint

because for $X \in Q_{\mathbb{R}}^{\theta} \cap (-\mathrm{Id})Q_{\mathbb{R}}^{\theta}$, $X = 0$ (6.1.4).

$$X = \begin{pmatrix} a_1^2 + b_1^2 & a_1c_1 + b_1d_1 \\ a_1c_1 + b_1d_1 & c_1^2 + d_1^2 \end{pmatrix} = \begin{pmatrix} -(a_2^2 + b_2^2) & -(a_2c_2 + b_2d_2) \\ -(a_2c_2 + b_2d_2) & -(c_2^2 + d_2^2) \end{pmatrix} \qquad (6.1.4)$$

$\square$

6.1.2 $\quad k = \mathbb{F}_q$

Let G be defined over $k = \mathbb{F}_q$. For all involutions θ, the extended symmetric and symmetric

spaces are equivalent, hence $A_{\mathbb{F}_q}/(A_{\mathbb{F}_q} \cap Q_{\mathbb{F}_q}^{\theta}) = \{\mathrm{Id}\}$ and $\widetilde{Q_{\mathbb{F}_q}^{\theta}}/Q_{\mathbb{F}_q}^{\theta} = \{\mathrm{Id}\}$.

6.1.3 $\quad k = \mathbb{Q}_p$

Let G be defined over $k = \mathbb{Q}_p$. We break this example up into several cases. First, for

each of the four involutions of G and when necessary, for $p \equiv 1 \mod 4$ and $p \equiv 3 \mod 4$.

Case 1: For the involution $\theta = \theta_1$, the extended symmetric and symmetric spaces are

equivalent, hence $A_{\mathbb{Q}_p}/(A_{\mathbb{Q}_p} \cap Q_{\mathbb{Q}_p}^{\theta}) = \{\mathrm{Id}\}$ and $\widetilde{Q_{\mathbb{Q}_p}^{\theta}}/Q_{\mathbb{Q}_p}^{\theta} = \{\mathrm{Id}\}$.

Case 2: Let $p \equiv 3 \mod 4$ and consider the involution θ_p. Because $-1 \notin (\mathbb{Q}_p^*)^2$, we let

$N_p = -1$. For $a = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in A_{\mathbb{Q}_p}$, we have $a = g\theta(g)^{-1} \in Q_{\mathbb{Q}_p}^{\theta}$ for $g = \begin{pmatrix} dx & \sqrt{\frac{d^2x^2-x}{p}} \\ \frac{p}{x}\sqrt{\frac{d^2x^2-x}{p}} & d \end{pmatrix}$.

We must determine when there exists $w, d \in \mathbb{Q}_p$ such that $xd^2 - \frac{p}{x}w^2 = 1$. This is equivalent

to determining the Hilbert's symbol $(x, -\frac{p}{x})_p$ (Table 6.1).

Thus, $a \in Q_{\mathbb{Q}_p}^{\theta}$ for $x \in \overline{1}$ or $x \in \overline{-p}$. If $x \in \overline{p}$, then $-x \in \overline{p}$ and if $x \in \overline{-1}$, then $-x \in \overline{1}$.

Therefore if $a \notin Q_{\mathbb{Q}_p}^{\theta}$, then $(-\mathrm{Id})a \in Q_{\mathbb{Q}_p}^{\theta}$ and $A_{\mathbb{Q}_p}/(A_{\mathbb{Q}_p} \cap Q_{\mathbb{Q}_p}^{\theta}) = \{\pm\mathrm{Id}\}$.

Theorem 6.1.6. *Let G be defined over $k = \mathbb{Q}_p$, $p \equiv 3 \mod 4$, with the involution $\theta = \theta_p$, then $\widetilde{Q_{\mathbb{Q}_p}^{\theta}}/Q_{\mathbb{Q}_p}^{\theta} = \{\pm\mathrm{Id}\}$. Furthermore, $\widetilde{Q_{\mathbb{Q}_p}^{\theta}} = Q_{\mathbb{Q}_p}^{\theta} \sqcup (-\mathrm{Id})Q_{\mathbb{Q}_p}^{\theta}$.*

Table 6.1: Hilbert symbol for $a \in Q_{\mathbb{Q}_p}^\theta$, $\theta = \theta_p$, $p \equiv 3 \mod 4$

$x \in \overline{m}$	$\left(x, -\frac{p}{x}\right)_p$
$x \in \overline{1}$	$(1, -p)_p = (-1|p)^0 (1|p)^1 (-1|p)^0 = 1$
$x \in \overline{p}$	$(p, -1)_p = (-1|p)^0 (1|p)^0 (-1|p)^1 = -1$
$x \in \overline{-1}$	$(-1, p)_p = (-1|p)^0 (-1|p)^1 (1|p)^0 = -1$
$x \in \overline{-p}$	$(-p, 1)_p = (-1|p)^0 (-1|p)^0 (1|p)^1 = 1$

Proof. Let $q = \begin{pmatrix} x & y \\ -py & \frac{1-py^2}{x} \end{pmatrix} \in \widetilde{Q_{\mathbb{Q}_p}^\theta}$. Note that $x \neq 0$ because $\det(q) = py^2 = 1$ has no

solution in $\mathbb{Q}_p$. Then $q = g\theta(g)^{-1} \in Q_{\mathbb{Q}_p}^\theta$ for $g = \begin{pmatrix} \sqrt{\frac{x}{1-py^2}} & y\sqrt{\frac{x}{1-py^2}} \\ 0 & \sqrt{\frac{1-py^2}{x}} \end{pmatrix}$. We must determine

when there exists $w, y \in \mathbb{Q}_p$ such that $py^2 + xw^2 = 1$. This is equivalent to determining the
value of the Hilbert's symbol $(p, x)_p$ (Table 6.2).

Table 6.2: Hilbert symbol for $q \in Q_{\mathbb{Q}_p}^\theta$, $\theta = \theta_p$, $p \equiv 3 \mod 4$

$x \in \overline{m}$	$(p, x)_p$
$x \in \overline{1}$	$(p, 1)_p = (-1|p)^0 (1|p)^0 (1|p)^1 = 1$
$x \in \overline{p}$	$(p, p)_p = (-1|p)^1 (1|p)^1 (1|p)^1 = -1$
$x \in \overline{-1}$	$(p, -1)_p = (-1|p)^0 (1|p)^0 (-1|p)^1 = -1$
$x \in \overline{-p}$	$(p, -p)_p = (-1|p)^1 (1|p)^1 (-1|p)^1 = 1$

Thus, $q \in Q_{\mathbb{Q}_p}^\theta$ for $x \in \overline{1}$ and $x \in \overline{-p}$. In the other cases, $(-\mathrm{Id})q = g\theta(g)^{-1} \in Q_{\mathbb{Q}_p}^\theta$ for

$g = \begin{pmatrix} \sqrt{\frac{x}{py^2-1}} & y\sqrt{\frac{x}{py^2-1}} \\ 0 & \sqrt{\frac{py^2-1}{x}} \end{pmatrix}$. We can verify that there exists $y, w \in \mathbb{Q}_p$ such that $py^2 - xw^2 = 1$,

or equivalently, $(p, -x)_p = 1$ (Table 6.3).

Hence $\widetilde{Q_{\mathbb{Q}_p}^\theta} \subset \left(Q_{\mathbb{Q}_p}^\theta \cup (-\mathrm{Id})Q_{\mathbb{Q}_p}^\theta\right)$ and the reverse containment is clear from Lemmas
4.1.2 and 6.1.4. $\qquad\square$

Case 3: Let $p \equiv 1 \mod 4$ and consider the involution θ_p. In this case $-1 \in (\mathbb{Q}_p^*)^2$, so

we let N_p represent the "smallest non-square" in $\mathbb{Q}_p$. For $a = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in A_{\mathbb{Q}_p}$, we have

Table 6.3:   Hilbert symbol for $(-\,\mathrm{Id})q \in Q^{\theta}_{\mathbb{Q}_p}$, $\theta = \theta_p$, $p \equiv 3 \mod 4$

$x \in \overline{m}$	$(p, -x)_p$
$x \in \overline{p}$	$(p, -p)_p = (-1|p)^1(1|p)^1(-1|p)^1 = 1$
$x \in \overline{-1}$	$(p, 1)_p = (-1|p)^0(1|p)^0(1|p)^1 = 1$

$a = g\theta(g)^{-1} \in Q^{\theta}_{\mathbb{Q}_p}$ for $g = \begin{pmatrix} dx & \sqrt{\frac{d^2 x^2 - x}{p}} \\ \frac{p}{x}\sqrt{\frac{d^2 x^2 - x}{p}} & d \end{pmatrix}$. We must determine when there exists $d, w \in \mathbb{Q}_p$ such that $xd^2 - \frac{p}{x}w^2 = 1$, or equivalently, when $(x, -\frac{p}{x})_p = 1$ (Table 6.4).

Table 6.4:   Hilbert symbol for $a \in Q^{\theta}_{\mathbb{Q}_p}$, $\theta = \theta_p$, $p \equiv 1 \mod 4$

$x \in \overline{m}$	$\left(x, -\frac{p}{x}\right)_p = \left(x, \frac{p}{x}\right)_p$
$x \in \overline{1}$	$(1, p)_p = (-1|p)^0(1|p)^1(1|p)^0 = 1$
$x \in \overline{p}$	$(p, 1)_p = (-1|p)^0(1|p)^0(1|p)^1 = 1$
$x \in \overline{N_p}$	$\left(N_p, \frac{p}{N_p}\right)_p = (-1|p)^0(N_p|p)^1\left(\frac{1}{N_p}|p\right)^0 = -1$
$x \in \overline{pN_p}$	$\left(pN_p, \frac{1}{N_p}\right)_p = (-1|p)^0(N_p|p)^0\left(\frac{1}{N_p}|p\right)^1 = -1$

Thus, $a \in Q^{\theta}_{\mathbb{Q}_p}$ for $x \in \overline{1}$ and $x \in \overline{p}$. If $a \notin Q^{\theta}_{\mathbb{Q}_p}$, then $\begin{pmatrix} N_p & 0 \\ 0 & N_p^{-1} \end{pmatrix} a \in Q^{\theta}_{\mathbb{Q}_p}$ (6.1.5).

$$x \in \overline{N_p}: \quad \begin{pmatrix} N_p & 0 \\ 0 & N_p^{-1} \end{pmatrix} a = \begin{pmatrix} \alpha^2 N_p^2 & 0 \\ 0 & (\alpha^2 N_p^2)^{-1} \end{pmatrix} \in Q^{\theta}_{\mathbb{Q}_p} \qquad (6.1.5)$$

$$x \in \overline{pN_p}: \quad \begin{pmatrix} N_p & 0 \\ 0 & N_p^{-1} \end{pmatrix} = \begin{pmatrix} \alpha^2 N_p^2 p & 0 \\ 0 & (\alpha^2 N_p^2 p)^{-1} \end{pmatrix} \in Q^{\theta}_{\mathbb{Q}_p}$$

Therefore, $A_{\mathbb{Q}_p}/(A_{\mathbb{Q}_p} \cap Q^{\theta}_{\mathbb{Q}_p}) = \left\{ \mathrm{Id}, \begin{pmatrix} N_p & 0 \\ 0 & N_p^{-1} \end{pmatrix} \right\}$. Note that $\widetilde{Q^{\theta}_{\mathbb{Q}_p}} \neq Q^{\theta}_{\mathbb{Q}_p} \cup \begin{pmatrix} N_p & 0 \\ 0 & N_p^{-1} \end{pmatrix} Q^{\theta}_{\mathbb{Q}_p}$ because $\begin{pmatrix} N_p & 0 \\ 0 & N_p^{-1} \end{pmatrix} Q^{\theta}_{\mathbb{Q}_p} \not\subseteq \widetilde{Q^{\theta}_{\mathbb{Q}_p}}$.

Case 4: Let $p \equiv 3 \mod 4$ and consider the involution $\theta = \theta_{N_p}$. Because $-1 \notin (\mathbb{Q}^*)^2$, let

$N_p = -1$. Then for $a = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in A_{\mathbb{Q}_p}$, we have $a = g\theta(g)^{-1} \in Q^{\theta}_{\mathbb{Q}_p}$ for $g = \begin{pmatrix} a & \sqrt{x-a^2} \\ \frac{\sqrt{x-a^2}}{x} & -\frac{a}{x} \end{pmatrix}$.

We must determine for which values of x, there exists $a, w \in \mathbb{Q}_p$ such that $\frac{1}{x}a^2 + \frac{1}{x}w^2 = 1$, or equivalently, when $\left(\frac{1}{x}, \frac{1}{x}\right)_p = 1$ (Table 6.5).

Table 6.5: Hilbert symbol for $a \in Q^{\theta}_{\mathbb{Q}_p}$, $\theta = \theta_{-1}$, $p \equiv 3 \mod 4$

$x \in \overline{m}$	$\left(\frac{1}{x}, \frac{1}{x}\right)_p$
$x \in \overline{1}$	$(1, 1)_p = (-1|p)^0 (1|p)^0 (1|p)^0 = 1$
$x \in \overline{p}$	$\left(\frac{1}{p}, \frac{1}{p}\right)_p = (-1|p)^1 (1|p)^{-1} (1|p)^{-1} = -1$
$x \in \overline{-1}$	$(-1, -1)_p = (-1|p)^0 (-1|p)^0 (-1|p)^0 = 1$
$x \in \overline{-p}$	$\left(-\frac{1}{p}, -\frac{1}{p}\right)_p = (-1|p)^1 (-1|p)^{-1} (-1|p)^1 = -1$

If $x \in \overline{1}$ or $x \in \overline{-1}$, then $a \in Q^{\theta}_{\mathbb{Q}_p}$. If $x \in \overline{p}$ or $x \in \overline{-p}$, then $\begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} a \in Q^{\theta}_{\mathbb{Q}_p}$ because $p^2 \in \overline{1}$ and $-p^2 \in \overline{-1}$. Therefore, $A_{\mathbb{Q}_p}/(A_{\mathbb{Q}_p} \cap Q^{\theta}_{\mathbb{Q}_p}) = \left\{ \mathrm{Id}, \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \right\}$. Note that $\widetilde{Q^{\theta}_{\mathbb{Q}_p}} \neq$ $Q^{\theta}_{\mathbb{Q}_p} \cup \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} Q^{\theta}_{\mathbb{Q}_p}$ because $\begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} Q^{\theta}_{\mathbb{Q}_p} \notin \widetilde{Q^{\theta}_{\mathbb{Q}_p}}$.

Case 5: Let $p \equiv 1 \mod 4$ and consider the involution $\theta = \theta_{N_p}$. Let N_p represent the "smallest non-square" in $\mathbb{Q}_p$. For $a = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in A_{\mathbb{Q}_p}$, we have $a = g\theta(g)^{-1} \in Q^{\theta}_{\mathbb{Q}_p}$ for $g = \begin{pmatrix} -dx & \sqrt{\frac{d^2x^2-x}{N_p}} \\ -\frac{N_p}{x}\sqrt{\frac{d^2x^2-x}{N_p}} & d \end{pmatrix}$. To determine if there exists $d, w \in \mathbb{Q}_p$ such that $xd^2 - \frac{N_p}{x}w^2 = 1$, we evaluate $\left(x, -\frac{N_p}{x}\right)_p$ (Table 6.6).

Thus, $a \in Q^{\theta}_{\mathbb{Q}_p}$ if $x \in \overline{1}$ or $x \in \overline{N_p}$. If $x \in \overline{p}$ or $x \in \overline{pN_p}$, then $\begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} a \in Q^{\theta}_{\mathbb{Q}_p}$ because $p^2 \in \overline{1}$ and $p^2 N_p \in \overline{N_p}$. Therefore, $A_{\mathbb{Q}_p}/(A_{\mathbb{Q}_p} \cap Q^{\theta}_{\mathbb{Q}_p}) = \left\{ \mathrm{Id}, \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \right\}$. Again, note that $\widetilde{Q^{\theta}_{\mathbb{Q}_p}} \neq Q^{\theta}_{\mathbb{Q}_p} \cup \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} Q^{\theta}_{\mathbb{Q}_p}$ because $\begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} Q^{\theta}_{\mathbb{Q}_p} \notin \widetilde{Q^{\theta}_{\mathbb{Q}_p}}$.

Table 6.6:   Hilbert symbol for $a \in Q^\theta_{\mathbb{Q}_p}$, $\theta = \theta_{N_p}$, $p \equiv 1 \mod 4$

$x \in \overline{m}$	$\left(x, -\frac{N_p}{x}\right)_p = \left(x, \frac{N_p}{x}\right)_p$
$x \in \overline{1}$	$(1, N_p)_p = (-1|p)^0(1|p)^0(N_p|p)^0 = 1$
$x \in \overline{p}$	$\left(p, \frac{N_p}{p}\right)_p = (-1|p)^{-1}(1|p)^{-1}(N_p|p)^1 = -1$
$x \in \overline{N_p}$	$(N_p, 1)_p = (-1|p)^0 \, (N_p|p)^0 \, (1|p)^0 = 1$
$x \in \overline{pN_p}$	$\left(pN_p, \frac{1}{p}\right)_p = (-1|p)^{-1}(N_p|p)^{-1}(1|p)^1 = -1$

Case 6: Let $p \equiv 1 \mod 4$ and consider the involution $\theta = \theta_{pN_p}$. Let N_p represent the "smallest non-square" in $\mathbb{Q}_p$. For $a = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in A_{\mathbb{Q}_p}$, we have $a = g\theta(g)^{-1} \in Q^\theta_{\mathbb{Q}_p}$ for $g = \begin{pmatrix} dx & \sqrt{\frac{d^2x^2-x}{pN_p}} \\ \frac{pN_p}{x}\sqrt{\frac{d^2x^2-x}{pN_p}} & d \end{pmatrix}$. To determine if there exists $d, w \in \mathbb{Q}_p$ such that $xd^2 - \frac{pN_p}{x}w^2 = 1$ we evaluate $(x, -\frac{pN_p}{x})_p$ (Table 6.7).

Table 6.7:   Hilbert symbol for $a \in Q^\theta_{\mathbb{Q}_p}$, $\theta = \theta_{pN_p}$, $p \equiv 1 \mod 4$

$x \in \overline{m}$	$\left(x, -\frac{pN_p}{x}\right)_p = \left(x, \frac{pN_p}{x}\right)_p$
$x \in \overline{1}$	$(1, pN_p)_p = (-1|p)^0(1|p)^1(N_p|p)^0 = 1$
$x \in \overline{p}$	$(p, N_p)_p = (-1|p)^0(1|p)^0(N_p|p)^1 = -1$
$x \in \overline{N_p}$	$(N_p, p)_p = (-1|p)^0 \, (N_p|p)^1 \, (1|p)^0 = -1$
$x \in \overline{pN_p}$	$(pN_p, 1)_p = (-1|p)^0(N_p|p)^0 \, (1|p)^1 = 1$

If $x \in \overline{1}$ or $x \in \overline{pN_p}$, then $a \in Q^\theta_{\mathbb{Q}_p}$. If $x \in \overline{p}$ or $x \in \overline{N_p}$, then $\begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} a \in Q^\theta_{\mathbb{Q}_p}$ because $p^2 \in \overline{1}$ and $pN_p \in \overline{pN_p}$. Therefore, $A_{\mathbb{Q}_p}/(A_{\mathbb{Q}_p} \cap Q^\theta_{\mathbb{Q}_p}) = \left\{ \mathrm{Id}, \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \right\}$. Again, note that $\widetilde{Q^\theta_{\mathbb{Q}_p}} \neq Q^\theta_{\mathbb{Q}_p} \cup \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} Q^\theta_{\mathbb{Q}_p}$ because $\begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} Q^\theta_{\mathbb{Q}_p} \notin \widetilde{Q^\theta_{\mathbb{Q}_p}}$.

Case 7: Let $p \equiv 3 \mod 4$ and consider the involution $\theta = \theta_{pN_p}$. Because $-1 \notin (\mathbb{Q}^*)^2$, we

let $N_p = -1$. For $a = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in A_{\mathbb{Q}_p}$, we have $a = g\theta(g)^{-1} \in Q^\theta_{\mathbb{Q}_p}$ for

$g = \begin{pmatrix} dx & \sqrt{\frac{x-d^2x^2}{p}} \\ -\frac{p}{x}\sqrt{\frac{x-d^2x^2}{p}} & d \end{pmatrix}$. To determine for which values of x there exists $d, w \in \mathbb{Q}_p$

such that $xd^2 + \frac{p}{x}w^2 = 1$, we evaluate $\left(x, \frac{p}{x}\right)_p$ (Table 6.8).

Table 6.8: Hilbert symbol for $a \in Q^\theta_{\mathbb{Q}_p}$, $\theta = \theta_{-p}$, $p \equiv 3 \mod 4$

$x \in \overline{m}$	$\left(x, \frac{p}{x}\right)_p$
$x \in \overline{1}$	$(1, p)_p = (-1|p)^0(1|p)^0(1|p)^1 = 1$
$x \in \overline{p}$	$(p, 1)_p = (-1|p)^0(1|p)^0(1|p)^1 = 1$
$x \in \overline{-1}$	$(-1, -p)_p = (-1|p)^0\,(-1|p)^1\,(-1|p)^0 = -1$
$x \in \overline{-p}$	$(-p, -1)_p = (-1|p)^0(-1|p)^0\,(-1|p)^1 = -1$

Thus, if $x \in \overline{1}$ or $x \in \overline{p}$, then $a \in Q^\theta_{\mathbb{Q}_p}$. If $x \in \overline{-1}$ or $x \in \overline{-p}$, then $(-\operatorname{Id})a \in Q^\theta_{\mathbb{Q}_p}$ because $(-1)^2 \in \overline{1}$ and $-(-p) \in \overline{p}$. Therefore, $A_{\mathbb{Q}_p}/(A_{\mathbb{Q}_p} \cap Q^\theta_{\mathbb{Q}_p}) = \{\pm\operatorname{Id}\}$.

Theorem 6.1.7. *Let G be defined over $k = \mathbb{Q}_p$, $p \equiv 3 \mod 4$, with the involution $\theta = \theta_{-p}$, then $\widetilde{Q^\theta_{\mathbb{Q}_p}}/Q^\theta_{\mathbb{Q}_p} = \{\pm\operatorname{Id}\}$. Furthermore, $\widetilde{Q^\theta_{\mathbb{Q}_p}} = Q^\theta_{\mathbb{Q}_p} \cup (-\operatorname{Id})Q^\theta_{\mathbb{Q}_p}$.*

Proof. Let $q = \begin{pmatrix} x & y \\ py & \frac{1+py^2}{x} \end{pmatrix}$. Note that $x \neq 0$ because $\det(q) = -py^2 = 1$ has no solution in

$\mathbb{Q}_p$. Then $q = g\theta(g)^{-1} \in Q^\theta_{\mathbb{Q}_p}$ for $g = \begin{pmatrix} \sqrt{\frac{x}{1+py^2}} & y\sqrt{\frac{x}{1+py^2}} \\ 0 & \sqrt{\frac{1+py^2}{x}} \end{pmatrix}$. We must determine when there

exists $y, w \in \mathbb{Q}_p$ such that $-py^2 + xw^2 = 1$, or equivalently, we evaluate $(-p, x)_p$ (Table 6.9).

71

Table 6.9:   Hilbert symbol for $q \in Q^\theta_{\mathbb{Q}_p}$, $\theta = \theta_{-p}$, $p \equiv 3 \mod 4$

$x \in \overline{m}$	$(-p, x)_p$
$x \in \overline{1}$	$(-p, 1)_p = (-1|p)^0(-1|p)^0(1|p)^1 = 1$
$x \in \overline{p}$	$(-p, p)_p = (-1|p)^1(1|p)^1(-1|p)^1 = 1$
$x \in \overline{-1}$	$(-p, -1)_p = (-1|p)^0 (-1|p)^0 (-1|p)^1 = -1$
$x \in \overline{-p}$	$(-p, -p)_p = (-1|p)^1(-1|p)^1 (-1|p)^1 = -1$

Thus, $q \in Q^\theta_{\mathbb{Q}_p}$ for $x \in \overline{1}$ or $x \in \overline{p}$. If $x \in \overline{-1}$ or $x \in \overline{-p}$, then $(-\,\mathrm{Id})q = g\theta(g)^{-1} \in Q^\theta_{\mathbb{Q}_p}$ for $g = \begin{pmatrix} \sqrt{\frac{-x}{py^2+1}} & -y\sqrt{\frac{-x}{py^2+1}} \\ 0 & \sqrt{\frac{py^2+1}{-x}} \end{pmatrix}$. We can verify that there exists $y, w \in \mathbb{Q}_p$ such that $-py^2 - xw^2 = 1$ by evaluating $(-p, -x)_p$ (Table 6.10).

Table 6.10:   Hilbert symbol for $(-\,\mathrm{Id})q \in Q^\theta_{\mathbb{Q}_p}$, $\theta = \theta_{-p}$, $p \equiv 3 \mod 4$

$x \in \overline{m}$	$(-p, -x)_p$
$x \in \overline{-1}$	$(-p, 1)_p = (-1|p)^0(-1|p)^0(1|p)^1 = 1$
$x \in \overline{-p}$	$(-p, p)_p = (-1|p)^1(1|p)^1(-1|p)^1 = 1$

Hence, $\widetilde{Q^\theta_{\mathbb{Q}_p}} \subset \left(Q^\theta_{\mathbb{Q}_p} \cup (-\,\mathrm{Id})Q^\theta_{\mathbb{Q}_p} \right)$ and the reverse containment is clear from Lemmas 4.1.2 and 6.1.4.

$\square$

We have shown that when $-1 \notin (k^*)^2$ and $-(\mathrm{Id}) \notin A_k \cap Q^\theta_k$, we can determine the quotient $\widetilde{Q^\theta_k}/Q^\theta_k$ as $\widetilde{Q^\theta_k}/Q^\theta_k = \{\pm\,\mathrm{Id}\}$. Note that for $a = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$, the only values of x such that $aQ^\theta_k \subset \widetilde{Q^\theta_k}$ are $x = \pm 1$.

Chapter 7

Results for $\mathrm{SL}_3(k)$

Ideally, most of the results for $\mathrm{SL}_2(k)$ in this thesis will extend nicely to $\mathrm{SL}_n(k)$ as the group is generated by multiple copies of $\mathrm{SL}_2(k)$. We begin by extending some results to $\mathrm{SL}_3(k)$.

7.1 Fixed-point group

Notation 7.1.1. In this chapter let $G = \mathrm{SL}_2(\overline{k})$ and $G_k = \mathrm{SL}_3(k)$ defined over a field k of characteristic not two. We will only consider the involution θ of G defined by $\theta(g) = (g^T)^{-1}$ for all $g \in G$. If k has a topology, then the fixed-point group H is k-anisotropic and θ is a Cartan involution. For a review of the involutions of $\mathrm{SL}_n(k)$ for $n > 2$, see Section 2.9.

For the Cartan involution θ, the fixed-point group H is the special orthogonal group $\mathrm{SO}(3)$. Note that $\mathrm{SO}(3)$ is generated by three copies of $\mathrm{SO}(2)$ (7.1.1).

$$\mathrm{SO}(3) = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{pmatrix}, \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ -b & 0 & a \end{pmatrix}, \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| a^2 + b^2 = 1 \right\rangle \qquad (7.1.1)$$

Notation 7.1.2. The 3×3 permutation matrices are contained in H and we will denote them with $h_i, i = 1 \ldots 6$ (7.1.2).

$$h_1 = \mathrm{Id}, h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, h_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (7.1.2)$$

$$h_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, h_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, h_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Lemma 7.1.3. *Let $g \in G$, then there exists $h \in H$ such that the $(1,1)$-entry of hg is nonzero.*

Proof. For $g \in G$, if the $(1,1)$-entry is nonzero, then for $h = \mathrm{Id}$ the $(1,1)$-entry of $hg = g$ is nonzero. Assume the $(1,1)$-entry of g is zero. Because $\det(g) = 1$, there exists a nonzero entry in the first column of g. Left multiplication by a permutation matrix exchanges the rows. By choosing the appropriate permutation matrix $h_i, i = 2\ldots 6$, we get the $(1,1)$-entry of $h_i g$ is nonzero.

$\square$

Definition 7.1.4. Let g be an $n \times n$ matrix. The (i,j)-*minor* of g is the $(n-1) \times (n-1)$ matrix obtained from deleting the ith row and j^{th} column from g, denoted M_{ij}. The (i,j)-*cofactor* of g is the value $g_{ij} = (-1)^{i+j}\det(M_{ij})$.

Lemma 7.1.5. *Let $g \in G$, then there exists $h \in H$ such that the $(3,3)$-cofactor of hg is nonzero.*

Proof. By abuse of notation, let $g = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. The $(3,3)$-cofactor of g is $ae - bd$. If $ae - bd \neq 0$, then we are done. If $ae - bd = 0$, then for h_2 as in Notation 7.1.2, the $(3,3)$-cofactor of $h_2 g$ is $ah - bg$ (7.1.3).

$$h_2 g = \begin{pmatrix} a & b & c \\ g & h & i \\ -d & -e & -f \end{pmatrix} \tag{7.1.3}$$

If $ah - bg \neq 0$, then we are done. If $ah - bg = 0$, then the $(3,3)$-cofactor of $h_6 h_2 g$ is $eg - dh$ (7.1.4).

$$h_6 h_2 g = \begin{pmatrix} -d & -e & -f \\ g & h & i \\ -a & -b & -c \end{pmatrix} \tag{7.1.4}$$

The determinant of $h_6 h_2 g$ is $-c(eg - dh)$, which must equal one and therefore $eg - dh$ is nonzero.

$\square$

7.2 Generalizing the decomposition

For G with a Cartan involution θ, the extended symmetric space is the set of symmetric matrices (7.2.1). The unipotent subgroup will be the upper triangular matrices with ones on the diagonal (7.2.2).

$$\widetilde{Q} = \{g \in G \mid (g^T)^{-1} = g^{-1}\} = \{g \mid g = g^T\} \tag{7.2.1}$$

$$U = \left\{ \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; \alpha, \beta, \gamma \in k \right\} \tag{7.2.2}$$

Theorem 7.2.1. *For G with the involution Cartan involution θ, $G_k = H_k \widetilde{Q}_k U_k$.*

Proof. Let $g \in G_k$. By Lemmas 7.1.3 and 7.1.5, there exists $h \in H_k$ such that the $(1,1)$-entry and $(3,3)$-cofactor of hg are nonzero (7.2.3).

$$hg = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \tag{7.2.3}$$

Choosing $u \in U_k$ appropriately (7.2.4), $hgu \in \widetilde{Q}_k$; thus $g \in H_k \widetilde{Q}_k U_k$ (7.2.5) .

$$u = \begin{pmatrix} 1 & \frac{d-b}{a} & \frac{-ace+adf-adh+aeg+bcd-cd^2}{(ae-bd)a} \\ 0 & 1 & \frac{-af+ah-bg+cd}{ae-bd} \\ 0 & 0 & 1 \end{pmatrix} \tag{7.2.4}$$

$$hgu = \begin{pmatrix} 1 & d & g \\ d & \frac{ae-bd+d^2}{a} & \frac{ah-bg+dg}{a} \\ g & \frac{ah-bg+dg}{a} & \frac{a^2ei-a^2fh+a^2h^2-abdi+abfg-2abgh+acdh-aceg+aeg^2+b^2g^2-bdg^2}{(ae-bd)a} \end{pmatrix} \tag{7.2.5}$$

$\square$

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