ABSTRACT

PAUL, RYAN CHRISTENSEN. Low-Order Post-Stall Aerodynamics Modeling for use in Flight Simulation and Stability Analysis. (Under the direction of Dr. Ashok Gopalarathnam.)

Aerodynamic models that can be used to rapidly compute forces and moments acting on a wing or aircraft configuration have applications in flight simulation, design, and flight dynamics characterization. For use in aerodynamic prediction on arbitrary aircraft configurations, these models are difficult to formulate for situations like post-stall conditions where flow separation causes the aerodynamics to be highly nonlinear. A decambering approach, which uses known airfoil section data as input, is studied for incorporation in an aerodynamics analysis method such as a vortex-lattice formulation. To perform post-stall aerodynamic analysis of a finite wing, it is necessary to formulate the decambering calculation as a Newton iteration. The application of the Newton iteration to the decambering problem is non-standard as no closed form expression exists to facilitate the residual calculation. Four residual calculation methods are evaluated on the basis of computational efficiency and robustness. Solutions for the post-stall behavior of a rectangular wing using the most appropriate residual calculation method are compared to computational fluid dynamics predictions. The comparison between the low-order analysis using decambering and the computational solutions is excellent. With increasing sweep angle, however, spanwise propagation of separated flow resulting from transverse pressure gradients causes the behavior of wing sections to depart from that of the corresponding airfoil. This flow phenomenon results in progressively inaccurate prediction with increasing sweep. Next, an architecture is presented for combining the post-stall aerodynamics capabilities of the decambering approach with a discrete-time implementation of an un-
steady vortex lattice method. The formulation of the vortex-lattice method in discrete
time is well suited for time marching simulation. Finally, by utilizing a linearized form
of the unsteady vortex lattice equations coupled with a rigid body model the stability of
wings and configurations is studied. The decambering approach lends itself to a natural
incorporation into the stability analysis framework. For post-stall flight dynamics, de-
cambering is introduced into the system matrices by treating it at each strip as a control
effector, whose deflection is linearly related to the change in flight velocities. This rela-
tionship between the change in flight velocities and change in decambering is appended
to the system in feedback form. Results for the post-stall stability analysis are validated
through a study of a free-to-roll wing as compared to experimental results. A full aircraft
configuration is studied to predict the changes in stability characteristics as the post-stall
model is introduced. The trends shown by the rigid body modes as the post-stall effects
are introduced agree well with experiments, but additional study is warranted to validate
the model.
Low-Order Post-Stall Aerodynamics Modeling for use in Flight Simulation and Stability Analysis

by
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DEDICATION

To my family for the support they have given me as I have pursued my education.
BIOGRAPHY

Ryan was born to Chris and Anne Paul in Monterey, California while his dad was stationed at Fort Ord as part of the Army 7th Infantry Division. Following a move, most of his childhood was spent between Downingtown, PA and Tulsa, OK, until Tulsa became a more permanent home for his family in 1993. He graduated Union High School in 2006, and subsequently attended Oklahoma State University graduating in 2010 with a B.S. in Aerospace Engineering and a B.S. in Mechanical Engineering. Ryan joined N.C. State in the Fall of 2010 and shortly thereafter began working with Dr. Gopalarathnam. The Summer of 2014, Ryan spent working with the Flight Dynamics Branch at NAVAIR, located in Patuxent River, MD. He looks forward to starting his career at NAVAIR after completing his degree.
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Through travel support provided by the UGPN at NC State, I had the opportunity to spend time working in the UK with Dr. Murua at the University of Surrey. Dr. Murua is gratefully acknowledged for hosting me in the UK, for providing me with guidance in the research, and for sharing his SHARP toolset for use in my research.

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LIST OF SYMBOLS, ABBREV., NOMENCLATURE

$\Gamma_b$ bound circulation vector

$\Gamma_w$ wake circulation vector

$\nu$ right eigenvector

$w$ non-circulatory velocity vector at collocation points

$\delta_1$ decambering variable

$\gamma_{fp}$ flight path angle

$\lambda_i$ continuous time eigenvalue

$\nu$ body-axis velocities

$\Theta$ orientation angles of the body fixed reference frame

$\zeta$ aerodynamic lattice coordinate

$a_0$ potential flow lift curve slope

$A_b$ bound aerodynamic influence coefficient matrix

$A_w$ wake aerodynamic influence coefficient matrix

$C_l$ airfoil lift coefficient

$G_{con}$ control input matrix

$K_b$ number of bound panels

$K_w$ number of wake panels
$K_{dec}$       decambering sensitivity matrix

$x$         state vector

$z$         discrete time eigenvalue

$u$         control input
Chapter 1

Introduction

Increasingly, training for flight crews is taking place in simulators rather than actual aircraft. For effective training to occur, the simulator must accurately replicate the behavior of the aircraft. Along with numerous other factors, such as providing realistic visual effects, control feel, and even motion queues, adequate simulation fidelity requires that the aerodynamic model be representative of the aircraft. High-fidelity aerodynamic models implemented in the form of look-up tables are commonplace in certified flight simulators and training devices. Force and moment information for these look-up tables is often developed using some combination of the following techniques: computational fluid dynamics (CFD), experimental testing of models in wind tunnels (static and dynamic), and/or experimental flight testing. Developing these data tables requires significant effort and expense. Each expected operating condition, which is generally defined by aerodynamic inflow angles, body axis velocities, and other explanatory variables must be considered using computational analysis, in an experimental environment, or in a flight test. Assembling the requisite number of test points for a database covering even a very narrow aerodynamic regime can quickly grow to require many thousands of eval-
uations, depending on how coarse of a grid between the explanatory variables is deemed acceptable.

Because of the significant amount of effort and expense required to develop the high fidelity aerodynamic data, often only a small range of the flight envelope is covered. Beyond the range of coverage, aerodynamic force and moment data is interpolated (if possible), held constant, or even extrapolated [1]. As aircraft are not expected to operate outside the bounds of their aerodynamic model coverage, this approach is normally acceptable.

However, the need to improve aerodynamic modeling at off nominal operating conditions can not be ignored. In fact, the majority of commercial aircraft accidents, as reported by Boeing [2] in the years 2004-2013, have been attributed to loss of control. The general aviation fleet has not fared much better, so much so that the National Transportation Safety Board has recognized general aviation loss of control in its Most Wanted improvement areas for 2015 [3]. By definition, loss of control accidents involve flight outside of the normal aerodynamic operating envelope [4], where the normal operating envelope is characterized by low angle of attack and sideslip excursions. Part of addressing loss of control as a cause of accidents involves expanding the capability of aerodynamic models to include high angle of attack regimes [5].

Research performed under the NASA Aviation Safety Program has endeavored, with admirable success, to expand the modeling capability beyond the nominal flight regime for transport type configurations. Researchers in this program have proven the capability of both CFD [6] and carefully designed experiments [7] to represent forces and moments where significant amounts of flow separation exist, such as aerodynamic stall. While it may be that the characteristic behaviors discovered in these studies can be modeled and appended to existing aerodynamic models of similar transport configurations, the general
applicability to other types of configurations is uncertain.

This research represents a departure from the high-fidelity data-based aerodynamics modeling being applied to transport configurations discussed above. Rather, a medium-fidelity aerodynamics analysis method is considered, based on vortex lattice aerodynamics in conjunction with the application of decambering to account for flow separation, when necessary. This sort of aerodynamics model has the capability to output force and moment predictions with only geometry and viscous data representing airfoil sections supplied as input. If formulated properly, the model is well suited to fast aerodynamics prediction even when being evaluated for post-stall conditions, and has applications in design, dynamics simulation, and flight dynamics characterization.

1.1 Literature Review

This section provides a brief description of some of past work relevant to the topics in this document.

1.1.1 Post-Stall Corrections in Potential Flow Aerodynamics Analysis Methods

For linear conditions such as in pre-stall flight, the ability of low-order linear aerodynamic methods such as lifting-line theory, Weissinger’s method, and vortex lattice methods (VLMs) to predict the aerodynamics of multiple lifting surface configurations is well established. For several decades, researchers have sought to extend these linear prediction methods to handle the aerodynamic analysis of wings in which nonlinear airfoil lift curves extending to stall or post-stall are used as inputs [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].
The motivation is that it is significantly easier to obtain aerodynamic characteristics for airfoils than for wings and configurations. The input airfoil data may be generated using CFD or experiment or by drawing on existing databases from a multitude of sources — from Abbott and von Doenhoff [20] and those from the University of Stuttgart [21] and University of Illinois at Urbana-Champaign [[22],[23],[24]], to modern computational approaches designed to predict sectional aerodynamic characteristics based on arbitrary input geometry, such as XFOIL [25]. The flow over a wing at post-stall conditions is highly three dimensional, and the use of a quasi-two-dimensional approach represents a significant approximation. However, for use in real-time and design-oriented applications, rapid, albeit approximate approaches that model the important effects of flow separation on lifting surfaces continue to be of interest.

Efforts at NC State University, initially by Mukherjee and Gopalarathnam [26], led to the development of a decambering concept which is capable of extending the validity of a classical VLM or similar to model the effects of separated flow at post-stall conditions. Research by Cho and Cho [27] has resulted in the incorporation of the decambering concept in a frequency-domain VLM. The post-stall solutions computed using decambering have shown great promise, and the methodology is further improved as described in this document.

1.1.2 Aerodynamics in the Loop

An alternative to the use of extensive look-up tables with pre-computed values of aerodynamic force and moment coefficients is to use an aerodynamic calculation method as the function evaluation to predict forces and moments in a flight dynamics simulation.

Two commercial desktop flight simulators use this approach: the X-Plane [28] for
piloted aircraft and the FS One [29] for radio-control aircraft. Both these simulators use variations of the so-called strip-theory and component-buildup methodologies. In the strip-theory approach, the lifting surfaces and the bodies are divided into strips. On each strip at every time step, the forces and moments are determined from the net velocity at the strip and integrated to get total force and moment on the component. With component-buildup methodology, aerodynamic forces and moments on different components of the vehicle are summed up to determine the total force and moment on the vehicle. Aerodynamic models based on semi-empirical methods are used for determining the forces and moments on the strips and components. The modeling approach used in the FS One has been described by Selig in [30]. This paper [30] illustrates the significant effort and experience necessary to develop efficient semi-empirical aerodynamic models for use in realistic, full-envelope flight simulation. The development of such semi-empirical models are often informed by data from myriad sources including wind-tunnel results, analytical predictions, CFD solutions, flight experiments, and observations of aircraft flight behavior.

While the approaches described above are able to model the effects of aerodynamic stall, there are drawbacks to the strip-theory approach besides necessitating empirical model corrections. The primary disadvantage is in the evaluation of spanwise variations. Spanwise variations may be modeled correctly using lifting surface methods, such as vortex lattice models. Vortex lattice aerodynamic models have been combined with the equations of motion by researchers such as Bunge and Kroo [31], who demonstrated the use of a VLM formulated in a compact form in real time dynamic simulation, and Obradovic and Subbarao [32], who used a vortex lattice aerodynamics to model the flight dynamic behavior of a morphing wing configuration. The aerodynamics in [31] and [32] featured no correction for post-stall conditions.
1.1.3 Stability Analysis

Flight dynamics equations are directly obtained by applying the Newton-Euler equations to the center of gravity of an aircraft. For stability analysis, these nonlinear dynamic equations are linearized about an equilibrium configuration (corresponding to aircraft trim) through a Taylor series expansion [33]. In the linearization, the loads generally are from aerodynamics, weight, and possibly thrust. For a typical aircraft configuration where it is reasonable to assume that longitudinal and lateral states are decoupled, the longitudinal stability quartic and lateral stability quintic are obtained, and the modes may be computed. Stability analysis routines have been implemented using aerodynamic loads from a vortex lattice method. The linear relationship between the circulations computed in the VLM and the flight velocities precludes the use of the standard VLM based stability analysis techniques in situations where flow separation is present.

1.2 Layout of the Document

Chapter 2 begins by describing the decambering approach in two-dimensional flow past an airfoil. An example calculation for this simple test case is carried out. Next, the iterative process by which the decambering calculation is applied to a finite wing is discussed. This discussion brings out the principal challenge that must be overcome to perform the post-stall analysis rapidly, mainly the residual calculation. Four assumptions, leading to four different residual calculation schemes, and the motivation behind each are presented in Chapter 3. The residual calculation schemes are compared against one another in their ability to successfully converge when analyzing a rectangular wing geometry. After determining the most robust and efficient residual calculation scheme, the implementation of the post-stall model into a time-marching simulation environment is described in
Chapter 4. Chapter 5 utilizes the toolbox provided by the SHARP routines to study the flight dynamic characteristics a free-to-roll wing and a full aircraft configuration in the post-stall regime. Conclusions are drawn in Chapter 6 and recommendations for future work are provided in Chapter 7.
Chapter 2

The Decambering Post-Stall Aerodynamic Model

In the following sections, the concept of decambering is explained. The discussion and figures in this chapter are adapted from Paul and Gopalarathnam [34]. Section 2.1 demonstrates decambering as it is applied first to a two-dimensional airfoil case. For an airfoil flow, the application of decambering is relatively straightforward, as demonstrated in a simple example case. Complications arise when decambering is applied to a finite wing. The interactions between sections that arise when a three-dimensional finite wing is considered in post-stall analysis cases are described in Section 2.2. Finally, posing the decambering problem for a finite wing into the framework of a Newton iteration is treated in Section 2.3.
2.1 Decambering Applied to a 2D Flow

At small angles of attack, the boundary layer on an airfoil is usually attached and thin. At these conditions, the predicted lift coefficient from potential-flow methods like thin airfoil theory agrees excellently with lift coefficient from experiments or viscous computations. Potential-flow methods do not model any viscous effects. Consequently they predict an almost linear lift curve having a slope close to $2\pi$ per radian, with lift increasing with angle of attack even beyond stall. As the angle of attack is increased to stall and beyond, the boundary layer on the upper surface thickens and separates. It is this flow separation that causes the viscous $C_l$ to deviate from the potential-flow theory prediction. Because the flow separation in effect causes a reduction in camber, the effect is sometimes referred to in the literature as “viscous decambering” [35] or “decambering.”

The idea behind the current decambering approach is that the potential-flow prediction for lift coefficient of the decambered airfoil at some $\alpha$ will match the viscous $C_l$ at that $\alpha$ even if that $\alpha$ is well beyond the angle of attack for stall. The decambering can be modeled as an effective reduction in the airfoil angle of attack using a decambering variable $\delta_1$ that is given (in radians) by $\delta_1 = (C_{l,\text{viscous}} - C_{l,\text{potential}})/2\pi$. This is a simplified version of the two-variable decambering approach described in [26]. Note that camber reduction due to upper-surface flow separation corresponds to a negative $\delta_1$ in this sign convention.

The overall idea behind decambering is illustrated in Fig. 2.1. Figure 2.1(a) compares the $C_l$-$\alpha$ curves from potential flow and viscous flow for an airfoil. Considering two example angles of attack of 2 degrees and 20 degrees, it is seen that the $C_l$ values from potential flow and viscous flow are nearly identical for the 2-degree case, but are significantly different for the 20-degree case. The difference in $C_l$ for the 20-degree case is
Figure 2.1: Decambering applied to an airfoil: (a) potential- and viscous-flow lift curves (lines) and operating points (symbols) for a cambered airfoil at \( \alpha \) of 2\(^\circ\) and 20\(^\circ\); potential-flow lift curves are shown for decambering of \( \delta_1 = 0^\circ \) and \(-9.65^\circ\), (b) airfoil with boundary-layer displacement thickness for \( \alpha = 20^\circ \) added, (c) equivalent decambering using a linear function, and (d) original and modified camberlines.

related to the flow separation, shown in Figure 2.1(b) by overlaying the boundary-layer displacement thickness for \( \alpha = 20 \) degrees on the airfoil contour. Using the difference in \( C_l \) between the viscous and potential flow predictions for \( \alpha = 20 \) degrees in Figure 2.1(a) to calculate the effective decambering for this angle of attack results in \( \delta_1 = -9.65^\circ \). Figures 2.1(c) shows the decambering angle of \( \delta_1 = -9.65^\circ \), which when added to the original camberline of the airfoil results in an effective camber reduction, as shown in Figure 2.1(d). Shown in Figure 2.1(a) as one of the dashed lines is the potential-flow \( C_l-\alpha \) curve for the decambered airfoil with \( \delta_1 = -9.65^\circ \). This \( C_l-\alpha \) curve crosses the viscous \( C_l-\alpha \) curve at \( \alpha = 20 \) degrees, confirming that \( \delta_1 = -9.65^\circ \) is the appropriate amount of decambering for this airfoil at this angle of attack. Note that the linear decambering function defined by \( \delta_1 \) is not intended to exactly match the viscous decambering resulting from the separated flow shown in Figure 2.1(b); it is only intended to capture the loss in lift on the airfoil due to the separation.
2.2 Decambering in a 3D Flow

The primary goal of the decambering approach is to apply it to post-stall prediction of finite wings and multiple lifting-surface configurations. The aerodynamic prediction for a finite wing, whether it is for a pre-stall or post-stall angle of attack, has to necessarily satisfy two conditions:

**Condition 1:** The boundary conditions of zero normal flow on all the wing sections are correctly satisfied such that the resulting spanwise lift distributions are consistent with the distribution of the induced flow angles, and

**Condition 2:** The resulting operating point on each section, given by \((\alpha, C_l)\) for that section, falls on the airfoil \(C_l-\alpha\) curve for that section.

As an illustration of how these conditions are satisfied for the simpler example of a finite wing operating in a pre-stall condition, it is useful to briefly review Prandtl’s lifting-line theory (see [36, 37, 38]), which is applicable to unswept finite wings of moderate to high aspect ratio. In this theory, the first condition is satisfied by using Biot-Savart’s law to relate the induced angle of attack, \(\alpha_i\), at any section to the spanwise distribution of the gradient of the bound-circulation strength, \(d\Gamma/dy\), and by expressing the effective angle of attack, \(\alpha_{eff}\), as the difference between geometric angle of attack and the induced angle of attack, i.e., \(\alpha_{eff} = \alpha - \alpha_i\). The second condition is satisfied by ensuring that the local \(C_l\) and the effective angle of attack, \(\alpha_{eff}\), at any \(y\) location are related by the potential-flow linear lift curve for the airfoil (defined by the lift-curve slope, \(a_0\), and the zero-lift angle of attack, \(\alpha_{0l}\)) as follows:

\[
C_l(y) = a_0(y) [\alpha_{eff}(y) - \alpha_{0l}(y)]
\]  

These two conditions together result in the fundamental monoplane equation of Prandtl’s
lifting-line theory, which is relatively straightforward to solve because the lift curve is linear.

The challenge with post-stall aerodynamic analysis of finite wings arises because the section lift curves in viscous flows are nonlinear. As a result, an iterative procedure is typically required if both the conditions discussed in the preceding two paragraphs are to be satisfied for all sections of all the lifting surfaces. In the current formulation, decambering is applied to a multiple lifting-surface configuration using a strip-theory approach by dividing each lifting surface into sections (or “strips”). Figure 2.2 shows an example of a wing-tail configuration divided into sections. Each section \( j \) has a decambering variable \( \delta_{1j} \) for defining the local decambered geometry at that section. The overall objective of the post-stall aerodynamic calculation is to determine the values of \( \delta_1 \) on all the sections so that the two conditions are simultaneously satisfied on all sections. It is also seen that the current approach of using decambering to model the effects of flow separation is essentially the same as applying a spanwise twist distribution, \( \delta(y) \), to the wing. While the twist distribution remains unchanged for a rigid wing, the spanwise distribution of decambering angles changes with flight condition and needs to be determined for each condition.

The formulation of such a procedure is not straightforward because of how the induced flow and hence the decambering at any section is affected by the operating points and the decambering of the other sections on the wing. To better illustrate the issues involved with the formulation of a suitable procedure, the effect of decambering on the operating conditions of a section (section 5) of the wing of Figure 2.2 is considered at a multiple angles of attack. It is assumed that the wing is not twisted and has the same airfoil throughout the span; the airfoil is assumed to have the characteristics shown in Fig. 2.1. Figure 2.3 shows the potential-flow operating conditions for section 5 at two an-
gles of attack: a pre-stall angle of attack of 2 degrees, and a post-stall angle of attack of 25 degrees. To provide context, the figure also shows the potential-flow and viscous-flow $C_l-\alpha$ curves for the airfoil.

Taking the pre-stall case first, it is seen that the potential-flow operating point for the section occurs at an angle of attack that is less than the wing angle of attack of 2 degrees. This difference is because of the induced flow resulting from the lift distribution on the wing. Such induced-flow calculations are routinely handled by any standard finite-wing analysis method such as lifting-line theory, vortex lattice, Weissinger or similar methods. At this low angle of attack the boundary layers are thin and the viscous-flow operating point coincides with the potential-flow operating point. Thus, there is no need for any decambering, and $\delta_1 = 0$ degrees works well.

Considering the post-stall case next, it is seen again that the potential-flow operating point for the section occurs at an angle of attack that is less than the wing angle of attack of 25 degrees due to induced flow. At this high angle of attack, the potential-flow operating point does not fall on the viscous $C_l-\alpha$ curve of the airfoil section because of separated
Figure 2.3: Potential-flow operating points for section 5 of the wing are shown for wing angles of attack of 2° and 25°. Potential-flow lift curves are shown for two values of δ₁: 0° (dash-dot line) and −4° (dashed line). Viscous-flow lift curve for the airfoil is shown as a solid line. For wing α of 25° are the starting (s) and perturbed (p) operating points which define the trajectory line are shown. The intersection of this line with the viscous lift curve gives the target operating point (t). Vertical lines in gray at α of 2° and 25° are shown for providing context.

boundary layers at this stalled condition. The crux of the problem is to determine the amount of decambering (i.e., the values of δ₁) needed so that the operating point for the section falls on the viscous Cl-α curve for this and every other section of the wing. In seeking an appropriate procedure, a negative perturbation to δ₁ is considered at this section to study the effect of a small amount of boundary layer separation. The potential-flow Cl-α curve for the airfoil with a small negative value of δ₁ = −4 degrees is shown as a dashed line in Figure 2.3. While it is clear that, with δ₁ = −4 degrees, the operating point for this section will fall somewhere on this dashed line, it is not known a priori where exactly the point will lie on this line. The location of the point on this line will depend on the induced flow, which in turn will depend on the unknown decambering on all the other sections of the wing.

To start the iteration, an assumption is made for how the operating point in the Cl-α space moves with changes in decambering, allowing a point (αₚ, Clₚ) to be selected on the
dashed line for the perturbed $\delta_1$ value. The line joining the original, or starting, operating point $(\alpha_s, C_{ls})$ and the perturbed operating point is referred to as the “trajectory line” in this document. This trajectory line can be extrapolated to intersect with the viscous $C_l-\alpha$ curve to determine the approximate “target” operating point given by $(\alpha_t, C_{lt})$. Plausible examples for trajectory line and target operating point are co-plotted in Figure 2.3. The residual for this section of the wing is the difference in lift coefficient between the starting value from potential flow and the assumed target value on the viscous curve: $\Delta C_l = C_{ls} - C_{lt}$. As seen, the residual at any section depends on the decambering at all the other sections, which are also being determined as a part of the procedure. While such a coupled nonlinear problem can be solved using a multi-dimensional Newton iteration by relying on strong under-relaxation for convergence, the objective of the current research is to utilize the post-stall model for flight simulation, which makes it essential that the procedure is both robust and highly efficient. Four different assumptions for calculating the trajectory line, which offer four different residual calculation methods, are studied in detail in Chapter 3. In that Chapter, the computational efficiency and robustness of each trajectory line calculation method is evaluated, and a recommendation is made for the scheme to be implemented in the methodology for simulation of post-stall flight dynamics which is presented in Chapter 4.

2.3 Solution Procedure for Applying Decambering to a 3D Wing

This section discusses the solution procedure for applying decambering to a wing at some angle of attack. The procedure is implemented in a multidimensional Newton-type
iteration, which generally takes the form shown in Equation 2.2. The goal in the general formulation of a Newton iteration is to drive the residual, $F$, toward zero.

### 2.3.1 The General Form of a Newton Iteration

The multi-dimensional Newton-Raphson method is described in detail in several references such as [39] and [40], in which the following matrix equation is solved to determine the correction vector $\delta x$ at every step using the known Jacobian matrix $J$ and the residual vector $F$:

$$J \cdot \delta x = -F$$  \hspace{1cm} (2.2)

The correction vector is then used to determine the next best approximation to the vector of zeros $x$ with an appropriate under-relaxation, or damping, factor $D$ which is necessary for highly nonlinear functions:

$$x_{\text{new}} = x_{\text{old}} + D\delta x$$  \hspace{1cm} (2.3)

The application of the Newton-Raphson method to the decambering approach in the current problem differs from the standard implementation in some important aspects. This is most easily explained using the case of a single-variable iteration, for which the standard Newton update to go from iteration $n$ to $n + 1$ is calculated as follows:

$$x_{n+1} = x^n - D \frac{f(x^n)}{f'(x^n)}$$  \hspace{1cm} (2.4)

In the above expression, the update to the estimate of the root is given by an evaluation of the function, $f(x^n)$, and its derivative, $f'(x^n)$, at iteration $n$. When applying the
Newton-Raphson method to a function expressed analytically, the update is straightforward to implement. When the Newton-Raphson method is applied in the decambering scheme, questions arise as to how to calculate the residual. Depending on the method of residual calculation, the equation for calculating the update will also change.

2.3.2 Applying the General Form of a Newton Iteration to Post-Stall Wing Aerodynamics

In this section, an overview of how the decambering calculation is carried out for a finite wing is provided. Making the procedure robust and efficient is largely dependent on how the residual is computed. The residual computation details are left out and treated in detail in Chapter 3. The decambering procedure may be described as follows:

1. **Assume initial decambering**

   Begin with an assumed starting $\delta_1$ value for each section, denoted by $\delta_{1,s}(j)$ for section $j$. Typically, no decambering is applied initially and $\delta_{1,s}(j) = 0$ for all $j$.

2. **Calculate current operating points**

   A wing analysis method is used to calculate the current operating point for each section with the current values for $\delta_1$. Various potential flow analysis techniques are suitable for use in this calculation, as long as an output of the technique includes the current section lift coefficient, $C_{l_{sec}}$, for each section, which corresponds to a spanwise strip on the lifting surface. The effective angle of attack, $\alpha_{eff}$, at this lift coefficient, which is taken as the current value of the operating angle of attack, $\alpha_{sec}$, for section $j$ is determined as follows:

   $$\alpha_{sec}(j) = \alpha_{eff}(j) = C_{l_{sec}}(j)/a_0(j) - \delta_1(j) + \alpha_0(j) \quad (2.5)$$
In Equation 2.5, \( a_0 \) is the slope of the linear portion of the airfoil lift curve, which is typically close to \( 2\pi \) per radian. The point \( (\alpha_{sec}(j), C_{l_{sec}}(j)) \) defines the current operating point of section \( j \) for the decambering procedure. At the start of the iterative procedure \( (iter = 0) \), the current operating point is the starting operating point and is denoted by \( (\alpha_s(j),C_{ls}(j)) \). Figure 2.4 shows these initial operating points for both a low and a high angle of attack decambering computation on a 3-D wing. In the low angle of attack case, it is seen that the starting points lie on the input viscous data, as the solution exists in the linear aerodynamic regime. In the high angle of attack case, the starting points lie on the potential-flow \( C_l-\alpha \) curve for the airfoil, which is significantly different from the viscous \( C_l-\alpha \) curve because of the separated boundary layer on the upper surface.

![Figure 2.4: Initial section operating points with \( \delta_1 = 0 \) degrees for low and high values of the wing angle of attack.](image)

(a) Low angle of attack, \( \alpha = 5 \) degrees  
(b) High angle of attack, \( \alpha = 25 \) degrees
3. Calculate residual

The residual is calculated for each section. In the decambering formulation, the residual for section \( j \) is the \( \Delta C_l(j) \) between the current operating point, \((\alpha_{sec}(j), C_{lsec}(j))\), and some unknown target point, \((\alpha_t(j), C_{lt}(j))\), which lies on the viscous input data:

\[
F(j) = \Delta C_l(j) = C_{lsec}(j) - C_{lt}(j)
\] (2.6)

Unlike in a typical Newton iteration, where the residual calculation is generally via a straightforward function evaluation, it is not clear in the current procedure how to select the target point on the viscous curve for computing the residual. The target point for section \( j \) will depend on the unknown decambering at other sections.

4. Modify \( \delta_1 \) vector

After the target points are identified for all sections and the residual vector \( \mathbf{F} \) is determined, the vector of corrections is calculated. Experience with the decambering schemes has shown that, for a wing with \( N \) sections, rather than solve the \( N \times N \) matrix equation (Equation 2.2) and apply the update equation (Equation 2.3) for calculating the updates to \( \delta_1 \) values, it is just as effective to determine the correction vector by independent application of Equation 4.13 to the \( N \) sections. For section \( j \), the update to \( \delta_1 \) from iteration \( n \) to \( n+1 \) is as follows:

\[
\delta_1^{n+1}(j) = \delta_1^n(j) - D \frac{\Delta C_l^n(j)}{(\frac{\partial C_l}{\partial \delta_1})^n}(j)
\] (2.7)

This update approximation tends to bring down the computational cost with no noticeable effect on convergence for any of the four residual calculation schemes.
After initialization, the final three steps of the decambering procedure are repeated until each operating point, defined to be on a strip of a lifting surface, are brought within a specified tolerance of the input viscous $C_l$-$\alpha$ data.

Using a wing analysis method inside the solution procedure ensures that condition 1 of section 2.2 is satisfied at every iteration for the values of $\delta_1$ used in that iteration. Once convergence is achieved, condition 2 of section 2.2 is also satisfied.

### 2.4 Chapter Summary

A description of the decambering concept has been provided. The method allows the effects of flow separation to be modeled by applying a twist distribution to an airfoil or finite wing such that the potential flow $C_l$ matches the viscous $C_l$. In a two-dimensional flow, the amount of decambering applied is straightforward to calculate. Complications arise when applying decambering to a finite wing because of the interactions between all sections when decambering is changed on any one section. The problem is necessarily formulated into a Newton iteration to seek the proper amount of decambering to apply to all sections simultaneously. The application of the Newton iteration to the problem is non-standard due assumptions that must be made to perform the residual calculation. Schemes for computing the residual are studied next in chapter 3.
Chapter 3

Comparison of Residual Calculation Schemes

This chapter compares four residual calculation schemes as implemented in the decambering approach to predict post-stall aerodynamics for finite wings. As with Chapter 2, much of the discussion and figures in this chapter has also been adapted from Paul and Gopalarathnam [34]. While later chapters in this document focus on implementing the decambering calculation in an unsteady vortex lattice method, throughout the course of this research it has been found to be enlightening to study the residual calculation schemes using an alternate aerodynamic analysis method based on the superposition of basic and additional aerodynamic loadings. The superposition approach is an equivalent representation of a vortex lattice (or similar) method, and is described in Section 3.1. Following the brief description of the superposition approach, the four separate approaches for the trajectory line, and the motivation behind each are presented in Section 3.2. After the trajectory line approximations and the corresponding equations that are used for the residual calculations are presented for separate approaches, the residual calculation
schemes are compared against one another on the basis of computational efficiency and rate of convergence. Two of the approaches are found to converge to nearly the same solutions for the test case considered, while two are shown to be unsuitable for the problem at hand. Due to higher computational efficiency, a particular scheme for implementation in the vortex lattice method. The Chapter concludes by highlighting a limitation of the decambering methodology that is independent of the residual calculation scheme utilized.

3.1 Aerodynamic Analysis Method Used to Study the Residual Calculation Schemes

To use the decambering approach for calculating the forces and moments acting on an aircraft, a lifting-surface aerodynamic analysis method is needed. This can be a vortex lattice method (VLM) (as used in [26], and in subsequent chapters of this document), Weissinger method, or similar tool. The analysis method is used to determine the aerodynamics for a given set of aircraft states and a given distribution of $\delta_1$ on all the sections of the surfaces. In the superposition approach, elementary basic and additional lift distributions are pre-computed using an analysis method and stored. For any given flight condition, the lift distributions and aerodynamic forces and moments are computed in the approach by linear superposition of the elementary loadings. The superposition of basic and additional lift distributions is described in several references [36, 41, 20] and has been used effectively for several decades in wing design [41, 42], and most recently in flap optimization of adaptive wings [43, 44] and multiple-wing configurations [45]. The results of [43], [44], and [45] show that the superposition approach is highly effective in determining the aerodynamics of a lifting-surface configuration with an accuracy compa-
rable to the original analysis method used to compute the elementary lift distributions. A brief description of the superposition approach is provided here.

Within the assumption of linear aerodynamics (linear \( C_l-\alpha \) variation and linear \( C_l-\Gamma \) relationship), the spanwise distribution of \( C_l \) over a wing or configuration can be expressed as a sum of two contributions: i) basic distribution, \( C_{lb}(y) \), and ii) additional distribution, \( C_{la}(y) \):

\[
C_l(y) = C_{lb}(y) + C_{la}(y)
\]  

(3.1)

The basic distribution, \( C_{lb} \), is the \( C_l \) distribution at \( C_L = 0 \), and is the result of spanwise variations in twist and flap/control deflections. Because the spanwise decambering is modeled using decambering variable \( \delta_1 \), it is similar to spanwise flap-angle variation, and hence can be used to generate basic loadings.

The additional \( C_l \) distribution, \( C_{la} \), is due to changes to \( \alpha \) for the wing with zero twist and zero decambering. The additional \( C_l \) distribution is, therefore, independent of geometric or aerodynamic twist and it scales with wing \( C_L \). Thus, the additional \( C_l \) distribution for \( C_L = 1 \), written as \( C_{la,1} \), can be precomputed for a wing and used to compute the \( C_{la} \) for any \( C_L \), as follows:

\[
C_{la}(y) = C_L C_{la,1}(y)
\]  

(3.2)

The advantage of using the superposition concept is that the net \( C_l \) distribution for a particular wing \( C_L \) can be posed in terms of the unknown decambering variables, \( \delta_1 \). Assuming \( N \) sections on the wing and denoting the \( \delta_1 \) at section \( j \) by \( \delta_{1,j} \), the expression for the net \( C_l \) distribution is:
\[ C_l = C_L C_{lb,0} + C_{lb,1} \delta_{1,1} + C_{lb,2} \delta_{1,2} + \cdots + C_{lb,N} \delta_{1,N} \] (3.3)

where, \( C_{lb,0} \) is the zero-decambering basic \( C_l \) distribution due to geometric and aerodynamic twist resulting from spanwise changes to the wing airfoil. The increment in basic \( C_l \) distribution due a unit \( \delta_1 \) for section \( j \) is denoted by \( C_{lb,j} \).

While Equation 3.3 is expressed in terms of the wing \( C_L \), for post-stall computations at a given wing angle of attack, \( \alpha_w \), it is necessary to write the wing \( C_L \) in terms of wing \( \alpha_w \), as follows:

\[ C_L = a(\alpha_w - \alpha_{b,0} - (\alpha_{b,1} \delta_{1,1} + \alpha_{b,2} \delta_{1,2} + \cdots + \alpha_{b,N} \delta_{1,N})) \] (3.4)

where, \( a \) is the wing lift-curve slope \( (a = 1/\alpha_{a,1}) \), \( \alpha_{b,0} \) is the wing angle of attack corresponding to the zero-decambering basic \( C_l \) distribution and \( \alpha_{b,j} \) is the wing angle of attack corresponding to the increment in basic \( C_l \) distribution due a unit \( \delta_1 \) for section \( j \).

The use of the superposition approach instead of a VLM-like analysis method for the decambering method was first proposed in [46] where additional details are available. The approach was subsequently used for flight-dynamics simulation in [47]. More details of the use of the superposition technique in the decambering approach and flight-dynamics simulation are discussed in [47].

For the results presented in the Chapter, the WINGS Weissinger code was used to compute the elementary basic and additional loadings for the superposition approach. To illustrate the effectiveness of the superposition approach, Figure 3.1 compares the spanwise \( C_l \) distributions for the wing-tail configuration of Figure 2.2 from the superposition approach with those from direct analysis using the WINGS Weissinger code. For
this illustration, symmetric loadings are assumed, although the formulation is applicable to the modeling of asymmetric loadings or asymmetric stall. Figure 3.1(a) shows the additional loading \( C_{la,1} \) for the wing-tail configuration. Two example basic loadings due to decambering, \( C_{lb,j} \), are shown, with each basic loading shown for a strip on the left side of the aircraft added to that from its mirror-image strip on the right side to ensure symmetry of loadings. The two basic loadings, magnified by three times for clarity, are for strip pairs (2, 19) at the wing tips and (10, 11) at the wing root. Figure 3.1(b) compares the total \( C_l \) distribution from superposition with direct analysis (Weissinger) for \( \alpha = 10 \) degrees with and without prescribed decambering. For the prescribed decambering case, \( \delta_1 \) is set to a value of \(-3\) degrees for the pair of strips (10, 11) at the wing root. When decambering is specified at the wing root, the loss in lift there affects the lift distribution on the downstream tail surface. It is seen that the predicted loadings from the superposition approach agree excellently with those from direct analysis. Of interest as it relates to the post-stall aerodynamics is that the effect of wing decambering on the tail loading is also correctly predicted by the superposition approach.
Figure 3.1: Comparison of spanwise $C_l$ distributions from the superposition approach and direct analysis (using Weissinger method) for wing-tail configuration: (a) Additional loading and two example basic loadings, with the basic loading scaled by 3 times for clarity, (b) Total loading from superposition and direct analysis for $\alpha = 10^\circ$ with no decambering and with $\delta_1 = -3^\circ$ applied to wing sections 10,11 near the root.

3.2 Residual calculation schemes

This section presents the details of the four residual calculation schemes. In the discussion for each scheme, the concept of the “trajectory line” from Section 2.2 is used to explain how the target operating point on the viscous $C_l$-$\alpha$ curve is selected. The trajectory lines for the four schemes are compared in Figure 3.2 for a section (section 5) of the rectangular wing (shown in Figure 2.2) operating at $\alpha = 20$ degrees with zero initial decambering. The four trajectory lines are co-plotted from the initial operating point for this section, obtained using $\delta_1 = 0$ degrees, in Figure 3.2.
Figure 3.2: Comparison of the trajectory lines from the four schemes for section 5 on the wing. The initial operating point (with $\delta_1 = 0$ degrees) is for wing angle of attack of 20 degrees.
3.2.1 Scheme A

The trajectory line used in scheme A is similar to that used in previous efforts by other researchers [15, 18]. In this method, a vertical trajectory line is used to find the intersection with the viscous input data, as shown in Figure 3.2. The implicit assumption is that the operating point for a section changes with decambering as if the section was in 2D flow, with no induced flow due to other portions of the wing. Thus, in the calculation of the residual, this scheme does not take into consideration any effect of the wing aspect ratio or the spanwise location of the section on the wing. Using a wing section $j$ for this discussion, the decambering variable $\delta_1(j)$ is updated from iteration $n$ to $n+1$ as follows:

$$\delta_{1}^{n+1}(j) = \delta_{1}^{n}(j) - D \frac{C_{l_{sec}}^{n}(j) - C_{l_{t}}^{n}(j)}{a_0}$$ (3.5)

As discussed later in Section 3.3.2, this scheme is very robust in determining a converged solution, but requires considerable under-relaxation for convergence at some post-stall conditions. Owing to the fact that the conditions at which under-relaxation is needed are not known \textit{a priori}, a conservative decision to use under-relaxation at all conditions is taken, whether the conditions correspond to pre-stall or post-stall angle of attack. The large number of iterations results in increased computational time for convergence.

3.2.2 Scheme B

Scheme B uses a sloped trajectory line following the approach presented in [26] in which a VLM was used as the aerodynamic analysis method. This scheme was subsequently used in [46] in which the superposition approach was used in lieu of a full aerodynamic analysis method with the aim of decreasing computational time. The use of the superposition approach also resulted in the insight [46] that the Jacobian and trajectory-line
slope in this scheme are invariant with angle of attack and decambering. They need to be calculated only once for a given geometry and discretization. To determine the slope of this trajectory line, a small perturbation is made to the decambering of the current section as illustrated in Figure 2.3 and the operating point for the perturbed condition is determined by assuming that there is no change in the decambering for all the other sections of the wing. The resulting trajectory-line slope for section \( j \), for which the derivation was provided in [46], is given as:

\[
\left( \frac{dC_l}{d\alpha} \right)_{\text{traj},j} = -\frac{\alpha_{b,j} C_{la,1}(j)/\alpha_a + C_{lb,j}(j)}{C_{lb,j}(j)/a_0 - \alpha_{b,j} C_{la,1}(j)/(\alpha_a a_0) - 1}
\]  (3.6)

The \( \delta_1 \) vector is updated from iteration \( n \) to \( n + 1 \) as follows:

\[
\delta_{1}^{n+1}(j) = \delta_{1}^{n}(j) - D \frac{C_{l \text{sec}}^{n} - C_{l l}^{n}}{-a_0 \alpha_{b,j} C_{la,1}(j) + C_{lb,j}(j)}
\]  (3.7)

Scheme B was the first scheme [26] to take three-dimensional effects into account in computing the residuals for post-stall calculations. However, this scheme has drawbacks of the trajectory line being sensitive to the number of sections on the wing. This scheme also lacks robustness for certain combinations of wing aspect ratios and airfoil lift curves.

### 3.2.3 Scheme C

Scheme C was developed [47] with the aim of taking into consideration the effect of decambering on all sections of the wing on the trajectory line at any given section. In this scheme, using section \( j \) for illustration, the first iteration is made using a vertical trajectory line (as described in scheme A) from the starting operating point with zero decambering, \( (\alpha_s(j), C_{ls}(j)) \). As a result of the first iteration, a new operating point is calculated. Because this new operating point, labeled \( (\alpha_p(j), C_{lp}(j)) \), is a result of the
Newton iteration step, it partially takes into consideration the changes in decambering of all sections on the wing. The line joining \((\alpha_s(j), C_{ls}(j))\) and \((\alpha_p(j), C_{lp}(j))\) is the trajectory line for section \(j\), and is shown in Figure 3.2. The slope of the trajectory line is determined as follows:

\[
\frac{dC_l}{d\alpha}_{\text{traj},j} = \frac{C_{lp} - C_{ls}}{\alpha_p - \alpha_s} \tag{3.8}
\]

The update to \(\delta_1\) at iteration \(n\) that will bring the operating point closer to the airfoil \(C_l-\alpha\) curve is as follows:

\[
\delta_1^{n+1}(j) = \delta_1^n(j) - D \frac{C_{lsn}^n - C_{lt}^n}{(C_{lp} - C_{ls})/(\delta_1^p - \delta_1^s)} \tag{3.9}
\]

As discussed later in Section 3.3.2, scheme C has issues of being unable to converge because of significantly incorrect trajectory-line slopes when some sections of the configuration have operating points that are converged (on the viscous \(C_l-\alpha\) curve) while other sections are well away from convergence.

### 3.2.4 Scheme D

Scheme D was developed because of issues with the robustness of schemes B and C and large computational times of scheme A. In scheme D, the trajectory lines are determined by assuming that all sections have the same change in decambering, which is essentially like a change in the wing angle of attack. To determine the slope of this trajectory line, a small perturbation is made to the decambering of the current section as illustrated in Figure 2.3 and the operating point for the perturbed condition is determined by assuming that the same perturbation is also applied to all the other sections of the wing. The resulting trajectory-line slope for section \(j\) is:
These trajectory lines are then used to determine the target operating points for each section. The update to \( \delta_1 \) at iteration \( n \) that will bring the operating point closer to the viscous \( C_l-\alpha \) curve of the airfoil is as follows:

\[
\delta_1^{n+1}(j) = \delta_1^n(j) - D \frac{C_l_{sec}^n - C_l_i^n}{C_l_{a,1}(j)/\alpha_{a,wing}}
\] (3.11)

As discussed later in Section 3.3.2, the approximation for the trajectory-line slope in this scheme results in a very good estimation of the target operating point on the viscous \( C_l-\alpha \) curve, resulting in excellent convergence even for post-stall conditions. The estimated trajectory-line slope takes into consideration both the wing geometry (such as aspect ratio) and the spanwise location of the section along the wing. Additionally, it does not exhibit any undesirable sensitivity due to the discretization (number of sections). The calculated slopes are invariant with angle of attack and decambering, because of which they need to be determined only once, and this may be done as a pre-processing step after which many aerodynamic inflow conditions may be evaluated. It is also shown that no under-relaxation factor is needed for this scheme. Among the four schemes considered in this paper, this scheme requires the minimum number of iterations for convergence at any angle of attack. This scheme has proven to be both fast and robust.

### 3.3 Results and Discussion

Results are presented in this section from pre- and post-stall calculations to compare the effectiveness of the four schemes. The schemes were used to analyze the wing-tail...
configuration shown in Figure 2.2. The wing will be the primary focus of the study. The tail surface has been included to illustrate convergence problems with one of the schemes. Because the decambering approach uses airfoil lift curve as input and provides the wing lift curve as output, validation of the approach requires the availability of pre- and post-stall data for both airfoil and wing from the same source. Owing to the dearth of experimental data having consistent airfoil and wing lift characteristics at post-stall conditions, a recent effort was undertaken at NC State to develop a database of results for airfoils and wings using CFD with a Reynolds Averaged Navier Stokes (RANS) code in time-accurate mode. For this effort, the NASA TetrUSS CFD software [48] was used with the Spalart-Allmaras turbulence model [49], with care taken to ensure that the grids were consistent between the simulations for the 2D (airfoils) and the 3D (wings) geometries. More details of the approach and early results from the effort are documented in Ref. [50]. The RANS CFD computations were performed for the wing alone (without the tail surface) to keep computational costs low. Because of the negligible influence of the tail on the wing aerodynamics, the comparison of wing-only $C_L$ results from the four residual-computation schemes applied to wing+tail geometries with CFD results for the wing-alone geometries is adequate for validation of the low-order method.

The comparison of the predicted lift curves for the wing from the four schemes with CFD results is presented in Section 3.3.1 along with results comparing the four schemes based on the number of iterations needed for convergence. It is shown that schemes B and C do not converge for certain situations even with under-relaxation. Section 3.3.2 next presents an explanation of the observed behavior of the four schemes by comparing the trajectory lines and the “evolution” of the solution from the initial to the converged solution. Section 3.3.3 then compares the effects of initial value of the decambering on the convergence properties of schemes A and D. It is shown that scheme D is not only
robust, but also the most computationally efficient. Section 3.3.4 compares the spanwise distributions of lift coefficient from scheme D with those from CFD for several angles of attack, highlighting interesting features of the solution. Finally, section 3.3.5 presents the effect of sweep angle on the stall characteristics to illustrate the limitations of the current decambering approach.

3.3.1 Comparison of residual calculation schemes

The results from CFD were computed at a Reynolds number of 3.0 million for the NACA 4415 airfoil and the aspect ratio 12 constant-chord wing. Figure 3.3 shows the wing-only $C_L$ vs. $\alpha$ curves for the wing-tail configuration computed using each residual calculation scheme. For these decambering calculations, the 2D lift curves from CFD were used as inputs. Also shown in Figure 3.3 are the 2D input data and 3D wing data from CFD. For
all schemes, the initial condition of $\delta_1 = 0.0$ was used for starting the Newton iteration for every angle of attack. At every pre-stall angle of attack, all schemes converge to the same $C_L$ value. Scheme B fails to converge for $\alpha > 16$ degrees. Scheme C fails to converge for $\alpha > 19$ degrees if the tail is present in the model. Schemes A and D successfully converge for all angles of attack from 0 to 40 degrees for this wing without the need for any under-relaxation (i.e., $D = 1$), except for the $\alpha$ values of 24 and 25 degrees at which scheme A is successful in converging only if an under-relaxation factor of $D \leq 0.6$ is used. At every post-stall angle of attack, schemes A and D converge to $C_L$ values that are within 0.05 of each other. This small discrepancy between the two schemes is not surprising—the occurrence of multiple converged solutions for a post-stall condition have been discussed by several researchers [10, 11, 12, 13, 14, 16, 26]. The results from schemes A and D show good agreement with CFD in predicting $C_L$ for angles of attack from 0 to 25 degrees, including the $C_{L,max}$ and the $\alpha_{stall}$ for the wing. Beyond $\alpha = 25$ degrees, both schemes predict converged $C_L$ values that match up with the $C_l$ values of the viscous airfoil lift curve from CFD provided as input. Because the CFD prediction for the airfoil $C_l$ is approximately 0.15 higher than the CFD prediction for the wing $C_L$ at these angles of attack, the two schemes over-predict the wing $C_L$ compared to CFD results for the wing. The reason for this over-prediction is not known. Given that, at these challenging post-stall conditions, even CFD and experimental predictions tend to have uncertainty and discrepancies that are larger than 0.15, the predictions from the current low-order method are considered acceptable.
Table 3.1: Number of iterations required for convergence for three angles of attack. Under-relaxation factor of $D \leq 0.6$ was needed for $\alpha$ of 24 degrees with scheme A. The numbers in parenthesis for scheme A show the number of iterations required for convergence with $D = 0.6$ at $\alpha$ of 16 and 30 degrees. Schemes B and C did not converge at the high angles of attack even with damping. Scheme D did not require damping at any angle of attack.

<table>
<thead>
<tr>
<th>Scheme</th>
<th>$\alpha = 16$ deg.</th>
<th>$\alpha = 24$ deg.</th>
<th>$\alpha = 30$ deg.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Damping</td>
<td>Iterations</td>
<td>Damping</td>
</tr>
<tr>
<td>A</td>
<td>1 (0.6)</td>
<td>3 (6)</td>
<td>0.6</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>11</td>
<td>–</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>3</td>
<td>–</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

3.3.2 Solution evolution for the four schemes

The observation that all schemes converge to the nearly the same final $C_L$ value (unless they fail to converge like with scheme B for $\alpha > 16$ degrees and scheme C for $\alpha > 19$ degrees) shows that even if the gradient (Jacobian) information is approximate, the Newton iteration works well. However, when seeking computational efficiency, it is desirable to compute the gradients as accurately as possible provided that no significant additional computational time is needed for the more accurate gradient calculation. To compare the relative performance of the four schemes, Table 3.1 presents the number of iterations required for convergence at three example angles of attack: a pre-stall angle of attack of 16 degrees, a near-stall angle of attack of 24 degrees, and a post-stall angle of attack of 30 degrees. Because the function evaluation in each iteration is the practically the same for all schemes, the number of iterations to convergence is a good measure of the computational time. At the pre-stall angle of attack of $\alpha = 16$ degrees, all four schemes converge successfully without the need for any damping. At the post-stall condition of $\alpha = 30$ degrees, at which schemes B and C do not converge even with damping, scheme A
requires more than twice the number of iterations as scheme D. The near-stall condition of \( \alpha = 24 \) degrees is seen to be the most challenging case. At this condition, strong cross-coupling is observed between the Newton equations for the different sections on the wing and tail. As a result, scheme A converges successfully only with damping of \( D \leq 0.6 \), and needs 160 iterations for convergence. Owing to the fact that the conditions at which under-relaxation is needed are not known \textit{a priori}, a conservative decision to use under-relaxation at all conditions is taken, whether the conditions correspond to pre-stall or post-stall angle of attack. While scheme D does not need any damping, \( \alpha = 24 \) degrees is the angle of attack at which it needs the maximum number of iterations to achieve convergence. Overall, it is seen that scheme D needs the smallest number of iterations to converge at any angle of attack.

The reason for the different computational efficiency and robustness of the schemes can be better understood by comparing, for each scheme, the trajectory line with the evolution of the operating points as the iteration progresses. Figure 3.4 presents such a comparison for wing section 5 at a wing angle of attack of 16 degrees. For this angle of attack, all four schemes converge successfully without the need for under-relaxation. Each of the four insets shows how, starting from the initial operating point for this section corresponding to zero decambering, the operating point in the \( \alpha-C_l \) space approaches the final converged solution on the viscous curve with increasing iteration number (with no under-relaxation). It must be remembered, however, that such a plot provides the evolution of only one of the many \( \delta_1 \) variables in the multi-dimensional iteration.

\textbf{Scheme A:} Considering scheme A, it is seen that the vertical trajectory line is a poor approximation to the evolution path followed by the operating points. Although the scheme converges within 3 iterations for this condition, an under-relaxation factor is required to ensure convergence at some other angles of attack.
Figure 3.4: Evolution of the operating points from initial ($\delta_1 = 0$ degrees) to converged for section 5 compared for schemes A–D. Wing angle of attack is 16 degrees and no under-relaxation was used with any scheme.
Scheme B: In scheme B, the sloped trajectory line is much more horizontal compared to the path followed by the evolving operating points. The poor estimation of the trajectory-line slope results in increased number of iterations. An even more severe problem is that no intersection is found between the inclined trajectory line and the viscous airfoil post-stall lift curve for higher angles of attack, which causes a termination of the iterative procedure without convergence for $\alpha > 16$ degrees. This makes scheme B unreliable for certain airfoil lift-curve shapes.

Scheme C: The inspiration for scheme C came from the observation that the path taken by the evolving operating points in scheme A is practically linear and would result in an excellent estimate for the trajectory line. Thus in scheme C, the trajectory line is determined by connecting the first two operating points resulting from a single iteration with scheme A and using that trajectory line for the remainder of the procedure until convergence. The problem with this approach is that, on wing-tail configurations, there are usually some sections on the tail that are unstalled and have operating points close
to the viscous lift curve, while simultaneously sections on the wing may have operating points near the stall region and have large residuals between the operating point and the target point. As illustrated by the plot in Figure 3.5, scheme C predicts a highly incorrect trajectory line for such tail sections. When this problem occurs, it results in a failure to converge. In the current example, convergence is possible for the wing-alone geometry without any under relaxation, but no amount of under-relaxation was able to achieve convergence on the wing-tail configuration for $\alpha > 19$ degrees.

**Scheme D:** As seen for scheme C, the trajectory line in scheme D is also close to the path followed by the evolving operating points. It is seen that the target operating point calculated prior to the very first iteration and the intermediate operating point resulting from the first iteration are both close to the final converged solution. Unlike with scheme C, however, this behavior was consistent over all the sections of the wing-tail configuration and over the entire range of angles of attack. As a consequence, no under-relaxation was found to be necessary for the entire range of angles of attack. Convergence was seen to be rapid, with no more than 3 iterations required at pre-stall conditions and typically no more than 10 iterations required for post-stall conditions. For a few angles of attack in the near-stall region such as $\alpha = 24$ degrees, as many as 78 iterations were needed for convergence. The trajectory line, being invariant with angle of attack and decambering, can be pre-computed and reused throughout the simulation. This scheme is both efficient and robust, with the target operating point and the intermediate solution from the very first iteration being close to the converged solution.
Figure 3.6: Effect of initial $\delta_1$ value on the evolution of the operating points from initial ($\delta_1 = 0$ degrees) to converged for section 5 for schemes A and D. Wing angle of attack is 30 degrees.
3.3.3 Effect of initial value of decambering for schemes A and D

Because of the convergence difficulties with schemes B and C, only schemes A and D are selected for further study. In this subsection, the convergence properties of schemes A and D are compared to determine the effect of the initial value of $\delta_1$ used to start the iteration. Under-relaxation factor, $D$, was set to 1.0 (no damping) for all cases in this sub-section. Three initial $\delta_1$ values of $-10$, $0$, $10$ degrees were used for this study. In each case, the initial $\delta_1$ value was used for all sections of the wing-tail configuration. The convergence of the operating point for wing section 5 is examined at a post-stall angle of attack of 30 degrees in Figure 3.6. It is seen that both schemes converge to the same final operating point independent of the starting value of $\delta_1$. The number of iterations required for convergence of scheme A depends on the initial $\delta_1$ value: 18, 24, and 31 iterations for initial $\delta_1$ values of $-10$, $0$, and $10$, respectively. In contrast, the number of iterations for convergence of scheme C is 9 for all three values of the starting $\delta_1$. The insensitivity of scheme D compared to that of scheme A can be explained by comparing the trajectory lines for the three starting points in Figure 3.6. For scheme A, depending on the starting value of $\delta_1$, it is seen that the vertical trajectory line intersects the airfoil lift curve at a different location. As a result, the residual for the first iteration in scheme A depends on the choice of the initial value of $\delta_1$. In turn, this dependence affects the number of iterations required for convergence. In contrast, the trajectory line for scheme D is seen to be independent of the starting $\delta_1$ value. As mentioned in section 3.2.4, the trajectory-line slope for scheme D is determined by assuming that all sections have the same change in decambering, which makes the trajectory-line slope independent of the starting $\delta_1$ value. As a consequence, the convergence of scheme D is not affected by the choice of initial $\delta_1$. 

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This behavior is true for all the angles of attack.

It is also seen that, with scheme D, both the target operating point determined just prior to the first iteration and the solution point from the first iteration are also unaffected by the initial $\delta_1$ value. These points are both close to the final converged operating point even for this post-stall condition. Although convergence with scheme D is achieved with a relatively small number of iterations, it is worthwhile to further examine the results obtained from using the target point and the solution point from just the first iteration of scheme D. With this objective, two variations of scheme D are introduced: schemes D0 and D1.

**Scheme D0:** In scheme D0, the target operating point determined prior to the first iteration is taken as the final solution. The results of scheme D0 are determined without even a single Newton iteration. Thus scheme D0 is a zero-iteration variant of scheme D.

**Scheme D1:** In scheme D1, the solution after a single iteration is taken as the final solution without checking for convergence to a given tolerance. Thus scheme D1 is a one-iteration variant of scheme D.

Figure 3.7 compares the lift curves from the schemes D, D0, and D1 with 2D and 3D CFD-predicted lift curves. It is seen that the $C_L$ values from schemes D, D0, and D1 are identical to each other for pre-stall conditions and acceptably close for post-stall conditions. Also plotted in Figure 3.7 is the lift curve obtained after a single iteration with scheme A, computed without under-relaxation. The $C_L$ values for post-stall angles of attack from this lift curve are substantially over-predicted compared to scheme D and CFD. This result highlights how scheme D, in contrast to scheme A, is successful in closely approximating the converged solution even with the first iteration. The maximum error in $C_L$ between schemes D0 and D is 0.08 and the maximum error between D1 and D is 0.12. It is interesting to note that these results from D0 and D1, achieved without full
Figure 3.7: Wing-only lift coefficient vs. angle of attack from schemes D, D0, and D1 compared with 2D and 3D CFD results. Also shows are the results from one iteration of scheme A.

convergence, are close to the fully-converged results from scheme D. A potential benefit of this interesting result is that the D0 or D1 scheme could be used to develop a fast, approximate version of scheme D, which could be used in flight-dynamics simulation in which the run-time uncertainty with number of iterations required for convergence is removed. As seen from Table 3.1, the number of iterations for convergence with scheme D varies from 3 to 78, depending on the angle of attack. By performing only a single iteration, the time required for the aerodynamic analysis at any time step of the simulation can be made independent of the flight condition.

3.3.4 Spanwise lift distributions

In assessing schemes D, D0, and D1, for post-stall aerodynamic prediction of finite wings, it is important to determine the effectiveness of the methods in capturing the spanwise
distribution of lift. The objective is to determine if the methods are successful in providing a reasonable prediction of the stall progression along the wing span. For this purpose, using CFD results as the benchmark against which to compare the results from the low-order method has significant advantages. Spanwise lift distributions are often hard to obtain from experiments as they require a large number of surface pressure measurements. With CFD, on the other hand, such information is generated as a part of the flow solution. In this subsection, spanwise $C_l$ distributions from schemes D, D0, and D1 are compared with results from the CFD analysis.

Figure 3.8 compares the spanwise lift-coefficient distributions from schemes D, D0, and D1 with CFD for a pre-stall ($\alpha = 14$ degrees) and a post-stall ($\alpha = 20$ degrees) angle of attack for the rectangular wing. The comparison with CFD results for $\alpha = 14$ degrees is seen to be excellent for all three schemes. For $\alpha = 20$ degrees, while the spanwise extent of stalled flow from the three schemes and CFD show good agreement, there is some over-prediction of lift in the stalled region with the decambering approach compared to
Figure 3.9: Comparison of wing $C_l$ distributions using 20 and 80 strips along the wing span, with CFD results for $\alpha = 24$ and 26 degrees.

the CFD solution. Nevertheless, the comparison is reasonable for a low-order method that is intended for use in simulation.

Figure 3.9 compares the spanwise lift-coefficient distributions from scheme D with CFD for angles of attack that are well beyond stall: at $\alpha = 24$ degrees and $\alpha = 26$ degrees. These two angles of attack were chosen because the fully-converged solutions from scheme D for these conditions exhibit spanwise “sawtooth variations” in the lift dis-

Figure 3.10: Upper-surface flow visualizations on the wing from CFD for 24 and 26 degrees angle of attack. Right side of geometry is shown.
tributions. These sawtooth variations persist even when the spanwise discretization is increased by using more sections along the wing, as shown in Figure 3.9 for scheme D with increased spanwise discretization using 80 strips along the wing span. Such spanwise sawtooth variations have also been observed by other researchers [12, 13, 14, 16] who studied the behavior of low order aerodynamic models augmented to handle nonlinear sectional input data. It has been speculated that these sawtooth variations might be an artifact of the numerical method [26]. This may be the case, and interestingly, very recent research derives a numerical explanation for the appearance of stall cells based on the equations of classical lifting-line theory [51]. If similar expressions could be arrived at for the present aerodynamics analysis method, these sawtooth patterns may be better understood. In either event, it is curious to observe that the higher-order CFD lift distributions also exhibit spanwise waviness, although not of a sawtooth nature. Although the match between the low-order method and CFD is not exact, the variations are roughly over the same spanwise region of the wing. The spanwise variations in CFD can be explained by studying the upper-surface flow on the wing at these angles of attack shown in Figure 3.10. It is seen that the wing is not cleanly stalled—that is, the wing contains alternating regions with varying extents of chordwise separation. These flow features are referred to as stall cells; they have also been studied experimentally [52] and theoretically [53, 54]. This flow pattern provides an explanation for the spanwise lift variations in the CFD solution. It remains to be seen if the occurrence of the sawtooth variations in the converged solution from decambering is simply a coincidence or is connected with the occurrence of the stall cells in some way.

Figure 3.11 compares the spanwise lift-coefficient distributions from schemes D, D0, and D1 with CFD for $\alpha = 24$ and 26 degrees. It is seen that the approximate variants, D0 and D1, predict spanwise distributions that closely match those seen from scheme D.
While the approximate schemes D0 and D1 do not exhibit the sawtooth variations seen with scheme D, they do show loss in lift due to stall over approximately the same spanwise portions of the wing as do scheme D and CFD. In general, it is seen that scheme D and its approximate variants, D0 and D1, are computationally efficient and robust for the aerodynamic prediction of wings at pre- and post-stall angles of attack. Given that the aim of this decambering formulation is to rapidly compute the aerodynamics, the prediction of lift and spanwise lift distributions from schemes D, D0, and D1 compare reasonably well with CFD solutions. These schemes, therefore, show excellent promise for further development toward use in full-aircraft configurations for design, simulation, and control applications.

3.3.5 Limitations: effects of sweep angle

In this subsection, the effects of wing sweep are considered to illustrate a limitation of the current decambering approach. The swept-wing CFD results presented in this subsection
are from Ref. [50]. These CFD results are in qualitative agreement with the results from wind-tunnel experiments of Harper and Maki [55]. It must be mentioned that the flow physics of swept-wing stall has been well understood for several decades. The primary goal of this subsection is use the effects of sweep to bring out and explain an important limitation of the current decambering approach.

The wing-only geometry of the wing-tail configuration from Fig. 2.2 was modified by considering three aft-sweep angles: 10, 20, and 30 degrees. No other changes have been made to the wings. The swept-wing geometries were created using the NACA 4415 sections at the root and tip chords aligned in the streamwise direction. The results for the three swept-wing geometries from schemes D, D0, and D1 are compared with CFD results.

Figure 3.12(a)–(d) show the predicted lift curves from schemes D, D0, and D1 and from CFD for the rectangular and three swept wings. Also shown in each plot is the airfoil lift curve from CFD, which was used as input to the three schemes. The results for the unswept wing in Fig. 3.12(a) are the same as those shown in the previous subsections, but are repeated here to assist comparison with the swept-wing results.

Examining the lift curves for the four wings from CFD, it is seen that the predicted $C_{L,max}$ decreases progressively with sweep from approximately 1.62 for the zero-sweep wing to approximately 1.25 for the 30-degree swept wing. The stall behavior also changes progressively, with the $\alpha$-range over which the $C_L$ drops beyond stall progressively increasing with sweep angle. For the 30-degree sweep case, the $C_L$ stays nearly constant at the $C_{L,max}$ value of approximately 1.25 for almost 15 degrees beyond stall. Examining the lift curves from the schemes D, D0, and D1 next, it is seen that, for all sweep angles, the $C_L$ values from the three schemes compare excellently with each other at pre-stall angles of attack and acceptably well at post-stall angles of attack. These comparisons
Figure 3.12: Wing-only $C_L$-$\alpha$ curves from schemes D, D0, and D1 compared with CFD for the four aft-sweep angles.
Figure 3.13: Upper-surface flow visualizations on the wings from CFD at $C_{L,max}$. Right side of each wing is shown.

show that the zero-step and one-step implementations of scheme D are viable alternatives to the fully-converged implementation even for the swept-wing geometries. Comparison of the lift curves from schemes D, D0, and D1 with those from CFD, however, brings out the limitations of the current decambering approach. The significant reduction in $C_{L,max}$ with increasing sweep angle, seen in the CFD-predicted lift curves, is absent in the predictions from schemes D, D0, and D1. Instead the three schemes predict a small increase in $C_{L,max}$ with increasing aft-sweep angle.

The CFD predicted $C_{L,max}$ reduction with increasing sweep angle can be better un-
Figure 3.14: Comparison of wing $C_l$ distributions for $\alpha = 4$ and 26 degrees from schemes D, D0, and D1 with CFD results for the four sweep angles.
nderstood by examining the upper-surface flow visualizations for the four wings, shown in Figs. 3.13(a)–(d). It is seen that, as the sweep angle is increased, the surface skin-friction lines, which are mostly in the streamwise direction for the zero-sweep wing, become progressively more tilted in the spanwise direction. The transverse pressure gradients on the aft-swept wings result in a transport of the boundary-layer and separated flow toward the tip. This behavior causes premature stall on the outboard portions of the aft-swept wings. On wings with high aft sweep, the tipward transport of the separated flow results in the root section being uninstalled even well beyond wing $C_{L,max}$.

Figures 3.14(a)–(d) show the spanwise distributions of local $C_l$ for the four wings at a pre-stall and a post-stall angle of attack. Also shown as a horizontal line on each plot is the airfoil $C_{l,max}$. The spanwise $C_l$ distributions are shown from CFD and from the three implementations of scheme D. It is to be noted that the local $C_l$ in this study is defined using wing strips that are aligned in the streamwise direction. For all cases, it is seen that the three implementations of scheme D agree closely with each other. For the zero-sweep case, it is seen that the spanwise $C_l$ distributions from schemes D, D0, and D1 agree well with CFD. The section $C_l$ values never exceed the airfoil $C_{l,max}$ even at post-stall conditions. The correlation between CFD and the schemes D, D0, and D1 continues to be good at pre-stall conditions for all the swept wings. At post-stall conditions for the swept wings, however, there is significant disagreement in the $C_l$ distributions between CFD and the three schemes. For example, for the 30-degree swept wing at $\alpha = 26$ degrees, the CFD-predicted spanwise $C_l$ distribution shows that the $C_l$ in the root region is well above airfoil $C_{l,max}$ and the stalled outboard regions have $C_l$ less than $C_{l,max}$. For this condition, the three implementations of scheme D show a spanwise $C_l$ variation that is everywhere less than the airfoil $C_{l,max}$ and in disagreement with the CFD results. In all cases, the premature stall of the outboard portions and postponement of stall in the
inboard portions seen in the CFD predictions for the swept wings is in overall agreement with the surface flow visualizations in Fig. 3.13 which show a tipward transport of the separated flow on aft-swept wings at stall.

The spanwise transport of separated flow is not accounted for in the current decambering method. This flow phenomenon invalidates the main assumption in the current decambering method that the relationship between local-\(C_l\) and effective \(\alpha\) on all sections of a finite wing are practically the same as the relationship between \(C_l\) and \(\alpha\) for the corresponding airfoil. This assumption holds good for unswept wings at both pre-stall and post-stall conditions and for swept wings at pre-stall conditions, but is invalidated at post-stall conditions on swept wings. More generally, this assumption is invalidated at conditions when there is separated flow in combination with transverse pressure gradients on the wing surface. Because the assumption is invalidated, the decambering approach converges to an incorrect prediction of the stall.

### 3.4 Chapter Summary

In this chapter, four residual calculation schemes were compared on the basis of computational efficiency and robustness. Two of the schemes were found to be unsuitable for use when analyzing multiple lifting surface configurations. The other two were found to perform well when analyzing a finite wing at post-stall angles of attack. One scheme, scheme D, approached the final converged solution much more rapidly than the others with out the need for relaxation, and as such was selected for further use. Solutions for a rectangular wing were presented for \(1 \leq \alpha \leq 40\) degrees using 2-D CFD as input data supplied to the decambering approach. Outputs from the decambering calculations compared very favorably to 3-D CFD results produced for the same geometry. While it was
shown that more work is needed to apply the decambering calculations to wings that have significant amounts of spanwise pressure gradient, the current approach is clearly suitable for use analyzing configurations with un-swept wing geometries.
Chapter 4

Nonlinear Simulation

This chapter begins by giving a description of the unsteady vortex lattice method, implemented in discrete time form, in Section 4.1. The discrete time form of the method is well suited for flight simulation. Next, Section 4.2 describes how the post-stall model is included in the aerodynamics analysis framework, where required. Further, a methodology is presented whereby the rigid body states, which may be considered as inputs to the aerodynamic model, are propagated in time. Finally, the framework is exercised for a representative light aircraft configuration in Section 4.3.

4.1 Description of the Numerical Implementation of the Unsteady Vortex Lattice Method

The unsteady vortex lattice method (UVLM) is an efficient computational technique to solve 3-D potential-flow problems about moving lifting surfaces. The basics of the standard UVLM algorithm are described by Katz and Plotkin [56], but an alternative formulation by Murua et. al will be used herein [57], whereby the governing equations
are written in discrete-time form, which is ideally suited for flight simulation.

4.1.1 Non-Penetration Boundary Condition

![Aerodynamics model showing the lifting-surface and wake discretization using vortex-ring elements.](image)

In the UVLM, elementary solutions are distributed over lifting surfaces and the non-penetration boundary condition is imposed at a number of control (collocation) points. The elementary solution is the vortex ring, distributed over the mean surface, ignoring thickness. The leading segment of the vortex ring lies on the panel’s quarter-chord line, and a collocation point is placed at the three-quarter-chord line, which falls at the center of the vortex ring (see Figure 4.1). Placing the leading segment of the vortex ring at the quarter-chord of the panel satisfies the two-dimensional Kutta-Joukowskii condition along the chord. As the surface moves along its flight path, a wake is obtained as part of the solution procedure. The wake is also represented by vortex rings. By imposing the non-penetration boundary condition at a number of collocation points, a system of algebraic equations is obtained. At discrete time step \( n + 1 \), the vorticity distribution of the bound vortex elements is determined by
\[ A_b \Gamma_b^{n+1} + A_w \Gamma_w^{n+1} + w^{n+1} = 0, \]  

(4.1)

where \( \Gamma_b \) and \( \Gamma_w \) are the column vectors with the circulation strengths in the bound and wake vortex-rings; \( A_b = A_{cb}(\zeta_b) \) and \( A_w = A_{cw}(\zeta_b, \zeta_w) \) are the wing-wing and wing-wake aerodynamic influence coefficient matrices, computed at the collocation points; and \( \zeta_b \) and \( \zeta_w \) are the column vectors with the bound and wake grid coordinates. \( w \) in Eq. (4.1) is the column vector of normal components of all velocities except those induced by bound and wake vorticity, which may encompass deployment of control surfaces, gust induced velocities, and rigid-body motions. It also includes the contribution of the decambering, when applicable in the post-stall regime. The velocity induced at a collocation point by a vortex ring (located on the lifting surface or in the wake) is obtained from the Biot-Savart law. For a unit circulation strength, the Biot-Savart law is given by Equation 4.2.

\[
(\vec{v}_{\text{ind}})_{k,l} = \oint d\vec{s}_l \times \vec{r}_{kl} \over 4\pi r_{kl}^3
\]

(4.2)

In Equation 4.2, \( d\vec{s}_l \) represents the vortex ring segments making up vortex ring \( l \), \( \vec{r}_{kl} \) is the vector from collocation point \( k \) to the midpoint of the relevant vortex segment being considered, and \( r_{kl} \) is the magnitude of the vector \( \vec{r}_{kl} \). Note that, numerically, four evaluations of the Biot-Savart law are required to compute the induced velocity at one collocation point due to one vortex ring. Clearly, this is computationally expensive when the effects of a large number of vortex rings are to be calculated at several time steps.

The entries in the aerodynamic coefficient matrices, \( A_{cb} \) and \( A_{cw} \), consist of the induced velocities projected along the bound panel normal vectors. This is represented as:
\[(A_b)_{k,l} = (\vec{v}_{ind})_{k,l} \cdot \vec{n}_k, \quad k, l = 1...K_b \quad (4.3)\]
and
\[(A_w)_{k,l} = (\vec{v}_{ind})_{k,v} \cdot \vec{n}_k, \quad k = 1...K_b, \quad v = 1...K_w. \quad (4.4)\]

In Equations 4.3 and 4.4, \(k\) and \(l\) are bound panel counters, \(v\) is a wake panel counter, \(K_b\) is the total number of bound panels, \(K_w\) is the total number of wake panels, and \(\vec{n}\) is a normal vector for a vortex ring.

The velocities included in the vector \(\mathbf{w}\) in Equation 4.1 are non-circulatory and projected along the normal vector. This is written as:

\[\mathbf{w}^{n+1} = W_b \cdot \hat{\zeta}_b^{n+1} \quad (4.5)\]

In Equation 4.5, \(W_b\) is a matrix size \([K_b, 3 \times K_b]\), that stores bound panel normal vectors. The multiplication of \(W_b\) with a vector of size \([3 \times K_b, 1]\) containing the current velocity at the collocation points, denoted \(\hat{\zeta}_b\), projects the non-circulatory velocity along the normal vectors. By updating the \(W_b\) matrix entries, control surface effects can be implemented. After identifying the panels included in the control, the relevant normal vectors in matrix \(W_b\) are rotated using the rotation matrix calculated from the Rodriguez rotation formula, as shown in Equation 4.6.

\[R = I + [u]_x \sin(A) + (1 - \cos(A))[u]^2_x \quad (4.6)\]

In Equation 4.6, \(I\) is a 3x3 identity matrix, \(A\) is the amplitude of the control deflection, and \([u]_x\) is the skew-symmetric form of the unit vector describing the axis of rotation.
For panels included in control surfaces, the axis of rotation is taken to be the leading edge of the relevant vortex ring. If the $k$th normal vector is affected by the control surface deflection, it is updated as

$$\vec{n}_{rot} = R\vec{n}$$

(4.7)

to account for the deployment of the control.

### 4.1.2 Wake Propagation

At each time step, as the circulation of the wing changes, a new row of vortex rings will be shed into the wake from the trailing edge of each lifting surface. In addition to this, the existing wake may be displaced following the local flow velocity (when using the free-wake model). An accurate description of a force-free wake might be essential when it impinges on a body, such as if the wake interaction with a trailing aircraft or downstream lifting surface or blade is being modeled (e.g. blade-vortex interactions on rotorcraft). However, its impact on the fixed-wing-aircraft flight dynamics considered in this work has been shown to be minimal [58]. As a result, and as it is a computationally very expensive process that requires the evaluation of the local velocities at the wake grid-points, a fixed wake model was used for the current work. The propagation equations for the wake circulation can be written in discrete time as

$$\Gamma_{n+1}^w = C_{\Gamma_b}\Gamma_n^b + C_{\Gamma_w}\Gamma_n^w,$$

(4.8)

where $C_{\Gamma_b}$ and $C_{\Gamma_w}$ map the circulation of the previous time step to the current one, and they are very sparse constant matrices which account for Kelvin’s circulation theorem and Helmholtz’s vortex theorem.
4.1.3 Aerodynamic Loads Computation

Once the distribution of vorticity has been obtained at each time step, the inviscid aerodynamic loads can be computed. Different methods exist, such as the expressions given by Katz and Plotkin [56], which are based on the unsteady Bernoulli equation but use the small-angle assumption and introduce an approximation for induced drag in order to account for leading edge suction. The method used in the current work is the Joukowski theorem applied at each bound vortex segment midpoint. To account for apparent mass effects due to the time varying circulation acting on the lifting surfaces, the unsteady form of the expression is used [59]. While the Joukowski theorem is computationally more expensive than the unsteady Bernoulli [60], it provides a more general approach and avoids the small-angle assumption. The use of the Joukowski theorem is, therefore, more appropriate for the high angle of attack work. The expression for the aerodynamic force acting on a panel is

\[
\Delta F = \rho_\infty \Gamma_\delta l (U \times \delta l) + \rho_\infty \left( \frac{d\Gamma_p}{dt} \Delta S \right) \hat{n}
\]  

(4.9)

where \( \Gamma_\delta l \) is the net circulation acting on the leading edge of the bound vortex ring, \( U \) is the total velocity (kinematic + induced) at the point of lift-calculation, and \( \delta l \) is the vector describing the leading-edge segment of the panel vortex ring. The second term in Equation 4.9 describes the unsteady force components, commonly called the apparent mass effects. In the second term, \( \Gamma_p \) is the vortex strength whose derivative is approximated in the numerical implementation by a backwards finite difference, \( \Delta S \) is the vortex ring panel area, and \( \hat{n} \) is the panel unit normal vector. The net velocity at the point of lift calculation, \( U \), may be written as
\[
U = -\begin{bmatrix}
u \\
v \\
w
\end{bmatrix} - \begin{bmatrix}
p \\
q \\
r
\end{bmatrix} \times \vec{r}_{lc} + \begin{bmatrix}
H_{lb_x} \Gamma_b \\
H_{lb_y} \Gamma_b \\
H_{lb_z} \Gamma_b
\end{bmatrix} + \begin{bmatrix}
H_{lw_x} \Gamma_w \\
H_{lw_y} \Gamma_w \\
H_{lw_z} \Gamma_w
\end{bmatrix}
\] (4.10)

where

\[
\vec{r}_{lc} = \begin{bmatrix}
x_{lc} - x_{CG} \\
y_{lc} - y_{CG} \\
z_{lc} - z_{CG}
\end{bmatrix}.
\]

The \(H_{lb}\) and \(H_{lw}\) matrices are analogous to aerodynamic influence coefficient matrices. They contain induced-velocity information computed via the Biot-Savart Law (Equation 4.2) at the lift-calculation points, projected along the body-axis \(x\), \(y\), or \(z\) direction, as appropriate. As the forces are computed for each vortex ring, \(\vec{r}_{lc}\) is updated to account for effects of angular velocity.

### 4.2 Simulation of Post-Stall Flight Dynamics

The objective of the current research is to incorporate the post-stall model of Section 2.3 into the unsteady aerodynamics analysis method presented in Section 4.1 and integrate the resulting aerodynamic model with the aircraft equations of motion for a flight dynamics simulation framework. The methodology for the combined model is presented as a flowchart in Figure 4.2.
The following sections describe the steps of the combined unsteady, post-stall aerodynamics model in the order they are presented in the flowchart.
4.2.1 Inputs to the aerodynamics model

The aerodynamic model receives inputs related to the configuration, viscous data for use in the post-stall model, and initial values to facilitate time-marching the non-linear flight dynamic equations. Configurations are made up of groups of lifting surfaces. Configuration information provided includes both reference areas and geometry definitions for each individual wing. Geometry input files allow for easy definition of each wing, including support for twist, sweep, taper, and varying camber. Control surfaces geometry is also specified at this point.

Viscous $C_l - \alpha$, $C_d - \alpha$, and $C_m - \alpha$ curves are input for later use in the post-stall iteration. If control surface(s) are defined for a wing, viscous curves with controls deflected to the minimum and maximum values must also be provided. At a given time step, depending on the specified control deflection, the model will interpolate the viscous input data for wing sections that have a control surface. These interpolated curves will be used for the post-stall model when necessary.

Initial values of the flight dynamic state vector, $y$ are provided as input. For a full six degree of freedom, nonlinear simulation with either 12 or 13 initial values are provided. The 12 element initial state vector, 

$$y_0 = [x_{e0} \ y_{e0} \ z_{e0} \ u_0 \ v_0 \ w_0 \ \phi_0 \ \theta_0 \ \psi_0 \ \phi_0 \ \theta_0 \ \psi_0 \ p_0 \ q_0 \ r_0]^T,$$

contains initial values for inertial position, body axis velocity, body axis orientation relative to the inertial frame, and body axis angular rates. The model also allows for specification of a 13-state state vector whereby quaternions are used for attitude representation. When quaternions are used for attitude representation, initial values of the Euler angles $[\phi_0 \ \theta_0 \ \psi_0]$ are replaced with quaternion parameters, $[\eta_{10} \ \eta_{20} \ \eta_{30} \ \eta_{40}]$. 

4.2.2 Pre-process geometry

Geometry data is discretized and vortex ring elements are placed on each wing as described in Section 4.1. UVLM matrices $A_b$, $A_w$, $H_{lb}$, and $H_{lw}$ are computed and stored. Wake propagation matrices $C_{Gb}$ and $C_{Gw}$ are generated and stored at this step. As discussed in Section 4.1, these matrices are required to enforce the Kutta-condition and for efficiently time-marching the wake vorticity. With the configuration completely defined and discretized, steady vortex lattice runs are made as described in Appendix A in order to calculate $(\frac{dC_l}{d\alpha})_t$ and $\frac{dC_l}{d\delta_1}$. This is an equivalent representation of the methodology in Section 3.2.4.

4.2.3 Solve inviscid aerodynamics

Given the current value of the flight dynamic state vector, $y$, the pre-computed aerodynamics matrices, and the current value of decambering vector $\delta_1$, the non-penetration boundary condition (Eqn. 4.1) and wake circulation propagation (Eqn. 4.8) equations may be solved to find the gamma distribution at time step $n + 1$. Equations 4.1 and 4.8 are written together, and modified to incorporate the effect of decambering, as

$$\begin{pmatrix} \Gamma_b \\ \Gamma_w \end{pmatrix}^{n+1} = \begin{pmatrix} A_b & A_w \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ C_{Gb} & C_{Gw} \end{pmatrix} \begin{pmatrix} \Gamma_b \\ \Gamma_w \end{pmatrix}^n + \begin{pmatrix} w + w_{dec} \end{pmatrix}^{n+1}$$ (4.11)

Solving Eqn. 4.11 for the $n+1$ gamma distribution simultaneously satisfies the boundary condition and unsteady wake propagation equations while accounting for the effects of decambering on the configuration. Given the current circulation distribution acting on the configuration, the total inviscid aerodynamic loads may be computed. In the current
implementation, the loads are calculated using the Kutta-Joukowski theorem, as shown in Equation 4.9.

### 4.2.4 Calculate current operating points

Initialization of the decambering algorithm requires knowledge of the current operating point of each ‘strip’ along each wing. The operating points exists in a $C_l$-$\alpha$ space, and are not directly computed when inviscid aerodynamic loads are found using Kutta-Joukowski or other load calculation techniques. An important consideration when computing the operating points is that they will be compared to the appropriate input viscous $C_l$-$\alpha$ curve, which comes from 2-D airfoil aerodynamics. Each strip is approximated as a 2-D airfoil section.

![Wing-tail geometry with ring vortices (bound only) shown. The ring vortices corresponding to one spanwise section, or ‘strip’, are highlighted in blue.](image)

Figure 4.3: Wing-tail geometry with ring vortices (bound only) shown. The ring vortices corresponding to one spanwise section, or ‘strip’, are highlighted in blue.

In the current model, the $C_l$ calculation is undertaken by first calculating the total aerodynamic force acting on a ‘strip’. A strip of a wing is illustrated in Figure 4.3 as
the panel elements which have a depiction of vortex ring elements highlighted in blue. Equation 4.12 shows the expression used to calculate the local $C_l$ values.

$$C_{l_{sec}}(i) = \frac{|\vec{F}_{cl}|}{1/2 \rho |\vec{V}_{local}|^2 dS_{strip}}$$

(4.12)

The operating $\alpha$ for each strip is calculated as given in Equation 2.5.

4.2.5 Check convergence of operating points

After the operating points have been computed, they are projected onto the viscous input data along the trajectory line, as discussed in Section 3.2.4. The difference between the $C_{l_{sec}}$ and computed target $C_{l_t}$ is defined as the residual. If the residual is less than a pre-defined tolerance, no iteration is necessary as the points are converged. If the residual is greater than the predefined tolerance, each element of the $\delta_1$ vector is altered according to Equation 4.13 and the aerodynamics are re-evaluated until the operating points are converged on the viscous input data.

$$\delta_{1}^{new}(i) = \delta_{1}^{old}(i) - C_{l_{sec}} - C_{l_t} \frac{\partial C_l}{\partial \delta_1}$$

(4.13)

4.2.6 Calculate total force and moment vectors

Inviscid aerodynamic loads are known from the steps described in Section 4.2.3. After convergence of the decambering algorithm, each sectional operating point $(\alpha_{sec}, C_{l_{sec}})$ falls on the viscous airfoil $C_l-\alpha$ curve for that section. The sectional profile drag and moment coefficients are found by querying the input $C_d-\alpha$ and $C_{m^*}-\alpha$ curves at the converged operating point. For strip $i$, the contributions from these coefficients are given by Equations 4.14 and 4.15. The total contributions from all sections are added to the
inviscid forces and moments according to Equations 4.16 and 4.17.

\[
\Delta \vec{F}(i) = \frac{1}{2} \rho |\vec{V}_{strip}(i)| c(i) dS(i) C_{d_{sec}}(i) \begin{bmatrix} \vec{V}_{strip}(i) \cdot \hat{i} \\ \vec{V}_{strip}(i) \cdot \hat{j} \\ \vec{V}_{strip}(i) \cdot \hat{k} \end{bmatrix} 
\]

\[
\Delta \vec{M}(i) = \frac{1}{2} \rho |\vec{V}_{strip}(i)|^2 c(i)^2 dS(i) C_{m_{sec}}(i) \begin{bmatrix} 0 \\ \vec{N}_{strip}(i) \cdot \hat{k} \\ \vec{N}_{strip}(i) \cdot \hat{j} \end{bmatrix} 
\]

\[
\vec{F}_{tot} = \vec{F}_{inv} + \sum_i \Delta \vec{F}(i) 
\]

\[
\vec{M}_{tot} = \vec{M}_{inv} + \sum_i (\vec{r} \times \Delta \vec{F}(i)) 
\]

where

\[
\vec{r} = \begin{bmatrix} x_{qc} - x_{CG} \\ y_{qc} - y_{CG} \\ z_{qc} - z_{CG} \end{bmatrix} 
\]
4.2.7 Time marching of the equations of motion

For a full nonlinear 6 degree of freedom simulation (using quaternion parameters), Equation 4.18 provides the flight dynamic state vector derivative, $f_n$.

\[
\begin{align*}
\dot{x} &= u(\eta_4 \eta_4 + \eta_1 \eta_1 - \eta_2 \eta_2 - \eta_3 \eta_3) + v(2\eta_2 \eta_1 - 2\eta_4 \eta_3) + w(2\eta_3 \eta_1 + 2\eta_4 \eta_2) \\
\dot{y} &= u(2\eta_1 \eta_2 + 2\eta_4 \eta_3) + v(\eta_4 \eta_4 - \eta_1 \eta_1 + \eta_2 \eta_2 - \eta_3 \eta_3) + w(2\eta_3 \eta_2 - 2\eta_4 \eta_1) \\
\dot{z} &= u(2\eta_1 \eta_3 - 2\eta_4 \eta_2) + v(2\eta_2 \eta_3 + 2\eta_4 \eta_1) + w(\eta_4 \eta_4 - \eta_1 \eta_1 - \eta_2 \eta_2 + \eta_3 \eta_3) \\
\dot{u} &= rv - qw + g(2\eta_4 \eta_3 - 2\eta_4 \eta_2) + (X_{tot} + X_{thrust})/mass \\
\dot{v} &= -ru + pw + g(2\eta_2 \eta_3 + 2\eta_4 \eta_1) + Y_{tot}/mass \\
\dot{w} &= qu - pv + g(\eta_4 \eta_4 - \eta_1 \eta_1 - \eta_2 \eta_2 + \eta_3 \eta_3) + Z_{tot}/mass \\
\dot{\eta}_1 &= 1/2(\eta_4 p - \eta_3 q + \eta_2 r) \\
\dot{\eta}_2 &= 1/2(\eta_3 p + \eta_4 q - \eta_1 r) \\
\dot{\eta}_3 &= 1/2(-\eta_2 p + \eta_1 q + \eta_4 r) \\
\dot{\eta}_4 &= 1/2(-\eta_1 p - \eta_2 q - \eta_3 r) \\
\dot{p} &= (c_1 r + c_2 p - c_4 h_{eng})q + (c_3 R_{tot} + c_4 N_{tot}) \\
\dot{q} &= (c_5 p + c_7 h_{eng})r - c_6(p^2 - r^2) + c_7 M_{tot} \\
\dot{r} &= (c_8 p - c_2 r - c_9 h_{eng})q + (c_4 R_{tot} + c_9 N_{tot})
\end{align*}
\]
where the following constants have been defined [61]:

\[
c_1 = \frac{(I_y - I_z)I_z - I_{xz}^2}{I_xI_z - I_{xz}^2},
\]
\[
c_2 = \frac{(I_x - I_y + I_z)I_{xz}}{I_xI_z - I_{xz}^2},
\]
\[
c_3 = \frac{I_z}{I_xI_z - I_{xz}^2},
\]
\[
c_4 = \frac{I_{xz}}{I_xI_z - I_{xz}^2},
\]
\[
c_5 = \frac{I_z - I_x}{I_y},
\]
\[
c_6 = \frac{I_x}{I_y},
\]
\[
c_7 = \frac{1}{I_y},
\]
\[
c_8 = \frac{(I_x - I_y)I_{xz} - I_{xz}^2}{I_xI_z - I_{xz}^2},
\]
\[
c_9 = \frac{I_x}{I_xI_z - I_{xz}^2}.
\]

The flight dynamic state vector, \( y \), is updated using a numerical method for the solution of differential equations. In the current methodology, only explicit methods are considered since only previous values of the state time history are available. Flight dynamic simulation results presented in this paper have all been computed using the fourth-order Adams-Bashforth method, whose state update equation is given by Equation 4.19 [62].

\[
y_{n+1} = y_n + \frac{dt}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}) \quad (4.19)
\]

The fourth-order Adams-Bashforth method requires that a starter formula is used to generate the first few points. In this work forward Euler is used to generate these points.

### 4.3 Results

The methods presented above for flight dynamics simulation with unsteady and post-stall aerodynamics are exercised next for a configuration representative of a light general aviation aircraft.
4.3.1 Post-Stall Modeling

This section presents simulation results that demonstrate the capability that the post-stall model offers for flight dynamics simulation. Results were generated using a rigid aircraft geometry, the unsteady aerodynamics model, and the fully non-linear time-marching scheme for the flight dynamics. The test configuration in this example is loosely adopted from a light general aviation aircraft from Roskam [63]. The main wing is rectangular with a camberline from a NACA4415 airfoil and 2° dihedral. The horizontal and vertical tail planes are also rectangular lifting surfaces with a symmetric airfoil cross-sections, and feature full span control surfaces of quarter-chord-length. The wings in this example have a constant chord with no aerodynamic twist, thus the input viscous airfoil data that is assigned to each section is constant throughout each wing. The viscous input data with zero-flap deflection used in the simulation is obtained from computational fluid dynamics simulation, described in Reference [50]. The sectional viscous data used when considering control deflections were generated by combining predicted coefficients from XFOIL [25] which were merged with the high α data beyond the convergence capability of XFOIL. Input viscous airfoil data for each airfoil used in the post-stall test case is shown in Figures 4.4 and 4.5. Relevant configuration details are presented in Table 4.1, and mass properties and reference areas are given in Table 4.2.
Table 4.1: Post-stall example aircraft geometric properties.

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Main wing</th>
<th>HTP</th>
<th>VTP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chord</td>
<td>1.5 m</td>
<td>1 m</td>
<td>1 m</td>
</tr>
<tr>
<td>Span</td>
<td>11 m</td>
<td>3.5 m</td>
<td>1.8 m</td>
</tr>
<tr>
<td>Airfoil Cross Section</td>
<td>NACA 4415</td>
<td>Symmetric</td>
<td>Symmetric</td>
</tr>
</tbody>
</table>

For the simulation example, the initial speed of the aircraft is set at $V_\infty = 40$ m/s at sea level, where the density is $\rho_\infty = 1.225$ kg/m$^3$. A time step of $\Delta t = 0.02$ s, corresponding to an initial non dimensional time step of $\Delta t^* = V_\infty \Delta t/c = 0.53$, was used throughout the simulation and was found to be sufficiently small. The configuration was
Table 4.2: Post-stall example aircraft mass properties and reference areas.

<table>
<thead>
<tr>
<th>Mass</th>
<th>1200 kg</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_x$</td>
<td>1285 kg-m$^2$</td>
</tr>
<tr>
<td>$I_y$</td>
<td>1824 kg-m$^2$</td>
</tr>
<tr>
<td>$I_z$</td>
<td>2666 kg-m$^2$</td>
</tr>
<tr>
<td>$I_{XZ}$</td>
<td>0 kg-m$^2$</td>
</tr>
<tr>
<td>$S_{ref}$</td>
<td>16.5 m$^2$</td>
</tr>
<tr>
<td>$b_{ref}$</td>
<td>11.0 m</td>
</tr>
<tr>
<td>$c_{ref}$</td>
<td>1.5 m</td>
</tr>
<tr>
<td>$X_{cg}$</td>
<td>$-0.7$ m</td>
</tr>
<tr>
<td>$X_{NP}$</td>
<td>$-0.9$ m</td>
</tr>
</tbody>
</table>

discretized using two panels chordwise, with 20 spanwise panels in the wing and 8 in the tail surfaces.

As seen in the trajectory drawn as Figure 4.6, the simulation example begins with the aircraft gliding near a trim condition at $V_\infty = 40$ m/s traveling in the positive $x$ direction. After 0.4 s of simulation time, the elevator is deflected from the trim condition to pitch the aircraft up. Figure 4.7 shows an immediate response to the up elevator command in the $\alpha$ signal. After this immediate reaction to the elevator deflection, the angle of attack gradually increases as the elevator input is held until the aircraft stalls. A post-stall lift distribution from simulation time = 7.6s is given in Figure 4.9. The post-stall lift distribution shown is asymmetric, as around the same time the stall is reached the rudder is deflected to its maximum value, as can be seen by the plot of the rudder command and $\beta$ signal in Figure 4.8. The stall is not maintained for long - the aircraft pitches down nearly immediately after the stall. Control inputs are relaxed and the aircraft proceeds to enter a spiral like motion. A lift distribution from a point in the spiral motion is shown in Figure 4.10.
Figure 4.6: Post-stall simulation example trajectory.
Figure 4.7: Elevator command and $\alpha$ response.

Figure 4.8: Rudder command and $\beta$ response.

Figure 4.9: Post-stall $C_l$ distribution at Sim Time = 7.6s.

Figure 4.10: Unstalled $C_l$ distribution from Sim Time = 20s.
4.4 Chapter Summary

This chapter describes the integration of the decambering post-stall model into the unsteady vortex lattice method. The equations describing the unsteady vortex lattice method are written in discrete time form. The UVLM transforms the circulations from one time step to the next by satisfying the boundary condition and wake circulation propagation equations simultaneously. The boundary condition equation has been re-written to take into account the decambering. Application of the decambering correction does require iteration internal to the time stepping that takes place to integrate the rigid body equations of motion. The rigid body equations of motion are advanced in time via an Adams-Bashforth integration scheme. The next chapter explores freezing the general nonlinear implementation described here at a trim condition, performing a linearization that retains the aerodynamic loads from the full vortex lattice set of equations, and analyzing the stability of the resulting system at pre- and post-stall conditions.
Chapter 5

Integration of the Decambering Post-Stall Model in the SHARP Framework for Stability Analysis

This chapter describes the integration of the post-stall model, based on iterative decambering, into the analysis framework established by the SHARP tool. The capabilities of SHARP, and how it is used in this work, are described in Section 5.1. After a description of the framework for stability analysis created by SHARP in Section 5.2, the proposed methodology for including decambering is described in Section 5.3. Following a description of the inclusion of the stall model in the SHARP framework, this model is exercised in numerical results for two example configurations. The first is a free-to-roll wing model, as may be found in a wind tunnel. The resistance to roll, called roll damping, is studied and the results from the aerodynamic model are compared to experimental results. Finally, the stability of a full aircraft configuration is presented for angles of attack ranging from pre-stall, where no decambering is necessary, through the post-stall regime. The
effects of the stall model are noted, and compared qualitatively to experimental results.

5.1 Introduction to the SHARP Tool

Simulation of High Aspect Ratio Planes (SHARP) [57, 64] is a medium-fidelity tool designed originally for the simulation of large, flexible aircraft. It couples an aerodynamics model, based on the UVLM, and a structural model, based on a finite-element representation of a nonlinear beam. SHARP is capable of performing both aerodynamic and aeroelastic analysis, in addition to flight dynamic analysis for a full aircraft configuration. Using the framework established by SHARP, many different types of simulations may be performed, including:

- **Nonlinear Time Marching**
  In a nonlinear time-marching simulation, the coupled aerodynamic/structural dynamic system is propagated in time, accounting for three-dimensional aerodynamic effects, changes in the aircraft geometry (if flexibility is enabled), and rigid body motions. This type of simulation is computationally very expensive, particularly when the aerodynamic wake remains free and the full effects of wake rollup are simulated. This type of simulation is similar to what has been described in Chapter 4, with the important difference being the way rigid body motion is taken into account. The implementation in Chapter 4 updates rigid body states via the standard Newton-Euler equations of motion, as does the implementation in SHARP. However the rigid body states are not propagated in time until after the inertia tensors have been updated considering structural deformations.
• Linear Analysis About a Static Condition

From an equilibrium condition, SHARP allows the capability to carry out linearization of the UVLM and nonlinear beam equations with monolithic coupling, resulting in a compact linear model expressed in state space form. The state vector includes circulation states and derivatives from the vortex lattice, structural states that can track deformations, and full configuration flight velocities and orientation. The linear state space system that results is an efficient tool which can be time marched by simply performing matrix multiplication.

5.1.1 Motivation for Including Post-Stall Aerodynamics in the SHARP Linear Analysis Framework

For low speed and low Reynolds number aerodynamic in-flow conditions, potential flow methods provide excellent tools for aerodynamic analysis. The linear analysis capabilities in SHARP, based on UVLM aerodynamics, have been shown to be accurate in attached flow cases, and has been particularly well validated when compared to other aerodynamic panel methods and experiment for the prediction of flutter boundaries for T-Tail configurations [65]. Additionally, the flight dynamic prediction capabilities have been shown to agree well with classical methods in several instances [64], and perhaps even offer additional insight due to the inclusion the unsteady aerodynamic effects [66].

Potential flow methods, however, are not appropriate for all flight regimes. Aircraft, especially those with low wing-loading and high aspect ratio wings, are highly susceptible to gusts which may lead to excursions beyond stall. Further, many unmanned aircraft trim in post-stall conditions, where the standard SHARP potential-flow based aerodynamics are not appropriate. With these limitations in mind, the goal of this Chapter is to describe
the implementation of the decambering post-stall aerodynamic model within the SHARP framework for stability analysis.

The work described in Section 5.3 is focused on including the decambering model within the stability analysis capabilities of SHARP, and all results and discussion will be presented as it relates to a rigid aircraft configuration. For a rigid aircraft, the linearization performed by SHARP is equivalent to a small perturbation analysis on Equations 4.18, with aerodynamic loads coming from the UVLM. In general, the structural degrees of freedom present in SHARP could be allowed. However this initial work is focused on generating a methodology for including the effects of the post-stall model for the rigid aircraft case and examining how the stability of rigid aircraft test cases are altered when the stall model is included.

5.2 Stability of the SHARP Homogeneous System

This section describes how SHARP is utilized to perform stability analysis for a given configuration. Symbolically, this process may be represented using the block diagram shown in Figure 5.1.

Figure 5.1: Flowchart describing the process to perform stability analysis cases.
From equilibrium, SHARP routines are utilized to assemble the linear system matrices. An eigenvalue problem is formulated from the system matrices, from which the eigenvalues are studied to look at modal behavior and stability. The following sections give an overview of the homogeneous system matrices and proceed to describe the stability analysis problem in detail.

5.2.1 Homogeneous System Matrices

SHARP routines are utilized to perform a linearization on the aerodynamic and rigid-body equations. Full details on the linearization are presented in Murua et. al [57]. The linearized UVLM and rigid-body equations are assembled into a unified framework with an analytical coupling between the aerodynamic and rigid body states. The resulting system can be cast into a discrete-time state-space formulation with the form

$$E_{sys} \Delta x^{n+1} = A_{sys} \Delta x^n$$

(5.1)

where the state vector is

$$x = \left[ x_A^T \mid x_R^T \right] = \left[ \Gamma_b^T \quad \Gamma_w^T \quad \dot{\Gamma}_b^T \mid \nu^T \quad \Theta^T \right]^T$$

(5.2)

that describes the system completely. The full state vector concatenates the aerodynamic circulation states, the rigid-body linear and angular velocities ($\nu = [u \quad v \quad w \quad p \quad q \quad r]$), and the rigid-body orientation angles ($\Theta = [\phi \quad \theta \quad \psi]$). Note that the axis system used in the SHARP linear stability analysis is a body-fixed system, though not one coincident with traditional X-forward, Y-starboard, and Z-down orientation utilized in many
aircraft flight dynamics applications [67]. Rather, a different body-fixed frame is used, as depicted in Figure 5.2, with $X$-axis pointing aft of the aircraft, $Y$-axis point to the starboard, and $Z$-axis pointing to the top of the aircraft. This axis system corresponds to that used in the UVLM as described in Reference [56], and is the most natural frame from which to obtain the aerodynamic forces.

![Figure 5.2: Axis system used in the SHARP Linear Stability Analysis](image)

The system matrices themselves may represented in a partitioned form, with entries that look like

$$
E_{sys} = \begin{bmatrix}
E_A & E_{A,RB} \\
E_{RB,A} & E_{RB}
\end{bmatrix}, \quad
A_{sys} = \begin{bmatrix}
A_A & A_{A,RB} \\
A_{RB,A} & A_{RB}
\end{bmatrix}
$$

where the subscript ‘$A$’ refers to the aerodynamic model and the subscript ‘$RB$’ refers to the rigid body model. Interface terms are required to map between the two models.

**Linearized Aerodynamic Subsystem**

As this work focuses on modifying the aerodynamic model to include the effect of stall using decambering; terms in the aerodynamic subsystem are explained next. The equations
that describe the aerodynamics are based on a linearized form of the UVLM equations presented in Chapter 4. Following a small-perturbation analysis, the UVLM boundary condition equation (Equation 4.1) and wake propagation equation (Equation 4.8), are re-cast as:

\[
A_b^o \Delta \Gamma_b^{n+1} + A_w^o \Delta \Gamma_w^{n+1} + W_b^o \Delta \zeta_b^{n+1} = 0, \quad (5.3)
\]

where the term \(W_b^o\) is used to store the vortex-ring normal vector information for the equilibrium configuration, and

\[
\Delta \Gamma_w^{n+1} = C_b \Delta \Gamma_b^n + C_w \Delta \Gamma_w^n. \quad (5.4)
\]

A mid-point integration scheme is used for the computation of the derivative of the bound circulation as:

\[
\Delta \Gamma_b^{n+1} - \frac{1}{2} \Delta t \Delta \dot{\Gamma}_b^{n+1} = \Delta \Gamma_b^n + \frac{1}{2} \Delta t \Delta \dot{\Gamma}_b^n. \quad (5.5)
\]

Output equations for the aerodynamic subsystem are derived in closed form by Murua [64]. With \(\gamma = U_c \Lambda_c + U_s \Lambda_s\) for brevity, the equations are included below for completeness.

\[
\Delta L^n = \rho_{\infty} \left\{ G_c^o \gamma^o \Delta \Gamma_b^n + G_c^o \left( \frac{\partial \gamma}{\partial \Gamma_w} \right) \Gamma_b^o \Delta \Gamma_w^n + G_c^o \Delta \dot{\Gamma}_b^n \right. \\
\left. + \left[ G_c^o \left( \frac{\partial \gamma}{\partial \zeta_b} \right) \Gamma_b^o + \left( \frac{\partial G_c^o}{\partial \zeta_b} \right) \left( \gamma^o \Gamma_b^o + \dot{\Gamma}_b^o \right) \right] \Delta \zeta_b^n \right\} \quad (5.6)
\]
\[ \Delta D^n = \rho_\infty \left\{ - \left[ \left( \frac{\partial \hat{U}}{\partial \Gamma_b} \right)_o \Lambda_c \Gamma_b^o + \hat{U}^o \Lambda_c \right] \Delta \Gamma_b^n - \left( \frac{\partial \hat{U}}{\partial \Gamma_w} \right)_o \Lambda_c \Gamma_w^o \Delta \Gamma_w^n \right\} + G_s^o \Delta \hat{\Gamma}_b^n + \left( \frac{\partial G_s}{\partial \zeta_b} \right)_o \hat{\Gamma}_b^o \Delta \zeta_b^n \right\} \] (5.7)

In the output equations, \( \Lambda_c \) is a matrix filled with 1 and \(-1\) in the appropriate locations to account for adjacent panels. The matrices \( G_c \) and \( G_s \) are dependent on the panel geometry, and \( U_c \) and \( U_s \) are diagonal matrices that store weighted velocities. Exact definitions of these terms, and the analytically computed partial derivatives, are given in full detail in Murua [64].

The linearized UVLM Equations 5.3–5.7 define the aerodynamic subsystem, where the inputs are taken to be the change in the aerodynamic lattice velocities and outputs are given as the change in aerodynamic loads. Two important features regarding the aerodynamic model implemented in the system matrices must be emphasized for the upcoming discussion in Section 5.3: (1) All terms in Equations 5.3–5.7, except for those retained as states (circulations) or as inputs (aerodynamic grid velocities), are computed only at the equilibrium condition (with these terms denoted by the ‘o’ superscript), and (2) there exists an analytical relationship between the inputs and outputs of the linearized aerodynamic model, such that the outputs (loads) may be computed by pre-multiplying elements of the state vector by pre-stored constant matrices, such as those in expressions 5.3–5.7, that are placed in the appropriate locations in the \( E_{sys} \) and \( A_{sys} \) matrices.
Interfacing the Aerodynamic and Rigid-Body Subsystems

The coupling between the aerodynamic subsystem and rigid-body model is shown schematically in Figure 5.3.

![Diagram of coupling between aerodynamic and rigid-body subsystems]

**Figure 5.3**: Schematic of the coupling between the linearized aerodynamics and rigid-body systems.

The outputs of the aerodynamic subsystem, the incremental changes in the aerodynamic loads, are passed to equations that transfer these loads to the aircraft center of gravity. The net change in loads is passed as an input into the linearized rigid-body model. Output from the rigid-body model are the change in flight velocities at the center of gravity. A linear mapping exists links the velocity change at the CG to the change in velocity on the aerodynamic lattice, which serves as input to the linearized aerodynamic model.
5.2.2 Stability Analysis for the Homogeneous System

After the system described by Equation 5.1 has been assembled, the discrete time eigenvalue problem is defined as:

\[ E_{sys} \psi_i = z_i A_{sys} \psi_i \]  

(5.8)

In Equation 5.8, \( z_i \) is the \( i^{th} \) discrete time eigenvalue and \( \psi_i \) is the corresponding right eigenvector. To guarantee stability, the discrete time eigenvalues must fall within the unit circle, that is for all discrete time eigenvalues the inequality

\[ |z_i| < 1 \]

must be strictly satisfied. As is the case for typical flight dynamic modes of standard aircraft, it is possible to have some unstable poles, as will be shown in the numerical studies later in this Chapter. The eigenvalues may be transformed to the more familiar continuous time version using the following relationship

\[ \lambda_i = \ln(z_i)/\Delta t \]  

(5.9)

where \( \Delta t \) is the time step used in the discretization of the aerodynamic and rigid-body models.

When the continuous time eigenvalues are considered, the condition necessary for stability requires that the real part of the eigenvalue, \( \lambda_i \), be negative. Because continuous time eigenvalues are easier to interpret, the rest of this document will report continuous eigenvalues unless it is noted otherwise.

It is worth noting that the size of a typical system is quite large—stability analysis
cases have between 2000 to 4000 elements in the state vector (with most of these corres-
dponding to wake circulation states). The eigenvalue problem is solved in MATLAB
using the function eig. As stated in the MATLAB documentation [68], the eigenvalues
are computed using QZ factorization. With such large systems, it is prudent to verify
that the matrix of eigenvectors pre-multiplied with the system matrix is equal to the di-
agonal matrix containing the eigenvalues pre-multiplied with the matrix of eigenvectors.
Verification of this quantity shows that the QZ factorization method used to compute
the eigenvalues is indeed reliable for these very large matrices.

After the eigenvalues have been found for a given system, rigid body modes are to be
identified. Given that there are as many eigenvalues as there are states, with the number of
states in the thousands, this process can be daunting. It has been seen that the rigid-body
modes are the dominant modes in the full, coupled, system. The dominant modes are
identified by sorting eigenvalues by magnitude, while keeping track of the corresponding
eigenvectors. The dominant eigenvalues must then be examined manually. Rigid-body
modes are identified by examining elements of the eigenvectors.

5.3 Incorporation of the Stall Model in the Linear
State Space Formulation

In this section, two options are proposed for the inclusion of decambering into the methodology for stability analysis described in Section 5.2.2. The options explored are:

- **Fixed decambering model**
  The fixed decambering model includes the rotation of the panel normal vectors due to decambering in the linear stability analysis formulation only in the establishment
of the equilibrium condition.

**Variable decambering model**

The stall model also includes the rotation of the panel normal vectors due to decambering in the establishment of the reference condition. In addition, this model takes into account the effects of body-axis motion on the decambering, and the resulting amount of aerodynamic force, on each strip of the lifting surfaces. The impetus for attempting to include a mechanism that alters the amount of aerodynamic force acting on a lifting surface from the pure potential flow prediction near the stall angle-of-attack may be seen in discussions in References [33] and [69] regarding roll rate damping. A similar discussion, as it pertains to the problem at hand, is presented below.

Consider an aircraft undergoing a steady roll-rate, as shown in Figure 5.4(a). The roll rate causes an angle of attack distribution, shown in Figure 5.4(b), which varies linearly across the span from \( pb/2u \) on the left wingtip to \( -pb/2u \) on the right wingtip. Figure 5.4(c) shows two different wing angle of attack cases, one low and one high, at which the roll rate may be considered. For the low-\( \alpha \) case, the roll rate causes a anti-symmetric increment in the angle of attack on the left and right wing. The right wing experiences a lower angle of attack and produces less lift. Conversely, the left wing experiences a higher angle of attack and a higher amount of lift production. Both wings operate in the linear regime on the \( C_l-\alpha \) curve, and no decambering is required (\( \delta_1 = 0 \)). The net result is that a restoring moment is created that acts to oppose the roll rate. Due to this effect, the roll rate is highly damped. Considering the high-\( \alpha \) equilibrium case next, we have a small amount of decambering required for the potential flow prediction to converge on the viscous
input data. The same anti-symmetric distribution of angle of attack is placed on the configuration due to roll rate. The right wing, experiences a lower angle of attack, and develops less lift. Although the left wing experiences a higher angle of attack, due to the nonlinearity in the $C_l$-$\alpha$ curve, the amount of lift may not necessarily increase. This results in a reduction in the roll damping in the high-$\alpha$ condition.

Although the discussion above applies to the changes in lift behavior due to roll rate, similar changes may occur as well when variations in other body-axis velocities occur. It is desired that these changes be correctly captured in the variable decambering model.
5.3.1 Implementation of the Fixed Decambering Model

The fixed decambering model simply requires the addition of the post-stall calculation to the establishment of the equilibrium condition prior to performing the full system linearization. This calculation is implemented by modifying the aerodynamic calculations according to decambering, to arrive at an equilibrium condition that satisfies Condition 1 and Condition 2 of Section 2.2.

Then, the matrix $W^0_b$, as discussed in Section 5.2.1, is updated with the rotated normal vector information for enforcement of the zero-penetration boundary condition. Based on this altered equilibrium configuration, the full system linearization is carried out. The stability analysis then proceeds as discussed in Section 5.2.2.

5.3.2 Implementation of the Variable Decambering Model

This model uses the same equilibrium condition as the fixed decambering model. The challenge in the implementation of this variable decambering model is the desire for the decambering to change as motion occurs, and for the aerodynamic force to be adjusted accordingly. Initial discussions focused on attempting to include the decambering variables as states in the aerodynamic system. This was never realized. In order to be included as a state, there would need to exist a relationship between the decambering at time step $n$ and time step $n+1$, that could be written analytically. By the nature of the decambering calculation, described in detail in Chapters 3 and 4, the amount of decambering required for a given strip on a lifting surface is dependent on the strip’s current operating point in the $C_l-\alpha$ space. Perturbing a body-axis velocity component will alter the amount of decambering present for a given strip if it is operating in the post-stall regime, but the amount that the decambering will vary is dependent on the nonlinear viscous input data.
The decambering calculation relies heavily on utilizing this viscous input data as a look-up table as the post-stall iterations approach convergence. The requirement to look up viscous data to determine the change in decambering prevents relating the decambering at time step \( n \) and time step \( n + 1 \) analytically, which rules out including the \( \Delta \delta_1 \) vector as a state.

Rather than including the decambering vector in the states, the change in decambering is considered a control input and introduced to the system matrices as such. Considering the discrete time system from Equation 5.1, a control input is added as:

\[
E_{sys} \Delta x_{sys}^{n+1} = A_{sys} \Delta x_{sys}^{n} + G_{con} \Delta u_{con}^{n}
\]  

(5.10)

The control input is taken to be the change in decambering from the equilibrium condition, \( \Delta \delta \). The change in decambering introduced to the system, and thus the change in the amount of circulation carried by the affected panels, is related to the change in the rigid-body velocities, as discussed in Section 5.3. In the local area right around the equilibrium configuration, the change in decambering is assumed to be linearly related to the change in flight velocities as

\[
\Delta u_{con}^{n} = \Delta \delta_1^{n} = K_{dec} \Delta \nu^{n}
\]

(5.11)

where
The $K_{dec}$ matrix is computed using finite differencing between steady vortex lattice runs. Depending on whether a one-sided or central finite difference is used, this requires six or twelve vortex lattice and an equal number of decambering evaluations for each trim condition being considered. Each column of the $K_{dec}$ matrix is filled out by varying one of the rigid-body velocity components from the equilibrium condition, transferring the motion induced velocity to the aerodynamic grid, and solving the system using the decambering iteration extensively described in Chapter 3. The gradient representing the change in decambering due to a small velocity perturbation for each strip is stored in the $K_{dec}$ matrix, which is valid only around the equilibrium condition being studied. For pre-stall conditions, the small velocity perturbations do not move the operating points off of the linear portion of the $C_l$-$\alpha$ input curves, and all of the $K_{dec}$ matrix entries evaluate to zero.

The Control Input Matrix

The control input matrix, $G_{con}$, is computed to introduce the effect of changing the decambering on the system matrices. The effect is introduced by taking into account the change in orientation of panel normal vectors on the non-penetration boundary condition. The main contribution of the decambering comes from the non-circulatory induced velocities, and no correction is made for the effect on influence coefficient matrices. Details of the computation of the control input matrix are contained in Appendix B.
With the control input defined, it is substituted into Equation 5.10.

\[
E_{sys} \Delta x_{sys}^{n+1} = A_{sys} \Delta x_{sys}^n + G_{con} K_{dec} C \Delta x_{sys}^n,
\]

Equation 5.13

In Equation 5.13, the matrix \( C \) is introduced which selects the rigid-body velocities from the full state vector, as shown in Equation 5.14, as these are needed to compute the control input at time step \( n \).

\[
\Delta \nu^n = C \Delta x_{sys}^n
\]

Equation 5.14

The Eigenvalue Problem for the Variable Decambering Model

Factoring the state vector out of Equation 5.13 yields

\[
E_{sys} \Delta x_{sys}^{n+1} = (A_{sys} + G_{con} K_{dec} C) \Delta x_{sys}^n
\]

Equation 5.15

and it is clearly seen that the discrete time eigenvalue problem for the system describing the variable decambering model may be written as shown in Equation 5.16.

\[
E_{sys} \nu_i = z_i [A_{sys} + G_{con} K_{dec} C] \nu_i
\]

Equation 5.16

5.4 Numerical Studies

Numerical studies are presented in this section for a free-to-roll wing and for a full aircraft configuration. The free-to-roll wing represents a simple test case. This is found
to be useful to consider, as some expectation for the roll damping behavior is known (as discussed in Section 5.3), and including only one rigid-body degree of freedom makes the problem more tractable. Further, the wing geometry and interia properties were replicated based on an experimental campaign that produced results for roll damping even beyond the stall. The wing is studied based on pre- and post-stall equilibrium conditions that were developed, and the capability of the fixed decambering and variable decambering stall model implementations to capture the roll damping characteristics from experimental results is examined. Subsequently, a full aircraft configuration is considered. Stability results for the full aircraft at level flight trim conditions yield the characteristic flight dynamic modes at low angles of attack. In this regime, no post-stall model is required and the flight dynamic modes are compared to other potential flow based analysis methods. As the trim angle of attack approaches and exceeds the stall, the effects of the fixed decambering and variable decambering model become apparent. Particularly for the longitudinal short period mode and the lateral-directional roll damping mode, the variable decambering model predicts quite different behavior approaching and beyond the stall. Qualitatively, these predictions match experimental results for aircraft flight dynamics.

5.4.1 Verification of the Implementation Using a Free to Roll Wing

In this section, the stability of a free-to-roll wing is considered. The free-to-roll wing is a wing geometry studied with only the roll degree of freedom active, as may be studied in a wind tunnel investigation. The full system linearization described in Section 5.2.1 is reduced for this study to the free-to-roll form by removing rows and columns of appro-
appropriate subsystem matrices and interface terms. This is a simpler case compared to that of a full aircraft, and one where roll damping results are available for comparison from References [70, 71].

The wing geometry comes from a Schweizer SGS 1-36 Sailplane, with an aspect ratio of 15, and Wortmann airfoil sections. The wing is modeled in the vortex lattice using the FX 61-163 airfoil for the camberline shape across the entire span, and a nondimensional time step of $\Delta t^* = V_{\infty} \Delta t/c = 0.25$, where $c$ is the characteristic chord length, and discretized using 4 panels chordwise and 20 panels spanwise. The geometry is shown in Figure 5.5a, with relevant geometric and mass properties of the wing in Table 5.1. The input viscous airfoil information, required by decambering, is shown in Figure 5.5b. CFD results, such as those used for the results in Chapter 3 were not available for the Wortmann airfoil. The viscous input data shown in Figure 5.5b were generated by relying on three sources: (1) XFOIL [25] was used to generate viscous $C_l$ input information up to $C_{l_{\text{max}}}$, (2) experimental data for the Wortmann series airfoils was consulted to accurately reproduce the immediate post-stall behavior [22], and (3) characteristic lift coefficient behavior was input beyond the range covered by the airfoil experimental data. The characteristic behavior comes from CFD studies that were carried out by this author and colleagues [50]. These CFD studies show that even for airfoils with vastly different behavior in the region immediately after stall, the $C_l$-$\alpha$ curves nearly coincide eventually, paticularly in the deep-stall region. Further, making this assumption is consistent with aerodynamic models proposed for airfoil aerodynamics beyond stall proposed by Lindenburg [72] and Viterna [73].
Table 5.1: Geometric Properties for the Free-to-Roll Wing

<table>
<thead>
<tr>
<th><strong>Geometry</strong></th>
<th>Main wing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Span</td>
<td>14 m</td>
</tr>
<tr>
<td>Root Chord</td>
<td>1.28 m</td>
</tr>
<tr>
<td>Tip Chord</td>
<td>0.576 m</td>
</tr>
<tr>
<td>Dihedral</td>
<td>0 deg</td>
</tr>
<tr>
<td>Airfoil Cross Section</td>
<td>FX 61-163</td>
</tr>
<tr>
<td>Viscous Input Data</td>
<td>$C_l$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Reference Data</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>82.6 kg</td>
</tr>
<tr>
<td>$I_{xx}$</td>
<td>1,345 kg-m$^2$</td>
</tr>
<tr>
<td>$S_{ref}$</td>
<td>13 m$^2$</td>
</tr>
<tr>
<td>$b_{ref}$</td>
<td>14 m</td>
</tr>
<tr>
<td>$c_{ref}$</td>
<td>1 m</td>
</tr>
</tbody>
</table>
Figure 5.5: The free-to-roll wing configuration: (a) isometric view, and (b) input viscous airfoil input information. The airfoil cross section is shown in the inset.

Following definition of the lifting surfaces in the vortex-lattice environment, equilibrium conditions are established for a range of angles of attack using the VLM + decambering. The cases studied used a freestream velocity of 10 m/s and $1 \leq \alpha \leq 40$ degrees. Next, SHARP (modified with the stall models) was used to perform the linearization and stability analysis on the free-to-roll system. The raw results from SHARP consist of the real pole which corresponds to the dimensional roll damping derivative, denoted $L_p$. A root locus of the pole corresponding to roll damping is plotted with increasing $\alpha$ is shown in Figure 5.6. At low angles of attack, the fixed and variable decambering models predict
Figure 5.6: Pole locations corresponding to roll damping plotted as the wing angle of attack is increased.

equivalent damping values. As the angle of attack increases, the results from the models begin to diverge. As the angle of attack approaches 40 degrees, the variable decambering model is showing much less roll damping than the fixed decambering prediction.

For comparison purposes the dimensional roll damping values output from the stability analysis are converted to a non-dimensional form, $C_{lp}$, through the relationship in Equation 5.17,

$$C_{lp} = \frac{2QI_{xx}}{\bar{\bar{q}}Sb^2} L_p$$  \hspace{1cm} (5.17)

where $Q$ is the freestream speed and $\bar{\bar{q}}$ is the dynamic pressure.

Figure 5.8 presents the non-dimensional roll damping results from the UVLM imple-
mentations as well as the results by Sim in References [70] and [71]. Sim presents two sets of results for the non-dimensional roll damping coefficient — predicted and from flight test. Details regarding the predicted results are not clearly described in the test reports, so it is unclear if the roll damping predictions come from the wind tunnel test campaign that was completed or an alternate methodology. Sim’s flight test results were extracted by performing a parameter estimation analysis to flight data. In the results given by Sim’s, there is a discrepancy between the predicted and flight-test roll damping derivative, with the flight-test data showing significantly less damping than the predicted values, and the onset of reduction of the damping coefficient (in absolute value) occurring at a lower angle of attack. It is noted, however, that the two methods show consistent trends. For $12 \leq \alpha \leq 34$ degrees, flight test data is not reported, although a predicted value of roll damping is provided. At low angles of attack, it is noted once again that the fixed decambering and variable decambering models implemented in SHARP give the same results for the roll damping coefficient, which is to be expected. As the angle of attack is increased to 10 degrees, the variable decambering model results diverge from the fixed decambering results — this transition corresponds with operating points on the wing beginning to diverge significantly from pure potential flow predictions, with some points approaching the stall as can be seen in Figure 5.7. Beyond 10 degrees, the roll damping predicted by the variable decambering continues to move towards the stability boundary until it essentially settles at a very small, but still negative, value. Meanwhile the prediction by the fixed decambering model remains nearly constant across the entire range of angle of attack studied.
Figure 5.7: Operating points for the port wing at various equilibrium configurations for the free to roll wing example.

Good agreement is seen between the results provided by Sim’s and both UVLM stability prediction methodologies at low angles of attack, with the UVLM methodologies falling between the predicted and experimental results in the NASA Technical Paper by
Figure 5.8: Variation of non-dimensional roll damping derivative with increase in the wing angle of attack.

Sims. As the wing begins to stall around 10 degrees angle of attack, the need for the variable decambering stall model comes to light. Not only does the variable decambering stall model predict the loss of roll stability, but it shows the roll stability progressing from very stable to abutting against the stability boundary over very nearly the same wing angle of attack range as the NASA TP data. Further, the quantitative values of roll damping at high-\( \alpha \) from the variable decambering model agree very well with the flight test data. The variable decambering model appears very capable of capturing the salient effects on the rolling dynamics as the angle of attack is increased through and even well beyond stall. Further, these results suggest that the model form chosen, of taking the change in decambering being linearly related to the change in flight velocities, is appropriate to the problem at hand.
5.4.2 Full Aircraft Stability Results

A full aircraft configuration with external geometry, weights, and inertias similar to a commercial UAS is studied in this section. The first challenge working towards examining the stability and flight dynamic characteristics of such a configuration is to arrive at an equilibrium condition about which to perform the linearization. The equilibrium condition corresponds to aircraft trim, in which the aircraft will continue on its steady level trajectory in the absence of external disturbances. After trim is reached, the linearization is performed and the stability calculations are carried out.

Test Case

The aircraft studied is shown in Figure 5.9(a) with details of the geometry, weights, and inertias in Table 5.2. Viscous airfoil input information representing the NACA 4415 airfoil from CFD from Reference [50] shown in Figure 5.9(b), is applied to the main wing of the configuration, while the tail surfaces are assumed to operate in purely potential flow. The aircraft consists of a main wing with a moderately large aspect ratio with and a small amount of dihedral to enhance directional stability, and a conventional horizontal and vertical tail. The entire horizontal tail plane acts as a control surface, making it a stabilator. The propulsive thrust acts along the body-axis $X$ direction. All numerical studies using the vortex lattice aerodynamics use a nondimensional time step of $\Delta t^* = V_\infty \Delta t/c = 0.25$ where $c$ is the characteristic chord length. The main wing is discretized using 4 panels chordwise and 12 panels spanwise. The tail surfaces have a shorter chord than the main wing and as such are discretized using 3 panels chordwise with 8 total spanwise panels across the horizontal tailplane and 4 spanwise panels on the vertical tailplane.
Figure 5.9: The configuration for the full aircraft stability test case: (a) Isometric view of the aircraft, and (b) Viscous airfoil input information applied to the main wing.
Table 5.2: Aircraft geometric properties, inertial properties, and reference information for the configuration used in the full aircraft used for stability results.

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Main wing</th>
<th>HTP</th>
<th>VTP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Span</td>
<td>2.8 m</td>
<td>0.7 m</td>
<td>0.35 m</td>
</tr>
<tr>
<td>Root Chord</td>
<td>0.27 m</td>
<td>0.2 m</td>
<td>0.2 m</td>
</tr>
<tr>
<td>Tip Chord</td>
<td>0.18 m</td>
<td>0.14 m</td>
<td>0.14 m</td>
</tr>
<tr>
<td>Dihedral</td>
<td>5 deg</td>
<td>0 deg</td>
<td>–</td>
</tr>
<tr>
<td>Airfoil Cross Section</td>
<td>NACA 4415</td>
<td>Symmetric</td>
<td>Symmetric</td>
</tr>
<tr>
<td>Viscous Input Data</td>
<td>$C_l$, $C_d$, $C_m$</td>
<td>None</td>
<td>None</td>
</tr>
</tbody>
</table>

<table>
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<td>Mass</td>
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<tr>
<td>$I_{xx}$</td>
</tr>
<tr>
<td>$I_{yy}$</td>
</tr>
<tr>
<td>$I_{zz}$</td>
</tr>
<tr>
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</tr>
<tr>
<td>$b_{ref}$</td>
</tr>
<tr>
<td>$c_{ref}$</td>
</tr>
</tbody>
</table>

**Trim Calculation**

An aircraft is trimmed when it is following a steady level flight trajectory in the absence of external disturbances. The level-flight trajectory is enforced by constraining the pitch orientation angle and freestream aircraft angle-of-attack to be equal. By Equation 5.18, this constraint corresponds to a constant altitude with the flight path angle, $\gamma_{fp}$, being zero.

$$\gamma_{fp} = \theta - \alpha$$  \hspace{1cm} (5.18)

The trim cases considered are for cases with zero sideslip and zero angular velocities.
With these conditions enforced, and by providing symmetric initial conditions for the
decambering vector $\delta$ for surfaces on the port and starboard sides of the aircraft, the
resultant aerodynamic loads computed either in purely potential flow or those modified
by decambering remain symmetric laterally. Therefore, only longitudinal trim is needed.
This trimming is done by fixing the free-stream angle of attack, $\alpha$, by providing trim
inputs for the elevator deflection, $\delta_e$, thrust, $T$, and free-stream speed, $Q$. Trim iterations
follow a standard Newton-Raphson method and are described by the flowchart in Figure
5.10. Given initial guess trim input variables, the aerodynamics are evaluated using the
vortex lattice method including decambering iterations run to convergence when required.
The total vertical force, horizontal force, and pitching moment are computed. If the
residual force and moment have been driven below a pre-defined tolerance, trim has been
established and the current values of the trim variables are saved. If not, subiterations
are run to establish a relationship between the trim inputs and the change in horizontal
force, vertical force, and pitching moment. Each subiteration consists of an evaluation
of the configuration aerodynamics using VLM + decambering when a small change has
been applied to a trim input variable. The output change in horizontal and vertical force
and pitching moment due to a change in trimming variable are assembled into Jacobian
matrix, from which the new guess values of trim inputs are computed. The routine repeats
until trim is achieved.

In pre-stall cases, where no aerodynamic non-linearities due to flow separation occur,
the trim routine converges very rapidly (three to four iterations are generally sufficient)
even when the initial guess values are poor. More iterations are required when the aero-
dynamic solution requires decambering, and generally convergence is reached in less than
20 iterations without the need for relaxation. Figure 5.11 shows the trimmed aircraft in
a low- and high-$\alpha_{AC}$ condition. At a low $\alpha_{AC}$, the aircraft pitch angle is very low, all the
Figure 5.10: Trimming routine outline.

Figure 5.11: Aircraft orientation and operating points for the main wing strips shown for $\alpha = 1$ deg. (left) and $\alpha = 20$ deg. (right).

sectional operating points on the main wing fall on the linear portion of the input $C_l$-$\alpha$ curve, and no decambering is required. For the high $\alpha_{AC}$, the aircraft pitch angle is high, decambering is required during the trim iterations, and the main wing operating points fall on the non-linear portion of the $C_l$-$\alpha$ curve.

Figure 5.12 shows the freestream speed, elevator deflection, and thrust required to trim the aircraft for $\alpha_{AC} = 1$ through 24 degrees. As the $\alpha_{AC}$ is increased, the trim speed for the aircraft initially becomes lower until some minimum is reached. The minimum
speed corresponds to the maximum lift coefficient that may be developed by the aircraft. Beyond the maximum configuration $C_L$, as portions of the main wing begin to stall, the trim speed actually must increase to satisfy the trim requirement of steady level flight. A substantial contribution to the pitching moment moment comes from the horizontal tail plane. As the angle of attack of the aircraft is increased, more stabilator deflection is required to maintain trim. At low $\alpha_{AC}$, little thrust is required to overcome drag and maintain steady flight. As the angle of attack is increased, both the induced and profile drag increase, necessitating additional thrust to maintain level flight. A rapid rise in thrust required is seen after the stalling angle of attack is exceeded due to the much larger amount of profile drag acting on the lifting surfaces beyond stall.

### Stability Results

Once the trim conditions have been determined, the linearization (Sections 5.2.1-5.2.1) and stability analysis with and without the stall model (Section 5.2.2 and Section 5.3) may be carried out. The stability analysis, described by either Equation 5.8 or 5.16 depending on whether the fixed decambering or variable decambering model is being used, gives as
many eigenvalues as the system has states and will include both lateral and longitudinal modes. As discussed in Section 5.2.2, the rigid body modes are identified by sorting the continuous-time eigenvalues of the system and examining the shape of the corresponding eigenvectors. As the rigid-body modes typically appear in the first 15 eigenvalues when sorted by magnitude, manual interpretation is kept to a minimum.

At low angles of attack, the continuous-time eigenvalues that describe the modes for the fixed-decambering and variable decambering model are essentially the same as the lifting surfaces operate with the airflow attached and no stall model is needed. These results may be compared with other potential flow analysis methods. In this work, the Athena Vortex Lattice (AVL) code was used for comparison of pre-stall flight dynamic modes. AVL is a steady VLM code with the capability to perform stability analysis to analyze flight dynamic characteristics [74]. Unlike SHARP, the stability analysis in AVL retains only the flight velocities and orientation angles as states, and therefore the stability results themselves are easier to interpret. Comparison with the AVL results serves to verify that SHARP is producing reasonable results and also checks that the eigenvalues that are being chosen to describe the aircraft stability characteristics are indeed rigid body modes. This comparison is only valid in attached flow conditions, and as such the results from AVL are not plotted for the full range of $\alpha$.

For clarity, the stability results are presented separately for the longitudinal and lateral-directional modes. Figure 5.13 shows a root locus of the longitudinal modes as angle of attack is increased. Results are presented for the fixed decambering model and the variable decambering model for the range $2 \leq \alpha \leq 25$ degrees. Results from AVL analysis are presented for $2 \leq \alpha \leq 10$ degrees. Four branches appear on the root locus. These branches correspond to four characteristic longitudinal modes of a rigid aircraft. Except for the very high angle-of-attack cases predicted by the variable decambering
model, these modes are manifested as two pairs of complex conjugate roots: one with high frequency and high damping and the other with low frequency and low damping.

The high frequency pair of eigenvalues are associated with what is traditionally called the short-period mode. At low angles of attack, the short-period is highly damped, and the fixed and variable decambering model results are essentially the same. The comparison between the results from the UVLM implementation and AVL for $\alpha = 2$ degrees is shown in Table 5.3. The short-period eigenvalues from each method show excellent agreement. As the angle of attack is increased and the trim speed decreases, the fixed decambering, variable decambering, and AVL results predict a decrease in damping along with a modest decrease in natural frequency. AVL predicts the short-period frequency and damping
Table 5.3: Longitudinal continuous-time eigenvalues compared to other analysis method.

<table>
<thead>
<tr>
<th></th>
<th>α = 2 deg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short-Period (UVLM)</td>
<td>−9.92 ± 11.36i</td>
</tr>
<tr>
<td>Short-Period (AVL)</td>
<td>−9.57 ± 11.46i</td>
</tr>
<tr>
<td>Phugoid (UVLM)</td>
<td>0.0296 ± 0.54i</td>
</tr>
<tr>
<td>Phugoid (AVL)</td>
<td>−0.0039 ± 0.54i</td>
</tr>
</tbody>
</table>

decreasing nearly linearly up to the maximum angle of attack for which it is plotted (10 degrees). Considering the fixed decambering model, the natural frequency nearly levels off around 18 degrees angle of attack, which corresponds to the minimum trim speed for the configuration. The damping ratio continues to decrease slightly as the angle of attack is increased to 25 degrees. The behavior of the variable decambering model results for the short-period is quite different. Beyond the maximum configuration $C_L$, increasing the angle of attack has little effect on the natural frequency, but the damping drops (in absolute value) sharply.

The low-frequency pair of eigenvalues correspond to the phugoid mode of the aircraft. These eigenvalues are located very close to the origin in Figure 5.13, and are shown in an expanded view in the figure inset in order to aid the interpretation. At low angles of attack, the eigenvalues associated with the fixed decambering and variable decambering models are nearly indistinguishable. For the 2 degree angle of attack case, these are compared to the AVL results in Table 5.3. Both the UVLM and AVL results show the phugoid roots to have very low (absolute values) damping, and the agreement in frequency is exceptional. It is noted, however, that although the damping for each prediction method is small, the UVLM and AVL predictions show the phugoid on opposite sides of the imaginary axis, meaning that AVL predicts the phugoid to be stable while the UVLM predicts the phugoid to be unstable. The degree of instability predicted by the UVLM is
small and it is possible to have an aircraft that is statically stable but with an undamped phugoid oscillation [75]. The disagreement between the two analysis methods is not well understood at this time although plausible reasons for it include how the drag is modeled in AVL and UVLM and the fact that thrust is included in the trim calculation for the UVLM model, but not with the AVL model. Despite the discrepancy in the predicted damping values for the phugoid mode, the trend that is exhibited as the trim speed decreases (and angle of attack increases) is the same for both the UVLM and AVL predictions. As the angle of attack is increased, the damping of the phugoid mode remains relatively unchanged while the natural frequency increases. For the AVL predictions up to 10 degrees and the fixed decambering model predictions all the way to 25 degrees, this trend continues. The variable decambering model, however, behaves differently. Beyond the maximum configuration $C_L$, the variable decambering model begins to predict two zero-frequency longitudinal modes. These modes occur after the wing is nearly fully stalled.

The trends shown by the longitudinal eigenvalues as the trim angle of attack is increased are shown to be the same for the UVLM predictions and AVL in the low angle of attack range. Beyond stall, however, the AVL comparison is no longer valid (and thus not plotted). Further, the longitudinal modal characteristics are not necessarily intuitively predicted as the stalling angle of attack is approached and exceeded. The fixed and variable decambering models implemented in the SHARP stability analysis offer a prediction in this range but the validity of the prediction is not well vetted. The trends displayed by the fixed decambering and variable decambering models, however, may be compared qualitatively to results presented by Stengel, who studied the longitudinal flight dynamic characteristics for a general aviation aircraft as stall is approached [76]. In Stengel’s results, the short-period frequency and damping decrease as the speed is reduced, until
the minimum power point is reached. At angles of attack beyond the minimum power point, particularly as the stall is approached, the natural frequency of the phugoid mode is nearly constant and the damping drops dramatically. Synchronously, the phugoid natural frequency is shown to increase while the damping remains nearly constant. While not enough configuration information was available from the Stengel report to reproduce the configuration and results, the trends align quite well with the predictions provided by the UVLM variable decambering model.

The continuous time eigenvalues describing the lateral-directional modes of the aircraft are plotted in a root-locus in Figure 5.14. The continuation shown as the angle of attack is increases was deduced by examining the shape of the eigenvectors. Four branches of the root locus are shown, with results presented from AVL for $2 \leq \alpha \leq 10$ degrees, and the fixed decambering and variable decambering models for $2 \leq \alpha \leq 25$ degrees. The four branches correspond to three characteristic modes of an aircraft.

At low angles of attack, there is a very highly damped zero-frequency mode, usually referred to as the roll mode. At $\alpha = 2$ degrees, the roll mode from UVLM is compared against the AVL prediction in Table 5.4. The predictions from the two methods are in good agreement, with approximately a 10% difference. As the angle of attack is increased,
Figure 5.14: Eigenvalues for the lateral-directional modes plotted as angle of attack is increased.
the damping of the roll mode decreases in the AVL predictions as well as in the fixed
decambering and variable decambering models. The reduction in roll damping is much
more pronounced with the variable decambering model.

The other zero-frequency mode corresponds to the spiral mode of the aircraft. At
2 degrees angle of attack, the spiral mode is predicted to be unstable by the UVLM
methods and AVL. This mode is located very close to the origin on the root locus, and as
such is shown on the inset in Figure 5.14. As the angle of attack is increased, the spiral
mode becomes more unstable as predicted by both UVLM models and AVL. The degree
of instability is relatively close in all three analysis methods, and hence not affected by
stall, it would appear.

The final mode plotted for the lateral-directional stability results is a complex conju-
gate pair, typically called the Dutch Roll mode. For the 2 degree angle-of-attack case, the
UVLM and AVL predictions are compared in Table 5.4. The agreement between the AVL
and UVLM models is excellent at low angles of attack. As the angle of attack is increased,
all three models predict that the dutch roll damping remains nearly the same, and the
natural frequency decreases until the maximum configuration $C_L$ is reached. Beyond the
maximum configuration $C_L$, the damping is shown to increase slightly.

At low angles of attack, the SHARP stability results for the lateral-directional modes
compare favorably with the AVL predictions. Additional confirmation that the results are
correct comes from the observation that as the angle of attack is increased, the stability
predictions from SHARP and AVL move in the same manner. At high angles of attack,
the roll mode damping predicted by the variable decambering model moves much closer
to the stability boundary as compared to the fixed decambering model. Qualitatively,
the very lightly damped roll mode seen in the model matches the behavior of the results
of the free-to-roll wing from Section 5.4.1 which were corroborated with experiment.
5.5 Chapter Summary

In this chapter, two methodologies were presented by which a post-stall model can be introduced into a stability analysis framework which uses aerodynamic loads from the unsteady vortex lattice method. The first stall model introduced, the fixed decambering model, brings the effects of flow separation into the model only in the establishment of the equilibrium condition, from which the linearization and stability analysis proceed. The equilibrium condition is set by simultaneously satisfying the requirements of the equilibrium (usually aircraft trim) and the decambering calculation which is included as an integral part of the loads calculation within the UVLM. The second post-stall model supposes that not only must the equilibrium condition be altered to account for the effects of stall, but also the aerodynamic loads about that equilibrium. In a novel approach, a dependency is introduced in the linearized UVLM boundary condition equation that relates the changes in circulations about a trim condition to the change of decambering, where the change in decambering is appended to the system as a control input in feedback form.

Numerical studies tested the two post-stall stability analysis models. The first study considered the damping on a free-to-roll wing used in past experiments. The fixed decambering model predicted nearly constant roll damping throughout a large angle of attack range. The variable decambering model showed phenomenal agreement with the experimental results, not only in trends but even in numerical values at high $\alpha$’s. The second study considered a full aircraft configuration. No direct comparison results were available, however excellent qualitative agreement was seen between experimental trends and the variable decambering model results when studying the longitudinal modes. Regarding the lateral-directional modes, the roll mode damping followed the same trends observed
in the free to roll example. The results suggest that the variable decambering model, whose only inputs are aircraft geometry and viscous data describing the airfoil sections, is a powerful tool that can extend stability prediction methodologies to post-stall cases.
Chapter 6

Summary and Conclusions

This document presents a low-order post-stall aerodynamics model suitable for use in flight simulation and stability analysis. This study focuses on the decambering approach, which is utilized to rapidly calculate wing and configuration aerodynamics at post-stall conditions using airfoil lift curves that are provided as input. Decambering is implemented using a multi-dimensional Newton iteration. At each step of the Newton iteration, a residual calculation is performed with the aim of bringing the sectional operating points closer to the input data as required for convergence. The application of the Newton method to the decambering procedure is non-standard because the residual for any wing section depends on the target operating point chosen on the viscous lift curve, and the target point is not known \textit{a priori}. The identification of the target point itself is dependent on the decambering at the other sections of the wing, which are also being determined as a part of the Newton solution.

Four schemes were presented for selecting the target points. Each scheme requires its own residual calculation. The residual calculation schemes proposed were compared based on computational efficiency, sensitivity to different starting values, and robust-
ness in successfully converging for a wide range of angles of attack. One of the schemes (scheme D) was seen to be the most capable of successfully converging for a wide range of angles of attack without needing any under-relaxation even with multiple lifting surfaces present. Visualization of the evolution of the solution during the iterative procedure provided the insight into the reason for the behavior of the four schemes. It was seen that the assumption made in identifying the target point in scheme D results in the least number of iterations required for convergence. Indeed, for all spanwise locations in the aerodynamic geometry, the target operating points computed before the first iteration and the operating points resulting from the very first iteration of scheme D were both seen to be exceptionally close to the operating points from the fully-converged solution even at post-stall conditions.

Results from the decambering approach were compared with high-order CFD solutions for total lift and spanwise lift distributions on rectangular wings. An excellent match was seen at pre-stall conditions. At post-stall angles of attack, the results from the decambering approach tend to over-predict the total $C_L$ compared to the CFD solutions for the wing by approximately 0.15. The discrepancy between the CFD and low-order results is acknowledged. However, considering that even CFD and experimental predictions tend to have uncertainty larger than 0.15 in high-$\alpha$ flow conditions, the predictions from a low-order method are considered acceptable, especially because they are developed with a computational burden that is orders of magnitude less than CFD. Comparison of the results from the decambering approach with CFD for aft-swept wings brings to light a limitation of the approach. With increasing sweep angle, CFD solutions show a spanwise transport towards the wingtip of the separated flow on aft-swept wings at stall. This flow behavior affects the stall progression along the span and causes loss in $C_{L_{\text{max}}}$. This spanwise flow invalidates the assumption in the decambering method that the relationship
between local-$C_l$ and effective $\alpha$ on all sections of a finite wing are practically the same as the relationship between $C_l$ and $\alpha$ for the airfoil section at that spanwise wing station. The predictions from the current decambering method are inaccurate when swept wing geometries or geometries with large amounts of spanwise flow, are considered.

Following the study that determined the most reliable and efficient residual calculation approach to be used in the post-stall model, attention turned to implementing the decambering computations into an environment for flight simulation. The unsteady vortex lattice method was used for aerodynamics analysis, and unlike a standard UVLM, a discrete-time implementation is used. The discrete-time form of the UVLM allows for efficient simulation, although this does require that the wake remains fixed relative to the aircraft. Time marching the simulation is performed using standard integration techniques, with a requirement imposed at every time step that the aerodynamics computed satisfy both the equations of the UVLM and the conditions for convergence of a decambering solution. An example time marching simulation is shown, with both pre- and post-stall lift distributions highlighted.

Finally, an application of decambering was considered wherein the decambering calculations were employed in the SHARP framework for linear stability analysis. In SHARP, following establishment of an equilibrium condition, a linearization of the aircraft dynamic equations is carried out leading to a state space system that may be written in the form $x^n = Ax^{n+1}$ where $x$ includes the aerodynamic circulation states, from the UVLM, and rigid-body states. The eigenvalue analysis of the system yields the flight dynamic modes of the vehicle. For post-stall stall flight dynamics, decambering is introduced to the system in two ways. The first, denoted the “fixed decambering” model, alters only the equilibrium configuration. The reference values of the configuration normal vectors are modified by rotating them according to the appropriate amount of decambering
such that, following aerodynamic load calculation, the corresponding sectional operating points fall on the sectional input data. Following establishment of the reference condition the stability analysis proceeds based on this geometry. In the second implementation, decambering is introduced to the system matrices by treating it at each strip as a control effector, whose deflection is linearly related to the change in vehicle flight velocities. This relationship between the change in flight velocities and change in decambering is appended to the system in feedback form, whose stability may be determined by an eigenvalue analysis of a modified system matrix.

Numerical studies are carried out to test the implementation of decambering into the SHARP stability analysis. The first study considers the damping in roll of a wing-only configuration. For this example, predicted and flight test data describing the roll damping is available for comparison. Pre-stall, the results from the fixed decambering and stall model both output the same non-dimensional value for roll damping, which fall between the predicted and flight test data sets used for comparison. As the angle of attack is increased, and the stall is approached, roll-damping results from the fixed decambering and stall model diverge. The fixed decambering model continues to predict a nearly constant value of roll damping throughout the entire angle of attack range studied. In contrast, the roll damping predicted by the stall model moves towards the stability boundary as the wing stalls. The stall model never predicts that the roll damping becomes positive, and as the angle of attack continues to increase the roll damping settles at a very small negative value where it remains until very high angles of attack. The trends predicted by the stall model closely replicate the behavior of the prediction and flight test data offered by the comparison results.

The final numerical example demonstrates the effects of including decambering on the post-stall flight dynamic modes for a representative aircraft configuration. The aircraft
configuration is trimmed at angles of attack ranging from pre-stall to post-stall. At each trim condition the stability is evaluated using the fixed decambering model and the stall model. Evaluation of the stability for pre-stall cases yields the expected rigid-body modes for the aircraft. These compare well with other potential flow methods. As the angle of attack is increased and the stall is approached, the predictions from the fixed decambering approach and the stall model begin to differ. The most notable differences are seen in the short period and roll modes. Stall model predictions show the short period damping decreases drastically as the stalling angle of attack is approached and exceeded, while the natural frequency of this mode remains nearly unchanged, and that the roll mode becomes very lightly damped at high angles of attack. The predictions computed using the fixed decambering model do not show the drastic reduction in short period damping or much of a decrease in roll damping. Qualitative comparison with experimental results reveals that the modal characteristics predicted by the stall model follow expected high-α trends for a conventional aircraft configuration much more closely than those of the fixed decambering model or un-augmented vortex aerodynamics method.

In the work described by this document, iteration schemes for a simplified version of the decambering post-stall calculation have been studied thoroughly. One scheme, scheme D, is seen to converge most reliably and to be more robust than the others, particularly for multiple lifting surface configurations. An important contribution represented by this work is that by using input data consisting of two-dimensional CFD $C_l$ v. $\alpha$ curves in conjunction with the reliable scheme D iteration, results from the decambering calculation applied to finite wings compare excellently with three-dimensional CFD solutions at a much lower computational cost for many wing geometries.

With the most suitable iteration scheme established, it is shown that the decambering approach can be used to extend the modeling capability of the unsteady vortex lattice
method to post-stall conditions. The combination of the unsteady vortex lattice method and decambering, where required, is an efficient tool for flight dynamics simulation. By freezing the aerodynamic geometry as discretized in the UVLM, the flight dynamic stability of the configuration may be examined utilizing the SHARP tool. The decambering approach lends itself to a natural incorporation into a framework suitable for stability analysis. This results in a novel tool that, with very minimal input data consisting of only basic aircraft geometry and inertia information, supplemented by sectional airfoil data, excellent results for post-stall flight dynamic characteristics may be predicted.
Chapter 7

Future Work

This chapter provides recommendations for future work that could be undertaken, either to improve the current modeling methodology or to address known shortcomings.

1. Extension of the decambering methodology to handle swept wings

   Section 3.3.5 demonstrated clearly that when significant amounts of spanwise flow are present, fundamental assumptions of decambering are violated, resulting in poor predictions of the aerodynamics. For analysis of swept wings (or any planform geometry with spanwise flow), it may be possible to use finite-wing CFD predictions to empirically develop corrections that can be applied to the section lift curves used as input for the decambering method. The need for finite-wing CFD analyses clearly makes such a proposed empirically-modified decambering approach unsuitable for design predictions. However, such an approach, if successful, will still be useful for flight dynamics simulation. A better approach, which is also a more challenging research problem, is to augment the decambering method to account for the spanwise redistribution of separated boundary layer by taking into account the transverse pressure gradients.
2. **Modeling the effects of bodies**

   The work described in this document only models wings and other lifting surfaces. Further, the decambering approach requires that the sectional aerodynamics of these lifting surfaces is accurately described by input airfoil data. Aerodynamic effects of bodies, such as a fuselage, are ignored. This may be a reasonable approximation for highly “wing dominated” configurations such as a HALE UAS, but is obviously a huge omission for many aircraft configurations.

3. **Modeling the momentum deficit in the wake**

   The UVLM models the vortex wake by propagating the circulation from the trailing edge of the wing downstream to infinity, or as implemented numerically, until the wake itself is truncated. Propagating the circulations downstream models the variation in downwash seen by downstream lifting surfaces as lift behavior changes. The effect of wake shadowing, however, is not modeled by the UVLM, or any panel method type formulation. This effect relates to the momentum deficit in the wake due to the flow separation present on upstream surfaces. If the wake impinges on downstream surfaces, the aerodynamics may be significantly altered, changing the flight dynamics of the vehicle. Expressions exist [77] that describe the wake width and momentum deficit behind lifting surfaces, and perhaps these could be somehow utilized in conjunction with the aerodynamic wake grid already present in UVLM to capture the salient effects caused by momentum loss on downstream surfaces.
REFERENCES


Appendix A

Trajectory Line Calculation Using Superposition and the Vortex Lattice Method for Residual Calculation Scheme D

This appendix documents the derivation of the residual calculation referred to as “Scheme D” throughout this document, and the application of this scheme when used in a vortex lattice type aerodynamics analysis method.

The trajectory line slopes may be calculated using the superposition approach, as in Chapter 3, or by an equivalent representation from vortex lattice aerodynamics. Whichever calculation method is used, the slopes are invariant for a fixed aerodynamic geometry.
Computing the Trajectory Lines for Scheme D with Superposition

Using superposition, the trajectory line slopes may be computed from considering the movement of an operating point due to the change in additional load distribution.

We begin by defining some terms:

- $\hat{\alpha}_{w,1}$ — The angle of attack where the additional loading gives $C_L = 1$
- $\alpha_{w,0.8}$ — The angle of attack where the additional loading gives $C_L = 0.8$
- $C_{l_{a,1}}(y)$ — Vector containing additional $C_l$ distribution across the wing for $C_L = 1$
- $\alpha_{a,1}(y)$ — Vector containing additional $\alpha$ distribution across the wing for $C_L = 1$
- $C_{l_{a,0.8}}(y)$ — Vector containing additional $C_l$ distribution across the wing for $C_L = 0.8$
- $\alpha_{a,0.8}(y)$ — Vector containing additional $\alpha$ distribution across the wing for $C_L = 0.8$

Taking wing strip $i$, the local trajectory line slope is calculated as:

$$
\left[ \frac{dC_l}{d\alpha} \right]_{\text{traj}} (i) = \frac{C_{l_{a,1}}(i) - C_{l_{0.8}}(i)}{\alpha_{a,1}(i) - \alpha_{\text{new}}(i)}
$$

where

$$
\alpha_{\text{new}}(i) = \alpha_{a_{0.8}}(i) + \Delta \alpha
$$

$$
\Delta \alpha = \alpha_{w,1} - \alpha_{w,0.8}
$$

$$
\Delta \alpha = 0.2 \alpha_{w,1}
$$
Substitution yields:

\[
\left[ \frac{dC_l}{d\alpha} \right]_{\text{traj}}(i) = \frac{0.2C_{l,1}(i)}{\alpha_{a,1}(i) - \alpha_{0.8}(i) - \Delta\alpha}
\]

\[
\left[ \frac{dC_l}{d\alpha} \right]_{\text{traj}}(i) = \frac{0.2C_{l,1}(i)}{0.2\alpha_{a,1}(i) - 0.2\alpha_{w,1}}
\]

The trajectory line slopes are then seen to be:

\[
\left[ \frac{dC_l}{d\alpha} \right]_{\text{traj}}(i) = \frac{C_{l,1}(i)}{\alpha_{a,1}(i) - \alpha_{w,1}}
\]

**Computing the Trajectory Lines for Scheme D with Vortex Lattice**

Superposition yields a simple equation that can be used to compute the trajectory line slope for scheme D. When using a vortex lattice method for aerodynamic analysis, the terms used in the above expressions are not readily available. Further, in the superposition approach, the trajectory line slopes are related solely to variations in the additional loading. The additional loading is directly related to changes in angle of attack, therefore for vertical surfaces, such as the vertical tailplane, the trajectory line slopes computed using only the additional loadings are not defined. Using the vortex lattice method allows the trajectory line slopes from Scheme D to be generalized to work for arbitrary aircraft configurations.

The Scheme D trajectory line slopes are computed assuming the lifting surface is subject to a change in angle of attack. To calculate the Scheme D trajectory line slopes using vortex lattice aerodynamics, each lifting surface (without its twist distribution), is individually given a rotation that mimics altering the angle of attack of the surface. This rotation can be applied by physically rotating the surface, or by decambering the
surface normal vectors. Aerodynamic symmetry must be considered when this rotation is applied. For example, if a wing is modeled using two surfaces, the left and right wing for example, the rotation needs to be applied to each wing simultaneously to compute trajectory line slopes equivalent to those computed using superposition. The pseudo-code below automates this procedure:

<table>
<thead>
<tr>
<th>Algorithm 1: Calc Traj Lines</th>
<th>finds the trajectory line slopes</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong> Configuration Geometry, Baseline VLM computation, operating condition</td>
<td></td>
</tr>
<tr>
<td><strong>Output:</strong> Trajectory line slopes</td>
<td></td>
</tr>
</tbody>
</table>

```
1  for mm = 1 to N_{\text{surface}} do
2      for nn = 1 to N_{\text{surface}} do
3          \vec{n}_1 = \vec{n}_{mm};
4          \vec{n}_2 = \vec{n}_{nn};
5          (\delta_1)_{nn} = -(\vec{n}_1 \cdot \vec{n}_2) ;
6          Run VLM;
7          Find \Delta C_l on Wing mm;
8          Find \Delta \alpha_{\text{eff}} on Wing mm;
9          Compute \left[ \frac{dC_l}{d\alpha} \right]_{\text{traj}} (i) for each strip on wing mm;
10         return slopes;
```

In the algorithm, $N_{\text{surface}}$ is the total number of lifting surfaces, $\delta_{1_{nn}}$ represents the decambering vector for wing $nn$, and $\vec{n}$ is a surface unit normal vector.
Appendix B

Description of Entries in the Control Input Matrix

This Appendix documents the entries in the control input matrix discussed in Section 5.3.2.

First, some definitions are defined.

- $K_b$ – Total number of bound panels
- $K_w$ – Total number of wake panels
- $N_{AeroStates}$ – This is the number of circulation states retained in the aerodynamic model. The circulation states describing the aerodynamics include the bound circulations, the wake circulations, and the derivative of the bound circulation states.

Thus, the total number of aerodynamic states is written as:

$$N_{AeroStates} = K_b + K_w + K_b.$$
• $N_{\text{RigidBodyStates}}$ – This is the number of rigid body states. The rigid body states are given by

$$
\begin{bmatrix}
\nu^T \\
\Theta^T
\end{bmatrix}
$$

where $\nu$ contains the rigid body velocities (3 linear, 3 angular), and $\Theta$ contains the three rigid body orientation angles. Thus, there are 9 rigid body states tracked.

• $N_{\text{states}}$ – This is the total number of states. The total number of states is the number of aerodynamic states plus the rigid body states.

$$
N_{\text{states}} = N_{\text{AeroStates}} + N_{\text{RigidBodyStates}}.
$$

• $N_{\text{strips}}$ – This is the number of spanwise strips for the configuration.

• $W^0_b$ – This is a matrix containing normal vector information at the equilibrium configuration. It is sized $[K_b \times 3K_b]$.

• $\dot{\zeta}^0_b$ – This is a column vector, sized as $[3K_b \times 1]$ with velocity information computed at the collocation points at equilibrium.

Each spanwise strip is assigned a decambering variable, $\delta_{1,ii}$ which is treated like a control input. Thus, there are considered to be as many control inputs as number of strips defined in the configuration. The partial derivative of the vector containing the non-circulatory velocities due to a change in the decambering on strip $ii$ may be expressed as:

$$
\frac{\partial w}{\partial \Delta \delta_{1,ii}} = W^0_b \frac{\partial R}{\partial \Delta \delta_{1,ii}} \dot{\zeta}^0_b 
$$

(B.1)

In Equation B.1, $\frac{\partial R}{\partial \Delta \delta_{1,ii}}$ establishes the dependence of the vortex ring normal vectors on
the $\Delta \delta_{1,ii}$ input. This term is sized as $[3K_b \times 3K_b]$. The entries in $\frac{\partial R}{\partial \Delta \delta_{1,ii}}$ are best visualized as being made up of $[3 \times 3]$ sub-matrices. The sub-matrices along the diagonal of $\frac{\partial R}{\partial \Delta \delta_{1,ii}}$ are associated with a normal vector entry in $W_w^o$. For a normal vector $jj$ that is affected by the input $\delta_{1,ii}$, the sub-matrix on the diagonal contains elements computed using the derivative of Rodrigues’ rotation matrix formula to take into account the normal vector rotation, as shown in Equation B.2.

$$\frac{\partial R_{ij}}{\partial \Delta \delta_{1,ii}} = [\cos(\Delta \delta_{1,ii})[u]_x + \sin(\Delta \delta_{1,ii})[u]^2_x]^T$$  \hspace{1cm} (B.2)

In Equation B.2, as in Equation 4.6, $[u]_x$ is the skew-symmetric form of the unit vector describing the axis of rotation which is taken to be the leading segment of the relevant vortex ring.

For each $\delta_{1,ii}$, $\frac{\partial w}{\partial \Delta \delta_{1,ii}}$ is computed. Each $\frac{\partial w}{\partial \Delta \delta_{1,ii}}$ has size $[K_b \times 1]$. These are assembled to the form:

$$\frac{\partial w}{\partial \Delta \delta_1} = \left[ \frac{\partial w}{\partial \Delta \delta_{1,1}} \frac{\partial w}{\partial \Delta \delta_{1,2}} \cdots \frac{\partial w}{\partial \Delta \delta_{1,N_{strips}}} \right]$$  \hspace{1cm} (B.3)

The control input matrix is sized as $[N_{States} \times N_{strips}]$. Many terms in this matrix are zero, as the effect of changing decambering is taken into account solely through how it affects the boundary condition equation. Keeping consistency with the order of the state vector, the $G_{con}$ matrix is filled out as:

$$G_{con}[1 : K_b, 1 : N_{strip}] = \frac{\partial w}{\partial \Delta \delta_1}$$