ABSTRACT

HUNNELL, MARK CONSTANTINE. Orbits of Minimal Parabolic k-subgroups on Symmetric k-varieties. (Under the direction of Aloysius Helminck.)

Symmetric k-varieties generalize classical symmetric spaces to extend their applications to arbitrary fields. Parabolic subgroups play an important role in the study of symmetric k-varieties, in this dissertation the action of minimal parabolic k-subgroups on symmetric k-varieties is studied in the context of a generalized complexification map. This map embeds the orbits over the base field into the corresponding orbits over the algebraic closure. There are many natural questions related to this map, including surjectivity and the cokernel. We develop a condition of the generalized complexification map applied to the orbits of minimal parabolic k-subgroups acting on symmetric k-varieties.

Orbits of Minimal Parabolic k-subgroups on Symmetric k-varieties

by
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DEDICATION

To my parents.

BIOGRAPHY

Mark Constantine Hunnell was born in Milwaukee, Wisconsin on October 31, 1986. He moved to Raleigh, North Carolina in 1988 where he has lived ever since. In 2005 he started his undergraduate studies at North Carolina State University in electrical engineering before seeing the light and transferring to mathematics in his junior year. He continued at NC State for graduate school, and is an enthusiastic homebrewer in his spare time.

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Chapter 1

Introduction

Given a group G one can construct the generalized symmetric space corresponding to G. Applications of symmetric spaces permeate numerous branches of mathematics as well as the applied sciences, particularly in physics. Originally studied by Cartan, symmetric spaces arose in the context of Riemannian manifolds and Lie groups. The globally Riemannian symmetric spaces of differential geometry are in fact a special case of the algebraic definition common in Lie theory. Let θ be an involutorial automorphism $\theta : G \to G$, $\theta^2 = \mathrm{id}$, and let $H = \{g \in G \mid \theta(g) = g\}$ be the set of fixed points of the involution θ. The generalized symmetric space is then the homogeneous space G/H, which is isomorphic to the set $Q = \{g\theta(g)^{-1} \mid g \in G\}$. For an algebraic group G/H is also called the symmetric variety. For a group N defined over a field k, denote by N_k the set of k-rational points of N. If θ preserves the k-rational points of G, i.e. $\theta(G_k) = G_k$, then one can define an analogous object defined over k. The quotient G_k/H_k is known as a symmetric k-variety, and it is one object belonging to a larger class of generalizations known collectively as generalized symmetric spaces. In the specific case $k = \mathbb{R}$, one obtains the usual symmetric space.

Let $M_n(k)$ denote the set of $n \times n$ matrices with entries in k. Then

$$\mathrm{GL}_n(k) = \{A \in M_n(k) \mid \det(A) \neq 0\}$$

is the general linear group. Let $G = \mathrm{GL}_n(k)$ and define an involution $\theta : G \to G$ by $\theta(g) = (g^T)^{-1}$. Then $H = \{g \in G \,|\, g^T = g^{-1}\}$, the set of $n \times n$ orthogonal matrices. The quotient G/H is then the set of symmetric matrices, hence the motivation for the name symmetric space.

For many years work concentrated on a particular class of symmetric k-varieties known as real reductive symmetric spaces. The real reductive symmetric spaces are the homogeneous spaces $G_{\mathbb{R}}/H_{\mathbb{R}}$ where $G_{\mathbb{R}}$ is a reductive real Lie group and $H_{\mathbb{R}} = G_{\mathbb{R}}^{\theta}$ is the set fixed points of an involution θ. Decomposing representations of these spaces in to their irreducible components has brought the attention of many prominent mathematicians including Cartan [9], who studied the compact groups and their representations, and Harish-Chandra [14], who gave a Plancherel formula for the Riemannian symmetric spaces in which the fixed point group is compact and a Placherel formula in the groups case. A proof of the general Plancherel formula was not completed until 1996 by Delorme [12]. Beginning in the 1980's, Helminck and Wang [19] commenced a study of the rationality properties of the symmetric k-varieties for arbitrary fields. The symmetric k-varieties have many applications, such as the study of arithmetic subgroups [34], character sheaves [13], and geometry [1],[10, 11]. The most well known application, however, is in representation theory where the parabolic subgroups play a fundamental role. Most of the representations occuring in the Plancherel formula are induced from a parabolic k-subgroup and thus it is important to have an understanding of the action of parabolic k-subgroups acting on symmetric k-varieties. This dissertation aims to describe this action in the context of a map relating the symmetric k-variety to its corresponding symmetric $\bar{k}$-variety.

Springer gave several equivalent characterizations of the B-orbits on G/H, where B is a Borel subgroup of G. The number of orbits is finite, thus the symmetric varieties form a class of spherical varieties. One of these characterizations is to identify the orbits with double cosets $B\backslash G/H$. When k is algebraically closed these are the orbits of a minimal parabolic subgroup acting on G/H. Helminck and others extended this characterization to cover general parabolics and arbitrary fields. A natural question is then to determine how the $P\backslash G/H$ orbits break up in to $P_k\backslash G_k/H_k$ orbits over the base field. Not all of the algebraically closed orbits contribute to the k-orbits. Even in the real numbers this phenomenon occurs, as demonstrated in Example 5.3. Instead of determining how algebraically closed orbits break up over the k-rational points one can reverse the process via an embedding map $P_k\backslash G_k/H_k \hookrightarrow P\backslash G/H$ which we call generalized complexification. The surjectivity of the generalized complexification map is then equivalent to all orbits over the algebraic closure contributing to the k-orbits. Given a group N, k-rank(N)

denotes the dimension of a maximal k-split torus of N. When the group G is k-split we have a characterization of the surjectivity:

Theorem 5.8.6. *Let G be a k-split group, H the set of fixed points of an involution θ, and P a minimal parabolic k-subgroup. Then the generalized complexification map*

$$\varphi : P_k \backslash G_k / H_k \to P \backslash G / H, \quad \varphi(P_k x H_k) = PxH$$

is surjective if and only if $k\text{-rank}(H) = k\text{-rank}(G)$.

There are several direct consequences of the surjectivity of the generalized complexification map. For instance, the Iwasawa decomposition of a real Lie group decomposes the group in to a product of the fixed points of the Cartan involution and a Borel subgroup, i.e. if we use the notation above and let B be a Borel subgroup of G, we have $G_{\mathbb{R}} = H_{\mathbb{R}} B_{\mathbb{R}}$. This is equivalent to having only one orbit in $B_{\mathbb{R}} \backslash G_{\mathbb{R}} / H_{\mathbb{R}}$, however surjectivity of the generalized complexification map will imply a maximal number of orbits for a fixed field k. Thus surjectivty implies a decomposition that is in some sense as far as possible from the Iwasawa decomposition.

Additionally, a corollary to Theorem 5.8.6 suggests a tool for inductive proofs in the case of a symmetric k-variety whose generalized complexification map is surjective. Many results for symmetric k-varieties defined over algebraically closed can be proven with such an inductive step, thus surjectivty implies a context for the generalization of these results.

Chapter 2

Background

Much of the theory needed to study the generalized complexification of orbits over algebraically closed fields is due to Springer and Richardson. Later Helminck and Wang extended this work to general fields in [19]. This work has been continued by several of Helminck's students, particularly Beun and Mason.

It is frequently convenient to work with elements of the general linear group. This involves no loss of generality, since from [31] we have:

Theorem 2.0.1. *Let G be an algebraic group.*

(a) G is isomorphic to a closed subgroup of $\mathrm{GL}(n, \bar{k})$.

(b) If G is defined over k, then the isomorphism is defined over k.

2.1 The Morphism $\boldsymbol{\tau}$

We follow the notation established in [5], [31], and [23]. Throughout the paper G will denote a connected reductive algebraic group, θ a group involution of G that leaves the k-rational points invariant and H a k-open subgroup of $G^\theta = \{g \in G \mid \theta(g) = g\}$. The variety G/H is called the symmetric variety and G_k/H_k is the symmetric k-variety. Define a map

$$\tau : G \to G,\ \tau(x) = x\theta(x)^{-1}$$

Denote the image τ by Q, then τ induces an isomorphism between G/H and Q as well as an isomorphism between G_k/H_k and Q_k. It is sometimes more convenient in calculation to let H act from the left, in this case $\tau(x) = x^{-1}\theta(x)$. This change does not affect the results since $H\backslash G$ is isomorphic to G/H.

Let $\mathrm{Aut}(G)$ denote the set of group automorphisms $G \to G$, and for a subgroup $K \subset G$ we will use $\mathrm{Aut}(G, K)$ to denote the set of automorphisms of G which leave K invariant. In particular we are concerned with the order 2 elements of $\mathrm{Aut}(G, G_k)$. For $g \in G$, we will use $\mathrm{Int}(g)$ to denote the inner automorphism corresponding to g, i.e. $\mathrm{Int}(g)(a) = gag^{-1}$ for all $a \in G$. An involution is outer if it is not inner.

2.2 Tori

Recall that a torus T of G is a connected semisimple abelian subgroup. Much of the structure of symmetric k-varieties can be described by way of tori and their associated root systems. Here we recall some basic facts about tori that will be useful in what follows. Let T be a torus, then $N_G(T) = \{g \in G \mid gTg^{-1} = T\}$ will denote the normalizer of T in G, and $Z_G(T) = \{g \in G \mid gtg^{-1} = t \text{ for all } t \in T\}$ will denote the centralizer of T in G. The elements of the normalizer that nontrivially permute the elements of T are given by the Weyl group, denoted $W_G(T) = N_G(T)/Z_G(T)$. The classification of orbits of minimal parabolic subgroups acting on symmetric k-varieties relies on a quotient of the Weyl group by elements having representatives in the fixed point group H, and we denote this set by $W_H(T) = \{w \in W_G(T) \mid w \text{ has a representative in } H\}$. These groups all have analogues for a group defined over k, replacing the group in the definition of in $N_G(T),\ Z_G(T),\ W_G(T),$ and $W_H(T)$ with its associated k-rational points, we obtain definitions for $N_{G_k}(T),\ Z_{G_k}(T),\ W_{G_k}(T),$ and $W_{H_k}(T)$. These groups will contribute to the combinatorial description of the orbits of interest described in what follows.

Let $\varphi \in \mathrm{Aut}(G, T)$, then T can be decomposed via its φ (Lie algebra) eigenspaces, i.e. $T = T_\varphi^+ T_\varphi^-$ where $T_\varphi^+ = \{t \in T \mid \theta(t) = t\}^\circ$ and $T_\varphi^- = \{t \in T \mid \theta(t) = t^{-1}\}^\circ$, where K° denotes the identity component of subgroup $K \subset G$. The product map

$$\mu : T_\varphi^+ \times T_\varphi^- \to T,\ \mu(t_1, t_2) = t_1 t_2$$

is a separable isogeny. In fact, $T_\varphi^+ \cap T_\varphi^-$ is an elementary abelian 2-group. Of particular interest will be the case when $\varphi = \theta$, in this case we will use T^+ for T_θ^+ and T^- for T_θ^-.

Maximal tori play the fundamental role in the description of the structure of symmetric k-varieties. A torus is maximal if it is properly contained in no other torus. It is a fact that all such tori are conjugate under G, i.e. for two maximal tori $T_1,\ T_2 \subset G$ there exists $g \in G$ such that $gT_1g^{-1} = T_2$.

A torus is called θ-split if $\theta(t) = t^{-1}$ for all $t \in T$. From [26] we know that if A is a maximal θ-split torus then $\Phi(G, A)$ is a root system with Weyl goup $W(A) = N_G(A)/Z_G(A)$. Recall that a k-torus is k-split if it can be diagonalized over the base field k. We will call a k-torus (θ, k)-split if it is θ-split and k-split. These tori yield a natural root system for the symmetric k-variety G_k/H_k since a maximal (θ, k)-split torus A of G has a root system $\Phi(G, A)$ [19] with Weyl group $N_{G_k}(A)/Z_{G_k}(A)$. Additionally this root system can be obtained by restricting the roots of a maximal torus of G containing A. We denote the root system of T by $\Phi(T)$, its positive roots by $\Phi^+(T)$, and a basis by Δ.

The rank of a group G is the dimension of a maximal torus $T \subset G$ and the k-rank of a group is the dimension of maximal k-split torus. A group is called k-split if the k-rank is equal to the rank. For examples, in $\mathrm{SL}_2(\mathbb{R})$ the diagonal matrices have dimension equal to 1, which is equal to the dimension of the diagonal matrices in $\mathrm{SL}_2(\mathbb{C})$. Since the $\mathbb{R}$-rank is equal to the rank of $\mathrm{SL}_2(\mathbb{C})$, we have that $\mathrm{SL}_2(\mathbb{R})$ is an $\mathbb{R}$-split group.

2.2.1 The Root Space Decomposition

For a group G, we will denote its Lie algebra by $\mathfrak{g}$, and similarly for T, Q, and H, their Lie algebras will be denoted by $\mathfrak{t}$, $\mathfrak{q}$, and $\mathfrak{h}$ respectively. If a torus T is maximal then $\mathfrak{t}$ is a maximal cartan subalgebra and thus we have the associated root space decomposition of the Lie algebra,

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Phi(T)} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid [t, g] = \alpha(t)g \text{ for all } t \in \mathfrak{t}\}$, and each $\mathfrak{g}_\alpha$ is one dimensional.

2.3 Isomorphy Classes of Involutions

It will be shown later that the double cosets $P\backslash G/H$ are characterized up to the isomorphy class of the involution of G. In this section we define the isomorphy classes and provide examples for the special linear group which is used as the principle example throughout.

Two involutions $\sigma, \varphi \in \mathrm{Aut}(G, G_k)$ are isomophic if there exists $\gamma \in \mathrm{Aut}(G, G_k)$ such that $\gamma\varphi\gamma^{-1} = \sigma$. Additionally φ, σ are $\mathrm{Int}(G, G_k)$-isomorphic if $\gamma \in \mathrm{Int}(G, G_k)$. A

Table 2.3.1: Summary of square classes

Field	$\lvert k^*/(k^*)^2\rvert$	**Representatives**
$\mathbb{R}$	2	1, -1
$\mathbb{Q}_p$, $p \equiv 1 \mod 4$	2	1, p
$\mathbb{Q}_p$, $p \equiv 3 \mod 4$	4	1, p, N_p, pN_p
$\mathbb{Q}_2$	8	1, ± 1, ± 2, $\pm 3, \pm 6$
$\mathbb{Q}$	∞	1, 2, ...
$\mathbb{F}_q$, $q \neq 2^r$	2	1, N_p
$\mathbb{F}_{2^r}$	1	1

characterization of the orbits $P\backslash G/H$ rests on the characterization of the involutions given in [17]. For some fields, notably algebraically closed fields and the real numbers, a full classification exists and was given in [20]. Additionally the results are known for some special cases, notably $\mathrm{SL}(n, k)$ [21, 22], $\mathrm{SO}(n, k)$ [2], and $\mathrm{SP}(n, k)$ [3].

Involutions of $\mathbf{SL}(2, \boldsymbol{k})$

Our main example for this paper will be $\mathrm{SL}(2, k)$, so we give the explicit result here. Borel proved that every automorphism of $\mathrm{SL}(2, k)$ is inner, and the explicit form was found in [21]. Let k^* be the set of nonzero elements of the field k, and let $(k^*) = \{a^2 \mid a \in k*\}$ be the set of nonzero squares in k. By abuse of notation we use $m \in k^*/(k^*)^2$ to denote a representative of the square class of m.

Theorem 2.3.1. *The k-isomorphy classes of* $\mathrm{SL}(2, k)$ *are represented by* $\mathrm{Int}(A)$*, where* $A = \left(\begin{smallmatrix} 0 & 1 \\ m & 0 \end{smallmatrix}\right)$ *for some* $m \in k^*/(k^*)^2$.

Thus it will be useful to have a summary of the representatives of square classes for some common fields, which we now present. N_p is used to denote the smallest nonsquare in the finite field $\mathbb{F}_p$.

2.3.2 Involutions of $\mathbf{SL}(\boldsymbol{n}, \boldsymbol{k})$

For arbitary n the classification of involutions of $\mathrm{Aut}(G, G_k)$ is more complicated and is available in [4]. For convience we summarize the results here.

The classification of involutions of $\mathrm{SL}(n,k)$ can be split in to two subproblems, namely one can approach inner and outer involutions separately. The results in this section are heavily field dependent, but for algebraically closed fields the two subproblems are closely related thanks to the following lemma.

Theorem 2.3.3. *Let $G = \mathrm{SL}(n,k)$ and $k = \bar{k}$.*

(a) $|\mathrm{Aut}(G)/\operatorname{Int}(G)| = 2$

(b) For a fixed outer automorphism η, every element of $\mathrm{Aut}(G)$ can be written in the form $\operatorname{Int}(M)\eta$.

Thus classifying outer automorphisms requires only the classification of the inner involutions and an explicit outer involution. For the latter we give the usual transpose inverse involution, $\eta(A) = (A^T)^{-1}$.

The classification of inner automorphisms is relatively simple but will require some additonal notation. To this end, let:

$$I_{n-i,i} = \begin{pmatrix} I_{n-i,n-i} & 0 \\ 0 & -I_{i,i} \end{pmatrix}$$

$$J_{2m} = \begin{pmatrix} 0 & I_{m,m} & 0 \\ -I_{m,m} & 0 & \end{pmatrix}, \quad n = 2m$$

$$M_{n,x,y,z} = \begin{pmatrix} I_{n-3,n-3} & & & \\ & x & & \\ & & y & \\ & & & z \end{pmatrix}$$

$$L_{n,x} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ x & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 0 & 1 \\ 0 & \dots & 0 & x & 0 \end{pmatrix}$$

With this notation in place, we are ready to summarize the isomorphy classes of inner involutions for $\mathrm{SL}(n,k)$ for the specific fields k given in Table 2.3.1. This summary is the subject of Table 2.3.2.

Table 2.3.2: Isomorphy classes of inner involutions of $\mathrm{SL}(n,k)$

Field	Number of Isomorphy Classes	Representative Matrix A, $\theta = \mathrm{Int}(A)$
n odd		
$k =$ any field	$\frac{n-1}{2}$	$A = I_{n-i,i}\ i = 1, 2, \ldots, \frac{n-1}{2}$
n even		
$k = \overline{k}$	$\frac{n}{2}$	$A = I_{n-i,i}\ i = 1, 2, \ldots, \frac{n}{2}$
$k = \mathbb{R}$	$\frac{n}{2} + 1$	$A = I_{n-i,i}\ i = 1, 2, \ldots, \frac{n}{2}$ $A = L_{n,-1}$
$k = \mathbb{Q}$	∞	$A = I_{n-i,i}\ i = 1, 2, \ldots, \frac{n}{2}$ $A = L_{n,\alpha}\ \alpha \not\equiv 1 \mod (\mathbb{Q}^*)^2$
$k = \mathbb{F}_p$, $p \neq 2$	$\frac{n}{2} + 1$	$A = I_{n-i,i}\ i = 1, 2, \ldots, \frac{n}{2}$ $A = L_{n,N_p}$
$k = \mathbb{Q}_p$, $p \neq 2$	$\frac{n}{2} + 3$	$A = I_{n-i,i}\ i = 1, 2, \ldots, \frac{n-1}{2}$ $A = L_{n,\alpha}\ \alpha \in \{p, N_p, pN_p\}$
$k = \mathbb{Q}_2$	$\frac{n}{2} + 7$	$A = I_{n-i,i}\ i = 1, 2, \ldots, \frac{n-1}{2}$ $A = L_{n,\alpha}\ \alpha \in \{\pm 1, \pm 2, \pm 3, \pm 6\}$

Similarly, the outer involutions of $\mathrm{SL}(n,k)$ are summarized in Table 2.3.3.

2.4 Fixed Point Groups

The fixed point group of an involution θ plays an important role in certain isomorphy classes of k-split tori, as will be seen in a later section. Additionally, the nature of the fixed point group is crucial to the surjectivity of the generalized complexification map, therefore a summary of results is necessary for the development of a rich set of examples.

Given an involution θ, the fixed point group is $G^\theta = \{g \in G \mid \theta(g) = g\}$. Let H be a k-open subgroup of G^θ. In the original applications for which symmetric space theory was developed, the cases in which H is compact played a vital role. For instance, the Cartan decomposition and Iwasawa decomposition rely on a compact fixed point group. If the fixed point group is compact then θ is called the Cartan involution, which is unique up to $\mathrm{Aut}(G)$-isomorphism.

Table 2.3.3: Isomorphy classes of outer involutions of $\mathrm{SL}(n,k)$

Field	Number of Isomorphy Classes	Representative Involution
$k=\bar{k}$		
n odd	1	η
$n=2m$ even	2	η $\mathrm{Int}(J_{2m})\eta$
$k=\mathbb{R}$		
n odd	$\frac{n-1}{2}$	$\mathrm{Int}(I_{n-i,i}\eta \ i=1,2,\ldots,\frac{n-1}{2}$
$n=2m$ even	$\frac{n}{2}+1$	$\mathrm{Int}(I_{n-i,i}\eta \ i=1,2,\ldots,\frac{n}{2}$ $\mathrm{Int}(J_{2m})\eta$
$\mathbb{F}_p \ p\neq 2$		
n odd	2	η $\mathrm{Int}(M_{n,1,1,N_p})\eta$
$n=2m$ even	3	η $\mathrm{Int}(M_{n,1,1,N_p})\eta$ $\mathrm{Int}(J_{2m})\eta$

2.4.1 Fixed Point Groups of $\mathrm{SL}(2,k)$

Example 2.4.2. Let $G=\mathrm{SL}(2,\mathbb{R})$, then in light of Theorem 2.3.1 and Table 2.3.1, there are two k-ismorphy classes of involutions represented by $\mathrm{Int}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\mathrm{Int}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

(a) Let $\sigma=\mathrm{Int}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $H=\{\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right) \mid a^2-b^2=1\}$. H is clearly noncompact since it is isomorphic to a hyperbola in $\mathbb{R}^2$. Therefore σ does not represent the Cartan involution.

(b) Let $\theta=\mathrm{Int}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then $H=\{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \mid a^2+b^2=1\}$. Thus $H\cong S^1$, therefore the fixed point group is compact and θ is the Cartan involution.

With the development of the theory of symmetric k-varieties came a need to generalize the notion of compactness; this was accomplished with the notion of k-anisotropy. A group is said to be k-anisotropic if contains no nontrivial k-split torus, otherwise it is isotropic. For example, the fixed point group of 2.4.2(a) is a one dimensional semisimple abelian group and therefore a torus. It is k-split since symmetric matrices are diagonalizable via orthogonal matrices, explicitly one can conjugate by $\frac{\sqrt{2}}{2}\left(\begin{smallmatrix}1&1\\-1&1\end{smallmatrix}\right)$ to diagonalize the fixed point group. Thus the fixed point group of Example 2.4.2(a) is isotropic. The fixed point group of Example 2.4.2(b) is also semisimple and abelian and therefore a torus, however the eigenvalues are complex and therefore it cannot be diagonlized over $\mathbb{R}$. Thus, in this case, H is k-anisotropic.

This example illustrates a more general result of Beun and Helminck:

Theorem 2.4.3. *Let $G = \mathrm{SL}(2,k)$ with involution $\theta = \left(\begin{smallmatrix}0&1\\m&0\end{smallmatrix}\right)$ for some $m \in k^*/(k^*)^2$. Then the fixed point group H is k-anisotropic if and only if $m \neq 1$.*

Therefore one observes that for each field k there is precisely one isomorphy class of k-involutions of $\mathrm{SL}(2,k)$ yielding an isotropic fixed point group, and therefore for most base fields the number of involutions corresponding to k-anisotropic fixed point groups will outnumber the number of involutions corresponding to isotropic fixed point groups. For instance, if $k = \mathbb{Q}$ there are infinitely many k-anisotropic fixed point groups, for $k = \mathbb{Q}_p,\ p \neq 2$ there are 1 or 3 k-anisotropic fixed point groups.

2.4.4 Fixed Point Groups of $\mathrm{SL}(n,k)$

$\mathrm{SL}(n,k)$ provides an intuitive model for the generalized complexification of the orbits of minimal parabolic k-subgroups acting on symmetric k-varieties, so we shall continue with the development of their theory initiated in the last section. The fixed point groups for most involutions have been computed so we summarize those results here.

For ease of computation one first replaces $\mathrm{Int}(I_{n-i,i})$ by another representative of its isomorphy class, namely $\mathrm{Int}(\mathcal{I}_{n-i,i})$, where

$$\mathcal{I}_{n-i,i} = \begin{pmatrix} A & 0 \\ 0 & I_{n-2i\times n-2i} \end{pmatrix},\ \text{where } A \text{ is the } 2i \times 2i \text{ matrix with ones on the antidiagonal}$$

We can then present the form of the fixed point group for $\theta = \mathrm{Int}(\mathcal{I}_{n-i,i})$.

Lemma 2.4.5. *Let $G = \mathrm{SL}(n,k)$ and $\theta = \mathrm{Int}(\mathcal{I}_{n-i,i})$. Then $H_k = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\}$, where*

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,2i} \\ \vdots & & & \vdots \\ a_{i,1} & \dots & \dots & a_{i,2i} \\ a_{i,2i} & \dots & \dots & a_{i,1} \\ \vdots & & & \vdots \\ a_{1,2i} & \dots & a_{2,1} & a_{1,1} \end{pmatrix},$$ *B is $2i \times n-2i$ and symmetric about the ith row, C is $n-2i \times 2i$ and symmetric about the ith column, and $D \in \mathrm{GL}(n-2i,k)$, with all entries of A, B, C, and D in k.*

Lemma 2.4.6. *Let $G = \mathrm{SL}(2m,k)$ and $\theta = \mathrm{Int}(L_{2m,x})$. Then* $H_k = \begin{pmatrix} A_{1,1} & \dots & A_{1,m} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,m} \end{pmatrix}$ *where* $A_{i,j} = \begin{pmatrix} a_ij & b_{ij} \\ xb_{ij} & a_{ij} \end{pmatrix}$ *satisfying the determinant condition.*

2.5 Parabolic and Borel Subgroups

A Borel subgroup $B \subset G$ is a maximal closed and connected solvable subgroup. Borel subgroups are self-normalizing, i.e. $N_G(B) = B$, and every Borel subgroup contains a θ-stable maximal torus T. Every Borel subgroup is conjugate over G. Suppose $T_1 \subset B_1$, $T_2 \subset B_2$ are maximal tori, then they are conjugate under G, say $gT_1g^{-1} = T_2$. The same element $g \in G$ also conjugates B_1 to B_2, $gB_1g^{-1} = B_2$. By a theorem of Lie-Kolchin, if G is a closed subgroup of GL_n, then all Borel subgroups of G are conjugate to the group of upper triangular matrices. Fix a maximal torus T and consider its associated root system $\Phi(T)$. One has multiple choices for a Borel subgroup $B \supset T$, fixing a Borel subgroup containing T is equivalent to choosing $\Phi^+(T)$, a system of positive roots for $\Phi(T)$.

The following result is due to Borel and is fundamental to a characterization of the orbits $B\backslash G/H$:

Theorem 2.5.1. *Let $\theta \in \mathrm{Aut}(G)$ be an involution and $B \subset G$ a Borel subgroup. Then B contains a θ-stable maximal torus and all such tori are conjugate under $U \cap H$.*

Parabolic subgroups $P \subset G$ contain a Borel subgroup. Fix a torus T and a containing Borel subgroup $B \supset T$. Given a basis $\Delta \subset \Phi(T)$, one can choose a subset $\Gamma \subset \Delta$; to each root $\alpha \in \Gamma$ one identfies the Lie subalgebra corresponding to α, $\mathfrak{g}_\alpha$. Let $\mathfrak{p} = \mathfrak{b} \oplus_{\alpha \in \Gamma} \mathfrak{g}_{-\alpha}$. Lifting $\mathfrak{p}$ to the group one obtains a parabolic subgroup. This parabolic subgroup is unique up to isomorphism, therefore there is a one-to-one correspondence between parabolic subgroups and subsets of the basis of a system of roots. Parabolics subgroups can also be defined as subgroups of G such that G/P is a complete variety.

Parabolic subgroups yield a result close to Theorem 2.5.1.

Theorem 2.5.2. *Let $P \subset G$ be a minimal parabolic k-subgroup and $\theta \in \mathrm{Aut}(G)$ an involution. Then P contains a maximal k-split torus that is θ-stable. Furthermore, all of these tori are conjugate under $(U \cap H)_k$.*

A parabolic k-subgroup is a parabolic defined over k, and the orbits of minimal parabolic k-subgroups on symmetric k-varieties have extra structure. Over an algebraically closed field Borel subgroups are minimal parabolic subgroups in the sense that G/B is larger than G/P for any P that is not a Borel subgroup, but over other fields proper minimal parabolic k-subgroups may not exist.

Example 2.5.3. Let $G = \mathrm{SL}(2, \mathbb{C})$ with $\mathbb{R}$-form $G_\mathbb{R} = \mathrm{SL}(2, \mathbb{R})$.

(a) Let T_1 be the set of 2×2 determinant one diagonal matrices. Then $T_1 \subset P$, where P is the set of 2×2 determinant one upper (or lower) triangular matrices.

(b) Let $T_2 = \mathrm{SO}(2, \mathbb{R})$, then T_2 is not contained in a proper parabolic defined over $\mathbb{R}$, so the minimal parabolic $\mathbb{R}$-subgroup containing T_2 is G.

Parabolic subgroups are also self-normalizing, and can be decomposed via $P = LU$, where U is the unipotent radical $R_u(P)$ and L is the Levi factor of P. In the case that $P = B$ is a Borel subgroup, this decomposition simplifies to $B = TU$, where $T \subset B$ is a maximal torus and $U \subset B$ is unipotent.

2.5.4 The Bruhat Decomposition

The Bruhat decomposition is a fundamental tool in the theory of algebraic groups since it allows one to reduce questions about the structure of a group to questions about the fine structure of the group, namely tori and Weyl groups. This decomposition generalizes the well known $PA = LU$ factorization of a nonsingular matrix A, where P is a permuation matrix, L is lower triangular, and U is upper triangular. The Bruhat decomposition motivates the study of the double cosets $P\backslash G/H$ since the latter in fact generalize the Bruhat decomposition.

Let $B \subset G$ be a Borel subgroup containing a maximal torus T, then it is well known [23] that the double cosets $B\backslash G/B$ are in one-to-one correspondence with $W(T)$, the Weyl group. Briefly:

$$B\backslash G/B \cong W(T)$$

Equivalently, we have

$$G \cong \bigcup_{\omega \in W(T)} B\dot{\omega}B$$

where $\dot{\omega}$ is a representative of ω in $N_G(T)$. Given $\omega \in W(T)$, $C(\omega) = B\dot{\omega}B$ is called a Bruhat cell. To complete the analogy with the $PA = LU$, each cell corresponds to the matrices requiring a pivot before the typical Gauss-Jordan elimination algorithm is implemented. Thus we can regard the Bruhat decomposition as the precise statement of Gauss-Jordan elimination.

One can consider the double cosets $B\backslash G/B$ from different perspectives, including B-orbits on G/B and $B \times B$ orbits on G. One observes immediately that the B-orbits on G/B are finite since Weyl groups are finite, thus G/B is a spherical variety.

Example 2.5.5. Let $G = \mathrm{SL}(2,\mathbb{C})$, $T = \{\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right) \mid x \in \mathbb{C}\}$ a maximal torus, and $B = \{\left(\begin{smallmatrix} x & y \\ 0 & x^{-1} \end{smallmatrix}\right) \mid x, y \in \mathbb{C}\}$ a Borel subgroup containing T. Then $G = B \cup B\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)B$.

There is also an order associated with the Bruhat decomposition, called the (strong) Bruhat order, that encode data about the geometry of the orbits. This order will be generalized to the $P\backslash G/H$ double cosets, thus we omit the definition here.

The Bruhat decomposition extends to general fields. Let P be a minimal parabolic k-subgroup containing a maximal k-split torus A. Then

$$G_k \cong \bigcup_{\omega \in W(A)} P_k\dot{\omega}P_k$$

where $\dot{\omega}$ is a representative of ω in $N_{G_k}(A)$.

The substitutions of maximal tori for maximal k-split tori, minimal parabolic k-subgroups for Borels, etc. used in generalizing the Bruhat decomposition will occur again in later sections, when we generalize the $P\backslash G/H$ double cosets to $P_k\backslash G_k/H_k$ double cosets.

2.5.6 Generalized Bruhat Decomposition

If we replace the Borel subgroup in the Bruhat decomposition with an arbitrary parabolic subgroup, we obtain the generalized Bruhat decomposition. Fix a parabolic P, a Borel subgroup $B \subset P$, and a maximal torus $T \subset B$. This is equivalent to choosing a set of positive roots $\Phi^+(T)$ and a basis Δ for $\Phi(T)$. Recall that P corresponds to a subset Γ of a basis for the root system Φ. Each $\alpha \in \Gamma$ corresponds to a reflection s_α, denote by W_P the group generated by these reflections.

The generalized Bruhat decomposition is usually given as the disjoint union of double cosets $P\omega P$, $\omega \in W_P \backslash W / W_P$. Briefly:

$$G = \bigcup_{w \in W_P \backslash W / W_P} P\dot{\omega}P$$

where $\dot{\omega}$ is representative of ω in $N_G(T)$.

Example 2.5.7. Let $G = \mathrm{SL}(3, \mathbb{C})$, $\Gamma = \{\alpha_1\}$, B the set of upper triangular matrices, and T the set of diagonal matrices. Then

$$P = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$$

and W_P has representative

$$\dot{\omega} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Furthermore

$$W_P \backslash W / W_P \cong \{s_{\alpha_2}\} \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Therefore $G = P \cup P \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{smallmatrix}\right) P$.

Chapter 3

Double Cosets $P\backslash G/H$

The double cosets $P\backslash G/H$ generalize the Bruhat decomposition, as shown in Example 3.1.4. They were studied first by Springer and later by Brion, Helminck, and Wang. One can move to general fields to consider double cosets $P_k\backslash G_k/H_k$ where many of the results are similar. We seek to understand the relationship between $P\backslash G/H$ and $P_k\backslash G_k/H_k$ for a fixed involution and parabolic P. Our approach exploits the generalized complexification map which will embed the $P_k\backslash G_k/H_k$ double cosets inside the $P\backslash G/H$ double cosets, allowing us to observe the relation ship directly. We now provide the relevant background for these objects that will be the objects of study for the remainder.

3.1 $k = \bar{k}$, Borel Subgroups Acting on G/H

Springer studied this case extensively in [30] where he proved several equivalent characterizations. His approach took advantage of the following result due to Steinberg [32].

Theorem 3.1.1. *Let θ be an involution of G. Then there exists a θ-stable Borel subgroup B.*

To obtain a characterization of the orbits $B\backslash G/H$, one can separately consider B-orbits on G/H, H-orbits on G/B, or (B, H)-orbits on G. We outline these results in this section. Let $T \subset G$ be a maximal torus. To denote the set of Weyl group elements with representatives in H we use $W_H(T) = N_H(T)/Z_H(T)$. Consider first the H-orbits. Let $\mathcal{B}$ denote the variety of all Borel subgroups of G, then we can identify G/B with $\mathscr{B}$ since all Borel subgroups are conjugate over G, and let $\mathscr{C}$ denote the set of pairs

(B', T') where T' is a maximal torus contained in the Borel subgroup B'. G acts on both $\mathscr{B}$ and $\mathscr{C}$ by conjugation, denote these orbits by $\mathscr{B}/H$ and $\mathscr{C}/H$ respectively. The H-orbits on $\mathscr{C}$ consist of two parts, namely the H-conjugacy classes of maximal tori and the H-conjugacy classes of Borel subgroups containing them. The H-conjugacy classes of maximal tori will have representatives $\{T_i\}_{i \in I}$, thus the H-orbits on $\mathscr{C}$ are in correspondence with $\cup_{i \in I} W_G(T)/W_H(T)$.

G acts on $G/H \cong Q$ (from the left) via the *θ-twisted action*, i.e. $g * q = gq\theta(g)^{-1}$. Thus the B-orbits on G/H can be viewed as B-cosets in Q, which we denote $B \backslash Q$.

The (H, B)-orbits on G are the same as the $B \times H$-orbits on G and the action is given by $(b, h) * g := bgh^{-1}$. From [19] we know that every U orbit on G/H, where U is the unipotent component of B, meets $N_G(T)$. Let $\mathcal{V} = \{g \in G \,|\, \tau(g) \in N_G(T)\}$, then $\mathscr{V}$ is stable under left multiplication by $N_G(T)$ and right multiplication by H. We denote by V the $T \times H$-orbits on $\mathcal{V}$, which in fact parameterize the $(B \times H)$ orbits on G.

Borel showed that all of these characterizations are isomorphic:

Theorem 3.1.2 ([30])**.** *Let B be a Borel subgroup of G and $\{T_i\}_{i \in I}$ a set of representatives of the H-conjugacy classes of θ-stable maximal tori in G. Then*

$$B \backslash G/H \cong \mathscr{B}/H \cong \bigcup_{i \in I} W_G(T_i)/W_H(T_i) \cong \mathscr{C}/H \cong B \backslash Q \cong V$$

Example 3.1.3. Let $G = \mathrm{SL}(2, \mathbb{C})$, $B \subset G$ the set of upper triangular matrices of determinant 1, and define an involution θ by $\theta(g) = (g^T)^{-1}$. Since every element of $\mathbb{C}^*$ is a square, all involutions on G are isomorphic by Theorem 2.3.1. The dimension of any torus inside G is 1. The fixed point group of θ is $H = \mathrm{SO}(2, \mathbb{C})$, which is connected and abelian and therefore a torus. Let T denote the set of diagonal matrices with determinant 1, then T is also a torus. There are two H-conjugacy classes of tori in G, and $\{T, H\}$ is a set of representatives. The Weyl groups are given by

$$W_G(T) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

$$W_G(H) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}$$

We observe that the non-indentity element in $W_G(T)$ is an element of H, while the non-identity element of $W_G(H)$ is not. Therefore $|W_G(T)/W_H(T)| = 1$ and $|W_G(H)/W_H(H)| = 2$. Thus one orbit $B\backslash G/H$ corresponds to the diagonal matrices and two correspond to the fixed point group.

The next example shows that this characterization is in fact a generalization of the Bruhat decomposition.

Example 3.1.4. Let $\mathbf{G}$ be a connected reductive algebraic group, $\mathbf{B}\subset\mathbf{G}$ a Borel subgroup of $\mathbf{G}$, and $\mathbf{T}\subset\mathbf{G}$ a maximal torus of $\mathbf{B}$. Define $G = \mathbf{G}\times\mathbf{G}$ with the involution $\theta(g_1, g_2) = (g_2, g_1)$. Then $H = \{(g, g) \mid g \in \mathbf{G}\}$, the diagonal subgroup of $\mathbf{G}$, and $B = \mathbf{B}\times\mathbf{B}$, $T = \mathbf{T}\times\mathbf{T}$ are both θ-stable. Consider the map $\xi : G \to \mathbf{G}$ given by $\xi(g_1, g_2) = g_1^{-1}g_2$. Then $\xi(H) = \{\mathrm{id}\}$, $\xi(G) \cong \mathbf{G}$, and $\xi(B) \cong \mathbf{B}$. Thus ξ induces a map $H\backslash G/B \to \mathbf{B}\backslash\mathbf{G}/\mathbf{B} \cong W(\mathbf{T})$. Thus $H\backslash G/B$ is equivalent to the Bruhat decomposition of the group $\mathbf{G}$, and therefore in the groups case the results about the double cosets can be derived from the Bruhat decomposition.

3.2 Orders Associated with the Orbit Decomposition

The versatility of Theorem 3.1.2 is that the different characterizations of the orbits allow one to study their structure from multiple perspectives. In this section we develop the tools to study the orbits combinatorially. We define two orders, one a refinement of the other, that have a connection to the geometry of the orbits.

3.2.1 Bruhat order on the orbits $B\backslash G/H$

One can endow the set of double cosets $B\backslash G/H$ with a partial order that generalizes the usual Bruhat order on a connected reductive algebraic group. This was studied in [27] and [28]. The partial order is given by inclusion of closures in the Zariski topology. Explicitly, let $\mathscr{O}_1 = Bg_1H$ and $\mathscr{O}_2 = Bg_2H$, then $\mathscr{O}_1 \leq \mathscr{O}_2$ if $\mathscr{O}_1 \subset \mathrm{cl}(\mathscr{O}_2)$. One can also give this ordering combinatorially and this was the approach taken by Richardson and Springer.

3.2.2 I-poset

The Bruhat order is a refinement of the order on the I-poset. Recall that the set I parameterizes the H-conjugacy classes of the θ-stable maximal tori in G, let $\{T_i\}_{i\in I}$ be a set of representatives for these conjugacy classes. Suppose T_i and T_j are maximal tori such that $T_i^- \subset T_j^-$. Then we can introduce an order defined by $T_i \leq T_j$ if $\dim(T_i^-) \leq \dim(T_j^-)$. It will be shown later that all such tori are conjugate to tori satisfying this property, so this order can be extended to an order on all maximal tori of G. Furthermore, we can associate poset diagrams to the orbit decompositions. Since the Bruhat poset is a refinement of the I-poset, we will call the Bruhat diagram obtained from an I diagram the expansion of the I diagram.

Example 3.2.3. Let $G = \mathrm{SL}(3,\mathbb{C})$, $\theta(A) = (A^T)^{-1}$ for all $A \in G$. Then $H = \mathrm{SO}(3,\mathbb{C})$. Let T be the set of diagonal matrices, then T is θ-split. Furthermore, a maximal torus of H is given by

$$\left\{\begin{pmatrix}\cos(\nu) & -\sin(\nu) & 0\\ \sin(\nu) & \cos(\nu) & 0\\ 0 & 0 & 1\end{pmatrix}\right\}$$

[33]. Thus there are precisely two nodes in the I-poset diagram, illustrated in Figure 3.2.1.

Figure 3.2.1: I-poset diagram for $\mathrm{SL}(3,\mathbb{C})$

3.3 $k = \bar{k}$, P a parabolic subgroup acting G/H

Theorem 3.1.1 is not true for general parabolic subgroups, but Helminck and Brion [8] showed that this condition is not necessary; this section summarizes their results. The important difference in this case is that for a fixed parabolic P we are not assured that the set of G-conjugates of P includes all parabolic subgroups of G. In fact, if we identify parabolic subgroups P_1 and P_2 with their associated subsets of a basis for the root space $\Gamma_1, \Gamma_2 \subset \Delta$, we see that P_1 and P_2 are G-conjugate if and only if $\Gamma_1 = \Gamma_2$. Let $\mathscr{P}$ denote the variety of all parabolic subgroups of G and let $\mathscr{D}$ denote the set of triples (P, B, T), where B is a Borel subgroup of P such that $(P \cap H)B$ is open in P and T is a θ-stable maximal torus of B. Fix a parabolic subgroup P and let $\mathscr{P}^P$ denote the set of G conjugates of P and let $\mathscr{D}^P$ denote the set of triples $(P', B', T') \in \mathscr{D}$ such that $P' \in \mathscr{P}^P$. G acts on $\mathscr{P}$, $\mathscr{P}^P$, $\mathscr{D}$, and $\mathscr{D}^P$ via conjugation; denote the H-orbits on these sets by $\mathscr{P}/H$, $\mathscr{P}^P/H$, $\mathscr{D}/H$, and $\mathscr{D}^P/H$ respectively.

Theorem 3.3.1. *There is a bijective map $\mathscr{P}/H \to \mathscr{D}/H$.*

Every maximal k-split torus of G is conjugate under G_k, the k-rational points of G. Furthermore, every minimal parabolic k-subgroup of G contains a maximal k-split torus. Let A be a maximal k-split torus. We wish to characterize all minimal parabolic k-subgroups which contain A. As a generalization of the algebraically closed case, we have the following lemma:

Proposition 3.3.2. *Suppose A_1 and A_2 are two maximal k-split tori, and P_1 and P_2 are two minimal parabolic k-subgroups containing A_1 and A_2 respectively. Then the element of G_k that conjugates A_1 to A_2 also conjugates P_1 to P_2.*

We now fix a parabolic subgroup P, and let T be the θ-stable maximal torus occurring in the image of P under the above bijection. Let $x_1, x_2, ..., x_r \in G$ such that $T_1 = x_1 T x_1^{-1}, ..., T_r = x_r T x_r^{-1}$ are representatives for the H-conjugacy classes of the elements of $\mathscr{D}$. Let $P_1 = x_1 P x_1^{-1}, ..., P_r = x_r P x_r^{-1}$ and denote by $W_{P_i}(T_i)$ the Weyl group of P_i. One then sees that for each T_i the H-conjugacy classes of $(P', B', T_i) \in \mathscr{D}^P$ are in bijection with $W_{P_i}(T_i) \backslash W_G(T_i) / W_H(T_i)$. As before let $\mathscr{V} = \{g \in G \mid g^{-1}\theta(g) \in N_G(T)\}$ and let $\mathscr{V}^P = \{g \in \mathscr{V} \mid BgH \text{ is open in } PgH\}$. Then the actions of B, H on $\mathscr{V}$ extend to actions of P, H on $\mathscr{V}^P$, and every $P \times H$-orbit on G meets $\mathscr{V}^P$ in a unique (T, H) double coset. We can now generalize Springer's theorem charactering $B \backslash G / H$ double cosets to the case of a general parabolic subgroup.

Theorem 3.3.3. *There is a bijective map from the set of H-orbits in $\mathscr{P}$ onto the set of H-conjugacy classes of triples $(P, B, T) \in \mathscr{D}$. Moreover for a fixed parabolic subgroup P, we have*

$$P \backslash G / H \cong \mathscr{P}(G)^P / H \cong \mathscr{D}^P / H \cong \bigcup_{i=1}^{r} W_{P_i}(T_i) \backslash W_G(T_i) / W_H(T_i) \cong V^P$$

3.4 Orbits Over Non-algebraically Closed Fields

We now turn to the double cosets which we study for the remainder. If we restrict our attention to parabolic subgroups defined over k that are minimal we can obtain a characterization very similar to the case of a Borel subgroup acting on G/H. In this case we obtain are assured of the existence of θ-stable Levi factor.

Theorem 3.4.1 ([19])**.** *Let P be a minimal parabolic k-subgroup with unipotent radical U, then P contains a θ-stable maximal k-split torus, unique up to an element of $(H \cap U)_k$.*

Let P be a minimal parabolic k-subgroup and let G and H be defined as before. With a few adjustments, one can construct a characterization of the $P_k \backslash G_k / H_k$ orbits in several equivalent ways by considering P_k orbits on Q_k, H_k orbits on $P_k \backslash G_k$, or $P_k \times H_k$ orbits on G_k.

We start with the H_k-orbits on $P_k \backslash G_k$. Let $\mathscr{P}_k$ denote the variety of all minimal parabolic k-subgroups of G, then we have that $P_k \backslash G_k$ is isomorphic to $\mathscr{P}_k$. G_k acts on $\mathscr{P}_k$ by conjugation, so we can identify the double cosets with the H_k-orbits on $\mathscr{P}_k$, denote these orbits by $\mathscr{P}_k / H_k$.

Let $\mathscr{C}_k$ denote the set of all pairs (P'_k, A'_k), where P'_k is a minimal parabolic k-subgroup and A'_k is a θ-stable maximal k-split torus contained in P_k. G_k acts on $\mathscr{C}_k$ by conjugation in both coordinates, i.e. $g * (P'_k, A'_k) = (gP'_k g^{-1}, gA'_k g^{-1})$. We can analyze the H_k orbits on $\mathscr{C}_k$ (denoted $\mathscr{C}_k / H_k$) in two steps; first we consider the H_k-conjugacy classes of θ-stable maximal k-split tori and choose a set of representatives for these conjugacy classes, and second for each representative of an H_k-conjugacy class we consider the set of minimal parabolic k-subgroups that contain the representative but are not conjugate via H_k. This allows one to identify $\mathscr{C}_k / H_k$ with $\cup_{i \in I} W_{G_k}(A_i) / W_{H_k}(A_i)$, where $\{A_i\}_{i \in I}$ is a set of representatives for the H_k conjugacy classes of θ-stable maximal k-split tori.

P_k acts on G_k / H_k via the θ-twisted action. Let $A \subset P$ be a maximal k-split torus. Then as in the case of a Borel subgroup acting on the symmetric space over an algebraically closed field, we have that the orbit of the unipotent radical of P_k meets $N_{G_k}(A)$. Let $\mathscr{V}_k = \{x \in G_k \mid \tau(x) \in N_{G_k}(A)\}$, then we can identify the P_k-orbits on G_k / H_k with the $Z_{G_k}(A) \times H_k$-orbits on $\mathscr{V}_k$. We denote these orbits by V_k.

Theorem 3.4.2. *For a minimal parabolic k-subgroup P and $\{A_i\}_{i \in I}$ a set of representa-*

tives for the H_k-conjugacy classes of θ-stable maximal k-split tori, we have

$$P_k\backslash G_k/H_k \cong \mathscr{P}_k/H_k \cong \bigcup_{i\in I} W_{G_k}(A_i)/W_{H_k}(A_i) \cong P_k\backslash Q_k \cong V_k$$

We observe that this characterization is in direct analogy with the case of a Borel subgroup acting on the symmetric space and simpler than the case of a general parabolic subgroup acting on the symmetric space. This is because a minimal parabolic k-subgroup contains a θ-stable maximal k-split torus in light of Lemma 3.4.1.

Example 3.4.3. Let $G = \mathrm{SL}(2,\mathbb{C})$ with real form $G_\mathbb{R} = \mathrm{SL}(2,\mathbb{R})$. Let T denote the group of diagonal matrices and P the set of upper triangular matrices. Recall from Example 2.4.2 that there are two isomorphy classes of involutions.

(a) Let $\sigma = \mathrm{Int}(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$. Then $H_\mathbb{R} = \{(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}) \mid a^2 - b^2 = 1 \; a, b \in \mathbb{R}\}$ is connected and abelian and is diagonalizable by an orthogonal matrix. We compute the Weyl group elements

$$W_{G_\mathbb{R}}(T) = W_{G_\mathbb{R}}(H) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

The nonindentity element has a representative in H, given by $(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix})$, but this representative is not in $H_\mathbb{R}$. Thus we conclude $|W_{G_\mathbb{R}}(T)/W_{H_\mathbb{R}}(T)| = |W_{G_\mathbb{R}}(H)/W_{H_\mathbb{R}}(H)| = 2$, and there are 4 orbits in $P_\mathbb{R}\backslash G_\mathbb{R}/H_\mathbb{R}$.

(b) Let $\theta = \mathrm{Int}(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$. Then $H = \{(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}) \mid a^2 + b^2 = 1,\; a, b \in \mathbb{R}\}$. While $H_\mathbb{R}$ is connected and abelian, its eigenvalues are complex and thus it not an $\mathbb{R}$-split torus. Therefore there is only $H_\mathbb{R}$-conjugacy class of θ-stable maximal k-split tori and only one orbit in $P_\mathbb{R}\backslash G_\mathbb{R}/H_\mathbb{R}$.

As with the Bruhat decomposition, we can reformulate Theorem 3.4.2 to obtain

$$G_k = \bigcup_{i\in I} \bigcup_{w\in W_{G_k}(A_i)} P_k \dot{w} H_k, \text{ where } \dot{w} \text{ is a representative of } w \in W_{G_k}(A_i)$$

Applying this observation to Example 3.4.3(b) and the fact that $\mathrm{id} \in N_{G_\mathbb{R}}(T)$, we have that $G_\mathbb{R} = H_\mathbb{R} P_\mathbb{R}$, which is the well known Iwasawa decomposition of a real reductive group with Cartan Involution θ.

While the characterization is similar, many of the properties from the algebraically closed do not hold. For instance, the orbits over the algebraic closure are always finite, but over general fields this condition is frequently not satisfied. The next example illustrates one such case.

Example 3.4.4. Let $G = \mathrm{SL}(2,\mathbb{C})$ with $\mathbb{Q}$-form $G_{\mathbb{Q}} = \mathrm{SL}(2,\mathbb{Q})$. In this case it is computationally easier to let H act from the left, so we will be analyzing the double cosets $H_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/P_{\mathbb{Q}}$. The morphism $\tau : G \to G$ of Section 2.1 becomes $\tau(g) = g^{-1}\theta(g)$. We fix the involution θ defined by $\theta(g) = (g^T)^{-1}$ and $P \subset G$ as the set of upper triangular matrices. Note that P contains the maximal $\mathbb{Q}$-split torus $A \subset G$ consisting of the diagonal matrices. We will compute Q_k/P_k, each element of which must meet $\mathscr{V}_k = \{g \in G_k \mid g^{-1}\theta(g) \in N_{G_k}(A)\}$. For an arbitrary element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{Q}}$, we compute $\tau(g) = \begin{pmatrix} b^2+d^2 & -ab-cd \\ -ab-cd & a^2+c^2 \end{pmatrix}$. For this matrix to be in the normalizer $N_{G_{\mathbb{Q}}}(A)$ then it must be the case that $ab+cd = 0$. Therefore $g \in \mathscr{V}_{\mathbb{Q}} \iff ab+cd = 0$. In this case we can identify this image with an equivalent formulation, $\tau(\mathscr{V}_{\mathbb{Q}}) = \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \mid r = b^2+d^2,\ (b,d) \in \mathbb{Q}^2 - (0,0) \right\}$. Since we have candidates for representatives of the orbits, we want to know which of these candidates lie in the same orbit. To this end, we will to observe when the action of $P_{\mathbb{Q}}$ is capable of moving one element of $\tau(\mathscr{V}_{\mathbb{Q}})$ to another. Consider an arbitrary $p = \begin{pmatrix} v & u \\ 0 & v^{-1} \end{pmatrix} \in P_{\mathbb{Q}}$. Then $P_{\mathbb{Q}}$ acts on $Q_{\mathbb{Q}}$ via the θ-twisted action. Let $x = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \in \tau(\mathscr{V}_{\mathbb{Q}})$, then $p * x = \begin{pmatrix} rv^{-2}+u^2r^{-1} & -uvr^{-1} \\ -uvr^{-1} & v^2r^{-1} \end{pmatrix}$, which lies in $N)G_k(A)$ if and only if $u = 0$. $\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$ and $\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$ are in the same twisted $P_{\mathbb{Q}}$-orbit if there exists $v \in \mathbb{Q}$ such that $\begin{pmatrix} rv^{-2} & 0 \\ 0 & v^2r^{-1} \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \iff r^{-1}s \in (\mathbb{Q}^*)^2$ (seen by equating entries). It is a fact that such pairs (r,s) satisfying this condition can be identified with $\oplus_{p\equiv 1(4)}\mathbb{Z}/2\mathbb{Z}$. Therefore, the set $H_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/P_{\mathbb{Q}}$ is infinite.

In several cases, the number of orbits is finite. For algebraically closed fields this was proved by Springer [30], for $k = \mathbb{R}$ it was shown by Matsuki [25], Rossman [29], and Wolf [35], and for general local fields the result is due to Helminck and Wang [19].

We can associate diagrams to the partial orders placed on the orbit decompositions of our group in Section 3.2. This is done by exploiting the poset structure corresponding the index set of the θ-stable maximal k-split tori I and the Bruhat order on the orbits themselves.

Similar to the algebraically closed case, we can place an order on the I-poset. Suppose A_i and A_j are maximal k-split tori such that $A_i^- \subset A_j^-$. Then $A_i \leq A_j$ if $\dim(A_i^-) \leq \dim(A_j^-)$. As with maximal tori, all maximal k-split are G_k-conjugate to tori with this property, so the order extends to all maximal k-split tori. Thus each representative of an H_k-conjugacy class of θ-stable maximal k-split tori with the same dimension of its split component correspond to nodes in the same level in the diagram associated to the order on the I-poset.

Example 3.4.5. Consider $\mathrm{SL}(2, \mathbb{Q}_p)$, $p \equiv 1 \mod 4$. Let $\theta = \mathrm{Int}(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$. Then by [4] we know that there are four H_k-conjugacy classes of θ-stable maximal $(\theta, \mathbb{Q}_p)$-split tori and one H_k-conjugacy classes of θ-stable, θ-fixed maximal $\mathbb{Q}_p$-split tori. Thus the diagram for the I-poset is:

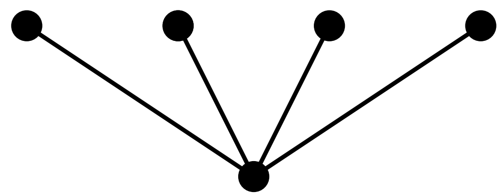

Figure 3.4.1: I-poset diagram for $\mathrm{SL}(2, \mathbb{Q}_p)$, $p \equiv 1 \mod 4$

This diagram can be expanded to the order on the orbits given by closure in the Zariski topology. Each node in the I-poset diagram is expanded to the number of orbits corresponding to the torus A_i, obtained by looking at $|W_{G_k}(A_i)/W_{H_k}(A_i)|$.

Example 3.4.6. Consider the setting of the previous example. Each $(\theta, \mathbb{Q}_p)$-split torus corresponds to one orbit, while the θ-fixed torus corresponds to two. Thus the orbit diagram is given in Figure 3.4.2.

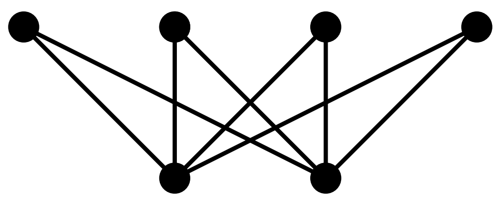

Figure 3.4.2: Orbit diagram for $\mathrm{SL}(2, \mathbb{Q}_p)$, $p \equiv 1 \mod 4$

Chapter 4

Properties of Tori

In this chapter we collect some properties of tori that will be relevant to the proof of the main result. Of particular interest is the concept of standard tori, which align the θ-fixed and θ-split portions of a torus.

4.1 Algebraically Closed Fields

This case was studied in [15] and will be related to the description of the I-poset for image of the generalized complexification map.

4.1.1 Standard Pairs

Let $\mathscr{T}$ denote the set of θ-stable maximal tori in G.

Definition 4.1.2. Let $T_1, T_2 \in \mathscr{T}$. Then (T_1, T_2) is called a standard pair if $T_1^- \subset T_2^-$ and $T_2^+ \subset T_1^+$. T_1 is said to be standard with respect to T_2.

The θ-stable maximal tori of G can be arranged in to a chain of standard pairs.

Proposition 4.1.3 ([15]). *(a) Let $T_1, T_2 \in \mathscr{T}$ such that $T_2^+ \subset T_1^+$. Then there exists $x \in Z_H(T_2^+)$ such that (T_1, xT_2x^{-1}) is a standard pair.*

(b) Let $T_1, T_2 \in \mathscr{T}$ such that $T_1^- \subset T_2^-$. Then there exists $x \in Z_H(T_1^-)$ such that (T_1, xT_2x^{-1}) is a standard pair.

(c) If T_1^+ and T_2^+ (resp. T_1^- and T_2^-) are H-conjugate, then so are T_1 and T_2.

The proof of the lemma makes use of the following theorem, which will be of use in its own right.

Theorem 4.1.4. *All maximal θ-split tori of G are conjugate under H and so are all maximal tori containing a maximal θ-split torus.*

Example 4.1.5. Let $G = \mathrm{SL}(3, \mathbb{C})$ with involution $\theta(g) = (g^T)^{-1}$, then $H = \mathrm{SO}(3, \mathbb{C})$. Let T denote the torus consisting of diagonal matrices. Consider the torus

$$T_1 = \left\{ \begin{pmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix} \middle| \, (a^2 - b^2)c = 1 \right\}$$

Then T, T_1 are maximal θ-split tori. Let $g = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H$. Then $gT_1g^{-1} = T$.

4.1.6 $\boldsymbol{\theta}$-singularity

An involution θ of a connected reductive group M is called split if there exists a θ-split maximal torus of M.

Definition 4.1.7. Let $T \in \mathscr{T}$ and $w \in W(T)$ such that $w^2 = \mathrm{id}$ and $\theta w = w\theta$. Let $G_w = Z_G(T_w^+)$. Then w is called θ-singular if

(a) $\theta|_{[G_w, G_w]}$ is split

(b) $\mathrm{rank}([G_w, G_w]) = \mathrm{rank}([G_w, G_w] \cap H)$

A root $\alpha \in \Phi(T)$ is called θ-singular if its corresponding reflection s_α is θ-singular.

θ-singular roots are precisely the roots of $\Phi(T)$ for which the corresponsing root groups $G_\alpha := G_{s_\alpha}$ contain both a θ-split and a θ-fixed torus. This means that inside the root group one can ‘flip’ a torus from θ-split to θ-fixed and vice versa.

For a maximal torus T, the roots $\Phi(T)$ can be classified in analogy with the case $k = \mathbb{R}$. θ acts on the Weyl group, and therefore on the reflections corresponding to the roots. Thus θ acts on Φ. If $\theta(\alpha) = -\alpha$ then α is called a real root, if $\theta(\alpha) = \alpha$ then

α is called imaginary, and α is called complex if $\theta(\alpha) \neq \pm\alpha$. If α is imaginary and θ-singular then it is called imaginary noncompact, if imaginary α is not θ-singular it is called imaginary compact.

Definition 4.1.8. Two roots $\alpha,\ beta \in \Phi(T)$ are called strongly orthogonal if $(\alpha, \beta) = 0$ and $\alpha \pm \beta \notin \Phi(T)$.

Theorem 4.1.9. *Let T be a θ-stable maximal torus of G and $\Psi = \{\alpha_1, \ldots, \alpha_r\} \subset \Phi(T)$ a set of strongly orthogonal roots. Let $G_\Psi = G_{\alpha_1} \cdots G_{\alpha_r}$. Then*

$$[G_\Psi, G_\Psi] = \prod_{i=1}^{r} [G_{\alpha_i}, G_{\alpha_i}]$$

Moreover, if $\alpha_1, \ldots, \alpha_r$ are θ-singular, then $\theta|_{[G_\Psi, G_\Psi]}$ is split and $\operatorname{rank}([G_\Psi, G_\Psi]) = \operatorname{rank}([G_\Psi, G_\Psi] \cap H)$.

4.2 Non-algebraically Closed Fields

This more general setting was the subject of [16], for our purposes it will be used to determine the domain of the generalized complexification map restricted to the I-poset.

4.2.1 Standard Pairs

Let $\mathscr{A}$ denote the set of θ-stable maximal k-split tori.

Definition 4.2.2. Let $A_1, A_2 \in \mathscr{A}$. Then (A_1, A_2) is called a standard pair if $A_1^- \subset A_2^-$ and $A_2^+ \subset A_1^+$. A_1 is said to be standard with respect to A_2.

The θ-stable maximal k-split tori can be arranged in to chains of standard pairs as in the algebraically closed case.

Theorem 4.2.3. *Let A be a maximal θ-stable k-split torus with A^- maximal (θ, k)-split. Then there exists a θ-stable maximal k-split torus S standard with respect to A such that S^+ is a maximal k-split torus of H.*

4.2.4 (θ, k)-singularity

The one dimensional subgroups containing both a θ-fixed k-split torus and a (θ, k)-split torus depend heavily on the k-structure of the group but can still be parameterized by tori. Therefore we refine the definition of the previous section to describe these groups. An involution defined over k of a connected reductive group M is called split if there exists a maximal (θ, k)-split torus.

Definition 4.2.5. Let $A \in \mathscr{A}$ and for each $\alpha \in \Phi(A)$ let $\ker(\alpha) = \{a \in A \mid s_\alpha(a) = a\}$. Set $G_\alpha = Z_G(\ker(\alpha)$. Then α is called (θ, k)-singular if

(a) $\theta|_{[G_\alpha, G_\alpha]}$ is split

(b) $\operatorname{rank}([G_\alpha, G_\alpha]) = \operatorname{rank}([G_\alpha, G_\alpha] \cap H)$

Theorem 4.2.6. *Let A be a θ-stable maximal k-split torus of G and $\Psi = \{\alpha_1, \dots, \alpha_r\} \subset \Phi(A)$ a set of strongly orthogonal roots. Let $G_\Psi = G_{\alpha_1} \cdots G_{\alpha_r}$. Then*

$$[G_\Psi, G_\Psi] = \prod_{i=1}^{r} [G_{\alpha_i}, G_{\alpha_i}]$$

Moreover, if $\alpha_1, \dots, \alpha_r$ are (θ, k)-singular, then $\theta|_{[G_\Psi, G_\Psi]}$ is k-split and k-$\operatorname{rank}([G_\Psi, G_\Psi]) = k$-$\operatorname{rank}([G_\Psi, G_\Psi] \cap H)$.

Chapter 5

Generalized Complexification

Consider the orbits $B\backslash G/H$ over an algebraically closed field. These are the orbits of a minimal parabolic subgroup acting on a symmetric variety, which can be related to the orbits $P_k\backslash G_k/H_k$ of a minimal parabolic k-subgroup acting on a symmetric k-variety. A description of how the algebraically closed orbits break up over a subfield is a fundamental question related to the representation theory of the symmetric k-varieties. In this chapter we approach this problem from the reverse angle, namely by embedding the orbits over a subfield k into the orbits over its algebraic closure $\bar{k}$. When $k = \mathbb{R}$ this process is the complexification of the real orbits, thus we call the map yielded by the embedding generalized complexification.

The primary goal is to obtain a condition for which the generalized complexification map is surjective. The approach taken here will not yield a full description of how algebraically closed orbits break up over a subfield, this problem requires a full characterization of the H_k-conjugacy classes of θ-stable maximal k-split tori that has not yet been completed. However, a study of the generalized complexification map requires only knowledge of the H-conjugacy classes of θ-stable maximal tori, as shall be described in this chapter.

5.1 The Generalized Complexification Map

Let $P \subset G$ be a minimal parabolic k-subgroup and $A \subset P$ a maximal k-split torus. We define the generalized complexification map:

$$\varphi : P_k \backslash G_k / H_k \to P \backslash G / H$$
$$P_k g H_k \mapsto PgH$$

Recall from Theorem 3.4.2 that there are several equivalent characterization of the double cosets $P_k \backslash G_k / H_k$. The generalized complexification map φ induces maps across all of these equivalent formulations. Let A be a maximal k-split torus contained in P, then given $v \in V_k$, let $x(v)$ be representative in $N_{G_k}(A)$ such that $v = Z_{G_k}(A)x(v)H_k$. Then we have an induced map:

$$\varphi_V : V_k \to V$$
$$Z_{G_k}(A)x(v)H_k \mapsto Z_G(A)x(v)H$$

5.2 k-split Groups

For the remainder we will concern ourselves only with the case of k-split groups. This restriction will allow us to simplify the description of the orbits of minimal parabolic k-subgroups acting on symmetric k-varieties. This allows for the

Let G be a k-split group. Then minimal parabolic k-subgroups are Borel k-subgroups. In this case we have a simpler generalized complexification map:

$$\varphi : B_k \backslash G_k / H_k \to B \backslash G / H$$
$$B_k g H_k \mapsto BgH$$

The corresponding induced maps are also simpler in this case since the maximal k-split tori are in fact maximal tori. Let A be a maximal k-split torus and $x(v) \in N_G(A)$ Thus the generalized complexification of orbits corresponding to the $B_k \times H_k$ action on G_k

becomes:

$$\varphi : V_k \to V$$
$$A_k x(v) H_k \mapsto A x(v) H$$

The greatestest simplification, however, occurs in the induced map among the union of quotients of Weyl groups. Let $\{A_i\}_{i\in I}$ be a set of representatives of the H_k-conjugacy classes of θ-stable maximal k-split tori. Then $\{A_i\}_{i\in I}$ corresponds to $\{B_i\}_{i\in I'}$, a set of representatives for the H-conjugacy classes of θ-stable maximal tori. This is done in the following manner. Among the $\{A_i\}$ that correspong to the same H-conjugacy class of θ-stable maximal tori, a representative is chosen. This set is then extended with arbitary representatives of the H-conjugacy classes of θ-stable maximal tori not obtained from the $\{A_i\}$. Therefore the generalized complexification map acts as the identity:

$$\varphi : \bigcup_{i\in I} W_{G_k}(A_i)/W_{H_k}(A_i) \to \bigcup_{i\in I'} W_G(A_i)/W_H(A_i)$$
$$gW_{H_k}(A_i) \mapsto gW_H(A_i)$$

Remark 5.2.1. For groups that are not k-split, φ still induces a map on the union of Weyl group quotients. This map is more complicated and involves the introduction of another quotient.

5.3 Some Examples

In general, the surjectivity of the generalized complexification map depends on both the choice of involution θ and th field of definition k. The first example of this section illustrates the dependence of the surjectivity of the generalized complexification map on the choice of involution.

Example 5.3.1. We return to the setting of Example 3.4.3, namely let $G = \mathrm{SL}(2, \mathbb{C})$ with real form $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$. Note that in this case G is $\mathbb{R}$-split. Let T denote the set of diagonal matrices and $P = B$ the set of upper triangular matrices.

(a) Let $\sigma = \mathrm{Int}(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$. Then from Example 3.4.3 we have that $|B_{\mathbb{R}}\backslash G_{\mathbb{R}}/H_{\mathbb{R}}| = 4$. Two orbits correspond to each representative of the $H_{\mathbb{R}}$-conjugacy class of σ-stable maximal

$\mathbb{R}$-split tori, denote the orbits corresponding to T by $\mathscr{O}_1$ and $\mathscr{O}_2$. There is only one orbit corresponding to T over $\mathbb{C}$, therefore the complexification maps $\mathscr{O}_1$ and $\mathscr{O}_2$ to the same orbit over $\mathbb{C}$. Consider $\varphi : B_{\mathbb{R}}\backslash G_{\mathbb{R}}/H_{\mathbb{R}} \to B\backslash G/H$. We can represent the action of φ diagrammatically, as in Figure 5.3.1.

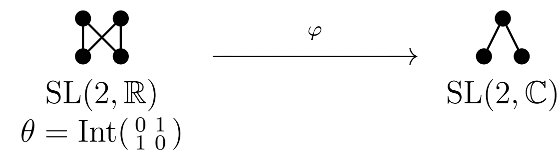

Figure 5.3.1: Generalized complexification of $\mathrm{SL}(2,\mathbb{R})$, $\theta = \mathrm{Int}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$

(b) Let $\theta = \mathrm{Int}\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. Then from Example 3.4.3 we have that $|B_{\mathbb{R}}\backslash G_{\mathbb{R}}/H_{\mathbb{R}}| = 1$. The complexification map has a cokernel, indicated by the empty nodes in Figure 5.3.2

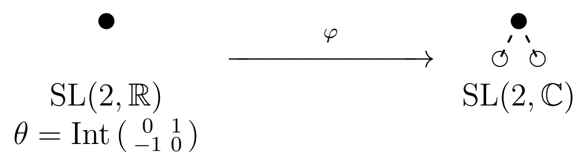

Figure 5.3.2: Generalized complexification of $\mathrm{SL}(2,\mathbb{R})$, $\theta = \mathrm{Int}\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$

Recall that there infinite number of orbits $B_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/H_{\mathbb{Q}}$ for $G_{\mathbb{Q}} = \mathrm{SL}(2,\mathbb{Q})$. The complexification of these orbits is quite similar to the complexification of the real orbits, as shown in Figure 5.3.3 and Figure 5.3.4.

Furthermore, the surjectivity is also determined by the nature of the base field k.

Example 5.3.2. Let $k = \mathbb{Q}(i)$, $G_k = \mathrm{SL}(2,k)$, and $\theta = \mathrm{Int}\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. Then there are still an infinite number of H_k-conjugacy classes of (θ,k)-split tori. However, the fixed point group $H = \mathrm{SO}(2,k)$ is diagonalizable via the matrix

$$\begin{pmatrix} \frac{1}{2}-\frac{1}{2}i & -\frac{1}{2}+\frac{1}{2}i \\ \frac{1}{2}+\frac{1}{2}i & \frac{1}{2}+\frac{1}{2}i \end{pmatrix}$$

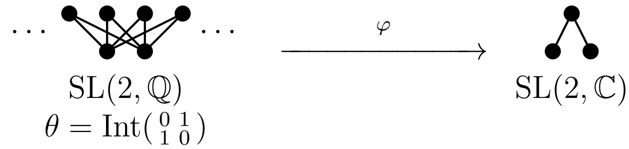

Figure 5.3.3: Generalized complexification of $\mathrm{SL}(2,\mathbb{Q})$, $\theta = \mathrm{Int}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$

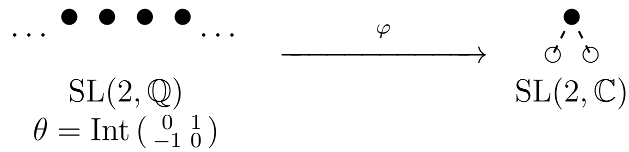

Figure 5.3.4: Generalized complexification of $\mathrm{SL}(2,\mathbb{Q})$, $\theta = \mathrm{Int}\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$

This torus has only one Weyl group element with representatives in H, so there are two orbits corresponding to a θ-fixed torus. The complexification diagram is given in Figure 5.3.5.

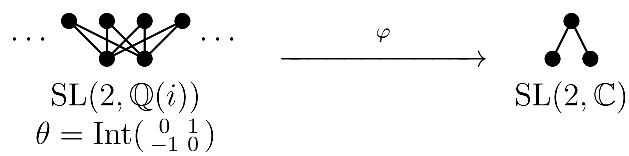

Figure 5.3.5: Generalized complexification of $\mathrm{SL}(2,\mathbb{Q}(i))$, $\theta = \mathrm{Int}\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$

5.4 Double Cosets of Isomorphic Involutions

We begin the study of the surjectivity of the generalized complexification map by reducing the complexity of the problem. The examples of Section 5.3 demonstrate the dependence of surjectivity on the involution, therefore we demonstrate that the double coset decompostion is dependent only the isomorphy class of involution.

Proposition 5.4.1. *Let $\theta = \gamma\sigma\gamma^{-1} \in \mathrm{Aut}(G, G_k)$ be isomorphic involutions with fixed point groups $H_1 = G^\theta$ and $H_2 = G^\sigma$. Then $H_2 = \gamma^{-1}(H_1)$.*

Proof. Let $h \in H_1$. Then $\sigma(\gamma^{-1}(h)) = \gamma^{-1}\theta\gamma(\gamma^{-1}(h)) = \gamma^{-1}\theta(h) = \gamma^{-1}(h)$. Therefore $\gamma^{-1}(H_1) \subset H_2$. Since γ is one-to-one, we have that $H_2 = \gamma^{-1}(H_1)$. $\square$

Corollary 5.4.2. *Assume the hypotheses of Proposition 5.4.1. If θ, γ are* $\mathrm{Int}(G)$*-isomorphic, then H_1, H_2 are conjugate.*

Proof. Apply Proposition 5.4.1 to the case $\gamma = \mathrm{Int}(g)$ for some $g \in G$. $\square$

Theorem 5.4.3. *Let θ, $\sigma \in \mathrm{Aut}(G)$ be involutions isomorphic by an element of* $\mathrm{Int}(G)$*. If $H_1 = G^\theta$ and $H_2 = G^\sigma$, then θ and σ admit isomorphic double coset decompositions $B_1\backslash G/H_1 \cong B_2\backslash G/H_2$. Furthermore, B_1 and B_2 are G-conjugate.*

Proof. Suppose $\theta = \mathrm{Int}(g)\sigma\,\mathrm{Int}(g)^{-1}$. From Corollary 5.4.2 we have that $H_2 = g^{-1}H_1g$. Let $B_2 = g^{-1}B_1g$. Given a double coset $B_1xH_1 \in B_1\backslash G/H_1$, we compute $\mathrm{Int}(g^{-1})(B_1xH_1) = \mathrm{Int}(g^{-1})(B_1)\,\mathrm{Int}(g^{-1})(x)\,\mathrm{Int}(g^{-1})(H_1) = B_2g^{-1}xgH_2 \in B_2\backslash G/H_2$. The inverse map is given by $\mathrm{Int}(g)$, so we have $B_1\backslash G/H_1 \cong B_2\backslash G/H_2$. $\square$

The isomorphy of the double coset decompositions extends to k-isomorphy, using the same proofs.

Theorem 5.4.4. *Let θ, $\sigma \in \mathrm{Aut}(G, G_k)$ be* $\mathrm{Int}(G, G_k)$*-isomorphic involutions with fixed point groups H_1 and H_2 respectively. Then θ and σ admit k-isomorphic double coset decompositions $(B_1)_k\backslash G_k/(H_1)_k \cong (B_2)_k\backslash G_k/(H_2)_k$*

5.5 Reduction to the *I*-poset

Recall that in light of Theorem 3.4.2, the double cosets $B_k\backslash G_k/H_k$ are parameterized by the H_k-conjugacy of θ-stable maximal k-split tori. The H_k-conjugacy classes have not been fully classified except in a number of specific cases, notably $\mathrm{SL}(2, k)$. However, for algebraically closed fields a complete classfication was achieved in [15]. Fortunately, the characterization of surjectivity of the generalized complexification map depends only on the H-isomorphy classes of θ-stable maximal tori, which we show in this section.

The following is a result of Borel [6, 7].

Lemma 5.5.1. *Let A be a k-torus of G. Then $W_{G_k}(A) = W_G(A)$.*

If we fix θ-stable maximal k-split tori A, we can restrict the generalized complexification map

$$\varphi : \bigcup_{i \in I} W_{G_k}(A_i)/W_{H_k}(A_i) \to \bigcup_{i \in I'} W_G(A_i)/W_H(A_i)$$

to the Weyl group quotient corresponding to A:

$$\varphi_A : W_{G_k}(A)/W_{H_k}(A) \to W_G(A)/W_H(A) \tag{5.1}$$

Then we can consider the surjectivity of φ_A. The following lemma shows that this map is in fact surjective in all cases.

Lemma 5.5.2. *The map φ_A of Equation 5.1 is surjective.*

Proof. It is clear that $W_{H_k}(A) \subset W_H(A)$. Given $gW_H(A) \in W_G(A)/W_H(A)$, the fiber is nonempty since g has a representative in $W_{G_k}(A)$. $\square$

Therefore surjectivity of the map between indexing sets of θ-stable maximal tori is sufficient to ensure surjectivity of the generalized complexification map.

5.6 Cayley Transforms

Having restricted our attention to the I-poset, we now develop tools for working within the I-poset. Over the real numbers Cayley transforms have been used to construct a new torus from given torus that differs by one dimension in the θ-split and θ-fixed compenents of the torus.

5.6.1 Cayley Transforms for $k = \mathbb{R}$

In this section we summarize the construction in [24]. We begin by fixing $G = \mathrm{SL}(2, \mathbb{C})$. Note that the Lie algebra of G is $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, the set of trace zero 2×2 matrices. Choose the standard basis vectors for $\mathfrak{g}$, $\{h, e, f\}$. If we restrict the span to real multiples we have a basis for $\mathfrak{g}_\mathbb{R} = \mathfrak{sl}(2, \mathbb{R})$. Consider the alternate basis $\{\tilde{h}, \tilde{e}, \tilde{f}\}$ where

$$\tilde{h} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \tilde{e} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \ \tilde{f} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Observe that $\tilde{h}$ is a maximal Cartan subalgebra and is fixed by the involution $\theta = \mathrm{Int}(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$., while h is the maximal Cartan subalgebra that is mapped to its negative under θ. The Cayley transform is the map $c_\beta = \mathrm{Ad}(exp(\frac{\pi}{4})(\tilde{f} - \tilde{e}))$. Computing, we find $c_\beta(\tilde{h}) = h,\ c_\beta(\tilde{e}) = e,\ c_\beta(\tilde{f}) = f$. Therefore the Cayley transform maps a θ-stable maximal $\mathbb{R}$-split Cartan subalgebra to a $(\theta, \mathbb{R})$-split maximal Cartan subalgebra.

5.6.2 The General Cayley Transform

Here we generalize this notion to arbitrary fields, but the construction is quite similar to the real case. First we want to lift to the group level, where the adjoint action in the Lie algebra is replaced by conjugation.

Fix a maximal (θ, k)-split torus A of G. Then A has a root system $\Phi(A)$ and for each $\alpha \in \Phi(A)$ we can define the root group $G_\alpha = Z_G(\ker(\alpha))$. Then the commutator $[G_\alpha, G_\alpha]$ is a semisimple group of rank 1, isomorphic to $\mathrm{SL}(2, k)$. Therefpre $\theta|_{[G_\alpha, G_\alpha]}$ is inner. If $[G_\alpha, G_\alpha]$ contains a nontrivial θ-fixed torus S, we define a map $\eta = \mathrm{Int}\, \frac{\sqrt{2}}{2} (\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix})$ that acts on S. In analogy with the real case, η maps S to a θ-split torus of $[G_\alpha, G_\alpha]$.

Remark 5.6.3. It may be the case that $\sqrt{2} \notin k$, in which case the action of η does not necessarily preserve the k-rational points. However, since all maximal k-split tori are G_k-conjugate, there exists an element of G_k whose action is equivalent to that of η, in which case we call conjugation by the latter element the Cayley transform.

Lemma 5.6.4. *S is k-split if and only if $\theta|_{[G_\alpha, G_\alpha]} \cong \mathrm{Int}(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$*

Proof. Since $[G_\alpha, G_\alpha] \cong \mathrm{SL}(2, k)$, this follows directly from Theorem 2.4.3. $\square$

Furthermore, S is k-split implies $\eta(S)$ is also k-split. We extend the action of η to the rest of the torus trivially.

Example 5.6.5. Let $G = \mathrm{SL}(4, \mathbb{C})$ with $\mathbb{R}$-form $\mathrm{SL}(4, \mathbb{R})$ and

$$\theta(A) = \mathrm{Int} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ((A^T)^{-1}) \text{ for all } A \in G$$

. Consider the torus

$$S = \left\{ \begin{pmatrix} a & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \right\}$$

Then $\dim(S^+) = 1$ and $\dim(S^-) = 2$. Consider the $\alpha \in \Phi(S)$ such that $\theta(\alpha) = \alpha$. Then $[G_\alpha, G_\alpha] \cong \mathrm{SL}(2, \mathbb{R})$ which contains the $\theta|_{[G_\alpha, G_\alpha]}$-fixed torus $\tilde{S} = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right\}$. $\eta(\tilde{S}) = \left\{ \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix} \right\}$ which is $\theta|_{[G_\alpha, G_\alpha]}$-split. Extending to the whole torus we find

$$\eta(S) = \begin{pmatrix} a+b & 0 & 0 & 0 \\ 0 & a-b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

which is θ-split.

5.7 I-posets for $k = \bar{k}$

In order to discuss the surjectivity of the generalized complexification map we should have a description of the I-posets in the image of the generalized complexfication map, namely the I-posets for algebraically closed fields. This was carried out in [15].

Theorem 5.7.1. $\theta \in \mathrm{Int}(G)$ *if and only if* $\mathrm{rank}(G) = \mathrm{rank}(H)$.

5.8 Surjectivity of φ

We are now ready to prove the characterization of the surjectivity of φ, which will require several lemmas. We have already reduced the problem to surjectivity in the I-poset, we now show when this occurs. The main tool will be the Cayley transforms of Section 5.6.2, which allow us to move through the I-poset.

Lemma 5.8.1. *Suppose H contains a nontrivial k-split torus S. Then $\Phi(S)$ consists θ-singular imaginary noncompact roots.*

Proof. Consider $\alpha \in \Phi(S)$, then $\theta(\alpha) = \alpha$. Construct the corresponding root group $G_\alpha = Z_G(\ker(\alpha))$. Then by Lemma 5.6.4 we have that $\theta|_{[G_\alpha, G_\alpha]} \cong \mathrm{Int}(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$. Therefore $[G_\alpha, G_\alpha]$ contains a θ-fixed k-split torus then can be flipped to a (θ, k)-split torus in $[G_\alpha, G_\alpha]$ via the Cayley transform η. Therefore α is θ-singular and imaginary noncompact. Since the choice of α was arbitrary we have the result. $\square$

Remark 5.8.2. Given a k-split torus $S \subset G$ contained in H, the proof constructs a new torus that is standard to S whose split part is one dimension higher.

Corollary 5.8.3. *Let $S \subset H$ be a k-split torus. There exists a maximal orthogonal subset of roots of torus lying in H such that each root is (θ, k)-singular.*

We can iterate this process, performing successive Cayley transforms in the root groups of a set of strongly orthogonal roots.

Lemma 5.8.4. *Assume* $\mathrm{rank}(G) = n$ *and let $S \subset H$ be a k-split torus of H and suppose $\Psi(S) = \{\alpha_1, \dots, \alpha_r\} \subset \Phi(S)$ is a maximal set of strongly orthogonal roots. Then G contains a (θ, k)-split torus of dimension $n - r$.*

Proof. We use induction on r. We may assume the $\Psi(S)$ consists of (θ, k)-singular roots. The case $r = 1$ is carried out explicitly in the proof of Lemma 5.8.1. Now assume $r > 1$ and let ${}^{\alpha_1}S \subset H$ be the subtorus lying in $[G_{\alpha_1}, G_{\alpha_1}]$. Then $S = ({}^{\alpha_1}S)\tilde{S}$, where $\tilde{S} \subset S$ denotes the factor of S such that $[G_{\alpha_1}, G_{\alpha_1}] \cap \tilde{S} = \pm\,\mathrm{id}$. Then $\tilde{S}$ is a k-split torus in H so Lemma 5.8.1 applies and $|\Psi(S)| = r - 1$. $\square$

Lemma 5.8.5. *Let S be a maximal k-split torus of H. Then $Z_{G_k}(S)$ contains a maximal k-split torus of G_k.*

Proof. Let $S \subset H$ be a maximal k-split torus of H. Consider

$$A_1 = \left(\bigcap_{\substack{\alpha \in \Phi(T) \\ \alpha \perp \Phi(S)}} \ker(\alpha) \right)^{\circ}$$

Since the -1-eigenspace of θ is orthogonal to H, $\pi(A_1)$ (where π is the usual projection) contains a maximal torus of G/H, let A_1^- be the inverse image of this torus. Then there exists a subtorus of A_1^- that is maximal k-split in G/H. Therefore $S \cdot A_1^-$ is a maximal k-split torus of G. $\square$

Theorem 5.8.6. *Let G be a k-split group. Then the generalized complexification map φ is surjective if and only if $k\text{-rank}(G) = k\text{-rank}(H)$.*

Proof. First assume that φ is surjective and suppose $k\text{-rank}(G) \neq k\text{-rank}(H)$. Let S be a maximal k-split torus of H, then by assumption S is not maximal k-split. Let $\tilde{S} \subset Z_{G_k}(S)$ be the maximal k-split torus containing S. Since G is k-split we have that $\text{rank}(G) = k\text{-rank}(G)$. Furthermore since $k\text{-rank}(H) \leq \text{rank}(H)$, our assumption yields $\text{rank}(H) = \text{rank}(G)$, and thus by Theorem 5.7.1 we have that θ is inner. $\theta \in \text{Int}(G)$ implies H contains a maximal torus of G, denote this torus by A. $\tilde{S}$ cannot be H-conjugate to any torus lying in H, therefore it is not H_k-conjugate. A does not have have a premiage under generalized complexification in the I-poset, and therefore φ is not surjective.

Next assume that $k\text{-rank}(G) = k\text{-rank}(H)$. Then let S be a maximal k-split torus of H, and thus a maximal torus of G. Then the $\Phi(S)$ and and the restricted root system $\Phi_0(S)$ consist of the same roots, thus they have the same maximal orthogonal set of roots and all such roots are (θ, k)-singular. Thus we have surjectivty in the I-poset and thus φ is surjective. □

Definition 5.8.7. A quasi k-split torus of G is a torus conjugate under G to a k-split torus of G.

All maximal k-split tori of G are conjugate, therefore all quasi k-split tori of G are conjugate.

Let A be a θ-stable maximal k-split torus. We define:

$$\begin{aligned}
\mathcal{V}_A &= \{x \in G/|x\theta(x)^{-1} \in N_G(A)\} \\
\mathcal{V}_k &= \{x \in G_k|x\theta(x)^{-1} \in N_{G_k}(A)\} \\
V_A &= \{Z_G(A) \times H \text{ orbits on } \mathcal{V}_A\} \\
V_k &= \{Z_{G_k}(A) \times H \text{ orbits on } \mathcal{V}_k\} \\
\mathcal{A} &= \{\text{ maximal quasi } k\text{-split tori of } G\} \\
\mathcal{A}^\theta &= \{\theta\text{-stable maximal quasi } k\text{-split tori of } G\} \\
\mathcal{A}^\theta_0 &= \{A \in \mathcal{A}^\theta \text{ that are } H\text{-conjugate to a maximal } k\text{-split torus of } G\} \\
\mathcal{A}^\theta_k &= \{\theta\text{-stable maximal } k\text{-split tori of } G\}
\end{aligned}$$

The surjectivity of φ implies surjectivty of the induced maps. Consider

$$\varphi_V : V_k \to V, \text{ given by } Z_{G_k}(A)gH_k \mapsto Z_G(A)gH$$

Since each element of V_k corresponds to a double coset $B_k g H_k$, this map is closely related to the complexification map φ. φ_V is W-equivariant, which induces a map $\delta : V_k/W \to V_A/W$. By Proposition 12 in [18] we have bijections $\gamma : V_A/W \to \mathcal{A}^\theta/H$ and $\gamma_k : V_k/W \to \mathcal{A}_k^\theta/H_k$. Let $\gamma_0 = \gamma\big|_{V_0/W}$, then γ_0 maps V_0/W onto $\mathcal{A}_)^\theta/H$. Finally denote by ζ the map that takes H_k-conjugacy classes of maximal k-split tori to the H conjugacy classes of these tori. Hence we get the following diagram:

$$\begin{tikzcd} V_k/W \arrow[r, "\gamma_k"] \arrow[d, "\delta"] & \mathcal{A}_k^\theta/H_k \arrow[d, "\zeta"] \\ V_0/W \arrow[r, "\gamma_0"] & \mathcal{A}_0^\theta/H \end{tikzcd}$$

Corollary 5.8.8. *If φ is surjective, then φ_V is surjective.*

Proof. Let $G_1 = Z_G(S)$, $H_1 = G_1^\theta$. Denote by $G_{1,k}$ (resp. $H_{1,k}$) the k-rational points of G_1 (resp. H_1). We will use the isomorphisms $B_k\backslash G_k/H_k \cong V_k$ and $B\backslash G/H \cong V_A$. Note that φ induces a complexification map from V_k to V_A that is surjective if and only if φ is. Consider an orbit v of G_1. Then there exists a maximal quasi k-split torus A of G_1 and an element $g \in G_1$ such that $v = Z_{G_1}(A)gH_1$. To this orbit we can associate an orbit of G, namely $v_1 = Z_G(A)gH$. Since the complexification map $G_k \to G$ is complete we have that $\mathrm{im}(\eta) = V_A$, therefore we can find an orbit (possibly renaming g) $v_2 = Z_{G_k}(A)gH_k$ (need to check independence of representative). Observe that $G1, k \subset G_k$, $A \subset A_1$, and $H_{1,k} \subset H_k$. Then appropriate restrictions are defined, so we can associate an orbit inside of G_1 to v_2, call it $v_3 = Z_{G_{1,k}}(A)gH_{1,k}$. Then the complexification of v_3 is v, hence surjectivity. $\square$

Example 5.8.9. Let $G = \mathrm{SL}(n, \mathbb{C})$, $G_\mathbb{R} = \mathrm{SL}(n, \mathbb{R})$, $\theta(g) = (g^T)^{-1}$ for all $g \in G$. Then $H_\mathbb{R} = \mathrm{SO}(2, \mathbb{R})$, which is compact. Therefore $\mathbb{R}$-rank$(H) = 0$, so φ is not surjective. In fact, from the Iwasawa decomposition we can deduce the following generalized complexification diagram on the I-poset:

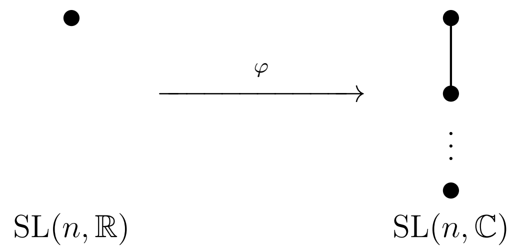

Figure 5.8.1: Generalized complexification of $\mathrm{SL}(n,\mathbb{R})$, $\theta(g) = (g^T)^{-1}$

5.9 Explicit Surjectivity

It remains to determine which combinations of group, field, and involution yield surjective generalized complexification. This requires an explicit description of the fixed point groups of involutions and the k-split tori therein for k-split groups. Such descriptions exist for the split forms of all semisimple algebraic groups except E_7 and E_8.

5.9.1 Type $\mathbf{A}_{n-1}$

Let $G = \mathrm{SL}(n,\bar{k})$, $G_k = \mathrm{SL}(n,k)$. For θ inner, Table 2.3.2 gives the isomorphy classes of involutions. By fixing a field, we can then treat each isomorphy class of involutions separately to determine surjectivity.

$\theta = \mathrm{Int}(I_{n-i,i})$

Recall that $\mathrm{Int}(I_{n-i,i}) \cong \mathrm{Int}(\mathcal{I}_{n-i,i})$, and that the fixed point group was given in Lemma 2.4.5. In light of Theorem 5.8.6 surjectivity is determined completely by the dimension of a maximal k-split torus contained in H. It was shown in [4] that a maximal k-split torus of H in this case is given by:

$$A = \begin{pmatrix} B & 0 & \dots & 0 \\ 0 & x_1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & x_{n-2i} \end{pmatrix}$$

where B is the $2i \times 2i$ matrix:

$$\begin{pmatrix} a_1 & 0 & \dots & \dots & 0 & b_1 \\ 0 & \ddots & & & \iddots & 0 \\ \vdots & & a_i & b_i & & \vdots \\ \vdots & & b_i & a_i & & \vdots \\ 0 & \iddots & & & \ddots & 0 \\ b_1 & 0 & \dots & \dots & 0 & a_1 \end{pmatrix}$$

Then $\dim(A) = n - 1 = k\text{-rank}(G)$, hence we have that the generalized complexification map is surjective in this case. Note that this result is independent of the field k.

$n = 2m$, $\theta = \mathrm{Int}(L_{2m,x})$

The fixed point group is given in Lemma 2.4.6. Recall that $x \not\equiv 1 \mod (k^*)^2$. The maximal k-split torus of H is then:

$$A = \left\{ \mathrm{diag}\{a_1, a_1, \dots, a_{\frac{n}{2}} a_{\frac{n}{2}}\}, \;\middle|\; a_1^2 \cdots a_{\frac{n}{2}}^2 = 1 \right\}$$

Therefore we do not have surjectivity in these cases. Consider the centralizer in H_k of A:

$$Z_{H_k}(A) = \begin{pmatrix} K_1 & 0 & \dots & 0 \\ 0 & K_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & K_{\frac{n}{2}} \end{pmatrix}$$

where each K_i is a 2×2 matrix:

$$\begin{pmatrix} a_i & b_i \\ xb_i & a_i \end{pmatrix}$$

Then $Z_{H_k}(A)$ is diagonalizable over $\tilde{k} = k(\sqrt{x})$ since its eigenvalues are $a_i \pm b_i\sqrt{x}$, $i = 1,\ 2,\ \dots,\ \frac{n}{2}$. Thus we have surjectivity of the generalized complexification over the quadratic extension field $\tilde{k}$.

5.10 Centralizer Lemma

Surjectivty of generalized complexification permeates much of the structure of the group, and in particular it implies that surjectivity can be restricted to centralizers of tori, which we show in this section. This allows for the use of inductive arguments on the double cosets $B_k \backslash G_k / H_k$ of a k-split group, which are a key facet of many results in the case of algebraically closed fields.

Lemma 5.10.1. *Let G be a k-split group, suppose the generalized complexification map $\varphi : B_k \backslash G_k / H_k \to B \backslash G / H$ is surjective, and let $A \subset G$ be a k-split torus. Define $G_1 = Z_G(A)$, $B_1 \subset G_1$ a Borel sugroup, and $H_1 = H \cap G_1$. Then the restriction $\varphi|_{G_1}$ is surjective.*

Proof. G_1 is connected and reductive since A is a k-split torus. Moreover, B_1 is contained in a Borel subgroup of G, i.e. $B_1 = B \cap G_1$ for B a Borel subroup, and therefore $(B_1)_k = (B \cap G_1)_k$. Thus the orbits $(B_1)g(H_1) \in (B_1) \backslash (G_1)/(H_1)$ can be embedded in the orbits $B_k \backslash G_k / H_k$. Therefore there is a preimage of $B_1 g H_1$ in $B_k \backslash G_k / H_k$. Then $\varphi^{-1}(B_1 g H_1) \cap (G_1)_k$ is nonempty, so surjetivity is achieved.

$\square$

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