ABSTRACT

MISKIEWICZ, MATTHEW NILE. Computer Generated Geometric Phase Holograms. (Under the direction of Michael Escuti.)

This dissertation concerns the fabrication, analysis, and simulation of computer generated geometric phase holograms (CGHs). The current knowledge of CGHs is advanced to enable the creation of new sophisticated optical elements with unique characteristics. These elements enable new technologies related to displays, astronomy, sensing, beam-steering, beam-shaping, and more.

First, a novel direct-write system for CGH creation is presented. A mathematical description of the system is developed which allows the result of a given scan pattern to be predicted. The accuracy of the model is validated with various scan patterns, then a high-quality direct-write polarization grating and q-plate are fabricated for the first time.

With a system capable of creating CGHs, the most common and useful CGHs are explored in depth: the polarization grating, the geometric phase lens, and the Fourier geometric phase hologram. For each element, the possible scan patterns and parameters and their effect on the resulting element’s quality are studied. Ultimately, the optimal scan patterns and parameters are found, then best-quality elements of each type are created and characterized.

Finally, a new tool for simulating periodic CGHs is developed. This begins with the derivation of the algorithm, which is based on the finite-difference time-domain (FDTD) method. Next tool’s capabilities are verified by simulating many test structures and comparing the results to known solutions. The tool is used to simulate, for the first time, a CGH multiple beam splitter and a GPL array.
Computer Generated Geometric Phase Holograms

by
Matthew Nile Miskiewicz

A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Electrical Engineering

Raleigh, North Carolina
2014

APPROVED BY:

________________________  ________________________
John Muth               Michael Kudenov

________________________  ________________________
Russell Philbrick        Michael Escuti
Chair of Advisory Committee
BIOGRAPHY

Matthew Miskiewicz was born and raised in Clearwater Florida. After moving to North Carolina, he attended Wake Technical Community College for a semester before joining North Carolina State University (NCSU). Initially enrolled in civil engineering, he switched to electrical and computer engineering (ECE) where he stayed for the remainder of his collegiate education.

While working towards his undergraduate degree, he worked for Duke Energy (then Progress Energy) assisting with probability risk analysis, fire protection, and electrical metering. While there, he learned many valuable business-related skills and also increased his technical skill set.

After obtaining B.S. degrees in both electrical engineering and computer engineering, Matthew entered the graduate program at NCSU and soon joined the Opto-Electronics and Lightwave Engineering Group. There he participated in numerous projects, collaborating with industry partners and other university research groups including Boulder Nonlinear Systems, Raytheon, ImagineOptix, Technical University of Eindhoven, University of Leiden, and many researchers at NCSU. Through these research projects, he learned the fundamentals of conducting high-quality research. In addition, while a graduate student he was an officer of the electrical engineering honor society for three years and became a prominent member of the NCSU ECE student body.

After graduating, he plans to try his hand at running a software company making applications for mobile devices. In his spare time, he enjoys disc golf, electronic music production, juggling, programming, and the occasional video game.
ACKNOWLEDGEMENTS

First, I must thank God for creating this world for us to be part of, and for giving us the tools and abilities to study and learn from it. I would like to thank Dr. Escuti for the guidance, support, and commitment he has given me over the years. I would also like to thank my fellow OLEG members for putting up with my constant questions and crazy ideas, and helping me whenever needed. Next, I want to give a big thanks to my parents for always encouraging me through this journey. Finally, I would like thank all the various researchers I have worked with, both at NCSU and elsewhere, who have made this experience that much better.
TABLE OF CONTENTS

**LIST OF TABLES** .......................................................... v

**LIST OF FIGURES** ....................................................... vi

Chapter 1 Introduction .................................................. 1
  1.1 Light in Everyday Life .............................................. 1
  1.2 Geometric Phase Holograms .......................................... 2
  1.3 Dissertation Structure ............................................. 3
  1.4 Publications From This Work ....................................... 4

Chapter 2 Background .................................................. 6
  2.1 Geometric Phase Holograms ......................................... 6
    2.1.1 Geometric Phase ................................................ 6
    2.1.2 Summary of Important GPH Concepts ................................. 10
    2.1.3 Polarization Grating ............................................ 10
    2.1.4 GPH Fabrication Techniques ...................................... 11
  2.2 Liquid Crystals .................................................... 12
    2.2.1 Physical Properties of Liquid Crystals ............................ 12
    2.2.2 Alignment of Liquid Crystals ..................................... 15
    2.2.3 Fabrication of LC Films ......................................... 16
  2.3 Electromagnetic Waves ............................................ 17
    2.3.1 The Electromagnetic Wave Equations ............................... 17
    2.3.2 Infinite Plane Wave Solution .................................... 18
    2.3.3 Diffraction ..................................................... 19
    2.3.4 General Solution ................................................. 19
  2.4 Electromagnetic Finite-Difference Time-Domain Simulation ........... 20
    2.4.1 Mathematical Basis ............................................. 21
    2.4.2 The Yee Cell .................................................. 22
    2.4.3 FDTD Summary .................................................. 23

Chapter 3 Direct Write System for CGH Creation ......................... 24
  3.1 Introduction ....................................................... 24
  3.2 System Design ..................................................... 25
  3.3 System Description ................................................ 27
    3.3.1 Approach ........................................................ 27
    3.3.2 System Inputs and Electromagnetic Output ....................... 28
    3.3.3 Adjusted Electromagnetic Output ................................ 28
    3.3.4 LC Response ................................................... 30
    3.3.5 Physical Interpretation of $H$ and $T$ ............................ 31
  3.4 System Description Validation ...................................... 31
    3.4.1 Fluence Threshold ............................................. 33
    3.4.2 DOLP Threshold ................................................. 34
    3.4.3 Alignment to Average Polarization Angle ........................ 34
LIST OF TABLES

Table 4.1 Summary of PG non-idealities (top) and corresponding causes (left). 59
Table 4.2 Summary of the deduced causes of non-idealities in fabricated samples. 59
Table 5.1 Scan parameters used for fabricated lenses. 72
Table 6.1 Various performance metrics for the target image, predicted image, and experimental images. 88
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Images of various GPHs made possible by this dissertation. These elements enabled new technologies related to astronomy, displays, and sensing.</td>
<td>2</td>
</tr>
<tr>
<td>2.1</td>
<td>Geometric phase via a birefringent retarder. The input and output polarization is shown in the form of an orthogonal vector sum (the red arrow indicates the phase of each component) and also in standard circular polarization notation. The half-wave plate (HWP) has a dynamic phase term that is modulo $2\pi$. The accumulated geometric phase $\phi$ is equal to twice the orientation angle $\theta$.</td>
<td>9</td>
</tr>
<tr>
<td>2.2</td>
<td>The NCSU mascot in various polarization states images directly (a,d) and through a PG (b,c,e,f).</td>
<td>11</td>
</tr>
<tr>
<td>2.3</td>
<td>a) Example rod-like LC molecule and anisotropy in index of refraction. b) LC in the nematic phase. c) A macro-scale analog to LCs, matchsticks in a box exhibit partial order.</td>
<td>13</td>
</tr>
<tr>
<td>2.4</td>
<td>Cross section of LC films fabricated in this dissertation.</td>
<td>17</td>
</tr>
<tr>
<td>2.5</td>
<td>Fields propagating through a PG; red and blue colors indicates positive and negative fields respectively. This result was obtained using a version of the FDTD method developed in this work (see Chap. 7) and is extremely difficult if not impossible to obtain with analytic methods.</td>
<td>21</td>
</tr>
<tr>
<td>2.6</td>
<td>The grid scheme used in this work is the 3D Yee grid; shown is a unit cell with the location of the $E$ and $H$ fields.</td>
<td>22</td>
</tr>
<tr>
<td>3.1</td>
<td>High-level schematic of the kind of polarization direct-write system studied in this paper.</td>
<td>26</td>
</tr>
<tr>
<td>3.2</td>
<td>High-level view of our direct-write system description, which starts with the system inputs and ends with the LC response. The transformation functions $H$ and $T$ relate to material properties of the LPP and LC respectively; all other operations are purely mathematical and make no assumptions about material properties.</td>
<td>27</td>
</tr>
<tr>
<td>3.3</td>
<td>Graphical representation of some properties of our chosen $H$, which takes the average of all input Stokes vectors. Each diagram is a 2D cross-section of the Poincare sphere (since we are only concerned with linear polarization), and the ring in each diagram represents a DOLP of 1. a) Non-causality and orientation averaging. b) Depolarization by superposition of orthogonal polarizations.</td>
<td>29</td>
</tr>
<tr>
<td>3.4</td>
<td>Threshold functions used to predict the magnitude of the LC anchoring vector from key parameters. a) Fluence threshold function $T_F$. b) DOLP threshold function $T_D$.</td>
<td>30</td>
</tr>
</tbody>
</table>
Figure 3.5  a) Polarizing optical microscope setup used to capture the images in this paper. The Full-Wave Plate (FWP) is used to distinguish between $0^\circ$ and $90^\circ$ polarized light. b) The full-color colormap for differently aligned liquid crystals, acquired when using the FWP. c) The grayscale colormap acquired without the FWP ($0^\circ$ and $90^\circ$ are indistinguishable). Orientation angle profiles of films were measured by matching the film’s color with b) using the least squares algorithm.

Figure 3.6  The first validation experiment, designed to test the effect of fluence on LC alignment. Pattern a) was created by scanning adjacent, uniform vertical lines and varying the fluence in the horizontal dimension. Pattern c) is the reverse of pattern a), and pattern b) is the superposition of a) and c). From this experiment we determined a fluence threshold to be used in subsequent simulations.

Figure 3.7  The second validation experiment, designed to test 1) the effect of DOLP on LC alignment quality, and 2) if the LC aligns to the average exposure angle of the LPP. Pattern a) was created by scanning adjacent, uniform vertical lines and varying the polarization angle in the horizontal dimension from $0^\circ$ to $90^\circ$. Pattern c) is the same as a), except the polarization varies from $0^\circ$ to $-90^\circ$. Pattern b) is the superposition of a) and c) with half fluence each (equivalent total fluence). From this experiment we determine a DOLP threshold to be used in subsequent simulations and confirm the aligning properties of the LC.

Figure 3.8  Adjacent, uniform scan lines with polarization orientations of $\psi_0 - 50^\circ$, $\psi_0$, and $\psi_0 + 50^\circ$. The distance between the scan lines decreases from a) to c), and we observe an averaging of the LC orientation angle. By sufficiently overlapping neighboring scan lines, we can create continuously varying orientation profiles.

Figure 3.9  Two intersecting scan lines with polarization orientations $0^\circ$ and $90^\circ$. The average polarization angle at every location is either $0^\circ$, $90^\circ$, or undefined where the fluence delivered by each line is equal. As a result, discrete domains are produced. a) Simulated $F$, b) $D_{avg}$, c) $|A|$, d) $\psi_{avg}$ using the color bar in Fig. ??(b), e) $A$, created by overlaying c) and d). f) Actual image of fabricated film, color bar in Fig. ??(b).

Figure 3.10  A simulated and fabricated $q$-plate, which is a kind of azimuthal waveplate. a) Simulated $F$, b) $D_{avg}$, c) $|A|$, d) $\psi_{avg}$ using the color bar in Fig. ??(b), e) $A$, created by overlaying c) and d). f) Actual image of fabricated film, color bar in Fig. ??(b).

Figure 3.11  A PG fabricated by scanning adjacent scan lines, where the polarization rotates from line to line. The target PG pitch is 50 $\mu$m and the scan lines were spaced 7 $\mu$m apart using a beam width of 7.5 $\mu$m. a), b), and c) are images obtained using different magnification objective lenses and the plots below show the desired orientation profile compared to our measured profile.

Figure 4.1  a) The scan pattern used to record PGs. The arrows indicate polarization orientation. b) The writing beam is assumed to have a gaussian profile.
Figure 4.2 Simulated results for a scan pattern with $B = 0.6$, $L = 0.4$, and $F_{ave} = 200$ mJ/cm$^2$. a) Intensity cross section for each scan line influencing a single pitch. b) Total fluence. c) Average degree of linear polarization (DOLP). d) Alignment quality calculated using b) and c). e) Phase profile, found as twice the LC orientation angle. f) Diffraction efficiencies found by applying the NTFFT on e). From the results, we can evaluate these scan parameters as unfavorable.

Figure 4.3 Simulated results for a scan pattern with $B = 0.8$, $L = 0.2$, and $F_{ave} = 200$ mJ/cm$^2$. a) Intensity cross section for each scan line influencing a single pitch. b) Total fluence. c) Average degree of linear polarization (DOLP). d) Alignment quality calculated using b) and c). e) Phase profile, found as twice the LC orientation angle. f) Diffraction efficiencies found by applying the NTFFT on e). From the results, we can evaluate these scan parameters as favorable.

Figure 4.4 The predicted alignment quality $|\mathbf{A}|$ vs. the normalize beam size $B$ and the normalized line spacing $L$. Poor alignment quality results in LC defects, incorrect retardation, and incorrect alignment orientation; all of these things reduce $\eta_{+1}$. 

Figure 4.5 The predicted $\eta_{+1} |\mathbf{A}|$ vs. the normalize beam size $B$ and the normalized line spacing $L$ assuming $|\mathbf{A}| = 1$. These efficiencies assume the profile is perfectly aligned and so represent an upper limit.

Figure 4.6 Predicted optimal scan parameters for creating high-efficiency PGs. Parameters inside the white triangular region have high efficiency. Parameters with larger line spacing result in a shorter scan time. Parameters farther from the edges of the high-efficiency region have the maximum yield, i.e., the highest likelihood of success.

Figure 4.7 Polarizing microscope images of fabricated samples. Different shades correspond to different alignment orientations. Samples g and h represent high-quality alignment.

Figure 4.8 Images of a 633 nm beam after passing through samples A-I. Samples d, e, g, and h are the highest quality of the nine samples, but there still significant power diffracted into the 0 and +2 orders; since the alignment quality is good, this indicates the alignment profiles are nonlinear. Samples a and b show many other higher orders, indicating these profile are very nonlinear. The background noise in samples c, f, and i is caused by the massive quantity of LC defects which scatter the incident beam and greatly reduce the efficiency.

Figure 4.9 a-b) Examples of super-grating profiles. c-d) The simulated $\eta_{+1}$ associated with the profiles in a-b). The super-grating profile is characterized by some periodicity in the profile with a pitch greater than the intended pitch of the PG. The diffraction caused by such profiles always results fractional orders, when considering the intended diffraction order as the +1 order.

x
Figure 4.10  a) Examples of noisy PG phase profiles. b) Simulated $\eta_{+1}$ vs. noise level. c-d) Simulated diffraction efficiencies of noisy phase profiles (pure noise, no linear PG profiles). Noise in the phase profile results in diffraction into all directions, similarly to scattering. .................................................. 56

Figure 4.11  a) Examples of sawtooth-grating profiles. b) Simulated $\eta_{+1}$ vs. the scale factor $\xi$. c-d) The simulated D.E.s associated with the profiles in a). The sawtooth-grating profile is characterized by a linear phase with a slope of $2\pi\xi/\Lambda$. The sawtooth profile may be caused by errors in the direct-writing system or LC fabrication errors, and the profile results in power directed into orders other than $+1$. .................................................. 58

Figure 4.12  Very high quality direct-write fabricated PG with a pitch of 30 $\mu$m and $\eta_{+1} > 99.7\%$. a) Polarizing microscope image of the sample. b) Image of a 633 nm laser after diffracting through the sample. ................................. 60

Figure 5.1  Demonstration of the polarization properties of a GPL. Different circular polarizations converge or diverge; unpolarized light is equivalent to linearly polarized light and acts as the superposition of both circular polarizations. 62

Figure 5.2  Comparison of the ideal lens profile, parabolic approximation, and spheric approximation. The lens f-number is a) 5, b) 2.5, and c) 0.5. The parabolic profile is a good approximation when the f-number is high, and only at low f-number does the profile differ significantly from the ideal. The spherical approximation is only good at high f-number. .................................................. 64

Figure 5.3  a) Linear Archimedean spiral. b) Nonlinear Archimedean spiral. The nonlinear spiral is more efficient in terms of scan time, but can only be used if the beam size can be changed dynamically, as in our system. ................................. 66

Figure 5.4  The spiral pattern sampled with a) uniform time, b) uniform arc length, and c) uniform angular distance. The sampled points are assumed to be connected by straight lines. A key observation is that sampled points of adjacent lines in a) and b) are misaligned to each other, but sampled points of adjacent lines in c) are aligned to each other. .................................................. 68

Figure 5.5  Polarizing microscope images of fabricated GPLs. a) Uniform arc length sampling with $d_0 = 200$ $\mu$m. b) Uniform arc length sampling with $d_0 = 100$ $\mu$m. c) Uniform angular distance sampling with $\theta_0 = 2\pi/50$. c) Uniform angular distance sampling with $\theta_0 = 2\pi/200$. .................................................. 71

Figure 5.6  Polarizing microscope images of samples A-D. Sample A and B both show very good alignment. Sample C suffers from a staircase profile. Sample D has many LC defects, except for the center of the pattern where the beam size could not be increased any further and so B was lowered. ................................. 73

Figure 5.7  High-magnification polarizing microscope images of the edge of samples A-D. Here the phase profile and defects can be seen more clearly. ................................. 74

Figure 5.8  Images of the fabricated GPLs in operation. A 633 nm laser was sent through each sample and the beam was allowed to propagate $\sim 60$ cm before hitting a screen. Also shown is the result of the laser passing through a comparable spherical refractive lens to serve as a reference. ................................. 75

xi
Figure 5.9 Measured cross section of a 633 nm laser a) without alteration, b) after GPL with RCP input, c) after GPL with LCP input.

Figure 6.1 a) Target FGPH image used in this section. b) Needed phase profile calculated for the Gerchberg-Saxton algorithm. c) Predicted reconstructed image for RCP input. d) Predicted reconstructed image for LCP input.

Figure 6.2 Scan pattern used to record FGPHs.

Figure 6.3 Example target and actual exposure polarizations when using a Pockels cell; the polarization is blurred between between each polarization state. This example assumes a pixel pitch of 10 µm and that a polarization modulation takes 1 µm to finish.

Figure 6.4 Polarizing microscope images with various zoom of fabricated FGPH with the target phase profile shown in Fig. ??(b). All samples had the same scan pattern parameters, except for the beam size, which varied from 10 µm (B = 0.5) to 43 µm (B = 2.2). When the beam size is smaller, the pixels are more clearly defined. When the beam size is larger, more averaging takes place. Extremes at either end cause many LC defects; a beam size of ~ B = 1 seems optimal.

Figure 6.5 The FGPH characterization setup. A 633 nm laser passes through an iris to clean up the beam, a polarizer to control beam power and ensure polarization purity, a quarter-wave plate (QWP) to set the polarization, the FGPH element, and a series of lenses which focus the image onto a CCD.

Figure 6.6 The Fourier images produced by FGPHs with various beam sizes. Polarizing microscope images of the FGPHs are in Fig. ??, and the setup used to capture these images is in Fig. ??.

Figure 6.7 Images produced by sample B (B = 1.1) with both a 633 nm and 523 nm laser. Images were acquired using a beam-splitter and multiple QWPs to set beam polarizations. The green image is a smaller scaled version of the red image. a) RCP red and LCP green. b) RCP red and RCP green.

Figure 7.1 (a)) The simulate space used for Wolfsim 3D. b) The grid scheme used in this work is the 3D Yee grid; shown is a unit cell with the location of the E and H fields (which are identical to the positions of the P and Q fields).

Figure 7.2 Comparison of possible diffraction orders of an arbitrary 2D periodic structure in a) k-space, and b) real space. This figure demonstrates conical diffraction.

Figure 7.3 a) A glass etalon with index n = 1.5 and thickness d surrounded by air. b) Simulated (circles) and analytic (dashed/solid) transmission. The source was incident at \{\theta = 40^\circ, \phi = 90^\circ\}.

Figure 7.4 Conoscopic contour plot of the transmission through a pair of crossed dichroic polarizers. a) Our result. b) Berreman 4x4 result. Contour lines start at 0.001 (centermost) and increase in steps of 0.005.

Figure 7.5 FDTD simulation results for an LC cell between crossed polarizers compared with Berreman 4x4. a) Normalized Stokes parameters. b) Total transmittance.
Figure 7.6  a) A Bragg grating with $d = 5\ \mu m$, $\Lambda = 0.45\ \mu m$, $\epsilon_1 = 2.2525$, $\epsilon_2 = 0.15$, and $\sigma_1 = \sigma_2 = 250\ S/m$. b) The simulated transmission of the $m = -1$ order using Wolfsim 3D and RCWA (dashed). For each case, the source was TE and incident at $\theta = 37.67^\circ$. ................................................................. 102

Figure 7.7  a) A photonic band gap structure with $a = 9\ mm$, $d = 4\ mm$, and $\epsilon_r = 4.2$. Solid lines are our results, dashed lines are results from [61]. b) The transmission coefficient for the incidence directions indicated. The source was TM polarized for the oblique cases and polarized in the $y$ direction for the normally incident case. ................................................................. 103

Figure 7.8  a) The geometry of the infinite 2D array of square perfect electric conductor particles. b) Reflectance through the 2D array. Inset: a close up of the 100% reflectance maxima for normal incidence and $\phi = 45^\circ$ and $\theta = 5^\circ$. ........................................ 104

Figure 7.9  a) Structure of the simulated 2D grating. b) Far-Field orders resulting from Wolfsim 3D and an RCWA code for an $x$-polarized linear input. ........................................ 105

Figure 7.10  a) Target k-space image for an $M = 5$ MBS. Orders have equal intensity but unspecified phase. b) A calculated possibly optimal phase profile to produce this k-space image. ................................................................. 106

Figure 7.11  Analytic and simulated k-space images (far-field orders) resulting from the MBS profile given in Fig. ??(b) and $\Lambda = 5\ \mu m$. Analytic output with a) RCP, and b) linear (all linear polarizations are the same). FDTD simulated output with input a) RCP, b) $0^\circ$ linear, c) $45^\circ$, and d) $90^\circ$. ........................................ 107

Figure 7.12  Simulated k-space images (far-field orders) resulting from the MBS profile given in Fig. ??(b) and $\Lambda = 10\ \mu m$. FDTD simulated output with input a) $0^\circ$ linear, b) $45^\circ$, and c) $90^\circ$. These results match the analytic results closer than when $\Lambda = 5\ \mu m$. ................................................................. 108

Figure 7.13  Polarizing microscope image of a GP MLA made with The Howlographer. Image courtesy Jihwan Kim. ................................................................. 109

Figure 7.14  a) Unit cell of the target MLA phase profile. Simulated phase profile of 550 nm light directly after the MLA for b) $E_x$ fields, c) $E_y$ fields, and d) $E_z$ fields. These are the inputs to the vacuum wave propagation algorithm. ........................................ 110

Figure 7.15  Calculated total field power after the MLA for $xz$ and $yz$ cross sections. The field values were calculated using the general diffraction solution and the complex field values after the MLA (shown for 550 nm in Fig. ??). ........................................ 111
Chapter 1

Introduction

1.1 Light in Everyday Life

The ability to control light is a cornerstone of 21st century society. We use incandescent bulbs, fluorescent bulbs, and LEDs to light schools, office buildings, and sports fields. Liquid crystal displays are present in mobile smartphones, desktop monitors, gas station pumps, cash registers, clocks, and countless other devices. Eyeglasses and contact lenses allow millions of people to see the world more clearly. Lasers are used in fiber optic and telecommunication technology giving us high-speed internet, as well as being foundational to basics science research. Cameras allow us to capture and preserve images and scenes, while optical disc readers and projectors allow us to replay scenes to a wide audience.

We are continuing to realize things that were once science fiction, such as 3D holograms, laser weapon systems, and coin-sized projectors. However, these and other technologies and end-products are the result of decades of fundamental research by the scientific community into the generation and manipulation of light. These researchers include optical scientists, physicists, astronomers, electrical engineers, computer engineers, material engineers, and others who have explored the basic phenomena surrounding electromagnetic waves and their interaction with physical media.

This dissertation aims to aid in the never-ending quest of understanding light by studying a newly recognized class of optical elements called Geometric Phase Holograms (GPHs). GPHs have many unique and interesting properties, and existing applications include aiming systems for laser-based weapons, countermeasures, and communications [1], low-power displays [2], imaging of planets in other solar systems [3], highly precise measurement tools which might aid in detection of pollution and toxic chemicals [4], and numerous fundamental science experiments. Some example GPHs, created over the course of this work and which enabled new technologies, are shown in Fig. 1.1.
1.2 Geometric Phase Holograms

So what is this amazing GPH element? First and foremost, it is a hologram...but what exactly is that? One definition of a hologram is an optical element with a higher information density than a photograph. For example, when you take a picture of a bed of flowers, you are storing information about the scene from a single perspective with a single focus. With a hologram, you can store information about the scene from multiple perspectives (i.e., multiple viewing angles) and/or multiple focuses (i.e., focusing on the foreground, the flowers, or the background) all in a single element. This property of high information density means that holograms have many uses that extend beyond 3D intergalactic communication with Darth Vader. Perhaps the most successful commercial use of holograms is extremely fast and compact encrypted data storage [5].

There are many technologies for recording and replaying holograms and each has a different physical operating principle. GPHs use something called the geometric phase of light to record and replay information about a scene. Use of the geometric phase gives GPHs their unique and advantages features. These features are summarized as follows:

- Superior optical characteristics when compared with other holograms
- Fabrication using low-cost commercial materials
- Ability to manipulate polarized light
- Ability to respond to different wavelengths of light (i.e., different colors) in a controlled manner

Combined, these attributes are what enable the powerful applications previously mentioned.
This dissertation focuses on a specific class of GPHs: Computer Generated Geometric Phase Holograms (CGHs); computer generated in the sense that a computer controls the generation and fabrication of the hologram. Going back to the example of the flower bed, one might take sophisticated and expensive equipment to the actual flower bed to record a GPH of the scene. Or, a computer could be used to simulate the geometry, lighting, and color of the flower bed and calculate the hologram that would be recorded from the scene; then use a holographic printer to turn this calculated hologram into an actual element. The later demonstrates the core concept of the CGH, although none of the CGHs studied and created in this dissertation are as complex as this.

1.3 Dissertation Structure

Chapter 2 contains technical background information to equip the reader with the knowledge and terminology needed for the proceeding chapters. The topical coverage begins with the fundamentals of geometric phase and GPHs. Next, the key properties of liquid crystals are discussed and the fabrication processes used in this dissertation are described. The concept of diffraction is introduced and various solutions for the general diffraction problem are discussed. Finally, the electromagnetic finite-difference time-domain (FDTD) method is reviewed.

In Chapter 3, we address the following questions: What system configuration will enable the recording of arbitrary CGHs with quality well beyond the prior art? What mathematical system description will allow us to accurately model the system? To answer these questions, we study a new tool, called The Howlographer, for CGH fabrication using a direct-write laser system, photoalignment polymers, and liquid crystals. We lay out a thorough mathematical description of the system that allows us to predict the physical properties of fabricated CGHs based on system inputs. After verifying the accuracy of the system description, we use it to fabricate and study several interesting patterns. The primary contribution of this chapter is the design, system description, and validation of a new direct-write system that is the first tool capable of creating arbitrary CGHs with continuously varying profiles.

In Chapters 4, 5, and 6, we address the following questions: What are design parameters for fabricating important CGH elements, and what impact do these parameters have on the element’s properties? What are the optimal parameters for fabricating the highest quality CGHs? To this end, we design, fabricate, and study the most important CGH elements: polarization gratings, geometric phase lenses, and arbitrary Fourier GPHs. For each element, we study the various system inputs that can be used to generate the element.

1While this dissertation uses the plural 1st person voice as is standard in the scientific community, almost the entirety of its contents are a direct result of the author’s efforts. Each Chap. contains comments on additional contributors.
and use our system description to predict the advantages and disadvantages of each. Then we fabricate many elements using a wide range of input parameters, and characterize them with metrics such as diffraction efficiency, zero-order leakage, scattering efficiency, and polarization contrast. The primary contribution of these chapters is the creation and characterization of the first high quality CGH polarization gratings, geometric phase lenses, and arbitrary Fourier GPH, and the determination of the optimal parameters for fabricating these elements.

In Chapter 7, we address the following question: **What numerical simulation tool is ideal for simulating the physical phenomena present in GPHs?** Seeing no such tool in prior art, we develop a new algorithm based on the FDTD algorithm, a popular method based on Maxwell’s equations. We derive the key mathematical equations of the algorithm, validate its accuracy and ability to simulate GPHs, and simulate for the first time several GPHs. The tool is open source and available online [6]. The primary contribution of this chapter is the creation of the first tool capable of simulating arbitrary, 3D, anisotropic structures that are periodic in two dimensions with sources incident from anywhere in the hemisphere (conductive materials are supported as well).

### 1.4 Publications From This Work

**Published Refereed Journal Manuscripts**


**In Press Refereed Journal Manuscripts**


**Published Refereed Conference Proceedings**


Journal Manuscripts In Preparation


Chapter 2

Background

2.1 Geometric Phase Holograms

Geometric Phase Holograms (GPHs) are a class of holograms utilizing geometric phase. The kind of GPHs studied in this dissertation are birefringent thin-film elements with an optical axis that varies in one or two dimensions. In this section, we will review the concept and properties of the optical geometric phase, demonstrate these properties in a polarization gratings, and review prior methods to realize GPHs.

2.1.1 Geometric Phase

Geometric phase is a general term that refers to the phase acquired by a system that travels over a cycle or closed path, where the amount of acquired phase depends on the path traveled. Pancharatnam first explained the geometric phase in 1956 [7] as it relates to polarized light. In 1984, Berry rediscovered the geometric phase in the general context of quantum mechanics [8]. Though it has a broad meaning, we will always be using geometric phase in the context of the optical geometric phase, specifically the Pancharatnam-Berry phase.

As a basic definition, the geometric phase is the phase accumulated by a lightwave as it changes from polarization $A$ to polarization $B$ over path $P_1$, compared to a reference wave changing from polarization $A$ to polarization $B$ over path $P_2$. The phase is calculated from the combination of $P_1$ and $P_2$, which by definition is a closed path. The precise amount of phase accumulated is equal to half the area (solid angle) of the closed path when the path is described on the Poincare sphere [7, 9]. In practice, it is common to choose a fictitious reference polarization path $P_1 = P_r$ and compare all subsequent paths to this reference path. Since we are many times interested only in the relative phase between different paths, $i.e.$, some $P_2$ and $P_3$,
the effect of the reference path is nullified. Thus, simplifying the situation further, we usually ignore the reference polarization path altogether or choose a reference polarization path with a value of zero, allows us to use the geometric phase as if it was dependent on a single, unclosed polarization path.

The geometric phase should be compared and contrasted to the dynamic phase, which is a phase accumulated through an optical path distance. Dynamic phase is phase accumulation method we are most familiar with, as it is responsible for the operation of most optical elements such as refractive lenses. The dynamic phase acquired through a medium of refractive index $n$ and thickness $d$ is found as:

$$\phi_d = \frac{2\pi n d}{\lambda} = ndk$$  \hspace{1cm} (2.1)

where $k$ is the wavenumber.

**Geometric Phase via. a Birefringent Retarder**

We now examine the change in polarization to a plane wave traveling in $z$ after passing through a birefringent medium (a retarder) located at $z = 0$ with thickness $d$, birefringence $\Delta n$, and optical axis $\theta$. We assume the input wave is right circular polarized (RCP) with a phase constant of $\phi = 0$. Given these assumptions, the output wave can be described using a Jones vector as follows:

$$J_{out} = \frac{e^{j\phi_o}}{2\sqrt{2}} \left[ (e^{j2\pi\zeta} + 1) \left( \begin{array}{c} 1 \\ j \end{array} \right) + (e^{j2\pi\zeta} - 1)e^{j2\theta} \left( \begin{array}{c} 1 \\ -j \end{array} \right) \right]$$  \hspace{1cm} (2.2)

where $\phi_o = 2\pi n_o d/\lambda$ and $\zeta = \Delta nd/\lambda$ is the optical retardation. For the remainder of this work, we consider $\zeta$ to be the modulo retardation, i.e., it varies between 0 and 1. This equation can be arrived at in a number of different ways (see Appendix A for a derivation using Jones Calculus), and we will now examine its properties.

**Dynamic Phase Term**

The outermost coefficient of the equation shows that the input wave accumulates a phase equal to $\phi_o = 2\pi n_o d/\lambda$ regardless of the retardation or orientation angle of the retarder. This is recognized as a dynamic phase term via Eq. 2.1.

**Zero-Order Leakage**

If the $\zeta$ is zero, then the second term will also be zero and the output will be RCP (i.e., the same polarization as the input). As the retardation is increased up to half-wave, the power
in the output transfers from RCP (first term) to left-circular polarized (LCP) (second term). Notice that the RCP term at the output has no dependence on the orientation angle $\theta$ and is identical to the input except for the dynamic phase term. Hence, this first term is called the zero-order leakage term. If the retardation of the retarder is half-wave, the zero-order leakage term disappears [10].

**Geometric Phase Term**

Recall that the optical geometric phase is a result of different paths traveled between two different polarization states. In 2.4, polarization A is RCP (the input) and polarization B is LCP (the second output term). Assuming the retardation is half-wave, then regardless of the angle $\theta$, the output polarization will always be LCP. However, the polarization path of the light as it travels from one side of the retarder to the other does depend on $\theta$. Thus, to find the geometric phase $\phi_g$, we find the difference between an RCP test beam passing through a half-wave retarder retarder having angle $\theta$, and accumulating phase $\phi_t$, and an RCP reference beam passing through a half-wave retarder having angle $\theta_r = 0$ and accumulating phase $\phi_r$:

$$\phi_g = \phi_t - \phi_r = (\phi_o + 2\theta) - (\phi_o + 2\theta_r) = 2\theta.$$  \hspace{1cm} (2.3)

This is a key result: the acquired geometric phase when a wave passes through a birefringent retarder is equal to twice the orientation angle of the retarder [10]. This is depicted in Fig. 2.1

**Polarization Dependence**

The above analysis was assuming that the input beam was RCP. It can be easily shown that if the input is LCP, the output retains the same form, being:

$$J_{out} = \frac{e^{j\phi_o}}{2\sqrt{2}} \left[ (e^{j2\pi\zeta} + 1) \left( \begin{array}{c} 1 \\ -j \end{array} \right) + (e^{j2\pi\zeta} - 1)e^{-j2\theta} \left( \begin{array}{c} 1 \\ j \end{array} \right) \right].$$ \hspace{1cm} (2.4)

The key differences from RCP input are 1) the handedness of the zero-order term and of the geometric phase term is flipped, and 2) the geometric phase accumulated is negative of that for RCP input.

Note that an input which is linearly or elliptically polarized can be broken down into constituent LCP and RCP components that are each analyzed individually with the outputs being summed.
<table>
<thead>
<tr>
<th>Input</th>
<th>HWP</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Input Image" /></td>
<td><img src="image2" alt="HWP Image" /></td>
<td><img src="image3" alt="Output Image" /></td>
</tr>
</tbody>
</table>

| θ=0° | θ=45° | θ=90° | θ=135° |
| φ=0° | φ=90° | φ=180° | φ=270° |

Figure 2.1: Geometric phase via a birefringent retarder. The input and output polarization is shown in the form of an orthogonal vector sum (the red arrow indicates the phase of each component) and also in standard circular polarization notation. The half-wave plate (HWP) has a dynamic phase term that is modulo $2\pi$. The accumulated geometric phase $\phi$ is equal to twice the orientation angle $\theta$.

**Wavelength Dependence**

The geometric phase is independent of wavelength; thus, the phase accumulated by different wavelengths passing through the same birefringent thin-film will be the same. Even if the geometric phase varies spatially as in a GPH (i.e., the spatial derivative of the phase is non-zero), the phase accumulated by different wavelengths is still the same. However, because different wavelengths have different wavevectors, the same accumulated phase causes each wavelength to diffract differently. In general, we can say that a GPH will affect larger wavelengths more than smaller wavelengths. If the phase slope of some region of a GPH is $\Lambda = 2\pi (rad/m)$, we can approximate the diffraction angle of light in this region as:

$$\theta = \sin^{-1} \left( \frac{\lambda}{\Lambda} \right).$$

Thus, while the geometric phase is itself wavelength-independent, the propagation effects of GPHs vary with wavelength. Conversely, the dynamic phase is wavelength-dependent, and the propagation effects of dynamic phase elements may or may not vary with wavelength depending on the exact phase profile (see Appendix B).
2.1.2 Summary of Important GPH Concepts

- The geometric phase accumulated is $2\theta$ where $\theta$ is the optical axis of a birefringent media
- Opposite handed circular polarizations accumulate negative phase
- The zero-order leakage has the same polarization as the input
- There will be no zero-order leakage if the retardation is half-wave
- A GPH causes different wavelengths to propagate differently

2.1.3 Polarization Grating

We will examine the polarization grating (PG) and observe the key GPH principles.

Ideal PG Profile and Operation

The PG is the geometric phase analog of the dynamic phase prism. The ideal PG phase profile has a linear slope in $x$ and is uniform in $y$. Thus, the optical axis of a PG fabricated as a birefringent film linearly varies in $x$ and is uniform in $y$ [11]. When viewed through a polarizing microscope, the optical axis can be visualized as in Fig. 1.1(b). The linear phase slope causes the optical axis to be periodic over some distance $\Lambda$, called the pitch or period of the PG. The phase of a PG can be expressed as $\phi_{PG} = Cx$ for some $C$. The optical axis profile $\theta_{PG}$, the corresponding pitch $\Lambda$, and the constant $C$ are related as follows:

$$\Lambda = \frac{\phi_{PG}}{2\pi x} = \frac{\theta_{PG}}{\pi x} = \frac{C}{2\pi}.$$  \hspace{1cm} (2.6)

Like a prism, the ideal operation of a PG is to redirect incident light into one and only one other direction. In grating terminology, the ideal PG sends all incident light into the +1 (RCP input) or -1 (LCP input) grating order.

Actual PG Operation

Figure 2.2(a) shows the NCSU mascot in a solid red color, displayed on a laptop’s LCD. In Fig. 2.2(b), the mascot is linearly polarized and viewed through a PG. Both the +1 and -1 orders appear in equal intensity, and we also see a zero order which looks identical to the original image. This is caused by a retardation of the PG that varies slightly from half-wave. We also note that the +1 and -1 orders are blurry; this is due to the fact that the displayed image contains multiple wavelengths, and each wavelength diffracts at slightly different angles. When we send in RCP or LCP (Fig. 2.2(c) and f), we see that the output consists entirely of the +1 or -1 order.
Next, we look at the full-color mascot in Fig. 2.2(d). When viewed through the PG, we can clearly see the different wavelengths \( i.e., \) red, green, blue) are diffracting to different angles \( i.e., \) the ±1 orders are spaced farther apart. We see that larger wavelengths such as red are diffracted more than smaller wavelengths such as blue. This figure demonstrates all the important GPH properties.

2.1.4 GPH Fabrication Techniques

Here, we review relevant fabrication techniques to create birefringent thin-films with spatially varying optical axis, \( i.e., \) GPHs. There are two main ways to create a birefringent film. The first is by using anisotropic materials such as liquid crystals (LC). The second is by using isotropic materials with sub-wavelength structures that result in a macro-scale birefringence, a phenomena known as form birefringence [12]. In either case, the key fabrication step is the patterning of the optical axis.

Polarization holography is a method that utilizes the interference of differently polarized beams to create a desired polarization pattern at a specific plane [11, 13]. This polarization pattern is then used to align the optical axis of a polarization-sensitive film placed at the interference plane. In the case of a PG, two orthogonally circularly polarized beams are interfered, creating a linear polarization pattern with a linearly varying polarization angle. This technique
has been highly successfully in fabricating PGs [14, 15].

For form-birefringent GPHs, the most common fabrication method involves UV lithography [10]. Various binary optical masks are used to selectively expose photoresist followed by etching, deposition, or growth processes to create sub-wavelength structures.

The most advanced fabrication methods involve a direct-write system, capable of scanning of a focused laser beam over a polarization-sensitive film [16, 17, 18, 19, 20, 21]. At each pixel location, the polarization angle of the beam is set and recorded, thus writing the optical axis one pixel at a time. In this way, more complex patterns can be created than with polarization holography. GPHs fabricated in this way are dynamically created with the aid of a computer controlled system (unlike the static one-shot polarization holography and lithography techniques) and are called computer generated geometric phase holograms (CGHs). We note that CGH may also refer to a GPH that is created in real-time, for example by using a combination of spatial light modulators; however in the context of this dissertation, CGH always refers to the former definition.

In any of the prior art, there have been limitations to one or more aspects of the patterning process, such as a limited number of possible patterns that can be recorded or requiring the use of inefficient materials. In this dissertation, we develop, study, and use a new kind of direct-write system for creating GPHs of unparalleled complexity and efficiency.

2.2 Liquid Crystals

Liquid crystals (LCs) were discovered in 1888 by Friedrich Reinitzer. Since that time, they have been studied extensively by the scientific community. In the 1960’s, the first liquid crystal displays (LCDs) were demonstrated, which promised a cheaper, more compact, and more efficient alternative to cathode ray tube (CRT) displays. Today, LCDs have almost completely replaced CRTs and are by far the dominate display technology. In addition to LCDs, LCs are used in a vast number of other applications, ranging from microscopy [22], to light-powered motors [23], to sensing atmospheric turbulence [24], to aiming laser weapon system [1]. In this section, we will review 1) basic properties of LCs relevant to this work, 2) methods to control the alignment of LCs, and 3) the fabrication process used in this work for creating LC based CGH.

2.2.1 Physical Properties of Liquid Crystals

There are many complex properties of LCs; here we will review only those properties relevant to this work.
Anisotropy

Broadly speaking, anisotropy is a property of directional dependence. LC molecules are typically shaped like rods or discs, and this asymmetry causes the LC molecules to exhibit certain anisotropic properties. The most important anisotropy is in the index of refraction and is called the birefringence $\Delta n$, defined as:

$$\Delta n = n_e - n_o$$  \hspace{1cm} (2.7)

where $n_e$ is the extraordinary index of refraction and $n_o$ is the ordinary index of refraction. In a uniaxial crystal, $n_o$ is the index of refraction seen by light polarized in two of the crystal axes and $n_e$ is the index of refraction seen by light polarized in the remaining axis, as shown in 2.3(a).

Dichroism is another anisotropic property of LCs, in which light polarized along different axes experiences different absorption. However, dichroism is generally negligible in LCs, and so our attention will be focused on their birefringence.

As both birefringence and dichroism are intrinsically related to polarization, it is only natural that LCs are useful for manipulation and detection of polarized light via elements such as GPHs.

Partial Order and The Nematic Phase

The anisotropic shaping of LCs gives rises a phenomena known as partial order, which is where LCs get their name. It is commonly known that matter can exist in three phases: gas, liquid, and solid. However, matter can also exist in phases in-between the liquid and solid phases. The
liquid phase is characterized by molecules that are not structured or ordered; molecules are free to move about or be reoriented as forces dictate. The solid phase is characterized by molecules that are very structured or ordered; molecules are spaced at regular intervals in a chemically bonded lattice and are not able to be moved or reoriented. Matter in a liquid crystal phase (a phase in-between the liquid and solid (crystal) phases) has molecules that are somewhat free to move and be reoriented, but are also somewhat structured and predictable. This latter property is called partial order.

There are many liquid crystal phases that exhibit different degrees and different kinds of partial order. In this dissertation, we exclusively utilize the nematic phase. In the nematic phase, molecules do not posses any degree of positional order; i.e., the spacing between molecules is random and unpredictable. However, the molecules posses orientational order; i.e., the orientation of the molecules is not entirely random and is somewhat predictable. In the nematic phase, the LC molecules will be oriented in an average direction, specified by a vector \( \mathbf{n} \) called the nematic director. An example of LC in the nematic phase is shown in 2.3(b). A macro-scale analog of partial order and the nematic phase is matchsticks in a box, shown in 2.3(c). While the orientation of each individual matchstick is somewhat random, the matchsticks as a whole point in the same general direction. (We note that these matchsticks might be classified more accurately as being in a smectic phase rather than a nematic phase.)

The degree to which molecules are oriented to \( \mathbf{n} \) is specified by a parameter \( S \), called the order parameter. If the order parameter is 0, it indicates that the molecules are randomly oriented and are in the isotropic phase (equivalent to the liquid phase). A low non-zero order parameter indicates that the molecules are very loosely oriented to \( \mathbf{n} \); a higher order parameter indicates the molecules are strongly orientated to \( \mathbf{n} \). An order parameter of 1 will only occur if the molecules are perfectly ordered equivalently to a crystal lattice. The LCs used in this work have order parameters of \( \sim 0.6 \).

**Polymerization**

In this work, we use a special class of LCs called liquid crystal polymers (LCPs). In contrast to typical liquid crystals such as those found in LCDs, the molecules in an LCP film are solidified and unable to move. Essentially, the nematic director is frozen in place, and the film becomes mechanically robust and relatively immune to external forces such as electric fields or mechanical pressure. This is highly advantageous for films subject to a variety of atmospheric conditions or which do not need to be altered (static elements).

Polymerization is accomplished through the use of specific photosensitive LC molecules or other dopant molecules. Upon being struck by photons with the required energy (typically UV), these photosensitive molecules form chemical bonds to each other and/or to LC molecules [25].
2.2.2 Alignment of Liquid Crystals

Understanding the alignment of liquid crystals, i.e., what the value of $n$ is, is of vital importance to creating functional LC elements. A vast amount of research has been done into understanding how material parameters, boundary conditions, applied electric or magnetic fields, dopant concentrations, and more affect $n$. We will focus here on methods developed to align LCs (i.e., set $n$) to desired in-plane angles. We define an in-plane orientation as an $n$ that is parallel to a substrate and with some polar angle $\theta$ with respect to the substrate plane (i.e., if $z$ is normal to the substrate, $n$ lies is in the $x - y$ plane).

The physics of LC alignment can be easily understood from the vantage of minimization of energy states. Consider a handful of matchsticks like those in 2.3(c), which are shaped much like LCs. The matchsticks are put in a matchstick box without any care. Some of the matchsticks will fall into the box completely, some will be half in the box and half propped up by the walls of the box, and some will sit completely on top of the box. If you agitate the box by shaking it slightly, the matchsticks will all fall into the box completely (ignoring those that fall off the box completely) and all be aligned in the same general direction as dictated by the walls of the narrow box. This example demonstrates that the minimum energy state of the matchsticks is set by the boundary conditions imposed by the box, and when some amount of free energy is applied to the system (i.e., shaking) the matchsticks naturally go to the minimum energy state.

So it is with LCs: as the molecules are not bound together, there exists some amount of free energy (e.g., thermal energy) enabling the molecules to move and reorient themselves [26]. The molecules will naturally “fall” into the minimum state as dictated by surrounding boundary conditions or by other external forces (e.g., an applied electric field [27]).

Surface Rubbing

One of the oldest and still most common methods of aligning LCs is by surface rubbing. In this method, a substrate is coating with a special alignment material designed to strongly influence the LC director. The substrate is then rubbed repeatedly with an abrasive material such as felt. The rubbing causes physical grooves or striations to form in the alignment material; essentially, aligning the alignment material. LC is then placed on the substrate and it aligns according to the alignment material (i.e., according to the rubbing direction). In the case of an LC cell such as in LCDs, two substrates are prepared, alignment material is aligned on each substrate individually, the substrates are placed together to form a cell, and the cell is filled with LC.

Photoalignment Dopants

Another way to align LCs is by doping them with photosensitive materials or by attaching photosensitive molecular groups to the desired LC molecules [28]. These dopants/side groups
will align themselves to incident polarized light of specific wavelengths; if the dopant is already chemically attached the LC molecules, then aligning the dopant molecule directly aligns the LC molecules as well. If the dopants are not attached to the LC molecules, but are rather dispersed in the LC, then the dopant molecules will indirectly align the LC molecules by imposing boundary conditions on the LC molecules and influencing the minimum energy states.

**Photoalignment Films**

Still another alignment technique is to use a photosensitive alignment material on the surface of a substrate [29, 30]. The alignment material is aligned by incident polarized light, then LC is placed on the alignment material and aligns accordingly. This is the method of alignment used throughout this dissertation.

**Comparison of Alignment Methods**

Both photoalignment dopants and photoalignment films allow for complex alignment patterns to be recorded which are otherwise impossible or practically infeasible with surface rubbing. Examples are the recording of polarization holograms such as the polarization grating [11] or the capture of entire holographic scenes [31]. Between these two photoalignment methods, using dopants is in principle more attractive. In practice, however, photoalignment films have better characteristics; namely, they enable LC films with higher birefringence and lower defects than LC films using photoalignment dopants. The work in this dissertation uses photoalignment films with linear photoalignment polymers (LPP), which are molecules sensitive only to linearly polarized light.

**2.2.3 Fabrication of LC Films**

Here we describe the basic steps used in this dissertation to fabricate LC films. Fabrication begins by coating a clean substrate with LPP, which coating is accomplished via spin coating. Spin coating is a thin-film coating technique in which a material is dropped onto a substrate and the substrate is spun at high speeds. The spinning causes most of the material to fly off of the surface, but intermolecular forces (i.e., surface tension and adhesion) cause a thin film to remain. By adjusting the acceleration and velocity of the spin, the thickness of the film can be controlled to within tens of nanometers in a highly repeatable fashion. Combined with simplicity of use and relatively low cost, spin coating is an extremely popular coating technique and is the coating technique used in this dissertation.

After being coated, the LPP is exposed to polarized UV light to align it. One focus of this dissertation is exposing LPP with a UV laser using a computer controlled scanning system, which allows us to create complex alignment patterns.
After LPP alignment, LCP is coated, again via spin coating. After being coated, the LCP is polymerized via a UV light source. At this point, additional layers may be added on top of the existing LCP film. For subsequent layers, the polymerized LCP serves as the alignment layer instead of the initial LPP. By adding more layers with specific spin parameters, any desired retardation may be achieved, typically half-wave retardation for some design wavelength. The final structure is shown in Fig. 2.4.

To summarize, LC films are fabricated as follows:

1. Coat LPP
2. Expose LPP
3. Coat LCP
4. Polymerize LCP
5. Repeat 3-4 as needed

2.3 Electromagnetic Waves

It is well known that light is an electromagnetic wave, composed of oscillating, orthogonal electric and magnetic fields. In this section, we review some key properties of electromagnetic waves which will be used throughout this dissertation.

2.3.1 The Electromagnetic Wave Equations

First discovered by Maxwell [32], the electromagnetic wave equations describe the propagation of electromagnetic waves through media with permittivity $\epsilon$ and permeability $\mu$. One way to express them is:
\[ \nabla^2 E = \mu \frac{\partial^2 E}{\partial t^2} \]  
\[ \nabla^2 B = \mu \frac{\partial^2 B}{\partial t^2} \]  

where \( \nabla^2 \) is the Laplacian, \( E \) is the electric field, and \( B \) is the magnetic flux. In this work, we only consider materials with zero magnetic susceptibility, making \( \mu = \mu_0 \). Thus, we only need consider the electric field, and the magnetic flux can later be derived via Ampere’s law:

\[ \nabla \times B = \epsilon \mu_0 \frac{\partial E}{\partial t}. \]

For most of this dissertation, we are concerned with waves which are monochromatic and have steady-state solutions. If the spatial extent of a wave is long compared to its wavelength, we can assume the wave has a steady-state solution. This is almost always the case. For example, a visible light source which is turned on for only 1 ns still has a spatial extent over one million times its wavelength. In addition, light sources which contain many wavelengths can in general be analyzed as a superposition of many monochromatic waves. For monochromatic waves of wavelength \( \lambda \) with sinusoidal steady-state solutions, a general solution to the wave equation is:

\[ E(r, t) = E(r) \cos(\omega t) = E(r) \cos(k c_0 t) = E(r) \Re \{e^{-j k c_0 t} \} \]

where \( \omega \) is the angular frequency, \( k = 2\pi/\lambda \) is the wavenumber, \( r = x \hat{x} + y \hat{y} + z \hat{z} \) is a position vector, and \( c_0 \) is the speed of light in a vacuum. Thus, the time dependent component is trivial and only \( E(r) \) needs solved for.

### 2.3.2 Infinite Plane Wave Solution

A simple and highly instructive solution to the wave equation is an infinite plane wave. In complex exponential form, an infinite plane wave in a vacuum is:

\[ E(r) = E_0 e^{-j(k \cdot r + \phi)} \hat{p} \]

where \( k \) is the wavevector, \( E_0 \) is the wave amplitude, \( \phi \) is an arbitrary phase constant, and \( \hat{p} \) is a vector describing the polarization of the wave. The wavevector is related to the wavelength \( \lambda \) by \( |k| = 2\pi/\lambda \).
2.3.3 Diffraction

When a wave encounters an inhomogeneous medium, the solution to the wave equation quickly becomes very complex. If we consider a plane wave hitting an opaque sphere (i.e., the wave does not pass through the sphere), we can qualitatively describe the wave as bending or deforming around the sphere (in addition to reflecting back). While the initial plane wave propagated in a single direction, $\hat{k}$, the wave after the object is now traveling in many directions; indeed, the wave is now traveling in all possible directions to some extent. This phenomena, the redirection of light after encountering an obstacle, is called diffraction.

As the subject of diffraction is extremely broad, we will limit out discussion by introducing the following assumptions: 1) diffraction only occurs at the plane $z = 0$, 2) all space beyond $z = 0$ is a vacuum, and 3) the fields at $z = 0$ are known (this is nearly equivalent to knowing the structure before $z = 0$, i.e., the incident wave has passed through the structure but has not yet diffracted, and is a good approximation if the structure is thin). We also define the diffraction aperture $D$ as the diameter of the physical structure; the structure may be an object in a vacuum or a void or structure in an otherwise homogeneous plane of material.

2.3.4 General Solution

A general solution to the diffraction problem may be obtained using Huygens’ principle. Huygens’ principle states that the propagation of any wave may be determined by decomposing a known wavefront into an infinite number of spherical wave sources. A solution to the diffraction problem stated above is thus:

$$E(x, y, z) = \frac{z}{j\lambda} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x', y', 0) e^{-j\frac{2\pi r}{\lambda z}} \frac{dr dx' dy'}{r^2}$$

where $r = ((x' - x)^2 + (y' - y)^2 + z^2)^{0.5}$. This equation is nearly always solved numerically.

Mathematical Approximations

A large number of mathematical approximations have been developed which can provide answers very close to the real solution. These various approximations are each based on different assumptions, and are thus only valid under certain conditions. In general, the most important parameter when choosing the appropriate mathematical model is the Fresnel number, defined as:

$$F = \frac{D^2}{L\lambda}$$

where $L$ is the distance from the diffraction aperture to the plane of interest.
Near Field

When \( F > \sim 1 \), the waves are in what is called the Near Field or Fresnel zone. This zone is the most complex. The fields in this zone can be approximated using the Fresnel diffraction integral:

\[
E(x, y, z) = \frac{e^{jkz}}{j\lambda z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x', y', 0) e^{-j \frac{2\pi}{\lambda z} ((x-x')^2 + (y-y')^2)} dx' dy' \quad (2.14)
\]

This integral is still very hard to solve analytically, but is easier to solve numerically than the general solution.

Far Field

When \( F \ll 1 \), the waves are in what is called the Far Field or Fraunhofer zone. This is the zone we will be working in for the majority of this work. In this zone, the relative field values do not change with \( z \), but are scaled versions of each other, \( i.e. \), for any plane \( z_1 \) and \( z_2 \) in the far-field, there is some \( \xi \) such that \( E(x, y, z_1) = E(\xi x, \xi y, z_2) \). A more convenient way to express this, is that the amount of power propagating in a given direction is constant. Thus, we usually speak of the Far Field in terms of wavevectors, not spatial coordinates. The Far Field can be approximated using what is called the Near-To-Far-Field Transformation (NTFFT):

\[
E(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x', y', 0) e^{-j(k_x x' + k_y y')} dx' dy' \quad (2.15)
\]

where \( k_x = 2\pi \alpha / \lambda, k_y = 2\pi \beta / \lambda, \) and \( \alpha \) and \( \beta \) are the direction cosines defined by the wavevector:

\[
k = \frac{2\pi}{\lambda} (\alpha \hat{x} + \beta \hat{y} + \gamma \hat{z}). \quad (2.16)
\]

Critically important is the fact that Eq. 2.15 has the form of a Fourier transform. Thus, this equation tells us that the near fields (fields directly after the structure) and far fields (fields infinitely far beyond the structure) are related via the Fourier transform; a fact that provides invaluable insight into the design and operation of diffractive elements.

2.4 Electromagnetic Finite-Difference Time-Domain Simulation

A variety of mathematical tools exist for simulating an element’s optical properties. Each of these methods makes some assumptions or approximations about the element itself or about how it interacts with light. The more assumptions that are made, the more limited the simulation method is in what elements or optical phenomena that can be studied. Examples of various methods, in order of most to least assumptions, are: Jones calculus [33], Berreman’s 4x4 method
Figure 2.5: Fields propagating through a PG; red and blue colors indicate positive and negative fields respectively. This result was obtained using a version of the FDTD method developed in this work (see Chap. 7) and is extremely difficult if not impossible to obtain with analytic methods.

[34], rigorous coupled wave analysis [35], finite-element methods [36], and finite-difference time-domain (FDTD) methods [37]. In this work, we focus on the FDTD method.

The FDTD method is advantageous because it makes what are arguably the least number of assumptions about the electromagnetic fields or elements in question when compared to other simulation methods. Because of this, almost any arbitrary structure can be simulated and many phenomena may be observed, such as plasmon interactions, charge accumulation, evanescent waves, reflection, scattering, interference, and coherence. For example, with FDTD the fields propagating through a PG can be relatively easily obtained (Fig. 2.5). However, this diversity comes at a cost: FDTD simulations are generally very time-consuming, requiring vast amounts of computing power. For this reason, many specific FDTD algorithms are developed to simulate specific classes of elements at higher efficiency than with a generic FDTD algorithm.

2.4.1 Mathematical Basis

The FDTD method is based on approximating derivatives using linear algebraic expressions. The first-order central-difference approximation of a first and second order derivative are:

\[
\frac{\partial f(t)}{\partial t} \approx \frac{f(t + \Delta t/2) - f(t - \Delta t/2)}{\Delta t} \quad (2.17a)
\]

\[
\frac{\partial^2 f(t)}{\partial t^2} \approx \frac{f(t + \Delta t) - 2f(t) + f(t - \Delta t)}{\Delta t^2} \quad (2.17b)
\]

where \(\Delta t\) is the time step. It can be shown that the error of the approximation is related to the size of the time step, with smaller time steps resulting in less error.

The FDTD method for electromagnetic waves is based around using the finite-difference approximation of the derivatives in Maxwell’s equations. The key expressions are:
The grid scheme used in this work is the 3D Yee grid; shown is a unit cell with the location of the $E$ and $H$ fields.

\[
\frac{\partial}{\partial t} (\epsilon_0 \epsilon_r E) + \sigma E = \nabla \times H \quad (2.18a)
\]
\[
\frac{\partial}{\partial t} (\mu_0 \mu_r H) = -\nabla \times E. \quad (2.18b)
\]

The FDTD algorithm works by substituting the finite-difference approximations for the time derivatives, and solving for future time steps $E(t + \Delta t)$ and $H(t + \Delta t)$ based on current time steps $E(t)$ and $H(t)$. The curl operator is also approximated, using spatial equations analogous to 2.17b. Thus, the fields are solved for on a discrete grid of points spaced $\Delta u$ apart.

### 2.4.2 The Yee Cell

In 1966, Yee proposed the first use of finite-difference approximations to solve Eq. 2.18b. One of Yee’s key insights was to use an asymmetric grid for the different components of $E$ and $H$. The following describes a single unit cell using Yee’s convention, called the Yee cell, also shown in Fig. 2.6:
\[
E = \hat{x} E_x((i + 1/2)\Delta u, j\Delta u, k\Delta u) + \\
\hat{y} E_y(i\Delta u, (j + 1/2)\Delta u, k\Delta u) + \\
\hat{z} E_z(i\Delta u, j\Delta u, (k + 1/2)\Delta u)
\]

\[
H = \hat{x} H_x(i\Delta u, (j + 1/2)\Delta u, (k + 1/2)\Delta u) + \\
\hat{y} H_y((i + 1/2)\Delta u, j\Delta u, (k + 1/2)\Delta u) + \\
\hat{z} H_z((i + 1/2)\Delta u, (j + 1/2)\Delta u, k\Delta u)
\]

(2.19a)

(2.19b)

where \(i, j,\) and \(k\) are integers. When applying the finite-difference approximations as previously described using this grid scheme, the \(E\) and \(H\) fields are solved for one after the other in subsequent time-steps using a leap-frog technique; \(E(\Delta t + 1)\) is solved for using \(E(\Delta t)\) and \(H(\Delta t + 1/2)\), then \(H(\Delta t + 3/2)\) is solved for using \(H(\Delta t + 1/2)\) and \(E(\Delta t + 1)\). The details of how this algorithm works may be found in a multitude of literature [38].

2.4.3 FDTD Summary

The FDTD method is based on finite-difference approximations of the time derivatives and curl operators in Maxwell’s equations. This allows the differential equations, otherwise intractable, to be reduced to a set of linear expression and efficiently solved. The generality of the algorithm allows many complex, arbitrary structures to simulated and understood to a degree otherwise impossible.
Chapter 3

Direct Write System for CGH Creation

This chapter takes the form of a manuscript accepted to Optics Express, whose authors are Matthew Miskiewicz and Michael Escuti.

Abstract

We report on a direct-write system for patterning of arbitrary, high-quality, continuous liquid crystal (LC) alignment patterns. The system uses a focused UV laser and XY scanning stages to expose a photo-alignment layer, which then aligns a subsequent LC layer. The exposure of the alignment layer utilizes overlapping exposures, which sometimes results in interesting responses by the LC and ultimately enables us to create continuous alignment patterns with feature sizes less than the beam size. We present our system design and a thorough mathematical system description, starting from the direct-write system inputs and ending with the alignment of the LC. We fabricate a number of patterns to test the accuracy of our predictions, then design and fabricate a number of interesting patterns including a $q$-plate and polarization grating.

3.1 Introduction

Liquid crystals (LCs) are most prominently used in liquid crystals displays (LCDs), where they function as the active elements used to control pixel brightness. In these cases, the alignment of the LC is simple, being homogeneous over relatively large areas. However a number of other

---

1While this chapter uses the plural 1st person voice as is standard in the scientific community, almost the entirety of its contents are a direct result of the author’s efforts. The other primary contributor was Michael Escuti, who assisted in the development of the system description, planning of the experiments, and writing of the manuscript document.
applications require complex LC alignment patterns for correct operation. Examples of such elements are polarization gratings (PGs) [1], $q$-plates [39], beam shaping elements [18], phase plates [3], and LC actuators [40].

With the rise of photo alignment materials [30, 41], the alignment patterns required for such elements has become easier to realize using polarization holography [42]. However, while polarization holography is quite suitable for periodic patterns such as the PG, it does not solve the problem of easily aligning more arbitrary patterns.

In recent years, various photo-alignment techniques have been developed to efficiently record specific patterns. These include scanning a cylindrical beam while changing the polarization to record a PG [43], scanning a cylindrical beam in an azimuthal pattern while changing the polarization to record a $q$-plate [39, 44], using a reconfigurable spatial light modulator (SLM) to record arbitrary patterns including a lens array [45], and using a point-by-point direct-write system with a polarization modulator to record a complex beam-shaping element [18]. It is notable that this last kind of direct-write system is actually quite old, as the first reported direct-write system with polarization modulation dates back to 1976 [16].

In this paper, we present, analyze, and demonstrate a direct-write system capable of creating arbitrary and continuous LC patterns (in-plane orientation profiles) with high quality and resolution. The system design and operation is discussed in Section 2. In Section 3, we develop a flexible system description starting with the direct-write scan inputs and ending with the LC response. Although we use the system with photoalignment materials and LCs, the system description is applicable to any material set sensitive to linear polarization. In Section 4, we fabricate a few patterns to test the accuracy of our system description and its ability to predict the LC response. In Section 5, we use the parameters obtained from Section 4 to study some interesting patterns, including adjacent scan lines to explore the continuous nature of the LC alignment, intersecting scan lines to study non-intuitive LC responses, and a highest quality $q$-plate and PG.

All of our results indicate that this approach is by far the most flexible and diverse LC patterning approach, allowing the creation of arbitrary continuous LC patterns. In addition, the approach can be easily scaled for large areas and volume manufacturing, where multiple high resolution LC patterns can be created on the same substrate.

### 3.2 System Design

A high level schematic of the system is shown in Fig. 3.1. The optical train begins with a continuous wave 325 nm HeCd laser. The laser then passes through an active polarization modulator. Since we are using a photoalignment material which only responds to linear polarization, we are only interested in changing the linear polarization angle of the beam. Thus, possible polariza-
Figure 3.1: High-level schematic of the kind of polarization direct-write system studied in this paper.

Polarization modulators include a rotating HWP or a variable retarder combined with a quarter wave plate (QWP) (when configured properly, these two elements rotate the polarization angle as the retardation is changed). Our system uses the later, with a KD*P Pockels cell by ConOptics as the variable retarder. After this, the laser passes through a spatial filter and a collimating lens (not shown in figure). Last, the beam is focused to the substrate with a 40X objective lens. The minimum beam waist [46] we have achieved with this setup is $\sim 1 \mu m$.

The substrate itself is coated with a linear photopolymerizable polymer (LPP) and is placed on two stages for XY translation. The stages are from Newport (ILS200LM) and have 10 nm resolution. Since even the smallest beam waist is 100 times this resolution, the resolution and accuracy of the XY stages is neglected in the analysis. The entire system is controlled via a Newport XPS controller.

After exposure, the substrate is removed and a liquid crystal polymer (LCP) layer is spin cast on top of the LPP layer, then polymerized by UV light. If needed, multiple layers of LCP can coated. Alternatively, two LPP coated substrates forming a cell can be used in place of a single LPP coated substrate. In this case, a non-polymerizing LC may be used to fill the cell, allowing the pattern to be toggled on or off with an applied voltage. However, we restrict the current discussion to LCP coated on a single substrate with LPP, although we expect the principles should apply to cells as well.

We note that the design of our system is similar to some prior art such as [3]. However, the prior art employs a point-by-point exposure method where overlapping exposures are detrimental. In our approach, we intentionally utilize overlapping exposures in our scan patterns, which allow us to achieve orientation profiles otherwise impossible. In particular, our approach enables the creation of continuously varying orientation profiles from a discrete scan pattern, allowing us to accurately record important elements such as the $q$-plate and PG.
3.3 System Description

3.3.1 Approach

Starting with the inputs to the scanning system, we would like to be able to predict the LC response. By LC response, we specifically mean the in-plane LC optical axis orientation (or nematic director) and the quality of LC alignment. We define quality of alignment as the degree to which the LC is aligned in-plane; LC that is aligned out of plane (i.e., poorly aligned) is characterized by point defects, line defects, and a reduced birefringence.

Finding the LC response is not trivial, even for relatively simple alignment patterns. Sophisticated mechanics-based models such as elastic continuum theory have been developed to analyze specific LC patterns. This model is most commonly applied to study the response of uniformly aligned LCs under applied voltages, but has also been applied to PGs [47] to gain insight into the physical limits of PG fabrication. However, using elastic continuum theory for patterns more complex than the PGs or other simple periodic structures [48] present serious difficulty when approached from either an analytic or numeric perspective. Thus, our approach here is to develop a new model tailored specifically for the problem of LC on complexly patterned LPP, based on fundamental, observed interactions between an LPP exposure and the resulting LC response.

One option to do this would be to a) find the electromagnetic output of the system, b) find the physical response of the LPP to the electromagnetic output, and then c) find the physical response of the LC to the LPP. However, finding the response of the LPP is difficult to do and cumbersome to describe. Instead, we a) find the electromagnetic output, b) adjust it based on assumed properties of the LPP resulting in what we call the adjusted electromagnetic output, and then c) find the LCs physical response from the adjusted electromagnetic response. A high-level view of this process is shown in Fig. 3.2.

The electromagnetic output of the system is fully described by \( S(r, t) \), a Stokes vector with spatial and temporal dependence. The adjusted electromagnetic output is fully described by

![Diagram of system input, electromagnetic output, adjusted electromagnetic output, and liquid crystal response]
S(r), a time-independent Stokes vector with spatial dependence. The response of the LC is thus predicted based on the Stokes vector S(r). This approach allows us to avoid unneeded complications and also to predict the response of the LC using familiar metrics such as fluence and degree of linear polarization (DOLP).

### 3.3.2 System Inputs and Electromagnetic Output

We begin with the independent inputs to the system: the beam intensity profile \( I_b(r) \), the XY scan path \( r_b(t) \), and the instantaneous exposure polarization \( \psi_b(t) \). We will only consider linear polarization, such that \( \psi_b(t) \) is the orientation of the linear polarization. However, the model can easily be used to describe systems with elliptically polarized beams as well. Next we find the instantaneous exposure intensity over the entire scan space:

\[
I(r, t) = I_b(r - r_b(t)) \quad (3.1)
\]

Then we find the instantaneous exposure Stokes vector over the entire scan space, which is the electromagnetic output:

\[
S(r, t) = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = I(r, t) \begin{bmatrix} 1 \\ \cos 2\psi(t) \\ \sin 2\psi(t) \\ 0 \end{bmatrix} \quad (3.2)
\]

### 3.3.3 Adjusted Electromagnetic Output

We would like to adjust \( S(r, t) \) using some transformation function \( H \) that embodies the relevant properties of the LPP and removes the time-dependence. While it may not be valid for all LPPs, our experimental work with at least some supports the assumption that \( H \) is a simple average over the entire exposure:

\[
S(r) = H(S(r, t)) = \frac{1}{\tau} \int_0^\tau S(r, t)dt \quad (3.3)
\]

where \( \tau \) is the duration of the entire exposure.

We can say that \( H \) is a valid transformation from \( S(r, t) \) to \( S(r) \) if and only if all \( S(r, t) \) with the same \( S(r) \) produce the same response in the LC. Figure 3.3 shows some different \( S(r, t) \) with the same \( S(r) \) for our chosen \( H \), which may provide insight into what \( H \) implies physically. We interpret Fig. 3.3(a) as implying that the LPP is non-causal, an assumption we believe is valid if the exposure fluence is low enough, keeping the LPP away from saturation. Figure 3.3(a) also implies that it is the average alignment of the LPP that matters, not the individual exposures.
We interpret Fig 3.3(b) as implying that orthogonal exposure orientations are equivalent to an unpolarized input, i.e. they cancel each other out and have no influence on the alignment of the LC. Our $H$ also implies that the LPP responds at the point $r$ only to incidence fluence delivered to the point $r$ (i.e., neighboring regions do not affect each other), and also that the LPP’s response does not change with incident power.

We will transform $S(r)$ into three terms which we call the key parameters that allow for easy interpretation of the final LPP alignment. They are the total fluence $F(r)$, the average polarization orientation $\psi_{avg}(r)$, and the average DOLP $D_{avg}(r)$. The total fluence is found as:

$$F(r) = S_0(r)\tau$$  \hfill (3.4)

The average polarization orientation is easily found from $S_1(r)$ and $S_2(r)$ using the $\text{atan2}$ function (the unwrapped arctangent):

$$\psi_{avg}(r) = \text{atan2}(S_2(r), S_1(r))$$  \hfill (3.5a)$$

$$\text{atan2}(y, x) = 2\tan^{-1}\frac{y}{\sqrt{x^2 + y^2} + x}$$  \hfill (3.5b)

The average DOLP goes from 0 to 1 and the calculation is straightforward:
Figure 3.4: Threshold functions used to predict the magnitude of the LC anchoring vector from key parameters. a) Fluence threshold function $T_F$. b) DOLP threshold function $T_D$.

\[
D_{avg}(r) = \frac{\sqrt{S_1(r)^2 + S_2(r)^2}}{S_0(r)}
\]  

(3.6)

### 3.3.4 LC Response

From the key parameters we find what we define as the LC anchoring vector $A(r)$. The angle of the vector is the in-plane angle we expect the LC to align to. The magnitude of the vector, which varies from 0 to 1, represents our confidence that the LC will align well to $\angle A(r)$. To elaborate, if $|A(r)|$ at some $r$ is a high value such as 0.99, it means we have high confidence that the LC will align to $\angle A(r)$ and that the LC will be well aligned, *i.e.*, without scattering or defects. If $|A(r)|$ at some $r$ is a low value such as 0.01, it means that we do not have confidence that the LC will align to $\angle A(r)$ or that the LC will be well aligned. When LC is not well aligned, it usually means that the azimuthal angle is random and will result in a defect. We emphasize that $|A(r)|$ is our prediction confidence, and when our prediction confidence is low it means we do not know how the LC will respond. Thus, when $|A(r)|$ is low, the LC may be well aligned to $\angle A(r)$, well aligned to some other angle, poorly aligned to some angle, or have no apparent alignment at all depending on a multitude of variables which this model does not account for.

In general, $A(r)$ is found using some function $T$ which itself is a function of the key parameters:

\[
A(r) = T\{\psi_{avg}(r), F(r), D_{avg}(r)\}
\]  

(3.7)

We make the assumption that the angle and magnitude information of $T$ are independent, which simplifies the function considerably. A further assumption is that the magnitude of $T$ has the form of two multiplied threshold functions. The hyperbolic tangent is a convenient threshold function to use, and combined with a simple assumption for the angle information, $A(r)$ is
found as follows:

\[
\langle A(\mathbf{r}) \rangle = \psi_{\text{avg}}(\mathbf{r}) \tag{3.8a}
\]

\[
|A(\mathbf{r})| = T_F\{F\}T_D\{D_{\text{avg}}\} \tag{3.8b}
\]

\[
T_F\{F\} = \frac{1}{2} \left( \tanh\left( \frac{F(\mathbf{r}) - F_T}{w_T^F} \right) + 1 \right) \tag{3.8c}
\]

\[
T_D\{D_{\text{avg}}\} = \frac{1}{2} \left( \tanh\left( \frac{D(\mathbf{r}) - D_T}{w_T^D} \right) + 1 \right) \tag{3.8d}
\]

where $F_T$ and $D_T$ are the threshold values and $w_T^F$ and $w_T^D$ are the widths of the threshold functions. The widths specify the distance from the threshold value where the function is equal to $0.5 \pm 0.33$. Examples of the functions $T_F$ and $T_D$ are shown in Fig. 3.4. We use this form of $T$ for the remainder of this paper.

### 3.3.5 Physical Interpretation of $H$ and $T$

Since the material transfer functions we have introduced ($H$, $T$) are unique to this formalism, providing a physical interpretation of them may give the reader a better understanding of their meaning. As mentioned previously, $H$ embodies the material properties of the LPP. Physically, this function accounts for the photo-induced molecular reorientation of the LPP (see [49, 50]). The precise reorientation mechanism is abstracted away, allowing us to treat different classes of photoalignment materials (e.g., trans-cis or photocrosslinkable) similarly. $T$ embodies the materials properties of the LC in a similarly abstracted way. In particular, $T$ describes the response of the LC when in contact with an LPP layer exposed with a given fluence, DOLP, and orientation angle. It encompasses metrics the reader is probably familiar with such as the LC elastic constants and the Frank free energy density, although in a limited way. Depending on the complexity and sophistication of the functions $H$ and $T$, a wide range of physical phenomena occurring in LPP and LC may be accounted for with this model.

### 3.4 System Description Validation

We have done a series of experiments to test the accuracy of our system description in predicting the LC response. The experiments validate the primary assumptions of $H$ (non-causality, alignment to average exposure angle, and depolarization by orthogonal polarizations) and the basic form of $T$ (a threshold fluence function multiplied by a threshold DOLP function). Where applicable, all experiments were simulated using a numerical implementation of our system.
description with a discrete spatial grid and discrete time steps. The experimental films were fabricated using commercial LPP (Rolic ROP-108) and LCP (Merck RMS03-001C). All films were creating by spin-coating an LPP layer approximately 200 nm thick, then after exposure spin-coating two LC layers with a combined thickness of about 1.3 µm. All images were captured using the setup shown in Fig. 3.5. No post-processing was done to the images, and the alignment of the films was determined by least squares error color matching to the colormap in Fig. 3.5(b).

The light source for the microscope was a white-light lamp. Different wavelengths experience different retardation for a given thickness and birefringence of an LC film. Because of this, the polarization state of the light after any LC film will in general be elliptical and will vary by wavelength. The exception is when the LC optical axis is parallel to or orthogonal to the input polarization, in which case no wavelength experiences any retardation and the polarization output is identical to the input. By inserting a full waveplate oriented at 45° (compared to 25° of the input), we cause the light to experience chromatic retardation even in these cases, making them distinguishable. In combination with an analyzing polarizer that is not parallel to or orthogonal to the input polarizer, the final result is that each possible LC orientation angle (0° to 180° due to rotational symmetry) produces a different, non-grayscale transmitting color as shown in Fig. 3.5(b).

In none of our tests have either the power of the beam or the scanning speed affected the LC response (as reflected by their absence in our model). However, some practical details about these parameters are provided for the benefit of the reader. In our scanning system, the ideal
Figure 3.6: The first validation experiment, designed to test the effect of fluence on LC alignment. Pattern a) was created by scanning adjacent, uniform vertical lines and varying the fluence in the horizontal dimension. Pattern c) is the reverse of pattern a), and pattern b) is the superposition of a) and c). From this experiment we determined a fluence threshold to be used in subsequent simulations.

maximum single-direction scanning speed is 200 mm/s, the maximum beam power is 17 mW, and the minimum beam power is in the nW range. For a given beam width, there are many combinations of beam power and scanning speed that result in the same delivered fluence and identical LC response. For a given pattern, the speed is usually chosen first (the fastest speed possible), and then the needed beam power is calculated. In practice, the scanning speed is limited by the desired fidelity of the scan pattern and digital control limitations. For simple patterns such as the PG, the scan speed may be as high as 150 mm/s; for complex patterns such as the $q$-plate, the scan speed may be as low as 100 nm/s near the center of the pattern.

### 3.4.1 Fluence Threshold

We wish to know if the use of the fluence threshold function $T_F$ in Eq. (3.8) accurately reflects the LC response when the LPP is exposed to various fluence levels; that is to say, if the LC aligns poorly below and well above some fluence value. To test this, we scanned three large areas using many adjacent stripes with a beam of width $\sim 12 \, \mu m$. An image of the LCP response and corresponding exposure fluence is shown in Fig. 3.6. The scan patterns were uniform in the vertical dimension, and varied slowly in the horizontal dimension. In Fig. 3.6(a) and 3.6(c), the fluence varied from low (50 nJ/cm$^2$) to high (50 mJ/cm$^2$), and high to low respectively. Figure 3.6.b is a combination of Fig. 3.6(a) and 3.6(c) written on top of each other in the same location.

From Fig. 3.6(a) and Fig. 3.6(c) we see that poor alignment occurs at lower fluence and good alignment occurs at higher fluence. Figure 3.6(b) has higher fluence everywhere and the
alignment is also good everywhere. In addition, since $\psi_{\text{avg}} = 0^\circ$ everywhere, $D_{\text{avg}} = 1$ everywhere according to Eq. (3.6). Therefore, we can confidently attribute the decrease in alignment quality to a decrease in fluence, and conclude that the use of $T_F$ in Eq. (3.8) is reasonable. Based on these results, we identify the threshold fluence as 19 mJ/cm$^2$. This value is the same order of magnitude as the manufacture’s recommended dose (60 mJ/cm$^2$).

We note that the LCP aligns poorly for very low fluence ($<1$ mJ/cm$^2$), then aligns somewhat well with low fluence (1-10 mJ/cm$^2$), then aligns poorly again with higher fluence (10-19 mJ/cm$^2$), then finally aligns well for even higher fluence (>19 mJ/cm$^2$). These effects can be seen in Fig. 3.6 and are highly consistent and reproducible but lie outside the scope of this paper. The fluence threshold function we use treats all fluence values less than the threshold as resulting in poor alignment, which is an adequate approximation for our purposes.

### 3.4.2 DOLP Threshold

Next, we address whether or not the DOLP threshold function $T_D$ in Eq. (3.8) accurately reflects the LC response when the LPP is exposed to various DOLP levels. If our assumptions about $H$ and $T$ are correct, then a low DOLP should result in poor alignment and a high DOLP should result in good alignment. To test this, we scanned three large areas again using many adjacent stripes with a beam of width $\sim12$ $\mu$m. An image of the LCP response and corresponding DOLP and average polarization is shown in Fig. 3.7. The scan patterns were uniform in the vertical dimension, and varied slowly in the horizontal dimension. In Fig. 3.7(a) and 3.7(c), the fluence is constant but $\psi$ varied linearly from 0° to 90°. However, importantly, the sign of the rotation of $\psi$ is opposite for Fig. 3.7(a) and 3.7(c). Figure 3.7(b) is a combination of Fig. 3.7(a) and 3.7(c) written on top of each other in the same location, except they were each half fluence so that the total fluence would be the same as the other patterns. For each pattern, an appropriate beam power and scan speed was chosen to give an average fluence of 50 mJ/cm$^2$.

In Fig. 3.7(b), $D_{\text{avg}}$ varies from 1 at the far left (both patterns have $\psi = 0^\circ$) to 0 at the middle (the difference between $\psi$ of the two patterns is 90°) and back to 1 at the far right (both patterns have $\psi = 90^\circ$). Of the three patterns in Fig. 3.7, poor alignment only occurs near the center of Fig. 3.7(b) where the DOLP is lowest. Since the fluence is constant, we can attribute the decrease in alignment quality to a decrease in DOLP, and conclude that the use of $T_D$ in Eq. (3.8) is reasonable. Based on these results, we identify the DOLP threshold as 32.1%.

### 3.4.3 Alignment to Average Polarization Angle

Lastly, we wish to confirm that the alignment angle of the LC is equal to the average exposure angle of the LPP, as assumed in Eq. (3.8). The experiment just discussed for testing $T_D$ serves this purpose as well. In Fig. 3.7(a) and Fig 3.7(c) we see that the alignment angle varies linearly
Figure 3.7: The second validation experiment, designed to test 1) the effect of DOLP on LC alignment quality, and 2) if the LC aligns to the average exposure angle of the LPP. Pattern a) was created by scanning adjacent, uniform vertical lines and varying the polarization angle in the horizontal dimension from 0° to 90°. Pattern c) is the same as a), except the polarization varies from 0° to -90°. Pattern b) is the superposition of a) and c) with half fluence each (equivalent total fluence). From this experiment we determine a DOLP threshold to be used in subsequent simulations and confirm the aligning properties of the LC.

along with the average polarization angle. In Fig 3.7(b), \( \psi_{avg} \) forms two discrete domains, being \( \psi_{avg} = 0° \) to the left of the middle and \( \psi_{avg} = 90° \) to the right of the middle. The alignment angle of the LCP in the fabricated film is remarkably close to \( \psi_{avg} \). We suspect that differences in the results can be attributed to very small errors in the polarization purity and accuracy of the scanning system, which prevent two discrete domains from being formed and instead induce an averaging from one angle to the other near the center.

3.5 Analysis of Selected Scan Patterns

We will now analyze a few interesting scan patterns. For the remainder of this paper, we will use \( F_{TH} = 19 \text{ mJ/cm}^2 \), \( D_{TH} = 32.1\% \), \( w_F^{TH} = F_{TH}/4 \), and \( w_D^{TH} = D_{TH}/4 \). The threshold functions with these values are shown in Fig. 3.4.
3.5.1 Adjacent Scan Lines

We consider a pattern that comprises three vertical lines spaced a distance \( d \) apart, where the lines have \( \psi_{b1} = \psi_0 - 50^\circ \), \( \psi_{b2} = \psi_0 \), and \( \psi_{b3} = \psi_0 + 50^\circ \) respectively. Since our system is assumed to be non-causal, we can describe the pattern as three lines being scanned simultaneously as follows:

\[
I_b(r) = I_0 \exp \left( -2 \frac{x^2 + y^2}{w_0^2} \right) 
\]

\[r_{b1}(t) = t\hat{y} + d\hat{x}\]  
\[r_{b2}(t) = t\hat{y}\]  
\[r_{b3}(t) = t\hat{y} - d\hat{x}\]

where \( w_0 = 23 \, \mu m \), \( \psi_0 = -20^\circ \), and \( d = 18 \, \mu m \), 36 \, \mu m, and 72 \, \mu m. Using these input parameters, we numerically solve for the total fluence and expected LC orientation angle in the \( x \) dimension. Shown in Fig. 3.8 are images of films fabricated using these parameters, the fluence of each scan line, the resulting total fluence, and the predicted and measured LCP orientation angles. We note that our method of measuring the orientation angle only allows us to measure angles in discrete 10° steps. In addition, this film seemed to have slightly different characteristics (i.e., thickness) than that of the other patterns, resulting in slightly different transmitting colors and a less accurate color mapping. Nevertheless, the measurements are accurate enough for the purposes of this experiment.

We make three important observations. First, when the scans are farther apart (Fig. 3.8(a) and 3.8(b)), the orientation angle plateaus at \( \psi_0 \) near the second scan line \( (x = 0) \). Second, when the lines are very close (Fig. 3.8(c)), the presence of the second scan line cannot be detected, as the orientation angle varies smoothly from from \( \psi_0 - 50^\circ \) to \( \psi_0 + 50^\circ \). Third, when the scans are overlapping to any degree (Fig. 3.8(b) and 3.8(c)) the orientation angle does not have discrete boundaries, but rather continuously varies from the orientation of one scan line to the next. This is quite amazing, though not altogether surprising.

Therefore, by controlling the spacing and polarization of adjacent scan paths, we can for the first time create arbitrary, continuously varying orientation profiles, subject to some limitations by the size of the recording beam and material response parameters. This feature is incredibly useful, allowing a discrete scan to create a continuous profile, and is a large incentive to use a direct-write system to pattern LC elements. We also point out that the orientation plateau in Fig. 3.8(b) is smaller than the beam width; hence creation of features smaller than the beam is also possible with this system.
Figure 3.8: Adjacent, uniform scan lines with polarization orientations of $\psi_0 - 50^\circ$, $\psi_0$, and $\psi_0 + 50^\circ$. The distance between the scan lines decreases from a) to c), and we observe an averaging of the LC orientation angle. By sufficiently overlapping neighboring scan lines, we can create continuously varying orientation profiles.

### 3.5.2 Intersecting Scan Lines

Next we examine a simple pattern that produces some interesting results. The pattern is two intersecting scan lines with polarizations orientated $90^\circ$ apart. We might qualitatively expect that where the lines meet there will not be good alignment, as the LC will have no preferred direction. However, without a quantitative description of the system such as the one we have developed, it is hard to ascertain where exactly well and poor alignment will occur. We can express the scan as follows:

\[
I_b(r) = I_0 \exp \left( -\frac{2x^2 + y^2}{w_0^2} \right) \tag{3.10a}
\]

\[
r_{b1}(t) = tx \tag{3.10b}
\]

\[
r_{b2}(t) = ty \tag{3.10c}
\]

\[
\psi_{b1}(t) = 0 \tag{3.10d}
\]

\[
\psi_{b2}(t) = \frac{\pi}{2} \tag{3.10e}
\]

where $w_0 = 23 \ \mu$m. The power was set such that the fluence at the center of each line was 50
Figure 3.9: Two intersecting scan lines with polarization orientations 0° and 90°. The average polarization angle at every location is either 0°, 90°, or undefined where the fluence delivered by each line is equal. As a result, discrete domains are produced. a) Simulated $F$, b) $D_{avg}$, c) $|A|$, d) $\psi_{avg}$ using the color bar in Fig. 3.5(b), e) $A$, created by overlaying c) and d). f) Actual image of fabricated film, color bar in Fig. 3.5(b).

mJ/cm². Using these input parameters, we numerically solved for the key parameters and the final LC response.

All of the relevant parameters are shown in Fig. 3.9, as well as the actual result of experimentally scanning the pattern. The plot of $F$ (Fig. 3.9(a)) needs no explanation. The average DOLP $D_{avg}$ (Fig. 3.9(b)), is very high near the center of each scan line and drops to 0 at points which are equidistant from each scan line. The alignment prediction confidence $|A|$ (Fig. 3.9(c)), combines $F$ and $D_{avg}$ and the threshold functions. The average polarization angle $\psi_{avg}$ (Fig. 3.9(d)), is not intuitive at first glance, but the concept here is similar to Fig. 3.7(b). Since the scan lines have orthogonal polarizations, they cannot be averaged together. Therefore, at any given point one of the scan lines will dominate and solely determine $\psi_{avg}$.

We have combined Fig. 3.9(c) and Fig. 3.9(d) giving a complete picture of the predicted result in Fig. 3.9(e). Upon comparing this to the actual result shown in Fig. 3.9(f), we see that the results match very well, including the X shape in the center. Last, we note that the random line defects that appeared in this pattern are unusual; they did not appear in any of the other
Figure 3.10: A simulated and fabricated $q$-plate, which is a kind of azimuthal waveplate. a) Simulated $F$, b) $D_{avg}$, c) $|A|$, d) $\psi_{avg}$ using the color bar in Fig. 3.5(b), e) $A$, created by overlaying c) and d). f) Actual image of fabricated film, color bar in Fig. 3.5(b).

patterns fabricated, even though all patterns were created on a single substrate with the same LPP and LCP fabrication steps.

### 3.5.3 Q-Plate

A $q$-plate is a kind of azimuthal waveplate used to affect the orbital angular momentum of light. High quality $q$-plates have been fabricated with a variety of material sets including LPP/LC [39, 44, 51]. Of particular interest in $q$-plate fabrication is the minimum possible size of the defect at the center of the element, also called the singularity defect. We can fabricate $q$-plates by scanning the beam in a spiral pattern while continually changing $\psi$. This pattern can be expressed as follows:
\[ I_b(r) = I_0 \exp \left( -\frac{2x^2 + y^2}{w_0^2} \right) \]
\[ r_b(t) = \left( m + \frac{rt}{2\pi} \right) (\cos(2\pi t) \hat{x} + \sin(2\pi t) \hat{y}) \]
\[ \psi_b(t) = qt \]

where \( w_0 = 1 \, \mu m \), \( q \) is the charge of the \( q \)-plate, \( m \) is the minimum radius of the spiral, and \( r \) is the spacing between the rings of the spiral. The power was set such that \( F = 200 \, \text{mJ/cm}^2 \).

In practice, the scan pattern will be discretized by some time step as required by the scanning hardware. Here we assume the time discretization of the pattern is sufficiently small, so have neglected it in the above description.

In order to create the smallest possible singularity defect for a given beam width and \( q \)-plate charge, we used a numerical implementation of our model to simulate and optimize \( m \) and \( r \) for a charge 4 \( q \)-plate. The key parameters are shown in Fig. 3.10 for one of our best designs \((m = 2 \, \mu m \text{ and } r = 0.2 \, \mu m)\), along with an image of the predicted and the actual result of experimentally scanning the pattern. (We fabricated a small size \( q \)-plate because the central region is the only particular region of interest and it was easier to simulate; larger sized \( q \)-plates can easily be fabricated in the same way.)

The singularity defect predicted is about 5 \( \mu m \) in diameter. However, the actual film produced a kind of rectangular defect about 2 by 10 \( \mu m \) large. This demonstrates a case where our system description is not entirely accurate and more sophisticated \( H \) and \( T \) functions would be beneficial. Nevertheless, this \( q \)-plate has a singularity defect comparable to the smallest reported to date [51].

### 3.5.4 Polarization Grating

The polarization grating (PG) is one of the most useful liquid crystal elements, being used in imaging, beam-steering [1], spectroscopy [4], and many other fields. When light hits an ideal PG, the incident light is diffracted into a single order. The ideal PG has a linear phase profile which corresponds to a linearly varying optical axis. If the optical axis does not vary linearly, then other diffraction orders are produced decreasing the efficiency of the element. Thus, we are primarily interested in producing a perfectly linearly varying optical axis profile. The most convenient way to do this is to scan lines of constant polarization with some spacing between the lines, just like in Section 5.1 but with many more lines.

Our target PG has a pitch of 50 \( \mu m \), meaning the optical axis rotates 180° in 50 \( \mu m \). We must choose the size of the beam to scan and the spacing between the lines. If the beam size is
Figure 3.11: A PG fabricated by scanning adjacent scan lines, where the polarization rotates from line to line. The target PG pitch is 50 µm and the scan lines were spaced 7 µm apart using a beam width of 7.5 µm. a), b), and c) are images obtained using different magnification objective lenses and the plots below show the desired orientation profile compared to our measured profile.

very large, e.g. 90 µm, then every point will be exposed by many polarizations and we expect that the DOLP will be very low, causing poor alignment. Therefore, we expect a smaller beam size will be better and we chose a beam with a width of 15 µm. As for the line spacing, if the spacing is too far apart then space between the lines will not receive enough fluence. Thus, adjacent beams should be overlapping somewhat and we chose a line spacing of 7 µm. We chose a beam power and scan speed such that the average fluence delivered was 200 mJ/cm², well-above the required threshold.

The fabricated film using these parameters is shown in Fig. 3.11. Below the image of the film we show the ideal and measured optical axis profile, and below that the unwrapped phase profile. The phase of the PG is equal to twice the optical axis orientation [18]. The measured profile looks like a staircase because our colormap only contains 18 samples, causing the measured angle to jump in steps of 10°. Taking this into account, we see that the fabricated film is extremely close to the ideal PG profile. Using a laser, we measured >99.5% of the power going into the +1 and 0 orders. If the retardation was half-wave (which can be achieved relatively easily by optimizing the fabrication), we would expect the +1 order diffraction efficiency to be >99.5% [1, 52, 53]. Using this approach, we can very easily set the pitch of the PG and create large-area PGs that have the exact same pitch over the entire area, a major challenge for traditional PG
fabrication methods [54].

3.6 Conclusion

We have presented a direct-write system for fabrication of complex LC patterns. We developed a complete system description which appears to be very accurate in predicting the alignment angle and alignment quality of LC given a set of input scan parameters. We studied some patterns which highlighted the advantages of this system, including the continuous nature of the LC alignment and the creation of features smaller than the beam size. Using the system, we were able to fabricate a high-quality $q$-plate with minimal effort. Finally, we created a PG with a near-perfect linear optical axis profile which would result in a diffraction efficiency equivalent to prior best results if the retardation were half-wave.

We note that the system description, as we have presented it, is limited in a number of ways. For example, it does not account for causal effects in the photoalignment material (e.g., saturation), the effect of neighboring LC regions on each other, or the thickness of the LC film. It also does not predict anything regarding the out-of-plane LC orientation angle (i.e., pre-tilt). However, we believe that by choosing different functions for $H$ and $T$, it is possible to account for more complex alignment responses in the LC such as those just mentioned. The presented formalism is also flexible enough that it can be used with many different photoalignment and liquid crystal materials. In addition, We also envision that the adjusted electromagnetic response of this system may be used as an input to an elastic continuum model to provide an even more accurate prediction of the LC response in very complex situations. To conclude, this work enables for the first time a simple means of fabricating sophisticated LC elements, beneficial to a wide set of applications including displays, holography, imaging, spectroscopy, sensing, beam-steering, beam-shaping, and astronomy.

Acknowledgment

The authors gratefully acknowledge the support of the National Science Foundation (NSF grant ECCS-0955127).
Chapter 4

Computer Generated Polarization Gratings

Equipped with a direct-write fabrication tool and an accurate description of the system, we turn our attention to CGHs themselves. It is known that CGHs possess some unique attributes which have made them useful in various applications [18, 3, 14, 39]. However, with a thorough understanding of how to design CGHs and how they operate, the usefulness of CGHs greatly increases. In this chapter, we study the Polarization Grating (PG), perhaps the most important GPH element. We address the following questions: What are the optimal design parameters for fabricating high-quality PGs? What impact do these parameters have on the element’s properties? What is the highest quality direct-write PG that can be fabricated?

First, we describe and analyze the PG design parameters in an analytical manner. These design parameters are related to the scanning system and are called the scan parameters. Next, we experimentally fabricate PGs and study the effects of various scan parameters, with the ultimate goal of optimizing the scan parameters. Finally, we experimentally characterize the highest quality PG fabricated with our system.

4.1 Polarization Gratings

PGs are used in a multitude of applications such as beam steering systems [1], polarization conversion systems [2], interferometers [4], and variable optical attenuators [53]. The key PG specifications vary by application, but are typically the first-order diffraction efficiency $\eta_{+1}$, the active area of the element, the accuracy of the grating pitch, and the wavefront distortion.

While this chapter uses the plural 1st person voice as is standard in the scientific community, almost the entirety of its contents are a direct result of the author’s efforts.
Figure 4.1: a) The scan pattern used to record PGs. The arrows indicate polarization orientation. b) The writing beam is assumed to have a gaussian profile.

With the classic polarization holography recording setup, it is possible to achieve extremely high values for $\eta_{+1}$. However, it is difficult to achieve high-quality characteristics for the remaining specifications.

In the classic polarization holography setup, setting the grating pitch $\Lambda$ involves physically adjusting the optical setup until the desired $\Lambda$ is obtained. While obtaining a very accurate pitch ($i.e., \pm 10$ nm) is possible, it requires a very precise calibration of the setup. In the case of PGs with large active areas ($e.g., 150x150$ mm), a number of challenges present themselves. It was previously shown that larger area PGs usually result in a non-uniform $\Lambda$, $i.e., \Lambda$ is not constant but is a function of spatial position [54]. This non-uniformity is highly undesirable. In addition, the exposure setups for recording such large area PGs require large, expensive optics, and the setup is in general very susceptible to external vibrations.

Lastly, achieving extremely low wavefront distortion is the hardest specification to meet with the classical recording setup. This is because any wavefront distortion introduced into the recording beams will be recorded into the PG. Thus, the only solution to avoid wavefront distortion is extremely precise optics in the recording setup.

We propose the use of a direct-write system as the simplest and most effective means of recording PGs of any size with extremely accurate and uniform $\Lambda$, very low wavefront distortion, and very high $\eta_{+1}$.
4.2 Scan Pattern and Parameters

When recording the PG, we chose the scan pattern shown in Fig. 4.1(a). In our tests, this scan pattern has proven most successful. The pattern consists of adjacent, uniform, evenly spaced parallel scan lines. The only difference between lines is the linear polarization angle of the recording beam. Due to the averaging properties of the LPP/LC materials, the LC orientation angle smoothly varies from one scan line to the next. This scan pattern can be described by the line spacing \( l \), the beam width \( w_0 \), the PG pitch \( \Lambda \), and the average delivered fluence \( F_{\text{ave}} \).

The polarization of each of line is constant, and the difference in polarization between adjacent lines is \( \psi_{\text{diff}} = \frac{\pi l}{\Lambda} \). To aid in analysis, we define three additional parameters: the normalized beam size \( B \), the normalized line spacing \( L \), and the beam overlap \( O \):

\[
L = \frac{l}{\Lambda} \quad \text{(4.1a)}
\]
\[
B = \frac{2w_0}{\Lambda} \quad \text{(4.1b)}
\]
\[
O = \frac{B}{L}. \quad \text{(4.1c)}
\]

We illustrate the meaning of these parameters with some examples: \( L = 0.1 \) means that the line spacing is one tenth of the pitch and that there are ten scan lines per pitch; \( B = 0.1 \) means that the beam diameter is one tenth the size of the pitch; and \( O = 1 \) means that the beam size and line spacing are the same size, hence minimal overlap occurs but the entire pattern is still exposed. An example scan pattern with \( L = 1/3 \) and \( B = 2/3 \) is shown in Fig. 4.1.

The line spacing directly affects the scan time (i.e., the time required to complete a scan). For the case of large area PGs with small \( \Lambda \), the scan time can be substantial. If it takes 0.25 s to scan a single line of a 150x150 mm PG, then for \( \Lambda = 2 \, \mu\text{m} \) and \( l = 0.5 \, \mu\text{m} \) the total scan time will be 42 hr. Such long scan times are highly undesirable for a number of reasons. Thus, we wish to set the line spacing as large as possible without sacrificing PG quality.

The minimum \( \Lambda \) that can be created will be limited by the minimum beam size that can be achieved. Since the minimum beam size is a direct result of the laser and optics in the direct-write system, it is very desirable to know the relationship between the minimum \( \Lambda \) and minimum beam size; in other words, what is the largest \( B \) that still results in a high-quality PG. In our system the minimum beam diameter is \( \sim 2 \, \mu\text{m} \).

The average delivered fluence should be higher than the threshold fluence in order to achieve good LC alignment. But even if the average fluence is higher than the threshold, some parts of the pattern may still receive fluence less than the threshold due to the Gaussian profile of the recording beam. Therefore, the average fluence should be chosen as much higher than the
threshold. However, if the delivered fluence is too high, then the LPP saturates, which prevents the LC from aligning according to the theory of Chap. 3. Thus, the average fluence should be as high as possible without saturating the LPP. Experimentally, we found a value of 200 mJ/cm² to be a good value, and is the fluence used for the remainder of this chapter.

4.3 Scan Pattern Optimization

4.3.1 Scan Pattern Evaluation Strategy

The system description given in Chap. 3 allows the LC’s response to be predicted from a given scan pattern. The LC’s predicted response is characterized by the orientation angle \( \angle A \) and alignment confidence \( |A| \). The predicted orientation angle is straightforward to understand and needs no explanation. The alignment confidence ranges from 0 to 1 and represents our confidence that the LC will align at \( \angle A \). For the purposes of this chapter, we treat the alignment confidence as equivalent to alignment quality; since we are only interested in PGs with good alignment quality, we are thus only interested in scan patterns which result in high \( |A| \).

After obtaining \( A \), the diffraction properties of the PG can be simulated by assuming the resultant phase profile is twice the orientation angle: \( \phi_{PG} = 2\angle A \). This result follows from theory regarding the geometric phase (Pancharatnam-Berry phase) in Chap. 2. With \( \phi_{PG} \), the near-to-far-field transformation (NTFFT) is applied (i.e., the Fraunhofer diffraction approximation) to obtain the far-field diffraction orders. From these orders \( \eta_{+1} \) can be determined.

Our strategy for quantifying the performance of a set of scan parameters is summarized as follows:

1. Determine the scan pattern from chosen scan parameters
2. Simulate the scan pattern to find \( A \)
3. Find the phase profile of the PG as \( \phi_{PG} = 2\angle A \)
4. Apply the NTFFT to \( \phi_{PG} \) to find \( \eta_{+1} \)
5. Evaluate the scan parameters based on \( |A| \) and \( \eta_{+1} \)

The entire process is demonstrated for an example scan pattern with \( B = 0.6 \) and \( L = 0.4 \). The intensity cross section for each scan line that influences a single pitch is given in Fig. 4.2(a). The intensity cross section for each scan line that influences a single pitch is given in Fig. 4.2(a). The scan pattern is simulated to find \( A \), resulting in Fig. 4.2(b)-(d) as well as the orientation profile (not shown). The phase is found as twice the orientation profile in Fig. 4.2(e). The diffraction efficiency (D.E.) for all orders is calculated using the NTFFT in Fig. 4.2(f). Finally, we judge the effectiveness of the scan parameters: from Fig. 4.2(d) we see that the alignment...
Figure 4.2: Simulated results for a scan pattern with $B = 0.6$, $L = 0.4$, and $F_{\text{ave}} = 200$ mJ/cm$^2$. a) Intensity cross section for each scan line influencing a single pitch. b) Total fluence. c) Average degree of linear polarization (DOLP). d) Alignment quality calculated using b) and c). e) Phase profile, found as twice the LC orientation angle. f) Diffraction efficiencies found by applying the NTFFT on e). From the results, we can evaluate these scan parameters as unfavorable.

confidence is low which may yield LC defects; furthermore, from Fig. 4.2(f) the maximum $\eta_{+1}$ is low. Thus, these are unfavorable scan parameters.

To contrast this, 4.3 shows a scan pattern with $B = 0.8$ and $L = 0.2$. With this larger sized beam and smaller line spacing, much more averaging of the orientation profile takes place. This results in a virtually constant fluence, DOLP, and alignment quality, a linear phase profile, and a calculated $\eta_{+1} > 99.999\%$. Since the alignment quality is high, we trust the calculated $\eta_{+1}$ and we judge these as favorable scan parameters.
Figure 4.3: Simulated results for a scan pattern with $B = 0.8$, $L = 0.2$, and $F_{\text{ave}} = 200$ mJ/cm$^2$. a) Intensity cross section for each scan line influencing a single pitch. b) Total fluence. c) Average degree of linear polarization (DOLP). d) Alignment quality calculated using b) and c). e) Phase profile, found as twice the LC orientation angle. f) Diffraction efficiencies found by applying the NTFFT on e). From the results, we can evaluate these scan parameters as favorable.

### 4.4 Parameter Space Analysis

We applied this evaluation strategy to all permutations of $B$ and $L$ where $B$ was varied from 0.05 to 0.5 in steps of 0.025 and $L$ was varied from 0.05 to 1.2 in steps of 0.05. For all cases, we used $F_{\text{ave}} = 200$ mJ/cm$^2$.

The minimum predicted alignment quality, shown in Fig. 4.4, will be low only if at some part of the PG the delivered fluence is too low or if the average degree of linear polarization $D_{\text{ave}}$ is too low. Considering the fluence first: since we have fixed $F_{\text{ave}}$, the only way some part of the PG will receive too low of a fluence is if the beam size is small relative to the line spacing, i.e. if $O$ is low. Int Fig. 4.4, we see this effect as a border between high and low confidence on the left of the plot at $O \simeq 1$. Now considering $D_{\text{ave}}$: as $L$ gets closer to 1, the polarization change
between each scan line increases. Assuming \( O \) remains constant, this results in a decreased \( D_{ave} \). This effect can be seen in the plot as a border between high and low confidence around \( L = 0.4 \). Finally, if \( B > 1 \) (the beam size is larger than the pitch), then \( D_{ave} \) will be low, regardless of \( L \), seen in the figure as a border between high and low confidence around \( B = 1 \).

Even with good alignment, however, the D.E. may still be low. The predicted \( \eta_{+1} \), shown in Fig. 4.5, will decrease whenever the LC alignment profile deviates from the ideal linear profile. The essential feature of this plot is that the diffraction efficiency will improve if \( B \) is larger or if \( L \) is lower. A reasonable explanation for this is that a larger \( B \) results in a higher \( O \) and a lower \( L \) results in more scan lines per pitch, both of which result in a more linear profile.

Typically, we are only interested in scan parameters which result in \( \eta_{+1} > 99\% \). We can approximate the valid parameter space as a triangle by combining Fig. 4.4 and Fig. 4.5 as as shown in Fig. 4.6. Thus, we might expect a very high \( \eta_{+1} \) when \( O \geq 2.5 \), and \( B \leq 0.85 \). If optimizing total scan time is the primary concern, choosing \( B = 0.85 \) and \( L = 0.33 \) is ideal (unity divisible \( L \) values are desirable, as discussed later). If maximum yield is a concern (i.e., the sample should have a very high chance of success), choosing \( B = 0.5 \) and \( L = 0.05 \) could be considered ideal, as the values are very far from the edge of the valid parameter space. This may be important when the beam size cannot be controlled precisely or when processing conditions might cause the simulation model to break down (e.g., coating very thick LC layers).
Figure 4.5: The predicted $\eta_{+1} |A|$ vs. the normalize beam size $B$ and the normalized line spacing $L$ assuming $|A| = 1$. These efficiencies assume the profile is perfectly aligned and so represent an upper limit.

4.5 Experimental Results

To test these hypothesis, we fabricated a number of PG samples and measured the resulting D.E. The LC thickness was optimized to be half-wave, so we expect D.E. $> 99.9\%$ if the orientation profile is linear. We fabricated 9 samples, with $B = 0.45, 0.8, 1.3$ and $L = 0.1, 0.25, 0.45$. All samples had a fluence of 200 mJ/cm$^2$. According to the simulation results of the previous section, we would expect only samples d, e, g, and h (see Fig. 4.7) to result in good alignment and high $\eta_{+1}$.

Images of the the samples viewed through a microscope with crossed polarizers are shown in Fig. 4.7. Matching the theory, we see that samples d, e, g and h (which all have reasonable scan parameters) have the best alignment and most linear profiles. Sample e has the most aggressive parameters of these four, and its alignment is the worst with many defects showing up. Samples c, f, and i all use $B = 1.3$, and the patterns are very chaotic with many massive LC defects, making the PG profile barely distinguishable. Samples a and b both use $L = 4.5$, and this large of a line spacing results in a very discrete looking profile.

We characterized the samples using an RCP 633 nm HeNe laser. Images of the transmitting light are shown in Fig. 4.8. Samples c, f, and i show a very large zero-order leakage and immense scattering, resulting in very low $\eta_{+1}$. Samples a and b show slight scattering and also much light in orders other than $+1$. Also present are half-orders, discussed in the next section. Again matching the theory, we can see that samples d, e, g and h have the highest $\eta_{+1}$ of the nine
Figure 4.6: Predicted optimal scan parameters for creating high-efficiency PGs. Parameters inside the white triangular region have high efficiency. Parameters with larger line spacing result in a shorter scan time. Parameters farther from the edges of the high-efficiency region have the maximum yield, i.e., the highest likelihood of success.

samples. However, the efficiencies are notably less than what we expected. After analysis and additional experiments, we realized the less-than-stellar efficiency was a result of coating an LCP layer that was too thick, likely causing the DOLP threshold parameter to be increased. Even so, the relative difference between the samples show that the trends identified in the previous section are accurate.

4.6 Analysis of Non-Ideal PG Diffraction

The ideal polarization grating has a linear phase profile and half-wave retardation everywhere, leading to 100% D.E. into a single diffraction order. In practice, the best PGs can achieve efficiencies remarkably close to this [1, 53, 52]. However, as seen from the results of the prior section, there are many factors which can prevent a PG from performing this well. The high-level causes of poor efficiency can be described as LPP fabrication error and LCP fabrication error. These high-level causes lead to a number of physical phenomena which are directly responsible for the poor efficiency. They are incorrect retardation, LC defects, and a nonlinear phase profile. We will discuss the fundamental physical phenomena at play in these PG non-idealities and correlate the related high-level causes, and lastly attempt to identify the reasons for the various failure in the fabricated samples.
Figure 4.7: Polarizing microscope images of fabricated samples. Different shades correspond to different alignment orientations. Samples g and h represent high-quality alignment.

4.6.1 Incorrect Retardation

The first and also most common reason for poor PG efficiency is incorrect retardation. As described in Chap. 2, to achieve 0% zero-order leakage the retardation should be half-wave. If the retardation varies from half-wave, the zero-order leakage increases. Incorrect retardation is caused by LCP fabrication errors in the majority of cases. The most common cause is that the LCP thickness is not correct, due to improperly chosen spin-coating parameters. The spin coating process can also cause the thickness to be non-uniform over the area of the element, which will cause the zero-order leakage to vary according to position. The retardation may also be incorrect if the thickness is uniform and as desired, but the birefringence of the liquid crystal
Figure 4.8: Images of a 633 nm beam after passing through samples A-I. Samples d, e, g, and h are the highest quality of the nine samples, but there still significant power diffracted into the 0 and +2 orders; since the alignment quality is good, this indicates the alignment profiles are nonlinear. Samples a and b show many other higher orders, indicating these profile are very nonlinear. The background noise in samples c, f, and i is caused by the massive quantity of LC defects which scatter the incident beam and greatly reduce the efficiency.

differs from what is expected at some or all locations. This is a rare case, but may occur if the anchoring strength is low (LPP fabrication error) or the material is not homogenized properly (LCP fabrication error).

4.6.2 LC Defects

LC defects are the second common reason for non-ideal PG efficiency. Defects are typically small localized regions (< 5 µm) where the alignment varies chaotically and/or where the LC is aligned homeotropically (i.e., out-of-plane). The primary result of defects is the random scattering of incident photons. Depending on the type of defect (e.g., the defect may reside on top or within the bulk LCP layer), the retardation and possibly phase-profile may also be affected. The high-level causes of defects are many: improper mixing or homogenization of materials (LCP fabrication error), improper spin parameters (LCP fabrication error, e.g., attempting to coat a layer that is extremely thick), or poor anchoring strength (LPP fabrication error) caused by
the factors discussed in Chap. 3 (low fluence, low DOLP).

4.6.3 Nonlinear Phase Profiles

The third and most complicated reason for poor PG efficiency is a nonlinear phase profile. In this case, even though the retardation may be half-wave everywhere and the LC may be well aligned everywhere, the $\eta_{+1}$ may still be low. If the phase profile deviates from perfectly linear, then some amount of light will be sent into the orders other than +1, including higher positive orders, the zero-order, and negative orders. We can easily predict the efficiency of any nonlinear phase profile by using the NTFFT, which is simply a Fourier transform as discussed in Chap. 2. We consider in detail a couple of different types of nonlinear profiles which occur in practice.

**Nonlinear Profile: Staircase**

The first profile considered is a staircase profile. This profile is simply a discrete approximation of the linear PG profile and has been studied before [55, 56]. According to the prior art, the +1 order efficiency will be >99% if the number of discrete sections is $\geq \sim 14$ The lower the number of discrete sections, the more power will be directed into primarily the +2 and 0 orders. In our case, these kind of discrete phase sections rarely occur due to the averaging nature of the LPP. Nevertheless, regions similar to discrete phase sections may occur if the overlap parameter is too small, which prevents significant averaging from occurring. This helps explains the result of samples a and b, where discrete sections are apparent from the microscope images.

**Nonlinear Profile: Super-Grating**

The next profile considered is the super-grating profile, and is similar to the staircase. In this profile, the phase varies linearly with $\Lambda$, but it also has periodic features of a different pitch $\Lambda' = S\Lambda$, where $S$ is an integer greater than one. Some example super-grating profiles are shown in Fig. 4.9(a) and (b). Since the periodicity is now greater than the desired $\Lambda$, the intended +1 order is now the $+S$ order. However, this tends to confuse characterization; instead, we scale all order numbers by $S$ so that the intended +1 order (corresponding to pitch $\Lambda$) is still the +1 order, and the actual +1 order (corresponding to pitch $\Lambda'$) becomes the +1/$S$ order.

We simulated the two phase profiles shown in Fig. 4.9(a) and (b) and show the resulting diffraction orders in Fig. 4.9(c) and (d). As explained, we see power going into orders in between the integer orders.

The most likely cause for this type of profile is an error in the scan pattern being written (LPP fabrication error) due to imperfect calibration of the polarization or power components of the direct-write system. In such situations, certain scan patterns become asymmetric. This may occur, for example, when $1/L$ is not an integer, such as $L = 0.4$. For this value of $L$, the absolute
polarization of the writing beam only repeats every two periods (assuming the system has a range of 180°): the polarizations of each scan line will be $\psi_b = 0, 72, 144, (second\ period)\ 216, 288, 0$. While typical calibration errors would not likely result in profiles as dramatic as that in Fig. 4.9, the concept is the same and explains well the half-orders that appear for samples a and b where $L = 0.4$.

**Nonlinear Profile: Noisy Linear**

Another profile considered was a perfect linear profile with added white gaussian noise. White noise is characterized by having a flat spectral response, that is to say the Fourier transform of ideal white noise is a constant power $P$ dB at every frequency. In terms of a noisy alignment direction, this means the white noise profile has a mean-squared power of $P$ dB relative to 1 radian. Examples of noisy profiles are shown in Fig. 4.10(a). This or similar types of profiles may occur if the anchoring strength in a pattern is poor or if the material is not properly homogenized, causing chaotic local alignment but still having the correct alignment on average. Applying white noise $wn(P)$ to a phase profile equates to convoluting the Fourier transform of
Figure 4.10: a) Examples of noisy PG phase profiles. b) Simulated $\eta_{+1}$ vs. noise level. c-d) Simulated diffraction efficiencies of noisy phase profiles (pure noise, no linear PG profiles). Noise in the phase profile results in diffraction into all directions, similarly to scattering.

the white noise phase with the Fourier transform of the noiseless phase profile. In the case of a PG and circular input:

\[
I(k) = \int_{-\infty}^{\infty} w_n(P) e^{j(\phi(r) + w_n(P))} e^{-2\pi j k \cdot r} dr
\]  
\[
I(k) = \int_{-\infty}^{\infty} w_n(P) e^{2\pi r} e^{jw_n(P)} e^{-2\pi j k \cdot r} dr
\]
\[
I(k) = \delta \left(k - \frac{1}{\Lambda}\right) \ast \mathcal{F} \left[e^{jw_n(P)}\right]
\]  

Since the NTFFT of the PG is a delta, the effect of white noise can be determined by applying the NTFFT to the white noise phase term alone. The result of the NTFFT for some noisy phase profiles is shown in Fig. 4.10(c) and (d) for $P = -26$ and $-11.5$ dB (these are the NTFFT of the noise phase only, without the linear PG profile). The results shows that a noisy phase term has a single delta at the origin, and all other power is equally distributed into the remaining orders. This makes sense intuitively; if the noise is 0, i.e., $P = -\infty$, then the phase will be constant and only the zero-order will be present (no diffraction occurs). As
the noise grows, the phase becomes more random, power is randomly distributed into other orders, and the zero-order is decreased. This kind of randomly distributed diffraction is similar to scattering, but caused by diffraction and not photon scattering.

When convoluted with a linear phase for the PG, the delta at 0 is multiplied by the delta at the +1 order of the PG, reducing $\eta_{+1}$. It is worth noting that depending on the wavelength, only a portion of the orders shown in Fig. 4.10(c) and (d) will actually diffract; many will be reflected due to total internal reflection. (Also, the noise will occur in the direction normal to the periodicity of the PG, so diffraction will occur in all directions. However, the power into the zero-order is constant whether the simulated noise is 1D or 2D.)

We simulated noisy PG profiles with noise powers ranging from $P = -25$ to 15 dB, and the results are shown in Fig. 4.10(b). Below $P = -26$ dB, the $\eta_{+1}$ is >99%. After this, the power drops off in a manner similar to an error function. After 0 dB, the presence of the PG profile is completely negated. These results tell us that even small amount of randomness in the PG profile will quickly reduce the efficiency. Thus, very good alignment is critical to achieving high-efficiency PGs.

**Nonlinear Profile: Sawtooth**

The last kind of nonlinear profile considered was a sawtooth profile. In this profile, the phase (twice the optic axis) varies linearly, but with an incorrect slope such that the total phase over one pitch is $\xi 2\pi$ (the correct profile has $\xi = 1$). Some example sawtooth profiles are shown in Fig. 4.11. The sawtooth profile usually sends extra light mainly into the 0 and +2 orders.

We simulated sawtooth profiles with $\xi$ equals 0.7 to 1.3 and calculated the diffraction efficiencies. The results for $\eta_{+1}$ are shown in Fig. 4.11(b). The key point of the results is that $\eta_{+1}$ will be >99% even up to $\xi = 1 \pm 0.055$, telling us that a slight sawtooth profile is not a major concern. In Fig. 4.11(c) and (d) all diffraction orders up to ±4 are shown for the cases of $\xi = 1.055$ ($\eta_{+1} = 99\%$) and $\xi = 1.25$ ($\eta_{+1} = 75\%$).

The sawtooth profile can appear in practice under a few conditions. The first is if the power or polarization components of the system are not calibrated well. If a Pockels cell is used as the polarization modulator, as in our system, calibration can be difficult because the Pockels effect is temperature dependent; the polarization output of the system may be as much as 1.4% off per deg if it is not calibrated to the current temperature. The second cause of sawtooth-like profiles is when the anchoring strength is very low or when a very thick LC layer is coated. In both of these cases, the system description of Chap. 3 is inadequate, and the LC may align in unintended ways. This is likely due to slightly differing minimum energy states of the alignment directions 0° and 90° (alignment direction parallel and perpendicular to the grating vector), something that is completely ignored in Chap. 3.
Figure 4.11: a) Examples of sawtooth-grating profiles. b) Simulated $\eta_{+1}$ vs. the scale factor $\xi$. c-d) The simulated D.E.s associated with the profiles in a). The sawtooth-grating profile is characterized by a linear phase with a slope of $2\pi\xi/\Lambda$. The sawtooth profile may be caused by errors in the direct-writing system or LC fabrication errors, and the profile results in power directed into orders other than +1.

**Nonlinear Profile Summary**

To conclude the discussion of nonlinear phase profiles, we note three key points: 1) periodic deviations in the phase profile will lead to an increase in D.E. into orders other than +1, 2) periodic deviations in the phase profile with a pitch larger than $\Lambda$ lead to orders $<1$, and 3) non-periodic deviations in the phase profile lead to scattering.

**4.6.4 Non-Ideal PG Diffraction Summary**

Table 4.1 summarizes the primary non-idealities that can occur in PGs and the possible causes. Table 4.2 correlates the observed non-idealities in the fabricated samples to the most probable causes, which in total adequately explain all observed experimental results. We note that the key failure of these samples was the presence of a nonlinear orientation profile similar to the sawtooth profile in all patterns. It was later found that this was caused by coating an LCP layer that was too thick. When the LCP processing steps were changed, the PG efficiency improved dramatically, as shown in the next section.
Table 4.1: Summary of PG non-idealities (top) and corresponding causes (left).

<table>
<thead>
<tr>
<th></th>
<th>Scattering</th>
<th>0-Order</th>
<th>Higher-Orders</th>
<th>Half-Orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incorrect Retardation</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>LC Defects</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Staircase</td>
<td></td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Super-Grating</td>
<td></td>
<td></td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Noisy Linear</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Sawtooth</td>
<td></td>
<td></td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

Table 4.2: Summary of the deduced causes of non-idealities in fabricated samples.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>i</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incorrect Retardation</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LC Defects</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Staircase</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Super-Grating</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Noisy Linear</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Sawtooth</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

4.7 Highest-Quality Direct-Write Fabricated PG

We fabricated a PG with scan parameters in the $\eta_{+1} > 99\%$ parameter space, optimal retardation, and optimal fabrication processing steps. The scan parameters were $L = 0.1, B = 0.2, and F_{ave} = 200 \text{ mJ/cm}^2$ with a pitch of $\Lambda = 30 \text{ m}\mu \text{m}$. The thickness of the layer was tuned experimentally to achieve half-wave retardation at 633 nm. The LPP used was Rolic ROP-108, and the LCP was coated with 3 thin layers (compared to 1 thin plus 1 thick layer for the samples in the previous sections) of Merck RMS03-001C ($\Delta n \simeq 0.16$). The layers were each polymerized for $\sim 60 \text{ s}$ using a UV LED in a nitrogen environment.

A polarizing microscope image of the PG and the resulting diffraction are shown in Fig. 4.12. $\eta_{+1}$ was measured as $>99.7\%$. Given the discussion of the last section, it is apparent that to achieve this kind of efficiency the profile must be almost exactly linear, and indeed no problems with the LC alignment quality or profile can be discerned from the microscope image. Of the remaining 0.3% power, $\sim 0.1\%$ was measured in the zero-order, $\sim 0.05\%$ in the +2 order and -1 order, and $<0.1\%$ in the +3 order. The polarization contrast, defined as the ratio of the +1 to the -1 order, was over 1800:1. These characteristics represent the present best possible PG that can be fabricated using our direct-write system, and are similar or superior to PGs fabricated with other methods. We also note that this is the highest efficiency reported direct-write PG.
Figure 4.12: Very high quality direct-write fabricated PG with a pitch of 30 µm and $\eta_{+1} > 99.7\%$. a) Polarizing microscope image of the sample. b) Image of a 633 nm laser after diffracting through the sample.

4.8 Conclusions

In this chapter, we have addressed fundamental theoretical and practical issues surrounding fabrication of ideal CGH PGs. We studied the effect of line spacing and beam size on alignment quality and $\eta_{+1}$, and found the optimal scan parameters for different target metrics such as $\eta_{+1}$ or scan time. We experimentally fabricated many PGs and confirmed that our analysis was accurate, but some oddities existed in the PGs. We studied these unexpected behaviors and found the cause to be various non-idealities in the PG profiles. With this knowledge, we fabricated a best-quality PG with $\eta_{+1} > 99\%$ and otherwise outstanding properties. The primary contributions of this chapter are an understanding and optimization of the PG scan parameters, a thorough description of the causes and impact of PG non-idealities, and the demonstration and characterization of the first high-quality direct-write PG.
Chapter 5

Computer Generated Geometric Phase Lenses

The geometric phase lens (GPL) is a significantly more complex element than the PG in terms of its phase profile, the scan pattern needed, and the properties of the element. The defining characteristic of the GPL is its polarization sensitivity: light of one circular handedness is converged (positive focal length) while light of the opposite handedness is diverged (negative focal length). This is demonstrated in Fig. 5.1. Thus far, the GPL has only been used extensively in one application, an extremely efficient polarization conversion system, but other applications are in development at the time of this writing. The GPL is a very interesting element, and understanding how to fabricate it as well as better understanding its operation will provide insights into CGHs as a whole. In prior art, a few attempts have been made to produce GPLs. With LCs, the GPL has been recorded using polarization holography \cite{45, 57}. GPLs have also been created using form birefringence and sub-wavelength structures \cite{58}. In this chapter, we\textsuperscript{1} address the following questions: \textbf{What is the optimal scan pattern for fabricating GPLs? What are the optimal scan parameters for creating high-quality GPLs? What is the highest quality CGH GPL that can be created with our system?}

To answer these questions, the first part of this chapter is dedicated to determining the optimal scan pattern for the GPL. The second part of this chapter tests the hypothesis that the relationship between scan parameters and element quality are the same for the GPL as for the PG. The final part of this chapter relates to a thorough characterization of the highest quality GPL that was fabricated.

\textsuperscript{1}While this chapter uses the plural 1st person voice as is standard in the scientific community, almost the entirety of its contents are a direct result of the author's efforts.
Figure 5.1: Demonstration of the polarization properties of a GPL. Different circular polarizations converge or diverge; unpolarized light is equivalent to linearly polarized light and acts as the superposition of both circular polarizations.

5.1 GPL Phase Profile

5.1.1 Ideal GPL Profile

The phase (and thus orientation) of the ideal GPL profile varies with the distance \( r \) from the center of the lens. However, if the slope of the phase does not vary rapidly over a short distance, we can approximate the phase in any local region as being linear. The slope of the phase will then be locally equivalent to a PG with some pitch \( \Lambda_l \). Using simple geometry, the diffraction equation, and the relationship between phase and pitch, we can find the variation of the local pitch and then the phase of an ideal lens as:

\[
\Lambda_l(r) = \frac{\lambda_0 \sqrt{r^2 + f_0^2}}{r} = \lambda_0 \sqrt{\frac{f_0^2}{r_0^2}} + 1 \quad (5.1a)
\]

\[
\frac{\partial \phi_{gpl}(r)}{\partial r} = \frac{2\pi}{\Lambda_l(r)} = \frac{2\pi r}{\lambda_0 \sqrt{r^2 + f_0^2}} \quad (5.1b)
\]

\[
\phi_{gpl}(r) = \frac{2\pi \sqrt{r^2 + f_0^2}}{\lambda_0} \quad (5.1c)
\]

where the lens has focal length \( f_0 \) for some wavelength \( \lambda_0 \). It is useful to know the relationship between the focal length of the lens with respect to the minimum pitch of the lens:

\[
f_0 = r_0 \sqrt{\frac{\Lambda_l(r_0)^2}{\lambda_0^2}} - 1. \quad (5.2)
\]
We are also interested in the chromatic properties of the ideal lens and so attempt to solve for the focal length as a function of $\lambda$:

$$f(\lambda) = \frac{\lambda_0}{\lambda} \sqrt{f_0^2 - \left( \frac{\lambda^2}{\lambda_0^2} - 1 \right) r^2}. \quad (5.3)$$

This equation is surprising, however, in that it contains a spatial term $r$, indicating chromatic aberration. Thus, the ideal lens profile will focus a single wavelength ideally, and other wavelengths to a slightly lesser degree. It can also be seen from the equation that faster lenses with wavelengths farthest with the design wavelength will have the most aberration.

Let us consider the focusing power of the lens at a single point, such as the edge of a lens of radius $r_0$. In this case, we can remove the spatial dependence in Equation 5.3 and restate it in terms of the f-number $F = f/(2r_0)$, and design f-number $F_0 = f_0/(2r_0)$:

$$F(\lambda) = \frac{\lambda_0}{2\lambda} \sqrt{4F_0^2 + 1 - \frac{\lambda^2}{\lambda_0^2}} \quad (5.4)$$

if $F_0$ is large, then we can approximate $F(\lambda)$ as:

$$F(\lambda) \simeq \frac{\lambda_0}{\lambda} F_0. \quad (5.5)$$

So the general rule of thumb is that the f-number (or focal length) of a GPL scales with the ratio of the design and replay wavelengths.

### 5.1.2 Parabolic And Spheric Approximations

For dynamic phase lenses (i.e., refractive elements), the ideal phase (optical path difference) profile is a scaled version of the ideal GPL profile. However, in practice dynamic phase lenses will approximate the ideal profile with some lesser order equation, depending on the requirement of the lens and physical fabrication limitations. A common approximation is the second order parabolic lens profile. In this case, the phase and local pitch variation are:

$$\phi_{gpl}(r) = sr^2 \quad (5.6a)$$

$$\frac{\partial \phi_{gpl}(r)}{\partial r} = \frac{2\pi}{\Lambda_l(r)} = 2sr \quad (5.6b)$$

$$\Lambda_l(r) = \frac{\pi}{sr} \quad (5.6c)$$

$$s = \frac{\pi}{\lambda_0 \sqrt{r_0^2 + f_0^2}} = \frac{\pi}{r_0 \Lambda_l(r_0)}. \quad (5.6d)$$
Figure 5.2: Comparison of the ideal lens profile, parabolic approximation, and spheric approximation. The lens f-number is a) 5, b) 2.5, and c) 0.5. The parabolic profile is a good approximation when the f-number is high, and only at low f-number does the profile differ significantly from the ideal. The spherical approximation is only good at high f-number.

But even the parabolic profile is relatively difficult to fabricate, and so the most common lens profile used is actually the spheric profile. For a convex lens profile, the phase takes the following general form:

\[
\phi_{GPL}(r) = \frac{-2\pi(n-1)}{\lambda_0} \sqrt{R_0^2 - r^2} \tag{5.7a}
\]

\[
\frac{\partial \phi_{GPL}(r)}{\partial r} = \frac{2\pi}{\Lambda_t(r)} = \frac{2\pi r(n-1)}{\lambda_0 \sqrt{R_0^2 - r^2}} \tag{5.7b}
\]

\[
\Lambda_t(r) = \frac{\lambda_0 \sqrt{R_0^2 - r^2}}{r(n-1)} \tag{5.7c}
\]

\[
R_0 = \sqrt{(r_0^2 + f_0^2)(n-1)^2 + r_0^2} \tag{5.7d}
\]

where \(d\) is the thickness of the lens, \(n\) is the index of refraction, and \(R_0\) is the radius of curvature.

For a GPL with this profile, \(n\), \(d\), and \(R_0\) become arbitrary parameters, but for a dynamic phase element there will be limitations on the possible values, e.g. \(R_0\) cannot be smaller than \(r_0\). In
the above equation we have solved for one possible value of $R_0$, but other $R_0$ values may result in better performance for fast lenses.

In Fig. 5.2 we have plotted $\phi_{gpl}(r)$ and $\partial \phi_{gpl}(r) / \partial r$ for ideal, parabolic, and spheric lens profiles with target f-numbers of 5, 2.5, and 0.5. For the spheric profile, we imposed constraints of $n = 1.5$ and $R < r_0$ as would be the case for a refractive lens. As we are just interested in the difference between the profiles, $\lambda_0$ is ignored and we have also plotted the phases and phase slopes in arbitrary units (a.u.). Since $\partial \phi_{gpl}(r) / \partial r$ is related to the inverse pitch, deviations in the phase slope from the ideal indicate light will not be diffracted in the desired direction (i.e. optical aberration). It is seen that the parabolic approximation is good except for when the f-number is small. Even then, the aberrations produced are minimal for most applications. The spheric approximation is decent at large f-numbers, but quickly becomes unusable for small f-numbers.

With direct-write fabrication of GPLs, it is possible for us to create the ideal, parabolic, or spheric lens profile. As the parabolic profile is simpler than the ideal profile to understand, more commonly used in standard lens, and only significantly differs from the ideal profile in the case of very fast lenses, we have chosen to use the parabolic lens profile for the remainder of this chapter.

5.2 Scan Pattern and Parameters

5.2.1 Linear Archimedean Spiral

To create a GPL with *The Howlographer*, a basic but efficient scan pattern is a spiral with a slowly changing polarization. This scan pattern is shown in Fig. 5.3(a) and can be described in polar coordinates as:

$$r(t) = \frac{t}{t_{\text{ring}}(r)} l \quad \text{(5.8a)}$$
$$\theta(t) = \frac{2 \pi t}{t_{\text{ring}}(r)} \quad \text{(5.8b)}$$
$$\psi(t) = \frac{\phi_{gpl}(r)}{2} \quad \text{(5.8c)}$$

where $l$ is the distance between the rings of the spiral and $t_{\text{ring}}$ is the time taken to scan one ring of the spiral. In practice, $t_{\text{ring}}$ is usually a function $r$, as it is not efficient to take the same time to scan the smaller inner rings as the larger outer rings.

Treating the distance between adjacent rings of the spiral as the line spacing, and using the
Figure 5.3: a) Linear Archimedean spiral. b) Nonlinear Archimedean spiral. The nonlinear spiral is more efficient in terms of scan time, but can only be used if the beam size can be changed dynamically, as in our system.

definition of the local pitch in Eq. 5.7, we can use the same parameters as were used for the direct-write PG. The parameters are: line spacing $l$, beam width $w_0$, and the average delivered fluence $F_{ave}$. From these we can find derived parameters similar to the PG: the normalized beam size $B$, the normalized line spacing $L$, and the beam overlap $O$. These are defined as:

\begin{align}
L &= \frac{l}{\Lambda_l(r)} \quad \text{(5.9a)} \\
B &= \frac{2w_0}{\Lambda_l(r)} \quad \text{(5.9b)} \\
O &= \frac{B}{L}. \quad \text{(5.9c)}
\end{align}

Since the local pitch changes according to $r$, the normalized parameters will also be functions $r$. Assuming that the relationship between the scan parameters and the element quality are the same as what was found for the PG (discussed in a later section), the scan parameters should be chosen for the smallest pitch, $\Lambda_l(r_0)$. For all other $\Lambda_l(r)$ in the lens, $L$ and $B$ will be smaller but $O$ will remain constant, which will create equivalent or higher quality profiles assuming the element behaves similarly to the PG.
5.2.2 Nonlinear Archimedean Spiral

If the direct-write system can dynamically change the size of the beam, for example by raising or lowering the height of the final objective lens as in our system, then the derived parameters can be made constant for all \( r \). In this case, we see from Equation 5.9 that the absolute parameters will be functions of \( r \), and Equation 5.8(a) becomes:

\[
r(t) = \frac{t}{t_{\text{ring}}(r)} \frac{\Lambda_l(r)}{L}
\]  

(5.10)

Since \( \Lambda(r) \) varies inversely with \( r \), as \( r \) is increased the line spacing will decrease. Likewise, as \( r \) is increased the beam size decreases. In addition, as \( r \) increases either the power of the beam must be reduced or the scan speed must be increased to maintain a constant \( F_{\text{ave}}(r) \). Finally, since \( \Lambda_l = \infty \) at \( r = 0 \), in practice the normalized parameters can only be held constant for \( r > r_{\text{min}} \) for some \( r_{\text{min}} \) which is usually constrained by the maximum controllable beam size. An example of what this scan profile looks like is shown in Fig. 5.3(b). This is the basic lens scanning pattern used the remainder of this chapter.

5.3 Spiral Sampling

The use of a spiral pattern presents some unexpected difficulties. Such a complex scan pattern must be discretized in time and space to accommodate the limitations of any motion control system. Hence, the continuous spiral pattern in Equation 5.8 must be sampled. A standard motion control methodology for scanning arbitrary patterns is to supply the motion controller with a series of data points called a PVT file. Each entry in a PVT file specifies that at some point in time, the stages should be at some target position having some velocity. Where exactly the stages are located and what their velocities are in between the specified times is up to the controller; all that is guaranteed (in a working system) is that the stages will be at the positions specified with the velocities specified at the times specified. Thus, properly sampling the desired pattern is critical to achieving an accurate scan path and resulting phase profile.

For any sampling approach, there will be three parameters describing the relationship between sampled points: the sampling time interval \( t_s \), the sampling arc length \( d_s \), and the sampling angular distance \( \theta_s \). The three main methods of sampling are with uniform time intervals, uniform arc lengths, or uniform angular distances. Each approach has advantages and disadvantages. For convenience, we define \( N_{ff} = 2\pi/\theta_s \) as the fidelity factor, which describes the number of data points per ring of the spiral and is useful in judging the accuracy of a discretization scheme. We note that in our system the minimum discretization time for a scan pattern is 20 ms, which presents a hard lower limit on the sampling time interval.
Figure 5.4: The spiral pattern sampled with a) uniform time, b) uniform arc length, and c) uniform angular distance. The sampled points are assumed to be connected by straight lines. A key observation is that sampled points of adjacent lines in a) and b) are misaligned to each other, but sampled points of adjacent lines in c) are aligned to each other.

5.3.1 Uniform Time Sampling

The most straightforward sampling method is to sample the scan pattern in time using uniform time intervals \( t_0 \). This is demonstrated in Fig. 5.4(a). This results in sampling parameters of:

\[
N_{ff} = \frac{t_{ring}(r)}{t_0} \quad (5.11a)
\]
\[
t_s = t_0 \quad (5.11b)
\]
\[
d_s = \frac{2\pi rt_0}{t_{ring}(r)} \quad (5.11c)
\]
\[
\theta_s = \frac{2\pi t_0}{t_{ring}(r)}. \quad (5.11d)
\]

Thus, \( d_s \) and \( \theta_s \) are functions of \( r \). In addition, the fidelity factor depends on \( r \) as well.

There are two main problems that must be avoided with this approach. First, \( t_0 \) cannot be too large, or the fidelity factor will be low resulting in the spiral being under-sampled. This is likely to occur for inner rings of the spiral, where \( t_{ring}(r) \) is lowest. Second, \( t_0 \) cannot be too small, as 20 ms is the physical limit of the system as stated previously. Thus, for some \( t_{ring} \), there is no adequate \( t_0 \); for example, if \( t_{ring} = 100 \) ms and the minimum possible \( t_0 \) is 20 ms, the fidelity factor will be 5, resulting in a very under-sampled spiral. However, both of these problem can be corrected by ensuring \( t_{ring}(r) \) is never too small.
5.3.2 Uniform Arc Length Sampling

Another approach is to sample the spiral with uniform arc lengths $d_0$, with each data point being separated by some non-uniform time and angular distance. This scheme is demonstrated in Fig. 5.4(b), and the sampling parameters are found to be:

\[
N_{ff} = \frac{2\pi r}{d_0} \quad (5.12a)
\]
\[
t_s = \frac{2\pi r t_{ring}(r)}{d_0} \quad (5.12b)
\]
\[
d_s = d_0 \quad (5.12c)
\]
\[
\theta_s = \frac{d_0}{r}. \quad (5.12d)
\]

Like uniform time sampling, the fidelity factor changes with $r$.

There are two main problems that must be avoided with this approach. First, if $d_0$ is too small, then the time interval between data points may become smaller than the 20 ms limit. This first problem can be avoided by choosing $t_{ring}(r)$ properly. Second, if $d_0$ is too large, the inner rings will be under-sampled (low fidelity factor).

5.3.3 Uniform Angular Distance Sampling

The third approach is to sample with uniform angular distances $\theta_0$, and is demonstrated in Fig. 5.4(c). With this approach, we find the sampling parameters to be:

\[
N_{ff} = \frac{2\pi}{\theta_0} \quad (5.13a)
\]
\[
t_s = \frac{t_{ring}(r)\theta_0}{2\pi} \quad (5.13b)
\]
\[
d_s = r\theta_0 \quad (5.13c)
\]
\[
\theta_s = \theta_0. \quad (5.13d)
\]

The fidelity factor is now constant, unlike the previous approaches.

There are again two potential pitfalls of this approach. First, a large $\theta_0$ directly results in a lower fidelity factor and should be avoided. Second, if $\theta_0$ is too large, the time between data points may be below the minimum. This last problem can be avoided by properly choosing $t_{ring}$.
5.4 Spiral Sampling Comparison

We experimental fabricated portions of a spiral lens pattern using uniform arc length sampling and uniform angular distance sampling. Uniform time interval sampling is very similar to arc length sampling, so the results are assumed to be similar. The portion of the lenses fabricated and imaged in Fig. 5.5 are the outermost rings of lenses with \( r_0 = 1.25 \) mm, \( \Lambda_l(1.25\text{mm}) = 30 \) \( \mu \text{m} \), \( \tau_{\text{ring}} = 2 \) s, \( L = 0.2 \), \( B = 0.25 \), and \( F_{\text{ave}} = 0.2 \) J/cm\(^2\).

The left side in each image shows some thin regions of poorly aligned LC. We will call these regions “flipping defects”; they are caused due to the limited range of polarizations available to the system. In the ideal spiral pattern, the polarization is always changing, however the polarization modulator in our system (a Pockels cell and QWP) is only calibrated to provide \( \pm 90^\circ \) of polarization. So, if the current polarization state is \( 85^\circ \) and the next requested polarization state is \( 95^\circ \), the system will quickly flip to the other end of the polarization range and use \( -85^\circ \) instead. During this “flipping” all intermediate polarization states will be written, causing the profile seen. Also, the rapid change of polarization combined with the adjacent scan lines results in a low DOLP, creating the point defects at the ends of the flip defects. Since the polarization change per ring of the spiral is constant and only depends on \( L \), the azimuthal location of the flip defects always very similar, as seen in the images.

Samples A and B used uniform arc length sampling, with values of \( d_0 = 200 \) \( \mu \text{m} \) and \( 100 \) \( \mu \text{m} \). The first observation is that the size of the flip defect scales with \( d_0 \). Inspecting the quality of the alignment profile in A, we see that the different colored lines (i.e., different profile orientations) are not very smooth, but are instead somewhat jagged. In sample B, the problem is much more pronounced, and the lines appear to wiggle and change size erratically.

The wiggly lines are caused by a combination of two effects. The first is the slight unpredictability of the stage position in between specified data points; the second is the misalignment of sampled points in adjacent rings, as seen in Fig. 5.4(b). If the stages are assumed to travel in a straight line between data points, then the scan patterns will be similar to what is plotted in Fig. 5.4(b). It is seen that the lines deviate and get closer and farther in a random fashion. When this happens, it results in regions that receive more or less fluence, regions that receive increased or decreased DOLP, and regions that receive the wrong average polarization. In the samples fabricated here, the alignment quality was still high, but if a larger \( B \) is used, the pattern will not align well in addition to having an incorrect orientation profile.

Samples C and D used uniform angular distance sampling, with values of \( \theta_0 = 2\pi/50 \) and \( 2\pi/200 \) (\( N_{ff} = 50 \) and 200). We first note that the flipping defects in D are smaller than C, since a higher \( N_{ff} \) results in a smaller \( d_0 \). We also note that the flip defects are now aligned to exactly the same azimuthal angle; indeed, all sampled points in adjacent rings are aligned as shown in Fig. 5.4(c). This mostly eliminates the problem of jagged/wiggly lines seen in samples.

70
A and B. In sample C, it is apparent that the profile is not altogether circular; rather, flat edges are obvious, and it is clear where the sampled points lie. However in sample D, with four times the fidelity factor, the profile appears very circular and no flat edges are apparent.

It is clear from the results that uniform angular distance sampling results in the best approximation of a continuous spiral pattern. In addition, we found that a fidelity factor of 50 or less results in an under-sampled spiral and that values of 200 or more should be used. In the case of a very large diameter lens, it may be appropriate to use even higher fidelity factors. For the remainder of this work, we assume a nonlinear Archimedean spiral pattern with uniform angular distance sampling and a fidelity factor of 200.
Table 5.1: Scan parameters used for fabricated lenses.

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>B</th>
<th>$F_{\text{ave}}(J/cm^2)$</th>
<th>$r_0$ (mm)</th>
<th>f@633 nm, (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (relaxed)</td>
<td>1/7</td>
<td>1/5</td>
<td>0.2</td>
<td>1.25</td>
<td>∼60</td>
</tr>
<tr>
<td>B (aggressive)</td>
<td>1/3</td>
<td>1/2</td>
<td>0.2</td>
<td>1.25</td>
<td>∼60</td>
</tr>
<tr>
<td>C (large L)</td>
<td>1/2.5</td>
<td>1/2.5</td>
<td>0.2</td>
<td>1.25</td>
<td>∼60</td>
</tr>
<tr>
<td>D (large B)</td>
<td>1/5</td>
<td>1</td>
<td>0.2</td>
<td>1.25</td>
<td>∼60</td>
</tr>
</tbody>
</table>

5.5 Parameter Space Analysis

One of our main hypothesis is that the GPL and PG behave very similarly, and that the same parameter guidelines can be used for both elements. We test that hypothesis in this section by fabricating four GPLs with different parameters, shown in Table 5.1. Each GPL has an expected result based on the prior PG studies: A) this sample uses relaxed scan parameters and we expect to be the highest quality; B) this sample uses aggressive parameters and we expect is slightly less ideal than A); C) this sample uses a line spacing that is too large and we expect the result to be poor; D) this sample uses a beam size that is too large and we expect the result to be poor.

5.5.1 Orientation Profile and Alignment Quality

Images of the four lenses in a polarizing microscope with a FWP are shown in Fig. 5.6 and 5.7. The results match what we expected to a high degree. Sample A is the highest quality, without any scattering or noticeable problems in the profile. Sample B is similar in quality, but the profile is not quite as smooth. It appears that the phase profile has a slight staircase profile similar to what was analyzed in the PG section. Sample C has a clearly non-ideal phase profile, with a severe staircase profile and defects forming in between adjacent alignment regions. Sample D is the worst, with many LC defects, and the phase profile is barely apparent. The central region of D is however aligned well; this is due the scan parameters deviating from what was previously stated near the center of the lens. The beam size can only be increased so much, so that the central region in all samples actually used a lower $B$, and then also a lower $L$ to main a constant $O$. This causes the alignment to be good in the central region of all the samples.

5.5.2 Comparison of Lens Operation

We characterized the lenses with a 633 nm HeNe laser approximately 2.2 mm in diameter. The laser passed through an iris, a linear polarizer to control the power, a quarter-wave plate (QWP) to set the polarization, the GPL, and was propagated about 60 cm to a screen for imaging.
Figure 5.6: Polarizing microscope images of samples A-D. Sample A and B both show very good alignment. Sample C suffers from a staircase profile. Sample D has many LC defects, except for the center of the pattern where the beam size could not be increased any further and so $B$ was lowered.

Recall from Chapter 2 that orthogonal circularly polarized inputs into a GPL have negative focal distances (60 mm and -60 mm for these lenses). With a 60 cm propagation distance from the element and a collimated input beam, the positive and negative focal lengths will both produce a large spot on the screen, and the spot belonging to the negative focal length will be slightly bigger.

The spots produced by the lenses in this configuration were captured for both RCP and LCP inputs and are shown in Fig. 5.8. In addition, we captured an image with a spherical refractive lenses that had a similar focal length to serve as a reference. Since the laser has a Gaussian intensity profile, the ideal lens in the configuration should magnify the profile and produce a larger Gaussian profile. The RCP input case has negative focus and results in a larger...
Figure 5.7: High-magnification polarizing microscope images of the edge of samples A-D. Here the phase profile and defects can be seen more clearly.

spot than the LCP input case, which has positive focus.

Sample A with relaxed parameters resulted in a uniform and Gaussian intensity profile that is very similar to the reference. Sample B has more aggressive parameters, but also produced a fairly uniform spot. The only obvious difference from A is a slightly increased zero-order (there was a particle on the surface of some optic which produced the airy ring pattern near the center, which is the other obvious difference).

In Sample C, which used a large $L$, the spot produced is no longer uniform; instead, there are many bright and dark rings. (The rings look similar to interference fringes, but are not. The wave is circularly polarized at this point and destructive interference cannot occur.) These rings are caused by the severe staircase profile of this lens. The ideal lens requires a smoothly varying profile to produce a smoothly varying spot, so it is only natural that a discretized profile results in a discretized spot. We also note that the zero-order leakage of this lens is higher than A and
Figure 5.8: Images of the fabricated GPLs in operation. A 633 nm laser was sent through each sample and the beam was allowed to propagate $\sim 60$ cm before hitting a screen. Also shown is the result of the laser passing through a comparable spherical refractive lens to serve as a reference.

B.

The result of Sample D, which used a large $B$, was the worst. There are numerous problems. First, the zero-order leakage is the highest of all the lenses. Second, there was massive scattering of light, as evidenced by the decreased total intensity compared to A-C. Third, the spots produced are smaller than A-C. Fourth, the LCP and RCP results are asymmetric. The center of the lens, which was well aligned as seen in Fig. 5.6, is likely responsible for most of what is seen in the image.

To conclude, the samples performed exactly as expected: samples A and B aligned well and with accurate profiles, and functioned well as lenses; samples C and D did not align with an accurate profile, and did not function as lenses nearly as well as A and B. Thus, we conclude that our hypothesis was correct that the same relationships between scan parameters and element
5.6 Characterization of Highest Quality GPL

Sample A represents the highest quality GPL that can be fabricated with our system. In addition to the prior characterization, some additional measurements were taken on the sample. The zero-order leakage at 633 nm was measured by taking two power measurements with a 633 nm laser. The first measurement was with RCP input and an LCP polarizer (QWP + linear polarizer) at the output. This measures diffracted light (+1 order, -1 order, higher orders, etc.), since all diffracted light is assumed to be LCP (orthogonal to input) and the zero-order is assumed to be RCP (same as input). The second measurement is with RCP input and an RCP polarizer at the output. By the same reasoning, this measures the zero-order leakage directly. The ratio of the two measurements yields the percent zero-order leakage, which was found to be only 0.4%.

We quantitatively measured the focusing capabilities of the GPL using a 633 nm laser and a beam profiler. At this wavelength, the 2.5 mm diameter GPL has a focal distance of \( \sim 59 \text{ mm} \) and an f-number of 24. The beam profiler measures both \( x \) and \( y \) cross sections; both cross sections were very similar, so we show only the \( x \) cross section in Fig. 5.9. When acting as a diverging lens, the beam expands into a wider Gaussian beam. In this case, we also observe the conjugate order (focused beam) which contains very low total power. When acting as a converging lens, the GPL focused the input beam to a very Gaussian 33 µm spot. A similar spherical lens was also able to focus the beam to a minimum 33 µm spot. For reference, the

![Figure 5.9: Measured cross section of a 633 nm laser a) without alteration, b) after GPL with RCP input, c) after GPL with LCP input.](image)
diffraction limited spot size is 18 µm. Differences from this absolute minimum may be caused by imperfections in the HeNe laser itself.

5.7 Conclusions

In this chapter, we have addressed fundamental theoretical and practical issues surrounding fabrication of ideal computer generated GPLs. The GPL is a significantly more complex element than the PG. We introduced a number of new scan parameters and analyzed different possible scan patterns. We experimentally determined that a nonlinear Archimedean spiral with uniform angular distance discretization was optimal. Then we fabricated various GPLs and found that the optimal scan parameters, if described correctly, are equivalent to the PG. With our best-quality lens, we measured important metrics and found that it performed just as well as a refractive lens. The primary contributions of this chapter are the determination of the optimal GPL scan pattern and scan parameters, and the demonstration and characterization of the first high-quality direct-write GPL.
Chapter 6

Fourier Geometric Phase Holograms

A Fourier hologram is a hologram in which the goal is to produce a specific image \( I(\mathbf{k}) \), also called a k-space image, in the Fourier plane. The Fourier plane is equivalent to angular space, k-space, the far-field, and the Fraunhofer zone. A PG can be considered a Fourier GPH (FGPH) where the k-space image produced is a delta function located at some non-zero \( \mathbf{k} \). By choosing a specific phase profile for an FGPH, almost any k-space image can be produced. This chapter addresses these question: **What is the theoretical behavior of FGPHs? What are the optimal scan parameters for producing high-quality arbitrary FGPHs? What are the properties of a high-quality FGPHs fabricated with our system?**

To this end, we\(^1\) describe and analyze the FGPH scan parameters, namely the beam size, in an analytical manner. Next, we experimentally fabricate FGPHs and study the effects of different beam sizes, comparing the results to our predictions. Finally, we experimentally characterize the highest quality FGPH fabricated with our system.

### 6.1 Ideal FGPH Behavior

To analyze the ideal theoretical FGPH behavior, we consider the target k-space image shown in Fig. 6.1(a). To find the phase profile needed to create this k-space image, we use an iterative Fourier technique called the Gerchberg-Saxton algorithm [59]. Briefly described, the algorithm begins with a target Fourier image and iteratively solves for a phase profile that best relates the two via the Fourier transform (recall that the Fraunhauffer diffraction approximation is a Fourier transform). Using this algorithm, we find the phase profile \( \phi(\mathbf{r})_{\text{target}} \) needed, shown in Fig. 6.1(b). If the input polarization to the hologram is RCP, then the phase is applied to the input wave and directly inserted into the NTFFT to find the produced image. If however the

\(^{1}\)While this chapter uses the plural 1st person voice as is standard in the scientific community, almost the entirety of its contents are a direct result of the author’s efforts.
polarization is LCP, then, via Eq. 2.4, the phase applied is negative:

\[
I(k)_{\text{RCP}} = \int_{-\infty}^{\infty} e^{j\phi(r)_{\text{target}}} e^{-2\pi jk \cdot r} dr 
\]

(6.1a)

\[
I(k)_{\text{LCP}} = \int_{-\infty}^{\infty} e^{-j\phi(r)_{\text{target}}} e^{-2\pi jk \cdot r} dr. 
\]

(6.1b)

A negative phase term will have all the same characteristics as a positive phase term, but will diffract light in the negative \( \mathbf{k} \) direction, which can be shown using the conjugation property of the Fourier transform. This property states that if \( \mathcal{F}[f(r)] = F(k) \), then \( \mathcal{F}[\overline{f(r)}] = F(-k) \). Applying the property to the problem at hand:

Figure 6.1: a) Target FGPH image used in this section. b) Needed phase profile calculated for the Gerchberg-Saxton algorithm. c) Predicted reconstructed image for RCP input. d) Predicted reconstructed image for LCP input.
Thus, an opposite handed polarization results in an identical k-space image that is rotated by $180^\circ$. This image is called the conjugate image. If the incident light is linearly or elliptically polarized, then the output will be some superposition of the primary and conjugate images.

What are the wavelength-dependent properties of the FGPH? We see that the phase term does not contain any wavelength dependence; the phase applied to all incident waves is the same. In Eq. 6.1, the final k-space image is the same regardless of wavelength. However, as we have written it the k-space itself is wavelength dependent, since the wavevector contains a wavelength term. Expanding the wavevector, we find the impact on different wavelengths:

$$I(k) = I \left( \frac{2\pi}{\lambda} (\alpha \hat{x} + \beta \hat{y} + \gamma \hat{z}) \right).$$

(6.3)

This tells us that the shape of the image produced is the same regardless of wavelength, but that a smaller wavelength will result in a smaller scaled image and that a larger wavelength will result in a larger scaled image. This is the only wavelength dependent feature of FGPHs, aside from a potential difference in zero-order leakage similar to other GPH.

## 6.2 FGPH Scan Pattern

We are interested in FGPHs composed of a uniform rectangular grid of square pixels. There are two scan patterns that we have used to create an arbitrary 2D grid of discrete phases (orientation angles). The first is to expose one pixel, use a shutter to stop the beam while moving to the next pixel, expose the next pixel, then repeat. The problem with this method is that using a shutter and stopping and starting the movement of the stages slows down the scan rate (pixels/s) tremendously. In our system, such an approach results in about 5 pixels/s, and a 1000x1000 pixel pattern would take over 55 hours.

The second method, which is the one used for the remainder of this work, is to move the beam continuously through one line of pixels, changing the polarization rapidly as the beam enters the position of each new pixel. This scan pattern is shown in Fig. 6.2. This method requires a very fast polarization modulator such as a Pockels cell. In our system, we can achieve a scan rate of 500 pixel/s, which allows us to scan a 1000x1000 pixel pattern in about 30 minutes.
The method has one serious drawback: the range of the Pockels cell is finite and the change in polarization is not instantaneous, thus, since a shutter is not being used, all of the incorrect polarizations between pixels will have some minimal effect on the pattern. This is demonstrated in Fig. 6.3. In the figure, the target polarization orientation starts at $0^\circ$ and is changed to $80^\circ$. However, as the polarization is changed to $80^\circ$, all of the polarizations in-between $0^\circ$ and $80^\circ$ are briefly exposed. While discrete pixels were desired, the polarization will actually blur between each pixel to some degree depending on how fast the polarization can be modulated.

In our system, the polarization can be modulated arbitrary in approximately 0.2 ms, and as mentioned already we can scan at a rate of 500 pixels/s (2 ms/pixel). Therefore, 10% of the scan time will be spent on “in-between” polarization states. However, the in-between polarizations by definition start and end at correct polarizations, so the effect of the the in-between states in not as bad as having poor alignment or a random orientation. In addition, since the phase slope produced by these rapid polarization changes is very high compared to the rest of the phase profile, any spurious diffraction that occurs from these transition regions will most likely be diffracted into very high angles and will not be apparent in the Fourier image. For these reasons, and based on our experimental results, we have neglected these in-between polarizations in our analysis.
Figure 6.3: Example target and actual exposure polarizations when using a Pockels cell; the polarization is blurred between each polarization state. This example assumes a pixel pitch of 10 µm and that a polarization modulation takes 1 µm to finish.

6.3 Theoretical Impact of Beam Size

Apart from choosing the scan rate which determines the percentage of in-between polarization states, the beam size is the only remaining parameter which has a significant effect on the quality of the produced image. We define the normalized beam size \( B \), as

\[
B = \frac{2w_0}{p}
\]

where \( p \) is the width of a single pixel.

6.3.1 Large Beam Size

If \( B > 1 \), it means that each pixel is fully exposed as intended, and is then also exposed by all neighboring pixels to some degree. Due to the orientation averaging property of the LPP/LC system, the net effect is a blurring of the phase profile, when the phase is represented as a complex exponential. With a Gaussian beam profile, this can be described mathematically as follows:

\[
e^{j\phi(r)_{\text{target}}} \rightarrow G_0(r) \ast e^{j\phi(r)_{\text{target}}}
\]

\[
G_0(r) = \frac{2e^{-\frac{r^2}{d^2}}}{\pi d^2}
\]

\[
d = \frac{Bp}{2}
\]
where $G_0$ is the Gaussian kernel and $d$ is the blur distance. (The Gaussian kernel can be substituted for any beam intensity profile.) Since the phase profile and $k$-space image are related via the Fourier transform, this is the same as the multiplication of the $k$-space image with the Fourier transform of $G_0$. Furthermore, the Fourier transform of $G_0$ is another Gaussian, $G_1$. Putting it all together, we have:

$$I(k)_{RCP} = \int_{-\infty}^{\infty} G_0(r) \ast e^{j\phi(r)_{\text{target}}} e^{-2\pi j k \cdot r} dr$$  \hspace{1cm} (6.5a)$$

$$I(k)_{RCP} = \int_{-\infty}^{\infty} G_0(r) e^{-2\pi j k \cdot r} dr \int_{-\infty}^{\infty} e^{j\phi(r)_{\text{target}}} e^{-2\pi j k \cdot r} dr$$  \hspace{1cm} (6.5b)$$

$$I(k)_{RCP} = G_1(k) I_{\text{target}}(k)_{RCP}.$$  \hspace{1cm} (6.5c)$$

It is well known that the width of the two Gaussians in a Fourier pair are inversely related. Thus, as $G_0$ gets larger (i.e., more blurring occurs), $G_1$ gets smaller, and the edges of $k$-space become much less visible. Lastly, consider the special case where $G_0$ is a delta (i.e., no blurring), in which case $G_1$ encompasses all $k$ (i.e., no effect on the $k$-space image). To summarize, when $B > 1$ we expect the $k$-space image to be multiplied by a Gaussian, increasing the intensity in the center of the image and reducing it at the borders.

### 6.3.2 Small Beam Size

If $B < 1$, it means that each pixel will not be fully exposed in the direction orthogonal to scan direction. The regions which are not fully exposed will have low anchoring strength, and thus will likely contain many defects and not be aligned to the intended direction, if at all. From a device perspective, this might be equivalent to pixels being reduced in size in the $x$ dimension without being brought closer together. This will create a periodicity that is smaller than the pixel pitch, which will result in higher orders being produced in the $k$-space image. In addition, the resulting space in-between the pixels might be considered to be filled with a random phase that does not obtain half-wave retardation. This will result in an increased zero-order leakage, scattering from defects, and random noise in the $k$-space image. (The difference between scattering and random noise, is that scattering is unpolarized while random noise is polarized the same as the $k$-space image.)

Still another effect is that the fluence that is delivered to some parts of the pattern will be higher, due to the beam intensity being larger. This may cause the LPP to saturate, resulting in less averaging, which may subsequently result in pixels that are more clearly defined in the direction of the scan ($y$) as well. The net effect of more clearly defined pixels is the opposite of
that when $B > 1$: the borders of the k-space image will be clearer, as will any higher orders.

To summarize, when $B < 1$, we expect more zero-order leakage, more scattering, more higher orders, and more clearly defined features at the edge of the k-space image.

6.4 Experimental Results

Experimentally, we have fabricated FGPHs with $B$ equal to 0.5, 1.1, 1.8, and 2.2. All samples used a pixel pitch of 20 $\mu$m, an average fluence of 0.2 J/cm$^2$, and had the target phase profile shown in Fig. 6.1(b). Polarized microscope images of the holograms are shown in Fig. 6.4.

6.4.1 LC Alignment Quality and Profile Observations

Sample A has many defects, located primarily along lines in the $y$ direction (the direction of the scan) but also some in the $x$ direction. The defects occur almost exclusively at the boundaries between pixels. The orientation within each pixel is mostly constant, but some gradient profiles can be seen, indicating some degree of averaging is taking place.

Sample B most resembles the target phase profile. The pixels are clearly defined as with A, but now the defects are almost entirely gone. A few point defects remain, which mostly occur at the corners of pixels where the most averaging is occurring and the DOLP is lowest. Gradient profiles can be seen in both the $x$ and $y$ directions, although the profile is averaged more in the $y$ direction.

In sample C, the averaging has become extreme. Individual pixels can barely be identified, although the location of the columns of pixels can still be discerned. There are many more defects than in B, still primarily located at the borders of pixels. The increase in defects is caused by a decrease in the DOLP brought on by the larger beam size.

In sample D, the averaging is even more extreme; it is not apparent that a discrete profile was intended to be written. The defects have increased even more and are now no longer confined to the borders of pixels.

6.4.2 FGPH Reconstruction Comparison

In a Fourier hologram, the reconstructed image is a virtual image located an infinite distance away. A simple method of viewing the holographic image is to shine a laser through the element, then let the beam propagate some distance and hit some screen and view the light that is scattered from the screen. As the viewing screen is placed farther and farther from the element, the image will gradually appear (come into focus). However, the image will also be increasing in size as the distance increases. Another way to view the image is to place a lens directly after element and image the focal plane of the lens. Since the lens acts as a Fourier transform for

84
Figure 6.4: Polarizing microscope images with various zoom of fabricated FPGH with the target phase profile shown in Fig. 6.1(b). All samples had the same scan pattern parameters, except for the beam size, which varied from 10 µm (B = 0.5) to 43 µm (B = 2.2). When the beam size is smaller, the pixels are more clearly defined. When the beam size is larger, more averaging takes place. Extremes at either end cause many LC defects; a beam size of \( \sim B = 1 \) seems optimal.
the incident wave, the FGPH image appears at the focus of the lens. This is convenient, as the resulting image may be made any size and captured directly with a camera, as opposed to imaging scattering light from a screen.

To observe the hologram produced by A-D, we used the characterization setup shown in Fig. 6.5. A 633 nm HeNe laser passed through an iris, a polarizer to control beam power and ensure polarization purity, a quarter-wave plate (QWP) to set the polarization, the FGPH element, a series of lenses, and finally a CCD placed at the focus of the lens elements. We recorded the images produced by the FGPH samples when the input beam was RCP, LCP, and linearly polarized. The results are shown in Fig. 6.6. We make these key observations:

1. Circular inputs of opposite handedness create $180^\circ$ rotated images; a linear input creates a superposition of the images produced by both circular inputs.

2. When the beam size was much smaller (A) or much larger (D) than the pixel pitch, the zero-order leakage was increased. This we expect is due to the increase in LC defects.

3. For samples with smaller beam size, higher orders are more visible, while for samples with larger beam sizes higher orders are not visible.

4. For samples with smaller beam size, the power over the main order is about equally distributed, while for samples with larger beam sizes, the power is clearly higher in the center of the sample and the edges of the images are faded.

In summary, the characteristics of the reconstructed images are exactly as predicted in the previous sections.
Figure 6.6: The Fourier images produced by FGPHs with various beam sizes. Polarizing microscope images of the FGPHs are in Fig. 6.4, and the setup used to capture these images is in Fig. 6.5.
Table 6.1: Various performance metrics for the target image, predicted image, and experimental images.

<table>
<thead>
<tr>
<th></th>
<th>k-Space RMSE (%)</th>
<th>0-Order (%)</th>
<th>Binary D.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Target</td>
<td>0</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>Predicted</td>
<td>0.003</td>
<td>0.01</td>
<td>79.0</td>
</tr>
<tr>
<td>A ($B = 1.1$)</td>
<td>0.010</td>
<td>3.6</td>
<td>57.1</td>
</tr>
<tr>
<td>B ($B = 1.6$)</td>
<td>0.011</td>
<td>2.9</td>
<td>55.6</td>
</tr>
</tbody>
</table>

6.4.3 Characterization of High-Quality FGPH

From qualitative observation, we decided that either sample B or C were optimal and deserved closer examination. We quantized the images in Fig. 6.6 corresponding to $B = 1.1$ and $B = 1.6$ for RCP input onto a grid of 128x128 pixels (the higher orders were excluded from this analysis).

Three metrics were calculated: 1) the root-mean-squared-error (RMSE) of the resultant k-space image, 2) the percentage intensity in the 0-order, and 3) a new metric called the binary D.E. The binary D.E. is a measure of how much power is going into the generally desired region and is designed to compensate for speckle in the image. It is defined as:

$$k_{mask} = \begin{cases} 
1 & : k_{target} \geq 0.5 \\
0 & : k_{target} < 0.5 
\end{cases}$$

$$D.E._{binary} = \frac{\sum(k_{mask} \cdot k_{actual})}{\sum k_{actual}}$$

These metrics were calculated for samples B and C and also the predicted image. The results of these measurements are shown in Table 6.1.

Both experimental samples performed similarly. The RMSE was about 10 times higher than expected, which we attribute largely to speckle [60]. The binary D.E. was much closer to the predicted result, showing that the holograms are generally working well. Interestingly, the sample with more averaging performed slightly better. With more optimization, we expect that even higher effective efficiencies are achievable. We note that this is most efficient FGPH ever reported [19].

Wavelength Dependence of FGPH

Lastly, we wanted to test the wavelength dependent nature of the FGPH. To do this, we setup sample B with both a 633 nm and 523 nm laser passing through it. This was accomplished
using a beam-splitter and multiple QWPs to set beam polarizations. The beam was allowed to propagate approximately 4 m beyond the sample before striking a screen where the Fourier image was observed via scattering and captured via a camera. In Fig. 6.7(a), the red laser was RCP and the green laser was LCP. Since the retardation was optimized for red, the zero-order leakage for the green laser is quite higher. In Fig. 6.7(b), the red laser was RCP and the green was also RCP. It is obvious from both images that the green palm tree is an identical but smaller scaled version of the red palm tree, experimentally confirming our predictions concerning the wavelength dependent nature of FGPHs.

6.5 Conclusions

In this chapter, we studied the FGPH, a sophisticated holographic element with a seemingly random phase profile. The operation of the FGPH is more complex than either the PG or GPL, but we found that there are few feasible scan patterns and few important scan parameters, the most notable being beam size. We analytically studied the impact of beam size on the k-space image produced by the FGPH, then fabricated various FGPHs to test our predictions. The experimental films confirmed that our analysis was accurate and allowed us to find near-optimal
scan parameters for FGPH. *The primary contributions of this chapter are an understanding of the impact of beam size on FGPH quality, the determination of the near-optimal beam size, and demonstration and characterization of the highest quality direct-write FGPH.*
Chapter 7

Electromagnetic Simulation of CGH

As discussed in Chap. 2, the Finite-Difference Time-Domain method (FDTD) is a powerful tool for obtaining accurate numerical solutions to Maxwell’s equations in arbitrary media. This makes it a viable tool for studying complex CGHs for which no analytic method can accurately analyze. One class of CGHs we are especially interested in are CGHs with periodic phase profiles. Example of such CGH are a polarization grating (PG), an array of geometric phase lenses (GPLs), and both 1 and 2D multiple beam splitters.

Similar kinds of periodic structures have been studied with FDTD in the past. The periodicity of the structures enables periodic boundary conditions to be employed and only one unit cell of the structure needing to be simulated, making the technique especially efficient. However, for the case of off-axis source incidence, the periodic boundary conditions fail for non-monochromatic sources. As a result, special methods must be used to simulate broadband oblique incidence sources. Multiple approaches have been used, but the most popular technique is referred to as the split-field method (SFM) [61], which is the basis of the algorithm in this chapter.

The SFM has been used to study arbitrary inhomogeneous 3D media periodic in two dimensions with isotropic permittivity and permeability [61]-[62], inhomogeneous 2D media periodic in one dimension (so-called 2.5D problems) with anisotropic permittivity [63], and 3D dispersive, metallic, isotropic media [64]. Unconditionally stable versions of the SFM have been derived [65]-[66], which serve to greatly increase its speed (as well as complexity).

In this chapter, we\textsuperscript{1} address the following questions: \textit{Can we develop an FDTD algorithm for simulation of 3D media periodic in 2D with anisotropic permittivity and conductivity? What steps are needed to obtain the far-field orders, including

----

\textsuperscript{1}While this chapter uses the plural 1st person voice as is standard in the scientific community, almost the entirety of its contents are a direct result of the author’s efforts. The other significant contributors are Michael Escuti and Stephan Schmidt, who provided document revisions and comments and who assisted in the description of the algorithm, the validation of its accuracy, and the creation of some figures.
polarization information, from the output of a simulation? Will this tool provide new insight into certain GPH/CGH?

To meet these goals, Section 1 of this chapter contains the algorithm derivation and comments on the implementation, which we call Wolfsim 3D. Section 2 presents the steps and equations for obtaining the far-field orders from the output of the algorithm. In Section 3 we validate the algorithm with several numerical simulations. Last, in Section 4 we simulate for the first time a GPH multiple-beam splitter and a GPH micro lens array.

### 7.1 Algorithm Derivation

This method assumes a simulation space that is finite in $z$, periodic in $x$, and periodic in $y$, with $z$ being the direction of normal incidence (Fig. 7.1). This method uses the standard Yee grid [67] shown in Fig. 7.1; however the formulation is presented in a way that is compatible with other grid types. While the final discretized update equations are in the time domain, our derivation will be in the frequency domain.

#### 7.1.1 Source Definition

We consider a soft dipole current source [68] that is a broadband planar wave traveling at an arbitrary angle of incidence. This source is defined as:

$$E_{in} = V_{in} \exp (-j k_{in} \cdot r) G(\omega)$$  \hspace{1cm} (7.1a)

$$k_{in} = \frac{2\pi}{\lambda} (\alpha_{in} \hat{x} + \beta_{in} \hat{y} + \gamma_{in} \hat{z})$$  \hspace{1cm} (7.1b)

where $\alpha_{in} = \sin \theta_{in} \cos \phi_{in}$, $\beta = \sin \theta_{in} \sin \phi_{in}$, and $\gamma_{in} = (1 - \alpha_{in}^2 - \beta_{in}^2)^{\frac{1}{2}} = \cos \theta_{in}$ are the direction cosines, and $r = \hat{x} x + \hat{y} y + \hat{z} z$. The polar and azimuth angles $\theta$ and $\phi$ are defined in Fig. 7.1 along with the other geometric assumptions. The $G(\omega)$ term is the Fourier transform of the source’s time-domain profile; ideally, the source is a pulse in the time-domain leading to a broadband $G(\omega)$, however other profiles may be used, for example, to make $G(\omega)$ monochromatic.

The complex amplitude vector $V_{in}$ describes the polarization and magnitude of the source. It is defined using TE/TM vectors as the basis vectors for a Jones vector, and can be constructed...
Figure 7.1: (a) The simulate space used for Wolfsim 3D. b) The grid scheme used in this work is the 3D Yee grid; shown is a unit cell with the location of the E and H fields (which are identical to the positions of the P and Q fields).

as follows:

\[ V_{TE} = \frac{-\hat{x}\beta_{in} + \hat{y}\alpha_{in}}{||-\hat{x}\beta_{in} + \hat{y}\alpha_{in}||} \] (7.2a)

\[ V_{TM} = V_{TE} \times k_{in} \] (7.2b)

\[ V_{in} = V_{TM}(J_{in} \cdot \hat{x}) + V_{TE}(J_{in} \cdot \hat{y}) \] (7.2c)

where \( J_{in} \) is a Jones vector using standard polarization angles \( \psi \) and \( \chi \). With this definition, \( \{\psi = 0; \chi = 0\} \) is TM and \( \{\psi = \pi/2; \chi = 0\} \) is TE [69].

Following the standard SFM, we employ the following field transformations:

\[ P = E \exp \left( \frac{j\omega}{c} (\alpha_{in}x + \beta_{in}y) \right) \] (7.3a)

\[ Q = \mu_0 H \exp \left( \frac{j\omega}{c} (\alpha_{in}x + \beta_{in}y) \right). \] (7.3b)

The incident wave can now be expressed as (using \( \omega/c = 2\pi/\lambda \)):

\[ P_{in} = V_{in} \exp \left( -j\mathbf{k}_{in}' \cdot \mathbf{r} \right) G(\omega) \] (7.4a)

\[ \mathbf{k}_{in}' = \frac{2\pi}{\lambda} \gamma_{in} \hat{z} \] (7.4b)
which is recognized as a plane wave traveling in the $z$ direction at a reduced speed due to $\gamma_{in}$. Standard periodic boundary conditions (PBC) do not work for the source described in Eq. (7.1) because of the wavelength dependent phase term in the direction of the PBC (in our case, the $x$ direction). The source in Eq. (7.4) does not possess this phase term, enabling standard PBC for broadband oblique incident sources.

### 7.1.2 Update Equations

The update equations begin with Maxwell’s curl equations, expressed fully using permittivity and conductivity tensors:

\[
\frac{\partial}{\partial t} \left( \epsilon_0 \epsilon_r \mathbf{E} \right) + \sigma \mathbf{E} = \nabla \times \mathbf{H} \tag{7.5a}
\]
\[
\frac{\partial}{\partial t} \left( \mu_0 \mu_r \mathbf{H} \right) = -\nabla \times \mathbf{E}. \tag{7.5b}
\]

Any material with anisotropy in the permittivity or conductivity can be fully described using the tensors $(\epsilon_r)$ and $(\sigma)$. These tensors can be obtained by rotating any body with linear dielectric and conductive properties according to Euler angles [70]. The relative permeability tensor $\mu_r$ is unity for nonmagnetic materials and therefore ignored in this analysis. In phasor form, where we have substituted $\partial/\partial t = j\omega$, Eq. (7.5) becomes:

\[
j\omega \epsilon_0 \epsilon_r \mathbf{E} + \sigma \mathbf{E} = \nabla \times \mathbf{H} \tag{7.6a}
\]
\[
j\omega \mu_0 \mathbf{H} = -\nabla \times \mathbf{E} \tag{7.6b}
\]

We now substitute the $\mathbf{P}$ and $\mathbf{Q}$ fields into Eq. (7.6). This produces extra time derivates, as in prior SFM [61, 62], expressed in matrix form as

\[
\frac{j\omega \epsilon_r}{c} \mathbf{P} + \sigma' \mathbf{P} = \nabla \times \mathbf{Q} + \frac{j\omega}{c} \mathbf{A} \mathbf{Q} \tag{7.7a}
\]
\[
\frac{j\omega}{c} \mathbf{Q} = -\nabla \times \mathbf{P} - \frac{j\omega}{c} \mathbf{A} \mathbf{P} \tag{7.7b}
\]

\[
\mathbf{A} = \begin{bmatrix}
0 & 0 & -\beta_{in} \\
0 & 0 & \alpha_{in} \\
\beta_{in} & -\alpha_{in} & 0
\end{bmatrix} \tag{7.8}
\]

where $\sigma' = c\mu_0 \sigma$. Because of the arbitrary incidence, the TE and TM modes cannot be decoupled as in [66], forcing us to solve for all field components.
To consolidate the time derivatives in Eq. (7.7), we follow the standard SFM approach and introduce a set of auxiliary variables:

\[
P = P_a + \frac{AQ}{\epsilon_r} \tag{7.9a}
\]

\[
Q = Q_a - AP. \tag{7.9b}
\]

Using Eq. (7.9) with Eq. (7.7), we obtain an update equation for \(P_a\) and \(Q_a\):

\[
\frac{j\omega}{c} P_a = \nabla Q - \sigma' P_a - \frac{\sigma' AQ}{\epsilon_r} \tag{7.10a}
\]

\[
\frac{j\omega}{c} Q_a = -\nabla P. \tag{7.10b}
\]

However, Eq. (7.10) also contains \(P\) and \(Q\), which must be solved for separately. Solving Eq. (7.9) for \(P\) and \(Q\) results in the following update equations:

\[
P = (I + A^2 \epsilon_r)^{-1} (P_a + \frac{AQ_a}{\epsilon_r}) \tag{7.11a}
\]

\[
Q = (I + A^2)^{-1} (Q_a - AP_a). \tag{7.11b}
\]

For stability reasons, Eq. (7.11) is only used to update \(P_z\) and \(Q_z\); the other components are updated afterwards using Eq. (7.9) as described in [38]. Since we include lossy materials, we time-average the loss terms using the weighted averaging:

\[
P^{n+\frac{1}{2}} = \zeta P^{n+1}_a + (1 - 2\zeta)P^{n+\frac{1}{2}}_a + \zeta P^n_a
\]

where \(\zeta = 0.5/\gamma^2_m\) (see Ref. [71]).

The update equations are discretized in time and space using standard finite-difference methods. The discretization will depend on the grid scheme chosen. As an example of the final discretized equations, the update equation for \(P_{ax}\) with our grid choice is:

\[
P^{n+1}_{ax,\frac{1}{2}00} = \frac{\xi}{c\Delta t} \left[ (\epsilon_r^{aa} - \zeta \sigma^{aa} c\Delta t)P^n_{ax,\frac{1}{2}00} + (\epsilon_r^{ab} - \zeta \sigma^{ab} c\Delta t)P^n_{ax,0\frac{1}{2}0} + (\epsilon_r^{ac} - \zeta \sigma^{ac} c\Delta t)P^n_{ax,00\frac{1}{2}} \right]
\]

\[
- \xi(1 - 2\zeta) \left[ \sigma^{aa} a P^{n+\frac{1}{2}}_{ax,\frac{1}{2}00} + \sigma^{ab} b P^{n+\frac{1}{2}}_{ax,0\frac{1}{2}0} + \sigma^{ac} c P^{n+\frac{1}{2}}_{ax,00\frac{1}{2}} \right]
\]

\[
- \xi \left( Q^{n+\frac{1}{2}}_{y,\frac{1}{2}1\frac{1}{2}} - Q^{n+\frac{1}{2}}_{y,\frac{1}{2}1\frac{1}{2}} \right)
\]

\[
+ \xi\beta \left[ \sigma^{ca} \epsilon_r^{ca} Q^{n+\frac{1}{2}}_{x,0\frac{1}{2}1\frac{1}{2}} + \sigma^{cb} \epsilon_r^{cb} Q^{n+\frac{1}{2}}_{y,0\frac{1}{2}1\frac{1}{2}} + \sigma^{cc} \epsilon_r^{cc} Q^{n+\frac{1}{2}}_{z,0\frac{1}{2}1\frac{1}{2}} \right]
\]

\[
\tag{7.12}
\]

where \(\xi = (c\Delta t)/(\epsilon_r + \zeta \epsilon_r' c\Delta t)\), \(\epsilon_r' = \epsilon_r^{-1}\), and \(\Delta t\) is the time step used. The superscript
denotes the associated time-step and the subscript denotes the field component and relative spatial location \((x, y, z)\). Non-collocated field components are computed by averaging of nearby components \([72]\). As with prior SFM implementations, it is necessary to record and compute the fields at half time steps as well as whole time-steps.

In summary, a single half time step proceeds as follows:

1. Update \(P_a\) and \(Q_a\) using Eq. (7.10).
2. Update \(P_z\) and \(Q_z\) using Eq. (7.11).
3. Update the remaining \(P\) and \(Q\) fields using Eq. (7.9).

We have not performed rigorous stability analysis of the algorithm, but the use of the stability criterion of the standard 3D SFM \((c\Delta t/\Delta u \leq \gamma^2/(2 + \gamma^2))\) has proven sufficient in our tests, where \(\Delta u\) is the distance between adjacent grid points and is assumed to be equal for all dimensions. There is some numerical dispersion in the results that is comparable to prior SFM algorithms (noticeable at very low or high frequencies). Finally, the boundaries of the \(z\) dimension are terminated with PML suitable for the standard 3D SFM \([61]\).

### 7.2 Near-to-Far-Field Transformation

For many simulations, the direction, transmittance, and polarization of the far-field (Fraunhofer) orders are the primary interest. The far-field for periodic structures is bounded to a discrete set of orders, denoted by the integers \(m\) and \(v\). We describe the far-field orders as plane-waves with Jones vectors \(J_{m,v}\) and wavevectors \(k_{m,v} = (2\pi/\lambda)(\alpha_{m,v}\hat{x} + \beta_{m,v}\hat{y} + \gamma_{m,v}\hat{z})\).

#### 7.2.1 Far-Field Wavevector

The wavevector \(k_{m,v}\) of diffraction order \([m, v]\) is found with the full form of the classic diffraction equation:

\[
\alpha_{m,v} = \frac{m\lambda}{\Lambda_x n_{out}} + \frac{\alpha_{in}}{n_{out}} \tag{7.13a}
\]

\[
\beta_{m,v} = \frac{v\lambda}{\Lambda_y n_{out}} + \frac{\beta_{in}}{n_{out}} \tag{7.13b}
\]

\[
\gamma_{m,v} = \sqrt{1 - \alpha_{m,v}^2 - \beta_{m,v}^2} \tag{7.13c}
\]

where \(\Lambda_x\) and \(\Lambda_y\) are the periodicity of the structure in the \(x\) and \(y\) directions, and \(n_{in}\) and \(n_{out}\) are the index of refraction of the incident and outgoing media. We can subsequently find
the related orientation angles as:

\[
\begin{align*}
\theta_{m,v} &= \cos^{-1} \gamma_{m,v} \\
\phi_{m,v} &= \tan^{-1} \frac{\beta_{m,v}}{\alpha_{m,v}}.
\end{align*}
\] (7.14a)

(7.14b)

Based on these equations, the possible orders are linearly spaced in k-space \((\alpha_{m,v}, \beta_{m,v})\) but are non-linearly spaced in real space \((\theta_{m,v}, \phi_{m,v})\). This is demonstrated in Fig. 7.2. From this figure, we observe that the output orders may have a different angle with respect to the \(y - z\) plane than the input does (i.e., the lines of Fig. 7.2(b) are not straight). This out-of-plane diffraction effect is called conical diffraction, and is a well known phenomenon [73].

### 7.2.2 Far-Field Polarization

The amplitude and polarization of each output order can be found if the complex near-field values are known in a plane normal to the periodicity. For our geometry, this is a plane in \(x - y\) located at some \(z = z_s\) beyond the structure of interested (see Fig. 1). This plane is the sampling plane, and the fields on it are \(P_s = P(x, y, z = z_s)\). The complex fields as function of wavelength, for either \(E\) or \(P\), can be computed during the simulation at the sampling line using the discrete Fourier transform [38]. However, as the complex \(P\) fields are more easily obtained,
we will proceed with those. After the simulation is complete, a Fourier transform is applied to get the far-field electric field vector $V_m$:

$$V_{m,v} = \frac{1}{\Lambda_x \Lambda_y} \int_{\Lambda_x} \int_{\Lambda_y} P_s \exp \left( -\frac{2\pi j x}{\Lambda_x} (\alpha_{m,v} - \alpha_{in}) - \frac{2\pi j y}{\Lambda_y} (\beta_{m,v} - \beta_{in}) \right) dx dy. \quad (7.15)$$

Equation (7.15) can be derived by starting with the standard near-to-far-field transformation that uses $E$ fields [38], substituting Eq. (7.3), and simplifying.

Next, we transform the 3D vector $V_{m,v}$ to the 2D vector $J_{m,v}$ in a process that is the inverse of that in (7.2):

$$V_{TE} = \frac{-\hat{x} \beta_{m,v} + \hat{y} \alpha_{m,v}}{||-\hat{x} \beta_{m,v} + \hat{y} \alpha_{m,v}||} \quad (7.16a)$$

$$V_{TM} = V_{TE} \times k_{m,v} \quad (7.16b)$$

$$J_{m,v} = \hat{x}(V_{m,v} \cdot V_{TM}) + \hat{y}(V_{m,v} \cdot V_{TE}). \quad (7.16c)$$

Equations (7.13) through (7.16) are used to obtain $J_{m,v}$ and $k_{m,v}$, which contain all the far-field information. These equations apply to general analysis of periodic FDTD simulations, not just the SFM. We note that the near-to-far-field transformation found in [63] can be obtained from these equations by imposing the constraints $\beta = 0$ and $\nu = 0$.

### 7.3 Algorithm Validation

We simulated a number of structures and compared our results to known solutions in order to confirm the accuracy and functionality of the algorithm.

#### 7.3.1 Glass Etalon

A simple glass etalon was simulated with index $n = 1.5$ surrounding by air ($n = 1.0$) (see Fig 7.3(a)). The transmission through an etalon is well-known and has an exact analytic equation [74]. We note that the transmission is a function of incidence angle, polarization, and wavelength. The key results are shown in Fig. 7.3(b), where we plot the TE and TM cases for 40° out-of-plane incidence ($\phi = 90°$). The results shown here match the analytic equations with a root mean squared error of 0.22% when $\Delta u = \lambda/50$. 

98
7.3.2 Crossed Polarizer Pair

A pair of crossed dichroic polarizers were simulated over a range of source incidences. The first polarizer was oriented with $T_\parallel = 0^\circ$ and the second polarizer was oriented with $T_\parallel = 90^\circ$. We modeled the polarizers with realistic material characteristics [75], having thickness $d = 100 \ \mu m$, ordinary index $\tilde{n}_o = 1.5 - i5 \times 10^{-5}$, and extraordinary index $\tilde{n}_e = 1.5 - i3.1 \times 10^{-3}$. Gradient index anti-reflection (AR) coatings were used on both sides of the polarizer pair [76]. The source was polarized at $90^\circ$ and was centered at 550 nm. The angle of source incidence varied over a wide range ($0^\circ \leq \theta \leq 60^\circ$ and $0^\circ \leq \psi < 360^\circ$). The results are shown in Fig. 7.4(a). As expected, the transmittance is the lowest at normal incidence. As the angle of incidence increases, the transmittance increases, particularly in the diagonal directions.

To validate the simulation quantitatively, we used the Berreman 4x4 method [34]. The Berreman method is capable of analyzing stacks of anisotropic materials with arbitrary angle of incidence. The results of the Berreman analysis, shown in Fig. 7.4(b), match closely with the FDTD results. The root mean square error between the two methods is 0.0046, with less error at normal incidence.

7.3.3 Twisted Nematic Cell

Liquid Crystals (LCs) are one the most commonly used anisotropic materials, being present in almost every modern display. The anisotropy of the LC depends on its material phase. In the isotropic phase, an LC does not posses any anisotropy, due to the LC molecules being randomly
oriented. In the nematic phase, the LC molecules become aligned to a certain direction, characterized by a vector called the nematic director. The basic LCD uses a Twisted Nematic (TN) LC cell. In the TN cell, the nematic director of the LC twists continually from one surface of the cell to the other. We can model this ideal TN cell as having twist $\phi_{\text{twist}}$ and thickness $d$ with a permittivity tensor given by:

$$\varepsilon = \mathbf{R}(z)^{-1} \begin{bmatrix} \epsilon_e & 0 & 0 \\ 0 & \epsilon_o & 0 \\ 0 & 0 & \epsilon_o \end{bmatrix} \mathbf{R}(z)$$  \hspace{1cm} (7.17)

where $\mathbf{R}(z)$ is a rotation matrix given by:

$$\mathbf{R}(z) = \begin{bmatrix} -\sin \left( \frac{z}{d} \phi_{\text{twist}} \right) & \cos \left( \frac{z}{d} \phi_{\text{twist}} \right) & 0 \\ \cos \left( \frac{z}{d} \phi_{\text{twist}} \right) & \sin \left( \frac{z}{d} \phi_{\text{twist}} \right) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  \hspace{1cm} (7.18)

We simulated a lossless TN cell having properties $n_o = 1.5$, $n_e = 1.6$, $d = \frac{\lambda}{2\Delta n}$ for $\lambda = 550$ nm, and $\phi_{\text{twist}} = 90^\circ$. AR coatings are placed on either side of the cell. The Stokes parameters of the transmitted light are shown in Fig. 7.5(a), and the total transmittance is shown in Fig. 7.5(b). Also shown are the Berreman 4x4 results of the same TN cell. The Stokes parameters indicate that the polarization properties of the transmitted light vary dramatically with wave-
length, as expected. Both the stokes and total transmittance results produced by the FDTD simulation are very close to those predicted by the Berreman 4x4, as well as derived equations for the transmittance [77].

7.3.4 Bragg Grating

We simulated a Bragg grating to demonstrate the FDTD algorithm and far-field calculation as a whole, and compared our result with that of a popular rigorous coupled-wave analysis (RCWA) method [35]. The grating was uniform in z with thickness d, and is periodic in x with pitch Λ as follows:

\[
\epsilon_r(x) = \epsilon_1 + \epsilon_2 \sin \frac{2\pi x}{\Lambda} \tag{7.19a}
\]

\[
\sigma(x) = \sigma_1 + \sigma_2 \sin \frac{2\pi x}{\Lambda} \tag{7.19b}
\]

with parameters \(d = 5 \mu m\), \(\Lambda = 0.45 \mu m\), \(\epsilon_1 = 2.2525\), \(\epsilon_2 = 0.15\), and \(\sigma_1 = \sigma_2 = 250 \text{ S/m}\) (see Fig 7.6(a)). The total cell dimensions including PML were 38x509 (\(\Delta u = 12.5 \text{ nm}\)) and the simulation was ran for 17400 time steps, taking 131 s (for each incidence case) on a 2011 laptop (2.2 GHz core i7). This many time steps is not always necessary or desired, but in this case we found it better matched the RCWA due to the multiple reflections having more time to settle. In comparison, the RCWA took < 20 s. We simulated the grating with TE polarization and an incidence angle of \(\theta = 37.67^\circ\) and \(\phi = 0^\circ\), 30°, and 45°. Note that the efficiency does not reach 100% due to the conductive nature of the grating. The results are shown in Fig. 7.6(b), where it is seen that that both methods produce very similar results.
While slower, the *Wolfsim 3D* method has advantages over RCWA in some cases. Perhaps most importantly, *Wolfsim 3D* allows us to view the propagation and transient interaction of the electromagnetic waves with the structure, while RCWA can only give the final steady-state solution. Another advantage is that using sources with arbitrary polarizations is simpler and more convenient in *Wolfsim 3D* than in RCWA, as most RCWA methods can only simulate a single TE or TM polarized source per simulation. In general, the accuracy of FDTD and RCWA methods are comparable [78]-[79].

### 7.3.5 Photonic Band Gap Structure

Next, a 2D photonic band gap (PBG) structure was simulated with different incidences. PBGs can act as band-pass filters and are applicable for much of the electromagnetic spectrum; simulated here is a millimeter wave PBG. The structure is shown in Fig. 7.7(a) and is the same structure simulated in [61] and [65]. It consisted of four layers of dielectric cylindrical rods with $\epsilon_r = 4.2$, diameter $d = 4$ mm, and placed in a square array with spacing $a = 9$ mm. Since the structure was periodic, we only needed to simulate one periodic unit, which consisted of four rods total. The simulation dimensions were 38x276 with $\Delta u = 0.25$ mm and the stability factor was 0.566 for the normal case and 0.261 for the oblique cases. The simulation was run for 18000 time steps for the normal incidence case and 60000 time steps for the oblique cases; it took 529 s to complete all three simulations. The transmission coefficient for the different incident cases is shown in Fig. 7.7(b). The normal and in-plane incidence results are in full agreement with
previously published measured and simulated results ([61] and [65]).

7.3.6 2D Array of Conducting Particles

There is interest in 2D arrays of sub-wavelength conducting or semiconducting particles for use as optical filters, detectors, photonic couplers, and more. We simulate light incident upon such an array of perfectly conducting particles. With the right parameters, the particle array will reflect incidence light of a certain wavelength with 100% efficiency. The particles simulated are rectangular and of equal height and width, with a width of Λ/5 where Λ is the period. The period is the same for both x and y dimensions. The geometry is shown in Fig. 7.8(a). We set Λ = 1 μm and chose a thickness d = 0.05 μm. Since we are employing periodic boundaries, we only needed to simulate one period of the system. We modeled the perfect electric conductors by setting the conductivity to a sufficiently high value. The simulated reflectance of the element is plotted in Fig. 7.8(b). The spectra contains the expected 100% reflectance peak. The wavelength at which the peak occurs, as well as the shape of the reflectance function agree with previous experimental and numerical results for similar structures [80]. We also simulate the array at an incidence of φ = 45° and θ = 5°, shown in the Fig. 7.8(b) inset. We see that the location of maximum reflectance shifts by a slight amount.
7.3.7 2D Asymmetric Grating

To validate the 2D NTFFT of our algorithm, we simulated a 2D periodic binary grating. The structure, shown in Fig. 7.9(a), consisted of asymmetric rectangularly patches with index $n = 1.5$, thickness $d = 0.6 \ \mu m$, spaced $5 \ \mu m$ and $8 \ \mu m$, and sized $3.5 \ \mu m$ and $4 \ \mu m$ in the $x$ and $y$ directions respectively. For comparison, we also simulated the structure using a popular RCWA code.

Some of the calculated far-field orders are shown in Fig. 7.9(b). It is seen that the results of the two methods are in very close agreement for all output orders. We note that the Wolfsim 3D result has smoother curves than the RCWA result. We believe that the Wolfsim 3D result is more accurate, and that the differences seen in the RCWA result are caused by non-convergence issues in the RCWA algorithm. This example demonstrates that our NTFFT is functioning properly and that our algorithm is fully capable of simulating asymmetric inhomogeneous 3D structures.

7.4 GPH Simulation

Having derived and validated the algorithm behind Wolfsim 3D, we will now use it to simulate for the first time two interesting GPH.
7.4.1 GPH Multiple Beam Splitter

The multiple beam splitter (MBS), also called a fan-out grating, array generator, or diffractive beam splitter, is a structure designed to split a single incident beam into $N$ output orders of equal intensity, where $N \geq 2$ and many times $N \geq 16$ [81, 82]. Typically, the output orders are on a rectangular $k$-space grid of size $M^2 = N$, but circular and 1D $k$-space patterns are also used [83]. The main applications for MBSs are in integrated optics [84], beam-combining [85], and ultra-fast optics [86]. MBSs are most commonly fabricated as binary phase gratings called Dammann gratings and can achieve very high efficiencies. However, as might be expected for a Fourier element, the efficiency depends on $N$, with higher $N$ typically resulting in higher efficiencies [87, 83]. In addition to diffraction efficiency, the uniformity of the output orders is usually important. We define the uniformity of an MBS with target orders $N$ as:

$$U = 1 - \frac{\sum_{n=1}^{N} |E_n - E_{avg}|}{NE_{avg}}$$  \hspace{1cm} (7.20a)

$$E_{avg} = \frac{\sum_{n=1}^{N} E_n}{N}.$$  \hspace{1cm} (7.20b)

The GPH MBS has these potential advantages: higher efficiency than Dammann gratings...
especially at lower $N$, on/off switching capability when using an LC cell, and spectral tunability (i.e., with specific LC coatings, one wavelength may be split by the MBS while another passes through unaltered). The MBS is essentially a Fourier hologram (i.e., the far-field is all that matters) with a target k-space image that is a uniform intensity square (or circle) of grid points centered at the origin. Thus, one way to calculate the needed phase profile is with the Gerchberg-Saxton algorithm as described in Chap. 6. Unlike the FGPHs studied in Chap. 6, the MBS is a periodic element, making Wolfsim 3D the idea simulation tool. The MBS under consideration has $M = 5$ (shown in Fig. 7.10) with a pitch of $\Lambda = 5 \mu$m.

**Analytic Predictions**

The Gerchberg-Saxton algorithm was used on the target k-space image (Fig. 7.10(a)) to calculate a possible phase profile to produce it (Fig. 7.10(b)). With this phase profile, the predicted output for RCP input and linear input are shown in shown in Fig. 7.11(a) and (b). In terms of order uniformity, the linear input case is predicted to produce a slightly more uniform output, with a uniformity of 92.0% vs. 88.5% for circular input. In terms of polarization, when the input is RCP, the output will be LCP; however, it is worth noting that the phase component of each output order is essentially random because the phase of the output orders is used as the free parameter in the Gerchberg-Saxton optimization.

However, when the input is linear, the predicted output polarization is more complex. The output of the linear input can be found by decomposing it into orthogonal circular inputs and then summing the resulting outputs. Since the phase component of the LCP/RCP output
orders is random, the summation of random phase LCP/RCP waves is a linear polarization with a random optic axis. In addition, the slight non-uniformity in intensity of the output will cause the orders to be slightly elliptical. Thus, the GPH MBS is likely to only be suitable for applications where polarization is not a concern or where different polarizations can be manipulated/utilized.

**Simulated Results**

The MBS was simulated with a grid resolution of $\lambda_0/\Delta u = 16$, a central wavelength $\lambda_0 = 550$ nm, a birefringent media with $n_o = 1.5$ and $n_e = 1.66$, and a thickness of 1.72 $\mu$m (half-wave at $\lambda_0$). The simulated k-space images (far-field orders) are shown in Fig. 7.11. For RCP input, the result is similar to the analytic predicted result, with an average intensity deviation of 87.7%. The most notable difference is the increased zero-order. This is most likely caused
by the finite thickness of the element, which goes against the paraxial approximation of the Gerchberg-Saxton algorithm.

For the linear input case, we found the results to vary significantly based on the absolute polarization angle. If the input angle is 45°, the result is actually slightly more uniform than both the RCP input or the predicted result, with only 88.7% average intensity deviation. However for input polarizations of 0° and 90°, the zero-order is dramatically increased or decreased respectively. The 0° case is not likely useful for any application, but the 90° case has an average intensity deviation of only 93.4% if the zero-order is neglected (U’), a configuration which may interesting for certain applications.

We propose that the odd behavior with linear input is due to near-field (non-paraxial) effects within the structure. To test this hypothesis, we simulated the same MBS structure but with twice the pitch. The larger pitch to wavelength ratio causes less diffraction and brings the scenario closer to paraxial. The results for linear input are shown in Fig. 7.12. All three inputs resulted in k-space images much closer to the analytic predictions than with \( \Lambda = 5 \, \mu\text{m} \). Thus, we conclude that our hypothesis was correct, and this odd behavior is a real phenomena due to complex wave interactions within the inhomogeneous anisotropic media.

This simulation demonstrates how analytic calculations are based upon many assumptions, and the actual electromagnetic results may significantly differ from what is predicted. We have also seen that the GPH MBS is a viable element that behaves predictably when the input is circularly polarized. When the input is linearly polarized however, the basic analytic models breakdown, and Wolfsim 3D gives us otherwise unknowable information about the element’s properties.

Figure 7.12: Simulated k-space images (far-field orders) resulting from the MBS profile given in Fig. 7.10(b) and \( \Lambda = 10 \, \mu\text{m} \). FDTD simulated output with input a) 0° linear, b) 45°, and c) 90°. These results match the analytic results closer than when \( \Lambda = 5 \, \mu\text{m} \).
7.4.2 GPH Micro Lens Array

Micro lens arrays (MLAs) are a common optical element that are used in a wide variety of applications such as displays, imaging [88], fiber optics [89], and LEDs [90]. One advantage of the MLA is its very short focal distance, even if the F/# of the individual lenses is large. GP MLAs can be made with polarization holography [45] or as CGHs using a direct-write system such as The Howlographer. An example GP MLA made with The Howlographer is shown in Fig. 7.13. We would like to understand better the optical properties of the GP MLA.

We simulated a GP MLA with a hexagonal lens array. While the array is hexagonal, Wolfsim 3D can only simulate rectangular unit cells, thus we must choose a rectangular unit cell of the MLA as shown in Fig. 7.14(a). The minimum pitch of each lens was 2 µm and the spacing between adjacent lenses, i.e., the lattice constant, was 6 µm. The simulated element had a 2 layer twisted nematic structure [∆n = 0.2, d1 = 1.375 µm, φ1 = −φ2 = 70°], which should result in a broadband half-wave condition that covers most of the visible range [14, 91].

After simulating the structure, we can view the phase profile of the exiting wave, as shown in Fig. 7.14(b)-(d) for 550 nm. The phase of the $E_x$ fields in Fig. 7.14(b) closely resembles the target phase profile. The $E_y$ fields in Fig. 7.14(c) are similar as well, but offset by 90°, indicating that the output polarization is circular as expected. The $E_z$ fields in Fig. 7.14(d) are quite chaotic; this is because its phase depends largely on propagation direction.

We are more interested in the fields at the focus of the MLA, which lies some distance...
beyond the element. Due to hardware limitations, it is not feasible to simulate the fields out that far in *Wolfsim 3D*; instead, we propagate the fields after the *Wolfsim 3D* simulation by starting with the exiting wave (phase and amplitude information) and applying the general diffraction equation solution described in Chap. 2. In this way, we can quickly calculate the fields at arbitrary locations beyond the MLA.

The vacuum propagation yields the total field powers shown in Fig. 7.15. There are many observations to be made. The zero-order leakage can be judged by the color of the background intensity in the figures. We see that for 650 and 550 nm light, the zero-order leakage is low, but it is substantial for 450 nm light.

The predicted focal lengths for 650, 550, and 450 nm should be 15.1, 18.2, and 22.5 µm. It is not entirely obvious from theory whether this length should be taken from the center or surface of the MLA, but given the 2-layer twisted nature of the film, from the center of the MLA seems reasonable. From the figures, we measure the focal lengths for 650, 550, and 450 nm as being 12.7 µm, 15.6 µm, and 22.4 µm. The focal lengths are shorter than anticipated for larger wavelengths, which focus the light tighter.

We also see many higher orders and some lower orders from the intensity plots. These other orders appear as spots focused closer or further from the intended focus. For the larger wavelengths, the other orders are minimal, but for 450 nm there are many other orders apparent. This seems to indicate that the shorter wavelengths are experiencing some kind of interaction within the media that the other wavelengths are not. It may also be indicative that the layer
Figure 7.15: Calculated total field power after the MLA for $xz$ and $yz$ cross sections. The field values were calculated using the general diffraction solution and the complex field values after the MLA (shown for 550 nm in Fig. 7.14).
parameters (index, thickness, twist) are not suitable for this wavelength.

Finally, we measured the spot size for the 650 and 550 nm cases as 1.27 and 1.01 µm. We approximated the diffraction limited spot sizes for these wavelengths as 1.01 and 1.05 µm, which are close to the measured sizes. Note that these lenses had a parabolic phase profile, so some variation from the diffraction limited size is expected.

To summarize, this is an initial study on the properties of small GP MLAs. We have made several interesting observations about zero-leakage, higher and lower orders, spot size, and the variation between wavelengths. We conclude that for small dimensions, the element behaves in some unanticipated ways, which only a numerical method such Wolfsim 3D can elucidate.

7.5 Summary and Conclusions

We have developed a new numerical simulation tool for studying GPHs. The tool is based on the split-field FDTD algorithm and enables broadband, off-axis simulation of 3D structures with two dimensions of periodicity and inhomogeneous, anisotropic permittivity and conductivity. In essence, the tool is ideally suited for studying electromagnetic wave interactions with GPHs. A primary contribution of this chapter is the derivation of the algorithms for an FDTD tool, Wolfsim 3D, ideally suited for studying GPH.

After developing the tool, we numerically implemented it using the C programming language and verified its accuracy with a number of test simulations. These tests ranged from simple glass etalons to complex Bragg gratings and PBG structures. The results of the test simulations were compared against analytic calculations, results from other numeric methods, and experimental results. Together, the tests validate the accuracy of the tool with respect to every feature described above. A primary contribution of this chapter is the validation of the accuracy and functionality of Wolfsim 3D.

Finally, we used Wolfsim 3D to simulate for the first time a GP MBS and a GP MLA. For the GP MBS, we found that the element does not always behave according to analytic calculations, and determined that the cause was related to near-field effects which violate the paraxial approximation used in the analytic calculations. For the GP MLA, we made several interesting observations about the unanticipated behavior of small GP MLAs. The final primary contributions of this chapter were the first-time simulation and analysis of a GP MBS and a GP MLA.
Chapter 8

Conclusion

This dissertation covered a wide range of novel research topics pertaining to CGHs. Broadly speaking, the three main research focuses were 1) CGH fabrication tools, 2) Analysis, fabrication, and evaluation of CGHs, and 3) Simulation of CGHs (and GPHs).

8.1 Direct-Write System for CGH Creation

In Chapter 3, a novel direct-write system was designed, modeled, and tested. The system was composed of a UV laser, a polarization modulation element, and an XY translation stage and was named The Howlographer. A thorough system description was developed that began with the scan parameters and predicted the alignment of the resulting liquid crystal (LC) based element. The accuracy of the model was verified, and several complex patterns were fabricated to demonstrate the useful properties of the system. The key contributions of Chapter 3 were:

- A novel direct-write system was designed for recording complex CGHs using LPP and LC
- For the first time, this kind of direct-write system was accurately described with a mathematical model
- Complex CGHs with continuously varying profiles were fabricated, analyzed, and explained for the first time
- The first high-quality direct-write polarization gratings and q-plate were demonstrated

8.2 Study and Optimization of CGHs

In Chapters 4, 5, and 6, being equipped with The Howlographer, a variety of CGH elements were fabricated and characterized. Before fabrication, the elements in question were thoroughly
analyzed from the standpoint of the scanning system, the predicted material response, and the predicted operation of the elements. After this, actual elements were fabricated and characterized, both to optimize scan parameters and verify the validity of the prior analysis. Finally, high-quality elements were made and evaluated. The elements studied were polarization gratings (PGs), geometric phase lenses (GPLs), and arbitrary Fourier GPH (FGPH). The key contributions of these chapters were:

- An analysis of potential non-idealities of direct-write fabricated PGs, GPLs, and FGPH and the subsequent impact on element operation
- The optimization of scan patterns and scan parameters for creating high-quality said elements
- Demonstration of high-quality said elements created for the first time with a direct-write scanning system
- Description of the limitations of direct-write systems like The Holwographer

8.3 FDTD for Simulation of CGHs and GPHs

In Chapter 7, a numerical tool was developed capable of simulating CGHs and other arbitrarily complex optical elements. The tool was called Wolfsim 3D and was based on the Finite-Difference Time-Domain (FDTD) method. The derivation for the Wolfsim 3D algorithm was given, and the algorithm was implemented in the C programming language and made available online for free [6]. Various test simulations were ran to validate the accuracy of the algorithm, including an etalon, a twisted nematic cell, crossed polarizers, a Bragg grating, a photonics band gap structure, a 2D array of conducting particles, and a 2D grating. Results were compared against analytic results, results of other simulation tools, and/or experimental results, and all verified the algorithm’s capabilities. After this, we simulated a GP multiple beam splitter (MBS) and a GP micro-lens array (MLA). These results showed that in some cases the analytic GPH models break down due to near-field (non-paraxial) effects. The key contributions of Chapter 7 were:

- Derivation of a unique FDTD algorithm for simulating CGHs and other complex elements
- Implementation of the algorithm in C and distribution as open source software
- Demonstration of the accuracy of the algorithm with various validation simulations
- Initial study of electromagnetic wave interactions with GP MBSs and GP MLAs
8.4 Future Research Directions

This work has explored many different facets of CGHs, and can be considered the initial study and investigation of certain problem sets; hence, many questions remain unanswered. In Chapter 3, we noted that our system description of *The Howlographer* was unable to accurately predict the results of some scan patterns. We expect that this can be remedied by introducing more complex transfer functions $H$ and $T$, which encompass a greater number of LPP and LC material properties. For example, we know that the LPP materials we use can become saturated and stop responding to subsequent exposure after being exposed to high fluences. We also know that at very small scales, the aligning behavior of LC is dictated not only by the LPP, but also by the alignment of neighboring LC molecules. Accounting for these and other material-related phenomena represent a significant potential extension of this research.

In this work, we exclusively considered elements which comprised LC layers of uniform thickness. However, given the flexibility of *The Howlographer*, this need not be the case. We have recently observed that *The Howlographer* may be used to selectively polymerize unpolymerized LCP or to selectively etch away already polymerized LCP. When combined with existing processing techniques, we have a wide variety of processing steps available which may be compared to a standard semiconductor process flow (*i.e.*, deposition, lithography, developing, etching). With a combination such processing steps, we expect that it is possible to create arbitrary, inhomogeneous LC landscapes, which in turn may enable many new exciting optical elements.

Lastly, our development of the *Wolfsim 3D* algorithm represents good progress in our capability to simulate and understand GPHs. However, due to lack of time and an abundance of other research questions, we were unable to conduct a thorough study into interesting GPH elements. One class of GPHs that especially deserves more study is the multi-twist retarder (MTR) GPH [91]. These elements enable a GPH to have a specially tuned spectral response. In the case of very complex MTR, the element may behave in strange ways, and the off-axis properties have yet to be thoroughly studied. Numerical simulation using *Wolfsim 3D* represents an easy path to understanding the operation and limitations of such elements.

8.5 Closing

The high-level goal of this dissertation research was advancing our ability to control and manipulate light via CGHs. This goal has been achieved in two primary ways. First, we have obtained a means of easily creating complex CGHs which we could previously only imagine. Second, the complex properties and behavior of CGHs have been illuminated as never before. This illumination was accomplished through a combination of analytical analysis, numerical simulation, and experimental results. All of these results enable this curious class of elements, Computer Gen-
erated Geometric Phase Holograms, to be used in new and exciting ways, ultimately changing
the world we live in.
REFERENCES


Appendix A

Geometric Phase via a Birefringent Retarder

We examine the change in polarization to a plane wave traveling in $z$ after passing through a birefringence medium (a retarder) located at $z = 0$ with thickness $d$, birefringence $\Delta n$, and optical axis $\theta$. We assume the input wave is right circular polarized (RCP) with a phase constant of $\phi = 0$. We also assume that some reference wave is also RCP with phase constant $\phi = 0$, but passes through an identical medium except the medium has optical axis $\theta_r = 0$.

Jones calculus is a convenient way to describe the polarization and phase of a lightwave as it travels through homogeneous media. The Jones vector of our input wave after passing through the retarder is:

$$ J_{\text{out}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ j \end{pmatrix} \begin{pmatrix} e^{j\phi_e} \cos^2 \theta + e^{j\phi_o} \sin^2 \theta & (e^{j\phi_e} - e^{j\phi_o}) \cos \theta \sin \theta \\ (e^{j\phi_e} - e^{j\phi_o}) \cos \theta \sin \theta & e^{j\phi_e} \sin^2 \theta + e^{j\phi_o} \cos^2 \theta \end{pmatrix} $$

(A.1)

where $\phi_e = 2\pi(n_o + \Delta n)d/\lambda$ and $\phi_o = 2\pi n_o d/\lambda$. We note that $\phi_e = \phi_o + 2\pi \zeta$, where $\zeta = \Delta nd/\lambda$ is the retardation. Expanding the matrix product and simplifying, we obtain:

$$ J_{\text{out}} = \frac{e^{j\phi_o}}{\sqrt{2}} \begin{pmatrix} e^{j2\pi \zeta}(\cos^2 \theta + j \cos \theta \sin \theta) + (\sin^2 \theta - j \cos \theta \sin \theta) \\ e^{j2\pi \zeta}(\cos \theta \sin \theta + j \sin^2 \theta) + (- \cos \theta \sin \theta + j \cos^2 \theta) \end{pmatrix}. $$

(A.2)

Now we apply a right circular polarizer to the output and simplify:
\[ \mathbf{J}_{RCP} = \mathbf{J}_{out} \frac{1}{2} \begin{pmatrix} 1 & j \\ -j & 1 \end{pmatrix} \]  \hspace{1cm} (A.3a)

\[ \mathbf{J}_{RCP} = \frac{e^{j\phi_0}}{2\sqrt{2}} (e^{j2\pi \zeta} + 1) \begin{pmatrix} 1 \\ j \end{pmatrix}. \]  \hspace{1cm} (A.3b)

Next, we apply a left circular polarizer to the output, then use Euler’s identity and simplify:

\[ \mathbf{J}_{LCP} = \mathbf{J}_{out} \frac{1}{2} \begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix} \]  \hspace{1cm} (A.4a)

\[ \mathbf{J}_{LCP} = \frac{e^{j\phi_0}}{2\sqrt{2}} \left( e^{j2\pi \zeta} (\sin \theta + j \cos \theta)^2 + (-j \sin \theta + \cos \theta)^2 \right) \]  \hspace{1cm} (A.4b)

\[ \mathbf{J}_{LCP} = \frac{e^{j\phi_0}}{2\sqrt{2}} \left( e^{j2\pi \zeta} (j \sin \theta - \cos \theta)^2 + (\sin \theta + j \cos \theta)^2 \right) \begin{pmatrix} 1 \\ -j \end{pmatrix} \]  \hspace{1cm} (A.4c)

\[ \mathbf{J}_{LCP} = \frac{e^{j\phi_0}}{2\sqrt{2}} (e^{j2\pi \zeta} - 1)e^{j2\theta} \begin{pmatrix} 1 \\ -j \end{pmatrix}. \]  \hspace{1cm} (A.4d)

Putting Eq. A.3 and A.4 together we obtain a simpler form for \( \mathbf{J}_{out} \):

\[ \mathbf{J}_{out} = \frac{e^{j\phi_0}}{2\sqrt{2}} \left[ (e^{j2\pi \zeta} + 1) \begin{pmatrix} 1 \\ j \end{pmatrix} + (e^{j2\pi \zeta} - 1)e^{j2\theta} \begin{pmatrix} 1 \\ -j \end{pmatrix} \right]. \]  \hspace{1cm} (A.5)
A key difference between dynamic phase and geometric phase (Pancharatnam-Berry phase) is the dependence on wavelength. The dynamic phase $\phi_d = 2\pi nd/\lambda$ is intrinsically wavelength dependent. The geometric phase $\phi_g = 2\theta$ has no wavelength dependence. If we insert the dynamic phase into the NTFFT, assuming a constant thickness but variable index of refraction, the result is:

$$E(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j2\pi n(x',y')d/\lambda} e^{-j(k_x x' + k_y y')} dx' dy'$$  \hspace{1cm} (B.1a) \\
$$E(k_x, k_y) = F[e^{j2\pi n(x',y')d/\lambda}].$$  \hspace{1cm} (B.1b) \\
where $F$ is the Fourier transform. We see that $E(k_x, k_y)$ depends on $\lambda$ in a non-trivial manner.

Now we insert a spatially varying geometric phase into the NTFFT:

$$E(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j2\theta(x',y')} e^{-j(k_x x' + k_y y')} dx' dy'$$  \hspace{1cm} (B.2a) \\
$$E(k_x, k_y) = F[e^{j2\theta(x',y')}].$$  \hspace{1cm} (B.2b) \\
Now $E(k_x, k_y)$ does not depend on $\lambda$. This is an important result which means that the wavevector is the same regardless of wavelength. For some wavevector and two different wavelengths $\lambda_1$ and $\lambda_2$, we can find the relationship between the corresponding direction cosines $\alpha$ and $\beta$ to be:
\[
E(k_x, k_y) = E\left(\frac{2\pi \alpha_1}{\lambda_1}, \frac{2\pi \beta_1}{\lambda_1}\right) = E\left(\frac{2\pi \alpha_2}{\lambda_2}, \frac{2\pi \beta_2}{\lambda_2}\right) \quad (B.3a)
\]

\[
\alpha_2 = \frac{\alpha_1 \lambda_2}{\lambda_1} \quad (B.3b)
\]

\[
\beta_2 = \frac{\alpha_1 \lambda_2}{\lambda_1} \quad (B.3c)
\]

Thus, for a given geometric phase profile \(2\theta(x', y')\), the direction cosines for different wavelengths will be scaled version of each other.

To illustrate the difference between these two phases, we examine a linear dynamic phase profile (a gradient index prism) and a linear geometric phase profile (a polarization grating). The prism’s result is:

\[
E(k_x, k_y) = \mathcal{F}[e^{j2\pi \frac{n(x', y')}{\lambda}}] = \mathcal{F}[e^{j2\pi x' \frac{d}{\lambda}}] \quad (B.4a)
\]

\[
E(k_x, k_y) = \delta(k_x - \frac{2\pi d}{\lambda}) \quad (B.4b)
\]

\[
E(\alpha, \beta) = \delta(\alpha - d) \quad (B.4c)
\]

and for a polarization grating of pitch \(\Lambda\), the result is:

\[
E(k_x, k_y) = \mathcal{F}[e^{j2\theta(x', y')} - \frac{2\pi x'}{\Lambda}] = \mathcal{F}[e^{j2\pi x' \frac{i}{\Lambda}}] \quad (B.5a)
\]

\[
E(k_x, k_y) = \delta \left(k_x - \frac{2\pi}{\Lambda}\right) \quad (B.5b)
\]

Hence, the prism diffracts light of any wavelength to the same direction cosine (i.e., the same direction), but the polarization grating diffracts light to different direction cosines that are determined by the incident wavelength.
Appendix C

Complex Phase Accuracy: A General, Quantitative GPH Metric

A complex phase describes a phase shift (real part) and a change in amplitude (imaginary part). A given element may be fully described by its complex phase profile; the real part corresponds to the standard phase profile, and the imaginary part corresponds to the transmittance profile. A complex phase takes the form $a + bj$, and a wave accumulating such a phase is multiplied by $\exp(j(a + bj)) = \exp(ja - b)$.

C.1 Complex Phase and GPHs

In an ideal GPH, the transmittance profile should be a constant 100%; hence, the imaginary component of the phase should be zero. As described in Chap. 2, if the retardation $\zeta = \Delta nd/\lambda$ of a GPH is not half-wave, then a zero-order leakage wave will be present at the output of the GPH. While the total transmittance of the GPH would still be 100%, the power into the +1 order is decreased, reducing the effective transmittance. If we treat a variation in $\zeta$ as a decrease in transmittance, we can find a corresponding imaginary phase profile.

From Eq. A.5, we find the effective transmittance $\tau_o$ (the total power transferred into the object wave) to be:

$$\tau_o = \frac{|(e^{j2\pi\zeta} - 1)^2|}{||(e^{j2\pi\zeta} + 1)^2 + |(e^{j2\pi\zeta} + 1)^2||}$$  \hspace{1cm} (C.1)

which, using trigonometric identities, can be reduced to:

$$\tau_o = \sin^2(\pi\zeta).$$  \hspace{1cm} (C.2)

We can now relate $\zeta$ to the complex component of the phase as follows:
\[ |\exp(ja - b)| = \tau_o \]  
\[ \exp(-b) = \sin^2(\pi\zeta) \]  
\[ b = -\log(\sin^2(\pi\zeta)). \]

Thus, the complex phase profile of a GPH with orientation angle \( \theta(r) \) and retardation \( \zeta(r) \) is:

\[ \phi(r) = 2\theta(r) + j\log(\sin^2(\pi\zeta(r))) \]

### C.2 Complex Phase Accuracy

Since the complex phase profile encompasses all relevant features of a GPH from a physical perspective (i.e., orientation angle and retardation) and from an optical perspective (i.e., first-order efficiency, zero-order leakage, higher orders, and scattering), we may compare an element's actual complex phase profile to its target complex phase profile to quantitively evaluate the quality of a GPH. To facilitate this, we define a parameter called the complex phase accuracy \( \xi \), derived from the components of the complex phase (\( \theta \) and \( \zeta \)):

\[ \xi = \xi_\theta \xi_\zeta \]  
\[ \xi_\theta = 1 - \frac{2}{\pi A} \iint_A |\theta_{\text{target}}(r) - \theta(r)|dr \]  
\[ \xi_\zeta = 1 - \frac{2}{\pi A} \iint_A |0.5 - \zeta(r)|dr \]

where \( A \) is the area of the GPH.

In practice, we can measure \( \theta(r) \) using a polarizing microscope, but \( \zeta(r) \) can be quite difficult to accurately ascertain. We can however approximate \( \xi_\zeta \) directly. Light that is “lost” due to the complex component of the phase is in reality directed into the zero-order leakage wave. Since by definition this wave does not diffract or change in polarization from the input, we can approximate \( \xi_\zeta \) from the percentage power in the zero-order leakage wave \( \eta_0 \) (which can be measured accurately using circular polarizers):
\[
\cos^2(\pi \zeta_{ave}) \simeq \eta_0 \quad \text{(C.6a)}
\]
\[
\zeta_{ave} \simeq \frac{\cos^{-1} \sqrt{\eta_0}}{\pi} \quad \text{(C.6b)}
\]
\[
\xi \zeta = 1 - |0.5 - \zeta_{ave}| \simeq 1 - \left| 0.5 - \frac{1}{\pi} \cos^{-1} \sqrt{\eta_0} \right| \quad \text{(C.6c)}
\]

where \( \zeta_{ave} \) is the average retardation.

There is one more phenomena to account for which will result in a more accurate approximation: in some locations on the sample where defects occur (and thus \( \zeta \) is not half-wave), light may scatter instead of being transmitted into the zero-order. Hence, a more accurate approximation of \( \xi \zeta \) can be obtained as follows:

\[
\xi \zeta \simeq 1 - \left| 0.5 - \frac{1}{\pi} \cos^{-1} \sqrt{\frac{\eta_0}{\tau - \eta_s}} \right| \quad \text{(C.7)}
\]

where \( \tau \) is the total transmitted light and \( \eta_s \) is the percentage of light that is scattered. With this approximation and a physical measurement of \( \theta(x) \), we can obtain an accurate value for the complex phase accuracy of a GPH element, allowing us to direct compare the quality of any GPHs.