

Abstract

POONTHANOMSOOK, CHANWUT. Quartic Loss Function and Its Application to the Parameter Design Problem. (Under the direction of Yahya Fathi.)

There are two functional forms that are commonly used to represent a quality loss function. One form is the conventional *step function* that is based on classifying the units as defective and non-defective, and the other form is the *quadratic loss function* that was proposed by Taguchi. Either of these functional forms has a specific functional profile, and as a result we do not have any degrees of freedom in choosing a desired functional profile in either context. This thesis propose a loss function that not only allows some degrees of flexibility in this regards, but it also can be applied in the case where a non-symmetric function is desirable. At one extreme the proposed loss function can be close to a quadratic loss function, and it can be close to a step function at the other extreme. We refer to this loss function as a *quartic loss function*. We also present the application of the quartic loss function to the parameter design problem in the case where the transfer function between a quality characteristic of interested and design parameters can be determined analytically.

**QUARTIC LOSS FUNCTION AND ITS APPLICATION TO THE
PARAMETER DESIGN PROBLEM**

By

CHANWUT POONTHANOMSOOK

A thesis submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Master of Science

INDUSTRIAL ENGINEERING

Raleigh

2002

APPROVED BY:

Marcia Gumpertz

Don Humphrey

Mahya Fathi
Chair of Advisor Committee

Dedication

To my parents, Kietchai and Suvimol, for their true love and invaluable support.

Biography

Chanwut Poonthanomsook was born on July 27, 1975 in Bangkok, the capital city of Thailand, where he was also raised. After graduating from Assumption College high school in 1993, he attended Kasetsart University. He received his Bachelor Degree of Engineering in Mechanical Engineering in 1997.

Right after he graduated, Chanwut started his first career at Thai Pure Drink, the Coca-Cola bottling plant in Thailand, where his passion for quality engineering began. After three years working with the company, he perceived that he needed to have advanced education to fulfill his eager in quality engineering.

In the spring of 2001, Chanwut began his graduate studies at North Carolina State University with a commitment to contribute his effort to a novel issue in the field of quality engineering that can be useful for people all over the world. While attending North Carolina State University, he was a teaching assistant for a graduate course in Quality Engineering and an undergraduate course in Engineering Economy. He was inducted into the honor society Phi Kappa Phi in 2002.

Acknowledgements

I would like to express my thanks to all members of my advisory committee. Dr. Yahya Fathi, without whom this thesis would have never existed, for being the chair of my advisory committee, and the great mentor in every angle over my educational career at North Carolina State University. Drs. David Humphrey, and Macia Gumpertz for their service as members of my advisory committee and for their careful review on this thesis.

I also thank Nourredine for his advice in constructing fine figures in this thesis, Yue Dai and Saowanee for their help in formatting the word processor for this thesis.

Finally, I thank my dearest wife Kanittha, who without her love, support, understanding, and delicious foods none of this would be possible.

Contents

List of Tables	vii
List of Figures	viii
1 Introduction	1
2 Literature Review	3
2.1 Continuous Loss Function	3
2.1.1 Taguchi's Derivation of the Quadratic Loss Function	5
2.1.2 Derivation of the Expected Loss Based on the Quadratic Loss Function	6
2.2 Moment Estimation Using Taylor Series Approximation	7
2.3 Nonlinear Programming Approach to the Parameter Design Problem . .	10
3 Quartic Loss Function	14
3.1 Derivation of the Quartic Loss Function	15
3.2 The Quality Loss Coefficients	16
3.2.1 Non-Symmetric Loss Function	16
3.2.2 Symmetric Loss Function	21
3.3 Examples	22
4 Expected Loss Based on the Quartic Loss Function	26
4.1 Derivation of the Expected Loss Formulas	26
4.2 Examples	28

5	Mathematical Model for the Parameter Design Problem and a Case Study	36
5.1	The Model	37
5.2	A Case Study	40
5.2.1	The Problem Statement	41
5.2.2	Symmetric Loss Function	44
5.2.3	Non-Symmetric Loss Function	48
6	Conclusions and Recommendations for Future Research	53
6.1	Conclusions	53
6.2	Recommendations for Future Research	54
	Bibliography	56

List of Tables

3.1	The Chosen Values of k_4 and Their Associated Values of k_2 (Symmetric Loss Function)	23
3.2	The Chosen Values of k_4 and Their Associated Values of k_2 (Non-Symmetric Loss Function)	24
4.1	Expected Losses at Some Values of Variance of Y (Symmetric Case with Mean = 10)	30
4.2	Expected Losses at Some Values of Variance of Y (Symmetric Case with Mean = 9.6)	31
4.3	Expected Losses at Some Values of Variance of Y (Non-Symmetric Case with Mean = 10)	33
4.4	Expected Losses at Some Values of Variance of Y (Non-Symmetric Case with Mean = 9.99)	34
4.5	Expected Losses at Some Values of Variance of Y (Non-Symmetric Case with Mean = 9.6)	35
5.1	Summary of the Results (Symmetric Loss Function)	47
5.2	Summary of the Results (Symmetric Loss Function with Embellishment) . .	48
5.3	Summary of the Results (Non-Symmetric Loss Function)	51
5.4	Summary of the Results (Non-Symmetric Loss Function with Embellishment)	51

List of Figures

2-1	A Symmetric Loss Function	4
3-1	A Non-Symmetric Loss Function (When $\Delta_1 \neq \Delta_2$).	17
3-2	A Non-Symmetric Loss Function (When $\Delta_1 = \Delta_2 = \Delta$)	21
3-3	A Family of Symmetric Loss Functions Corresponding to the Chosen Values of k_4	23
3-4	A Family of Non-Symmetric Loss Functions Corresponding to the Chosen Value of k_4	24
4-1	Symmetric Loss Functions representing the Scenario	29
4-2	Non-Symmetric Loss Functions representing the Scenario	32
5-1	Effects of k_4 on the Symmetric Loss Functions	45
5-2	Effects of k_4 on the Symmetric Loss Functions (Embellishment)	48
5-3	Effects of k_4 on the Non-Symmetric Loss Functions.	49
5-4	Effects of k_4 on the Non-Symmetric Loss Functions (Embellishment)	52

Chapter 1

Introduction

There are two approaches for measuring (assessing) quality that are commonly used in the open literature. One is a conventional approach based on the costs of defective units, i.e. units whose quality characteristics fall outside given specification limits. We refer to this approach as the *step function method*. The second approach is to use a *continuous loss function* which was first motivated by Taguchi [14] and later employed by other researchers. The most commonly used functional form in this context is the *quadratic loss function*. Recently, Leung and Spiring [8]; however, proposed another class of continuous loss functions based on the inversion of the standard beta probability density function which they referred to as the *inverted beta loss function*.

In this thesis we propose a different functional form in the same context which we refer to as the *quartic loss function*. The advantage of this functional form is its flexibility in the sense that it allows a user to choose, from among a family of loss functions, a specific loss function that can best describe the situation at hand. At one extreme a quartic loss function can be close to a quadratic loss function, and it can be close to a step function

at the other extreme. As a result it provides more flexibility to the user. Besides, this functional form can be employed to represent either a symmetric or non-symmetric loss function.

We also present a model for the parameter design problem based on this quartic loss function and solve a case study. We discuss only the nominal-the-best case in this thesis.

The remainder of the thesis is organized as follows. Chapter 2 contains a literature review, and we discuss three fundamental works that constitute the foundation for this thesis. The proposed quartic loss function, its derivation, and two illustrative examples are presented in Chapter 3. We derive the associated expected loss formula and present two simple examples illustrating the differences among expected losses attained by assuming the quadratic loss function, the quartic loss function, and the step function in Chapter 4. We discuss the parameter design problem based on the quartic loss function and provide a case study in Chapter 5. Finally in Chapter 6 we summarize the main conclusions of this thesis and present recommendations for future research in this area.

Chapter 2

Literature Review

In this chapter we discuss the general ideas of three fundamental works that constitute the foundation for this thesis.

- Continuous Loss Function
- Moment Estimation Using Taylor Series Approximation
- Nonlinear Programming Approach to the Parameter Design Problem

The following sections provide a brief review of each work.

2.1 Continuous Loss Function

The notion of loss function was introduced and successfully implemented by Taguchi [14], [15]. He defined the quality level of a product to be the loss incurred by society due to the deviation of its quality characteristic (denoted by Y) from a target value (denoted by τ), regardless of how small the deviation is. This measure of quality is somewhat

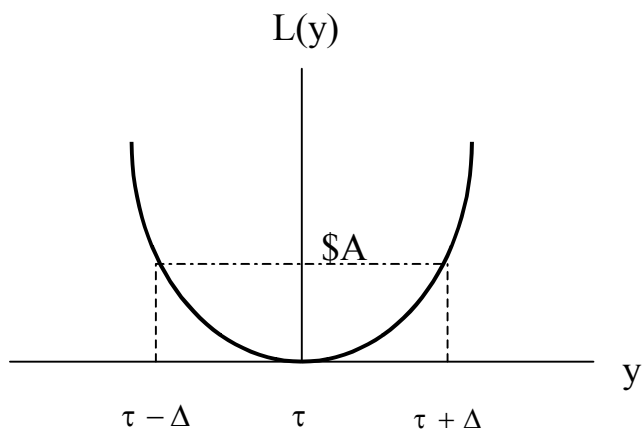


Figure 2-1: A Symmetric Loss Function

different from what we refer to as *fraction defective*, which implies that all products that fall within specification limits (denoted by $\tau \pm \Delta$) are equally good (nondefective), while those outside the specification limits are bad (defective). However, in reality, only the product whose response is exactly on target yields the best performance. As the product's response deviates from the target the quality becomes progressively worse.

Figure 2-1 displays a symmetric continuous loss function. It presents a relationship between the quality loss $L(Y)$ and the amount of deviation from the target. As shown in the figure the quality loss is zero when the quality characteristic is exactly on the target ($Y = \tau$); the quality loss keeps increasing as the quality characteristic deviates in either direction from the target. At each specification limit the quality loss is equal to A dollars. Note that A is the total cost of repair or replacement of a product. Taguchi [14], [15] derived the quadratic loss function, which we will discuss next, using the information from figure 2-1.

2.1.1 Taguchi's Derivation of the Quadratic Loss Function

Consider a quality characteristic whose value is represented by Y ; we assume that the target value of Y is τ . Deviations of Y from τ in either direction are considered undesirable. Let $L(Y)$ denote a function representing the monetary value (in dollars) of the losses incurred by an arbitrary customer of a unit produced where the quality characteristic is Y . We refer to $L(Y)$ as the loss function for Y . To approximately determine the function, we expand it in the Taylor series about the target value τ up to the quadratic term as:

$$L(Y) \approx L(\tau) + L'(\tau)(Y - \tau) + \frac{L''(\tau)}{2!}(Y - \tau)^2 \quad (2.1)$$

Clearly, as shown in figure 2-1, we have $L(\tau) = 0$. Also, since the minimum value of the function occurs at this point the first derivative of the function evaluated at τ is zero. Thus, the first two terms of (2.1) vanish and it would reduce to:

$$L(Y) = \frac{L''(\tau)}{2!}(Y - \tau)^2 \quad (2.2)$$

or

$$L(Y) = k(Y - \tau)^2 \quad (2.3)$$

where k is a constant called *quality loss coefficient*.

Equation (2.3) is the conventional form of the quadratic loss function. We then need to determine the proper value of the constant k . It is clear that the loss is A dollars when the quality characteristic deviates by the amount of Δ from the target in either direction

(exactly on each specification limit). By applying this fact to Eq. (2.3) we obtain:

$$k = \frac{A}{\Delta^2}$$

2.1.2 Derivation of the Expected Loss Based on the Quadratic Loss Function

Certainly, the quality characteristic Y varies from unit to unit, and from time to time. It is practical to represent its variation by a probability distribution function. Suppose that the probability density function of Y is $f(y)$. By having the quality loss function $L(Y)$ and the probability density function $f(y)$ the expected loss per unit can be written in a general form as:

$$E[L(Y)] = \int_{-\infty}^{\infty} L(y)f(y)dy \quad (2.4)$$

In the case of the quadratic loss function we can substitute $L(y)$ from Eq. (2.3) directly into Eq. (2.4). By doing so we get

$$\begin{aligned} E[L(Y)] &= \int_{-\infty}^{\infty} k(y - \tau)^2 f(y) dy \\ &= k [\sigma_Y^2 + (\mu_Y - \tau)^2] \end{aligned} \quad (2.5)$$

where μ_Y and σ_Y^2 are the mean and the variance of Y , respectively.

2.2 Moment Estimation Using Taylor Series Approximation

In this section we discuss a means to relate the probability distributions of random input variables X_i , for $i = 1, 2, \dots, n$, to the probability distribution of the response Y , where $Y = h(X_1, X_2, \dots, X_n)$. Moment is one of the statistical tools for describing the probability distribution of a random variable. Assume that the random input variables X_i , for $i = 1, 2, \dots, n$, have the joint probability density function $f(X_1, X_2, \dots, X_n)$; we define the r^{th} moments of Y about its mean, for $r \geq 2$, where $Y = h(X_1, X_2, \dots, X_n)$ as:

$$E[(y - \mu_Y)^r] = \mu_Y^r = \underbrace{\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty}}_n [h(x_1, x_2, \dots, x_n) - \mu_Y]^r \cdot f(x_1, x_2, \dots, x_n) dx_1 \dots dx_n \quad (2.6)$$

where μ_Y is the mean of Y , which is itself defined as

$$\mu_Y = \underbrace{\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty}}_n h(x_1, x_2, \dots, x_n) \cdot f(x_1, x_2, \dots, x_n) dx_1 \dots dx_n \quad (2.7)$$

As seen from Eqs. (2.6) and (2.7) evaluating the moments of Y directly from the definition would be computationally difficult, if not impossible. Tukey [17] provided generalized nonlinear propagation of error formulas for approximating up to the fourth moment of a function of random variables. In this section we present the results of Tukey's work. See Tukey [17] for detailed discussion.

Assume that all X_i , for $i = 1, 2, \dots, n$, are IID random variables with known μ_i, σ_i^2 ,

γ_i , Γ_i , and G_i , where

$$\begin{aligned}\gamma_i &= \frac{\mu_i^3}{(\sigma_i^2)^{\frac{3}{2}}} \\ \Gamma_i &= \frac{\mu_i^4}{(\sigma_i^2)^2} \\ G_i &= \frac{\mu_i^5}{(\sigma_i^2)^{\frac{5}{2}}}\end{aligned}$$

In the above definitions μ_i^3 , μ_i^4 , and μ_i^5 represent the 3rd, 4th, and 5th moment, respectively, of X_i around its mean. We can estimate the first four moments of Y around its mean, where $Y = h(X_1, X_2, \dots, X_n)$, by applying the following formulas. Note that the formulas presented here were originally derived by Tukey [17] and later revised by Evans [2] to make some notations look simpler.

$$\begin{aligned}\mu_Y &\approx h(\mu_1, \mu_2, \dots, \mu_n) + \frac{1}{2} \sum_i h_{ii} \sigma_i^2 + \frac{1}{6} \sum_i h_{iii} \gamma_i \sigma_i^3 + \frac{1}{24} \sum_i h_{iiii} \Gamma_i \sigma_i^4 \\ &+ \frac{1}{4} \sum_{i < j} h_{iijj} \sigma_i^2 \sigma_j^2 + \frac{1}{120} \sum_i h_{iiii} G_i \sigma_i^5 + \frac{1}{12} \sum_{i \neq j} h_{iijj} \sigma_i^3 \sigma_j^2 \\ &+ \text{terms of order higher than or equal to } \sigma^6\end{aligned}$$

$$\begin{aligned}\sigma_Y^2 &\approx \sum_i h_i^2 \sigma_i^2 + \sum_i h_i h_{ii} \gamma_i \sigma_i^3 + \frac{1}{3} \sum_i h_i h_{iii} \Gamma_i \sigma_i^4 + \frac{1}{4} \sum_i h_{ii}^2 (\Gamma_i - 1) \sigma_i^4 \\ &+ \sum_{i < j} (h_i h_{jj} + h_{ij}^2 + h_{ij} h_j) \sigma_i^2 \sigma_j^2 + \frac{1}{12} \sum_i h_i h_{iiii} G_i \sigma_i^5 \\ &+ \frac{1}{6} \sum_i h_{ii} h_{iii} (G_i - \gamma_i) \sigma_i^5 + \sum_{i \neq j} \left(\frac{1}{2} h_i h_{iijj} + \frac{1}{2} h_{ii} h_{ijj} + h_{ij} h_{iij} \right. \\ &\left. + \frac{1}{3} h_{iij} h_i \right) \gamma_i \sigma_i^3 \sigma_j^2 + \text{terms of order higher than or equal to } \sigma^6\end{aligned}$$

$$\begin{aligned}
\phi_Y &\approx \sum_i h_i^3 \gamma_i \sigma_i^3 + \frac{3}{2} \sum_i h_i^2 h_{ii} (\Gamma_i - 1) \sigma_i^4 + 6 \sum_{i < j} h_i h_j h_{ij} \sigma_i^2 \sigma_j^2 \\
&+ \frac{1}{2} \sum_i h_i^2 h_{iii} (G_i - \gamma_i) \sigma_i^5 + \frac{3}{4} \sum_i h_i h_{ii}^2 (G_i - 2\gamma_i) \sigma_i^5 \\
&+ 3 \sum_{i \neq j} \left(h_i h_{ij}^2 + \frac{1}{2} h_i^2 h_{ijj} + h_{ii} h_{ij} h_j + h_i h_{iij} h_j \right) \gamma_i \sigma_i^3 \sigma_j^2 \\
&+ \text{terms of order higher than or equal to } \sigma^6 \\
\varphi_Y &\approx \sum_i h_i^4 (\Gamma_i - 3) \sigma_i^4 + 2 \sum_i h_i^3 h_{ii} (G_i - 4\gamma_i) \sigma_i^5 + 12 \sum_{i \neq j} h_i^2 h_j h_{ij} \gamma_i \sigma_i^3 \sigma_j^2 \\
&+ \text{terms of order higher than or equal to } \sigma^6 + 3 (\sigma_Y^2)^2
\end{aligned}$$

where

- μ_Y is the mean of Y .
- σ_Y^2 is the variance of Y .
- ϕ_Y is the third moment of Y about its mean.
- φ_Y is the fourth moment of Y about its mean.
- $h_i = \frac{\partial h}{\partial x_i} \Big|_{\mu_1, \mu_2, \dots, \mu_n}$, $h_{ii} = \frac{\partial^2 h}{\partial x_i^2} \Big|_{\mu_1, \mu_2, \dots, \mu_n}$, $h_{ij} = \frac{\partial^2 h}{\partial x_i \partial x_j} \Big|_{\mu_1, \mu_2, \dots, \mu_n}$, etc.
- The summations are from 1 to n .
- All of the “terms of order higher than or equal to σ^6 ” are considered negligible.

2.3 Nonlinear Programming Approach to the Parameter Design Problem

The problem of determining the optimal set points for controllable parameters of a product or process is referred to as the *parameter design problem*. The problem was originally formulated and successfully implemented by Taguchi [13]. Most authors who explored this area employed the statistical design of experiment as the primary tools for studying the problems, for example, see Fowlkers [7], Montgomery [9], Phadke [11], and Taguchi [13], even in the cases where the analytical forms of the transfer functions are available.

However, in the case where the analytical form of a functional relationship between the input parameters and the output variable, i.e. the transfer function, is available, or at least can be well approximated, there is another strategy presented by Fathi [3], [5] to tackle the parameter design problem. He proposed constrained nonlinear programming models for this problem in which the expected loss or the variance of the response is minimized. We devote the rest of this section discussing this concept briefly.

Let X_1, X_2, \dots, X_n be IID random variables with $E(X_i) = \mu_i$ and $Var(X_i) = \sigma_i^2$, for $i = 1, 2, \dots, n$, and a response of interest can be defined analytically as $Y = h(X_1, X_2, \dots, X_n)$. Therefore, Y is also a random variable whose probability distribution function depends on the form of the function $h(X_1, X_2, \dots, X_n)$ and the probability distribution functions of X_1 through X_n .

The basic idea of the parameter design problem is to determine the nominal values of the controllable parameters $(\mu_1, \mu_2, \dots, \mu_n)$ that minimize the expected loss associated with a quality characteristic. However, in some instances, several technical constraints

may also be present and they need to be incorporated into the problem. By assuming the quadratic loss function, as discussed earlier, an appropriate mathematical model for the parameter design problem can be stated as:

$$\begin{aligned} \text{Minimize} \quad & E[L(Y)] = k [\sigma_Y^2 + (\mu_Y - \tau)^2] \\ \text{Subject to} \quad & (\mu_1, \mu_2, \dots, \mu_n) \in M \end{aligned}$$

where $(\mu_1, \mu_2, \dots, \mu_n) \in M$ represents possible constraints pertaining to the technical requirements, processing cost, etc. of a certain problem. Note that in the model the values of μ_Y and σ_Y^2 depend on the variables μ_1 through μ_n as stated in the previous section.

In general, the form of μ_Y or σ_Y^2 as a function of μ_1 through μ_n may be difficult or, in some circumstances, may be impossible to obtain. Usually, we have to approximate them as discussed earlier. There are several means to approximate these parameters, see Evan [2] and Fathi [5]. Several authors, for example, Fathi [3], [5], Montgomery [10], and Phadke [11], adopted the compact version of formulas presented in the previous section, by neglecting the higher-order terms, to approximate these parameters. The compact version of the formulas for approximating μ_Y and σ_Y^2 are in the forms:

$$\mu_Y \approx h(\mu_1, \mu_2, \dots, \mu_n) \tag{2.8}$$

$$\sigma_Y^2 \approx \sum_i h_i^2 \sigma_i^2 \tag{2.9}$$

where $h_i = \left. \frac{\partial h}{\partial x_i} \right|_{\mu_1, \mu_2, \dots, \mu_n}$ and the summation is from 1 to n .

As stated earlier the basic idea of the parameter design problem is to determine the

optimal values of μ_1 through μ_n that minimize the expected loss associated with the quality characteristic of interest. While keeping other parameters of the input variables fixed μ_Y and σ_Y^2 can be expressed as functions of μ_1 through μ_n . We use the notation $\boldsymbol{\mu}$ to represent the vector $(\mu_1, \mu_2, \dots, \mu_n)$. Therefore, μ_Y and σ_Y^2 as the functions of μ_1 through μ_n can be written as $\mu_Y(\boldsymbol{\mu})$ and $\sigma_Y^2(\boldsymbol{\mu})$, respectively. By applying (2.8) and (2.9) for approximating $\mu_Y(\boldsymbol{\mu})$ and $\sigma_Y^2(\boldsymbol{\mu})$, the above model can be restated as:

$$\begin{aligned}
\text{Minimize} \quad & E[L(Y)] = k [\sigma_Y^2(\boldsymbol{\mu}) + (\mu_Y(\boldsymbol{\mu}) - \tau)^2] \\
\text{Subject to} \quad & \mu_Y(\boldsymbol{\mu}) = h(\boldsymbol{\mu}) \\
& \sigma_Y^2(\boldsymbol{\mu}) = \sum_i \left(\left. \frac{\partial h}{\partial x_i} \right|_{\mu_1, \mu_2, \dots, \mu_n} \right)^2 \sigma_i^2 \\
& (\boldsymbol{\mu}) \in M
\end{aligned} \tag{PDME}$$

The mathematical model above is simply a constrained nonlinear programming problem that can be solved using an appropriate algorithm.

Nevertheless, while assuming the quadratic loss function, there is another way to model the parameter design problem as discussed by many authors, for example, see Fathi [5], Montgomery [10], and Phadke [11]. They suggested minimizing the variance of the quality characteristic directly, while keeping its mean on the target. This leads to the following mathematical model:

$$\begin{aligned}
\text{Minimize} \quad & \sigma_Y^2(\boldsymbol{\mu}) = \sum_i \left(\left. \frac{\partial h}{\partial x_i} \right|_{\mu_1, \mu_2, \dots, \mu_n} \right)^2 \sigma_i^2 \\
\text{Subject to} \quad & h(\boldsymbol{\mu}) = \tau \\
& (\boldsymbol{\mu}) \in M
\end{aligned} \tag{PDMV}$$

At this point an unavoidable question may come up as what are the differences between these two models. It can be shown that the optimal solution to the (PDME) problem would yield μ_Y which may be slightly different from τ , see Box [1]. This difference is referred to as the *aim-off factor*. In order to avoid the aim-off factor we can use (PDMV). However, the optimal solution to the problem would yield slightly higher value of the expected loss than what results from (PDME). See Fathi [3] for the discussion about this issue in more detail. Later in this thesis, we will employ (PDME) to the parameter design problem which is based on the assumption of our proposed quartic loss function.

Chapter 3

Quartic Loss Function

As stated earlier there are two functional forms that are commonly used to represent a quality loss function. One form is the conventional step function that is based on classifying the units as defective and non-defective, and the other form is the quadratic loss function that is proposed by Taguchi [14]. Either of these two functional forms has a specific functional profile, and as a result we do not have any degrees of freedom in choosing a desired functional profile in either context. This thesis proposes a loss function that not only allows some degrees of flexibility in this regard, but it also can be applied in the case where a non-symmetric function is desirable.

In this chapter we introduce a quartic loss function resulting from expanding the Taylor series up to its quartic term. We then examine the appropriate values of the quality loss coefficients in both cases where the function is symmetric and non-symmetric. Finally, we present two simple examples that illustrate how the quartic loss function can provide a family of loss functions for both the symmetric case and the non-symmetric case.

3.1 Derivation of the Quartic Loss Function

Consider a quality characteristic whose value is represented by Y , and we assume that the target value of Y is τ . Deviations of Y from τ in either direction are considered undesirable. Let $L(Y)$ denote a function representing the monetary value (in dollars) of the losses incurred by an arbitrary customer of a unit produced where the quality characteristic is Y . We refer to $L(Y)$ as the loss function for Y . To approximately determine this function we expand it in the Taylor series about the target value τ up to the fourth term as:

$$L(Y) \approx L(\tau) + \frac{L'(\tau)}{1!}(Y - \tau) + \frac{L''(\tau)}{2!}(Y - \tau)^2 + \frac{L'''(\tau)}{3!}(Y - \tau)^3 + \frac{L''''(\tau)}{4!}(Y - \tau)^4 \quad (3.1)$$

Clearly we have $L(\tau) = 0$. Also, since the minimum value of the function is attained at the target τ , the first derivative of the function at this point is zero. Therefore, the first two terms of (3.1) vanish. The equation would then reduce to

$$L(Y) = \frac{L''(\tau)}{2!}(Y - \tau)^2 + \frac{L'''(\tau)}{3!}(Y - \tau)^3 + \frac{L''''(\tau)}{4!}(Y - \tau)^4$$

or we can rewrite as

$$L(Y) = k_2(Y - \tau)^2 + k_3(Y - \tau)^3 + k_4(Y - \tau)^4 \quad (3.2)$$

where k_2 , k_3 , and k_4 are constants that we call *second*, *third*, and *fourth ordered quality loss coefficient*, respectively. For reasons that we explain later in this chapter, we also refer to k_4 as the *shape parameter*.

Equation (3.2) is the proposed quartic loss function. Instead of having only the quadratic term as in the case of the quadratic loss function, we obtain two more higher-order terms that represent the non-symmetry and the flexibility of the functional profile. We then need to determine the proper values of k_2 , k_3 , and k_4 that would make the loss function behave in the way that we desire.

3.2 The Quality Loss Coefficients

In this section we propose a methodology to determine the coefficients k_2 , k_3 , and k_4 based on either case where the loss function is symmetric and non-symmetric, separately. We initially examine the coefficients of a non-symmetric loss function; a symmetric loss function is the special case.

3.2.1 Non-Symmetric Loss Function

A non-symmetric loss function is shown in figure 3-1. The quality loss is zero when the quality characteristic is exactly on the target τ ; the quality loss keeps increasing nonsymmetrically as the quality characteristic deviates in either direction from the target. At the lower specification limit ($\tau - \Delta_1$) the quality loss is assumed to be A dollars, and at the upper specification limit ($\tau + \Delta_2$) it is assumed to be B dollars. Note that A and B are all of the costs associated with the repair or replacement of a nonconforming unit at either specification limit, respectively. We use the information from figure 3-1 to determine the proper values of the constants k_2 , k_3 , and k_4 .

Listed below are properties that we assume the loss function should have.

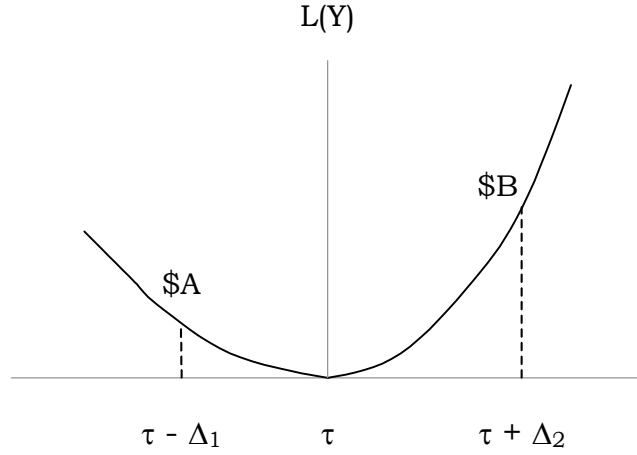


Figure 3-1: A Non-Symmetric Loss Function (When $\Delta_1 \neq \Delta_2$).

- $L(\tau - \Delta_1) = A$
- $L(\tau + \Delta_2) = B$
- $L''(Y) > 0 \quad -\infty < Y < \infty$

The last property holds because we assume that the loss function is convex everywhere in $(-\infty, \infty)$. We determine the values of k_2 , k_3 , and k_4 to satisfy these properties. At the specification limits we have

$$L(\tau - \Delta_1) = A = k_2\Delta_1^2 - k_3\Delta_1^3 + k_4\Delta_1^4 \quad (3.3)$$

$$L(\tau + \Delta_2) = B = k_2\Delta_2^2 + k_3\Delta_2^3 + k_4\Delta_2^4 \quad (3.4)$$

Having solved Eqs. (3.3) and (3.4) for k_2 and k_3 in term of k_4 we get

$$k_2 = \left(\frac{1}{1 + \frac{\Delta_1}{\Delta_2}} \right) \left(\frac{A}{\Delta_1^2} + \frac{\Delta_1 B}{\Delta_2^3} - (\Delta_1 \Delta_2 + \Delta_1^2) k_4 \right) \quad (3.5)$$

$$k_3 = \left(\frac{1}{1 + \frac{\Delta_1}{\Delta_2}} \right) \left(\frac{B}{\Delta_2^3} - \frac{A}{\Delta_1^2 \Delta_2} + \left(\frac{\Delta_1^2}{\Delta_2} - \Delta_2 \right) k_4 \right) \quad (3.6)$$

It is clear that the coefficients can not be uniquely specified at this point. For any value of k_4 , however, the corresponding values of k_2 and k_3 can be determined using these equations. We now consider the property that imposes $L''(Y) > 0$, $-\infty < Y < \infty$. By carrying out the second derivative of the quartic loss function (3.2) we have

$$L''(Y) = 2k_2 + 6k_3(Y - \tau) + 12k_4(Y - \tau)^2 \quad (3.7)$$

Equation (3.7) can be rearranged in the form

$$L''(Y) = (12k_4)Y^2 + (6k_3 - 24\tau k_4)Y + (2k_2 - 6\tau k_3 + 12\tau^2 k_4) \quad (3.8)$$

which is in the quadratic form

$$g(y) = ay^2 + by + c \quad (3.9)$$

Observation In order for $g(y)$ in Eq. (3.9) to be greater than zero for all values of y ,

two conditions below need to be satisfied:

1. $a > 0$

$$2. b^2 < 4ac$$

Proof. Equation (3.9) is in a parabolic form. To ensure that $g(y)$ is greater than zero, it is required that two conditions below are true:

1. The function is convex, which implies that the function has a minimum point.
2. The minimum point of the function lies above zero.

For condition 1 to be true we need $g''(y) > 0$, for all $-\infty < y < \infty$. This condition yields $a > 0$.

For condition 2 to be true we need $g(\bar{y}) > 0$, where \bar{y} is the global minimizer of $g(y)$. This condition yields $b^2 < 4ac$. ■

Therefore, in order for $L''(Y)$ to be greater than zero, we need to satisfy the following conditions:

1. $12k_4 > 0$
2. $(6k_3 - 24\tau k_4)^2 < 4(12k_4)(2k_2 - 6\tau k_3 + 12\tau^2 k_4)$

Which can be written as:

$$k_4 > 0 \tag{3.10}$$

$$k_3^2 < \frac{8}{3}k_2k_4 \tag{3.11}$$

Substituting for k_2 and k_3 from Eqs. (3.5) and (3.6), respectively, we obtain the following conditions for the value of k_4 .

1. $k_4 > 0$ and
2. k_4 is a value selected from the solutions to the following inequality:

$$\begin{aligned} & \left[\frac{(\Delta_1^2 - \Delta_2^2)^2}{\Delta_2^2} + \frac{8}{3} \left(\frac{\Delta_1 \Delta_2^2 + 2\Delta_1^2 \Delta_2 + \Delta_1^3}{\Delta_2} \right) \right] k_4^2 + \left[\left(\frac{2B}{\Delta_2^4} - \frac{2A}{\Delta_1^2 \Delta_2^2} \right) (\Delta_1^2 - \Delta_2^2) \right. \\ & \left. - \frac{8}{3} \left(\frac{A(\Delta_1 + \Delta_2)}{\Delta_1^2 \Delta_2} + \frac{\Delta_1 B(\Delta_1 + \Delta_2)}{\Delta_2^4} \right) \right] k_4 + \left[\frac{B^2}{\Delta_2^6} - \frac{2AB}{\Delta_1^2 \Delta_2^4} + \frac{A^2}{\Delta_1^4 \Delta_2^2} \right] < 0 \end{aligned} \quad (3.12)$$

Any value of k_4 that satisfies these inequalities leads to $L''(Y) > 0$. This typically leads to a range of allowable value for k_4 , and we can select the values of k_4 within this range. The corresponding values of k_2 and k_3 are then determined using Eqs. (3.5) and (3.6), respectively. The shape of the loss function depends on k_4 , which we previously mentioned as the shape parameter, that we select from the allowable range, and we show several examples illustrating the effect of k_4 on the functional profile later in this chapter.

Now, let's assume further that $\Delta_1 = \Delta_2 = \Delta$; in other words, the distances from the target τ to the lower specification limit ($\tau - \Delta$) and to the upper specification limit ($\tau + \Delta$) are the same. At the lower specification limit ($\tau - \Delta_1$) the quality loss is assumed to be A dollars, and at the upper specification limit ($\tau + \Delta_2$) it is assumed to be B dollars. This is a special case of what we derived above. Figure 3-2 depicts this situation.

In this situation Eqs. (3.5) and (3.6) reduce to

$$k_2 = \frac{A + B}{2\Delta^2} - \Delta^2 k_4 \quad (3.13)$$

$$k_3 = \frac{B - A}{2\Delta^3} \quad (3.14)$$

Note that in this case the value of k_3 does not depend on k_4 and it is uniquely de-

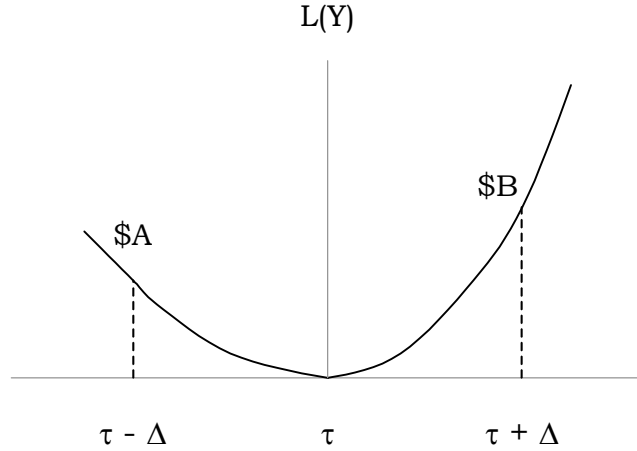


Figure 3-2: A Non-Symmetric Loss Function (When $\Delta_1 = \Delta_2 = \Delta$)

terminated in Eq. (3.14). The other two coefficients k_2 and k_4 can not be uniquely specified at this point. As before we need to include the property that requires $L''(Y) > 0$, $-\infty < Y < \infty$. In this case, we have $k_4 > 0$ and (3.12) reduces to:

$$(2\Delta^4)k_4^2 - (A + B)k_4 + \frac{3(B - A)^2}{8 \cdot 2\Delta^4} < 0 \quad (3.15)$$

Note that (3.15) is the special form of (3.12), when $\Delta_1 = \Delta_2 = \Delta$. After we select a desired value of k_4 we then use Eq. (3.13) to evaluate the corresponding value of k_2 .

3.2.2 Symmetric Loss Function

A symmetric loss function is the special case of what we presented in section 3.2.1, where at each specification limit the quality loss is the same (i.e., A dollars). We use the same procedure to evaluate the possible values of the coefficients k_2 , k_3 , and k_4 .

We can in fact apply Eqs. (3.13) and (3.14) where we replace B with A in this case.

The relationships would then be:

$$k_2 = \frac{A + A}{2\Delta^2} - \Delta^2 k_4 = \frac{A}{\Delta^2} - \Delta^2 k_4 \quad (3.16)$$

$$k_3 = \frac{A - A}{2\Delta^3} = 0 \quad (3.17)$$

The task would reduce to evaluating the feasible values of k_4 and then substitute a desired value of k_4 into Eq. (3.16) to obtain k_2 . By using the same derivation as in the previous section, the feasible range of k_4 is:

1. $k_4 > 0$ and
2. k_4 is a value selected from the solutions to the following inequality:

$$(2\Delta^4)k_4^2 - (2A)k_4 < 0 \quad (3.18)$$

It follows that $0 < k_4 < \frac{A}{\Delta^4}$.

3.3 Examples

In this section we present two simple examples that illustrate how the quartic loss function can provide a family of loss functions. The first example demonstrates the symmetric loss function and the second one does the non-symmetric loss function with $\Delta_1 = \Delta_2 = \Delta$.

Example 3.1 We assume that the target value for a quality characteristic of interest in a product is 10 and its specification limits are 10 ± 1 . We also assume that the cost for repairing or replacing an unacceptable product is \$5 at either specification limit.

Table 3.1: The Chosen Values of k_4 and Their Associated Values of k_2 (Symmetric Loss Function)

Pair	1	2	3	4
k_4	0.01	2.00	3.50	4.99
k_2	4.99	3.00	1.50	0.01

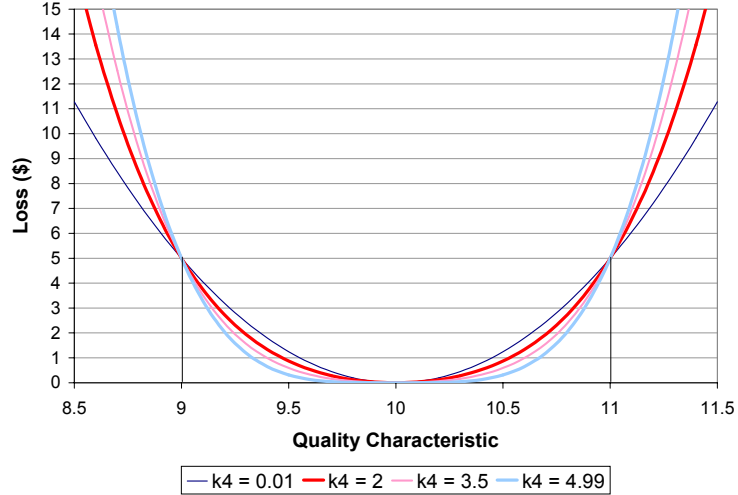


Figure 3-3: A Family of Symmetric Loss Functions Corresponding to the Chosen Values of k_4

Thus, this is the case where the loss function is symmetric with $A = \$5$. We know immediately from Eq (3.17) that k_3 is zero. We then need to evaluate the feasible range for k_4 and then k_2 can be solved by using Eq. (3.16). By using the procedure provided in the previous section, $k_4 \in (0,5)$. We present here four profiles of the quartic loss function based on four values of k_4 : 0.01, 2, 3.5, and 4.99. Table 3-1 summarizes the four values of k_4 and their associated values of k_2 . Note that when $k_4 = 0$, $L(Y)$ reduces to a quadratic loss function; and as we increase the value of k_4 the form of $L(Y)$ becomes closer to the form of the conventional step function within the specification limits.

Table 3.2: The Chosen Values of k_4 and Their Associated Values of k_2 (Non-Symmetric Loss Function)

Pair	1	2	3	4
k_4	0.16	2.00	4.00	5.30
k_2	5.34	3.50	1.50	0.20

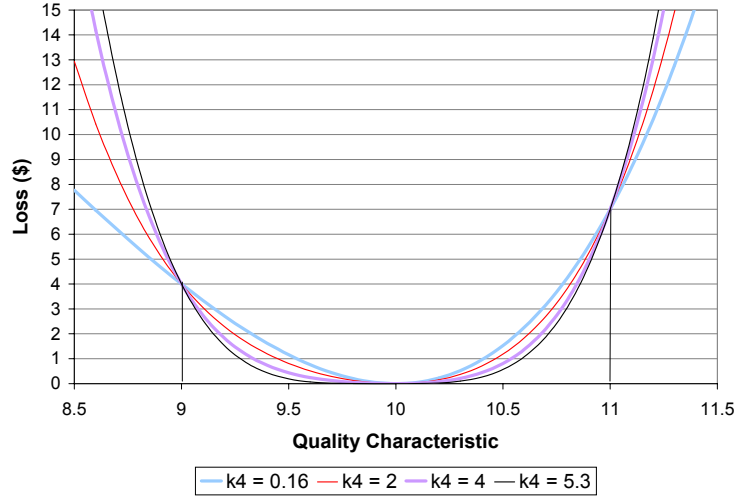


Figure 3-4: A Family of Non-Symmetric Loss Functions Corresponding to the Chosen Value of k_4

Example 3.2 This example is the embellishment of the previous example. Now let's assume that the cost for repairing or replacing an unacceptable product is not the same at both specification limits (10 ± 1), which implies the non-symmetric loss function with $\Delta_1 = \Delta_2 = \Delta$. Let's assume that the repair cost at the lower specification limit is $A = \$4$, and at the upper specification limit it is $B = \$7$. By using Eq. (3.14), $k_3 = 1.5$. Following the procedure in the previous section we have $k_4 \in (0.158, 5.342)$. We present here four profiles of the quartic loss function based on four values of k_4 : 0.16, 2, 4, 5.3. Table 3-2 summarize the four values of k_4 and their associated values of k_2 .

Conclusion Figures 3-3 and 3-4 depict various profiles of the symmetric and non-symmetric loss functions, respectively, based on k_4 . Observe that within the specification limits, as k_4 decreases, at the same value of a quality characteristic the quality loss increases. In other words, as k_4 decreases the function would get steeper. At a large value of k_4 the loss function gets close to a step function within the specification limits. At a small value of k_4 the function becomes close to a quadratic loss function. A decision maker can choose an appropriate value of k_4 in order to have a loss function that represents a situation at hand. This process of choosing the appropriate value of k_4 is somewhat subjective depending on the user's preference.

Chapter 4

Expected Loss Based on the Quartic Loss Function

In this chapter we develop expected loss formulas for a quartic loss function and present two simple examples that illustrate the differences among expected losses attained by assuming the quadratic loss function, the quartic loss function, and the step function.

4.1 Derivation of the Expected Loss Formulas

Because a quality characteristic Y varies from unit to unit and from time to time, it is practical to represent its variation by a probability distribution function. Let $f(y)$ be the probability density function of Y . By having the quality loss function $L(Y)$ and the probability density function $f(y)$, the expected loss per unit can be written in a general

form as

$$E[L(Y)] = \int_{-\infty}^{\infty} L(y)f(y)dy \quad (4.1)$$

The quartic loss function is in the form:

$$L(y) = k_2(y - \tau)^2 + k_3(y - \tau)^3 + k_4(y - \tau)^4$$

Therefore, the expected loss can be derived as:

$$\begin{aligned} E[L(Y)] &= \int_{-\infty}^{\infty} \{k_2(y - \tau)^2 + k_3(y - \tau)^3 + k_4(y - \tau)^4\}f(y)dy \\ &= k_2 \int_{-\infty}^{\infty} [(y - \mu_Y) + (\mu_Y - \tau)]^2 f(y)dy + k_3 \int_{-\infty}^{\infty} [(y - \mu_Y) + (\mu_Y - \tau)]^3 f(y)dy \\ &\quad + k_4 \int_{-\infty}^{\infty} [(y - \mu_Y) + (\mu_Y - \tau)]^4 f(y)dy \\ &= k_2 [\sigma_Y^2 + (\mu_Y - \tau)^2] + k_3 [\phi_Y + 3\sigma_Y^2(\mu_Y - \tau) + (\mu_Y - \tau)^3] \\ &\quad + k_4 [\varphi_Y + 4\phi_Y(\mu_Y - \tau) + 6\sigma_Y^2(\mu_Y - \tau)^2 + (\mu_Y - \tau)^4] \end{aligned} \quad (4.2)$$

where

- μ_Y is the mean of Y .
- σ_Y^2 is the variance of Y .
- ϕ_Y is the third moment of Y about its mean.

- φ_Y is the fourth moment of Y about its mean.

For the symmetric case, $k_3 = 0$, as shown earlier in the thesis. Thus, (4.2) would reduce to

$$E[L(Y)] = k_2 [\sigma_Y^2 + (\mu_Y - \tau)^2] + k_4 [\varphi_Y + 4\phi_Y(\mu_Y - \tau) + 6\sigma_Y^2(\mu_Y - \tau)^2 + (\mu_Y - \tau)^4] \quad (4.3)$$

4.2 Examples

In this section we present two simple examples that illustrate the differences of the expected loss per unit when we base our analysis on a quadratic loss function, a quartic loss function, and a step function.

Example 4.1 Let Y be a quality characteristic of interest of a product. Assume again that the desired value of Y is $\tau = 10$ and its specification limits are 10 ± 1 , or $\Delta = 1$. The cost for repairing or replacing an unacceptable product is \$5 at either specification limit, or $A = 5$.

Figure 4-1 depicts a quadratic loss function, a family of quartic loss functions, and a step function that represent the given situation. Note that the quartic loss functions are those between the quadratic loss function and the step function. The quartic loss coefficients applied here are the same as those we used in example 3.1.

Suppose that Y is normally distributed with mean $\mu_Y = 10$, and variance σ_Y^2 . We carry out the analysis for different values of σ_Y^2 as shown in the first column of table 4.1. We use Eq. (2.5) to determine the expected loss per unit based on the quadratic

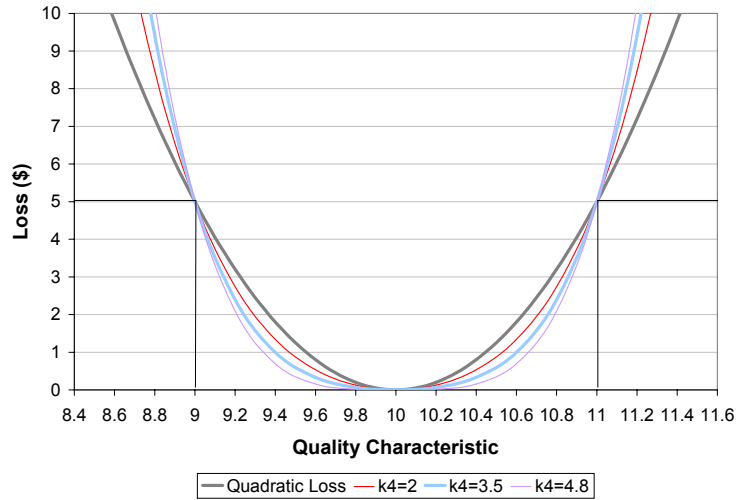


Figure 4-1: Symmetric Loss Functions representing the Scenario

loss function and Eq. (4.3) to determine the value based on the quartic loss function. Since Y is normally distributed $\phi_Y = 0$, and $\varphi_Y = 3\sigma_Y^4$. Note that $\Gamma_Y = 3$ for a normal distribution. We apply the following relationship to determine the expected loss per unit based on the step function:

$$E[L(Y)] = A \times (1 - \Pr\{\tau - \Delta \leq Y \leq \tau + \Delta\})$$

The results of the calculation for different values of σ_Y^2 are shown in table 4.1. Each row of the table corresponds to a different value of σ_Y^2 , and we present three different values of the shape parameter k_4 for the quartic loss function.

We make the following observations from table 4.1:

- At smaller value of σ_Y^2 the expected loss per unit based on the quartic loss functions is less than that based on the quadratic loss function. On the other

Table 4.1: Expected Losses at Some Values of Variance of Y (Symmetric Case with Mean = 10)

σ_Y^2	Expected Loss				
	Quadratic Function	Quartic Function			Step Function
		$(k_4 = 2.0)$	$(k_4 = 3.5)$	$(k_4 = 4.8)$	
0.001	0.005	0.003	0.002	0.000	0.000
0.01	0.050	0.031	0.016	0.003	0.000
0.1	0.500	0.360	0.255	0.164	0.008
0.2	1.000	0.840	0.720	0.616	0.127
0.3	1.500	1.440	1.395	1.356	0.339
0.4	2.000	2.160	2.280	2.384	0.570
0.5	2.500	3.000	3.375	3.700	0.787
1	5.000	9.000	12.000	14.600	1.587
2	10.000	30.000	45.000	58.000	2.398

hand, when the variance of Y is larger the expected loss per unit based on the quartic loss functions is greater than that based on the quadratic loss function. This incident can be best explained by figure 4-1. Within the specification limits less penalty would be charged to any products whose quality characteristics deviate from the target when we apply the quartic loss functions. However, beyond the limits, more penalty would be charged. When σ_Y^2 is small as compared with Δ most units fall within the specification limits and; therefore, less penalty is charged to the units whose quality characteristics deviate from the target. As σ_Y^2 gets larger there would be more units whose quality characteristics fall outside the limits, and these units are in fact charged with severe penalty. This penalty is much more than that inside the specification limits and tend to predominate on the value of the expected loss.

- For the quartic loss functions, at small value of σ_Y^2 the expected loss per unit decreases as k_4 increases. On the other hand, when σ_Y^2 is relatively large the

Table 4.2: Expected Losses at Some Values of Variance of Y (Symmetric Case with Mean = 9.6)

σ_Y^2	Expected Loss				
	Quadratic Function	Quartic Function			Step Function
		($k_4 = 2.0$)	($k_4 = 3.5$)	($k_4 = 4.8$)	
0.001	0.805	0.536	0.334	0.160	0.000
0.01	0.850	0.581	0.379	0.204	0.000
0.1	1.300	1.083	0.921	0.780	0.144
0.2	1.800	1.755	1.722	1.692	0.454
0.3	2.300	2.547	2.733	2.893	0.710
0.4	2.800	3.459	3.954	4.382	0.924
0.5	3.300	4.491	5.385	6.159	1.109
1	5.800	11.451	15.690	19.363	1.775
2	10.800	34.371	52.050	67.371	2.484

expected loss per unit increases as k_4 increases. The same argument as that in the previous observation can be used to explain this incident.

- The expected loss per unit based on the step function is the smallest no matter how large σ_Y^2 is since no penalty is charged to any unit whose quality characteristic falls within the specification limits. Moreover, a constant rate of \$5 is charged on the products whose quality characteristics fall outside the limits.

Now, assume that the mean of Y equals to 9.6, which deviates slightly from the target while we keep the other assumptions the same. Table 4.2 summarizes the expected loss per unit based on each kind of loss function as we vary σ_Y^2 . The same conclusion can be inferred. However, since the mean of Y is not exactly on target, the overall expected loss per unit for this situation is higher. Furthermore, it is obvious that the percentage difference between the expected loss per unit resulting from this situation and that resulting from the previous one is large at the small value of σ_Y^2 , and it gets smaller as σ_Y^2 increases.

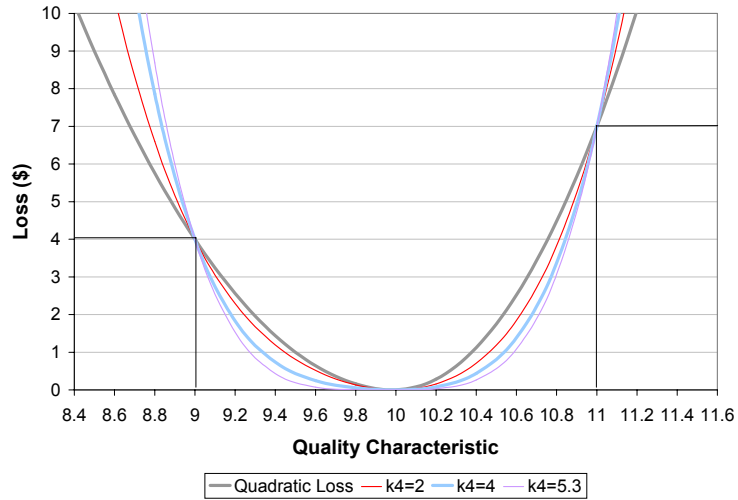


Figure 4-2: Non-Symmetric Loss Functions representing the Scenario

Example 4.2 We consider the case as described in the previous example, except that we now assume that the cost for repairing or replacing an unacceptable product is \$4 at the lower limit, or $A = 4$ and \$7 at the upper limit, or $B = 7$.

Figure 4-2 depicts a quadratic loss function, a family of quartic loss functions, and a step function that represent the given situation. Note that the quartic loss functions are those between the quadratic loss function and the step function within the specification limits. The quartic loss coefficients applied here are the same as those we used in example 3.2. We evaluated the quadratic loss coefficients using:

$$k_L = \frac{A}{\Delta^2}$$

$$k_U = \frac{B}{\Delta^2}$$

where we apply k_L when $Y \leq \tau$, and k_U when $Y > \tau$ to the quadratic loss function.

Table 4.3: Expected Losses at Some Values of Variance of Y (Non-Symmetric Case with Mean = 10)

σ_Y^2	Expected Loss				
	Quadratic Function	Quartic Function			Step Function
		$(k_4 = 2.0)$	$(k_4 = 4.0)$	$(k_4 = 5.3)$	
0.001	0.0055	0.004	0.002	0.000	0.000
0.01	0.055	0.036	0.016	0.004	0.000
0.1	0.550	0.410	0.270	0.179	0.009
0.2	1.100	0.940	0.780	0.676	0.139
0.3	1.650	1.590	1.530	1.491	0.373
0.4	2.200	2.360	2.520	2.624	0.626
0.5	2.750	3.250	3.750	4.075	0.865
1	5.500	9.500	13.500	16.100	1.745
2	11.000	31.000	51.000	64.000	2.637

In effect we assume that we have two different quadratic loss functions, one for each side of the target value.

Suppose that Y is normally distributed with mean $\mu_Y = 10$, and variance σ_Y^2 . We carry out the analysis for different values of σ_Y^2 as shown in the first column of table 4.3. Table 4.3 summarizes the expected loss per unit based on each kind of loss function as we vary σ_Y^2 . The same conclusion as that of the previous example can be inferred. Notice that the overall expected loss per unit is slightly higher than what resulted from the previous example since more penalty is charged to any products whose quality characteristics deviate to the right side of the target.

It may be of interested to know what would happen if the mean of Y deviates slightly to the left side of the target. Now, assume that the mean of Y equals 9.99. Table 4.4 summarizes the expected loss per unit based on each kind of loss function as we vary σ_Y^2 . Notice that the overall values of the expected loss per unit is less than the previous situation where the mean of Y was exactly on target. We can argue that

Table 4.4: Expected Losses at Some Values of Variance of Y (Non-Symmetric Case with Mean = 9.99)

σ_Y^2	Expected Loss				
	Quadratic Function	Quartic Function			Step Function
		($k_4 = 2.0$)	($k_4 = 4.0$)	($k_4 = 5.3$)	
0.001	0.0056	0.0038	0.0016	0.0002	0.0000
0.01	0.0543	0.0355	0.0159	0.0032	0.0000
0.1	0.5468	0.4060	0.2659	0.1748	0.0084
0.2	1.0952	0.9316	0.7716	0.6677	0.1374
0.3	1.6440	1.5772	1.5174	1.4785	0.3695
0.4	2.1930	2.3428	2.5031	2.6073	0.6210
0.5	2.7421	3.2285	3.7289	4.0541	0.8591
1	5.4886	9.4566	13.4576	16.0582	1.7381
2	10.9836	30.9128	50.9150	63.9164	2.6307

shifting the mean of Y slightly to the left side of the target, where the average cost at the lower specification limit is less than that at the upper limit, can help reduce the expected loss per unit. However, if we shift the mean of Y further to the left, the overall expected loss per unit will increase.

For example, let's assume that the mean of Y is now equal to 9.6. Table 4.5 summarizes the expected loss per unit based on each kind of loss function as we vary σ_Y^2 . This time, as we mentioned, the overall expected loss per unit increases, and shifting the mean of Y too far from the target can not help reduce the expected loss per unit.

Table 4.5: Expected Losses at Some Values of Variance of Y (Non-Symmetric Case with Mean = 9.6)

σ_Y^2	Expected Loss				
	Quadratic Function	Quartic Function			Step Function
		($k_4 = 2.0$)	($k_4 = 4.0$)	($k_4 = 5.3$)	
0.001	0.6440	0.519	0.250	0.075	0.000
0.01	0.6800	0.552	0.283	0.108	0.000
0.1	1.1203	0.937	0.720	0.579	0.116
0.2	1.6404	1.479	1.434	1.105	0.366
0.3	2.1610	2.141	2.388	2.549	0.584
0.4	2.6828	2.923	3.582	4.011	0.780
0.5	3.2059	3.825	5.016	5.791	0.959
1	5.8391	10.135	15.786	19.460	1.662
2	11.1584	31.755	55.326	70.648	2.470

Chapter 5

Mathematical Model for the Parameter Design Problem and a Case Study

In this chapter we present a mathematical model for the parameter design problem based on the quartic loss function assumption. Throughout the chapter we assume that the functional relationship of input parameters and a quality characteristic, i.e. $Y = h(X_1, X_2, \dots, X_n)$, is well defined in a closed form or at least could be well approximated. We then provide a case study for either case where the loss function is symmetric and non-symmetric. For the case of symmetric loss function we also compare the results with what we would obtain by applying the quadratic loss function.

5.1 The Model

The parameter design problem is the problem of determining optimal set points for design variables X_1 through X_n of a product or a process. The basic idea of the parameter design problem is to identify nominal values of the design variables μ_1 through μ_n that minimize the expected loss associated with a quality characteristic Y .

Let $X = (X_1, X_2, \dots, X_n)$ be a vector of design variables (components) for a product or a process, and let $Y = h(X_1, X_2, \dots, X_n)$ represents a quality characteristic of interest. We consider X_1 through X_n to be random variables with respective nominal values μ_1 through μ_n . Thus, $Y = h(X_1, X_2, \dots, X_n)$ is also a random variable whose probability density function and its associated parameters depend on the form of the transfer function and the probability density functions of X_1 through X_n , as well as on their corresponding parameters. As discussed in the previous chapter the expected loss of Y depends on some of its lower moments which, in fact, depend on the parameters of X_1 through X_n . However, for the parameter design problem we consider μ_1 through μ_n as controllable parameters and the other parameters are fixed.

We therefore define the parameter design problem to be the problem of determining an optimal set of values for μ_1 through μ_n that minimizes the expected loss associated with the quality characteristic Y . In most applications some constraints pertaining to the technical requirements, processing cost, etc. may be present. By assuming the quartic

loss function the problem can be expressed mathematically as the following model:

$$\begin{aligned}
\text{Minimize} \quad & E[L(Y)] = k_2 [\sigma_Y^2 + (\mu_Y - \tau)^2] + k_3 [\phi_Y + 3\sigma_Y^2(\mu_Y - \tau) + (\mu_Y - \tau)^3] \\
& + k_4 [\varphi_Y + 4\phi_Y(\mu_Y - \tau) + 6\sigma_Y^2(\mu_Y - \tau)^2 + (\mu_Y - \tau)^4] \\
\text{Subject to} \quad & (\mu_1, \mu_2, \dots, \mu_n) \in M
\end{aligned}$$

where $(\mu_1, \mu_2, \dots, \mu_n) \in M$ represents possible constraints pertaining to the technical requirements, processing cost, etc. of a certain problem.

Obviously the values of these parameters, μ_Y , σ_Y^2 , ϕ_Y , and φ_Y , depend on the design parameters μ_1 through μ_n . However, these functional relationships may be difficult, or in some cases even impossible to obtain in closed forms, and we usually have to approximate them. There are several means to approximate these parameters (see Evan [2]). In this thesis we adopt the approximation method presented by Tukey [17] as we discussed in Chapter 2, and we assume that X_1, X_2, \dots, X_n are IID random variables with known parameters σ_i^2 , γ_i , and Γ_i .

The formulas for estimating up to the fourth moment of the quality characteristic Y are presented below. Note that we include here all terms up to the terms of order σ^4 . The original work by Tukey [17] and the revised version by Evans [2] contained up to the terms of order σ^5 in their formulas. Including the terms of order σ^5 would require exhausting efforts; however, they can barely improve the accuracies of the approximation in the context of the parameter design problem. Thus, we decided that it is not worthwhile

to include them in our analysis.

$$\begin{aligned}\mu_Y &\approx h(\mu_1, \mu_2, \dots, \mu_n) + \frac{1}{2} \sum_i h_{ii} \sigma_i^2 + \frac{1}{6} \sum_i h_{iii} \gamma_i \sigma_i^3 \\ &\quad + \frac{1}{24} \sum_i h_{iiii} \Gamma_i \sigma_i^4 + \frac{1}{4} \sum_{i < j} h_{ijjj} \sigma_i^2 \sigma_j^2\end{aligned}\tag{5.1}$$

$$\begin{aligned}\sigma_Y^2 &\approx \sum_i h_i^2 \sigma_i^2 + \sum_i h_i h_{ii} \gamma_i \sigma_i^3 + \frac{1}{3} \sum_i h_i h_{iii} \Gamma_i \sigma_i^4 \\ &\quad + \frac{1}{4} \sum_i h_{ii}^2 (\Gamma_i - 1) \sigma_i^4 + \sum_{i < j} (h_i h_{ijj} + h_{ij}^2 + h_{ijj} h_j) \sigma_i^2 \sigma_j^2\end{aligned}\tag{5.2}$$

$$\phi_Y \approx \sum_i h_i^3 \gamma_i \sigma_i^3 + \frac{3}{2} \sum_i h_i^2 h_{ii} (\Gamma_i - 1) \sigma_i^4 + 6 \sum_{i < j} h_i h_j h_{ij} \sigma_i^2 \sigma_j^2\tag{5.3}$$

$$\varphi_Y \approx \sum_i h_i^4 (\Gamma_i - 3) \sigma_i^4 + 3 (\sigma_Y^2)^2\tag{5.4}$$

where

- $h_i = \frac{\partial h}{\partial x_i} \Big|_{\mu_1, \mu_2, \dots, \mu_n}$, $h_{ii} = \frac{\partial^2 h}{\partial x_i^2} \Big|_{\mu_1, \mu_2, \dots, \mu_n}$, $h_{ij} = \frac{\partial^2 h}{\partial x_i \partial x_j} \Big|_{\mu_1, \mu_2, \dots, \mu_n}$, etc.
- The summations are from 1 to n .

By applying these formulas the values of μ_Y , σ_Y^2 , ϕ_Y , and φ_Y can be expressed as functions of μ_1 through μ_n . We use the notation $\boldsymbol{\mu}$ to represent the vector $(\mu_1, \mu_2, \dots, \mu_n)$. Therefore, μ_Y , σ_Y^2 , ϕ_Y , and φ_Y as the functions of μ_1 through μ_n can be written as $\mu_Y(\boldsymbol{\mu})$, $\sigma_Y^2(\boldsymbol{\mu})$, $\phi_Y(\boldsymbol{\mu})$, and $\varphi_Y(\boldsymbol{\mu})$, respectively. This leads to the following model:

$$\begin{aligned}
\text{Minimize} \quad & k_2 \left[\sigma_Y^2(\boldsymbol{\mu}) + \{\mu_Y(\boldsymbol{\mu}) - \tau\}^2 \right] + k_3 \left[\phi_Y(\boldsymbol{\mu}) + 3\sigma_Y^2(\boldsymbol{\mu}) \{\mu_Y(\boldsymbol{\mu}) - \tau\} \right. \\
& \left. + \{\mu_Y(\boldsymbol{\mu}) - \tau\}^3 \right] + k_4 \left[\varphi_Y(\boldsymbol{\mu}) + 4\phi_Y(\boldsymbol{\mu}) \{\mu_Y(\boldsymbol{\mu}) - \tau\} \right. \\
& \left. + 6\sigma_Y^2(\boldsymbol{\mu}) \{\mu_Y(\boldsymbol{\mu}) - \tau\}^2 + \{\mu_Y(\boldsymbol{\mu}) - \tau\}^4 \right] \\
\text{Subject to} \quad & \mu_Y(\boldsymbol{\mu}) = h(\boldsymbol{\mu}) + \frac{1}{2} \sum_i h_{ii} \sigma_i^2 + \frac{1}{6} \sum_i h_{iii} \gamma_i \sigma_i^3 + \frac{1}{24} \sum_i h_{iiii} \Gamma_i \sigma_i^4 \\
& \quad + \frac{1}{4} \sum_{i < j} h_{ijj} \sigma_i^2 \sigma_j^2 \\
\sigma_Y^2(\boldsymbol{\mu}) &= \sum_i h_i^2 \sigma_i^2 + \sum_i h_i h_{ii} \gamma_i \sigma_i^3 + \frac{1}{3} \sum_i h_i h_{iii} \Gamma_i \sigma_i^4 \\
& \quad + \frac{1}{4} \sum_i h_{ii}^2 (\Gamma_i - 1) \sigma_i^4 + \sum_{i < j} \left(h_i h_{ijj} + h_{ij}^2 + h_{ijj} h_j \right) \sigma_i^2 \sigma_j^2 \\
\phi_Y(\boldsymbol{\mu}) &= \sum_i h_i^3 \gamma_i \sigma_i^3 + \frac{3}{2} \sum_i h_i^2 h_{ii} (\Gamma_i - 1) \sigma_i^4 + 6 \sum_{i < j} h_i h_j h_{ij} \sigma_i^2 \sigma_j^2 \\
\varphi_Y(\boldsymbol{\mu}) &= \sum_i h_i^4 (\Gamma_i - 3) \sigma_i^4 + 3 (\sigma_Y^2(\boldsymbol{\mu}))^2 \\
(\boldsymbol{\mu}) &\in M
\end{aligned} \tag{PDM}$$

where, as in the previous chapter

- $h_i = \frac{\partial h}{\partial x_i} \Big|_{\mu_1, \mu_2, \dots, \mu_n}$, $h_{ii} = \frac{\partial^2 h}{\partial x_i^2} \Big|_{\mu_1, \mu_2, \dots, \mu_n}$, $h_{ij} = \frac{\partial^2 h}{\partial x_i \partial x_j} \Big|_{\mu_1, \mu_2, \dots, \mu_n}$, etc.
- All summations are from 1 to n .

The model is a nonlinear constrained optimization problem which we can solve by a nonlinear programming software package, such as LINGO or the Excel Solver.

5.2 A Case Study

The example we have selected is the design of a coil spring which was adopted from Fathi [6] with slight modifications. Our objective is to demonstrate how the functional profile

of a quartic loss function, which depends on the shape parameter k_4 , has an effect on the optimal solutions. For the case of a symmetric function we also compare the results with what we would obtain by using a quadratic loss assumption.

5.2.1 The Problem Statement

We consider a coil spring loaded in tension or compression. We define the following notations and data for the problem formulation:

Shear modulus:	$G = 1.15 \times 10^7$ (lb/in ²)
Weight density of a spring material:	$V = 0.285$ (lb/in ³)
Gravitational constant:	$g = 386$ (in/sec ²)
Mass density of the spring material:	$\rho = \frac{V}{g} = 7.38342 \times 10^{-4}$ (lb-sec ² /in ⁴)
Deflection along the axis of the spring when it is loaded:	δ (in)
Shear stress:	S (lb/in ²)
Frequency of surge waves:	ω (Hz)
Applied load:	P (lbs)

The deflection (δ) is the performance characteristic of the coil spring with a desired target of 0.5 inches when a load of $P = 1000$ lbs is applied. The shear stress (S) and the surge waves (ω) are the technical requirements whose values should be within certain limits for a proper function of the coil spring. These issues will be discussed later in this subsection.

The following parameters represent the design variables in this problem:

Wire diameter: d (in.)

Coil diameter: D (in.)

Number of active coils: N

The relevant technical relationships are given below:

1. Load deflection equation:

$$P = \left(\frac{d^4 G}{8D^3 N} \right) \delta \quad (5.5)$$

2. Shear stress equation:

$$S = \frac{8PD}{\pi d^3} \left(\frac{4D - d}{4(D - d)} + \frac{0.615d}{D} \right) \quad (5.6)$$

3. Frequency of surge waves equation:

$$\omega = \frac{d}{2\pi D^2 N} \sqrt{\frac{G}{2\rho}} \quad (5.7)$$

We consider d , D , and N to be pairwise independent normal random variables with respective means μ_d , μ_D , and μ_N ; respective variances σ_d^2 , σ_D^2 , and σ_N^2 . We assume that $\sigma_d = 0.006$, $\sigma_D = 0.025$, $\sigma_N = 0.2$, and since all random variables are normally distributed we have $\gamma_d = \gamma_D = \gamma_N = 0$, and $\Gamma_d = \Gamma_D = \Gamma_N = 3$.

In the context of this problem μ_d , μ_D , and μ_N are the decision variables, and we want to determine their optimal set points in order to minimize the expected loss subject to some technical constraints at the design stage. Having defined the notations and data for the

problem, we are now ready to formulate the preliminary relationships for the parameter design problem in term of μ_d , μ_D , and μ_N .

The deflection of the spring under an applied load of 1000 lbs (the performance characteristic of the problem) can be formulated by using Eq. (5.5):

$$\delta = \frac{D^3 N}{1437.5 d^4} \quad (5.8)$$

The maximum allowable shear stress in this problem was given to be 80,000 lb/in². This requirement can be formulated by using Eq. (5.6):

$$\frac{636.62 D(4D - d)}{d^3(D - d)} + \frac{1566.084}{d^2} \leq 80,000 \quad (5.9)$$

To avoid serious resonance in dynamic applications the frequency of surge waves must be at least 100 Hz. This requirement can be formulated by using Eq. (5.7):

$$\frac{14045.1 d}{D^2 N} \geq 100 \quad (5.10)$$

The outer diameter of the coil spring is required to be no bigger than 2.5 inches. This requirement can be formulated as:

$$D + d \leq 2.5 \quad (5.11)$$

Finally, we have the following explicit bounds on the decision variables.

$$\begin{aligned} 0.2 &\leq \mu_d \leq 0.8 \\ 0.9 &\leq \mu_D \leq 2 \\ 2 &\leq \mu_N \leq 25 \end{aligned} \tag{5.12}$$

5.2.2 Symmetric Loss Function

As discussed in section 5.2.1 the desired target of deflection when a load of 1000 lbs is applied is $\tau = 0.5$ inches. We assume that the specification limits are 0.5 ± 0.04 ; and the losses at each limit is \$5. In order to solve the parameter design problem we first need to determine the proper value of the coefficients k_2 , k_3 , and k_4 . In this case k_3 is zero since the function is symmetric. Only k_2 and k_4 are left to be evaluated.

By applying the procedure discussed in chapter 3, $k_4 \in (0, 1953125)$. Recall that for the case of symmetric loss function, k_4 needs to satisfy two conditions; i) $k_4 > 0$, and ii) k_4 must satisfy the inequality (3.18). We solve five parameter design problems here with five values of the shape parameter; $k_4 = 0, 100, 100000, 1000000$, and 1953124 . By Using Eq. (3.16) we can determine their associated $k_2 = 3125, 3124.84, 2965, 1525$, and 0.0016 , respectively. When $k_4 = 0$ we have the quadratic loss function. Figure 5-1 depicts the various profiles of the loss functions when we vary the values of k_4 . Note that the profiles of the functions when $k_4 = 0, k_4 = 100$, and $k_4 = 100000$ are almost identical.

We are now ready to formulate a parameter design model. We demonstrate here only the model with $k_2 = 3124.84$ and $k_4 = 100$. The other models can be formulated in a

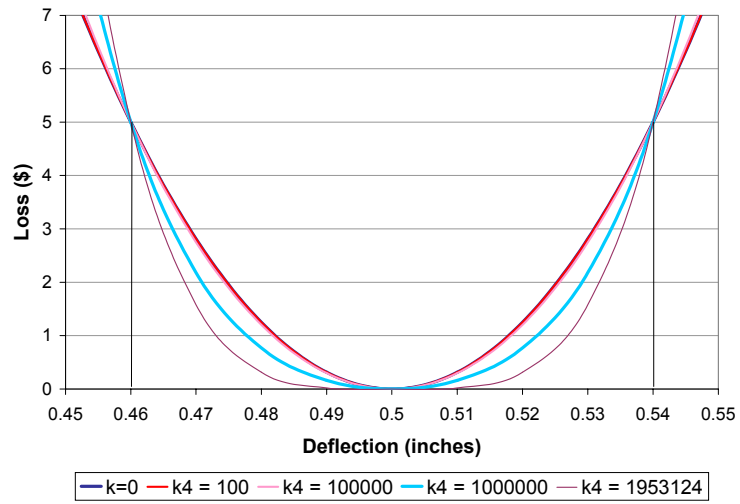


Figure 5-1: Effects of k_4 on the Symmetric Loss Functions

similar way. By applying the proposed model (PDM), the model for this specific problem can be written as:

$$\begin{aligned}
\text{Min} \quad & 3124.84 \left[\sigma_\delta^2(\boldsymbol{\mu}) + \{\mu_\delta(\boldsymbol{\mu}) - 0.5\}^2 \right] + 100 [\varphi_\delta(\boldsymbol{\mu}) + 4\phi_\delta(\boldsymbol{\mu}) \{\mu_\delta(\boldsymbol{\mu}) - 0.5\}] \\
& + 6\sigma_\delta^2(\boldsymbol{\mu}) \{\mu_\delta(\boldsymbol{\mu}) - 0.5\}^2 + \{\mu_\delta(\boldsymbol{\mu}) - 0.5\}^4 \\
\text{Subject to} \quad & \mu_\delta(\boldsymbol{\mu}) = \frac{1}{1437.5} \left\{ \left[\frac{\mu_D^3 \mu_N}{\mu_d^4} \right] + \left[\left(\frac{3\mu_D \mu_N}{\mu_d^4} \right) (0.025)^2 + \left(\frac{10\mu_D^3 \mu_N}{\mu_d^6} \right) (0.006)^2 \right] \right. \\
& \left. + \left[\left(\frac{105\mu_D^3 \mu_N}{\mu_d^8} \right) (0.006)^4 \right] + \left[\left(\frac{30\mu_D \mu_N}{\mu_d^6} \right) (0.025)^2 (0.006)^2 \right] \right\} \\
\sigma_\delta^2(\boldsymbol{\mu}) = & \frac{1}{1437.5^2} \left\{ \left[\left(\frac{3\mu_D^2 \mu_N}{\mu_d^4} \right)^2 (0.025)^2 + \left(-\frac{4\mu_D^3 \mu_N}{\mu_d^5} \right)^2 (0.006)^2 + \left(\frac{\mu_D^3}{\mu_d^4} \right) \right. \right. \\
& (0.2)^2 \left. \right] + \left[\left(\frac{3\mu_D^2 \mu_N}{\mu_d^4} \right) \left(\frac{6\mu_N}{\mu_d^4} \right) (0.025)^4 + \left(-\frac{4\mu_D^3 \mu_N}{\mu_d^5} \right) \left(-\frac{120\mu_D^3 \mu_N}{\mu_d^7} \right) \right. \\
& (0.006)^4 \left. \right] + \frac{1}{2} \left[\left(\frac{6\mu_D \mu_N}{\mu_d^4} \right)^2 (0.025)^4 + \left(\frac{20\mu_D^3 \mu_N}{\mu_d^6} \right)^2 (0.006)^4 \right] \\
& \left[\left(\left(\frac{3\mu_D^2 \mu_N}{\mu_d^4} \right) \left(\frac{60\mu_D^2 \mu_N}{\mu_d^6} \right) + \left(-\frac{12\mu_D^2 \mu_N}{\mu_d^5} \right)^2 + \left(-\frac{24\mu_D \mu_N}{\mu_d^5} \right) \left(-\frac{4\mu_D^3 \mu_N}{\mu_d^5} \right) \right) \right. \\
& (0.025)^2 (0.006)^2 + \left. \left(\left(\frac{3\mu_D^2}{\mu_d^4} \right)^2 + \left(\frac{6\mu_D}{\mu_d^4} \right) \left(\frac{\mu_D^3}{\mu_d^4} \right) \right) (0.025)^2 (0.2)^2 \right. \\
& \left. \left(\left(\frac{\mu_D^3}{\mu_d^4} \right) \left(\frac{20\mu_D^3}{\mu_d^6} \right) + \left(-\frac{4\mu_D^3}{\mu_d^5} \right)^2 \right) (0.2)^2 (0.006)^2 \right] \left. \right\} \\
\phi_\delta(\boldsymbol{\mu}) = & \frac{1}{1437.5^3} \left\{ 3 \left[\left(\frac{3\mu_D^2 \mu_N}{\mu_d^4} \right)^2 \left(\frac{6\mu_D \mu_N}{\mu_d^4} \right) (0.025)^4 + \left(-\frac{4\mu_D^3 \mu_N}{\mu_d^5} \right)^2 \left(\frac{20\mu_D^3 \mu_N}{\mu_d^6} \right) \right. \right. \\
& (0.006)^4 \left. \right] + 6 \left[\left(\frac{3\mu_D^2 \mu_N}{\mu_d^4} \right) \left(-\frac{4\mu_D^3 \mu_N}{\mu_d^5} \right) \left(-\frac{12\mu_D^2 \mu_N}{\mu_d^5} \right) (0.025)^2 (0.006)^2 \right. \\
& + \left. \left(\frac{3\mu_D^2 \mu_N}{\mu_d^4} \right) \left(\frac{\mu_D^3}{\mu_d^4} \right) \left(\frac{3\mu_D^2}{\mu_d^4} \right) (0.025)^2 (0.2)^2 + \left(\frac{\mu_D^3}{\mu_d^4} \right) \left(-\frac{4\mu_D^3 \mu_N}{\mu_d^5} \right) \left(-\frac{4\mu_D^3}{\mu_d^5} \right) \right. \\
& \left. (0.2)^2 (0.006)^2 \right] \left. \right\} \\
\varphi_\delta(\boldsymbol{\mu}) = & 3 (\sigma_\delta^2(\boldsymbol{\mu}))^2 \\
& \frac{636.62\mu_D(4\mu_D - \mu_d)}{\mu_d^3(\mu_D - \mu_d)} + \frac{1566.084}{\mu_d^2} \leq 80,000 \\
& \frac{14045.1\mu_d}{\mu_D^2 \mu_N} \geq 100 \\
& \mu_D + \mu_d \leq 2.5 \\
& 0.2 \leq \mu_d \leq 0.8, \quad 0.9 \leq \mu_D \leq 2, \quad 2 \leq \mu_N \leq 25
\end{aligned}$$

We solved this model and the other four models using the software package LINGO on a celeron 700 MHz personal computer. The results are displayed in Table 5-1. Notice that as we increase k_4 the optimal value of expected loss increases. We might suspect

Table 5.1: Summary of the Results (Symmetric Loss Function)

Assumption	μ_d	μ_D	μ_N	$\mathbf{E}[\mathbf{L}(\mathbf{y})]$	μ_δ	σ_δ
$k_4 = 0$ (Quadratic Loss Function)	0.6780	1.8220	25	\$2.35	0.4984	0.0274
$k_4 = 100$	0.6780	1.8220	25	\$2.35	0.4984	0.0274
$k_4 = 100000$	0.6781	1.8219	25	\$2.40	0.4982	0.0274
$k_4 = 1000000$	0.6783	1.8217	25	\$2.81	0.4974	0.0273
$k_4 = 1953124$	0.6784	1.8216	25	\$3.25	0.4970	0.0273

this incident at the first glance; however, with careful analysis, we found that the optimal solutions yielded σ_δ around 0.027, which is relatively large comparing with $\Delta = 0.04$. As the result, many springs whose quality characteristics fell outside the specification limits were charged with severe penalty. This penalty is much more than that inside the specification limits and tend to predominate on the value of the expected loss. Besides, as we increase k_4 the penalty charged on a nonconforming unit noticeably increases. Thus, these are the reasons why the optimal value of expected loss increases as we increase k_4 .

Now, consider the case when we assume that the specification limits are wider, and observe the result. Assume that the specification limits are now 0.5 ± 0.1 while we keep every parameter the same as before. By applying the procedure discussed in chapter 3 $k_4 \in (0, 50000)$. We solve five parameter design problems here with five values of the shape parameter; $k_4 = 0, 100, 10000, 30000, \text{ and } 49990$. By Using Eq. (3.16), we can determine their associated $k_2 = 500, 499, 400, 200, \text{ and } 0.1$, respectively. Figure 5-2 depicts the various profiles of the loss functions when we vary the values of k_4 . Note that the profiles of the functions when $k_4 = 0$ and when $k_4 = 100$ are almost identical.

We solved this embellishment using the software package LINGO on a celeron 700 MHz personal computer. The results of the parameter design problem based on this situation are displayed in Table 5-2. At this time the expected loss decreases as we increase k_4 . As

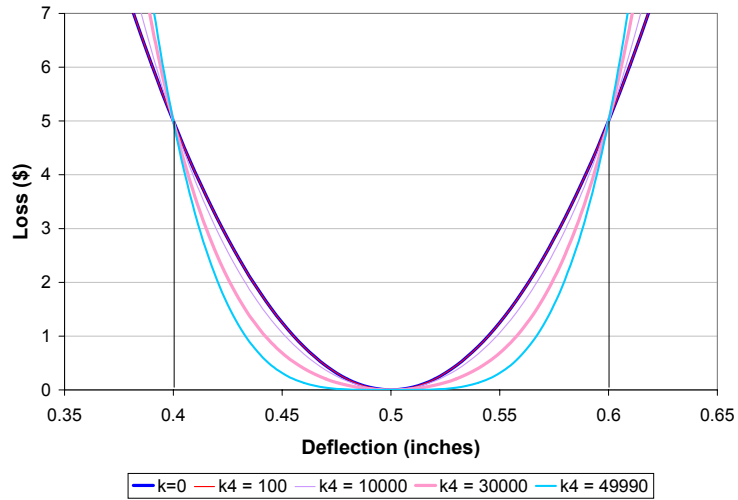


Figure 5-2: Effects of k_4 on the Symmetric Loss Functions (Embellishment)

Table 5.2: Summary of the Results (Symmetric Loss Function with Embellishment)

Assumption	μ_d	μ_D	μ_N	$\mathbf{E}[\mathbf{L}(\mathbf{y})]$	μ_δ	σ_δ
$k_4 = 0$ (Quadratic Loss Function)	0.6780	1.8220	25	\$0.38	0.4984	0.0274
$k_4 = 100$	0.6780	1.8220	25	\$0.38	0.4984	0.0274
$k_4 = 10000$	0.6781	1.8219	25	\$0.32	0.4983	0.0273
$k_4 = 30000$	0.6782	1.8218	25	\$0.20	0.4978	0.0273
$k_4 = 49990$	0.6784	1.8216	25	\$0.08	0.4970	0.0273

we widened the specification limits, while the optimal solutions yielded the same σ_δ , most of the produced springs would fall within the new limits. The penalty is not severe within the specification limits and as we increase k_4 the penalty is even less severe. Thus, these are the reasons why the optimal value of expected loss decreases as we increase k_4 .

5.2.3 Non-Symmetric Loss Function

As discussed in section 5.2.1 the desired target of deflection when a load of 1000 lbs is applied is $\tau = 0.5$ inches. However, we now assume that the specification limits are 0.5 ± 0.04 ; and the losses at the lower and upper specification limits are \$5 and \$7, respectively. In

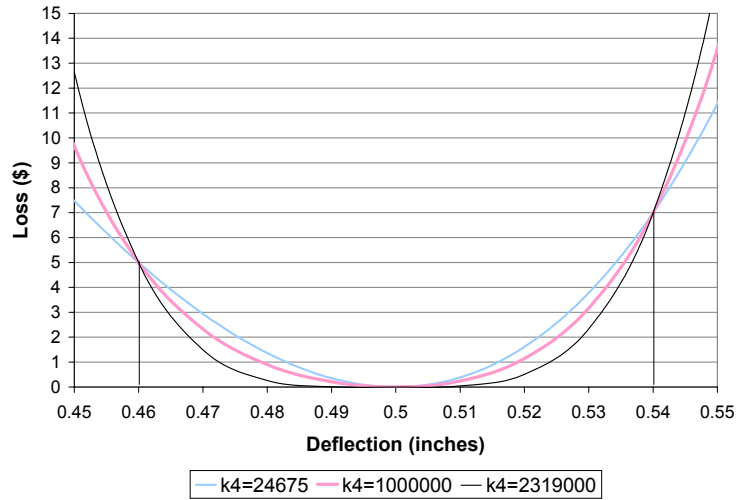


Figure 5-3: Effects of k_4 on the Non-Symmetric Loss Functions.

order to solve the parameter design problem we first need to determine the proper values of the quality loss coefficients k_2 , k_3 , and k_4 .

By applying the procedure discussed in chapter 3 $k_4 \in (24673.83, 2319076.17)$. Once again recall that for the case of non-symmetric loss function with $\Delta_1 = \Delta_2 = \Delta$, the shape parameter k_4 needs to satisfy the two conditions stated in the inequality (3.15). We solve three parameter design problems here with three values of the shape parameter; $k_4 = 24675$, 1000000 , and 2319000 . By using Eq. (3.13) we can determine their associated $k_2 = 3710.52$, 2150 , and 39.6 , respectively. By Using Eq. (3.14) we can determine $k_3 = 15625$. Figure 5-3 depicts the various profiles of the quality loss function when we vary the values of k_4 .

We are now ready to formulate a parameter design model. We exhibit only the model with $k_2 = 3710.52$ and $k_4 = 24675$. The other models can be formulated in a similar way. By applying the proposed model (PDM) the model for this specific problem can be written

as:

$$\begin{aligned}
\text{Min} \quad & 3710.52 \left[\sigma_\delta^2(\boldsymbol{\mu}) + \{\mu_\delta(\boldsymbol{\mu}) - 0.5\}^2 \right] + 15625 \left[\phi_\delta(\boldsymbol{\mu}) + 3\sigma_\delta^2(\boldsymbol{\mu}) \{\mu_\delta(\boldsymbol{\mu}) - 0.5\} \right. \\
& \left. + \{\mu_\delta(\boldsymbol{\mu}) - 0.5\}^3 \right] + 24675 \left[\varphi_\delta(\boldsymbol{\mu}) + 4\phi_\delta(\boldsymbol{\mu}) \{\mu_\delta(\boldsymbol{\mu}) - 0.5\} \right. \\
& \left. + 6\sigma_\delta^2(\boldsymbol{\mu}) \{\mu_\delta(\boldsymbol{\mu}) - 0.5\}^2 + \{\mu_\delta(\boldsymbol{\mu}) - 0.5\}^4 \right] \\
\text{subject to} \quad & \mu_\delta(\boldsymbol{\mu}) = \frac{1}{1437.5} \left\{ \left[\frac{\mu_D^3 \mu_N}{\mu_d^4} \right] + \left[\left(\frac{3\mu_D \mu_N}{\mu_d^4} \right) (0.025)^2 + \left(\frac{10\mu_D^3 \mu_N}{\mu_d^6} \right) (0.006)^2 \right] \right. \\
& \left. + \left[\left(\frac{105\mu_D^3 \mu_N}{\mu_d^8} \right) (0.006)^4 \right] + \left[\left(\frac{30\mu_D \mu_N}{\mu_d^6} \right) (0.025)^2 (0.006)^2 \right] \right\} \\
& \sigma_\delta^2(\boldsymbol{\mu}) = \frac{1}{1437.5^2} \left\{ \left[\left(\frac{3\mu_D^2 \mu_N}{\mu_d^4} \right)^2 (0.025)^2 + \left(-\frac{4\mu_D^3 \mu_N}{\mu_d^5} \right)^2 (0.006)^2 + \left(\frac{\mu_D^3}{\mu_d^4} \right) \right. \right. \\
& (0.2)^2 \left. \left. + \left[\left(\frac{3\mu_D^2 \mu_N}{\mu_d^4} \right) \left(\frac{6\mu_N}{\mu_d^4} \right) (0.025)^4 + \left(-\frac{4\mu_D^3 \mu_N}{\mu_d^5} \right) \left(-\frac{120\mu_D^3 \mu_N}{\mu_d^5} \right) \right. \right. \right. \\
& (0.006)^4 \left. \left. + \frac{1}{2} \left[\left(\frac{6\mu_D \mu_N}{\mu_d^4} \right)^2 (0.025)^4 + \left(\frac{20\mu_D^3 \mu_N}{\mu_d^6} \right)^2 (0.006)^4 \right] \right. \right. \\
& \left. \left[\left(\left(\frac{3\mu_D^2 \mu_N}{\mu_d^4} \right) \left(\frac{60\mu_D^2 \mu_N}{\mu_d^6} \right) + \left(-\frac{12\mu_D^2 \mu_N}{\mu_d^5} \right)^2 + \left(-\frac{24\mu_D \mu_N}{\mu_d^5} \right) \left(-\frac{4\mu_D^3 \mu_N}{\mu_d^5} \right) \right) \right. \right. \\
& (0.025)^2 (0.006)^2 + \left. \left(\left(\frac{3\mu_D^2}{\mu_d^4} \right)^2 + \left(\frac{6\mu_D}{\mu_d^4} \right) \left(\frac{\mu_D^3}{\mu_d^4} \right) \right) (0.025)^2 (0.2)^2 \right. \\
& \left. \left. \left(\left(\frac{\mu_D^3}{\mu_d^4} \right) \left(\frac{20\mu_D^3}{\mu_d^6} \right) + \left(-\frac{4\mu_D^3}{\mu_d^5} \right)^2 \right) (0.2)^2 (0.006)^2 \right] \right\} \\
& \phi_\delta(\boldsymbol{\mu}) = \frac{1}{1437.5^3} \left\{ 3 \left[\left(\frac{3\mu_D^2 \mu_N}{\mu_d^4} \right)^2 \left(\frac{6\mu_D \mu_N}{\mu_d^4} \right) (0.025)^4 + \left(-\frac{4\mu_D^3 \mu_N}{\mu_d^5} \right)^2 \left(\frac{20\mu_D^3 \mu_N}{\mu_d^6} \right) \right. \right. \\
& (0.006)^4 \left. \left. + 6 \left[\left(\frac{3\mu_D^2 \mu_N}{\mu_d^4} \right) \left(-\frac{4\mu_D^3 \mu_N}{\mu_d^5} \right) \left(-\frac{12\mu_D^2 \mu_N}{\mu_d^5} \right) (0.025)^2 (0.006)^2 \right. \right. \right. \\
& + \left. \left(\frac{3\mu_D^2 \mu_N}{\mu_d^4} \right) \left(\frac{\mu_D^3}{\mu_d^4} \right) \left(\frac{3\mu_D^2}{\mu_d^4} \right) (0.025)^2 (0.2)^2 + \left(\frac{\mu_D^3}{\mu_d^4} \right) \left(-\frac{4\mu_D^3 \mu_N}{\mu_d^5} \right) \left(-\frac{4\mu_D^3}{\mu_d^5} \right) \right. \right. \\
& \left. \left. (0.2)^2 (0.006)^2 \right] \right\} \\
& \varphi_\delta(\boldsymbol{\mu}) = 3 \left(\sigma_\delta^2(\boldsymbol{\mu}) \right)^2 \\
& \frac{636.62\mu_D(4\mu_D - \mu_d)}{\mu_d^3(\mu_D - \mu_d)} + \frac{1566.084}{\mu_d^2} \leq 80,000 \\
& \frac{14045.1\mu_d}{\mu_D^2 \mu_N} \geq 100 \\
& \mu_D + \mu_d \leq 2.5 \\
& 0.2 \leq \mu_d \leq 0.8, \quad 0.9 \leq \mu_D \leq 2, \quad 2 \leq \mu_N \leq 25
\end{aligned}$$

We solved this model and the other two using the software package LINGO on a celeron

Table 5.3: Summary of the Results (Non-Symmetric Loss Function)

Assumption	μ_d	μ_D	μ_N	$\mathbf{E}[\mathbf{L}(\mathbf{y})]$	μ_δ	σ_δ
$k_4 = 24675$	0.6793	1.8207	25	\$2.62	0.4937	0.0271
$k_4 = 1000000$	0.6790	1.8210	25	\$3.20	0.4948	0.0272
$k_4 = 2319000$	0.6788	1.8212	25	\$3.81	0.4954	0.0272

Table 5.4: Summary of the Results (Non-Symmetric Loss Function with Embellishment)

Assumption	μ_d	μ_D	μ_N	$\mathbf{E}[\mathbf{L}(\mathbf{y})]$	μ_δ	σ_δ
$k_4 = 635$	0.6785	1.8215	25	\$0.44	0.4965	0.0273
$k_4 = 25000$	0.6788	1.8212	25	\$0.30	0.4957	0.0272
$k_4 = 59300$	0.6795	1.8205	25	\$0.10	0.4929	0.0271

700 MHz personal computer. The results are displayed in Table 5-3. Notice that the same pattern as the previous case occurred. This phenomenon can be explained in the same way as we did in the previous case.

Now, once again, we consider the case with the wider specification limits, and observe the result. Assume that the specification limits are now 0.5 ± 0.1 while we keep every parameter the same as before. By applying the procedure discussed in chapter 3 $k_4 \in (631.65, 59368.35)$. We solve three parameter design problems here with three values of the shape parameter; $k_4 = 635, 25000$, and 59300 . By using Eq. (3.13) we can determine their associated $k_2 = 593.65, 350$, and 7 , respectively. By Using Eq. (3.14) we can determine $k_3 = 1000$. Figure 5-4 depicts the various profiles of the loss functions when we vary the values of k_4 .

We also solved this embellishment using the software package LINGO on a celeron 700 MHz personal computer. The results of the parameter design problem based on this situation are displayed in Table 5-4. At this time the optimal value of expected loss decreases as we increase k_4 .

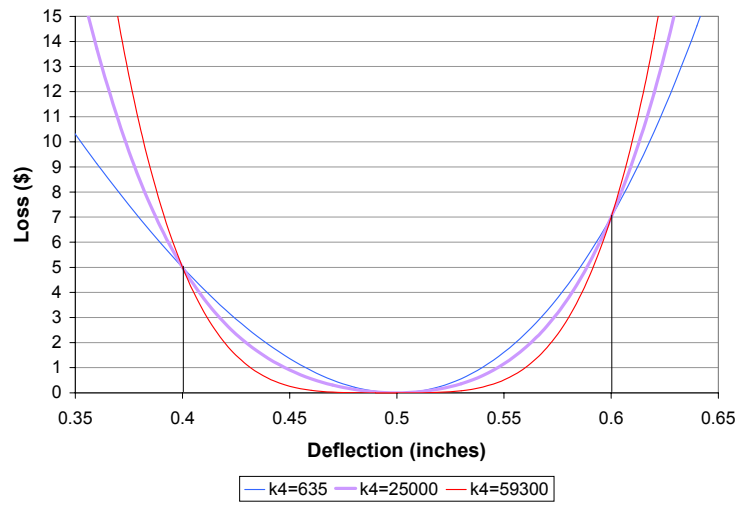


Figure 5-4: Effects of k_4 on the Non-Symmetric Loss Functions (Embellishment)

Chapter 6

Conclusions and Recommendations for Future Research

6.1 Conclusions

The quartic loss function is another functional form of the continuous loss function that can be applied to a certain situation where flexibility is desired. It can provide a family of symmetric and non-symmetric loss functions, and a user can choose, among the family of loss functions, a specific loss function that can best describe the situation at hand. The functional profile of the quartic loss function depends on the shape parameter (k_4), which can be any value in a feasible range that we discussed back in Chapter 3. At one extreme, where the shape parameter is relatively small, a quartic loss function is close to

a quadratic loss function. At the other extreme, where the shape parameter is relatively large, a quartic loss function is close to a step function. Based on this fact one would choose a large value of the shape parameter to make the function modestly penalize any product that deviates from its target but still fall within given specification limits. On the other hand, one would choose a small value of the shape parameter to make the function severely penalize that kind of product. The expected loss based on the assumption of the quartic loss function depends on several parameters of the response of interest, i.e. the mean, the variance, the third moment, and the fourth moment of the response.

We also applied the quartic loss function to the parameter design problem. We presented a mathematical model for this context based on minimizing the expected loss of a quality characteristic of interest. We assumed that the transfer function, i.e. $Y = h(X_1, X_2, \dots, X_n)$, is well defined in a closed form, and we assumed that X_1, X_2, \dots, X_n are IID random variables with known parameters σ_i^2 , γ_i , and Γ_i . We applied the non-linear propagation of error formulas presented by Tukey [17] to the parameter design problem for approximating several parameters of the quality characteristic of interest. We also provided a case study of a coil spring to illustrate this problem with symmetric and non-symmetric loss functions.

6.2 Recommendations for Future Research

While we attain satisfactory results with any products or processes that possess small variation, a discouraging pitfall that we came across in this work is the problem when we apply the quartic loss function to those whose variations are relatively large. For

example, a product whose quality characteristic falls outside given specification limits will be severely penalized since we assume our loss function to be convex for all values of Y . Consequently, the expected loss of a product or process with relatively large variation is unreasonably expensive. One alternative that may be possible for future research to eliminate this pitfall is to assume a constant loss outside given specification limits and a quartic loss function within the specification limits. Fathi [4] presented some mathematical models in this context.

In this thesis we cover only the nominal-the-base class of the loss function. It is possible for future research to apply the fundamental of the quartic loss function to the other classes of the loss function, i.e. the smaller-the-better, or the higher-the-better class. The quartic loss function may be benefit for the parameter design problem based on these classes of loss functions. For example, we may want to minimize the variance of the quality characteristic and to make its distribution skew to the right in the same time in order to minimize the expected loss for the smaller-the-better class of problem.

Bibliography

- [1] Box, G. E. P., “Signal-to-Noise Ratio: Performance Criteria, and Transformations”
(With Discussion), *Technometric*, Vol. 30 (1988), 1-17.
- [2] Evans, David H., “Statistical Tolerancing: The State of the Art Part II. Methods for
Estimating Moments”, *Journal of Quality Technology*, Vol. 7, No. 1 (1975), 1-12.
- [3] Fathi, Yahya, “Robust Design of Manufactured Products: An Analytical Approach”,
OR Report No. 244, North Carolina State University, Raleigh, NC, 1988.
- [4] Fathi, Yahya, “Producer-Consumer Tolerances”, *Journal of Quality Technology*, Vol.
22, No. 2 (1990), 138-145.
- [5] Fathi, Yahya, “A Nonlinear Programming Approach to the Parameter Design Prob-
lem”, *European Journal of Operational Research*, Vol. 53 (1991), 371-381.
- [6] Fathi, Yahya, “A Linear Approximation Model for the Parameter Design Problem”,
European Journal of Operational Research, Vol. 97 (1997), 561-570.
- [7] Fowlkers, William Y., and Creveling, Clyde M., *Engineering Methods for Robust
Product Design: Using Taguchi Methods in Technology and Product Development*,
Addison-Wesley, 1997.

- [8] Leung, Bartholomew P. K., and Spiring, Fred A., “The Inverted Beta Loss Function: Properties and Applications”, IIE Transactions, Vol. 34, No. 12 (2002), 1101-1109.
- [9] Montgomery, Douglas C., *Design and Analysis of Experiments*, 5th ed, John-Wiley & Sons, 2001.
- [10] Montgomery, Douglas C., *Introduction to Statistical Quality Control*, 4th ed, John-Wiley & Sons, 2001.
- [11] Phadke, Madhav S., *Quality Engineering Using Robust Design*, Prentice Hall, 1989.
- [12] Suhir, Ephraim, *Applied Probability for Engineers and Scientists*, McGraw-Hill, 1997.
- [13] Taguchi, Genichi, and Wu, Y., *Introduction to Off-Line Quality Control*, Central Japan Quality Control Association, 1980.
- [14] Taguchi, Genichi, Elsayed, Elsayeda A., and Hsiang, Thomas, *Quality Engineering in Production Systems*, McGraw-Hill, 1989.
- [15] Taguchi, Genichi, *Introduction to Quality Engineering: Designing Quality into Products and Processes*, Asian Productivity Organization, 1989.
- [16] Thomas, George B., Jr., and Finney, Ross L., *Calculus and Analytic Geometry*, 6th ed, Addison-Wesley, 1985.
- [17] Tukey, John W., “Propagation of Errors, Fluctuations and Tolerances, No. 1: Basic Generalized Formulas”, Technical Report No. 10, Princeton University, Princeton, NJ, 1957.