ABSTRACT

WANG, ZITENG. Newsvendor Games with Limited Supply. (Under the direction of Dr. Shu-Cherng Fang.)

This dissertation studies game models of newsvendors who place orders to a common supplier with limited supply which will be allocated according to a pre-announced rule if the total order quantity exceeds the supply capacity. A newsvendor may manipulate the order quantity to receive a preferred amount of inventory. Moreover, inventory transshipment among newsvendors after demand realization may generate extra profit. We study four different game models based on two settings – the number of newsvendors (two or more) and whether inventory transshipment is incorporated. In each game, we investigate the existence and uniqueness of the Nash equilibrium of newsvendors’ order quantities. We also develop mechanisms by which the newsvendors can be coordinated so that the supply chain profit can be maximized.

In the two-newsvendor game with limited supply and no inventory transshipment, Nash equilibrium of order quantities always exists and the equilibrium inventory allocation is unique. The newsvendors can be coordinated by a wholesale price contract. In the general case of multi-newsvendor game with limited supply, we show the specific form of Nash equilibrium and develop a general procedure for determining the coordinating wholesale prices.

In the two-newsvendor game with limited supply and inventory transshipment, Nash equilibrium exists under certain conditions and its uniqueness is closely related to the gaming effects of the supply constraint and inventory transshipment. The benefit of inventory transshipment to the newsvendors is investigated and a coordinating contract of negotiable transshipment prices is designed. For the multi-newsvendor case, we develop a coordinating mechanism that involves a transshipment fund through which the transshipment profit is appropriately allocated.
DEDICATION

To my family:
Zhenhai Wang
Ruifang Wu
Xiaoqian Lyu
BIOGRAPHY

Ziteng Wang was born and raised in a beautiful coastal city of Haiyang, Shandong, China. Ziteng received his B.S. degree in Mathematical Sciences at Tsinghua University in 2010, under the direction of Dr. Wenxun Xing. After that, Ziteng was admitted to the Edward P. Fitts Department of Industrial and Systems Engineering at North Carolina State University to pursue his Ph.D. degree under the direction of Dr. Shu-Cherng Fang.
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Chapter 1

Introduction

The newsvendor model has been successfully developed and applied to study single-period inventory problems. A newsvendor faces an uncertain demand with a known stochastic distribution. Before demand realization, the newsvendor places an order to an upstream supplier and receives that amount of inventory. The optimal order quantity is determined by maximizing the newsvendor’s expected profit.

In a supply chain of one supplier and multiple newsvendors, the supplier’s capacity could be limited, which means that the supplier may not be able to satisfy all newsvendors’ orders. This issue of limited supply capacity may arise when expanding production capability is considerably expensive or a quick response of supply capacity to substantial change in short-term demand is nearly infeasible. Related industries include automobiles, toys, electronics, apparel, fashion and media. Under this circumstance, the supplier has to allocate the limited supply to the newsvendors according to certain rules. Consequently, a newsvendor may tend to manipulate his order quantity strategically so that he may receive a preferred inventory allocation that maximizes his expected profit.

Inventory transshipment is often conducted in a supply chain to better match supply with demand and generate extra profit. In this dissertation, inventory transshipment means that after

\footnote{As conventionally in literature, we refer the newsvendors as male and the supplier as female.}
demand realization, the surplus inventory of one newsvendor may be transshipped to another newsvendor who has excess demand. This further complicates the decision making of each newsvendor. Moreover, it is not obvious whether inventory transshipment could be beneficial for all newsvendors under limited supply.

In this dissertation, we propose four types of game models to study newsvendors with limited supply. According to the number of newsvendors in a supply chain and whether inventory transshipment is incorporated or not, we have a two-newsvendor game with limited supply (and no inventory transshipment), a two-newsvendor game with limited supply and inventory transshipment, a multi-newsvendor game with limited supply (and no inventory transshipment), and multi-newsvendor games with limited supply and inventory transshipment. In all four games, each newsvendor makes order decision unilaterally with the purpose to maximize his own profit. The inter-relation of the models in this dissertation and relevant literature are shown in Figure 1.1, where blue blocks are existing models in literature and green and orange blocks are models in this dissertation. The arrows indicate directions of extension. From the classical newsvendor model, inventory transshipment between two newsvendors and among multiple newsvendors have been studied in literature. We consider the situation of limited supply that two or multiple newsvendors may face. Moreover, we integrate the settings of limited supply and inventory transshipment in this dissertation.

The primary research questions of interest include whether there exists a Nash equilibrium of order quantities and when it is unique, how the supply capacity constraint and inventory transshipment affect the newsvendors’ decision-making, when inventory transshipment is beneficial to the newsvendors and the supply chain, and how the newsvendors can be coordinated to jointly maximize the total profit of the supply chain. Our findings may help retailers make better decisions when they compete for limited supply capacity and when they have the option of transshipping inventory. This study also provides managerial insights to the supplier in capacity planning, wholesale pricing and retail store management.

In the remainder of this chapter, we briefly introduce the four games in Section 1.1, 1.2, 1.3,
and 1.4, respectively. Section 1.5 describes the organization of this dissertation.

1.1 Two-newsvendor Game with Limited Supply

In this game, we consider two newsvendors with limited supply and assume there is no inventory transshipment. The customer demand is stochastic at each newsvendor’s market. Before demand realization, newsvendors place orders to the supplier. The supplier’s capacity is limited and publicly known before the newsvendors submit orders. If the aggregate quantity of newsvendors’ orders exceeds the supplier’s capacity, the supplier will allocate her inventory to the newsvendors according to an allocation rule. Each newsvendor pays a wholesale price to the supplier for the received inventory, observes his customer demand and sells to customers. At the end of the period, leftover inventory will be salvaged.

In this supply chain, each newsvendor’s ordering quantity affects the supply capacity allocation and eventually the profits of both newsvendors. We assume that the supply capacity, allocation rule and wholesale prices are announced by the supplier before the newsvendors place orders. The retail price and salvage value of the product are determined by the market and are
exogenous to the newsvendors. Customer demand is assumed to be continuously distributed throughout this dissertation. Both newsvendors have complete information of the game before making ordering decisions. The newsvendors form a competitive game with payoff being their expected profits and the decision variables their order quantities.

We show that there always exists a pure Nash equilibrium of order quantity in this game. Moreover, we provide the explicit form of the Nash equilibrium as well as each newsvendor’s best response, i.e., the best order decision as a function of the other newsvendor’s order quantity. In most cases, the Nash equilibrium is unique. The only exception is that, when the sum of newsvendors’ optimal inventories is exactly the supplier’s capacity, there exist infinitely many Nash equilibria of orders. Nevertheless, these Nash equilibria correspond to the same inventory allocation. Therefore, the equilibrium inventory allocation is always unique.

With the newsvendor game well studied, we consider the total profit of the whole supply chain (supplier and newsvendors). The supplier gains profit by selling to retailers while incurring production cost. We apply the concept of channel coordination to find appropriate wholesale prices that can coordinate the newsvendors and optimize the supply chain profit.

1.2 Two-newsvendor Game with Limited Supply and Inventory Transshipment

The only difference of this model from the game in Section 1.1 is that, surplus inventory will be transshipped between newsvendors to meet as much demand as possible. That is, a newsvendor that is stocked out will buy surplus inventory, if any, from the other newsvendor and sell to his otherwise lost customers. The newsvendor with surplus inventory charges a transshipment price to the stock-out newsvendor and pays the logistics cost. Effectively, transshipment prices allocate to the newsvendors the extra profit generated from selling one newsvendor’s surplus inventory to the other newsvendor’s unmet customers. The transshipment prices are assumed to be exogenous and fixed. Therefore, a newsvendor can only decide his own order quantity.
After inventory transshipment, leftover inventory will be salvaged.

In this game, newsvendors face a more complicated situation. Besides supply capacity competition, inventory transshipment makes a second opportunity for each newsvendor to meet the demand in excess of his initial inventory level. The amount of transshipped inventory between the newsvendors is dependent on inventory allocation and demand realization.

A pure Nash equilibrium is not guaranteed to exist in this game. We provide sufficient conditions such that there exists a Nash equilibrium. Under these conditions, we characterize each newsvendor’s best response, which has different forms when the supply capacity is in different ranges. Subsequently, we calculate the Nash equilibrium explicitly and identify the range of supply capacity when the Nash equilibrium is unique.

Assuming the Nash equilibrium is unique and newsvendors place orders accordingly, we study the relation between the supply capacity and newsvendors’ total profit by considering two special cases. One case is for two symmetric newsvendors and the other case is for two symmetric newsvendors except one has a stochastically larger demand than the other. We prove that the total profit of the newsvendors is non-decreasing in the supply capacity, which implies that a larger supply may result in more profit for the newsvendors.

Compared with the equilibrium profits of newsvendors in the two-newsvendor game with limited supply as in Section 1.1, we find that it is not guaranteed that both newsvendors will be better off with inventory transshipment. One of the reasons is that one of the newsvendors may be allocated less inventory and he is not fully compensated by the profit from transshipment. Actually, at least one of the newsvendors is better off, and the largest possible total newsvendor profit is also more than that in the game without inventory transshipment. Therefore, an ideal scenario is that the total newsvendor profit is maximized and both newsvendors receive more than that in the game without inventory transshipment.

To achieve this ideal outcome, we develop a coordinating contract by allowing newsvendors to negotiate transshipment prices before placing orders. Consequently, the transshipment prices can vary according to the newsvendors’ orders. Under this contract, each newsvendor receives a
fixed proportion of their total profits that corresponds to his bargaining power. Therefore, the Nash equilibrium that yields largest profit for each newsvendor will maximize the newsvendors' total profits. Moreover, the supply chain profit will also be maximized as long as the wholesale prices fall in a certain range.

1.3 Multi-newsvendor Game with Limited Supply

To make our study more general and practical, we investigate a multi-newsvendor game with limited supply. When there is no inventory transshipment, the analysis and results in the two-newsvendor game with limited supply (Section 1.1) can be extended without much difficulty. A general solution method for the Nash equilibrium under general allocation rules is provided. We also derive the explicit forms of the Nash equilibrium of orders under proportional rule. Based on the properties of the Nash equilibrium, we develop a procedure to determine the wholesale prices that can coordinate the newsvendors in maximizing the supply chain profit.

1.4 Multi-newsvendor Games with Limited Supply and Inventory Transshipment

In a multi-newsvendor game with limited supply and inventory transshipment, one of the new difficulties is to determine the transshipping pattern of surplus inventory. When there are only two newsvendors, the surplus inventory at one newsvendor can only be transshipped to the other newsvendor. When there are more than two newsvendors, the surplus inventory may have more than one destination to be matched. Similarly, when a newsvendor experiences a stock out, he may have multiple choices of newsvendors from whom he can buy surplus inventory.

We take two approaches for modeling this problem. One is that every newsvendor specifies a deterministic portion of his excess demand or surplus inventory that he would share with each of the other newsvendors. The existence conditions for the Nash equilibrium are investigated. This model does not necessarily generate the largest transshipment profit for the supply
chain. The other approach is to transship the inventory in a way that maximizes total trans-
shipment profit. We develop a transshipment fund mechanism by which a fund collects initial
payments from newsvendors before they place orders and redistributes the transshipment profit
after inventory transshipment. We show that this mechanism can coordinate newsvendors with
payment methods being properly set.

1.5 Organization of the Dissertation

The rest of this dissertation is organized as follows. In Chapter 2, we review the existing litera-
ture that is relevant to our research. Chapter 3 studies the two-newsvendor game with limited
supply but without inventory transshipment. Chapter 4 considers inventory transshipment and
studies the two-newsvendor game with limited supply and inventory transshipment. Chapter 5
generalizes the game in Chapter 3 to the case of more than two newsvendors. Chapter 6 inves-
tigates multi-newsvendor games with limited supply and inventory transshipment. Chapter 7
concludes the dissertation and discusses future research directions.
Chapter 2

Literature Review

In this section, we review the literature that is related to our work. In Section 2.1, we briefly review game-theoretic models in supply chain management. In Section 2.2, we review the capacity allocation problem and briefly point out the difference between this dissertation and literature. In Section 2.3, inventory transshipment problem is reviewed. In Section 2.4, we review the coordination contracts in supply chain management that are closely related to this dissertation.

2.1 Game Theory in Supply Chain Management

Game theory analyzes the interaction among multiple parties in a decision process where each party’s payoff is affected by the other parties. Early developments of game theory include von Neumann and Morgenstern’s book “Theory of Games and Economic Behavior” (von Neumann and Morgenstern (1944)), Nash equilibrium (Nash (1950a)), cooperative games (Aumann (1959), Shapley (1953)) and auctions (Vickrey (1961)). In supply chain management, game theory has enjoyed numerous applications. Review papers include Mesterton-Gibbons (1992), Cachon and Netessine (2004), Nagarajan and Sošić (2008), Dror and Hartman (2011), Leng and Parlar (2005), etc. We review some important game models and their application in supply chain management.
In competitive (or non-cooperative) games, each player makes decision with the purpose of maximizing his/her own profit. When the players make decisions simultaneously, Nash equilibrium characterizes a situation that no player will obtain a better payoff by changing his/her own decision unilaterally. The existence and uniqueness of Nash equilibrium is of central importance. In a two-echelon supply chain, when a retailer sources his inventory from multiple suppliers, Li et al. (2010) investigate the strategies of both the retailer and suppliers. For a supply chain with multiple competing retailers, Bernstein and Federgruen (2005) investigate their equilibrium behavior. Cai et al. (2009) investigate the price discount contracts and pricing schemes on retailer competition. Parlar (1988) models two retailers with substitutable products as a competitive game. Lippman and McCardle (1997) study multiple competitive newsvendors when residual demand is allocated among them. Mahajan and van Ryzin (2001) accommodate customers’ dynamically choosing behavior. Supermodular games (Topkis (1998)) are often seen in supply chain models. Netessine and Shumsky (2005) investigate the supermodularity in an airline revenue management game. Dai et al. (2006) study the capacity allocation problem for competitive firms that have a local store and an online store. Moreover, Dai et al. (2005b) model multiple competing firms who make pricing decisions in a supermodular game. Other studies of competitive game models in supply chain management include Hall and Porteus (2000), Van Mieghem (1999), Cachon and Zipkin (1999), Dai (2003), etc.

Stackelberg game characterizes the situation where the players take actions sequentially. The player that makes decision first is called a leader while the other player is a follower. Lariviere and Porteus (2001) study the case with a wholesaler who sets the wholesale price and a newsvendor who decides his order quantity. Esmaeili et al. (2009) discuss a buyer-Stackelberg game and a seller-Stackelberg game model for a seller-buyer supply chain. Wang and Gerchak (2003) consider a decentralized assembly system where the assembler and supplier play a Stackelberg game. Chen et al. (2013) study the pricing policies for substitutable products by formulating a Stackelberg game. An extension of Stackelberg game in multiple periods can be found in Anand et al. (2008).
Cooperative game models are used to study the possible coalitions of the players and the payoff allocation within the coalitions. In a cooperative game, the “core” is an important solution concept which represents a stable payoff allocation of the grand coalition. Whether the core is non-empty becomes one of the central questions in a cooperative game model. Hartman et al. (2000) study a newsvendor centralization game and show it has a non-empty core under certain conditions. Müller et al. (2002) generalize this result and show that the core is always non-empty. Shapley value is another way of allocating payoffs and has seen applications in supply chain management (Kemahlioglu-Ziya and Bartholdi III (2011)). Fiestras-Janeiro et al. (2011) review the cooperative game theory in centralized inventory systems. Brandenburger and Stuart (2007) define the “biform” game (or competitive-cooperative game) which has been successfully adopted in supply chain management. In a biform game, players make decentralized decisions in the first stage (competitive game) and decide profit allocation in the second stage (cooperative game). Biform game models for inventory management problems can be found in Anupindi et al. (2001), Chakravarty and Zhang (2007), Granot and Sošić (2003) and Plambeck and Taylor (2005, 2007).

Bargaining games are used to model the scenario where the players can negotiate with each other about their decisions. Nash bargaining solution (Nash (1950b)) and generalized Nash bargaining solution (Roth (1979)) characterize the outcome of a bargaining game under certain axioms. Hanany and Gerchak (2008) use Nash bargaining solution in studying the inventory pooling contracts between several firms. Hezarkhani and Kubiak (2010a) develop a coordinating mechanism of two newsvendors by applying generalized Nash bargaining solution.

2.2 Capacity Allocation

Research on capacity allocation problems mainly focuses on the retailers’ strategic behaviors and the supply chain performance under different allocation rules. Lee et al. (1997) show that the newsvendors tend to increase their order quantities when capacity becomes uncertain. This “order inflation” phenomenon can result in the bullwhip effect. Assuming each retailer is a
local monopoly and the demand-price relation is linearly deterministic, Cachon and Lariviere (1999a) show that proportional and linear allocation are two “manipulatable” rules that lead the retailers to inflate their orders. Uniform allocation, on the other hand, is one of the “truth-telling” rules that eliminate the order inflation. Nevertheless, Cachon and Lariviere (1999c) show that the truth-telling rules can lower profits of the supplier, the retailers and the supply chain. The supplier tends to choose a higher level of capacity under a manipulable allocation rule than under a truth-telling rule. When each retailer’s demand depends on all retailers’ retail prices, Liu (2012) shows that uniform allocation becomes manipulatable and Cho and Tang (2014) construct a truth-telling “competitive allocation” rule. Other research of capacity allocation include the “turn-and-earn” rule (Cachon and Lariviere (1999b)) and Pareto allocation rules for competing retailers (Furuhata and Zhang (2006)). For a concise and introductory review of capacity allocation and its role in supply chain management, we refer the readers to Lariviere (2010).

This dissertation differs from literature in the following aspects: First, we model the retailers as newsvendors. The retail prices are exogenous and the customer demand is stochastic. Second, we consider individually responsive allocation rules as they preserve the monotonicity of order quantity. Third, we further investigate the benefit of introducing inventory transshipment in capacity allocation problems.

### 2.3 Inventory Transshipment

Supply chains with inventory transshipment have been extensively studied under the implicit assumption that the supplier’s capacity is unlimited. Rudi et al. (2001) consider two newsvendors in a decentralized supply chain where the transshipment prices are exogenous to the newsvendors. There exists a unique Nash equilibrium of ordering decisions in this game. However, the total newsvendor profits are not generally maximized at the equilibrium. It raises the question of whether there exist coordinating transshipment prices such that the order decisions of the decentralized newsvendors will jointly maximize their total profits. Hu et al. (2007) provide the
sufficient and necessary conditions for such coordinating transshipment prices to exist. Shao et al. (2011) show that the supplier prefers a high transshipment price and dealing with centralized newsvendors, whereas the newsvendors prefer a lower transshipment price and making decisions in a decentralized way. Hezarkhani and Kubiak (2010a) design a coordinating contract that can optimize the total newsvendor profits by which the newsvendors are allowed to negotiate a pricing mechanism of the surplus inventory before they place orders. Dong and Rudi (2004) investigate the benefits of transshipment for the manufacturer and retailers when the wholesale prices are exogenous and endogenous, respectively. Hu et al. (2008) consider production uncertainty in inventory and transshipment models. Dong et al. (2012) discuss transshipment incentives in a multi-level supply chain with information asymmetry. Slikker et al. (2005) study the cooperation of newsvendors by coordinating both orders and transshipment.

Transshipment price is not the only method to distribute the profit generated by inventory transshipment. Anupindi et al. (2001) propose a two-stage model of inventory transshipment involving two or more newsvendors. In the first stage, the newsvendors make order decisions unilaterally in a competitive way. In the second stage, the newsvendors cooperate to share the surplus inventory and excess demand through inventory transshipment. The transshipment pattern is determined by a transportation problem and the transshipment profit is allocated according to the dual optimal solutions. Anupindi et al. (2001) prove that the newsvendor profits allocated by this rule is in the core of the cooperative transshipment game. Granot and Sošić (2003) extend this framework to a three-stage model where the newsvendors can decide the portion of surplus inventory and excess demand to be shared before the transshipment stage. Granot and Sošić (2003) find that the dual allocation rule may induce the newsvendors to hold back or even not to share their surplus inventory at all. Shapley value and fraction allocation is then suggested in order to maximize the total profit from transshipment. Huang and Sošić (2010b) compare the performance of dual allocation with constant transshipment price. It turns out that neither allocation method dominates the other. Hezarkhani and Kubiak (2013) address the issue of cooperation costs for transshipment games with identical newsvendors. Herer et al.
(2006) study a transshipment model with one supplier and multiple retailers and show that an order-up-to policy is optimal in certain sense. Banerjee et al. (2003) study the transshipment problem through simulation experiments. For a comprehensive review of inventory transshipment within the same echelon, we refer to Paterson et al. (2011).

It is worth noting that demand substitution is a special case of inventory transshipment. Demand substitution, also called customer search in literature, means that the unsatisfied customers of one retailer change to another retailer instead of waiting for the transshipped inventory. This is equivalent to that the profit from inventory transshipment is fully allocated to the retailer with surplus inventory. Parlar (1988) studies two substitutable products with random demand, which is equivalent to two newsvendors with demand substitution. Parlar (1988) proves that there exists a unique Nash equilibrium in this game. Bassok et al. (1999) study the case where there are multiple demand classes and product classes. Lippman and McCardle (1997) consider more allocation methods of the excess demand and prove that the total inventory level would not be reduced under perfect substitutability. Netessine and Rudi (2003) obtain analytical solutions to centralized and decentralized newsvendors with deterministic fractional substitution. Dai et al. (2005a) use a competitive game model and a Stackelberg game model to study demand substitution under limited supply capacity. Moreover, Qi et al. (2015) compare the cases when the wholesale price is exogenous and endogenous. Avsar and Baykal-Gursoy (2002) apply a stochastic game model to the substitutable demand problem. Zhao and Atkins (2008) and Zhao and Atkins (2009) study the situation in which retailers can decide the retail price and inventory level simultaneously. Jiang and Anupindi (2010) characterize the cases when customer search or retailer search is preferred. Yang and Qin (2007) consider the “virtual” inventory transshipment which to some extent is a type of demand substitution.

The most significant difference between the inventory transshipment problem in this dissertation and in literature is that the supply capacity is limited. This setting considers more realistic scenarios in practice. The joint gaming effects of supply constraint and inventory transshipment may further complicate the decision-making of the newsvendors. One interesting ques-
tion is that whether inventory transshipment benefits both newsvendors, assuming they make decisions as the Nash equilibrium suggests. Moreover, coordinating mechanisms that make both newsvendors benefit from inventory transshipment are highly desired.

2.4 Supply Chain Coordination

It is one of the core objectives in supply chain management to make the decentralized decisions at different parties jointly optimize the whole supply chain in a centralized way. To achieve this, various kinds of contracts are designed for different models in supply chain management literature (Cachon (2003), Hezarkhani and Kubiak (2010b), Hennet and Arda (2008), Krishnan and Winter (2010)).

Wholesale price contracts are well studied in literature and commonly observed in practice for a simple model consisting of one supplier and one retailer. The supplier offers a wholesale price and the retailer decides his/her order quantity. Lariviere and Porteus (2001) model the retailer as a newsvendor. Bresnahan and Reiss (1985) discuss the case where customer demand is deterministic. Generally, if the wholesale price contract coordinates the supply chain, the supplier will have to earn a nonpositive profit (Cho and Gerchak (2005), Bernstein et al. (2006)). However, variants and extensions of wholesale prices contracts improve the supply chain performance. Discussion about the retailer and supplier’s relative power can be found in Messinger and Narasimhan (1995), Ailawadi (2001) and Bloom and Perry (2001). Models of multiple or infinite periods are studied by Anupindi and Bassok (1999), Cachon (2004), etc.

Buy-back contract means that the supplier will subsidize the retailer for the remaining inventory at the end of the period. Research on buy-back contracts include Pasternack (1985), Padmanabhan and Png (1997), Emmons and Gilbert (1998) and Taylor (2002), etc. Revenue-sharing contract specifies that the retailer pays the supplier a fixed part of his/her revenue. Analysis of the revenue-sharing contract can be found in Cachon and Lariviere (2005), Dana Jr and Spier (2001), Giannoccaro and Pontrandolfo (2004), Pasternack (2005), Pasternack (2005), etc. El Ouardighi (2014) compares the wholesale price contract and revenue share contract
between a manufacturer and a supplier in a two-stage game model. Chen et al. (2001) investigate coordination mechanisms for a two-echelon supply chain with one supplier and multiple retailers.

Coordination of two competing newsvendors require that the Nash equilibrium of decentralized orders jointly optimize the supply chain performance. Wholesale price contracts are developed in the case where the retail price is fixed and the demand is allocated between the newsvendors. See Wang and Gerchak (2001), Netessine and Shumsky (2005), Gans (2002), etc. Another type of contracts enjoys more flexibility by allowing the transfer payment between retailers to be contingent on the realization of some random variables after the contract is signed. Consequently, the transfer payment will be a function varying with parameters of the model. Such models are seen in Kouvelis and Lariviere (2000), Agrawal and Tsay (2002), Erkoc and Wu (2002) and Hezarkhani and Kubiak (2010a).
Chapter 3

Two-newsvendor Game with Limited Supply

3.1 Introduction

The newsvendor model is used for a single-period inventory decision problem. A newsvendor sells a product to customers in a given period. The customer demand is stochastic and its distribution is known to the newsvendor. In this dissertation, we assume the demand is characterized by a continuous random variable. That is, it has a probability density function. In this research, we assume that the probability density function is continuous. At the beginning of the period, the newsvendor places an order to the supplier and receives inventory, the same amount as the order quantity, at a wholesale price of \( w \) per unit. Then the demand is observed and the newsvendor sells his inventory to customers at a retail price of \( r \) per unit. The stochastic demand of customer is denoted by \( D \) with cumulative distribution function \( F_D(\cdot) \). The unsatisfied demand is lost at a penalty. Without loss of generality, we assume the lost-sale penalty is zero. Actually, the analysis of a newsvendor with lost-sale penalty \( p \) and retail price \( r \) is similar to a newsvendor with lost-sale penalty 0 and retail price \( r + p \). At the end of the period, the newsvendor disposes unsold inventory, if any, at a unit salvage value \( s \). Assume \( r \geq w \geq s \). In this model, the
newsvendor needs to decide his order quantity (or equivalently, inventory level) such that his expected profit is maximized.

Let $x$ denote the inventory of the newsvendor and $\pi^n(x)$ denote his expected profit function (The superscript “$n$” means “newsvendor”). We have

$$\pi^n(x) = -wx + rE[\min\{x, D\}] + sE[(x - D)^+]$$

where $(\cdot)^+ = \max\{\cdot, 0\}$ and $E[\cdot]$ denotes the expectation of a random variable. It is easy to verify that $\pi^n(x)$ is concave in $x$. The optimal inventory level $x^*$ that maximizes $\pi^n(x)$ is given by

$$x^* = F^{-1}_D \left( \frac{r - w}{r - s} \right)$$

(3.1)

In this chapter, we consider two newsvendors who submit orders to a common supplier and study the situation when the supplier’s capacity is limited. The structure of the supply chain is shown in Figure 3.1.

Figure 3.1: Supply chain structure of two-newsvendor game with limited supply
When the total order quantity of the two newsvendors is no more than the capacity, each newsvendor receives the same amount of inventory as his order. Otherwise, the total order quantity of the two newsvendors exceeds the supply, and the supplier has to allocate the inventory between the newsvendors by a certain allocation rule. Consequently, a newsvendor may receive less inventory than his order quantity. We assume that no newsvendor is allowed to place an order of more than the supply capacity. All information, such as the capacity, allocation rule, and stochastic demand distributions are available to both newsvendors before they make order decisions.

Assume the newsvendors are rational and self-interested. They may tend to manipulate their order quantities to compete for the limited supply. We apply a game theoretic approach to study the interaction of the newsvendors. We will show that the expected profit function of each newsvendor is quasi-concave in his own order decision. Therefore, there always exists a Nash equilibrium in this newsvendor game. We can also obtain the explicit form of the Nash equilibrium as well as each newsvendor’s best response function. The uniqueness of a Nash equilibrium is then characterized. We find that the inventory allocation at the Nash equilibrium is always unique.

One desirable objective in supply chain management is to maximize the supply chain profit, i.e., total profits of the supplier and both newsvendors (Chopra and Meindl (2007), Snyder and Shen (2011)). To achieve this, we use the method of channel coordination to coordinate the newsvendors. Note that the Nash equilibrium of order quantity depends on the wholesale prices. With proper wholesale prices, the Nash equilibrium may coincide with the “first-best” order quantity that maximizes the supply chain profit. In this way, the equilibrium inventory allocation in decentralized control will equal the first-best inventory allocation under centralized control.

The remainder of this chapter is organized as follows. In Section 3.2, we propose the two-newsvendor game with limited supply. The existence and uniqueness of Nash equilibrium of the order quantity are investigated in Section 3.3. In Section 3.4, we discuss the wholesale prices.
that coordinate the newsvendors and maximize the supply chain profit. Section 3.5 summarizes this chapter.

3.2 The Model

Consider two newsvendors who sell identical products and place orders to a common supplier before their stochastic demands are realized. The capacity of the supplier is limited and assumed to be no more than $K > 0$. If the total order quantity of the newsvendors exceeds $K$, the supplier will allocate the supply to the newsvendors by an inventory allocation rule. Let the newsvendors be indexed by $i$ and $j$ with $i, j = 1, 2$ and $i \neq j$. For the newsvendor $i$, we denote his order quantity by $y_i \geq 0$ and the inventory allocation he receives by $x_i \geq 0$. Let $y = (y_1, y_2)$ and $x = (x_1, x_2)$. An inventory allocation rule is a function $\varphi$ that assigns an inventory allocation vector $x$ to each order vector $y$. Denote $x = \varphi(y) = (\varphi_1(y_1, y_2), \varphi_2(y_2, y_1))$ where $\varphi_i(y_i, y_j)$ is newsvendor $i$’s allocated inventory as a function of $y_i$ and $y_j$. In this chapter, we assume the inventory allocation rule $\varphi$ satisfies the following conditions:

(I) (Feasible condition) $\varphi_1(y_1, y_2) + \varphi_2(y_2, y_1) \leq K$ and $0 \leq \varphi_i(y_i, y_j) \leq y_i, i, j = 1, 2, i \neq j$.

(II) (Efficient condition) If $y_1 + y_2 \leq K$, then $\varphi_i(y_i, y_j) = y_i, i, j = 1, 2, i \neq j$. If $y_1 + y_2 > K$, then $\varphi_1(y_1, y_2) + \varphi_2(y_2, y_1) = K$.

(III) (Increasing condition) $\varphi_i(\tilde{y}_i, y_j) \geq \varphi_i(y_i, y_j)$ for all $\tilde{y}_i > y_i$ and $y_j, i, j = 1, 2, i \neq j$.

(IV) (Individually responsive condition) If $0 < \varphi_i(y_i, y_j) < K$ and $\tilde{y}_i > y_i$, then $\varphi_i(\tilde{y}_i, y_j) > \varphi_i(y_i, y_j), i, j = 1, 2, i \neq j$.

(V) (Fair condition) If $y_i \geq y_j$, then $\varphi_i(y_i, y_j) \geq \varphi_j(y_j, y_i), i, j = 1, 2, i \neq j$.

(VI) (Continuous condition) $\varphi_i(y_i, y_j)$ is continuous in $(y_i, y_j), i, j = 1, 2, i \neq j$.

Conditions (I)-(IV) are adopted from Cachon and Lariviere (1999c) and condition (V) is assumed in this dissertation without loss of generality. The proportional rule and the linear rule
are two of the simple and often used inventory allocation rules. They both satisfy the conditions (I)-(VI). For the proportional rule,

\[
\varphi_i(y_i, y_j) = \begin{cases} 
    y_i, & \text{if } y_1 + y_2 \leq K, \\
    \frac{y_i}{y_1 + y_2} K, & \text{if } y_1 + y_2 > K.
\end{cases}
\]

For the linear rule,

\[
\varphi_i(y_i, y_j) = \begin{cases} 
    y_i, & \text{if } y_1 + y_2 \leq K, \\
    y_i - \frac{y_1 + y_2 - K}{2}, & \text{if } y_1 + y_2 > K.
\end{cases}
\]

The supplier announces the supply capacity \( K \) and the inventory allocation rule \( \varphi \) before the newsvendors place orders. Hence, \( K \) and \( \varphi \) are exogenous for the newsvendors. We assume that the newsvendors are not allowed to place arbitrarily large orders that may distort the picture. Without loss of generality, we assume that neither of the newsvendors can place an order larger than \( K \).

The newsvendors pay wholesale prices \( w_i \) for the received inventories and sell to customers at retail prices \( r_i \) for \( i = 1, 2 \). We assume the customer demand is stochastic and continuously distributed. Let \( D_i \) denote the demand of the newsvendor \( i \) with a cumulative distribution function \( F_{D_i} \) known to both newsvendors. Assume \( F_{D_i} \) is strictly increasing in the support of \( D_i \). In other words, the probability density function \( f_{D_i}(u) > 0 \) for every \( u \) in the support of \( D_i \). In this study, \( D_i \) and \( D_j \) are not required to be mutually independent in this study. At the end of the period, any leftover inventory at the newsvendors will be disposed at salvage values \( s_i \) and \( s_j \), respectively.

In this model, \( w_i, r_i \) and \( s_i, i = 1, 2 \), are exogenous information for both newsvendors. We make the natural assumption that \( r_i > w_i > s_i, i = 1, 2 \). Moreover, we assume the lost-sale penalty to be zero. Actually, it is easy to verify that the analysis for a model with lost-sale penalty \( p_i > 0 \) and retail price \( r_i \) is similar to that with lost-sale penalty 0 and retail price
Both newsvendors are assumed to have complete information about the parameters of the game, and they are rational with respective objectives to maximize their own profits by making ordering decisions unilaterally and simultaneously. We call this model a two-newsvendor game with limited supply.

### 3.3 Game Theoretic Analysis

In this section, we study the existence and uniqueness of Nash equilibrium of ordering decisions in the two-newsvendor game with limited supply. Let $\pi^n_i(x_i)$ denote the expected profit of the newsvendor $i$ as a function of the inventory allocation $x_i$ that he receives. (We use the superscript “$n$” to denote “newsvendor” game without inventory transshipment, to be distinguished with later chapters.) We use $\Pi^n_i(y_i, y_j)$ to denote the expected profit of the newsvendor $i$ with respect to order quantities $y_i$ and $y_j$. Then we have

$$
\pi^n_i(x_i) = -w_i x_i + r_i E[\min\{x_i, D_i\}] + s_i E[(x_i - D_i)^+]
$$

and

$$
\Pi^n_i(y_i, y_j) = \pi^n_i(\varphi_i(y_i, y_j)).
$$

Note that the profit of each newsvendor depends on the ordering decisions of both newsvendors.

In this dissertation, we only consider pure-strategy Nash equilibrium. From game theory (for example Theorem 1.2 in Fudenberg and Tirole (1991)), we know that there exists a (pure-strategy) Nash equilibrium in this game if (i) the decision space for each newsvendor is compact; (ii) the profit function $\Pi^n_i(y_i, y_j)$ is continuous in $y_i$ for any $y_j$, $i, j = 1, 2, i \neq j$; and (iii) $\Pi^n_i(y_i, y_j)$ is quasi-concave in $y_i$ for any $y_j$, $i, j = 1, 2, i \neq j$. Obviously, conditions (i) and (ii) are satisfied.

As to the quasi-concavity of $\Pi^n_i(y_i, y_j)$, we have the next Theorem.
Theorem 3.1. In the two-newsvendor game with limited supply, the newsvendor $i$’s profit function $\Pi^i_n(y_i, y_j)$ is quasi-concave with respect to $y_i$ for any given $0 \leq y_j \leq K$.

Proof. For a given $y_j$, $\Pi^i_n(y_i, y_j) = \pi_i(\varphi^i_n(y_i, y_j))$ is the composite of a concave function $\pi^i_n(x_i)$ with a non-decreasing function $\varphi_i(y_i, y_j)$ with respect to $y_i$. Therefore, $\Pi^i_n(y_i, y_j)$ is quasi-concave with respect to $y_i$.

Consequently, we have the next theorem about the existence of Nash equilibrium.

Theorem 3.2. There exists a Nash equilibrium in the two-newsvendor game with limited supply.

Proof. The strategy space for each newsvendor is the closed finite interval $[0, K]$. Note the continuous condition (VI) of the function $\varphi$, the continuity of $\Pi^i_n(y_i, y_j)$ is clear. The quasi-concavity of $\Pi^i_n(y_i, y_j)$ is proved by Theorem 3.1. By Theorem 1.2 in Fudenberg and Tirole (1991), the existence of Nash equilibrium follows.

We use $y^{N,n} = (y_1^{N,n}, y_2^{N,n})$ to denote the Nash equilibrium of order quantity and use $x^{N,n} = (x_1^{N,n}, x_2^{N,n}) = (\varphi_1(y_1^{N,n}, y_2^{N,n}), \varphi_2(y_2^{N,n}, y_1^{N,n}))$ to denote the equilibrium inventory allocation corresponding to $y^{N,n}$. (The superscript “N” denotes “Nash equilibrium”.) Given $0 \leq x_i \leq K$, we define $\psi_i(y_j|x_i)$, a function with respect to $y_j$, as follows. If $\varphi_i(K, y_j) \geq x_i$, then $\psi_i(y_j|x_i)$ is defined to be the solution $y_i$ to the equation $\varphi_i(y_i, y_j) = x_i$, which represents the quantity that the newsvendor $i$ should order to receive a desired amount $x_i$. If $\varphi_i(K, y_j) < x_i$, then let $\psi_i(y_j|x_i) = K$, which is the largest quality that the newsvendor $i$ can order in order to receive as close to $x_i$ as possible. $\psi_i(y_j|x_i)$ is well defined because $x_i \geq 0$ and $\varphi$ satisfies the increasing condition (III) and the individually responsive condition (IV). For example, if the supply is allocated by the proportional rule, then

$$
\psi_i(y_j|x_i) = \begin{cases} 
  x_i, & \text{if } 0 \leq x_i \leq K - y_j, \\
  \frac{y_j}{K - x_i} x_i, & \text{if } K - y_j < x_i \leq \frac{K^2}{K + y_j}, \\
  K, & \text{if } \frac{K^2}{K + y_j} < x_i \leq K.
\end{cases}
$$
For linear rule,

\[
\psi_i(y_j|x_i) = \begin{cases} 
  x_i, & \text{if } 0 \leq x_i \leq K - y_j, \\
  2x_i + y_j - K, & \text{if } K - y_j < x_i \leq K - \frac{y_j}{2}, \\
  K, & \text{if } K - \frac{y_j}{2} < x_i \leq K. 
\end{cases}
\]

To obtain the explicit form of the Nash equilibrium and study its uniqueness, we need to characterize the best response of each newsvendor. For any given ordering decision \(y_j\) of the newsvendor \(j\), the best response of the newsvendor \(i\) is defined as the ordering decision \(y_i\) that maximizes newsvendor \(i\)'s profit \(\Pi^i(y_i, y_j)\). Denote the best response of the newsvendor \(i\) with respect to the newsvendor \(j\)'s ordering decision by \(R^i(y_j)\). Therefore, \(R^i(y_j) = \arg\max_{0 \leq y_i \leq K} \Pi^i(y_i, y_j)\). Let \(x^*_i, i = 1, 2\), denote the desired amount of inventory of the newsvendor \(i\), which is determined by Equation (3.1), i.e.,

\[
x^*_i = F^{-1}_{D_i} \left( \frac{r_i - c_i}{r_i - s_i} \right).
\]

The next lemma gives the expression of \(R^i(y_j)\).

**Lemma 3.1.** In the two-newsvendor game with limited supply, the best response \(R^i(y_j)\) of the newsvendor \(i\) is of one of the following two cases:

(Case 1) If \(x^*_i < K\), then

\[
R^i(y_j) = \begin{cases} 
  x^*_i, & 0 \leq y_j \leq K - x^*_i, \\
  \psi_i(y_j|x^*_i), & K - x^*_i < y_j \leq \psi_j(K|K - x^*_i) \\
  K, & \psi_j(K|K - x^*_i) < y_j \leq K, 
\end{cases}
\]

(Case 2) If \(x^*_i \geq K\), then

\[
R^i(y_j) = K, \text{ for } 0 \leq y_j \leq K.
\]
Proof. Note that the newsvendor $i$ would like to have $x^*_i$ and his order is bounded by the supply capacity $K$. Therefore, $R^i_n(y_j) = \min\{\psi_i(y_j|x^*_i), K\}$. Consider the case that $x^*_i < K$. When $0 \leq y_j \leq K - x^*_i$, the newsvendor $i$ may place an order of $x^*_i$ and he will receive exactly $x^*_i$ because the supply $K$ is abundant. As $y_j$ increases, the newsvendor $i$ should manipulate his ordering decision by the function $\psi_i(y_j|x^*_i)$ to receive an inventory allocation of $x^*_i$. When $y_j > \psi_j(K|x^*_i)$, we know $\varphi_i(K, y_j) < x^*_i$ and the newsvendor $i$ is not allowed to order more than $K$. Consequently, he will order $K$.

For Case 2, the newsvendor will order as much as he can, which is $K$. □

With $R^i_n(y_j)$ being fully characterized, we can further provide the explicit form of the Nash equilibrium $y^{N,n} = (y^{N,n}_1, y^{N,n}_2)$ and equilibrium inventory allocation $x^{N,n} = (x^{N,n}_1, x^{N,n}_2)$. As we show in Table 3.1, the forms of $y^{N,n}$ and $x^{N,n}$ depend on the relations among $x^*_1, x^*_2$ and $K$.

<table>
<thead>
<tr>
<th>Case</th>
<th>$(y^{N,n}_1, y^{N,n}_2)$</th>
<th>$(x^{N,n}_1, x^{N,n}_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^<em>_1 + x^</em>_2 &lt; K$</td>
<td>$(x^<em>_1, x^</em>_2)$</td>
<td>$(x^<em>_1, x^</em>_2)$</td>
</tr>
<tr>
<td>$x^<em>_1 + x^</em>_2 = K$</td>
<td>$(y_1, \psi_2(y_1</td>
<td>x^<em>_2)), x_1^</em> \leq y_1 \leq K$</td>
</tr>
<tr>
<td>$x^<em>_1 + x^</em>_2 &gt; K$</td>
<td>$\min{x^<em>_1, x^</em>_2} \geq \frac{K}{2}$</td>
<td>$(K, K)$</td>
</tr>
<tr>
<td></td>
<td>$x^*_1 &lt; \frac{K}{2}$</td>
<td>$(\psi_1(K</td>
</tr>
<tr>
<td></td>
<td>$x^*_2 &lt; \frac{K}{2}$</td>
<td>$(K, \psi_2(K</td>
</tr>
</tbody>
</table>

Table 3.1: Nash equilibrium of the two-newsvendor game with limited supply

Summarizing Table 3.1, we have the next theorem about the uniqueness of the Nash equilibrium and equilibrium inventory allocation.

**Theorem 3.3.** In the two-newsvendor game with limited supply, there exists a unique Nash equilibrium unless $x^*_1 + x^*_2 = K$, in which case there are infinitely many Nash equilibria. The equilibrium inventory allocation is always unique.

We illustrate this game by the next two examples.
Example 3.3.1. Suppose \( w_1 = w_2 = 9, r_1 = r_2 = 10, s_1 = s_2 = 1 \) and the supplier uses the proportional allocation rule. Suppose the demand of the newsvendor 1 is uniformly distributed between 4 and 13, i.e., \( D_1 \sim U[4, 13] \). Suppose \( D_2 \sim U[2, 11] \) and \( K = 7 \). We have \( x^*_1 = 5 \) and \( x^*_2 = 3 \). Obviously, \( x^*_1 + x^*_2 > K \) and \( x^*_2 < \frac{K}{2} \). Therefore, \( y_{1}^{N,n} = 7, y_{2}^{N,n} = 5.25 \) is the unique Nash equilibrium. The inventory allocation at the Nash equilibrium is \( x_1^{N,n} = 4, x_2^{N,n} = 3 \).

Figure 3.2 shows the best responses of the newsvendors and the Nash equilibrium.

![Best responses and Nash equilibrium](attachment:image.png)

Figure 3.2: Best responses and Nash equilibrium in Example 3.3.1

Example 3.3.2. Suppose \( K = 8 \) in Example 3.3.1. Then, \( x^*_1 + x^*_2 = K \). Therefore, there exist infinitely many Nash equilibria forming the line segment \( \{(y_{1}^{N,n}, y_{2}^{N,n})|5 \leq y_{1}^{N,n} \leq 8, y_{2}^{N,n} = \frac{3}{5}y_{1}^{N,n}\} \). Nevertheless, the inventory allocation \( x_1^{N,n} = 5, x_2^{N,n} = 3 \) is the same at all Nash equilibria.

Figure 3.3 shows the best responses of the newsvendors and the Nash equilibria.
3.4 Channel Coordination

In this section, we apply the concept of channel coordination to find the coordinating wholesale prices with which the supply chain profit (total profit of both newsvendors and the supplier) can be maximized. First, we find the first-best inventory allocation if the supply chain is under centralized control. Then, we investigate what the wholesale prices $w_1$ and $w_2$ can be chosen so that the equilibrium inventory allocation $(x_{1,n}^{N,n}, x_{2,n}^{N,n})$, which depends on $w_1$ and $w_2$, will coincide with the first best allocation under centralized control.

3.4.1 Centralized Control

By “centralized control”, we suppose the order quantities of both newsvendors are determined by the same decision maker. Suppose the supplier incurs the production unit cost of $w_0$ and assume $w_0 > s_i, i = 1, 2$. We use $\pi_0(x)$ to denote the profit of the supplier when the inventory
allocation of newsvendors is \( x \). Then

\[
\pi_0(x) = (w_1 - w_0)x_1 + (w_2 - w_0)x_2.
\]

Let \( \pi_C(x) \) denote the supply chain expected profit when the inventory allocation of newsvendors is \( x \). (The subscript “C” means “Chain”.) Then

\[
\pi_C^n(x) = \pi_0(x) + \pi_1^n(x_1) + \pi_2^n(x_2).
\]

Note that \( \pi_C^n(x_1, x_2) \) is concave with respect to \( (x_1, x_2) \) because it is the sum of three concave functions. Let \( x_{C,n} = (x_{1,n}, x_{2,n}) \) be the centralized first-best inventory allocation which maximize \( \pi_C^n(x) \). (The superscript “C” means chain-optimal.) That means that \( (x_{1,n}, x_{2,n}) \) is an optimal solution of the following optimization problem \( (P_n) \):

\[
\max \quad \pi_C^n(x) \\
\text{s.t.} \quad x_1 + x_2 \leq K \\
\quad \quad \quad x_1, x_2 \geq 0.
\]

where \( x_1 \) and \( x_2 \) are decision variables. The KKT conditions of \( (P_n) \) are

\[
\begin{align*}
  r_1 - w_0 - (r_1 - s_1)F_{D_1}(x_1) - \mu + \lambda_1 &= 0, \\
  r_2 - w_0 - (r_2 - s_2)F_{D_2}(x_2) - \mu + \lambda_2 &= 0, \\
  \mu(x_1 + x_2 - K) &= 0, \\
  \lambda_1 x_1 &= 0, \\
  \lambda_2 x_2 &= 0, \\
  0 \leq x_1 &\leq K, \\
  0 \leq x_2 &\leq K.
\end{align*}
\]

Note the objective function \( \pi_C^n(x) \) is concave and the feasible domain is a polyhedron. The
KKT conditions are sufficient and necessary for \((x_1, x_2)\) to be optimal. Therefore, the first-best inventory allocation \(x_{C,n} = (x_{C,n}^1, x_{C,n}^2)\) can be one of the four cases characterized in the next theorem.

**Theorem 3.4.** In the two-newsvendor game with limited supply, the first-best inventory allocation \((x_{C,n}^1, x_{C,n}^2)\) is of one of the following cases:

(Case 1)

\[
\begin{align*}
& r_1 - w_0 - (r_1 - s_1)F_{D_1}(x_{C,n}^1) = 0, \\
& r_2 - w_0 - (r_2 - s_2)F_{D_2}(x_{C,n}^2) = 0, \\
& x_{C,n}^1 + x_{C,n}^2 < K, \\
& 0 \leq x_{C,n}^1 \leq K, \\
& 0 \leq x_{C,n}^2 \leq K.
\end{align*}
\]

(Case 2)

\[
\begin{align*}
& r_1 - (r_1 - s_1)F_{D_1}(x_{C,n}^1) = r_2 - (r_2 - s_2)F_{D_2}(x_{C,n}^2), \\
& x_{C,n}^1 + x_{C,n}^2 = K, \\
& x_{C,n}^1 > 0, \\
& x_{C,n}^2 > 0.
\end{align*}
\]

(Case 3)

\[
\begin{align*}
& r_2 - (r_2 - s_2)F_{D_2}(x_{C,n}^2) \geq r_1 - (r_1 - s_1)F_{D_1}(x_{C,n}^1), \\
& x_{C,n}^1 = 0, \\
& x_{C,n}^2 = K.
\end{align*}
\]

(Case 4)

\[
\begin{align*}
& r_1 - (r_1 - s_1)F_{D_1}(x_{C,n}^1) \geq r_2 - (r_2 - s_2)F_{D_2}(x_{C,n}^2), \\
& x_{C,n}^1 = K, \\
& x_{C,n}^2 = 0.
\end{align*}
\]
3.4.2 Coordinating Wholesale Prices

When the newsvendors make decentralized decisions, the equilibrium inventory $x^{N,n}$ is dependent on the wholesale prices $w_1$ and $w_2$. In this section, we rewrite $x^{N,n}$ as $x^{N,n}(w_1, w_2)$ to indicate this dependency relation. We investigate the wholesale prices $w_1$ and $w_2$ such that the equilibrium inventory allocation under decentralized control will be equal to the first-best inventory allocation under centralized control, i.e., $x^{N,n}(w_1, w_2) = x^{C,n}$. Such $w_1$ and $w_2$ are called coordinating wholesale prices and we denote them by $w_1^{C,n}$ and $w_2^{C,n}$.

Note that the explicit form of $x^{N,n}$ has been expressed in different cases. Based on the value of $x^{C,n}$, we can locate the corresponding case of $x^{N,n}$. Then, we can find $w_1^{C,n}$ and $w_2^{C,n}$ by solving $x^{N,n}(w_1, w_2) = x^{C,n}$. To be specific, the next theorem gives the coordinating wholesale prices.

**Theorem 3.5.** In the two-newsvendor game with limited supply, the coordinating wholesale prices $w_1^{C,n}$ and $w_2^{C,n}$ are determined as shown in Table 3.2.

<table>
<thead>
<tr>
<th>Case</th>
<th>$x_1^{C,n} + x_2^{C,n} &lt; K$</th>
<th>$x_1^{C,n} = 0, x_2^{C,n} = K$</th>
<th>$x_1^{C,n} = 0, x_2^{C,n} = K$</th>
<th>$x_1^{C,n} = K, x_2^{C,n} = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(x_1^<em>, x_2^</em>)$</td>
<td>$(x_1^<em>, K - x_1^</em>)$</td>
<td>$(x_1^<em>, K - x_1^</em>)$</td>
<td>$(K - x_2^<em>, x_2^</em>)$</td>
</tr>
<tr>
<td></td>
<td>$w_1^{C,n} = w_0$</td>
<td>$w_1^{C,n} = [w_0, r_2 - (r_2 - s_2)F_D(K)]$</td>
<td>$w_1^{C,n} = r_1$</td>
<td>$w_1^{C,n} = r_1$</td>
</tr>
<tr>
<td></td>
<td>$w_2^{C,n} = w_0$</td>
<td>$w_2^{C,n} = [w_0, r_2 - (r_2 - s_2)F_D(x_2^{C,n})]$</td>
<td>$w_2^{C,n} = [w_0, r_2 - (r_2 - s_2)F_D(x_2^{C,n})]$</td>
<td>$w_2^{C,n} = r_2$</td>
</tr>
</tbody>
</table>

**Proof.** If $x_1^{C,n} + x_2^{C,n} < K$, we know the equilibrium inventory allocation must be $(x_1^*, x_2^*)$. Hence, it holds that $\frac{\partial x^{n,c}}{\partial x_i} = \frac{\partial x^{n,c}}{\partial x_i} = 0$ for $i = 1, 2$. Therefore, $w_1^{C,n} = w_2^{C,n} = w_0$.

If $x_1^{C,n} = 0, x_2^{C,n} = K$, in order to have $x^{N,n}(w_1, w_2) = x^{C,n}$, the equilibrium inventory allocation must be $(x_1^*, K - x_1^*)$. Therefore, $x_1^* = 0$ and consequently, $w_1^{C,n} = r_1$. Moreover,
\( x_2^* \geq K \) leads to \( w_0 \leq w_2^{C,n} \leq r_2 - (r_2 - s_2)F_{D_2}(K) \).

If \( x_1^{C,n} + x_2^{C,n} = K \) and \( 0 < x_1^{C,n} < \frac{K}{2}, \frac{K}{2} < x_2^{C,n} < K \), in order to have \( x^{N,n}(w_1, w_2) = x^{C,n} \), the equilibrium inventory allocation must be \( (x_1^*, K - x_1^*) \). Therefore, it holds that \( r_1 - w_1^{C,n} - (r_1 - s_1)F_{D_1}(x_1^{C,n}) = 0 \) and then \( w_1^{C,n} = r_1 - (r_1 - s_1)F_{D_1}(x_1^{C,n}) \). The wholesale price \( w_2^{C,n} \) should satisfy that \( x_2^*(w_2^{C,n}) \geq x_2^{C,n} \). Then \( w_0 \leq w_2^{C,n} \leq r_2 - (r_2 - s_2)F_{D_2}(x_2^{C,n}) \).

The other cases can be analyzed similarly.

Note that in some cases, the coordinating wholesale price \( w_i^{C,n} \) may be chosen from an interval \([w_0, r_i - (r_i - s_i)F_{D_i}(\frac{K}{2})]\) rather than as a single value. This enables the supplier to adjust the share of the profit between herself and newsvendor \( i \). The smaller \( w_i^{C,n} \) is, the larger portion the newsvendor \( i \) will receive. The larger \( w_i^{C,n} \) is, the more the supplier will keep.

We use the next example to demonstrate the channel coordination.

**Example 3.4.1.** Suppose \( w_0 = 8 \) in Example 3.3.1. Solve problem \( P_n \) by checking the cases in Theorem 3.4 and we have \( x_1^{C,n} = 4.5 \) and \( x_2^{C,n} = 2.5 \). According to Table 3.2, the coordinating wholesale prices can be set as

\[
8 \leq w_1^{C,n} \leq 9.5 \quad \text{and} \quad w_2^{C,n} = 9.5.
\]

Note that if \( w_1^{C,n} = 8 \), the supplier does not make profit by selling to newsvendor 1. If \( w_1^{C,n} = 9.5 \), the supplier may make 1.5 per unit she sells to newsvendor 1, which is the largest possible unit profit if she wants the supply chain profit to be maximized at the same time.

### 3.5 Summary

We have proposed a game model to study two newsvendors with limited supply. In this game, the Nash equilibrium always exists. The explicit form of the Nash equilibrium can be calculated for any given inventory allocation rule. Accordingly, the uniqueness of the Nash equilibrium has been studied. To coordinate the newsvendors and maximize the supply chain profit, we discuss
appropriate wholesale prices that may make the equilibrium inventory allocation be equal to the first-best inventory allocation. This chapter formulates a basic model for newsvendors with limited supply. In the next chapter, we will investigate the effect of inventory transshipment.
Chapter 4

Two-newsvendor Game with Limited Supply and Inventory Transshipment

4.1 Introduction

When the stochastic demand is realized, a newsvendor may face stockouts while the other newsvendor could have surplus inventory. The inventory that is transshipped from a newsvendor who holds surplus inventory to a newsvendor with excess demand, may help satisfy more demand and generate extra profit to the supply chain. To divide this extra profit, the newsvendor with surplus inventory charges a transshipment price to the newsvendor with excess demand.

Inventory transshipment is commonly conducted in apparel, fashion goods (Bitran et al. (1998)), service parts, toys and electronics by companies such as Foot Locker, Ingram Micro and Teknosa (Özdemir et al. (2013)). Models of newsvendors with inventory transshipment have been studied by many researchers, including Rudi et al. (2001), Hu et al. (2007), Hezarkhani and Kubiak (2010a), Anupindi et al. (2001), Granot and Sošić (2003), Hanany et al. (2010), Huang and Sošić (2010a), and Huang and Sošić (2010b). An implicit assumption in these models is...
that, the newsvendors are not constrained by the product availability. That is, the newsvendors can stock their preferred amount of inventory.

In this chapter, we incorporate inventory transshipment into the game of two newsvendors with limited supply. The gaming effects of both limited supply and inventory transshipment further complicates the decision making of the newsvendors. As Özdemir et al. (2013) indicate, the problem of transshipment under limited supply is also motivated by the European banking system, where banks (retailers) not only borrow cash from the European central bank (supplier) whose money supply is tight and only available on a weekly or monthly basis, but also borrow from other banks (transshipment) to meet their short-term needs. The supply chain structure is shown in Figure 4.1.

![Figure 4.1: Supply chain structure of two-newsvendor game with limited supply and inventory transshipment](image)

Assuming the transshipment prices are exogenous and fixed, we investigate properties of the two-newsvendor game with limited supply and inventory transshipment. Unlike the case of unlimited supply studied by Rudi et al. (2001), we note that a pure Nash equilibrium of ordering decisions is not guaranteed to exist. Actually, the expected profit of each newsvendor may not
be quasi-concave in his own ordering decision. We provide sufficient conditions for the existence of a Nash equilibrium. These conditions reflect the structural effect of the supply constraint on each newsvendor’s profit and best response function. We characterize each newsvendor’s best response function and find its form related to the limited supply, and essentially, the relative gaming effects of inventory transshipment and limited supply. We also identify the ranges of supply capacity such that the Nash equilibrium is unique. In other ranges, there are infinitely many Nash equilibria.

Moreover, we study the effect of the limited supply on total newsvendor profit at the Nash equilibrium by considering two special cases. The first case is for two symmetric newsvendors (that is, they are the same in all prices and demand.). We show that both newsvendors ordering the same quantity forms a Pareto Nash equilibrium and both newsvendors will order equally in this case. The second case is two symmetric newsvendors but one has a stochastically larger demand than the other. In this case we only study the range of supply capacity in which there is a unique Nash equilibrium. For both cases, we prove that the total newsvendor profit is non-decreasing as the supply capacity increases.

When the supply is unlimited, Hu et al. (2007) show that in some cases there exist transshipment prices such that the newsvendors are coordinated in the sense that the Nash equilibrium maximizes the total profit of newsvendors. We call this Nash equilibrium “newsvendor-best”. If the supply is limited and the newsvendors order the newsvendor-best inventory stocks under unlimited supply, we investigate what inventory allocation rules the supplier may choose so that the inventory allocation continues to maximize the total newsvendor profit. We study special cases when the newsvendors are nearly symmetric in price parameters. Interestingly, we find that if the demands of the newsvendors are both uniformly distributed with lower bound zero, then the supplier should choose the proportional allocation rule.

Finally, we investigate whether inventory transshipment is beneficial to both newsvendors. The benchmark model is the two-newsvendor game with the same supply limit but without inventory transshipment, which is studied in Chapter 3. We compare the equilibrium profits
of both newsvendors with and without inventory transshipment. If the supply is unlimited, it is obvious that both newsvendors receive more profit with inventory transshipment. If the supply is limited, however, only one of the newsvendors is guaranteed to be better off in the game with inventory transshipment. Actually, a newsvendor may be worse off in the game with transshipment when his inventory allocation is less than in the game without transshipment and the consequent profit loss is not fully compensated by inventory transshipment. This means that inventory transshipment may not always benefit both newsvendors when the extra profit of inventory transshipment is not properly divided through transshipment prices. The ideal scenario would be that the total newsvendor profit is maximized and both newsvendors receive more profit than the case without inventory transshipment. To accomplish this, we consider and extend the coordinating contract proposed by Hezarkhani and Kubiak (2010a). In this contract, the newsvendors may negotiate and agree a transshipment pricing mechanism according to possible order quantities before they place orders. This mechanism is determined by using the general Nash bargaining solution. When the newsvendors fail to reach an agreement, the disagreement outcome becomes the equilibrium profits of the game without transshipment. We show that results similar to Hezarkhani and Kubiak (2010a) hold. As a consequence, the inventory allocation that maximizes the total newsvendor profit becomes a Nash equilibrium because it yields the largest profit for each newsvendor. Moreover, both newsvendors are better off than in the game without inventory transshipment. As for the supply chain profit, we prove that as long as the wholesale prices fall in a certain range, the newsvendor-best allocation also maximizes the supply chain profit.

The rest of this chapter is organized as follows. We define the proposed two-newsvendor game with limited supply and inventory transshipment in Section 4.2 and study the existence and uniqueness of the Nash equilibrium in Section 4.3. Illustrative examples are provided in Section 4.4. In Section 4.5, we investigate the effect of supply capacity on the total newsvendor profit. In Section 4.6, we discuss the benefit of inventory transshipment and design a coordinating contract. Section 4.8 summarizes the chapter.
4.2 The Model

We incorporate inventory transshipment to the two-newsvendor game with limited supply in Chapter 3. After each newsvendor fulfills his own demand, inventory transshipment may be conducted between them. That is, if the newsvendor $i$ has unsatisfied demand and the newsvendor $j$ has surplus inventory, then the newsvendor $i$ will buy inventory from the newsvendor $j$ at a certain transshipment price $c_{ji}$. (The subscript $ji$ indicates that the surplus inventory is transshipped from the newsvendor $j$ to the newsvendor $i$.) Without loss of generality, we assume the newsvendor $j$ incurs a unit cost of $t_{ji}$ for handling and shipping the surplus inventory transshipped from the newsvendor $j$ to the newsvendor $i$. At the end of the period, any leftover inventory at the newsvendors will be disposed at salvage values $s_i$ and $s_j$, respectively.

In this model, $w_i$, $r_i$, $s_i$, $c_{ij}$ and $t_{ij}$, $i, j = 1, 2, i \neq j$, are exogenous information for both newsvendors. Other than the existing assumption $r_i > w_i > s_i$, $i = 1, 2$, we also assume $r_i \geq c_{ji} \geq s_j + t_{ji}$, $i, j = 1, 2, i \neq j$, so that each newsvendor receives a nonnegative portion of the profit generated by inventory transshipment. To ensure that inventory transshipment happens only when one newsvendor has excess demand and the other has surplus inventory, we inherit the complete pooling assumptions introduced by Tagaras (1989): $w_i < w_j + t_{ji}$, $s_i < s_j + t_{ji}$ and $r_i < r_j + t_{ji}$, $i, j = 1, 2, i \neq j$.

In general, the retail price for transshipped inventory may not be equal to that for the initial inventory because, for instance, a retailer may offer a discount to the unsatisfied customers. Our model can be easily extended to this case by properly adjusting $c_{ij}$ and $t_{ij}$.

Both newsvendors are assumed to have complete information about the parameters of the game, and they are rational with respective objectives to maximize their own profits by making ordering decisions unilaterally and simultaneously. We call this model a \textit{two-newsvendor game with limited supply and inventory transshipment}. 
4.3 Game Theoretic Analysis

In this section, we study the existence and uniqueness of Nash equilibrium of ordering decisions in the newsvendor game defined in Section 4.2. Let $\pi_i(x_i, x_j)$ denote the expected profit of newsvendor $i$ with respect to inventory allocation $x_i$ and $x_j$. Then

$$
\pi_i(x_i, x_j) = -w_i x_i + r_i \mathbb{E}[\min\{D_i, x_i\}] + s_i \mathbb{E}[(x_i - D_i)^+]
$$

$$
+ (c_{ij} - t_{ij} - s_i) \mathbb{E}[\min\{(x_i - D_i)^+, (D_j - x_j)^+\}]
$$

$$
+ (r_i - c_{ji}) \mathbb{E}[\min\{(D_i - x_i)^+, (x_j - D_j)^+\}].
$$

The first three terms are the same as the classical newsvendor model. The fourth term represents the extra profit of transshipping surplus inventory to the newsvendor $j$. The fifth term is the extra profit of buying transshipped inventory from the newsvendor $j$ and selling to customers.

Using $\Pi_i(y_i, y_j)$ to denote the expected profit of newsvendor $i$ with respect to order quantities $y_i$ and $y_j$, we have

$$
\Pi_i(y_i, y_j) = \pi_i(\varphi_i(y_i, y_j), \varphi_j(y_j, y_i)).
$$

The efficient condition (II) of the inventory allocation rule leads to

$$
\Pi_i(y_i, y_j) = \begin{cases} 
\pi_i(y_i, y_j), & \text{if } y_i + y_j \leq K, \\
\pi_i(\varphi_i(y_i, y_j), K - \varphi_i(y_i, y_j)), & \text{if } y_i + y_j > K.
\end{cases}
$$

Note that the profit of each newsvendor depends on the ordering decisions of both newsvendors.

When the supply capacity is unlimited, the newsvendor game with inventory transshipment is studied in Rudi et al. (2001). The first and second order partial derivatives of $\pi_i(x)$ are

$$
\frac{\partial \pi_i}{\partial x_i} = (r_i - w_i) - (r_i - c_{ji}) \Pr(x_i \leq D_i \leq x_i + x_j - D_j)
$$

$$
- (r_i - s_i) \Pr(D_i \leq x_i) + (c_{ij} - t_{ij} - s_i) \Pr(x_i + x_j - D_j \leq D_i \leq x_i)
$$

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\[
\frac{\partial^2 \pi_i}{\partial x_i^2} = -(r_i - s_i)f_{D_i}(x_i) \\
- (r_i - c_{ji})[\Pr(x_i \leq D_i)f_{D_i + D_j|x_i \leq D_i}(x_i + x_j) \\
- \Pr(D_i + D_j \leq x_i + x_j)f_{D_i + D_j \leq x_i + x_j}(x_i)] \\
+ (c_{ij} - t_{ij} - s_i)[\Pr(x_i + x_j \leq D_i + D_j)f_{D_i + D_j \leq D_i}(x_i) \\
- \Pr(D_i \leq x_i)f_{D_i + D_j|x_i \leq D_i}(x_i + x_j)],
\]

respectively.

Note that

\[f_{D_i}(x_i) = \Pr(D_i + D_j \leq x_i + x_j)f_{D_i + D_j \leq x_i + x_j}(x_i) + \Pr(x_i + x_j \leq D_i + D_j)f_{D_i + D_j \leq D_i + D_j}(x_i)\]

and \(r_i - s_i > r_j - t_{ij} - s_i \geq c_{ij} - t_{ij} - s_i, r_i - s_i > r_i - (s_j + t_{ji}) \geq r_i - c_{ji}\). Therefore, \(\frac{\partial^2 \pi_i}{\partial x_i^2} \leq 0\) and, consequently, \(\pi_i(x)\) is concave with respect to \(x_i\) when \(x_j\) is fixed. Then, we have the following theorem about the existence and uniqueness of Nash equilibrium inventory levels.

**Theorem 4.1.** (*Rudi et al. (2001)*) If the supply is unlimited, there exists a unique Nash equilibrium in the two-news vendor game with inventory transshipment.

**Proof.** Existence: We may limit each newsvendor’s decision in a closed interval \([0, Q]\) where \(Q\) is a sufficiently large number. For each \(j = 1, 2\), when \(x_j\) is fixed, \(\pi_i(x_i, x_j)\) is continuous and concave with respect to \(x_i, i \neq j\). Then there exists a Nash equilibrium of inventory levels.

Uniqueness: Let \(R_i^u(x_j)\) denote the best response of Newservendor \(i\) when Newservendor \(j\)’s inventory level is \(x_j\). (The superscript \(u\) denotes “unlimited”.) Then \(R_i^u(x_j)\) satisfies the equation

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\[
\frac{\partial \pi_i}{\partial x_j} = 0.\] By calculation, we have

\[
\frac{\partial^2 \pi_i}{\partial x_j \partial x_i} = -(r_i - c_{ji}) \Pr(x_i \leq D_i)f_{D_i + D_j\mid x_i \leq D_i}(x_i + x_j)
\]

\[
- (c_{ij} - t_{ij} - s_i) \Pr(D_i \leq x_i)f_{D_i + D_j\mid D_i \leq x_i}(x_i + x_j).
\]

Then

\[
\left| \frac{dR_u^i}{dx_j} \right| = \left| \frac{\partial^2 \pi_i}{\partial x_j \partial x_i} \right| < 1.
\]

By Fudenberg and Tirole (1991), the Nash equilibrium is unique. \(\square\)

Suppose that the maximum profit of \(\pi_i(x_i, K - x_i)\) on \(0 \leq x_i \leq K\) is achieved at \(x_i^M\). (The superscript “M” means “maximum”.) We use \(R_u(y_j)\) to denote the best response of newsvendor \(i\)'s order as a function of \(y_j\), which characterizes the best order decision that the newsvendor \(i\) should make when the newsvendor \(j\) orders \(y_j\). Formally, for \(i, j = 1, 2, i \neq j\) and \(0 \leq y_j \leq K\), \(R_u(y_j)\) is defined by

\[R_u(y_j) = \arg \max_{0 \leq y_i \leq K} \Pi_i(y_i, y_j).\]

Define the line segment \(\{(x_1, x_2)\mid x_1 + x_2 = K, x_1, x_2 \geq 0\}\) as “supply line” (Dai et al. (2005a)).

If the function \(R_u^i(x_j)\) intersects with the supply line, we denote the value of \(R_u^i(x_j)\) at the intersection point as \(x_i^I\). (The superscript “I” denotes “Intersection”.) We have the next Theorem about the quasi-concavity of \(\Pi_i(y_i, y_j)\).

**Theorem 4.2.** In the two-newsvendor game with limited supply and inventory transshipment, the profit function \(\Pi_i(y_i, y_j)\) is quasi-concave with respect to \(y_i\) for any \(0 \leq y_j \leq K\) if the following two conditions are met:

(C1) \(\pi_i(x_i, K - x_i)\) is quasi-concave in \(0 < x_i < K\);

(C2) \(x_i^M \leq R_u^i(K - x_i^M)\).

**Proof.** When \(y_j = 0\), \(\Pi_i(y_i, 0) = \pi_i(y_i, 0)\). According to Rudi et al. (2001), \(\Pi_i(y_i, 0)\) is concave in \(y_i \geq 0\). We only need to consider \(0 < y_j \leq K\). Note that \(\Pi_i(y_i, y_j)\) is continuous in \(y_i\) and
$y_j$. We show the quasi-concavity of $\Pi_i(y_i, y_j)$ over $0 \leq y_i < K - y_j$, $K - y_j < y_i \leq K$ and at $y_i = K - y_j$, respectively.

If $0 \leq y_i < K - y_j$, then $x = y$ and $\Pi_i(y_i, y_j) = \pi_i(y_i, y_j)$. Rudi et al. (2001) proved that $\pi_i(y_i, y_j)$ is concave in $y_i$ for any given $0 < y_j \leq K$.

If $K - y_j < y_i \leq K$, then $\Pi_i(y_i, y_j) = \pi_i(\varphi_i(y_i, y_j), K - \varphi_i(y_i, y_j))$. Consequently,

$$\frac{\partial \Pi_i(y_i, y_j)}{\partial y_i} = \frac{\partial \pi_i(\varphi_i(y_i, y_j), K - \varphi_i(y_i, y_j))}{\partial \varphi_i(y_i, y_j)} \frac{\partial \varphi_i}{\partial y_i}.$$ 

The increasing condition (III) of $\varphi_i(y_i, y_j)$ implies that $\frac{\partial \varphi_i}{\partial y_i} \geq 0$. Therefore, $\Pi_i(y_i, y_j)$ is quasi-concave for $K - y_j < y_i \leq K$ if and only if $\pi_i(x_i, K - x_i)$ is quasi-concave for $K - y_j < x_i \leq \varphi_i(K, y_j)$. Note that the union of intervals $K - y_j < x_i \leq \varphi_i(K, y_j)$ over $0 < y_j \leq K$ is $0 < x_i < K$. Hence, $\Pi_i(y_i, y_j)$ is quasi-concave for $K - y_j < y_i \leq K$ with any $0 < y_j \leq K$ being given if and only if $\pi_i(x_i, x_j)$ is quasi-concave for $0 < x_i < K$.

At $y_i = K - y_j$, $\Pi_i(y_i, y_j)$ may not be differentiable with respect to $y_i$. We examine the left and right partial derivatives of $\Pi_i(y_i, y_j)$. If $R_i^u(x) = R_i^l(x)$ has no intersection with the supply line, then $\left.\frac{\partial \Pi_i(y_i, y_j)}{\partial y_i}\right|_{(K-y_j)} \geq 0$ and $\Pi_i(y_i, y_j)$ is quasi-concave at $y_i = K - y_j$. If $R_i^u(x) = R_i^l(x)$ intersects the supply line at $(x_i^l, K - x_i^l)$, we have two cases. If $K - y_j \leq x_i^l$, then $\left.\frac{\partial \Pi_i(K-y_j, y_j)}{\partial y_i}\right|_{(K-y_j)} \geq 0$ and $\Pi_i(y_i, y_j)$ is quasi-concave at $y_i = K - y_j$. Otherwise, $K - y_j > x_i^l$, then $\left.\frac{\partial \Pi_i(y_i, y_j)}{\partial y_i}\right|_{(K-y_j)} < 0$. By the condition $x_i^M \leq R_i^u(K - x_i^M)$, we know that $x_i^M \leq x_i^l$. Hence, $\left.\frac{\partial \Pi_i(\varphi_i(K-y_j, y_j), K - \varphi_i(K-y_j, y_j))}{\partial \varphi_i(y_i, y_j)}\right|_{(K-y_j)} \leq 0$ and $\left.\frac{\partial \Pi_i(y_i, y_j)}{\partial y_i}\right|_{(K-y_j)} = \left.\frac{\partial \pi_i(\varphi_i(K-y_j, y_j), K - \varphi_i(K-y_j, y_j))}{\partial \varphi_i(y_i, y_j)}\right|_{y_i=y_j} \leq 0$. In this case, $\Pi_i(y_i, y_j)$ is quasi-concave at $y_i = K - y_j$.

Therefore, $\Pi_i(y_i, y_j)$ is quasi-concave for $0 \leq y_i \leq K$ given any $0 \leq y_j \leq K$, $i, j = 1, 2, i \neq j$.

The meaning of the conditions (C1) and (C2) is not intuitive. To interpret these two conditions, we consider the function $\pi_i(x_i, K - x_i)$, which characterizes the expected profit of the newsvendor $i$ if the limited supply is completely allocated. In this case the newsvendor $i$ receives $x_i$ and all of the remaining supply $K - x_i$ is allocated to newsvendor $j$. The condition
(C1) then means that there is only one allocation that is favored by the newsvendor \(i\), both
globally and locally. Such allocation is corresponding to the maximum point \(x^M_i\) of the function
\(\pi_i(x_i, K - x_i)\). According to this allocation, the newsvendor \(j\) receives \(K - x^M_i\). Condition (C2)
asumes that \(x^M_i\) is not greater than the best response of the newsvendor \(i\) to the newsvendor
\(j\)'s inventory \(K - x^M_i\) in the unlimited supply case. This also means the constraint of limited
supply lowers the newsvendor \(i\)'s best decisions.

Now we have the next theorem about the existence of Nash equilibrium in our game.

**Theorem 4.3.** In the two-newsvendor game with inventory transshipment under limited supply,
there exists a Nash equilibrium of orders if conditions (C1) and (C2) hold for \(i, j = 1, 2, i \neq j\).

**Proof.** Note the decision space of each newsvendor is the closed interval \(0 \leq y_i \leq K\) and the
profit function \(\Pi_i(y_i, y_j)\) is continuous in \(y_i\) and \(y_j\). By Theorem 4.2, \(\Pi_i(y_i, y_j)\) is quasi-concave
when (C1) and (C2) are met for \(i = 1, 2\). Therefore, there exists a Nash equilibrium in this
game (Fudenberg and Tirole (1991)).

We use \(y^N = (y_1^N, y_2^N)\) to denote the Nash equilibrium of order quantity and use \(x^N =
(x_1^N, x_2^N) = (\varphi_1(y_1^N, y_2^N), \varphi_2(y_2^N, y_1^N))\) to denote the equilibrium inventory allocation corre-
sponding to \(y^N\). Given \(0 \leq x_i \leq K\), recall that we use \(\psi_i(y_j|x_i)\) to denote the solution \(y_i\)
to the equation \(\varphi_i(y_i, y_j) = x_i\). To study the uniqueness of the Nash equilibrium, we need to
characterize the best response of each newsvendor. The next lemma provides the expression of
\(R_i(y_j)\). Illustrative examples are given in Section 4.4.

**Lemma 4.1.** In the two-newsvendor game with limited supply and inventory transshipment, if
the conditions (C1) and (C2) are met, then the best response function \(R_i(y_j)\) of the newsvendor
\(i\) is of one of the following four cases:

(Case 1) If \(R^u_i(x_j)\) does not intersect with the supply line and \(x^M_i \leq \frac{K}{2}\), then

\[
R_i(y_j) = \begin{cases} 
K - y_j, & 0 \leq y_j \leq K - x^M_i \\
\psi_i(y_j|x^M_i), & K - x^M_i < y_j \leq K. 
\end{cases}
\]
(Case 2) If \( R_i(x_j) \) does not intersect with the supply line and \( x_i^M > \frac{K}{2} \), then

\[
R_i(y_j) = \begin{cases} 
K - y_j, & 0 \leq y_j \leq K - x_i^M \\
\psi_i(y_j|x_i^M), & K - x_i^M < y_j \leq \psi_j(K|K - x_i^M) \\
K, & \psi_j(K|K - x_i^M) < y_j \leq K.
\end{cases}
\]

(Case 3) If \( R_i(x_j) \) intersects with the supply line at \( x_i^l \) and \( x_i^M \leq \frac{K}{2} \), then

\[
R_i(y_j) = \begin{cases} 
R_i(y_j), & 0 \leq y_j \leq K - x_i^l \\
K - y_j, & K - x_i^l < y_j \leq K - x_i^M \\
\psi_i(y_j|x_i^M), & K - x_i^M < y_j \leq K.
\end{cases}
\]

(Case 4) If \( R_i(x_j) \) intersects with the supply line at \( x_i^l \) and \( x_i^M > \frac{K}{2} \), then

\[
R_i(y_j) = \begin{cases} 
R_i(y_j), & 0 \leq y_j \leq K - x_i^l \\
K - y_j, & K - x_i^l < y_j \leq K - x_i^M \\
\psi_i(y_j,x_i^M), & K - x_i^M < y_j \leq \psi_j(K|K - x_i^M) \\
K, & \psi_j(K|K - x_i^M) < y_j \leq K.
\end{cases}
\]

With \( R_i(y_j) \) being fully characterized, we can further provide an analytic expression of the Nash equilibrium \( y^N = (y_1^N, y_2^N) \) and equilibrium inventory allocation \( x^N = (x_1^N, x_2^N) \). The answer to the uniqueness of Nash equilibrium then follows. In Table 4.1, we list all combinations of cases of \( R_1(y_2) \) and \( R_2(y_1) \) with corresponding \( y^N \) and \( x^N \). It is also marked whether \( y^N \) is unique. To reduce the complexity of the table, we include only the combination (Case \( k \), Case \( l \)) and omit its counterpart (Case \( l \), Case \( k \)) for \( k < l \). The unique Nash equilibrium obtained when the supply capacity is unlimited (Rudi et al. (2001)) is denoted by \( x^{N,u} \).

We can see from Table 4.1 that the uniqueness of Nash equilibrium depends on the supply
capacity \( K \). As we will show in Section 4.4, both of the best response functions \( R_i(y_j), i, j = 1, 2, i \neq j \), may include a line segment. The Nash equilibrium is unique if and only if these two line segments intersect at no more than one point. Otherwise, there exist infinitely many Nash equilibria. Summarizing Table 4.1 and our analysis in this section, we have the next theorem.

**Theorem 4.4.** Assume that the conditions (C1) and (C2) hold for \( i, j = 1, 2, i \neq j \) in the two-news-vendor game with limited supply and inventory transshipment. There exists a unique Nash equilibrium if and only if \( K < x_1^M + x_2^M \) or \( K \geq x_1^I + x_2^I \). Moreover, there exist infinitely many Nash equilibria if and only if \( x_1^M + x_2^M \leq K < x_1^I + x_2^I \).

Theorem 4.4 indicates that the uniqueness/non-uniqueness of Nash equilibrium reflects the relative gaming effects of limited supply and inventory transshipment. When the supply capacity is low \((K < x_1^M + x_2^M)\), both newsvendors have a high priority to secure inventory because there may not be enough surplus inventory for transshipment. Consequently, they tend to place large orders and, in particular, one of the newsvendors places \( K \) at the unique Nash equilibrium. When the supply capacity is high \((K \geq x_1^I + x_2^I)\), competition for the limited inventory is mild and the newsvendors may safely order as that in the unique Nash equilibrium \( x_{N,u} \) under unlimited supply, which directly comes from the gaming effect of inventory transshipment. When \( x_1^M + x_2^M \leq K < x_1^I + x_2^I \), neither the competition for limited supply nor the cooperation via inventory transshipment can dominate the gaming effects on the newsvendors’ decisions. In this case, there may exist multiple (infinitely many) Nash equilibria.

### 4.4 Illustrative Examples

In this section, we use four examples to show the general shapes of the best response functions characterized in Lemma 4.1 and the uniqueness/non-uniqueness of Nash equilibrium characterized in Theorem 4.4 and Table 4.1. For illustration purposes, we adopt the proportional allocation rule.
Table 4.1: Nash equilibrium of the two-newsvendor game with limited supply and inventory transshipment

<table>
<thead>
<tr>
<th>$R_1(y_2)$</th>
<th>$R_2(y_1)$</th>
<th>Nash equilibrium $y^N$</th>
<th>$y^N$ unique?</th>
<th>$x^N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>Case 1</td>
<td>$(K, K)$</td>
<td>YES</td>
<td>$(K/K)$</td>
</tr>
<tr>
<td>Case 1</td>
<td>Case 2</td>
<td>$K &lt; x_1^M + x_2^M$</td>
<td>YES</td>
<td>$(K - x_2^N, x_2^M)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(K, \psi_2(K</td>
<td>x_2^M))$</td>
<td></td>
</tr>
<tr>
<td>Case 1</td>
<td>Case 3</td>
<td>$(K, K)$</td>
<td>YES</td>
<td>$(K - x_2^N, x_2^M)$</td>
</tr>
<tr>
<td>Case 2</td>
<td>Case 2</td>
<td>$K &lt; x_1^M + x_2^M$</td>
<td>YES</td>
<td>$(K - x_2^N, x_2^M)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(y_1^N, \psi_2(y_1^N</td>
<td>x_2^M)), x_1^M \leq y_1^N \leq K$</td>
<td>NO</td>
</tr>
<tr>
<td>Case 2</td>
<td>Case 3</td>
<td>$(y_1^N, K - y_1^N), x_1^M \leq y_1^N \leq K - x_2^M$</td>
<td>NO</td>
<td>$y^N$</td>
</tr>
<tr>
<td>Case 2</td>
<td>Case 4</td>
<td>$(y_1^N, K - y_1^N), x_1^M \leq y_1^N \leq K - x_2^M$</td>
<td>NO</td>
<td>$y^N$</td>
</tr>
<tr>
<td>Case 3</td>
<td>Case 3</td>
<td>$(K, K)$</td>
<td>YES</td>
<td>$(K/K)$</td>
</tr>
<tr>
<td>Case 3</td>
<td>Case 4</td>
<td>$(y_1^N, K - y_1^N), x_1^M \leq y_1^N \leq \min{K - x_2^M, x_1^M}$</td>
<td>NO</td>
<td>$y^N$</td>
</tr>
<tr>
<td>Case 3</td>
<td>Case 4</td>
<td>$(y_1^N, K - y_1^N), x_1^M \leq y_1^N \leq \min{K - x_2^M, x_1^M}$</td>
<td>NO</td>
<td>$y^N$</td>
</tr>
<tr>
<td>Case 4</td>
<td>Case 4</td>
<td>$(y_1^N, K - y_1^N), x_1^M \leq y_1^N \leq \min{K - x_2^M, x_1^M}$</td>
<td>NO</td>
<td>$y^N$</td>
</tr>
</tbody>
</table>
Example 4.4.1. In this example, the supply capacity $K = 9$, and $w_1 = w_2 = 4$, $r_1 = r_2 = 6$, $s_1 = s_2 = 1$, $t_{12} = t_{21} = 2.5$, $c_{12} = c_{21} = 4$. The demands of the newsvendors are mutually independent and uniformly distributed with $D_1 \sim U[4,10]$, $D_2 \sim U[0,10]$.

As shown in Figure 4.2, the best response of the newsvendor 1, $R_1(y_2)$, is of Case 4. $R_2(y_1)$ is of Case 3. Moreover, we find $x_1^M = 6.09$ and $x_2^M = 3.82$. Consequently, $K < x_1^M + x_2^M$. By Theorem 4.4 and Table 4.1, there exists a unique Nash equilibrium $(y_1^N, y_2^N) = (9.00, 6.64)$ with equilibrium inventory allocation $(x_1^N, x_2^N) = (5.18, 3.82)$.

Example 4.4.2. This example is the same as Example 4.4.1 except that the supply capacity $K = 10$. As shown in Figure 4.3, $R_1(y_2)$ is of Case 4 and $R_2(y_1)$ is of Case 3. We find that $x_1^M = 5.80$, $x_2^M = 1.33$, $x_1^I = 6.23$ and $x_2^I = 4.00$. Consequently, $x_1^M + x_2^M < K < x_1^I + x_2^I$.

By Theorem 4.4 and Table 4.1, there are infinitely many Nash equilibria and they form the line segment $\{(y_1^N, y_2^N)|6.00 \leq y_1^N \leq 6.23, y_2^N = 10 - y_1^N\}$.

Example 4.4.3. This example is the same as Example 4.4.1 except that $K = 12$. We observe that $R_1(y_2)$ is of Case 3, $R_2(y_1)$ is of Case 4, $x_1^M = 5.06$, $x_2^M = 0.00$, $x_1^I = 5.75$ and $x_2^I = 3.27$. Consequently, $K \geq x_1^I + x_2^I$. By Theorem 4.4 and Table 4.1, there exists a unique Nash equilibrium and $(y_1^N, y_2^N) = (x_1^{N,U}, x_2^{N,U}) = (6.20, 3.96)$ as shown in Figure 4.4.

Example 4.4.4. This example is the same as Example 4.4.3 except that $D_2 \sim U[4,10]$. The two newsvendors are symmetric. We may apply (Case 3, Case 4) of Table 4.1. Note that $x_1^M = x_2^M = 6.00$ and $x_1^I = x_2^I = 6.45$. Consequently, $K = x_1^M + x_2^M$ and there are infinitely many Nash equilibria forming the line segment $\{(y_1^N, y_2^N)|6.00 \leq y_1^N \leq 12, y_2^N = y_1^N\}$. The best response functions and the Nash equilibria are shown in Figure 4.5.

Example 4.4.1 demonstrates the order inflation phenomenon. That is, the newsvendors place large orders intentionally in order to receive their favorable inventory allocation. Example 4.4.2 illustrates the case of multiple Nash equilibria, which is not possible when the supply is unlimited. Example 4.4.3 indicates that if the supply is sufficient, the Nash equilibrium is the same as that under unlimited supply. Example 4.4.4 is a special case that there are multiple
Figure 4.2: Best responses and Nash equilibrium in Example 4.4.1

Figure 4.3: Best responses and Nash equilibria in Example 4.4.2
Figure 4.4: Best responses and Nash equilibrium in Example 4.4.3

Figure 4.5: Best responses and Nash equilibria in Example 4.4.4
Nash equilibria of order quantity but the equilibrium inventory allocation is unique. Finally, from examples 4.4.1, 4.4.2 and 4.4.3, we can see that the supply capacity $K$ is closely related to the uniqueness of the Nash equilibrium.

4.5 Total Newsvendor Profit under Limited Supply

In this section, assuming conditions (C1) and (C2) hold for $i, j = 1, 2, i \neq j$, we investigate two issues related to the total newsvendor profit in the two-newsvendor game with limited supply and inventory transshipment. In Section 4.5.1, we study how the total newsvendor profit at the Nash equilibrium changes as $K$ increases. In Section 4.5.2, we study what inventory allocation rule the supplier may choose to maximize the total newsvendor profit, given that the newsvendors order the newsvendor-best inventory stocks under unlimited supply. We use $\pi_R(x) = \pi_1(x_1, x_2) + \pi_2(x_2, x_1)$ to denote the total newsvendor profit when the newsvendors’ inventory stocks are $x_1$ and $x_2$, respectively. (The subscript “R” denotes “retailer”.) When the supply capacity is unlimited, we denote the newsvendor-best inventory stocks by $(x_1^{R,u}, x_2^{R,u})$. (The superscript “R” denotes “retailer”.)

4.5.1 Total Newsvendor Equilibrium Profit and Supply Capacity

Assume that the newsvendors make order decisions according to the Nash equilibrium when it is unique. We focus on the total newvendor equilibrium profit $\pi_R(x^N)$ and write it as $\pi_R(x^N(K))$ in this section because $x^N$ depends on $K$. Note that when $K \geq x_1^I + x_2^I$, the equilibrium allocation $(x_1^N(K), x_2^N(K)) = (x_1^{N,u}, x_2^{N,u})$ and thus $\pi_R(x^N(K))$ does not change as $K$ increases. When $x_1^M + x_2^M \leq K \leq x_1^I + x_2^I$, there are multiple Nash equilibria and the newsvendors may not have any stable joint decision. We analyze two special cases. In Section 4.5.1, we study two symmetric newsvendors. In this case, we can prove that $(\frac{K}{2}, \frac{K}{2})$ is a stable decision for all $K$ and $\pi_R(x^N(K))$ is non-decreasing in $K$. In Section 4.5.1, we consider two symmetric newsvendors but one with a stochastically larger demand. In this case, we show that $\pi_R(x^N(K))$ is non-decreasing in the range of $K \leq x_1^M + x_2^M$. 48
Symmetric Newsvendors with the Same Demand

When two newsvendors are symmetric, we know $\pi_R(x)$ is symmetric with respect to the line $x_1 - x_2 = 0$. Thus, there exists a maximum of $\pi_R(x)$ that satisfies $x_1 = x_2$. Moreover, when $x_1^M + x_2^M \leq K \leq x_1^I + x_2^I$, $(\frac{K}{2}, \frac{K}{2})$ is a Pareto Nash equilibrium, which yields the largest total newsvendor profit among all Nash equilibria. To see this, consider an arbitrary point $(x_1, x_2)$ on the supply line. By symmetry, we have $\pi_R(x_1, x_2) = \pi_R(x_2, x_1)$. By concavity, we know that $\pi_R(\frac{K}{2}, \frac{K}{2}) \geq \frac{1}{2}(\pi_R(x_1, x_2) + \pi_R(x_2, x_1)) = \pi_R(x_1, x_2)$. Hence $(\frac{K}{2}, \frac{K}{2})$ is the maximum point of $\pi_R(x_1, x_2)$ on the supply line. Assuming both of the symmetric newsvendors order $\frac{K}{2}$ when there are multiple Nash equilibria, we have the next theorem.

**Theorem 4.5.** In the two-newsvendor game with limited supply and inventory transshipment, if the newsvendors are symmetric, then the total newsvendor equilibrium profit $\pi_R(x^N(K))$ is non-decreasing in $K$.

**Proof.** Since $\pi_R(x_1, x_2)$ is concave in $(x_1, x_2)$ (Robinson (1990)), we see that $x_i^{N,u} = x_i^{B,u}$ because $(x_1^{B,u}, x_2^{B,u})$ maximizes $\pi_R(x_1, x_2)$ under unlimited supply. Therefore, for $K \geq 2x_i^{N,u}$, $x_i^N(K) = x_i^{N,u}$ and $\pi_R(x^N(K))$ remains constant. For $K \leq 2x_i^{N,u}$, $x_i^N = \frac{K}{2}$ and $\pi_R(\frac{K}{2}, \frac{K}{2}) = \max_{x_1+x_2\leq K} \pi_R(x_1, x_2)$. Then it is clear that $\pi_R(x^N(K)) = \max_{x_1+x_2\leq K} \pi_R(x_1, x_2)$ is non-decreasing in $K$. In this case, the newsvendors are symmetric in prices, market sizes and expected profits. A larger supply capacity meets the demand better via inventory transshipment, and each newsvendor receives a non-decreasing equilibrium profit.

Symmetric Newsvendors with Different Demands

We say the newsvendor 1 has a stochastically larger demand than the newsvendor 2 if $F_{D_1}(x) \leq F_{D_2}(x)$ holds for all $x \geq 0$, or equivalently, $\Pr(D_1 \geq x) \geq \Pr(D_2 \geq x)$ for all $x \geq 0$. This means that the newsvendor 1 enjoys a larger market than the newsvendor 2. We focus on the range...
\[ K \leq x_1^M + x_2^M \] when there is a unique Nash equilibrium \( y^N(K) \). The next lemma is needed to obtain the form of \( y^N(K) \) and \( x^N(K) \).

**Lemma 4.2.** In the two-newsvendor game with limited supply and inventory transshipment, if the two newsvendors are symmetric except that the newsvendor 1 has a stochastically larger demand than the newsvendor 2, then \( x_1^M \geq x_2^M \).

**Proof.** Notice that

\[
\frac{d\pi_i(x_i, K - x_i)}{dx_i} = t_{ij}\Pr(x_i \leq D_i \leq K - D_j) - t_{ij}\Pr(K - D_j \leq D_i \leq x_i)
\]

\[
+ (r_i - w_i) + (-r_i + c_{ij} - t_{ij})\Pr(D_i + D_j \leq K) - (r_i - c_{ij})\Pr(D_i + D_j \geq K)
\]

\[
- (c_{ij} - t_{ij} - s_i)\Pr(D_i \leq x_i, D_j \leq K - x_i)
\]

\[
+ (r_i - c_{ji})\Pr(D_i \geq x_i, D_j \geq K - x_i).
\]

Therefore,

\[
\frac{d\pi_1(x_1, K - x_1)}{dx_1} - \frac{d\pi_2(x_2, K - x_2)}{dx_2}
\]

\[
= t_{12}[\Pr(x_1 \leq D_1 \leq K - D_2) - \Pr(x_2 \leq D_2 \leq K - D_1)]
\]

\[
- t_{12}[\Pr(K - D_2 \leq D_1 \leq x_1) - \Pr(K - D_1 \leq D_2 \leq x_2)]
\]

\[
- (c_{12} - t_{12} - s_1)[\Pr(D_1 \leq x_1, D_2 \leq K - x_1) - \Pr(D_2 \leq x_2, D_1 \leq K - x_2)]
\]

\[
+ (r_1 - c_{21})[\Pr(D_1 \geq x_1, D_2 \geq K - x_1) - \Pr(D_2 \geq x_2, D_1 \geq K - x_2)].
\]

When \( x_1 = x_2 = x \), we have

\[
\Pr(x \leq D_1 \leq K - D_2) - \Pr(x \leq D_2 \leq K - D_1) \geq 0,
\]

\[
\Pr(K - D_2 \leq D_1 \leq x) - \Pr(K - D_1 \leq D_2 \leq x) \leq 0,
\]

\[
\Pr(D_1 \leq x, D_2 \leq K - x) - \Pr(D_2 \leq x, D_1 \leq K - x) \leq 0,
\]
and
\[ \Pr(D_1 \geq x, D_2 \geq K - x) - \Pr(D_2 \geq x, D_1 \geq K - x) \geq 0. \]

Therefore, \( \frac{d\pi_1(x_1, K - x_1)}{dx_1} \bigg|_{x_2^M} \geq \frac{d\pi_2(x_2, K - x_2)}{dx_2} \bigg|_{x_2^M} = 0. \) Consequently, \( x_1^M \geq x_2^M. \)

If the news vendor 1 had the full authority to allocate the supply \( K \), he would secure \( x_1^M \) for himself and allocate \( K - x_1^M \) to the news vendor 2. Similarly, the news vendor 2 would allocate \( x_2^M \) to himself if he had the allocating power. Lemma 4.2 indicates that, when the prices are symmetric, the news vendor with larger market desires no less inventory than the other news vendor. By Lemma 4.2, we know that if \( K \leq 2x_2^M \), then \( y^N = (K, K) \) and \( x^N = (\frac{K}{2}, \frac{K}{2}) \).

If \( 2x_2^M \leq x_1^M + x_2^M \), then \( y^N = (K, \psi(K \mid x_2^M)) \) and \( x^N = (K - x_2^M, x_2^M) \). This information leads to the next result.

**Theorem 4.6.** In the two-news vendor game with limited supply and inventory transshipment, if the news vendors are symmetric except that the news vendor 1 has a stochastically larger demand than the news vendor 2, then the total news vendor equilibrium profit \( \pi_R(x^N(K)) \) is non-decreasing in \( K \leq x_1^M + x_2^M \).

**Proof.** When \( K \leq x_1^M + x_2^M \) and \( x_2^M \geq \frac{K}{2} \), we have \( x_1^M \geq x_2^M \) and \( (x_1^N, x_2^N) = (\frac{K}{2}, \frac{K}{2}) \). It is not difficult to verify that \( \frac{\partial \pi_1}{\partial x_2} \geq \frac{\partial \pi_2}{\partial x_1} \). Consider the case \( K = 2x_2^M \). We know \( \frac{d\pi_2(x_2, K - x_2)}{dx_2} = \frac{\partial \pi_2}{\partial x_2} - \frac{\partial \pi_1}{\partial x_1} \geq 0 \) and \( \frac{\partial \pi_2}{\partial x_2} \geq 0 \) because \( x_2^M \geq x_2^M \). Then \( \frac{\partial \pi_1}{\partial x_2} \geq \frac{\partial \pi_2}{\partial x_1} \geq 0. \) Consequently,

\[ \frac{d\pi_R(x^N(K))}{dK} = \frac{1}{2} \left( \frac{\partial \pi_1}{\partial x_1} + \frac{\partial \pi_1}{\partial x_2} + \frac{\partial \pi_2}{\partial x_1} + \frac{\partial \pi_2}{\partial x_2} \right) \geq 0. \]

Therefore, \( \frac{d\pi_R(x^N(K))}{dK} \geq 0 \) when \( K \leq x_1^M + x_2^M \) and the total profit function \( \pi_R(x^N) \) is non-decreasing.
Therefore, \( \pi \)

Without loss of generality, we may assume that \( x \)

Note that at \( (x, x) \)

Consequently, \( d \)

\[
\frac{d\pi_R(x_1, x_2)}{dK} = \frac{\partial \pi_R(x_1, x_2)}{dx_1} \frac{dx_1}{dK} + \frac{\partial \pi_R(x_1, x_2)}{dx_2} \frac{dx_2}{dK} = (\frac{\partial \pi_1}{\partial x_1} + \frac{\partial \pi_2}{\partial x_2})(1 - \frac{dx_2}{dK}) + (\frac{\partial \pi_1}{\partial x_1} + \frac{\partial \pi_2}{\partial x_2}) \frac{dx_2}{dK} = (\frac{\partial \pi_1}{\partial x_1} + \frac{\partial \pi_2}{\partial x_1}) - (\frac{\partial \pi_1}{\partial x_1} + \frac{\partial \pi_2}{\partial x_2} - \frac{\partial \pi_2}{\partial x_2}) \frac{dx_2}{dK}.
\]

Note that at \( (x, x) = (K - x, x) \), we have

\[
\frac{\partial \pi_1}{\partial x_1} > 0, \frac{\partial \pi_2}{\partial x_2} > 0, \frac{\partial \pi_1}{\partial x_1} - \frac{\partial \pi_1}{\partial x_2} > 0 \text{ and } \frac{\partial \pi_2}{\partial x_2} - \frac{\partial \pi_2}{\partial x_1} = 0.
\]

Without loss of generality, we may assume that \( x \)

\[
\frac{\partial^2 \pi_2(x_2, K - x_2)}{\partial K \partial x_2} = t_21 \Pr(D_2 \geq x_2) f_{D_2+D_1 \mid D_2 \geq x_2}(K) + t_21 \Pr(D_2 \leq x_2) f_{D_2+D_1 \mid D_2 \leq x_2}(K) + (-r_2 + c_21 - t_21) f_{D_2+D_1}(K) + (r_2 - c_12) f_{D_2+D_1}(K)
\]

\[
= -(c_21 - t_21 - s_2) \Pr(D_2 \leq x_2) f_{D_1 \mid D_2 \leq x_2}(K - x_2) - (r_2 - c_12) \Pr(D_2 \geq x_2) f_{D_1 \mid D_2 \geq x_2}(K - x_2)
\]

\[
= 0.
\]

Consequently, \( \frac{dx_2}{dK} = -\frac{\partial^2 \pi_2(x_2, K - x_2)}{\partial K \partial x_2} / \frac{\partial^2 \pi_2(x_2, K - x_2)}{\partial x_2^2} \leq 0 \) and

\[
\frac{d\pi_R(x_1, x_2^N)}{dK} = (\frac{\partial \pi_1}{\partial x_1} + \frac{\partial \pi_2}{\partial x_1}) - (\frac{\partial \pi_1}{\partial x_1} - \frac{\partial \pi_2}{\partial x_2}) \frac{dx_2}{dK} \geq 0.
\]

Therefore, \( \pi_R(x^N) \) is non-decreasing in \( K \leq x_1^M + x_2^M \).

\[
\square
\]

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In this case, the two newsvendors differ only in market sizes. When \( x^N = (\frac{K}{2}, \frac{K}{2}) \), the newsvendor 2 has the same amount of inventory allocation as newsvendor 1. When \( x^N = (K - x_2^M, x_2^M) \), the newsvendor 2 receives his desired amount of inventory. Therefore, the equilibrium inventory allocation favors the newsvendor with a smaller demand. As we see in the proof of Theorem 4.6, \( x_2^M \) is non-increasing and the newsvendor 1’s equilibrium inventory \( K - x_2^M \) is non-decreasing in \( K \). Consequently, the newsvendor 1 may have more inventory for his larger market as \( K \) increases. In this sense, a higher supply capacity leads to a larger equilibrium profit of the two newsvendors.

4.5.2 Total Newsvendor Profit and Allocation Rules

Hu et al. (2007) showed that in the unlimited supply case, there may exist constant coordinating transshipment prices \( (c_{12}, c_{21}) \) such that \( x_i^{N,u} = x_i^{R,u}, i = 1, 2 \). (The superscript “R” denotes “retailer”. This means that, the Nash equilibrium of inventory decisions under unlimited supply maximizes the total profit of the newsvendors. We seek to answer the following question: if the supply is limited and the newsvendors order \( (x_1^{R,u}, x_2^{R,u}) \) (possibly due to delay of supply information), will the supplier be able to choose an appropriate allocation rule such that the inventory allocation remains optimal for the total newsvendor profit? We consider a special case where the newsvendors are “nearly” symmetric in terms of the price parameters. To be specific, we assume that \( r_1 - w_1 = r_2 - w_2, w_1 - s_1 = w_2 - s_2 \) and \( r_1 - t_{21} - s_2 = r_2 - t_{12} - s_1 \) in this section, which means that the revenue of selling initial inventory, the net loss of leftover inventory and the revenue of selling transshipped inventory are equal at both newsvendors’ locations. Clearly, two symmetric newsvendors belong to this case. We will show that the supplier may choose appropriate allocation rules based on the demands of the newsvendors.

To begin with, we study the optimality conditions for an inventory allocation to maximize the total profit of both newsvendors. Denote \( (x_1^{R}, x_2^{R}) \) as the first-best inventory allocation under limited supply. The next lemma provides a necessary (in some situations, sufficient, too) condition that is satisfied at \( (x_1^{R}, x_2^{R}) \).
Lemma 4.3. In the two-newsvendor game with limited supply and inventory transshipment, if \( r_1 - w_1 = r_2 - w_2, w_1 - s_1 = w_2 - s_2 \) and \( r_1 - t_{21} - s_2 = r_2 - t_{12} - s_1 \), then \( \Pr(D_1 \leq x_1^R) = \Pr(D_2 \leq x_2^R) \) holds at \( (x_1^R, x_2^R) \). Furthermore, if \( x_1^R + x_2^R = K \) and \( x_1 + x_2 = K \), then \( \Pr(D_1 \leq x_1) = \Pr(D_2 \leq x_2) \) is sufficient for \( (x_1, x_2) \) to maximize the total newsvendor profit \( \pi_R(x_1, x_2) \).

Proof. Note that \( \pi_R(x_1, x_2) \) is concave. The newsvendor-best inventories \( (x_1^R, x_2^R) \) form an optimal solution to the following problem:

\[
\max \pi_R(x_1, x_2) \\
\text{s.t.} \quad x_1 + x_2 \leq K, \\
\quad x_1, x_2 \geq 0.
\]

By the KKT conditions, \( (x_1, x_2) \) maximizes \( \pi_R(x_1, x_2) \) if and only if \( x_1 + x_2 < K \) and \( \frac{\partial \pi_R}{\partial x_1} = 0 \), or \( x_1 + x_2 = K \) and \( \frac{\partial \pi_R}{\partial x_1} = \frac{\partial \pi_R}{\partial x_2} \). Therefore, \( \frac{\partial \pi_R}{\partial x_1} = \frac{\partial \pi_R}{\partial x_2} \) is a necessary condition for \( (x_1, x_2) \) to be the newsvendor-best inventories. By calculation, we have

\[
\frac{\partial \pi_R}{\partial x_1} = (r_1 - w_1) - (r_1 - s_1)\Pr(D_1 \leq x_1) - (r_1 - t_{21} - s_2)\Pr(x_1 \leq D_1 \leq x_1 + x_2 - D_2) \\
+ (r_2 - t_{12} - s_1)\Pr(x_1 + x_2 - D_2 \leq D_1 \leq x_1),
\]

and

\[
\frac{\partial \pi_R}{\partial x_2} = (r_2 - w_2) - (r_2 - s_2)\Pr(D_2 \leq x_2) - (r_2 - t_{12} - s_1)\Pr(x_2 \leq D_2 \leq x_1 + x_2 - D_1) \\
+ (r_1 - t_{21} - s_2)\Pr(x_1 + x_2 - D_1 \leq D_2 \leq x_2).
\]
Noting that \( r_1 - w_1 = r_2 - w_2 \), we have

\[
- (r_1 - s_1)\Pr(D_1 \leq x_1) - (r_1 - t_{21} - s_2)\Pr(x_1 \leq D_1 \leq x_1 + x_2 - D_2) \\
+ (r_2 - t_{12} - s_1)\Pr(x_1 + x_2 - D_2 \leq D_1 \leq x_1)
\]

\[
= - (r_2 - s_2)\Pr(D_2 \leq x_2) - (r_2 - t_{12} - s_1)\Pr(x_2 \leq D_2 \leq x_1 + x_2 - D_1) \\
+ (r_1 - t_{21} - s_2)\Pr(x_1 + x_2 - D_1 \leq D_2 \leq x_2).
\]

Equivalently, we have

\[
- (r_1 - s_1)\Pr(D_1 \leq x_1) \\
+ (r_2 - t_{12} - s_1)[\Pr(x_2 \leq D_2 \leq x_1 + x_2 - D_1) + \Pr(x_1 + x_2 - D_2 \leq D_1 \leq x_1)]
\]

\[
= - (r_2 - s_2)\Pr(D_2 \leq x_2) \\
+ (r_1 - t_{21} - s_2)[\Pr(x_1 \leq D_1 \leq x_1 + x_2 - D_2) + \Pr(x_1 + x_2 - D_1 \leq D_2 \leq x_2)],
\]

and it can be further simplified as

\[
-(r_1 - s_1)\Pr(D_1 \leq x_1) + (r_2 - t_{12} - s_1)[\Pr(D_1 \leq x_1) - \Pr(D_1 \leq x_1, D_2 \leq x_2)]
\]

\[
= -(r_2 - s_2)\Pr(D_2 \leq x_2) + (r_1 - t_{21} - s_2)[\Pr(D_2 \leq x_2) - \Pr(D_1 \leq x_1, D_2 \leq x_2)].
\]

Consequently, we have

\[
(-r_1 + r_2 - t_{12})\Pr(D_1 \leq x_1) - (r_2 - t_{32} - s_1)\Pr(D_1 \leq x_1, D_2 \leq x_2)
\]

\[
= (-r_2 + r_1 - t_{21})\Pr(D_2 \leq x_2) - (r_1 - t_{21} - s_2)\Pr(D_1 \leq x_1, D_2 \leq x_2).
\]

From \( r_1 - w_1 = r_2 - w_2, w_1 - s_1 = w_2 - s_2 \) and \( r_1 - t_{21} - s_2 = r_2 - t_{12} - s_1 \), we have
\[ -r_1 + r_2 - t_{12} = -r_2 + r_1 - t_{21}. \text{ Note } r_1 - t_{21} - s_2 = r_2 - t_{12} - s_1. \text{ Therefore,} \]

\[ \Pr(D_1 \leq x_1) = \Pr(D_2 \leq x_2). \]

If \( x_1^R + x_2^R = K, \ x_1 + x_2 = K \) and \( \Pr(D_1 \leq x_1) = \Pr(D_2 \leq x_2) \), we can reverse the above analysis and have \( \frac{\partial \pi_R}{\partial x_1} = \frac{\partial \pi_R}{\partial x_2} \). By the concavity of \( \pi_R(x_1, x_2) \) and KKT conditions, \((x_1, x_2)\) is a maximum of \( \pi_R(x_1, x_2) \).

In this case, Lemma 4.3 indicates that at the newsvendor-best inventory allocation, both newsvendors have equal probabilities to fully satisfy their own customers before inventory transshipment. When the unlimited newsvendor-best inventory stocks add up to be more than the supply capacity \( K \), the inventory that each newsvendor receives is subject to the allocation rule.

The next two theorems state when to use the proportional rule and linear rule to maximize the total newsvendor profit.

**Theorem 4.7.** In the two-newsvendor game with limited supply and inventory transshipment, assume the following conditions hold:

(i) \( r_1 - w_1 = r_2 - w_2, \ w_1 - s_1 = w_2 - s_2 \) and \( r_1 - t_{21} - s_2 = r_2 - t_{12} - s_1 \);

(ii) the newsvendors order the newsvendor-best inventory stocks with unlimited supply, i.e., \((x_1^{R,u}, x_2^{R,u})\);

(iii) the supply capacity is less than the total orders, i.e., \( K < x_1^{R,u} + x_2^{R,u} \).

If the newsvendor 1’s demand is uniformly distributed between 0 and \( M_1 > 0 \) and the newsvendor 2’s demand is uniformly distributed between 0 and \( M_2 > 0 \), then the inventory allocation by the proportional rule maximizes the total newsvendor profit.

**Proof.** From Lemma 4.3, we have \( \Pr(D_1 \leq x_1^{R,u}) = \Pr(D_2 \leq x_2^{R,u}) \). From \( K < x_1^{R,u} + x_2^{R,u} \), we know that \( x_1^R + x_2^R = K \). By the proportional rule, the inventory allocation is \( x_1 = \frac{x_1^{R,u}}{x_1^{R,u} + x_2^{R,u}} K \) and \( x_2 = \frac{x_2^{R,u}}{x_1^{R,u} + x_2^{R,u}} K \). Therefore, \( \Pr(D_1 \leq x_1) = \frac{K}{x_1^{R,u} + x_2^{R,u}} \Pr(D_1 \leq x_1^{R,u}) = \frac{K}{x_1^{R,u} + x_2^{R,u}} = \frac{K}{x_1^{R,u} + x_2^{R,u}} \)
\[ \Pr(D_2 \leq x_2^{R,u}) = \Pr(D_2 \leq x_2). \]
Noting that \( x_1 + x_2 = K \) and using Lemma 4.3 again, we know that \((x_1, x_2)\) maximizes the total profit of the newsvendors. 

Actually, Theorem 4.7 is applicable to any case when the demand distributions of the newsvendors satisfy \( \frac{F_{D_1}(x_1)}{F_{D_2}(x_2)} = \frac{F_{D_1}(\lambda x_1)}{F_{D_2}(\lambda x_2)} \) for any \( \lambda > 0 \). A pair of uniform distributions \( U[0, M_1] \) and \( U[0, M_2] \) is a special case. The proportion rule makes the inventory allocation of both newsvendors be a same portion \( \lambda = \frac{K}{y_1+y_2} \) of their respective order quantity. Therefore, proportional rule can be used to maximize the total newsvendor profit.

### 4.6 Benefit of Inventory Transshipment

When the supply is unlimited, at the Nash equilibrium, inventory transshipment always benefits both newsvendors. To see this, recall that \( \pi^n_i(x_i) \) is the profit of the classic newsvendor model and note that at the Nash equilibrium \((x_1^{N,u}, x_2^{N,u})\), we have \( \pi_i(x_1^{N,u}, x_j^{N,u}) \geq \pi_i(x_i, x_j^{N,u}) \geq \pi^n_i(x_i) \) for all \( x_i \). This means that at the Nash equilibrium of unlimited supply case, each newsvendor has more profit with inventory transshipment than that without inventory transshipment.

In this section, we investigate whether inventory transshipment still benefits the supply chain under limited supply. The benchmark model that we compare with is the two-newsvendor game with the same amount of limited supply in Chapter 3. A coordinating contract that induces the newsvendors to order the newsvendor-best inventories will also be discussed. Throughout this section we assume all supply capacity is allocated to the newsvendors.

In the benchmark model, the equilibrium expected profits \((\Pi^n_1(y_1^{N,n}, y_2^{N,n}), \Pi^n_2(y_2^{N,n}, y_1^{N,n})) = (\pi^n_1(x_1^{N,n}), \pi^n_2(x_2^{N,n}))\) are unique. Next, we will compare the individual newsvendor equilibrium profits \( \Pi_i(y_i^{N}, y_j^{N}) \) and \( \Pi_i^{n}(y_i^{N,n}, y_j^{N,n}) \).
### 4.6.1 Individual Newsvendor Equilibrium Profit

Now we compare the individual newsvendor equilibrium profit in the two-newsvendor game with limited supply and inventory transshipment with that in the benchmark model. Throughout this section we assume the Nash equilibrium in the two-newsvendor game with limited supply and inventory transshipment is unique. If \( \Pi_i(y_i^N, y_j^N) \geq \Pi_i^n(y_i^{N,n}, y_j^{N,n}) \), then we say the newsvendor \( i \) is better off and inventory transshipment benefits the newsvendor \( i \). The next theorem implies that inventory transshipment can benefit at least one of the two newsvendors.

**Theorem 4.8.** If all supply is allocated to the newsvendors at the equilibrium in both of the game with inventory transshipment and the benchmark model, i.e., \( x_1^N + x_2^N = K \) and \( x_1^{N,n} + x_2^{N,n} = K \), then \( \Pi_1(y_1^N, y_2^N) \geq \Pi_1^n(y_1^{N,n}, y_2^{N,n}) \) or \( \Pi_2(y_2^N, y_1^N) \geq \Pi_2^n(y_2^{N,n}, y_1^{N,n}) \).

**Proof.** We discuss according to the value of \( x^N \).

If \( x_1^N = x_2^M \), then \( \Pi_1(y_1^N, y_2^N) = \pi_1(x_1^N, K-x_1^N) = \max_{x_1} (x_1, K-x_1) \geq \pi_1(x_1^{N,n}, K-x_1^{N,n}) = \pi_1^n(x_1^{N,n}, y_2^{N,n}). \) Similarly, if \( x_2^N = x_2^M \), then \( \Pi_2(y_2^N, y_1^N) \geq \pi_2^n(y_2^{N,n}, y_1^{N,n}). \)

If \( x_1^N = x_2^N = \frac{K}{2} \) and \( x_1^{N,n} = x_2^{N,n} = \frac{K}{2} \), then it is obvious that \( \pi_1(K/2, K/2) \geq \pi_1^n(K/2) \) for \( i = 1, 2. \)

If \( x_1^N = x_2^N = \frac{K}{2} \) and \( x_1^{N,n} > \frac{K}{2} > x_2^{N,n} \), then we know that \( x_2^M \geq \frac{K}{2} > x_2^{N,n}. \) By the quasi-concavity of \( \pi_2(x_2, K-x_2) \), we have \( \Pi_2(y_2^N, y_1^N) = \pi_2(K/2, K/2) \geq \pi_2(x_2^{N,n}, K-x_2^{N,n}) \geq \pi_2^n(x_2^{N,n}, y_1^{N,n}) = \Pi_2^n(y_2^{N,n}, y_1^{N,n}). \) Similarly, if \( x_1^N = x_2^N = \frac{K}{2} \) and \( x_2^{N,n} > \frac{K}{2} > x_1^{N,n} \), then \( \Pi_1(y_1^N, y_2^N) \geq \Pi_1^n(y_1^{N,n}, y_2^{N,n}). \)

Therefore, we have \( \Pi_1(y_1^N, y_2^N) \geq \Pi_1^n(y_1^{N,n}, y_2^{N,n}) \) or \( \Pi_2(y_2^N, y_1^N) \geq \Pi_2^n(y_2^{N,n}, y_1^{N,n}). \)

There are situations when both newsvendors can benefit from inventory transshipment as characterized in the next theorem.

**Theorem 4.9.** Assume that all supply is allocated to the newsvendors at the equilibrium in both of the game with inventory transshipment and the game without inventory transshipment, i.e., \( x_1^N + x_2^N = K \) and \( x_1^{N,n} + x_2^{N,n} = K. \) Then \( \Pi_1(y_1^N, y_2^N) \geq \Pi_1^n(y_1^{N,n}, y_2^{N,n}) \) and \( \Pi_2(y_2^N, y_1^N) \geq \Pi_2^n(y_2^{N,n}, y_1^{N,n}), \) if
(i) \( x_1^{N,n} = x_2^{N,n} = \frac{K}{2} \), or

(ii) \( x_1^{N,n} \leq x_1^N \) and \( x_2^N = x_2^M \), or

(iii) \( x_2^{N,n} \leq x_2^N \) and \( x_1^N = x_1^M \).

**Proof.** If \( x_1^{N,n} = x_2^{N,n} = \frac{K}{2} \) and \( x_1^N = x_2^N = \frac{K}{2} \), it is obvious that \( \Pi_i(y_i^N, y_j^N) = \pi_i(\frac{K}{2}, \frac{K}{2}) \geq \pi_i^n(\frac{K}{2}) = \Pi_i^n(y_i^{N,n}, y_j^{N,n}) \) for \( i, j = 1, 2, i \neq j \).

If \( x_1^{N,n} = x_2^{N,n} = \frac{K}{2}, x_1^N = K - x_2^M > \frac{K}{2} \) and \( x_2^N = x_2^M < \frac{K}{2} \), then \( \Pi_2(y_2^N, y_1^N) = \pi_2(x_2^M, K - x_2^M) \geq \pi_2(\frac{K}{2}, \frac{K}{2}) = \pi_2^n(y_2^{N,n}, y_1^{N,n}) \). Moreover, \( x_1^M \geq K - x_2^M > \frac{K}{2} \). By the quasi-concavity of \( \pi_1(x_1, K - x_1) \), we have \( \Pi_1(y_1^N, y_2^N) = \pi_1(K - x_2^M, x_2^M) \geq \pi_1(\frac{K}{2}, \frac{K}{2}) \geq \pi_1^n(\frac{K}{2}) = \Pi_1^n(y_1^{N,n}, y_2^{N,n}) \). Similar proof applies to the case when \( x_1^{N,n} = x_2^{N,n} = \frac{K}{2}, x_2^N = K - x_1^M > \frac{K}{2} \) and \( x_1^N = x_1^M < \frac{K}{2} \).

If \( x_1^{N,n} \leq x_1^N \) and \( x_2^N = x_2^M \), then \( \Pi_2(y_2^N, y_1^N) = \pi_2(x_2^M, K - x_2^M) \geq \pi_2(x_2^{N,n}, K - x_2^{N,n}) \geq \pi_2(x_2^{N,n}) = \Pi_2^n(y_2^{N,n}, y_1^{N,n}) \). Moreover, \( x_1^{N,n} \leq x_1^N \leq x_1^M \). We have \( \Pi_1(y_1^N, y_2^N) = \pi_1(K - x_2^M, x_2^M) \geq \pi_1(x_1^{N,n}, K - x_1^{N,n}) \geq \pi_1^n(x_1^{N,n}) = \Pi_1^n(y_1^{N,n}, y_2^{N,n}) \). Similar proof applies to the case when \( x_2^{N,n} \leq x_2^N \) and \( x_1^N = x_1^M \).

Theorem 4.9 guarantees that inventory transshipment benefits both newsvendors in two cases. (ii) and (iii) are essentially the same.) Generally, from Table 4.1, \( x_1^{N,n} = x_2^{N,n} = \frac{K}{2} \) happens when \( K \) is rather small. In this case, inventory transshipment contributes greatly in terms of moving the scarce inventory to match the relatively large demand. For (ii) and (iii), note that in the new game, one of the newsvendors receives his desired amount of allocation and consequently, achieves his best profit. The other newsvendor has no less inventory than that in the benchmark model and therefore, he has a larger chance of meeting his own demand in the new game than in the benchmark model. Hence both newsvendors are better off than in the benchmark model. Note that these conditions are characterized by the equilibrium inventory rather than in parameters. This is because the Nash equilibrium \( y_1^{N,n} \) and \( y_2^N \) are jointly determined by forming parameters of the problem in the first-order conditions of \( \pi_i(x_i, K - x_i) \) and taking the allocation rule into account. Therefore, many cases can result in the analytic
expressions of $y^{N,n}$ and $y^N$. To express the cases in Theorem 4.9 directly by the problem parameters is extremely complicated.

Interestingly, we find that a newsvendor may receive less equilibrium profit in the new game than in the benchmark model. The next example illustrates this situation.

**Example 4.6.1.** Let $w_1 = w_2 = 9$, $r_1 = r_2 = 10$, $s_1 = s_2 = 1$, $t_{12} = 5$, $t_{21} = 3$, $c_{12} = 7$, $c_{21} = 9$, $D_1 \sim U[4,18]$, $D_2 \sim U[2,9]$ and $K = 8$. We have $x_1^* = 5.5556$ and $x_2^* = 2.7778$. Therefore, $y_1^{N,n} = 8$, $y_2^{N,n} = 4.2554$, $x_1^{N,n} = 5.2222$ and $\Pi_1(y_1^{N,n}, y_2^{N,n}) = 4.7421$ in the benchmark model. In the new game, we have $y_1^N = 8$, $y_2^N = 6.4170$, $x_1^N = 4.4392$ and $\Pi_1(y_1^N, y_2^N) = 4.5544$. Therefore, $\Pi_1(y_1^N, y_2^N) < \Pi_1(y_1^{N,n}, y_2^{N,n})$, and the newsvendor 1 is worse off in the game with inventory transshipment than in the benchmark model. For the newsvendor 2, $x_2^{N,n} = 2.7778$, $\Pi_2(y_2^{N,n}, y_1^{N,n}) = 2.1173$, $x_2^N = 3.5608$ and $\Pi_2(y_2^N, y_1^N) = 2.8283$. Hence, the newsvendor 2 is better off.

In Example 4.6.1, we note that $c_{21} - s_2 - t_{21} = 5 > r_1 - c_{21} = 1$ and $r_2 - c_{12} = 3 > c_{12} - s_1 - t_{12} = 1$. This means that the newsvendor 2 receives a larger share out of the extra profit of inventory transshipment than the newsvendor 1. Consequently, the newsvendor 2 manages to obtain more inventory in the new game than in the benchmark model ($x_2^N > x_2^{N,n}$). For the newsvendor 1, his profit loss due to the reduced inventory ($x_1^N < x_1^{N,n}$) can not be fully compensated by the inventory transshipment. Therefore, the newsvendor 1 is worse off than in the benchmark model.

To verify the interpretation above and investigate whether the transshipment prices make a newsvendor better or worse off than in the benchmark model, we conduct a numerical study. We adopt all setting in Example 4.6.1 except letting $c_{12}$ and $c_{21}$ vary according to the relation that $\frac{c_{12} - s_1 - t_{12}}{r_2 - s_1 - t_{12}} = \frac{r_1 - c_{21}}{r_1 - s_2 - t_{21}}$. This fraction represents the portion that is effectively allocated to the newsvendor 1 out of the profit generated from inventory transshipment, and we denote it by $\delta$. Accordingly, $\frac{r_2 - c_{12}}{r_2 - s_1 - t_{12}} = \frac{c_{21} - s_2 - t_{21}}{r_1 - s_2 - t_{21}} = 1 - \delta$. We compute $\Pi_1(y_1^N, y_2^N)$ and $\Pi_2(y_2^N, y_1^N)$ for $\delta = \frac{4}{5}, \frac{3}{4}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, and compare with $\Pi_1(y_1^{N,n}, y_2^{N,n})$ and $\Pi_2(y_2^{N,n}, y_1^{N,n})$, respectively. The results are shown in Table 4.2.
Table 4.2: Numerical study: equilibrium profits in Example 4.6.1

<table>
<thead>
<tr>
<th>$c_{12}$</th>
<th>$c_{21}$</th>
<th>$\delta$</th>
<th>$\Pi_1(y_1^N, y_2^N)$</th>
<th>$\Pi_2(y_2^N, y_1^N)$</th>
<th>$\Pi_1^{n}(y_1^{N,n}, y_2^{N,n})$</th>
<th>$\Pi_2^{n}(y_2^{N,n}, y_1^{N,n})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.0000</td>
<td>5.5000</td>
<td>3/4</td>
<td>5.0486</td>
<td>2.4939</td>
<td>4.7421</td>
<td>2.1173</td>
</tr>
<tr>
<td>8.6667</td>
<td>6.0000</td>
<td>2/3</td>
<td>5.0124</td>
<td>2.5301</td>
<td>4.7421</td>
<td>2.1173</td>
</tr>
<tr>
<td>8.0000</td>
<td>7.0000</td>
<td>1/2</td>
<td>4.9400</td>
<td>2.6025</td>
<td>4.7421</td>
<td>2.1173</td>
</tr>
<tr>
<td>7.3333</td>
<td>8.0000</td>
<td>1/3</td>
<td>4.8677</td>
<td>2.6749</td>
<td>4.7421</td>
<td>2.1173</td>
</tr>
<tr>
<td>7.0000</td>
<td>8.5000</td>
<td>1/4</td>
<td>4.7162</td>
<td>2.7538</td>
<td>4.7421</td>
<td>2.1173</td>
</tr>
<tr>
<td>6.8000</td>
<td>8.8000</td>
<td>1/5</td>
<td>4.6375</td>
<td>2.7931</td>
<td>4.7421</td>
<td>2.1173</td>
</tr>
</tbody>
</table>

We observe that as $\delta$ decreases, $\Pi_1(y_1^N, y_2^N)$ is decreasing and $\Pi_2(y_2^N, y_1^N)$ is increasing. When $\delta \leq \frac{1}{4}$, $\Pi_1(y_1^N, y_2^N) < \Pi_1^{n}(y_1^{N,n}, y_2^{N,n})$ and the newsvendor 1 becomes worse off than in the benchmark model. In this range of $\delta$, the newsvendor 2 is better off. This numerical study indicates that the transshipment prices can affect whether a newsvendor benefits from participating in inventory transshipment, compared to his equilibrium profit in the benchmark model.

### 4.6.2 Total Newsvendor Profit

In this section, we compare the total newsvendor profit $\pi_R(x_1, x_2)$ in the two-newsvendor game with limited supply and inventory transshipment, not necessarily at the Nash equilibrium, with that in the benchmark model, which is denoted by $\pi_R^n(x_1, x_2)$. Even though the total newsvendor equilibrium profit in the game with inventory transshipment may not always be larger than that in the benchmark model, the next theorem states that in the best case, the largest possible total newsvendor profit in the game with inventory transshipment can be more than the best total newsvendor profit in the benchmark model. In this sense, inventory transshipment could still benefit the supply chain as long as the limited supply is properly allocated.

**Theorem 4.10.** The best total newsvendor profit with inventory transshipment is larger than
the best total newsvendor profit without inventory transshipment.

**Proof.** Note that for the inventory allocation \((x_1, x_2)\), we have \(\pi_i(x_i, x_j) \geq \pi^n_i(x_i, x_j)\). Therefore, \(\pi_R(x_1, x_2) \geq \pi^n_R(x_1, x_2)\) and, consequently, \(\max_{x_1 + x_2 \leq K} \pi_R(x_1, x_2) \geq \max_{x_1 + x_2 \leq K} \pi^n_R(x_1, x_2)\). □

If two newsvendors that are symmetric in price parameters, then the newsvendors can be together viewed as a big “newsvendor” with certain amount of profit loss due to the handling and shipping costs in inventory transshipment. The next theorem shows that as long as the handling and shipping costs are small enough, the total newsvendor profit with inventory transshipment is better off in the new game than that of any inventory allocation in the benchmark model. This is because inventory transshipment can better match the demand with the supply.

**Theorem 4.11.** If \(r_1 = r_2, w_1 = w_2, s_1 = s_2, t_{12} = t_{21}\) and

\[ t_{12} \leq \frac{1}{K} \left[ -w_1 K + r_1 E[\min\{D_1 + D_2, K\}] + s_1 E[(K - D_1 - D_2)^+] - \max_{x_1 + x_2 \leq K} \pi^n_R(x_1, x_2) \right], \]

then \(\pi_R(x_1, x_2) \geq \pi^n_R(x_1, x_2)\) for all \(x_1, x_2 \geq 0, x_1 + x_2 \leq K\).

**Proof.** Note that for two newsvendors symmetric in price parameters, the total newsvendor profit with inventory transshipment is

\[
\pi_R(x_1, x_2) = -w_1 K + r_1 E[\min\{D_1 + D_2, K\}] + s_1 E[(K - D_1 - D_2)^+] \\
- t_{12}(E[\min\{(D_i - x_i)^+, (x_j - D_j)^+\}] + E[\min\{(x_i - D_i)^+, (D_j - x_j)^+\}])
\]

To see this, consider putting the two newsvendors together to become a big newsvendor who in ours transportation and handling costs. Note that \(E[\min\{(D_i - x_i)^+, (x_j - D_j)^+\}] + E[\min\{(x_i - D_i)^+, (D_j - x_j)^+\}] \leq K\). The theorem then follows. □

The ideal scenario is that at an ordering decision \(y\), the total newsvendor profit \(\Pi_R(y)\) in the new game is larger than the total newsvendor equilibrium profit \(\Pi_R(y^{N,n})\) in the benchmark model. Moreover, the total profit \(\Pi_R(y)\) may be divided such that each newsvendor is better.
off than his equilibrium profit in the benchmark model, i.e., $\Pi_1(y_1^N, y_2^N) \geq \Pi_1^n(y_1^{N,n}, y_2^{N,n})$ and $\Pi_2(y_2^N, y_1^N) \geq \Pi_2^n(y_2^{N,n}, y_1^{N,n})$. Next, we develop a mechanism to achieve this desired result.

### 4.7 A Coordinating Contract

When the supply is unlimited, Hezarkhani and Kubiak (2010a) proposed a coordinating contract such that the Nash equilibrium of newsvendors’ inventory decisions jointly maximize their total profit. In this section, we derive a coordinating contract for the new game by adopting their key idea. That is, the transshipment prices $c_{12}$ and $c_{21}$ can vary according to the order quantity $y_1$ and $y_2$ and they are negotiated by the newsvendors before they place orders. Formally, the contract runs according to the following two stages:

(Stage 1) Before placing orders, the newsvendors negotiate and agree on a transshipment pricing mechanism $(c_{12}(y), c_{21}(y))$ for any feasible order quantity $y$.

(Stage 2) After the newsvendors place orders $y_1, y_2$, they use $(c_{12}(y), c_{21}(y))$ for inventory transshipment.

The negotiated transshipment prices $c_{12}(y)$ and $c_{21}(y)$ can be determined by using the general Nash bargaining solution to solve the newsvendors’ bargaining problem. In general, a bargaining problem consists of a set of utility pairs of both players and a disagreement point that corresponds to the utilities if the players fail to reach an agreement. In our model, associated with any given $(y_1, y_2)$, the utilities of the newsvendors are their expected profits ($\Pi_1(y_1, y_2, c), \Pi_2(y_2, y_1, c)$) as functions of $c = (c_{12}, c_{21})$. The disagreement point is $(\Pi_1^n(y_1^{N,n}, y_2^{N,n}), \Pi_2^n(y_2^{N,n}, y_1^{N,n}))$ because if the negotiation fails, there will be no inventory transshipment and the newsvendors will receive the equilibrium profits in the benchmark model. This is different from Hezarkhani and Kubiak (2010a), where the supply is unlimited and the disagreement point is the best profit of the classical newsvendor model.

According to the general Nash bargaining solution (Nash (1951), Roth (1979)), for any given orders $(y_1, y_2)$, the negotiated transshipment prices $(c_{12}(y), c_{21}(y))$ form an optimal solution
of the following optimization problem (P_y):

$$\max_c f_y(c) := [\Pi_1(y_1, y_2, c) - \Pi'_1(y_1^{N,n}, y_2^{N,n})]^{\alpha_1} [\Pi_2(y_2, y_1, c) - \Pi'_2(y_2^{N,n}, y_1^{N,n})]^{\alpha_2}$$

s.t.

$$\Pi_1(y_1, y_2, c) \geq \Pi'_1(y_1^{N,n}, y_2^{N,n}),$$  \hspace{1cm} (4.1)

$$\Pi_2(y_2, y_1, c) \geq \Pi'_2(y_2^{N,n}, y_1^{N,n}),$$  \hspace{1cm} (4.2)

$$r_2 \geq c_{12} \geq s_1 + t_{12},$$  \hspace{1cm} (4.3)

$$r_1 \geq c_{21} \geq s_2 + t_{21},$$  \hspace{1cm} (4.4)

where $$\alpha_1, \alpha_2 \in (0, 1)$$ are the newsvendors’ bargaining powers with $$\alpha_1 + \alpha_2 = 1$$.

Note that the total newsvendor profit $$\Pi_R(y)$$ is not dependent on the transshipment price vector $$c$$. If $$\Pi_R(y) < \Pi'_R(y_2^{N,n})$$, then constraints (4.1) and (4.2) cannot be simultaneously satisfied. Consequently, the negotiation will fail and there will be no inventory transshipment. Hence, we only need to consider those order quantities $$(y_1, y_2)$$ such that $$\Pi_R(y) \geq \Pi'_R(y^{N,n})$$.

Denote $$T_{ji}(y) = E[\min\{D_i - \varphi_i(y_i, y_j), \varphi_j(y_j, y_i) - D_j\}]$$ as the expected amount of inventory transshipped from newsvendor $$j$$ to newsvendor $$i$$, $$i, j = 1, 2, i \neq j$$. As we show in the next theorem, there exists $$c(y)$$ that maximizes $$f_y(c)$$ and satisfy constraints (4.1)-(4.4) as long as $$\Pi_R(y) \geq \Pi'_R(y^{N,n})$$.

**Theorem 4.12.** For any given order quantities $$y = (y_1, y_2)$$ such that $$\Pi_R(y) \geq \Pi'_R(y^{N,n})$$, the negotiated transshipment prices $$(c_{12}(y), c_{21}(y))$$ that solve problem (P_y) can be derived in the following manner:

(i) If $$T_{12}(y) \neq 0$$ and $$T_{21}(y) \neq 0$$, then $$c_{21}(y)$$ can be arbitrarily chosen in the interval

$$[\max\{t_{21} + s_2, L_{21}\}, \min\{r_1, U_{21}\}]$$

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where

\[ L_{21} = -\alpha_1 (r_2 - t_{12} - s_1) \frac{T_{12}(y)}{T_{21}(y)} + \alpha_2 r_1 + \alpha_1 (t_{21} + s_2) \]

\[ - \frac{1}{T_{21}(y)} [\alpha_1 \Pi_1^0 (y_2, y_1) - \Pi_2^2 (y_2^{N,n}, y_1^{N,n})] - \alpha_2 [\Pi_1^0 (y_1, y_2) - \Pi_1^0 (y_1^{N,n}, y_2^{N,n})], \]

\[ U_{21} = \alpha_2 (r_2 - t_{12} - s_1) \frac{T_{12}(y)}{T_{21}(y)} + \alpha_2 r_1 + \alpha_1 (t_{21} + s_2) \]

\[ - \frac{1}{T_{21}(y)} [\alpha_1 \Pi_1^0 (y_2, y_1) - \Pi_2^2 (y_2^{N,n}, y_1^{N,n})] - \alpha_2 [\Pi_1^0 (y_1, y_2) - \Pi_1^0 (y_1^{N,n}, y_2^{N,n})], \]

and

\[ c_{12}(y) = \frac{T_{21}(y)}{T_{12}(y)} c_{21}(y) + [\alpha_1 r_2 + \alpha_2 (t_{12} + s_1)] - \frac{T_{21}(y)}{T_{12}(y)} [\alpha_2 r_1 + \alpha_1 (t_{21} + s_2)] \]

\[ + \frac{1}{T_{21}(y)} [\alpha_1 \Pi_1^0 (y_2, y_1) - \Pi_2^2 (y_2^{N,n}, y_1^{N,n})] - \alpha_2 [\Pi_1^0 (y_1, y_2) - \Pi_1^0 (y_1^{N,n}, y_2^{N,n})]. \]

(ii) If \( T_{12}(y) = 0 \) and \( T_{21}(y) \neq 0 \), then

\[ c_{21}(y) = [\alpha_2 r_1 + \alpha_1 (t_{21} + s_2)] + \]

\[ \frac{1}{T_{21}(y)} [\alpha_2 [\Pi_1^0 (y_1, y_2) - \Pi_1^0 (y_1^{N,n}, y_2^{N,n})] - \alpha_1 [\Pi_2^2 (y_2, y_1) - \Pi_2^2 (y_2^{N,n}, y_1^{N,n})]] \]

and \( c_{12}(y) \) can be arbitrarily chosen in \( [t_{12} + s_1, r_2] \).

(iii) If \( T_{12}(y) \neq 0 \) and \( T_{21}(y) = 0 \), then

\[ c_{12}(y) = [\alpha_1 r_2 + \alpha_2 (t_{12} + s_1)] + \]

\[ \frac{1}{T_{12}(y)} [\alpha_1 [\Pi_2^2 (y_2, y_1) - \Pi_2^2 (y_2^{N,n}, y_1^{N,n})] - \alpha_2 [\Pi_1^0 (y_1, y_2) - \Pi_1^0 (y_1^{N,n}, y_2^{N,n})]] \]

and \( c_{21}(y) \) can be arbitrarily chosen in \( [t_{21} + s_2, r_1] \).

(iv) If \( T_{12}(y) = 0 \) and \( T_{21}(y) = 0 \), then \( c_{12}(y) \) can be arbitrarily chosen in \( [t_{12} + s_1, r_2] \) and \( c_{21}(y) \) can be arbitrarily chosen in \( [t_{21} + s_2, r_1] \).

Proof. Define \( f_y(c) = -\infty \) if \( \Pi_R(y) < \Pi_R^0 (y^{N,n}) \). It is not difficult to verify that \( f_y(c) \) is concave in \( c \). Note that no inventory transshipment is needed if \( \Pi_R(y) = \Pi_R^0 (y^{N,n}) \). Therefore,
we only consider the case of $\Pi_R(y) > \Pi^n_R(y^{N,n})$.

If $T_{12}(y) \neq 0$ and $T_{21}(y) \neq 0$, then $c$ maximizes $f_y(c)$ globally if and only if $c$ satisfies the first order conditions $\frac{\partial f_y(c)}{\partial c_{12}} = 0$ and $\frac{\partial f_y(c)}{\partial c_{21}} = 0$. Both are equivalent to the following condition:

$$T_{12}(c) c_{12} - T_{21}(c) c_{21} = \alpha_1 [r_2 T_{12}(c) - (t_{21} + s_2) T_{21}(c) + \Pi^n_2(y_2, y_1) - \Pi^n_2(y_2^{N,n}, y_1^{N,n})] - \alpha_2 [r_1 T_{12}(c) - (t_{12} + s_1) T_{21}(c) + \Pi^n_1(y_1, y_2) - \Pi^n_1(y_1^{N,n}, y_2^{N,n})].$$

From $\Pi_R(y) > \Pi_R^n(y^{N,n})$, we know that

$$r_2 T_{12}(y) - (t_{21} + s_2) T_{21}(y) + \Pi^n_2(y_2, y_1) - \Pi^n_2(y_2^{N,n}, y_1^{N,n}) > -r_1 T_{21}(y) + (t_{12} + s_1) T_{12}(y) - \Pi^n_1(y_1, y_2) + \Pi^n_1(y_1^{N,n}, y_2^{N,n}).$$

Together with (4.9), we have

$$T_{12}(y) c_{12} - T_{21}(y) c_{21} > -r_1 T_{21}(y) + (t_{12} + s_1) T_{12}(y) - \Pi^n_1(y_1, y_2) + \Pi^n_1(y_1^{N,n}, y_2^{N,n}),$$

(4.10)

$$T_{12}(y) c_{12} - T_{21}(y) c_{21} < r_2 T_{12}(y) - (t_{21} + s_2) T_{21}(y) + \Pi^n_2(y_2, y_1) - \Pi^n_2(y_2^{N,n}, y_1^{N,n}),$$

(4.11)

which are equivalent to constraints (4.1) and (4.2). Solving (4.9) for $c_{12}$, we have equation (4.7). By constraints (4.3) and (4.4), the range of $c_{21}$ is that $\max\{t_{21} + s_2, L_{21}\} \leq c_{21} \leq \min\{r_1, U_{21}\}$ where $L_{21}$ and $U_{21}$ are defined by equation (4.5) and (4.6), respectively. It is not difficult to verify that $L_{21} \leq U_{21}$, $L_{21} \leq r_1$ and $t_{21} + s_2 \leq U_{21}$. This means the interval $[\max\{t_{21} + s_2, L_{21}\}, \min\{r_1, U_{21}\}]$ is not empty. Therefore, $c_{12}(y)$ and $c_{21}(y)$ are given as in the theorem.

If $T_{12}(y) = 0$ and $T_{21}(y) \neq 0$, then $c_{12}$ is only constrained by (4.3). The first order condition $\frac{\partial f_y(c)}{\partial c_{21}} = 0$ leads to equation (4.8) and it is easy to verify that constraint (4.4) is satisfied. For cases when $T_{12}(y) \neq 0$ and $T_{21}(y) = 0$ or $T_{12}(y) = 0$ and $T_{21}(y) = 0$, the proof is similar.

Theorem 4.12 indicates that there may be multiple optimal solutions to the problem $(P_y)$. The newsvendors may choose any one of them to be $c(y)$. Actually, as shown in the next lemma,
the newsvendors’ expected profits are the same for all \( c \) specified in Theorem 4.12.

**Lemma 4.4.** For any given order quantities \( y_1, y_2 \) such that \( \Pi_R(y) \geq \Pi^n_R(y^{N,n}) \), if the transshipment prices \( c(y) \) is determined by the generalized Nash bargaining solution, then

\[
\Pi_i(y_i, y_j, c(y)) = \Pi^n_i(y_i^{N,n}, y_j^{N,n}) + \alpha_i[\Pi_R(y) - \Pi^n_R(y^{N,n})]
\]

for \( i, j = 1, 2, i \neq j \).

**Proof.** Substitute the \( c(y) \) as in Theorem 4.12 into \( \Pi_i(y_i, y_j, c) \). The lemma follows by simple calculation. \( \Box \)

Lemma 4.4 shows that each newsvendor’s profit is the equilibrium profit in the benchmark model plus a fixed portion of the total extra profit generated by transshipment. Consequently, the newsvendors may coordinate in placing orders to maximize \( \Pi_R(y) \). We complete this section by the next theorem.

**Theorem 4.13.** If the newsvendors may negotiate the transshipment prices before placing orders, then they will make the orders of \( y^R := \arg \max_{y_1, y_2} \Pi_R(y) \) and use \( c(y^R) \) for inventory transshipment.

**Proof.** For any \( y \neq y^R \), we have \( \Pi_R(y) \leq \Pi_R(y^R) \). By Lemma 4.4,

\[
\Pi_i(y_i, y_j, c(y)) \leq \Pi_i(y^R_i, y^R_j, c(y))
\]

holds for \( i, j = 1, 2, i \neq j \). Therefore, the newsvendor-best order quantity \( y^R \) is a Nash equilibrium for the newsvendors to order. \( \Box \)

### 4.7.1 Supply Chain Profit

Now we study the supply chain profit by assuming that the newsvendors are allowed to negotiate and set the transshipment prices as in Section 4.7. As we have known, the newsvendors will
cooperate in ordering and receive the inventory allocation that maximizes their total profits. We assume that the supplier offers the same wholesale price to the newsvendors, i.e., \( w_1 = w_2 = w \). We will prove that, for any given supply capacity \( K \), there exists a wholesale price \( w(K) \geq w_0 \) such that for all \( w \in [w_0, w(K)] \), the inventory allocation that maximizes the newsvendor total profit also maximizes the supply chain profit.

To begin with, we introduce the notation that will be used in this section. Assume \( w \geq w_0 \). We use \( \pi_0(x, w) = (w - w_0)(x_1 + x_2) \) to denote the supplier’s profit. Let \( x^C(K) = (x^C_1(K), x^C_2(K)) \) denote the inventory allocation that maximizes the supply chain profit \( \pi_C(x) = \pi_1(x, w) + \pi_2(x, w) + \pi_0(x, w) \). (The superscript “C” and subscript “C” both denote “Chain”.) Let \( x^R(K, w) = (x^R_1(K, w), x^R_2(K, w)) \) denote the inventory allocation that maximizes the newsvendors’ total profit \( \pi_R(x, w) \). Note that \( x^C(K) \) does not depend on the wholesale price \( w \) while \( x^R(K, w) \) does. The wholesale price \( w \) determines the allocation of profit between the supplier and the newsvendors.

The relation between \( x^C(K) \) and \( x^R(K, w) \) is worth studying. When \( w \) is relatively small, newsvendors tend to order more and \( x^R_1(K, w) + x^R_2(K, w) \) could be more than \( K \). In such case the supplier profit is \( \pi_0(x) = (w - w_0)K \) and maximized. Hence, \( x^R(K, w) = x^C(K) \). When \( c \) increases, the newsvendors may jointly order less than \( K \), in which case \( x^R(K, w) \) would not be equal to \( x^C(K) \). We have the following lemma:

**Lemma 4.5.** For any given \( K \), there exists \( w(K) \geq w_0 \) such that

\[
x^R(K, w) = \begin{cases} 
x^C(K), & \text{if } w_0 \leq w \leq w(K), \\
x^R(\infty, w), & \text{if } w > w(K).
\end{cases}
\]

**Proof.** Note that when \( w = w_0 \), \( \pi_0(x, w) = 0 \) and then \( \pi(x, w) = \pi_C(x) \). Hence, \( x^R(K, w_0) = x^C(K) \).

Recall that \( \pi_C(x) \) and \( \pi_R(x, w) \) are both concave in \( x \). It is easy to prove that \( x^R(\infty, w), i = 1, 2 \), is non-increasing in \( w \). Actually,
\[
\frac{dx^R(\infty, w)}{dw} = -\left[ \frac{\partial^2 \pi_R}{\partial x_1^2} \frac{\partial^2 \pi_R}{\partial x_1 \partial x_2} - \frac{\partial^2 \pi_R}{\partial x_2^2} \right]^{-1} \left( \frac{\partial^2 \pi_R}{\partial x_1 \partial x_2} \right) 
\]
\[
\leq 0.
\]

For any wholesale price \(w\), let \(K_w = x^R_1(\infty, w) + x^R_2(\infty, w)\). Then

\[
x^R(\infty, w) = \arg \max_{x_1 + x_2 = K_w} \{\pi_R(x, w)\} = \arg \max_{x_1 + x_2 = K_w} \{\pi_C(x)\} = x^C(K_w) = x^R(K_w, w_0).
\]

For any \(K\), let \(w(K)\) be the wholesale price such that \(x^R(\infty, w(K)) = x^R(K, w_0)\).

When \(w_0 \leq w \leq w(K)\), we have \(K_w \geq K\). Then

\[
x^R(K, w) = \arg \max_{x_1 + x_2 = K} \{\pi_R(x, w)\} = \arg \max_{x_1 + x_2 = K} \{\pi_C(x)\} = x^C(K).
\]

When \(w > w(K)\), we have \(K_w < K\). Then \(x^R(K, w) = x^R(\infty, w)\).

Therefore, as long as the wholesale price \(w\) is within the interval \(w_0 \leq w \leq w(K)\), the supply chain profit is maximized. Moreover, if the supplier wants to maximize her own profit while keeping the supply chain optimized, she may set \(w = w(K)\). To summarize, we have the next result.

**Theorem 4.14.** Assume that the newsvendors decide the transshipment prices by general Nash bargaining solution. If the wholesale prices of both newsvendors are equal, then there exists a range of this wholesale price within which the total profit of the supply chain is maximized, and the supplier receives a nonnegative profit.

Note that \(w(K) = w_0\) when \(K \geq x^C_1(\infty) + x^C_2(\infty)\). This implies that when the supply
capacity is sufficiently large, the supplier will receive no profit under the circumstance that
the newsvendors’ inventory allocation maximizes the supply chain profit. Equivalently, if the
supplier wants a positive profit, the supply chain profit will not be optimized. This phenomenon
is similar to the double marginalization in the one-supplier-one-newsvendor system, where a
wholesale price that optimizes the supply chain and brings the supplier positive profit has been
proved not to exist. When supply is not so large in the model, \( w(K) > w_0 \) and consequently,
double marginalization is eliminated since the supplier can extract a positive profit while keeping
the supply chain profit maximized.

4.8 Summary

In this chapter, we have studied the two-newsvendor game with limited supply and inventory
transshipment. We find that there may not exist a Nash equilibrium in this game, and we have
derived the existence conditions of the Nash equilibrium. The uniqueness of the Nash equi-
librium depends on the limit of supply capacity, which essentially connects to relative gaming
effects of limited supply and inventory transshipment. Moreover, a larger supply capacity allows
the newsvendors to generate more profits. The proportional rule and linear rule can result in
first-best inventory allocations in some special cases.

We have found that under limited supply, inventory transshipment may not always be bene-
ficial for both newsvendors. This may affect the transshipment incentive of the newsvendor who
is worse off and jeopardize the cooperation between the newsvendors. We have designed a coor-
dinating contract by suggesting that the transshipment prices be negotiated by the newsvendors
before placing orders, which results in a transshipment pricing mechanism. Under this contract,
the total newsvendor profit can be maximized and both newsvendors are better off. Based on
this mechanism, the supply chain profit can be maximized if the wholesale prices are within a
certain range.
Chapter 5

Multi-newsvendor Game with Limited Supply

5.1 Introduction

In this chapter, we generalize the two-newsvendor game with limited supply in Chapter 3 to the case that involves an arbitrary number of newsvendors, as shown in Figure 5.1.

Figure 5.1: Supply chain structure of multi-newsvendor game with limited supply
Accordingly, the inventory allocation rule needs to be generalized to fit the case with multiple newsvendors. We characterize the special form of the Nash equilibrium. Taking the proportional rule as an example, we demonstrate how to calculate the explicit form of Nash equilibrium. Based on the properties of the Nash equilibrium, we develop a procedure to find the coordinating wholesale prices so that the supply chain profit can be maximized.

In Section 5.2, we define the game model for multiple newsvendors with limited supply and introduce new notation. In Section 5.3, we investigate the existence and uniqueness of the Nash equilibrium of orders and equilibrium inventory allocations. In Section 5.4, we develop a procedure to calculate coordinating wholesale prices of multiple newsvendors for maximizing supply chain profit. Section 5.5 summarizes this chapter.

5.2 The Model

We adopt all settings of the two-newsvendor game with limited supply in Chapter 3 except that we suppose there are \( M \) newsvendors where \( M \) is an integer greater than or equal to 2. Denote the vector of order decisions by \( \mathbf{y} = (y_1, y_2, \cdots, y_M) \) and let \( \mathbf{y}_{-i} = (y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_M) \) denote order decisions of newsvendors other than the newsvendor \( i \). Similarly, we use \( \mathbf{x} = (x_1, x_2, \cdots, x_M) \) to denote the inventory allocation and \( \mathbf{x}_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_M) \) the inventory allocation of all other newsvendors except the newsvendor \( i \). The inventory allocation rule is a function

\[
\varphi(\mathbf{y}) = (\varphi_1(y_1, \mathbf{y}_{-1}), \cdots, \varphi_i(y_i, \mathbf{y}_{-i}), \cdots, \varphi_M(y_M, \mathbf{y}_{-M}))
\]

where \( \varphi_i(y_i, \mathbf{y}_{-i}) \) is newsvendor \( i \)'s allocated inventory as a function of \( y_i \) and \( \mathbf{y}_{-i} \). We assume that the inventory allocation rule \( \varphi \) satisfies the following conditions:

(I) (Feasible condition) \( \sum_{i=1}^{M} \varphi_i(y_i, \mathbf{y}_{-i}) \leq K \) and \( 0 \leq \varphi_i(y_i, \mathbf{y}_{-i}) \leq y_i, i = 1, \ldots, M \).

(II) (Efficient condition) If \( \sum_{i=1}^{M} y_i \leq K \), then \( \varphi_i(y_i, \mathbf{y}_{-i}) = y_i, i = 1, \ldots, M \). If \( \sum_{i=1}^{M} y_i > K \),
then $\sum_{i=1}^{M} \varphi_i(y_i, y_{-i}) = K$.

(III) (Increasing condition) $\varphi_i(\tilde{y}_i, y_{-i}) \geq \varphi_i(y_i, y_{-i})$ for all $\tilde{y}_i > y_i$ and $y_{-i}$, $i = 1, \ldots, M$.

(IV) (Individually responsive condition) If $0 < \varphi_i(y_i, y_{-i}) < K$ and $\tilde{y}_i > y_i$, then $\varphi_i(\tilde{y}_i, y_{-i}) > \varphi_i(y_i, y_{-i})$, $i = 1, \ldots, M$.

(V) (Fair condition) If $y_i \geq y_j$ in $y$, then $\varphi_i(y_i, y_{-i}) \geq \varphi_j(y_j, y_{-j})$, $i, j = 1, \ldots, M, i \neq j$.

(VI) (Continuous condition) $\varphi_i(y_i, y_{-i})$ is continuous in $(y_i, y_{-i})$, $i = 1, \ldots, M$.

The proportional rule for multiple newsvendors becomes

$$
\varphi_i(y_i, y_j) = \begin{cases} 
y_i, & \text{if } \sum_{i=1}^{M} y_i \leq K, \\
y_i - \frac{y_i}{\sum_{i=1}^{M} y_i} K, & \text{if } \sum_{i=1}^{M} y_i > K.
\end{cases}
$$

The linear rule becomes

$$
\varphi_i(y_i, y_j) = \begin{cases} 
y_i, & \text{if } \sum_{i=1}^{M} y_i \leq K, \\
y_i - \frac{\sum_{i=1}^{M} y_i - K}{M}, & \text{if } \sum_{i=1}^{M} y_i > K.
\end{cases}
$$

Both newsvendors are assumed to have complete information about the parameters of the game, and they are rational with respective objectives to maximize their own profits by making ordering decisions unilaterally and simultaneously. We call this model a multi-newsvendor game with limited supply.

5.3 Game Theoretic Analysis

In this section, we study the existence, uniqueness and the explicit forms of the Nash equilibrium in the multi-newsvendor game with limited supply. To begin with, we note that the profit
function of the newsvendor \( i \) is

\[
\Pi_i^n(y_i, y_{-i}) = -w_i \varphi_i(y_i, y_{-i}) + r_i E[\min\{\varphi_i(y_i, y_{-i}), D_i\}] + s_i E[(\varphi_i(y_i, y_{-i}) - D_i)^+].
\]

Similarly to Theorem 3.1, we have the next theorem about the quasi-concavity of \( \Pi_i^n(y_i, y_{-i}) \).

**Theorem 5.1.** In the multi-newsvendor game with limited supply, the profit function \( \Pi_i^n(y_i, y_{-i}) \) is quasi-concave with respect to \( y_i \) for any given \( y_{-i} \) with \( 0 \leq y_j \leq K, j \neq i \).

**Proof.** Note that \( \varphi_i(y_i, y_{-i}) \) is non-decreasing in \( y_i \). Then the proof of Theorem 3.1 can be applied. \( \square \)

Therefore, we have the next theorem about the existence of the Nash equilibrium.

**Theorem 5.2.** There exists a Nash equilibrium in the multi-newsvendor game with limited supply.

**Proof.** By Theorem 5.1, \( \Pi_i^n(y_i, y_{-i}) \) is quasi-concave in \( y_i \) for all \( i \). Note that \( \Pi_i^n(y_i, y_{-i}) \) is continuous and the decision space of each newsvendor is the closed interval \([0, K]\). The existence of Nash equilibrium then follows Fudenberg and Tirole (1991). \( \square \)

We use \( y^{N,n} = (y_1^{N,n}, \ldots, y_M^{N,n}) \) to denote the Nash equilibrium and use \( x^{N,n} \) to denote the equilibrium inventory allocation. Next we discuss the forms that \( y^{N,n} \) and \( x^{N,n} \) should have. Recall that \( x^*_i \) denotes the desired amount of inventory by the newsvendor \( i \). Without loss of generality, we may assume that \( x^*_1 \leq x^*_2 \leq \cdots \leq x^*_M \). The next lemma implies that there exists an integer \( L \) such that the first \( L \) newsvendors receive their desired amount of inventory, while the last \( M - L \) receive less than what they desire.

**Lemma 5.1.** Assume that \( x^*_1 \leq x^*_2 \leq \cdots \leq x^*_M \). If \( x^{N,n} \) is the equilibrium inventory allocation, then there exists an integer \( L \) such that \( x_i^{N,n} = x^*_i \) for all \( i = 1, \ldots, L \) and \( x_i^{N,n} < x^*_i \) for all \( i = L + 1, \ldots, M \).
Proof. Note that it is impossible to have $x_{i}^{N,n} > x_{i}^*$ for any $i = 1, \ldots, M$ because otherwise a newsvendor may reduce his order to have inventory allocation of $x_{i}^*$. If $x_{i}^{N,n} = x_{i}^*$, then $y_{i}^{N,n} \leq K$. If $x_{i}^{N,n} < x_{i}^*$, then $y_{i}^{N,n} = K$. Denote $I = \{ i | y_{i}^{N,n} \leq K \text{ and } x_{i}^{N,n} = x_{i}^* \}$ and $J = \{ j | y_{j}^{N,n} = K \text{ and } x_{j}^{N,n} < x_{j}^* \}$. For any $i \in I$ and $j \in J$, by the fair condition (V), we know $x_{i}^{N,n} \leq x_{j}^{N,n}$. Therefore, $x_{i}^* = x_{i}^{N,n} \leq x_{j}^{N,n} < x_{j}^*$. Let $L = |I|$ and we have $x_{i}^{N,n} = x_{i}^*$ for all $i = 1, \ldots, L$ and $x_{i}^{N,n} < x_{i}^*$ for all $i = L + 1, \ldots, M$.

To calculate the Nash equilibrium, we need to find $L$. To be consistent, we define $\sum_{i=1}^{0} x_{i}^* = 0$. Note that $K = \sum_{i=1}^{L} x_{i}^* + \sum_{j=L+1}^{M} x_{j}^{N,n}$ and $x_{L}^* \leq x_{L+1}^{N,n} < x_{L+1}^*$. Therefore, $\sum_{i=1}^{L-1} x_{i}^* + (M - L + 1)x_{L}^* \leq K < \sum_{i=1}^{L} x_{i}^* + (M - L)x_{L+1}^*$ holds for $K$ and $L$.

Given $K$, we can calculate the Nash equilibrium $y_{i}^{N,n}$ and the corresponding equilibrium inventory allocation $x_{i}^{N,n}$ by the following procedure:

Step 1. Calculate $x_{i}^* = F_{D_{i}}^{-1}(\frac{r_{i} - c_{i}}{r_{i} - s_{i}})$.

Step 2. Find $L$ that satisfies $\sum_{i=1}^{L-1} x_{i}^* + (M - L + 1)x_{L}^* \leq K < \sum_{i=1}^{L} x_{i}^* + (M - L)x_{L+1}^*$.

Step 3. Set $y_{i}^{N,n} = K$ for all $i = L + 1, \ldots, M$.

Step 4. Solve the system of equations

$$
\begin{align*}
\varphi_{1}(y^{N,n}) &= x_{1}^*, \\
& \vdots \\
\varphi_{L}(y^{N,n}) &= x_{L}^*,
\end{align*}
$$

where $y^{N,n} = (y_{1}^{N,n}, \ldots, y_{L}^{N,n}, K, \ldots, K)$ and the unknowns are $y_{1}^{N,n}, \ldots, y_{L}^{N,n}$.

Step 5. Find $x_{i}^{N,n} = \varphi(y^{N,n})$ according to the allocation rule $\varphi$.

The system of equations in Step 4 has a solution because

$$(x_{1}^*, \ldots, x_{L}^*, \frac{K - \sum_{i=1}^{L} x_{i}^*}{M - L}, \ldots, \frac{K - \sum_{i=1}^{L} x_{i}^*}{M - L})$$
is a feasible inventory allocation and \( x_1^*, \ldots, x_M^* \) are in an ascending order. That means that we can find \( 0 \leq y_1 \leq \cdots \leq y_L \leq K \), given \( y_{L+1} = \cdots = y_M = K \), such that the system of equations in Step 4 is satisfied. The procedure above applies to any allocation rule. Take the proportional allocation rule as an example. We give the explicit form of the Nash equilibrium (or equilibria if the Nash equilibrium is not unique) and the equilibrium inventory in the next theorem.

**Theorem 5.3.** Assume \( x_1^* \leq x_2^* \leq \cdots \leq x_M^* \) and the supply is allocated according to the proportional rule. The Nash equilibrium of orders and equilibrium inventory are

1. If \( 0 \leq K < Mx_1^* \), then
   \[
   y^{N,n} = (K, \ldots, K)
   \]
   and \( x^{N,n} = (K, \ldots, K \frac{K}{M}) \).

2. If \( Mx_1^* \leq K < x_1^* + (M - 1)x_2^* \), then
   \[
   y^{N,n} = \left( \frac{(M - 1)x_1^*}{K - x_1^*}, K, \ldots, K \right)
   \]
   and \( x^{N,n} = \left( x_1^*, \frac{K - x_1^*}{M - 1}, \ldots, \frac{K - x_1^*}{M - 1} \right) \).

3. For \( L = 2, \ldots, M \), if \( \sum_{i=1}^{L-1} x_i^* + (M - L + 1)x_L^* \leq K < \sum_{i=1}^{L} x_i^* + (M - L)x_{L+1}^* \), then
   \[
   y^{N,n} = \left( \frac{(M - L)x_1^*}{K - \sum_{i=1}^{L} x_i^*}, \ldots, \frac{(M - L)x_L^*}{K - \sum_{i=1}^{L} x_i^*}, K, \ldots, K \right)
   \]
   and \( x^{N,n} = \left( x_1^*, \ldots, x_L^*, \frac{K - \sum_{i=1}^{L} x_i^*}{M - L}, \ldots, \frac{K - \sum_{i=1}^{L} x_i^*}{M - L} \right) \).

4. If \( K > \sum_{i=1}^{M} x_i^* \), then
   \[
   y^{N,n} = (x_1^*, \ldots, x_M^*)
   \]
   and \( x^{N,n} = (x_1^*, \ldots, x_M^*) \).
(5) (Multiple Nash equilibria) If $K = \sum_{i=1}^{M} x_i^*$, then

$$y_{N,n}^* = (y_{N,n}^1, \frac{x_2^*}{x_1^*}y_{N,n}^1, \ldots, \frac{x_M^*}{x_1^*}y_{N,n}^M)$$

and $x_{N,n}^* = (x_1^*, \ldots, x_M^*)$.

Proof. Following the calculating procedure for Nash equilibrium, this theorem is then proved.

We conclude this section by the next theorem on the uniqueness of $y_{N,n}^*$ and $x_{N,n}^*$.

**Theorem 5.4.** In the multi-newsvendor game with limited supply, the Nash equilibrium is unique except when $K = \sum_{i=1}^{M} x_i^*$. The inventory allocation at the Nash equilibrium is always unique.

Proof. This theorem follows Lemma 5.1 and the calculation procedure of $y_{N,n}^*$ and $x_{N,n}^*$.

### 5.4 Channel Coordination

In this section, we provide a procedure to determine the wholesale prices that coordinate the newsvendors so that the supply chain profit can be maximized. We denote the profit of the supplier in the multi-newsvendor game with limited supply as

$$\pi_0(x) = \sum_{i=1}^{M} (w_i - w_0) x_i$$

where $w_0$ is the production cost of the supplier. Assume $w_0 > s_i, i = 1, \ldots, M$. The supply chain profit then becomes

$$\pi_C^n(x) = \pi_0(x) + \sum_{i=1}^{M} \pi_i^n(x_i).$$
The first-best inventory allocation \( x^{C,n} = (x^{C,n}_1, \ldots, x^{C,n}_M) \) is an optimal solution to the following problem (P\(_n\)):

\[
\begin{align*}
\text{max} & \quad \pi^n_C(x) \\
\text{s.t.} & \quad \sum_{i=1}^{M} x_i \leq K \\
& \quad x_i \geq 0, \quad i = 1, \ldots, M.
\end{align*}
\]

To find the coordinating wholesale prices \( w^{C,n}_1, \ldots, w^{C,n}_M \), we need to check the pattern of \( x^{C,n} \) against the forms of Nash equilibrium inventory allocation that is specified in Theorem 5.3. A complete procedure is as follows.

Step 1. Solve problem (P\(_n\)) and find \( x^{C,n} = (x^{C,n}_1, \ldots, x^{C,n}_M) \). Sort \( x^{C,n}_1, \ldots, x^{C,n}_M \) in ascending order. We still use \( x^{C,n} \) to denote it after sorted.

Step 2. If \( \sum_{i=1}^{M} x^{C,n}_i < K \), then solve \( x^{*,n}_i(w_i) = x^{C,n}_i \) for all \( i = 1, \ldots, M \). In this case, we have

\[
w^{C,n}_i = r_i - (r_i - s_i)F_{D_i}(x^{C,n}_i), \quad i = 1, \ldots, M.
\]

Otherwise, we know \( \sum_{i=1}^{M} x^{C,n}_i = K \). Find the smallest integer \( L \) such that \( x^{C,n}_{L+1} = \ldots = x^{C,n}_M \). Solve \( x^{*,n}_i(w_i) = x^{C,n}_i \) for all \( i = 1, \ldots, L \). We have

\[
w^{C,n}_i = r_i - (r_i - s_i)F_{D_i}(x^{C,n}_i), \quad i = 1, \ldots, L.
\]

For the newsvendor \( i = L + 1, \ldots, M \), the inequality \( K < \sum_{i=1}^{L} x^{C,n}_i + (M - L)x^{C,n}_i \) must hold. Therefore,

\[
w_0 \leq w^{C,n}_i \leq r_i - (r_i - s_i)F_{D_i}\left(\frac{K - \sum_{i=1}^{L} x^{C,n}_i}{M - L}\right), \quad i = L + 1, \ldots, M.
\]

We illustrate the multi-newsvendor game with limited supply and the channel coordination method by the next example.
Example 5.4.1. Consider a three-newsvendor game with limited supply and $K = 10$. $w_1 = w_2 = w_3 = 9$, $r_1 = r_2 = r_3 = 10$, $s_1 = s_2 = s_3 = 1$ and the supplier uses the proportional allocation rule. Suppose the demand of the newsvendors are uniformly distributed as $D_1 \sim U[4, 13]$, $D_2 \sim U[2, 11]$, $D_3 \sim U[3, 12]$. We have $x_1^* = 5$, $x_2 = 3$ and $x_3^* = 4$. The ascending order of $x_1^*, x_2^*, x_3^*$ is $x_2^*, x_3^*, x_1^*$.

Check the cases in Theorem 5.3 and we know $3x_2^* \leq K < x_2^* + 2x_3^*$. Therefore, $L = 1$ and the Nash equilibrium is $y^{N,n} = (K, \frac{2x_2^*}{K-x_2^*}K, K) = (10, 8.57, 10)$. The equilibrium inventory allocation is $x^{N,n} = (3.5, 3, 3.5)$.

Suppose $w_0 = 8$. We look for coordinating wholesale prices $(w_1^{C,n}, w_2^{C,n}, w_3^{C,n})$. First, by solving problem $P_n$, we have $(x_1^{C,n}, x_2^{C,n}, x_3^{C,n}) = (4.33, 2.33, 3.33)$. Second, we sort $x^{C,n}$ in ascending order as $(x_2^{C,n}, x_3^{C,n}, x_1^{C,n})$ and know that $L = 2$. Therefore,

$$w_2^{C,n} = r_2 - (r_2 - s_2)F_{D_2}(x_2^{C,n}) = 9.67,$$

and $w_3^{C,n} = 9.67$. Lastly, $w_1^{C,n}$ can be chosen in $8 \leq w_1^{C,n} \leq 9.67$.

5.5 Summary

This chapter extends Chapter 3 to a general case of two or more newsvendors. The Nash equilibrium exists and can be calculated by noticing its special form and solving a system of equations. Channel coordination is applicable to find coordinating wholesale prices. In the next chapter, we will see that inventory transshipment brings much complexity in modeling and investigating multiple newsvendors with limited supply.
Chapter 6

Multi-newsvendor Games with Limited Supply and Inventory Transshipment

6.1 Introduction

In this chapter, we consider multi-newsvendor games with limited supply and inventory transshipment. The supply chain structure is shown Figure 6.1 (taking three newsvendors for illustrative purpose).

When there are more than two newsvendors in the game with inventory transshipment, there may be more than one newsvendor who has surplus inventory and more than one newsvendor who has excess demand. A certain rule of inventory transshipment has to be specified to determine the amount of surplus inventory and excess demand that is shared between each pair of newsvendors for any inventory allocation and demand realization. Note that this is not an issue in the two-newsvendor game in Chapter 4 because when there are only two newsvendors, the direction of inventory transshipment is fixed. A newsvendor with surplus inventory may only sell to the other newsvendor.
We study two models in this chapter. In Section 6.2, we assume that the portion of surplus inventory and excess demand that a newsvendor would like to share with another newsvendor is deterministic and fixed. The analysis is similar to Chapter 4 with more complexity added in higher dimensional spaces. We provide conditions under which there exists Nash equilibrium.

In Section 6.3, we develop a transshipment fund mechanism to manage the profit that each newsvendor receives from the transshipment. By allocating the initial payments that the newsvendors make to the fund, each newsvendor’s profit differs from the total newsvendor profit by a constant. In this way, the newsvendors will coordinate to place the newsvendor-best orders and maximize their total profit. The supply chain profit is also maximized as long as the wholesale prices fall in a certain range.
6.2 Deterministic Transhipment Rates

In this section, we study a direct generalization of the two-newsvendor game with limited supply and inventory transshipment. Assume that the portion of surplus inventory and the portion of excess demand that a newsvendor would like to share with another newsvendor are deterministic. Denote by $\alpha_{ij}$ the portion of surplus inventory that the newsvendor $i$ is willing to share with the newsvendor $j$, and $\beta_{ij}$ the portion of excess demand that the newsvendor $i$ is willing to share with the newsvendor $j$. Assume $\sum_{j \neq i} \alpha_{ij} \leq 1$ and $\alpha_{ii} = 0$ for all $i$, $\sum_{j \neq i} \beta_{ij} \leq 1$ and $\beta_{ii} = 0$ for all $i$, and without loss of generality, $\alpha_{ij} > 0$ and $\beta_{ij} > 0$, for all $i, j, i \neq j$. We first obtain analytical results about the game with unlimited supply in Section 6.2.1. Then, we study the game with limited supply in Section 6.2.2.

6.2.1 Multi-newsvendor Game with Unlimited Supply and Deterministic Transshipment Rates

Before we study the limited supply problem, we consider the case when supply is unlimited. The profit function of the newsvendor $i$ is

$$
\pi_i(x_i, x_{-i}) = -w_i x_i + r_i E[\min\{D_i, x_i\}] + s_i E[(x_i - D_i)^+] \\
+ \sum_{j \neq i} (c_{ij} - t_{ij} - s_i) E[\min\{\alpha_{ij}(x_i - D_i)^+, \beta_{ji}(D_j - x_j)^+\}] \\
+ \sum_{j \neq i} (r_i - c_{ji}) E[\min\{\beta_{ij}(D_j - x_i)^+, \alpha_{ji}(x_j - D_j)^+\}].
$$
where \( x_{-i} \) denotes the vector of other newsvendors’ inventory. The first and second order derivatives of \( \pi_i(x_i, x_{-i}) \) with respect to \( x_i \) are

\[
\frac{\partial \pi_i(x_i, x_{-i})}{\partial x_i} = (r_i - w_i) - (r_i - s_i) Pr(D_i \leq x_i) \\
+ \sum_{j \neq i} \alpha_{ij} (c_{ij} - t_{ij} - s_i) Pr(D_i \leq x_i, \alpha_{ij} (x_i - D_i) \leq \beta_{ji} (D_j - x_j)) \\
- \sum_{j \neq i} \beta_{ij} (r_i - c_{ji}) Pr(x_i \leq D_i, \beta_{ij} (D_i - x_i) \leq \alpha_{ji} (x_j - D_j)),
\]

and

\[
\frac{\partial^2 \pi_i(x_i, x_{-i})}{\partial x_i^2} = - (r_i - s_i) f_{D_i}(x_i) \\
+ \sum_{j \neq i} \alpha_{ij} (c_{ij} - t_{ij} - s_i) \left[ Pr(\alpha_{ij} D_i + \beta_{ji} D_j \geq \alpha_{ij} x_i + \beta_{ji} x_j) f_{D_i | \alpha_{ij} D_i + \beta_{ji} D_j \geq \alpha_{ij} x_i + \beta_{ji} x_j}(x_i) \right] \\
- \alpha_{ij} Pr(D_i \leq x_i) f_{\alpha_{ij} D_i + \beta_{ji} D_j | D_i \leq x_i}(\alpha_{ij} x_i + \beta_{ji} x_j) \\
- \sum_{j \neq i} \beta_{ij} (r_i - c_{ji}) \left[ \beta_{ij} Pr(x_i \leq D_i) f_{\beta_{ij} D_i + \alpha_{ji} D_j | x_i \leq D_i}(\beta_{ij} x_i + \alpha_{ji} x_j) \right] \\
- Pr(\beta_{ij} D_i + \alpha_{ji} D_j \leq \beta_{ij} x_i + \alpha_{ji} x_j) f_{D_i | \beta_{ij} D_i + \alpha_{ji} D_j \leq \beta_{ij} x_i + \alpha_{ji} x_j}(x_i),
\]

respectively.

Suppose \( \alpha_{ij} = \beta_{ij} \) for all \( j \neq i \), which means that the portion of surplus inventory and the portion of excess demand that a newsvendor would like to share with another newsvendor are the same. In this case, the next theorem states that \( \pi_i(x_i, x_{-i}) \) is concave in \( x_i \) for any given \( x_{-i} \).

**Theorem 6.1.** If \( \alpha_{ij} = \beta_{ij} \) for all \( i, j \neq j \), then \( \pi_i(x_i, x_{-i}) \) is concave in \( x_i \) for any given \( x_{-i} \) and all \( i \).
Proof. When \( \alpha_{ij} = \beta_{ij} \), we have

\[
f_{D_i}(x_i) = \text{Pr}(\alpha_{ij}D_i + \beta_{ji}D_j \geq \alpha_{ij}x_i + \beta_{ji}x_j) f_{D_i}(\alpha_{ij}D_i + \beta_{ji}D_j \geq \alpha_{ij}x_i + \beta_{ji}x_j)(x_i) \\
+ \text{Pr}(\beta_{ij}D_i + \alpha_{ji}D_j \leq \beta_{ij}x_i + \alpha_{ji}x_j) f_{D_i}(\beta_{ij}D_i + \alpha_{ji}D_j \leq \beta_{ij}x_i + \alpha_{ji}x_j)(x_i)
\]

Note \( r_i - s_i > c_{ij} - t_{ij} - s_i \) and \( r_i - s_i > r_i - c_{ji} \). Therefore, \( \frac{\partial^2 \pi_i(x_i, x_{-i})}{\partial x_i^2} \leq 0 \) and then \( \pi_i(x_i, x_{-i}) \) is concave in \( x_i \).

Consequently, we have the next theorem about the existence of Nash equilibrium in the unlimited supply case.

**Theorem 6.2.** If \( \alpha_{ij} = \beta_{ij} \) for all \( i, j, i \neq j \), then there exists a Nash equilibrium in the multi-newsvendor game with unlimited supply and deterministic transshipment rates.

*Proof.* Without loss of generality, we may assume that the decision space of each newsvendor is the closed interval \([0, Q]\) where \( Q \) is a sufficiently large number. Moreover, the profit function \( \pi_i(x_i, x_{-i}) \) is continuous and concave in \( x_i \). Therefore, the Nash equilibrium exists. \( \square \)

Generally, in a competitive game, we have the next theorem about the uniqueness the Nash equilibrium. (See for example, Cachon and Netessine (2004).)

**Theorem 6.3.** In a competitive game of \( M \) players with \( \pi_i(x_i, x_{-i}) \) being the profit function of the player \( i \), if the Nash equilibrium exists, then it is unique if the matrix

\[
\begin{bmatrix}
\frac{\partial^2 \pi_1(x_1, x_{-1})}{\partial x_1^2} & \cdots & \frac{\partial^2 \pi_1(x_1, x_{-1})}{\partial x_M \partial x_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 \pi_M(x_M, x_{-M})}{\partial x_1 \partial x_M} & \cdots & \frac{\partial^2 \pi_M(x_M, x_{-M})}{\partial x_M^2}
\end{bmatrix}
\]

is strictly diagonally dominant.

A matrix \((a_{ij})_{n \times n}\) is called (strictly) diagonally dominant if \( \sum_{j=1, j \neq i}^n |a_{ij}| \leq (\leq) |a_{ii}| \) holds for all \( i = 1, \dotsc, n \). From now on, besides \( \alpha_{ij} = \beta_{ij} \) for all \( i, j, i \neq j \), we suppose \( \alpha_{ij} = \alpha_{ji} \) for all
This means that every pair of newsvendors have equal portions of shared surplus inventory and excess demand. The next theorem gives a sufficient condition for the uniqueness of Nash equilibrium.

**Theorem 6.4.** If \( \alpha_{ij} = \beta_{ij} \) for all \( i, j, i \neq j \) and \( \alpha_{ij} = \alpha_{ji} \) for all \( i, j \), the Nash equilibrium is unique.

**Proof.** Notice that

\[
\frac{\partial^2 \pi_i(x_i, x_{-i})}{\partial x_j \partial x_i} = -\alpha_{ij}\beta_{ji}(c_{ij} - t_{ij} - s_i)\Pr(D_i \leq x_i)f_{\alpha_{ij}D_i+\beta_{ji}D_j|D_i \leq x_i}(\alpha_{ij}x_i + \beta_{ji}x_j)]
- \beta_{ij}\alpha_{ji}(r_i - c_{ji})\Pr(x_i \leq D_i)f_{\beta_{ij}D_i+\alpha_{ji}D_j|x_i \leq D_i}(\beta_{ij}x_i + \alpha_{ji}x_j).
\]

If \( \alpha_{ij} = \beta_{ij} \) for all \( i, j, i \neq j \) and \( \alpha_{ij} = \alpha_{ji} \) for all \( i, j \), we can easily verify that

\[
\left| \frac{\partial^2 \pi_i(x_i, x_{-i})}{\partial x_i^2} \right| > \sum_{j \neq i} \left| \frac{\partial^2 \pi_i(x_i, x_{-i})}{\partial x_j \partial x_i} \right|
\]

holds for all \( i \). By Theorem 6.3, the Nash equilibrium is unique.

Based on the analysis and results for the case of unlimited supply, we are able to study the multi-newsvendor game with limited supply and deterministic transshipment rates.

### 6.2.2 Multi-newsvendor Game with Limited Supply and Deterministic Transshipment Rates

Now we turn to the limited supply case and investigate the conditions under which there exists a Nash equilibrium of orders. For simplicity, we assume the supply is allocated according to the proportional rule in this section. Cases with other allocation rules can be studied similarly. We introduce new definitions and notations that will be needed. The “supply surface” is defined to be the set

\[
S \triangleq \{ x \in \mathbb{R}^M | \sum_{i=1}^M x_i = K, x_i \geq 0, \forall i \}.
\]
For any given \( x^0_{-i} \) such that \( \sum_{j \neq i} x^0_j = K \), we define the allocation line associated with \( x^0_{-i} \) to be the set

\[
A(x^0_{-i}) \triangleq \{ x \in \mathbb{R}^M | 0 \leq x_i \leq K, x_{-i} = \frac{x^0_{-i}}{\sum_{j \neq i} x^0_j} (K - x_i) \}.
\]

The profit function \( \pi_i(x_i, x^0_{-i}) \) is assumed to achieve its maximum on \( 0 \leq x_i \leq K \) at \( x^M_i(x^0_{-i}) \). Denote the best response of the newsvendor \( i \) to \( x_{-i} \) in the game with unlimited supply \( R^u_i(x_{-i}) = \arg \max_{x_i \geq 0} \pi_i(x_i, x_{-i}) \).

The next theorem provides sufficient conditions under which a newsvendor’s profit function is quasi-concave.

**Theorem 6.5.** Assume \( \alpha_{ij} = \beta_{ij} \) for all \( i, j, i \neq j \) and the supplier uses the proportional allocation rule in the multi-newsvendor game with limited supply and deterministic transshipment rates. Then \( \Pi_i(y_i, y_{-i}) \) is quasi-concave with respect to \( y_i \) for any given \( y_{-i} \) if for each \( x^0_{-i} \) such that \( \sum_{j \neq i} x^0_j = K \), the following two conditions hold:

(C1) \( \pi_i(x_i, \frac{x^0_{-i}}{\sum_{j \neq i} x^0_j} (K - x_i)) \) is quasi-concave in \( 0 < x_i < K \).

(C2) \( x^M_i(x^0_{-i}) \leq R^u_i(\frac{x^0_{-i}}{\sum_{j \neq i} x^0_j} (K - x^M_i(x^0_{-i}))) \).

**Proof.** The proof is similar to the proof of Theorem 4.2.

Compared with the two-newsvendor game in Chapter 4, we find that the condition (C1) in Theorem 6.5 involves each \( x^0_{-i} \) such that \( \sum_{j \neq i} x^0_j = K \). This is because the allocation line \( A(x^0_{-i}) \) is in a multi-dimensional surface \( S \) rather than a one-dimensional “supply line”. We have the next theorem on the existence of Nash equilibrium of order quantities.

**Theorem 6.6.** Assume \( \alpha_{ij} = \beta_{ij} \) for all \( i, j, i \neq j \) in the multi-newsvendor game with limited supply and inventory transshipment. There exists a Nash equilibrium of orders if conditions (C1) and (C2) hold for \( i, j, i \neq j \).
Sharing surplus inventory and excess demand according to deterministic rates may not lead to the best match of supply and demand. Therefore, the transshipment rule in this model does not necessarily generate the largest transshipment profit possible. Next, we study a mechanism that enables the supply chain to match supply with demand as much as possible, with transshipment profit appropriately allocated.

### 6.3 Transshipment Fund Mechanism

In this section, we consider that the surplus inventory and excess demand of all newsvendors are matched to maximize the total transshipment profit. In this setting, we introduce a mechanism that sets up a transshipment fund and coordinates the newsvendor orders to maximize the total profit of all newsvendors and the fund.

To generate as much transshipment profit as possible, the best match of all surplus inventory and excess demand can be determined by solving the following transportation problem (P_T):

\[
\begin{align*}
\text{max} \quad & \sum_i \sum_j (r_j - s_i - t_{ij}) T_{ij} \\
\text{s.t.} \quad & \sum_{j \neq i} T_{ij} \leq (\varphi_i(y) - D_i)^+, \quad \forall i \\
& \sum_{j \neq i} T_{ji} \leq (D_i - \varphi_i(y))^+, \quad \forall i \\
& T_{ij} \geq 0, \quad \forall i, j.
\end{align*}
\]

where \( T_{ij}, i, j = 1, \ldots, M \) are decision variables. The newsvendors do not pay transshipment prices to each other. Instead, the total transshipment profit \( \sum_i \sum_j (r_j - s_i - t_{ij}) T_{ij} \) will be allocated among the newsvendors. Furthermore, this allocation method is desired to induce the newsvendors to place the first-best order quantities and maximize their total profits. When the supply is unlimited, Hanany et al. (2010) propose a transhipment fund mechanism to coordinate multiple newsvendors with inventory transshipment. We will develop a transshipment fund mechanism for the limited supply case. By this mechanism, a fund is financially supported by all newsvendors before they place orders. After inventory transshipment, the fund redistributes
a certain amount of money among the newsvendors according to each newsvendor’s contribution to the total profit of all newsvendors and the fund. In this way, a newsvendor’s profit is aligned with his contribution to the total profit of all newsvendors and the fund. Therefore, each newsvendor has the incentive to coordinate and place the first-best order quantities.

Formally, the transshipment fund mechanism runs as following:

Step 1. Each newsvendor pays \( m_i, i = 1, \ldots, M \) to the fund.

Step 2. The newsvendors place order \( y \).

Step 3. After demand realization, the newsvendors conduct inventory transshipment according to the optimal solution of problem \( (P_T) \).

Step 4. The fund pays each newsvendor amount of \( C_i(y, D) \) which depends on demand and orders.

We use \( D = (D_1, \ldots, D_M) \) to denote the vector of demands at the newsvendors. Denote \( y^N = (y^N_1, \ldots, y^N_M) \) the Nash equilibrium of order quantity and \( y^* = (y^*_1, \ldots, y^*_M) \) the first-best order quantity. For any order quantities \( y \) and demand \( D \), let \( T_{ij}(y, D) \) denote amount of inventory that is transshipped from the newsvendor \( i \) to the newsvendor \( j \) according to the optimal solution of the corresponding transportation problem \( (P_T) \). Recall that \( \pi_i^n(y^{N,n}) \) is newsvendor \( i \)'s expected profit in the multi-newsvendor game with limited supply. Let \( C(y, D) = (C_1(y, D), \ldots, C_M(y, D)) \).

Under the transshipment fund mechanism, the expected profit of the newsvendor \( i, i = 1, \ldots, M \), is

\[
\Pi_i(y, C_i, m_i) = -m_i + \Pi_i^n(y) + \mathbb{E} \left[ \sum_{j \neq i} r_{ij} T_{ji}(y, D) - \sum_{j \neq i} (s_i + t_{ij}) T_{ij}(y, D) \right] + \mathbb{E}[C_i(y, D)]
\]

where \( \Pi_i^n(y) = -w_i \varphi_i(y) + r_i \mathbb{E}[\min\{D_i, \varphi_i(y)\}] + s_i \mathbb{E}[(\varphi_i(y) - D_i)^+] \) is the part of profit that is not related to inventory transshipment. We include \( C_i \) and \( m_i \) in the notation of \( \Pi_i(y, C_i, m_i) \).
to indicate that the profit of a newsvendor depends on the transshipment profit allocation $C_i$ and initial payment $m_i$. The profit of the fund is

$$\Pi_F(y; C, m) = \mathbb{E} \left[ \sum_{i=1}^{M} (m_i - C_i(y, D)) \right].$$

The total profit of all newsvendors and the fund is

$$\Pi_R(y) = \sum_i \Pi^i(y) + \mathbb{E} \left[ \sum_i \sum_j [r_iT_{ji}(y, D) - (s_i + t_{ji})T_{ij}(y, D)] \right],$$

which does not depend on $C$ and $m$. Let the newsvendor-best order be

$$y^R = \arg \max_{y: 0 \leq y_i \leq K} \Pi_R(y).$$

The key of the mechanism is to choose $m_i$ and $C_i(y, D)$ such that the following three objectives are satisfied:

(O1) The Nash equilibrium $y^N(C, m)$ is equal to the first-best order quantity $y^R$.

(O2) The equilibrium profit $\Pi_i(y^N; C_i, m_i)$ is greater than or equal to the equilibrium in the benchmark model $\Pi^i(y^{N,n})$.

(O3) The fund’s profit $\mathbb{E} \left[ \sum_i^{M}(m_i - C_i(y^N, D)) \right]$ is nonnegative.

To achieve the first objective, we design functions $C(y, D)$ such that the marginal profit of each newsvendor with respect to his own order quantity, $\frac{\partial \Pi_i}{\partial y_i}$, is equal to the marginal profit of the total profit of the newsvendors and the fund, $\frac{\partial \Pi_R}{\partial y_i}$. To achieve the second and the third objectives, we need to choose appropriate $m = (m_1, \ldots, m_M)$ accordingly.

Given $y$, we use the vector $(0, y_{-i})$ to denote the ordering quantities that the newsvendor $i$ orders zero and the other newsvendors order the same quantity as in $y$. Given $y$ and $D$, if the newsvendor $i$ does not participate in inventory transshipment, we use $T_{kj}^{i}(y, D)$ to denote the
amount of inventory transshipped from the newsvendor \( k \) to the newsvendor \( j, k, j \neq i \). That is, \( T_{kj}^{-i}(y, D) \), \( k, j \neq i \) form the optimal solution to the following problem \( P_{R}^{-i} \):

$$
\begin{align*}
\text{max} & \quad \sum_{k \neq i} \sum_{j \neq i} (r_j - s_k - t_{kj})T_{kj}^{-i} \\
\text{s.t.} & \quad \sum_{j \neq i, j \neq k} T_{kj}^{-i} \leq (\varphi_k(y) - D_k)^+, \quad \forall k \neq i \\
& \quad \sum_{j \neq i, j \neq k} T_{jk}^{-i} \leq (D_k - \varphi_k(y))^+, \quad \forall k \neq i \\
& \quad T_{kj}^{-i} \geq 0, \quad \forall k, j \neq i.
\end{align*}
$$

The next theorem states that with certain form of \( C(y, D) \) and \( m \), all three objectives (O1), (O2) and (O3) can be satisfied, which indicates that the newsvendors can be coordinated under the transshipment fund mechanism.

**Theorem 6.7.** Let

$$
C_i(y, D) = \left[ \sum_{j \neq i} \Pi_j^n(y) - \sum_{j \neq i} \Pi_j^n(0, y_{-i}) \right] + \sum_{j \neq i} [r_j T_{ij}(y, D) - (s_j + t_{ji}) T_{ji}(y, D)] + \sum_{j, k \neq i, j \neq k} (r_j - s_k - t_{kj}) \left[ T_{kj}(y, D) - T_{kj}^{-i}(y, D) \right]
$$

for each \( i \). Then there exists initial payments \( m \), together with \( C(y, D) \), that coordinate the newsvendors.

Before proving Theorem 6.7, we provide useful interpretations. In this theorem, \( C_i(y, D) \) characterizes the changes that the newsvendor \( i \) causes to the profit of all other newsvendors. It contains three components:

1. \( \left[ \sum_{j \neq i} \Pi_j^n(y) - \sum_{j \neq i} \Pi_j^n(0, y_{-i}) \right] \): Change of the classical-newsvendor part of other newsvendors’ profit. This results directly from the supply competition caused by the newsvendor \( i \) ordering \( y_i \).

2. \( \sum_{j \neq i} [r_j T_{ij}(y, D) - (s_j + t_{ji}) T_{ji}(y, D)] \): Change of the other newsvendors’ profit corresponding to transshipment between the newsvendor \( i \) and other newsvendors.
3. $\sum_{j,k\neq i,j\neq k}(r_j - s_k - t_{kj}) \left[ T_{kj}(y, D) - T_{kj}^{-i}(y, D) \right]$: Change of the other newsvendors’ profit corresponding to transshipment within the other newsvendors. This results indirectly from the participation of the newsvendor $i$ in transshipment.

The above three terms will be paid to the newsvendor $i$ by the fund. Together with the profit that is generated purely at the newsvendor $i$’s market, he receives all his contribution to the system. Therefore, the marginal profit of each newsvendor in his own order quantity is the same as the marginal profit of the total profit of the newsvendors and the fund. Therefore, the objective (O1) will be achieved. The initial payments $m$ can be properly chosen to make sure that the other two objectives (O2) and (O3) are satisfied.

Now we prove Theorem 6.7.

Proof. The first-best order quantity $y^R$ is an optimal solution to the following problem $(P_R)$:

$$\max \quad \Pi_R(y)$$
$$\text{s.t.} \quad 0 \leq y_i \leq K, \quad i = 1, \ldots, N.$$  

Note that $\Pi_R(y)$ is quasi-concave. Then $y^R$ solves problem $(P_R)$ if $y^R$ satisfies the KKT conditions as below.

$$\begin{cases} 
\frac{\partial \Pi_R}{\partial y_i} + \mu_i - \lambda_i = 0, & \forall i, \\
\mu_i, \lambda_i \geq 0, & \forall i, \\
\mu_i y_i = 0, & \forall i, \\
\lambda_i (y_i - K) = 0, & \forall i.
\end{cases}$$

At the Nash equilibrium $y^N$, $y^N_i$ solves problems $(P_1), \ldots, (P_M)$ simultaneously, where for any given $y_{-i}$, the problem $(P_i)$ is as following:

$$\max \quad \Pi_i(y_i, y_{-i}; C_i, m_i)$$
$$\text{s.t.} \quad 0 \leq y_i \leq K.$$  

Note that $\Pi_i(y_i, y_{-i}; C_i, m_i)$ is quasi-concave with respect to $y_i$ for any given $y_{-i}$. Therefore, $y_i$
solves problem (P_i) if y_i satisfies the KKT conditions as below.

\[
\begin{align*}
\frac{\partial \Pi_i}{\partial y_i} + \mu_i - \lambda_i &= 0, \\
\mu_i, \lambda_i &\geq 0, \\
\mu_i y_i &= 0, \\
\lambda_i (y_i - K) &= 0.
\end{align*}
\]

It is easy to verify that \( \frac{\partial \Pi_i}{\partial y_i} = \frac{\partial \Pi_i^R}{\partial y_i} \) for any \( i \) and any \( y \). Therefore, \( y^R \) is a Nash equilibrium and any Nash equilibrium \( y^N \) is a first-best order quantity.

Note that \( \Pi_i(y^R) \geq \sum_i \Pi_i^n(y^{N,n}) \). The initial payments \( m \) should satisfy

\[
\begin{align*}
m_i &\leq \Pi_i(y^R; C_i, 0) - \Pi_i^n(y^{N,n}), \\
\sum_i m_i &\geq \sum_i \Pi_i(y^R; C_i, 0) - \Pi_i(y^R).
\end{align*}
\]  

(6.1)

We may let

\[
m_i = \Pi_i(y^R; C_i, 0) - \left[ \Pi_i^n(y^{N,n}) + \frac{\Pi_R(y^R) - \sum_i \Pi_i^n(y^{N,n})}{n} \right].
\]

In this case,

\[
\Pi_i(y^R; C_i, m_i) = \Pi_i^n(y^{N,n}) + \frac{\Pi_R(y^R) - \sum_i \Pi_i^n(y^{N,n})}{n}
\]

and the profit of the fund \( \sum_i m_i - \mathbb{E} \left[ \sum_i C_i(y^R, D) \right] = 0 \).

Note that there may exist other choices of \( m \) because the system of inequalities (6.1) may have more than one solution. It may happen that \( m_i < 0 \) for some \( i \) in which case the fund may pay money to a newsvendor to stimulate his incentive to participate in the game. Meanwhile, it is also possible that \( C_i(y^R, D) < 0 \). This can happen when a newsvendor generates a large portion of the transshipment profit. Based on the transshipment fund mechanism, he should subsidize the other newsvendors through the fund, which leads \( C_i(y^R, D) \) to be negative.

We conclude this section by the next theorem on wholesale prices that coordinate the supply chain.
**Theorem 6.8.** Assume that newsvendors and the fund operate according to the transshipment fund mechanism. If the wholesale prices of all newsvendors are equal, then there exists a range of this wholesale price within which the total profit of the supply chain is maximized, and the supplier receives a nonnegative profit.

*Proof.* Note that the total profit of the newsvendors and the fund is not related to their internal transactions. Similar arguments to Section 4.7.1 and Theorem 4.14 apply here. □

### 6.4 Summary

In multi-newsvendor games with limited supply and inventory transshipment, we have tackled the difficulty of determining transshipment pattern and profit allocation by considering two models. One assumes deterministic transshipment rates, in which case the Nash equilibrium has been proved to exist under certain conditions. The other model features a transshipment fund mechanism, which is capable of coordinating all newsvendors and improving the efficiency of allocating the transshipment profit.
Chapter 7

Conclusions

In this chapter, we summarize the research in this dissertation, highlight its contributions and point out potential directions of future research. Section 7.1 reviews Chapters 3, 4, 5 and 6, respectively. Contributions of each chapter and this dissertation are highlighted. Section 7.2 discusses future research directions.

7.1 Summary and Contributions

This dissertation has studied four newsvendor games with limited supply: a two-newsvendor game with limited supply (Chapter 3), a two-newsvendor game with limited supply and inventory transshipment (Chapter 4), a multi-newsvendor game with limited supply (Chapter 5), and a multi-newsvendor game with limited supply and inventory transshipment (Chapter 6). In each game, we have investigated the existence and uniqueness of the Nash equilibrium of the newsvendors’ ordering decisions. Different coordinating mechanisms are developed in the respective games such that the decentralized ordering decisions of newsvendors jointly maximize the total profit of the supply chain.

Chapter 3 proposes a newsvendor game to model two retailers in a two-echelon supply chain who face stochastic demands and share a limited capacity of the supplier. For a broad class of allocation rules, the explicit form of each newsvendor’s best response and the Nash equilibri-
um of the game are fully characterized. This analysis reveals the gaming effect of the supply constraint on the newsvendors’ strategic order decision. The uniqueness of the equilibrium inventory allocation indicates a stable outcome of the newsvendor game. Accordingly, we adopted the concept of channel coordination in order to find the coordinating wholesale prices, by which the decentralized decisions of newsvendors jointly maximize the total profit of all parties in the supply chain. Our study provides useful insights for the retailers as well as the supplier. On one hand, it assists the retailers in maximizing their own profits and deciding optimal responses to the competitors’ decisions. On the other hand, this work provides decision support for the supplier in capacity planning, wholesale pricing and retail store management. The game in Chapter 3 builds a basic framework for analyzing a newsvendor game with limited supply.

Chapter 4 incorporates the setting of inventory transshipment into the two-newsvendor game with limited supply. This game extends the existing literature of inventory transshipment, which implicitly assumes an unlimited supply. Analysis and results become complicated and are illustrated by examples. In this game, Nash equilibrium exists under certain conditions and its uniqueness is highly related to the relative gaming effects of inventory transshipment and supply constraint. By analytically investigating the case of two symmetric newsvendors and the case when one newsvendor has a stochastically larger demand than the other, we showed that the total equilibrium profit of the two newsvendors increases as the supply capacity increases. Moreover, we investigated the benefit of inventory transshipment to the newsvendors by comparing their profits at the Nash equilibrium with that in the benchmark game without inventory transshipment in Chapter 3. We found that at least one of the two newsvendors can receive a higher profit than that in the benchmark game. Actually, we demonstrated by example that a newsvendor may have a lower equilibrium profit. Nevertheless, in the sense of maximizing the total profit of newsvendors, inventory transshipment is always beneficial. A numerical study indicates that the transshipment prices play a critical role regarding to the allocation of the extra profit generated from inventory transshipment. Accordingly, we designed a contract that allows the newsvendors to negotiate transshipment prices before they place
orders. This contract coordinates the newsvendors in maximizing their total profit and ensures that both newsvendors’ profits are higher than that in the benchmark game. Moreover, the supply chain profit is maximized as long as the wholesale prices fall in a certain range. Through investigating the structural influence of supply capacity, this study brings valuable insights to the well-studied inventory transshipment problem of two newsvendors in a new environment. It helps the retailers evaluate options of whether to participate in inventory transshipment. The supplier, or more generally, the company who owns the supply chain may implement the coordinating contract and properly set wholesale prices in order to maximize the supply chain profit. The game in Chapter 4 is a basic model of two newsvendors that integrates inventory transshipment and limited supply in one setting.

Chapter 5 generalizes the game in Chapter 3 to consider the situation of two or more newsvendors. By extending the analysis and results in Chapter 3, we obtain the general form of the Nash equilibrium in the multi-newsvendor case. The newsvendors’ order quantities at the Nash equilibrium are sorted in the same order as their preferred amount of inventory. Based on this property, we have a general method to calculate the Nash equilibrium of order decisions. Moreover, we developed a procedure to determine the coordinating wholesale prices. This study assures the generality of the newsvendor game with limited supply and makes our research applicable for a supply chain with an arbitrary number of newsvendors.

Chapter 6 generalizes the newsvendor game in Chapter 4 to a multi-newsvendor game with limited supply and inventory transshipment. We considered two models that are different in determining the transshipment pattern among newsvendors. As a direct generalization of the game in Chapter 4, we assumed that each newsvendor specifies a deterministic and fixed portion of his surplus inventory (or excess demand) to be shared with another newsvendor. This transshipment pattern is easy to implement but does not necessarily generate the largest transshipment profit possible. Analysis shows that the Nash equilibrium exists under more restrictive conditions than in the two-newsvendor case. In the second model, the transshipment pattern is determined by solving a transportation problem of which the objective is to maximize the
total transshipment profit. Instead of using transshipment prices, we developed a transshipment fund mechanism to allocate the transshipment profit and effectively coordinate the newsvendors. By this mechanism, all newsvendors benefit from inventory transshipment, and the total profit of newsvendors and the fund can be maximized. Subsequently, the supply chain profit is maximized with appropriate wholesale prices. This study further extends the existing literature of multi-newsvendor inventory transshipment problems to the case with limited supply. The transshipment fund mechanism provides a new perspective from which a supplier may better manage the retail stores.

This dissertation provides new models with game-theoretic analysis and managerial insights for participants of a supply chain in their analytical decision-making under the circumstance of limited supply capacity. It shows new findings of retailers’ gaming behaviors and the benefit of inventory transshipment. Practical suggestions on coordinating mechanism design and supply chain optimization are also provided. Our research contributes useful analytical tools for supply chain analysis and expands the scope of existing literature on supply chain management. Moreover, this research may be used as a framework to investigate problems from different backgrounds but with similar structures.

### 7.2 Future Research

This research enables us to study complex supply chains by extending currently available models. Applications to problems with a similar structure in other areas are promising. It may further enrich the game-theoretic approaches in operations research and management. In this section, we suggest some directions for future research.

An immediate step to extend this research to a broader spectrum of applications is to consider a newsvendor model with discrete demand. For industries such as automobile, real estate and health care services, a discrete random variable could be more suitable for describing the customer demand than a continuous random variable. In this case, the optimal inventory level and the subsequent ordering strategy of a newsvendor should be characterized by studying
the finite difference, instead of the derivative, of the newsvendor’s expected profit function. The game models in this research are flexible for adapting the setting of discrete demands to be further studied. The accuracy of using continuous approximations to discrete distributions should be investigated.

When there are multiple suppliers in a supply chain, a newsvendor may diversify his order sources in order to increase inventory reliability and to reduce purchase costs. Models in this dissertation can be extended for this type of supply chain with multiple retailers and multiple suppliers. Potential applications may be seen in sustainable energy industries. For example, big biomass energy firms buy wood pellets and crops (raw materials) from several relatively smaller suppliers, produce fuel or electricity (product) and generate revenue by selling at market and receiving government subsidies (retail). The dynamics between energy firms and suppliers may affect the energy production costs and market prices. Therefore, energy firms’ order strategies and suppliers’ allocation rules are of critical interest in better understanding and management of the biomass industry. The games in Chapter 3 and 5 may be extended to study this problem.

Another extension is to consider retailers who make decisions of retail price and order quantity simultaneously in the environment of limited supply. This type of model characterizes the automobile industry and similar industries where a dealer has certain authority to determine his retail price in order to promote a product and stimulate the customer demand. In this situation, retailers are no longer modeled as newsvendors because demands may depend on retail prices. Existing literature covers the case with unlimited supply and assumes a linear demand-price relation with minor randomness. This dissertation provides analytical tools in investigating the limited supply case, and may be generalized to study retailers other than newsvendors.

The way of transshipping inventory after demand realization, as in this dissertation, is categorized as reactive transshipment in the literature. Other types of transshipment practices have been recognized and studied. Proactive transshipment, often seen in periodic review inventory systems, is to redistribute inventory among stocking locations at predetermined moments in
time and can be arranged in advance. Another example is preventive transshipment that may be conducted by retailers when some demand signals are observed during replenishment lead time. Models in Chapter 4 and 6 may be extended to investigate these transshipment methods in an environment with limited supply and to design corresponding coordinating mechanisms.
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