RAHBARI ASR, NAVID. Cooperative Distributed Energy Scheduling in Smart Grids. (Under the direction of Dr. Mo-Yuen Chow.)

The focus of the present thesis is to develop cooperative distributed algorithms for scheduling of electric energy in power grids consisting of responsive demands, dispatchable generators and storage devices within an enabled communications infrastructure for information exchange. The objective is to provide fully distributed solutions for energy management problems without using any control center, coordinator or leader. While most of the conventional scheduling approaches are based on centralized concept, distributed algorithms are more suitable for large scale systems such as the smart-grid because of their scalability, robustness to single point of failure, and resilience to communication failures. Five algorithms are developed each attacking a specific energy management problem. Cooperative Distributed Plug in Electric Vehicle Demand Management (CDPDM) algorithm was developed for distributed demand side management of large scale Plug in Hybrid /Plug in Electric Vehicles (PHEV/PEV) charging considering limit on the total available power. Incremental Welfare Consensus (IWC) algorithm was proposed for distributed energy management of a grid populated with distributed generators and responsive demands. Asynchronous Incremental Welfare Consensus (AIWC) algorithm was developed to relieve the synchronous communication/update requirement from IWC algorithm. Based on the concept of IWC algorithm, Distributed Real-Time Pricing for Charging and Generation Control (DRPCG) was proposed to manage PHEV/PEV charging considering multiple resources on the generation side. Finally, Cooperative Distributed Energy Scheduling for
Storage Devices (CoDES) algorithm was proposed to do a multi-time step scheduling in microgrids consisting of storage devices, dispatchable generation units, and renewables.
Cooperative Distributed Energy Scheduling in Smart-Grids

by
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To My Parents

Mahnaz Hashemi Mehrabani and Hamid Rahbari-Asr
BIOGRAPHY

Navid Rahbari-Asr was born in Tabriz, the capital of the East Azerbaijan province in Iran. He received his B.Sc. in Electrical Engineering from University of Tabriz, Iran, in 2008. In 2011, he received his M.Sc. in Control Engineering from Tarbiat Modares University, Tehran, Iran. In the same year, he joined Advanced Diagnosis, Automation, and Control Laboratory at North Carolina State University, Raleigh to pursue toward his PhD degree under the direction of Dr. Mo-Yuen Chow. His research interests include distributed control, distributed optimization, big data, and computational intelligence with applications to smart grids.
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# TABLE OF CONTENTS

**LIST OF TABLES** ........................................................................................................ viii

**LIST OF FIGURES** ....................................................................................................... ix

CHAPTER 1. INTRODUCTION ......................................................................................... 1

CHAPTER 2. Cooperative Distributed Demand Management for Community Charging of PHEV/PEVs Based on KKT Conditions and Consensus Networks........... 6

2.1 Introduction ............................................................................................................... 7

2.2 Problem Formulation ............................................................................................... 9

2.2.1 *Objective Function* .......................................................................................... 10

2.2.2 *Relationship between SoC and charging power* ............................................. 10

2.2.3 *Constraints* ..................................................................................................... 11

2.3 Distributed Charging Algorithm ............................................................................. 14

2.3.1 *KKT Conditions of Optimality* ...................................................................... 14

2.3.2 *Nonlinear Approximation of the Cost Function* ............................................. 14

2.3.3 *Solution to the Optimization Problem* ............................................................ 15

2.4 Distributed Implementation of the Algorithm ......................................................... 21

2.5 Simulation Results and Analysis .......................................................................... 24

2.5.1 *Case Study 1: Single–Step Optimization* ......................................................... 26

2.5.2 *Case Study 2: Dynamic Optimization* ............................................................ 29

2.5.3 *Scalability Analysis* ....................................................................................... 33

2.5.4 *Sub-Optimality of the Algorithm due to Nonlinear Approximation: Benchmarking against nonlinear optimization solvers* ................................................................. 36

2.6 Conclusion ............................................................................................................... 37
CHAPTER 3. Incremental Welfare Consensus Algorithm for Cooperative Distributed Generation/Demand Response in Smart Grid

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Introduction</td>
<td>39</td>
</tr>
<tr>
<td>3.2</td>
<td>Problem Formulation: Social Welfare Maximization in Energy Market</td>
<td>43</td>
</tr>
<tr>
<td>3.2.1</td>
<td>Welfare on the Demand Side</td>
<td>44</td>
</tr>
<tr>
<td>3.2.2</td>
<td>Welfare on the Generation Side</td>
<td>45</td>
</tr>
<tr>
<td>3.3</td>
<td>Incremental Welfare Consensus Algorithm</td>
<td>46</td>
</tr>
<tr>
<td>3.4</td>
<td>Numerical Results</td>
<td>60</td>
</tr>
<tr>
<td>3.4.1</td>
<td>IWC Algorithm on IEEE 39 Bus Test System</td>
<td>61</td>
</tr>
<tr>
<td>3.4.2</td>
<td>Monte Carlo Simulations for Scalability Analysis</td>
<td>63</td>
</tr>
<tr>
<td>3.5</td>
<td>Conclusion</td>
<td>64</td>
</tr>
</tbody>
</table>

CHAPTER 4. Distributed Real-Time Pricing Control for Large Scale Unidirectional V2G with Multiple Energy Suppliers

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Introduction</td>
<td>67</td>
</tr>
<tr>
<td>4.2</td>
<td>Problem Formulation</td>
<td>70</td>
</tr>
<tr>
<td>4.2.1</td>
<td>Objective Function</td>
<td>70</td>
</tr>
<tr>
<td>4.2.2</td>
<td>Constraints</td>
<td>76</td>
</tr>
<tr>
<td>4.3</td>
<td>Distributed Real Time Pricing for Charge/Generation Control</td>
<td>76</td>
</tr>
<tr>
<td>4.3.1</td>
<td>Simplifying the Objective Function</td>
<td>76</td>
</tr>
<tr>
<td>4.3.2</td>
<td>Augment the Objective Function with KKT Multipliers</td>
<td>77</td>
</tr>
<tr>
<td>4.3.3</td>
<td>Dual Decomposition</td>
<td>77</td>
</tr>
<tr>
<td>4.3.4</td>
<td>Distributed Observers to Estimate Global Information</td>
<td>78</td>
</tr>
<tr>
<td>4.3.5</td>
<td>Smart Transformers as Communication Nodes between Zones</td>
<td>79</td>
</tr>
<tr>
<td>4.4</td>
<td>Numerical Simulations</td>
<td>87</td>
</tr>
<tr>
<td>4.5</td>
<td>Conclusion</td>
<td>91</td>
</tr>
</tbody>
</table>
CHAPTER 5. Asynchronous Incremental Welfare Consensus algorithm .......... 93

5.1 Introduction .................................................................................. 93
5.2 Incremental Welfare Consensus Algorithm ........................................ 95
  5.2.1 Distributed Estimation of Power Imbalance ............................... 96
5.3 Local Price Update .......................................................................... 96
5.4 Power Regulation ........................................................................... 97
5.5 Sensitivity of IWC to Synchronous Updates ........................................ 99
5.6 Asynchronous Incremental Welfare Consensus Algorithm .................. 100
  5.6.1 Communicating nodes ............................................................ 102
  5.6.2 Non-communicating nodes ...................................................... 103
5.7 Numerical Results .......................................................................... 103
5.8 Conclusion ....................................................................................... 106

CHAPTER 6. Consensus Based Distributed Scheduling for Cooperative Operation
of Distributed Energy Resources and Storage Devices in Smart Grids ............ 107

6.1 Introduction ..................................................................................... 107
6.2 Problem Formulation ....................................................................... 110
  6.2.1 Objective function ..................................................................... 110
  6.2.2 Controllable Inputs ..................................................................... 111
  6.2.3 Uncontrollable Inputs .................................................................. 112
  6.2.4 System States ............................................................................ 112
  6.2.5 Operational Cost ......................................................................... 112
  6.2.6 Constraints ............................................................................... 113
6.3 Consensus based distributed iterative algorithm ................................. 114
  6.3.1 Primal-dual gradient descent method ....................................... 114
6.3.2 Distributed estimation of global information with consensus networks ................................................................. 117

6.4 Numerical Simulations ................................................................................................................................................. 128

6.4.1 Demonstrate the application of the algorithm ................................................................. 128

6.4.2 Robustness to communication link failures ..................................................................................... 132

6.4.3 Validate the convergence of the algorithm to global optimum using Monte Carlo simulations ........................................................................... 134

6.5 Conclusion ......................................................................................................................................................... 137

CHAPTER 7. Conclusion ............................................................................................................................................. 138

REFERENCES ............................................................................................................................................. 140
LIST OF TABLES

Table 2-1: Cooperative Distributed PHEV/PEV Demand Management (CDPDM) Algorithm .................................................................23

Table 2-2: Information About the Vehicles in the 1st Case Study .................................................................27

Table 2-3: Information About the Vehicles in the 2nd Case Study .............................................................30

Table 2-4: Monte Carlo Simulation Sampling Range .................................................................................34

Table 3-1: Generation and Demand Unit Cost/Utility Function Parameters Chosen for IEEE-39 Power Network ........................................................................................................61

Table 4-1. Dispatchable Unit Parameters Chosen for the Case Study .........................................................89

Table 4-2. Generated/Consumed Power By Non-Dispatchable Units .........................................................89

Table 5-1. Generation/Demand Units Parameters Chosen For Garver Network ................................99

Table 6-1. Node Types in the Network ......................................................................................................130

Table 6-2. Range of the random parameters for Monte Carlo simulations ...........................................136
LIST OF FIGURES

Figure 1-1: The general structure of the envisioned smart-grid ........................................1
Figure 1-1: Cooperative distributed systems in nature a) flock of birds b) neural cells ......3
Figure 2-1: Cooperatives distributed demand management for PHEV/PEV community
charging.................................................................................................................................9
Figure 2-2: The actual objective function and the approximated objective function .......16
Figure 2-3: Communication and power network topologies for case study 1 and 2 ..........25
Figure 2-4: Allocation of power and evolution of consensus states for case study (1)
with no link failure..................................................................................................................28
Figure 2-5: Allocation of power and evolution of consensus states for case study (1)
with single-link failure..........................................................................................................29
Figure 2-6: Charging profiles for case study (2) with no node failure .........................31
Figure 2-7: Charging profiles for case study (2) with node failure ...............................32
Figure 2-8: Computational complexity of CDPDM algorithm as a function of number
of nodes (average number of neighbors=3) .......................................................................35
Figure 2-9: Computational complexity of CDPDM algorithm as a function of number
of nodes (average number of neighbors=20) ....................................................................35
Figure 2-10: Distribution of optimization error for CDPDM algorithm due to nonlinear
approximation of the cost function with respect to a) the Interior Point method and b)
SQP ........................................................................................................................................37
Figure 3-1: General structure for running IWC algorithm..............................................42
Figure 3-2: The IEEE 39 bus power network with communications topology ..........60

x
Figure 3-3: Evolution of a) generation and demand powers, b) price, and c) social welfare in case study with IEEE 39 bus test system.................................................................62

Figure 3-4: Average number of iterations required for convergence as a function of the number of nodes for the IWC algorithm........................................................................64

Figure 4-1. General Concept for Distributed Real-Time Pricing for Charge/Generation Control (DRPCG) ........................................................................................................69

Figure 4-2. General Structure for Distributed Real-Time Pricing for Charge/Generation Control (DRPCG) ........................................................................................................80

Figure 4-3. Five bus power network with multiple energy devices................................88

Figure 4-4. DRPCG algorithm for case study a) Charging power for EVs, b) Generated power by dispatchable generation units, c) Evolution of prices among zones, d) Evolution of prices in zone 1, e) Power imbalance over the network, f) Objective value ..........................................................................................................................................................90

Figure 4-5. Response of DRPCG algorithm to changes in the loading condition of the system a) EV response, b) Generation response, c) Price response, d) Power imbalance ..........................................................................................................................................................91

Figure 5-1. Garver power network with communications topology.............................97

Figure 5-2. Evolution of power, social welfare and price of energy at different units for IWC algorithm ..............................................................................................................100

Figure 5-3. Evolution of generation, demand and social welfare for IWC algorithm under probability of asynchronous behavior for the updates a) $p = 0.1$, b) $p = 0.2$, and c) $p = 0.5$ ......................................................................................................................................................101

Figure 5-4. Automaton representing the event-driven dynamics of AIWC....................102
Figure 5-5. A typical communications scenario for AIWC algorithm running on the Garver power network ..............................................................................................................103
Figure 5-6. Evolution of a) generation/consumption powers, b) social welfare value, c) price and d) total generation/demand in AIWC algorithm ........................................104
Figure 5-7. Evolution of generation, demand and social welfare for AIWC algorithm under probability of asynchronous behavior for the pairwise updates a) \( p = 0.1 \), b) \( p = 0.2 \), c) \( p = 0.5 \) ..................................................................................................................................105
Figure 6-1. Structure of a microgrid with distributed controllers ...............................................................................111
Figure 6-2. General structure for each distributed controller .........................................................................................118
Figure 6-3. Five bus power network with multiple energy devices ..........................................................129
Figure 6-4. Input profiles for the case study ..............................................................................................................130
Figure 6-5. Convergence of dual variable estimations (a) for all time steps at node 1, (b) at all nodes for time step 2, and (c) discharging rate decisions of DESDs for different time steps ..................................................................................................................131
Figure 6-6. Resultant schedule of the proposed distributed scheduling approach and energy stored in each storage device ....................................................................................132
Figure 6-7. Convergence of the objective value (blue line) to the global optimum (red line) ...............................................................................................................................................133
Figure 6-8. Algorithm convergence with/without communication link failures in the 5 bus network .........................................................................................................................................134
Figure 6-9. Convergence time and objective value of the algorithm under failure of each communication link ..............................................................................................................134
Figure 6-10. Objective value under link failures that cause communications topology to be disconnected

Figure 6-11. Histogram for percentage difference in objective values between the centralized QP and proposed distributed algorithm
CHAPTER 1. INTRODUCTION

The future power system would be an aggregation of distributed generation units, responsive demands, and storage units each equipped with controllable power electronics devices connected to each other through the communications network [1], [2]. This Cyber-Physical System (CPS) is called “Smart-Grid”, and as shown in Figure 1-1, is geographically distributed and consists of a multitude of heterogeneous devices.

Distinguishing characteristics and benefits of the smart grid according to the Energy Independence and Security Act of 2007 (EISA) and National Institute of Technology (NIST) include [3]:

1. Dynamic optimization of grid operations and resources with full cyber-security,

Figure 1-1: The general structure of the envisioned smart-grid

---

1 Picture from http://www.consumerenergyreport.com/smart-grid/
2. Predictive maintenance and “self-healing” responses to system disturbances/changes,

3. Development and incorporation of demand response, demand-side resources, and energy-efficiency resources,

4. Deployment and integration of advanced electricity storage and peak-shaving technologies, including plug-in electric and hybrid electric vehicles, and thermal-storage air conditioning,

To reach these goals requires advanced control and optimization technologies. Managing the smart-grid, considering the number and variety of controllable devices necessitates a paradigm shift in traditional Energy Management Systems (EMSs), which are operated centrally through Supervisory, Control, and Data Acquisition (SCADA) systems [1], [4]. This paradigm shift needs to consider the local intelligence of the distributed devices, as well as their communications capability [5].

Self-organizing, distributed systems can provide useful solutions for developing scalable and robust algorithms for energy management in the smart grid. Many of the dynamic systems in the nature as shown in Figure 1-1 are composed of multiple smaller units which sense their environment locally, perform local computations, and coordinate with their neighbors to carry out complex tasks. Examples include ant colony [6], bird flock [7], vehicles in a transportation system, people in a social network, and biological cells inside our body. Distributed algorithms have been studied in several areas, such as social science, animal science [8], computer science [9], distributed estimation [10], [11], and distributed identification [12].

Distributed algorithms provide a suitable structure for the management and control of the smart-grid, because the envisioned future for the smart-grid is an intrinsically distributed
system consisting of multiple distributed devices with local computational and communications capabilities.

This thesis presents the author’s work in developing distributed energy management and scheduling techniques for resources, demands, and storage devices in smart grids. Multiple fully distributed algorithms are developed each for solving a particular energy management/scheduling problem. Chapters 2 to 6 are each dedicated to explaining one algorithm. The organization of chapters are as follows:

**Chapter 2** presents Cooperative Distributed Plug in Electric Vehicle Demand Management (CDPDM) algorithm for distributed demand side management of large scale Plug in Hybrid /Plug in Electric Vehicles (PHEV/PEV) charging considering limit on the total available power. The problem is to optimally allocate the available charging resources among multiple PHEV/PEVs integrated to the grid, based on their priority without employing any control center. In this structure the charging stations are capable of communicating with their local neighbors and can perform local computations.

Figure 1-1: Cooperative distributed systems in nature a) flock of birds b) neural cells
Chapter 3 addresses a more general problem: energy management for a smart grid populated with responsive demands and distributed generators using local communications and computational capabilities of distributed resources. The objective is to develop a distributed cooperative scheme such that the responsive demands and distributed generators can optimally change their consumption or generation level in real-time in response to the varying conditions of the grid (such as changes in the production level of renewables, increase in the nonresponsive demands, etc.). Incremental Welfare Consensus (IWC) algorithm is proposed to solve this problem.

Chapter 4 addresses the problem of managing PHEV/PEV charging considering multiple heterogeneous resources on the generation side. The problem considered in this chapter is different than the one in chapter 2. In chapter 2, we had only one limited source of energy, while in chapter 5, we consider multiple renewable and conventional energy resources each with different cost of providing the energy. Based on the framework of IWC algorithm, Distributed Real-Time Pricing for Charging and Generation Control (DRPCG) is proposed to solve this problem.

Chapter 5 proposes Asynchronous Incremental Welfare Consensus (AIWC) algorithm to relieve the synchronous update requirement from the IWC algorithm. The algorithm proposed in chapter 3, requires each device to update its states synchronously with all the other devices. Retaining synchronous behavior over thousands of devices in a geographically distributed system is not a realistic assumption. Therefore, this chapter extends the IWC algorithm into an asynchronous version obviating the necessity of synchronous updates.

Chapter 6 considers energy scheduling in microgrids grids consisting of storage devices, dispatchable generation units, and renewables. Cooperative Distributed Energy Scheduling
(CODES) algorithm was proposed to schedule the charging/discharging of storage devices and generation of dispatchable units to minimize the defined cost index of the system. While all the problems considered in the previous chapters were single time step optimization problems, this chapter addresses a multi-time step optimization problem, because by introducing storage devices, energy decisions at different time steps are no longer independent of each other.

**Chapter 7** concludes the thesis and proposes some future research directions as the continuation of the present work.
CHAPTER 2. COOPERATIVE DISTRIBUTED DEMAND MANAGEMENT FOR COMMUNITY CHARGING OF PHEV/PEVS BASED ON KKT CONDITIONS AND CONSENSUS NETWORKS

Abstract:
Efficient and reliable demand side management techniques for community charging of Plug-in Hybrid Electrical Vehicles (PHEVs) and Plug-in Electrical Vehicles (PEVs) are needed as large numbers of these vehicles are being introduced to the power grid. To avoid overloads and maximize customer preferences in terms of the time and cost of charging, a constrained nonlinear optimization problem can be formulated. In this chapter, we have developed a novel cooperative distributed algorithm for charging control of PHEVs/PEVs that solves the constrained nonlinear optimization problem using Karush-Kuhn-Tucker (KKT) conditions and consensus networks in a distributed fashion. In our design, the global optimal power allocation under all local and global constraints is reached through peer-to-peer coordination of charging stations. Therefore, the need for a central control unit is eliminated. In this way, single-node congestion is avoided when the size of the problem is increased and the system gains robustness against single-link/node failures. Furthermore, via Monte Carlo simulations, we have demonstrated that the proposed method is scalable with the number of charging points and returns solutions, which are comparable to solutions returned by conventional nonlinear optimization solvers with a maximum of 2% sub-optimality. Thus, the main advantages of our approach are to eliminate the need for a central energy management/coordination unit, gain robustness against single-link/node failures, and be scalable in terms of single-node computations.
2.1 Introduction

Technologies used to update utility electricity systems with computer-based automation and control through two-way communications structures constitute the core of the “smart grid” concept [13]. Enabling the transition to Plug-in Hybrid Electrical Vehicles (PHEVs) and Plug-in Electrical Vehicles (PEVs) is one of the anticipated benefits of the smart grid [3]. These vehicles provide many incentives for the transportation industry and the environment [14]. Large-scale integration of these vehicles has potential impacts and benefits for the grid. One of the apparent impacts is the increase of the peak demand [15] which can destabilize the grid if not managed properly. On the other hand, the PHEV/PEVs can benefit the grid by being treated as a flexible load through charge/discharge scheduling to shape the load profile [15], [16]. Moreover, on the customer side, the increased satisfaction from the charging process in terms of time and cost of charging as well as the State of Charge (SoC) of the vehicle when leaving the charging station, can improve the adoption rate of the electric vehicles. Therefore, employing efficient energy-management policies to control and optimize the charging process for electric vehicles is becoming a critical need for the future grid. The enabling technology is called Vehicle to Grid (V2G) technology [17]. In general, V2G technology is operated in large scale and involves optimization strategies with single or multiple objectives [18]. The process of optimal dispatch of PHEV/PEVs using the V2G technology is called “Smart-Charging” [19], [20]. The focus of this chapter is on a “Smart-Charging” method using unidirectional V2G technology.

The existing smart charging approaches for PHEV/PEV charging in the literature, in most cases, are centralized. The charging stations are required to transmit data to a control center called aggregator [18]. The control center performs the necessary computations and determines
the optimal power allocation for each unit and transmits it to the charging stations [21]–[24]. The centralized approach works well for small-scale problems with a small number of vehicles where each unit can send information to the central controller directly without congestion. However, as the number of PHEVs/PEVs increases and they spread over a wide zone, such as a large city, the centralized approach loses efficiency due to two main reasons: 1) the required communications and computational capacity of the control center grows as the dimension of the problem increases, 2) the system becomes fragile to single points of failure.

To solve the issues of centralized smart charging for large scale problems, research has started on decentralized PHEV/PEV demand control. In most of the proposed decentralized approaches in the literature, each charging station regulates its own power by responding to an external signal [25]–[30]. Therefore the computational burden is relieved from the control center. However, still a center/aggregator is required to send the coordinating/pricing signals. Specifically, in [25], low-voltage transformers coordinate with each other by communicating with a high-voltage transformer to reduce the imbalances; In [26]–[28] the utility sends a central price/event signal to all the charging stations to manage the aggregate charging load; In [29] the distributed units respond to changes in the price signal adjusted through a center and in [30] a hierarchical decentralized method is introduced in which central aggregator broadcasts update/average information back and forth to/from sub-aggregators and in some cases the global constraint to avoid overloads is violated.

In this chapter, we introduce a cooperative distributed optimal power-allocation algorithm to satisfy grid constraints and customer preferences for large-scale charging of PHEVs/PEVs.

Unlike the aforementioned decentralized approaches, our algorithm is completely center-free. It works based only on the local sharing of information among the charging stations and
has no central aggregator/coordinator unit. Consequently, single-node congestion is alleviated, and the system becomes more robust against single points of failures. Moreover, the computational and communications burden is divided among the distributed processors, so the proposed approach is scalable. Unlike approaches that are based on dual decomposition (e.g., [30]), in our algorithm prior to the convergence, the global constraint is never violated.

The idea of our algorithm is shown in Figure 2-1. Related work is done in [31]. The current chapter explains the algorithm in theoretical details and studies its scalability and sub-optimality and compares it with centralized algorithms. Similar concepts are also used in our research group to optimally manage distributed energy resources on the generation side [32].

Organization of this chapter is as follows: Section II provides the problem statement. In section III, we propose our distributed consensus-based algorithm. Section IV presents the numerical analysis and results, and the concluding remarks are made in Section V.

2.2 Problem Formulation

The problem of PHEV/PEV community charge allocation can be formulated as an optimization problem consisting of an objective function and appropriate constraints.
2.2.1 Objective Function

Different factors, such as battery longevity [21], the amount of charge and the tightness of the deadline [22], the flattening of the overall load profile and reducing the load variance [33], and the cost of charging for vehicle owners [28], can be considered in defining the objective function for PHEV/PEV charge coordination.

Satisfaction of the PHEV/PEV users from the charging service is directly related to the State of Charge (SoC) of the vehicle, when leaving the charging station. Therefore, in this chapter, similar to [23], the objective function is defined as the weighted sum of the State of Charge (SoC) of the vehicles at the next time step:

\[
\max J = \sum_{i=1}^{N} w_i(k) \text{SoC}_i(k+1),
\]

(2-1)

where \( P(k) = [P_i(k)]_{i=1,...,N} \) is the vector of the power allocated to the vehicles, \( N \) is the total number of vehicles, \( \text{SoC}_i(k+1) \) is the SoC of the battery pack of the \( i^{th} \) vehicle at the next iteration, and \( w_i(k) \) is the priority weighting for the \( i^{th} \) vehicle, as well as a nonnegative value. Depending on the desired performance, the weights can prioritize vehicles based on the remaining time of charge, remaining battery capacity to be charged, and/or the customer’s willingness to pay.

2.2.2 Relationship between SoC and charging power

The objective function is defined in terms of the SoC because, on the customer side, the SoC of the vehicle at the end of the charging period is the main issue of concern. To find the relationship between the allocated power for charging and the SoC, we need a model for the battery. In a certain operating range, the battery can be modeled as a capacitor circuit. In this range, the variation of the SoC is proportional to the variation of the battery voltage. More
complicated battery models can be considered in the optimization process, and this would be one of the future works of the authors. Considering the capacitance model, the equivalent capacitance of the $i^{th}$ battery, $C_i$, is then calculated as:

$$C_i = \frac{Q_i \left( SoC_{i,max} - SoC_{i,min} \right)}{V(SoC_{i,max}) - V(SoC_{i,min})},$$  \hspace{1cm} (2-2)$$

where $Q_i$ is the capacity of the $i^{th}$ battery in Coulombs and $[SoC_{i,min}, SoC_{i,max}]$ is the operating range of the battery. The change in the charge of the battery at each time step, $k$, would be $\Delta q_i(k) = \Delta C_i V_i(k)$. So, we can write:

$$SoC_i(k+1) - SoC_i(k) = \frac{C_i V_i(k+1) - C_i V_i(k)}{Q_i}.$$ \hspace{1cm} (2-3)

The energy provided to the battery during each time step, $\Delta t$, would be $P_i \Delta t$. Considering the efficiency of the charging to be $\eta_i$, the energy stored in the battery would equal $\eta_i P_i \Delta t$.

Applying conservation of energy,

$$\frac{1}{2} C_i V_i(k+1)^2 - \frac{1}{2} C_i V_i(k)^2 = \eta_i P_i \Delta t$$  \hspace{1cm} (2-4)$$

Substituting for $V_i(k+1)$ from (2-4) into (2-3) yields:

$$SoC_i(k+1) = SoC_i(k) + \frac{C_i}{Q_i} \left( -V_i(k) + \sqrt{V_i(k)^2 + \frac{2\eta_i P_i(k) \Delta t}{C_i}} \right).$$ \hspace{1cm} (2-5)$$

2.2.3 Constraints

Constraints represent the physical limits:

2.2.3.1 Global constraint

There is a limit on the total amount of power that the utility can allocate for charging purposes, $P_{total}$. It is modeled as the upper bound of the utility’s power delivery:
\[ \sum_{i=1}^{N} P_i(k) \leq P_{total}. \] (2-6)

2.2.3.2 Local constraint

Certain factors impose local bounds on the amount of allocated power for each unit. For instance, the maximum power output of the outlet (e.g., 4.6kW for a standard single-phase 230V outlet), level of charging (e.g., 1.33kW for level 1), the maximum tolerable charging current, \( I_{max} \), and the maximum allowable SoC, \( (SoC_{max}) \), to avoid overcharging. These constraints are mapped into one constraint for each unit:

\[ 0 \leq P_i(k) \leq P_{i,max}(k), \forall i = 1, ..., N, \] (2-7)

where \( P_{i,max}(k) \) is the upper bound of the allocated power considering all the local limitations. As the Vehicle-to-Grid (V2G) capability is not considered here, the lower bound is set to zero, so the PHEV/PEVs should not be discharged.

2.2.3.3 Optimization Problem Structure

Substituting for \( SoC_i(k+1) \) from (2-5) into (2-1),

\[
\text{Max } J = \sum_{i=1}^{N} w_i(k)SoC_i(k) - \sum_{i=1}^{N} \frac{C_i}{Q_i} V_i(k) + \sum_{i=1}^{N} w_i(k) \sqrt{V_i(k)^2 + \frac{2\eta_i P_i(k)\Delta t}{C_i}}. \] (2-8)

The first two summations in (2-8) are independent of the decision variables \( P_i(k) \) and therefore can be eliminated from the objective function,

\[
\text{Max } J = \sum_{i=1}^{N} w_i(k) \sqrt{V_i(k)^2 + \frac{2\eta_i P_i(k)\Delta t}{C_i}}. \] (2-9)
To simplify the notations we introduce $\alpha_i(k) = \frac{w_i(k)C_i/Q_i}{\sqrt{2\eta_i\Delta t/C_i}}$, and $\beta_i(k) = C_iV_i(k)^2/2\eta_i\Delta t$ and rephrase (2-9) as:

$$\Delta_t = \frac{\beta_i(k)}{2\eta_i\Delta t}$$

Considering the constraints, the optimization problem in terms of the allocated charging powers can be expressed as:

$$\min J = \sum_{i=1}^{N} \alpha_i(k)\sqrt{P_i(k) + \beta_i(k)}.$$  \hspace{1cm} (2-10)

subject to:

$$\sum_{i=1}^{N} P_i(k) \leq P_{\text{total}},$$

$$0 \leq P_i(k) \leq P_{i,\text{max}}(k), \quad \forall i = 1, ..., N.$$  \hspace{1cm} (2-11)

**Remark 2-1:**

More charging power for each unit results in a higher SoC for the vehicle and therefore higher satisfaction for the customers. That is why the objective function in (11) is a monotonically increasing function in terms of the charging power. If there was no limit on the total available charging power, the optimal allocation of power for all the vehicles would be the upper limit of the charging power at each charging station. However, as the total available power is limited by the grid constraints, the optimal allocation of power for each unit would not necessarily be equal to the boundary value.
2.3 Distributed Charging Algorithm

Our distributed charging algorithm is based on Karush-Kuhn-Tucker (KKT) conditions of optimality [23]:

2.3.1 KKT Conditions of Optimality

Consider the general optimization problem of the form:

\[
\begin{align*}
\text{Min } J(x) & \quad \text{subject to:} \\
& h_i(x) = 0, \forall i = 1, \ldots, n, \\
& g_j(x) \leq 0, \forall j = 1, \ldots, m.
\end{align*}
\]  

(2-12)

If \(x^*\) is a solution of (2-12), then there are constants \(\lambda_i (i = 1, \ldots, n)\) and \(\mu_i (i = 1, \ldots, m)\), called KKT multipliers [34], such that:

\[
\begin{align*}
\nabla J(x^*) + \sum_{i=1}^{n} \lambda_i \nabla h_i(x^*) + \sum_{j=1}^{m} \mu_j \nabla g_j(x^*) &= 0, \\
h_i(x^*) &= 0, \forall i = 1, \ldots, n, \\
g_j(x^*) \leq 0, \forall j = 1, \ldots, m, \\
\mu_i g_j(x^*) &= 0, \forall i = 1, \ldots, m, \\
\mu_i &\geq 0, \forall i = 1, \ldots, m.
\end{align*}
\]

(2-13) (2-14) (2-15) (2-16)

The conditions (2-13), (2-14), (2-15), and (2-16) are called KKT conditions and are necessary for optimality. When the objective function is convex and the constraints are affine (as in our case), the KKT conditions are sufficient for global optimality as well [34].

2.3.2 Nonlinear Approximation of the Cost Function

In order to simplify the optimization process, we approximate \(J_i = \sqrt{P_i(k)} + \beta_i(k)\) in (2-11) with \(\tilde{J}_i = K_{1i}(k)\sqrt{P_i(k)} + K_{2i}\) and choose \(K_{1i}(k)\) and \(K_{2i}(k)\) to minimize the norm of the difference between \(J_i\) and \(\tilde{J}_i\) for \(0 \leq P_i(k) \leq P_{i,\text{max}}(k)\):
\( \left[ K_{1i}(k)^* \quad K_{2i}(k)^* \right] = \arg \min_{K_{1i}(k),K_{2i}(k)} \left\| J_i(k) - \tilde{J}_i(k) \right\|_2^2. \) (2-17)

By solving (2-17):

\[
K_{1i}^*(k) = 9 P_{i,max}^{-1} \sqrt{P_{i,max}^{-2} + \beta P_{i,max}} - 8 P_{i,max}^{-3/2} \left( \beta + P_{i,max} \right)^{3/2} + 8 \beta^{3/2} P_{i,max}^{-3/2} \\
- 2.5 \beta^2 P_{i,max}^{-2} \ln \left( \beta / 2 + P_{i,max} + \sqrt{P_{i,max} \left( \beta + P_{i,max} \right)} \right) \\
+ 4.5 \beta P_{i,max}^{-2} \sqrt{P_{i,max}^{-2} + \beta P_{i,max}} + 2.5 \beta^2 P_{i,max}^{-2} \ln \left( \beta / 2 \right),
\]

\[
K_{2i}^*(k) = -3 \left( 2 P_{i,max}^{-1/2} \sqrt{P_{i,max}^{-2} + \beta P_{i,max}} + 2 P_{i,max}^{-1} \left( \beta + P_{i,max} \right)^{3/2} \\
+ 2 \beta^{3/2} P_{i,max}^{-1} - 0.5 \beta^2 P_{i,max}^{-3/2} \ln \left( \beta / 2 + P_{i,max} + \sqrt{P_{i,max} \left( \beta + P_{i,max} \right)} \right) \\
+ \beta P_{i,max}^{-3/2} \sqrt{P_{i,max}^{-2} + \beta P_{i,max}} + 0.5 \beta^2 P_{i,max}^{-3/2} \ln \left( \beta / 2 \right) \right),
\] (2-18)

With this approximation, we can write the optimization problem as (2-19) where \( \alpha'_i(k) = K_{1i}^*(k) \alpha_i(k) \). Note that as \( K_{2i}^*(k) \) is a constant addition, it can be eliminated from the objective function.

\[
\text{Min} \ J = -\sum_{i=1}^{N} \alpha'_i(k) \sqrt{P_i(k)},
\]

subject to:

\[
\sum_{i=1}^{N} P_i(k) \leq P_{\text{total}}, \ 0 \leq P_i(k) \leq P_{i,max}(k), \ \forall i = 1, \ldots, N.
\] (2-19)

Figure 2-2 shows the curves of \( J_i(k) \) and \( \tilde{J}_i(k) \) for \( \beta = 1000 \), and \( P_{i,max}(k) = 5 \text{ kW} \). The effect of this approximation on the optimality of the solutions will be studied in section IV.

2.3.3 Solution to the Optimization Problem

By applying the first KKT condition (2-13) for the optimization problem defined in (2-19), we get:
where $\lambda$ is the KKT multiplier associated with the global inequality constraint and $\mu_i$ and $\xi_i$ for $i=1,…,N$ are associated with the local constraints (one for each of the bounds).

If the local inequality constraints are strictly satisfied at the optimal point, based on (2-15), we would have $\mu_i = \xi_i = 0$. Applying this to (2-20), we get a positive value for $\lambda$, which indicates that $\sum_{i=1}^{N} P_i(k) = P_{total}$ based on the condition (2-15). Solving for $P_i(k)^*$:

$$P_i(k)^* = \frac{\alpha_i'(k)^2}{\sum_{j=1}^{N} \alpha_j'(k)^2} P_{total}, \forall i = 1,…,N.$$  \hfill (2-21)

The value for $P_i(k)^*$ already satisfies the non-negativity constraint. This trait is the direct result of the approximation in section II.B. In order to guarantee that the upper bound limits are also satisfied, we set those values of $P_i(k)^*$ that violate the local maximum bound to their maximum value and exclude their values from $P_{total}$. Then, we use the same procedure in (2-21) to allocate the remaining power to the remaining vehicles. This procedure is written as Algorithm 2-1.
Algorithm 2-1:
1. Set $I = \{1, 2, \ldots, N\}$, $n = 1$, $U^{1} = \emptyset$.

2. For all $i \in I - U^{n}$, calculate $P_{i}^{n}(k)^{**}$ as:

$$P_{i}^{n}(k)^{**} = \frac{\alpha'_{i}(k)^{2}}{\sum_{j \in I - U^{n}} \alpha'_{j}(k)^{2}} \left( P_{\text{total}} - \sum_{j \in U^{n}} P_{j, \text{max}} \right).$$  \hspace{1cm} (2-22)

3. Set $U_{1} = \emptyset$. For all $i \in I - U^{n}$, if $P_{i}^{n}(k)^{**} > P_{i, \text{max}}(k)$, update $U_{1} = U_{1} + \{i\}$.

4. If $U_{1} = \emptyset$, go to step 5. Otherwise, set $U^{n+1} = U^{n} + U_{1}$, $n = n + 1$, and Go to step 2.

5. Stop the algorithm. Set $U = U^{n}$. The optimal powers are assigned as follows:

   a. For all $i \in I - U$, $P_{i}(k)^{*} = \frac{\alpha'_{i}(k)^{2}}{\sum_{j \in I - U} \alpha'_{j}(k)^{2}} \left( P_{\text{total}} - \sum_{j \in U} P_{j, \text{max}} \right)$.

   b. For all $i \in U$, $P_{i}(k)^{*} = P_{i, \text{max}}(k)$.

These iterations are repeated until all vehicles receive a valid allocation of power. This method of allocating powers terminates in maximum of $N$ iterations, and the final point satisfies the KKT conditions of optimality.

**Theorem 2-1:**

The final point $P^{*}(k) = \left[ P_{i}^{*}(k) \right]$ resulted from Algorithm 2-1, satisfies the KKT conditions of optimality for optimization problem in (2-19).

**Proof:**

The Lagrangian for (2-19) can be written as follows:

$$L = -\sum_{i=1}^{N} \alpha'_{i}(k) \sqrt{P_{i}(k)} + \lambda \left( \sum_{i=1}^{N} P_{i}(k) - P_{\text{total}} \right) + \sum_{i=1}^{N} \mu_{i} \left( P_{i}(k) - P_{i, \text{max}}(k) \right) - \sum_{i=1}^{N} \xi_{i} P_{i}(k).$$  \hspace{1cm} (2-23)
Let’s denote the final point resulted from Algorithm 2-1 with \( P^*(k) = \left[ P_i^*(k) \right] \). KKT conditions hold if we find positive constants \( \lambda \) and \( \mu_i \) and \( \xi_i \) for all \( i = 1, \ldots, N \) such that:

\[
\nabla_p L |_{p_i} = 0 \Rightarrow \left( -\frac{\alpha'(k)}{2\sqrt{P_i^*(k)}} + \lambda + \mu_i - \xi_i = 0, \forall i = 1, \ldots, N, \right.
\]

(2-24)

\[
\lambda \left( \sum_{i=1}^{N} P_i^*(k) - P_{\text{total}} \right) = 0,
\]

(2-25)

\[
\mu_i \left( P_i(k)^* - P_{i,\text{max}}(k) \right) = 0, \forall i = 1, \ldots, N,
\]

(2-26)

\[
-\xi_i P_i(k)^* = 0, \forall i = 1, \ldots, N,
\]

(2-27)

\[
\sum_{i=1}^{N} P_i^*(k) \leq P_{\text{total}},
\]

(2-28)

\[
P_i^*(k) \leq P_{i,\text{max}}(k), \forall i = 1, \ldots, N,
\]

(2-29)

\[
P_i^*(k) \geq 0, \forall i = 1, \ldots, N.
\]

(2-30)

It is evident from the steps of Algorithm 2-1, that conditions (2-28), (2-29) and (2-30) are satisfied. To satisfy (2-27), we can take all \( \xi_i \) equal to zero. Now, two cases are considered:

1. \( \sum_{i=1}^{N} P_{i,\text{max}}(k) \leq P_{\text{total}} \): In this case \( P^*(k) = \left[ P_{i,\text{max}}(k) \right] \). Therefore, (2-26) will be satisfied.

Condition (2-25) can be satisfied by taking \( \lambda = 0 \). Then, (2-24) will be satisfied by taking \( \xi_i = \frac{\alpha'(k)}{2\sqrt{P_{i,\text{max}}(k)}} \geq 0 \) for all \( i \).

2. \( \sum_{i=1}^{N} P_{i,\text{max}}(k) > P_{\text{total}} \): In this case, \( \sum_{i=1}^{N} P_i^*(k) = P_{\text{total}} \). Therefore (2-25) will be satisfied.

Let’s define sets \( I \) and \( U \) as follows:
\[ I = \{1, 2, ..., N\}, \quad U = \{i \in I \mid P_i'(k) = P_{i,\text{max}}(k)\}. \] (2-31)

For all \( i \in U \), (2-26) is satisfied. For (2-26) to be satisfied for \( i \in I - U \), \( \mu_i \) should be zero. Thus, for (2-24) to be true,

\[
\forall i \in I - U : \frac{-\alpha_i'(k)}{2\sqrt{P_i'(k)}} + \lambda = 0, \quad (2-32)
\]

\[
\forall i \in U : \frac{-\alpha_i'(k)}{2\sqrt{P_{i,\text{max}}(k)}} + \lambda + \mu_i = 0. \quad (2-33)
\]

For (2-32) to be true,

\[
\forall i \in I - U : \lambda = \frac{\alpha_i'(k)}{2\sqrt{P_i'(k)}}, \quad (2-34)
\]

Then, as \( \lambda \) is unique,

\[
\forall i, j \in I - U : \frac{\alpha_i'(k)}{2\sqrt{P_i'(k)}} = \frac{\alpha_j'(k)}{2\sqrt{P_j'(k)}} \Rightarrow \frac{\alpha_i'(k)}{\alpha_j'(k)} = \frac{P_j'(k)}{P_j'(k)} \Rightarrow \frac{P_i'(k)}{P_j'(k)} = \frac{\alpha_i'(k)^2}{\alpha_j'(k)^2}. \quad (2-35)
\]

Condition (2-35) holds because based on step 5 of Algorithm 2-1, for all \( i, j \in I - U \),

\[
P_i'(k) = \left( \frac{\alpha_i'(k)^2}{\sum_{j \in I - U} \alpha_j'(k)^2} \right) \left( P_{\text{total}} - \sum_{j \in I - U} P_{j,\text{max}} \right) \quad \text{and thus for all} \quad i, j \in I - U, \quad \frac{P_i'(k)}{P_j'(k)} = \frac{\alpha_i'(k)^2}{\alpha_j'(k)^2}. \quad (2-36) \]

So (2-32) is satisfied.

For (2-33) to be true,

\[
\forall i \in U : \mu_i = \frac{\alpha_i'(k)}{2\sqrt{P_{i,\text{max}}(k)}} - \lambda. \quad (2-36)
\]

For this choice of \( \mu_i \) to be valid, \( \mu_i \) should be positive. Therefore,
\[ \forall i \in U : \frac{\alpha_i'(k)}{2 \sqrt{P_{i,\text{max}}(k)}} \geq \lambda. \quad (2-37) \]

And as \( \lambda \) is unique, based on (2-34) and (2-37), it is required that,

\[ \forall i \in U, m \in I - U : \frac{\alpha_i'(k)}{2 \sqrt{P_{i,\text{max}}(k)}} \geq \frac{\alpha_m'(k)}{2 \sqrt{P_m^*(k)}}. \quad (2-38) \]

For all \( i \in U \), as the algorithm has assigned \( P_{i,\text{max}}(k) \) as \( P_i(k)^* \), there was an iteration \( n \) of the algorithm, in which \( i \notin U^* \), and

\[
\left( \alpha_i'(k)^2 / \sum_{j \in I - U^*} \alpha_j'(k)^2 \right) \left( P_{\text{total}} - \sum_{j \in U^*} P_{j,\text{max}} \right) \geq P_{i,\text{max}}(k) \quad (2-39)
\]

Dividing both sides of (2-39) by \( P_m^*(k)^* \):

\[
\left( \alpha_i'(k)^2 / \sum_{j \in I - U^*} \alpha_j'(k)^2 \right) \left( P_{\text{total}} - \sum_{j \in U^*} P_{j,\text{max}} \right) \geq \frac{P_{i,\text{max}}(k)}{P_m^*(k)^*}, \quad (2-40)
\]

which yields,

\[
\forall i \in U, \forall m \in I, \exists n \leq n_f : \frac{\alpha_i'(k)^2}{\alpha_m'(k)^2} \geq \frac{P_{i,\text{max}}(k)}{P_m^*(k)^*}. \quad (2-41)
\]

Also, we note that each time the algorithm moves to the next iteration, it means some charging stations have reached their maximum bound and thus there is more power available for the ones that have not reached their maximum bound. So,

\[
\forall m \in I - U, \forall n \leq n_f : P_m(k)^* \geq P_m^n(k)^*, \quad (2-42)
\]

where \( n_f \) is the final iteration of the algorithm. Using (2-41) and (2-42),
\[
\forall i \in U, m \notin I - U : \frac{\alpha_i'(k)}{\alpha_m(k)} \geq \frac{P_{i,\text{max}}(k)}{P_m(k)}.
\]  

(2-43)

which is exactly the condition in (2-38). Therefore \( \mu_i \) selected as (2-36) is positive.

So, all the KKT conditions for \( P^*(k) = [P_{i,\text{max}}(k)] \) are satisfied and theorem is proven. 

2.4 Distributed Implementation of the Algorithm

The method explained in section II.C can be implemented in a distributed fashion. The required global information to assign the powers to the remaining vehicles would be the terms \( \sum_{j=1,j\notin U}^N \alpha_i'(k)^2 \) and \( P_{\text{total}} - \sum_{j\in U} \bar{P}_{j,\text{max}}(k) \) where \( U \) is the set of indices of all the vehicles allocated with their maximum allowable power. Each of these terms is composed of the summation of some local information. We introduce two consensus variables \( z_i(t) \) and \( q_i(t) \) for the \( i \)-th unit to access the global information using a local sharing of information with neighbors, based on consensus algorithms [35], [36].

Table 2-1 shows the step-by-step algorithm for finding the optimal power allocation at the \( i \)-th node. The first step is initialization. For all \( i=1,\ldots, N \), the \( z_i \) variable is initialized as \( \alpha_i'(k)^2 \) and \( q_i \) is initialized as zero, except in one of the nodes where the \( q \) variable is initialized as \( P_{\text{total}} \). Without loss of generality this node is indexed as 1.

The second step is the consensus phase. During this phase, each charging station updates its \( z \) and \( q \) variables according to:

\[
\begin{align*}
z_i(t+1) &= z_i(t) + \sum_{j\in N_i} w_{ij} \left( z_j(t) - z_i(t) \right), \\
q_i(t+1) &= q_i(t) + \sum_{j\in N_i} w_{ij} \left( q_j(t) - q_i(t) \right),
\end{align*}
\]  

(2-44)
where $N_i$ is the set of neighbors of node $i$, and $w_{ij}=w_{ji}$ are connectivity strengths, which are chosen such that $0 \leq w_{ij} < \left( \max_{i=1,\ldots,N} |N_i| \right)^{-1}$. It can be shown that with this choice of connectivity strengths, consensus values converge to the average of the initial values of all the nodes if the nodes form a connected group (i.e., if there is a path via neighbors between any two nodes) [16].

Equation (2-44) can also be represented in the matrix form as:

$$
\begin{bmatrix}
Z(t+1) \\
Q(t+1)
\end{bmatrix} = W \times \begin{bmatrix}
Z(t) \\
Q(t)
\end{bmatrix},
$$

(2-45)

where,

$$
Z(t) = \left[ z_1(t) \ldots z_N(t) \right]^T, \quad Q(t) = \left[ q_1(t) \ldots q_N(t) \right]^T.
$$

(2-46)

and,

$$
W(i,i) = 1 - \sum_{j \in N_i} w_{ij}, \quad W(i,j) = w_{ij}.
$$

(2-47)

At steps 3, 4, and 5, each individual charging station uses the global information from the consensus phase along with its own local information to decide about its power allocation by calculating $P_i(k)^{**}$ as:

$$
P_i(k)^{**} = \frac{q_i(T)}{z_i(T)} \alpha_i'(k)^2.
$$

(2-48)

Equation (2-48) basically evaluates (2-21) by noting that consensus values have converged to the average of their initial values:

$$
\frac{q_i(T)}{z_i(T)} \alpha_i'(k)^2 = \frac{P_{\text{total}}/N}{\sum_{j=1}^{N} \alpha_j'(k)^2/N} \alpha_i'(k)^2 = \frac{\alpha_i'(k)^2}{\sum_{j=1}^{N} \alpha_j'(k)^2} P_{\text{total}}.
$$

(2-49)

If $P_i(k)^{**}$ exceeds $P_{i,\text{max}}(k)$, the charging station allocates $P_{i,\text{max}}(k)$ as the optimal allocation of power and subtracts $\alpha_i'(k)^2$ from its $z$ variable and $P_{i,\text{max}}(k)$ from its $q$ variable to eliminate the information of charging station $i$ in subsequent consensus phases. The charging stations
Table 2-1: Cooperative Distributed PHEV/PEV Demand Management (CDPDM)

Algorithm

1. **Initialize variables**

\[ z_i(0) = \alpha'_i(k)^2, \quad \text{if } i = 1: q_i(0) = P_{total}^{\text{total}}, \quad \text{flag} = 0, \quad P_i^{**} = 0 \]

\[ \text{if } i \neq 1: q_i(0) = 0 \]

2. **Consensus phase**

\[ t = 0; \]

\[ \text{While } t \leq T \]

\[ z_i(t+1) = z_i(t) + \sum_{j \in N_i} w_{ij} \left( z_j(t) - z_i(t) \right); \]

\[ q_i(t+1) = q_i(t) + \sum_{j \in N_i} w_{ij} \left( q_j(t) - q_i(t) \right); \]

\[ t = t + 1; \]

End

3. **Check if** \[ \| z_i(T) - z_i(0) \| < \varepsilon_0: \]

\[ \begin{cases} \text{Yes: } P_i(k)^* = P_i^{**} & \rightarrow \text{Terminate} \\ \text{No: } & \rightarrow \text{Continue} \end{cases} \]

4. **Check if** \( \text{flag} = 0: \)

\[ \begin{cases} \text{Yes} \rightarrow P_i^{**} = \frac{q_i(T)}{z_i(T)} \alpha'_i(k)^2 \\ \text{No} \rightarrow P_i^{**} = P_{i,\text{max}}(k) \end{cases} \]

5. **Check if** \( \text{flag} = 0 \land \left( P_i^{**} \geq P_{i,\text{max}}(k) \right) \)

\[ \begin{cases} \text{Yes: } \text{flag} = 1; \quad P_i^{**} = P_{i,\text{max}}(k); \\ z_i(0) = z_i(T) - \alpha'_i(k)^2; \quad q_i(0) = q_i(T) - P_{i,\text{max}}(k); \end{cases} \]

\[ \text{No: } z_i(0) = z_i(T); \quad q_i(0) = q_i(T); \]

6. **Go back to 2. Consensus phase.**
that have already reached their power limit \( (P_{t,max}(k)) \) will not change their power allocation in subsequent iterations of the algorithm. Step 6 of the algorithm closes the loop; the algorithm goes back to step 2 and another consensus phase starts. The distributed algorithm for each node will terminate if no change in the consensus values occurs after the consensus phase. Due to the cooperative distributed nature of this algorithm, we call it the Cooperative Distributed PHEV/PEV Demand Management (CDPDM) algorithm.

Remark 2-2:

One of the requirements for the proposed cooperative distributed algorithm is connectivity for the communications network. Considering that the power grid network and the power feeders form a physically connected system, the connectivity for the communications network can be realized by having a communications network similar to the physical power network.

2.5 Simulation Results and Analysis

This section presents the performance of the proposed distributed algorithm in section II under different case studies. In all the case studies, each charging station is running the cooperative distributed algorithm shown in Table II. The algorithm is initialized every \( \Delta t \) seconds, which is the optimization step. During each optimization step, the algorithm keeps running an inner loop between steps 2 and 6. In all the case studies, the rationale for choosing the proper parameter values is as follows: Battery voltage ranges and capacities are typical values for the PHEV/PEV batteries taken from standard charging profiles such as the ones in [37]. SoC can be any value between 0 and 1 and for the case studies it is randomly selected for each EV. Arrival/departure times are also random and different PHEVs/PEVs can arrive and depart at different times and the maximum available power for each charging station \( (P_{max}) \) is
determined by the charging level (e.g., 3.3kW for single-phase, level 2 charging) and battery constraints.

The first case study shows how the algorithm works during each optimization step, and the second case study considers the successive usage of the algorithm in the long run. The robustness of the algorithm to link/node failures is also tested. To make the demonstrations tractable, the number of charging stations in the first two case studies is kept at five. The communications topology between charging stations is chosen similar to the power network topology as shown in Figure 2-3. If a single link/node fails in this topology, the rest of the graph will still remain connected and thus, the remaining nodes would be able to continue cooperation with others, via consensus.

The third experiment studies the scalability of the algorithm, and the fourth experiment studies the sub-optimality of the algorithm by benchmarking it against conventional centralized optimization methods. For these experiments, Monte Carlo simulations are used.
2.5.1 Case Study 1: Single–Step Optimization

A parking deck with five charging stations is considered. The objective is to optimally allocate the available power based on the priority of the vehicles. The priority for the \( i \)-th vehicle is set as:

\[
    w_i(k) = \frac{1}{\text{SoC}_i(k)T_i + \varepsilon}
\]

where \( T_i \) is the amount of time, the \( i \)-th vehicle is going to stay at the charging station and \( \varepsilon \) is a positive value introduced to avoid division by zero. Thus, the vehicles with lower SoC and sooner leaving time have higher priority for getting charged. At the \( k \)-th time step, the information regarding the battery side of the vehicles is given in Table 2-2.

The maximum number of neighbors of each node is two, so according to the condition 

\[
    0 \leq w_{ij} < \left( \max_{i=1, \ldots, N} |N_i| \right)^{-1}
\]

any connectivity strength between 0 and 0.5 can be chosen to ensure stability. Using a small value for the connectivity strengths results in a slow convergence of the consensus algorithm. We set all the connectivity strengths equal to 0.4 which is in the feasible range for stability (0 < 0.4 < 0.5) and is not small. In general, the optimal selection of the connectivity strengths to achieve fast convergence, is itself an optimization problem that has been described in reference [38] and is outside the scope of this chapter. The sampling time for the consensus phase is 1msec and each consensus phase consists of 50 iterations.

Figure 2-4 shows the allocated power and the evolution of the consensus variables over time. At the end of the first consensus phase \( t = 0.05 \), the fifth charging station reaches its maximum power boundary. So it subtracts its local information from its consensus values \( z_5 \) and \( q_5 \) according to step 5 of the algorithm. This causes the average of the consensus variables to
decrease. Consequently, the consensus variables converge to a new equilibrium point (their new average), which causes other charging stations to receive more power (because charging station 5 cannot consume any more power). Convergence to the new equilibrium is seen as big drops in the plots of the consensus variables. By the end of the second consensus phase ($t = 0.1$), charging station 2 reaches its maximum bound which allows other stations to receive more power. By the end of the third consensus phase ($t = 0.15$) all the available power is allocated and all the charging stations have received power in their feasible range. Therefore the consensus variables do not change any more and the algorithm settles down.

Now, we consider a link failure between node1 and node2 at time $t = 0.01$. Upon link failure, the two nodes can no longer exchange information with each other. When such a failure occurs, the connectivity strength between the two nodes becomes zero, namely $w_{12} = w_{21} = 0$.

Figure 2-5 shows the allocated power and the evolution of the consensus states. The charging
stations can still converge to the same power allocation as shown in Figure 2-4. As mentioned in Remark 2-2, the connectivity of the communications graph is the only requirement of the communications topology in order for the algorithm to work properly. This is the direct result of using consensus algorithms, which can work under switching topologies. After the link failure, the communications network remained connected, and the algorithm was able to converge. However, due to the loss of a connection link, the convergence during the consensus phases had been slowed down, especially in the second and third consensus phases.

Figure 2-4: Allocation of power and evolution of consensus states for case study (1) with no link failure
In this part, we study the behavior of the algorithm in long run. The vehicles arrive and departure at different times. Moreover, for the vehicles inside the charging stations, parameters such as SoC and voltage change due to the charging process. So the optimization becomes dynamic and the single-step optimization problem to be solved (as in Case Study 1) keeps

Figure 2-5: Allocation of power and evolution of consensus states for case study (1) with single-link failure

2.5.2 Case Study2: Dynamic Optimization

In this part, we study the behavior of the algorithm in long run. The vehicles arrive and departure at different times. Moreover, for the vehicles inside the charging stations, parameters such as SoC and voltage change due to the charging process. So the optimization becomes dynamic and the single-step optimization problem to be solved (as in Case Study 1) keeps
changing. Five vehicles with different initial SoCs and different arrival and departure times are considered as shown in Table 2-3.

The desired SoC is 0.9 and the maximum power output of each station is 3.3kW. The priority weights are defined similar to case study1. We simulate the distributed charging control model for 8 hours. The charging stations change their power allocation every 10 minutes. Figure 2-6 shows the charging profiles of the vehicles over time. At time \( t = 0 \), two PHEVs are in the charging stations. PHEV2 has an earlier departure time and, therefore, gets higher charging rate. At time \( t=1h \), PHEV3 arrives at the charging station. As PHEV3 needs to leave very soon, its priority for getting power is higher than those of PHEV1 and PHEV2. At this time PHEV1, which leaves later than the others, has the lowest priority. Therefore, its charging power drops to near zero, to allow PHEV3 to get more charging power. At time \( t=2h \), PHEV3 leaves with \( SoC=0.9 \). The allocation of power during the rest of the time can be explained similarly. We
can see that at all times the total consumed power is below 5kW. The current profiles are drawn to show that the charging currents resulted from the optimization process are within the feasible boundaries (i.e., \( I \leq I_{\text{max}} = 12\text{A} \)). This is done by translating the boundary on the charging current into a boundary on the charging power via multiplying it by the terminal voltage of the battery. This boundary then serves as a local constraint for the optimization problem.

Now, we consider the case where node2 fails to operate at \( t=0.5\text{h} \) and recovers at \( t=2\text{h} \). Figure 2-7 shows the charging profiles under this scenario. Upon failure, node2 loses its connection with all its neighbors. However the rest of the nodes remain connected. When node2 fails, its charging rate drops down to zero and no change happens in the SoC of the vehicle. When node2 recovers at \( t=3\text{h} \), PHEV2 starts to be charged again. Similar to the case without node failure, all the local and global constraints are satisfied.
Remark 2-3:

To see why the algorithm is robust against single-link/node failures, we note that information exchange is happening through the consensus algorithm:

\[ z_i(t+1) = z_i(t) + \sum_{j \in N_i} w_{ij} (z_j(t) - z_i(t)), i = 1...N. \]  

The global information is the summation of local information:

\[ \sum_{i=1}^{N} z_i(t+1) = \sum_{i=1}^{N} z_i(t) + \sum_{i=1}^{N} \sum_{j \in N_i} w_{ij} (z_j(t) - z_i(t)). \]  

The second summation on the right hand side of the equation under balanced weights (i.e., \( \forall i, j : w_{ij} = w_{ji} \)) always equals zero. Now if a link between two nodes \( i, j \) fails, the connectivity strength between those two nodes becomes zero (i.e., \( w_{ij} = w_{ji} = 0 \)). Then still

Figure 2-7: Charging profiles for case study (2) with node failure
\[ \sum_{j=1}^{N} \sum_{i \in N_j} w_{ij} (z_j(t) - z_i(t)) = 0 \] and as a result \[ \sum_{i=1}^{N} z_i(t+1) = \sum_{i=1}^{N} z_i(t) . \] Thus, the global information (summation of the local information) remains intact. Therefore, if the communications graph remains connected after the failure, the consensus values still converge to the true average (i.e., average of the global information). The proposed algorithm uses consensus networks to access the global information. Thus, it will be robust to any link failures that do not affect the connectivity of the communications networks. Also, when one node fails to communicate with its neighbors, then the rest of the group can continue their operations provided that their communications graph is connected.

2.5.3 Scalability Analysis

In this part of the article, we test the per-node computational scalability of the proposed distributed algorithm. To do so, we perform Monte Carlo simulations. Each node represents a charging station. The range for the number of nodes for the experiments is 10 to 450. For each value of the number of nodes, we ran 100 Monte Carlo simulations [39].

The parameters in each simulation are randomly sampled from their feasible ranges as shown in Table 2-4. A random network topology is chosen for each simulation. The number of operations each node needs to perform until reaching the optimal point, \( n_{\text{operations}} \), is calculated as follows:

\[ n_{\text{operations}} = n_{\text{out}} \times n_{\text{in}} \times \bar{k}, \]  

(2-53)

where \( n_{\text{out}} \) and \( n_{\text{in}} \) are the numbers of outer and inner loops, respectively, of the algorithm prior to convergence and \( \bar{k} \) is the average number of neighbors of each node. Basically, \( n_{\text{out}} \times n_{\text{in}} \) represents the total number of times the consensus operation is performed at each node, and
To count the number of outer loops, we count the number of times we have to repeatedly use (2-21) until all vehicles get a valid allocation of power. To count the number of inner loops, we look at the number of iterations the average consensus needs to reach within 2% of the final value. According to [38], this value can be calculated as:

\[ n_{in} = \frac{\ln(0.02)}{\ln(\rho(W))}, \] (2-54)

where \( \rho(W) \) is the spectral radius of update matrix \( W \) in (2-45).

Figure 2-8 and Figure 2-9 show the minimum, maximum, and average number of the operations among 100 random simulations as a function of the number of nodes. The case studies are repeated for \( \bar{k} = 3 \) and \( \bar{k} = 20 \). Each dot in the figure corresponds to a set of 100 simulation data. A logarithmic trend line is fitted to the data points using a least squares error approach. The coefficient of determination \( (R^2) \), with a maximum value of one, represents how well the data is explained by the curve [40]. It can be seen that for both of the figures, the logarithmic trend explains the increase in the average computational complexity with \( R^2=0.99 \).
In other words, by increasing the number of charging stations, the average computational complexity for the cooperative distributed algorithm grows almost logarithmically rather than exponentially. Therefore, the proposed distributed algorithm is scalable. By noting that each
multiplication operation in the algorithm corresponds to a data-packet transfer between two nodes, the per-node communicational burden would exhibit exactly the same scalable behavior.

2.5.4 Sub-Optimality of the Algorithm due to Nonlinear Approximation: Benchmarking against nonlinear optimization solvers

In this section, we study the sub-optimality of the solutions returned by the proposed algorithm due to the nonlinear approximation in section 2-3-2 by benchmarking it against two nonlinear optimization solvers: Sequential Quadratic Programming (SQP) and Interior Point methods. SQP methods are among the most successful nonlinear programming algorithms [41] and are used on nonlinear problems where the objective function and the constraints are both continuously differentiable. Interior Point methods are also conventional algorithms for solving constrained nonlinear optimization problems [42]. Both of these algorithms are provided by MATLAB through the fmincon optimization function.

For both of the centralized algorithms, the optimization problem to be solved is the problem represented by (2-11), prior to the nonlinear approximation of the cost function. To have a fair comparison, the optimality of the solutions of the algorithms should be measured using the same objective function. Therefore, we use the original objective function of interest in (2-1) to compare the optimality of the algorithms. We use Monte Carlo simulations and generate 100 different scenarios where the number of nodes is randomly selected between 10 and 100 and the parameters are randomly sampled from the ranges specified in Table 2-4. For each simulation scenario, the three algorithms are used and the value of the objective function is recorded. Then, the error of the proposed algorithm with respect to the $i$-th algorithm is calculated as:
where $J_{alg}$ is the objective function value returned using Algorithm 2-1, and $J_i$ is the objective function value from the $i$-th algorithm (SQP or Interior Point).

The distributions of errors are shown in Figure 2-10. We can see that the maximum deviation from optimality is around 2%.

2.6 Conclusion

In this chapter, we introduced a cooperative distributed energy management scheme for community charging of PHEV/PEVs. The main advantages of our approach are preventing the need for a central energy-management unit, being more robust against single-link/node failures, and being scalable in terms of single-node computations despite a small amount of sub-optimality (~2%). First, we formulated the charge allocation as a constrained nonlinear optimization problem and developed a procedure to solve the problem by checking KKT conditions. Then, we decomposed the charge information to local and global parts and
proposed a distributed consensus-based algorithm to find the optimal charging policy in each charging time frame using peer-to-peer communications capabilities among charging stations. The performance, scalability, and robustness of the algorithm to single-link/node failures were case-studied.

To apply the proposed algorithm in the real world, practical issues such as EV and charging station communications features, voltage and frequency deviations at the terminals, losses in the power lines and feeders as well as more complicated battery models should be considered in the distributed optimization process. This will be the future work of the authors.

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CHAPTER 3. INCREMENTAL WELFARE CONSENSUS ALGORITHM FOR COOPERATIVE DISTRIBUTED GENERATION/DEMAND RESPONSE IN SMART GRID

Abstract:
In this chapter, we introduce the Incremental Welfare Consensus (IWC) algorithm for solving the energy management problem in a smart grid environment populated with distributed generators and responsive demands. The proposed algorithm is distributed and cooperative such that it eliminates the need for a central energy-management unit, central price coordinator, or leader. The optimum energy solution is found through local peer-to-peer communications among smart devices. Each distributed generation unit is connected to a local price regulator, as is each consumer unit. In response to the price of energy proposed by the local price regulators, the power regulator on each generation/consumer unit determines the level of generation/consumption power needed to optimize the benefit of the device. The consensus-based coordination among price regulators drives the behavior of the overall system toward the global optimum, despite the greedy behavior of each unit. The primary advantages of the proposed approach are: 1) convergence to the global optimum without requiring a central controller/coordinator or leader despite the greedy behavior at the individual level and limited communications, and 2) scalability in terms of per-node computation and communications burden.

3.1 Introduction
The smart grid concept is revolutionizing the power grid. “Smart grid” refers to technologies used to update utility electricity systems with computer-based automation and control through two-way communications structures [13]. In light of the smart grid, the future power system
will be an aggregation of controllable power-electronics devices with the ability to exchange information through the communications network [1]. This Cyber-Physical System (CPS) is inherently geographically distributed, does not have a fixed topology, and embeds a variety of controllable energy devices, such as Distributed Generators (DGs) and Distributed Energy Storage Devices (DESDs), as well as controllable loads and responsive demands.

Managing the new power system with a variety of controllable devices requires a paradigm shift in traditional Energy Management Systems (EMSs), which are operated centrally through Supervisory, Control, and Data Acquisition (SCADA) systems [1], [4]. This paradigm shift needs to consider the local intelligence of the distributed devices, as well as their communications capability [5]. The new energy-management solution needs to be scalable in its computation and communications efforts and be robust enough when faced with single points of failures to survive the inevitable device/link failures of the smart grid.

Self-organizing distributed systems can provide useful solutions for developing scalable and robust algorithms for energy management in the smart grid. The recent trend, shown in the literature of energy management in the smart grid, has been moving toward distributed techniques [27], [30], [32], [43]–[53]. In [43] a distributed load shedding algorithm is introduced. In [32], [44]–[46], [53] distributed approaches are proposed to regulate the distributed energy resources optimally. In [27], a distributed framework for controlling user demand is proposed which requires a central coordinator to gather real-time demand information and update the price. A game theoretical approach is proposed in [48] to optimize the energy costs by forming a game among users on the demand side where the consumer units are required to be capable of two-way communications with all the other nodes. In [49], a multi-player methodology is considered where a set of independent players consisting of
distributed energy sources and responsive demands cooperate on common goals and compete on individual goals. This approach also requires different players to be aware of the state of all the other players. In [50] and [52], real-time pricing (RTP) algorithms are proposed to regulate the behavior of several subscribers and energy providers. The communication is between each subscriber and the energy providers and the energy provider acts as a coordinator for setting the price. Finally, [51] and [30] introduced distributed algorithms for the demand side management of PHEV/PEVs.

The aforementioned distributed energy management approaches in the literature have either of the following limitations:

1. They either consider solely the generation side [32], [44]–[46], [53] or the demand side [27], [43], [47], [51], [52]. However, in the future power system with integration of distributed energy resources, the traditional separation of the generation side and the demand side would not be valid anymore and the energy management system needs to consider both sides simultaneously.

2. They need either a central coordinator [27], [54],[50] and/or a leader [32] to gather specific information from the distributed devices and update certain global information (such as price) or the distributed devices need to have the ability to communicate with all the other devices [48], [49].

In this chapter we introduce the Incremental Welfare Consensus (IWC) algorithm for energy management in the smart grid considering the responsive distributed loads and dynamic distributed generators simultaneously. The general structure for running IWC algorithm is shown in Figure 3-1. Each responsive demand and dispatchable generation unit connected to the grid updates its generation/consumption level based on a pricing signal received from a
local price regulator. Each price regulator, monitors the generation/consumption level of one generation/demand unit and exchanges information with neighbor price regulators.

The IWC algorithm determines 1) how each generation/demand unit should update its consumption/generation level based on the local price signal, 2) what information is required to be exchanged among price regulators, and 3) how the price regulators should update the energy price. The goal is to ensure that the energy production/consumption level in the entire network is optimized and all the local/global constraints are satisfied. The detailed functionalities of each of the blocks of Figure 3-1 and the information flows will be explained throughout the chapter. We have published relevant work in [55], but it lacks theoretical proof, extensive case studies and scalability analysis.

The main features of the IWC algorithm are as follows:

1. Manages the demand side and the generation side simultaneously.

2. Distributed computation/communications: No matter how large the network is, each unit only performs a local optimization at each iteration and exchanges information only with its neighbors.
3. Privacy of information: As each unit carries out its own local optimization based on the locally offered price, no information about the utility/cost functions of consumer/generation units needs to be disclosed to others.

4. Convergence to the global optimum: Although each unit selfishly tries to choose the optimal local power consumption/generation to maximize its own benefit, the local price regulators adjust the prices such that the behavior of the overall system moves toward the global optimum. This is unlike uncoordinated approaches in which the greedy behavior of individuals results in sub-optimal solutions [56], [57].

5. Scalability: The IWC algorithm is scalable in terms of the computation and communications burden as the number of devices increases. The scalability of the algorithm is verified using the Monte Carlo simulations.

The chapter is organized as follows. Section II explains the social welfare maximization problem in the energy markets. Section III introduces the IWC algorithm as a cooperative distributed solution to the social welfare maximization problem, and Section IV brings the numerical simulation results. The concluding remarks come in Section VI.

3.2 Problem Formulation: Social Welfare Maximization in Energy Market

Social welfare maximization at time $t$ is defined as maximizing the summation of welfares of all the generation and consumer units [52], [55]:

$$
\text{Max}_{P(t), P} \left( \sum_{i \in D} W_{i,D}(P_i(t), p(t)) + \sum_{i \in G} W_{i,G}(P_i(t), p(t)) \right),$$

s.t. $\sum_{i \in D} P_i(t) = \sum_{i \in G} P_i(t), \forall i \in G: 0 \leq P_i(t) \leq P_{i,\text{max}}, \forall i \in D: P_{i,\text{min}} \leq P_i(t) \leq P_{i,\text{max}}$.

where $D$ is the set of indices of demand units, $G$ is the set of indices of generation units, $P(t) = [P_1(t), P_2(t), ..., P_{|D|+|G|}(t)]^T$ is the vector of all generation and consumption powers at time $t$,
and \( p(t) \) is the price of energy at time \( t \). Here, \( W_{i,D}() \) and \( W_{i,G}() \) are the welfare functions of the demand side and the generation side for the \( i \)-th unit respectively. The social welfare maximization problem is subject to the power balance constraint and local consumption/generation limits for each unit.

### 3.2.1 Welfare on the Demand Side

The welfare on the demand side for the \( i \)-th consumer unit is defined as the utility of the demand minus the cost of energy:

\[
W_{i,D}(P_i(t), p(t)) = U_i(P_i(t)) - p(t)P_i(t),
\]

where \( U_i(.) \) is the utility function for the \( i \)-th unit, representing the level of consumer satisfaction as a function of power consumption. Defining the proper utility functions for the \( i \)-th consumer unit requires satisfying three properties [52]:

1. The utility function should be non-decreasing:

   \[
   \frac{\partial U_i(P_i)}{\partial P_i} \geq 0.
   \]  

2. The level of satisfaction should saturate with using more amount of power:

   \[
   \frac{\partial^2 U_i(P_i)}{\partial P_i^2} \leq 0.
   \]

3. Zero power consumption has zero satisfaction:

   \[
   U_i(0) = 0.
   \]

In general to satisfy these conditions, \( U_i \) can be chosen as a concave differentiable function with value of zero at \( P_i = 0 \). A conventional form of the utility function satisfying these conditions is the quadratic utility function [52], [55]:

\[
U_i(P_i) = \begin{cases} 
\omega_i P_i - \alpha_i P_i^2 & P_i \leq \omega_i / 2\alpha_i \\
\omega_i^2 / 4\alpha_i & P_i \geq \omega_i / 2\alpha_i 
\end{cases},
\]
where $\omega$ and $\alpha$ are the parameters that differentiate consumers.

### 3.2.2 Welfare on the Generation Side

The welfare on the generation side for the $i^{th}$ generation unit can be defined as the net profit:

$$ W_{i,G}(P_i(t), p(t)) = p(t)P_i(t) - C_i(P_i(t)), $$

(3-7)

where $C_i(.)$ is the cost function for the $i$-th generation unit. The cost function for the $i$-th generation unit is usually approximated by a quadratic convex function [58]:

$$ C_i(P_i) = a_i P_i^2 + b_i P_i + c_i, $$

(3-8)

where $a_i$, $b_i$, and $c_i$ are predetermined constants and $P_i$ is the amount of generated power. In general, the cost function can be chosen as a convex differentiable function.

**Remark 3-1:** Throughout this chapter, we consider the lower limit for generation units to be zero. A more general formulation can be achieved by setting the lower limit of generation units to nonzero values, but allowing the generation units to be in the OFF state as well (i.e. not producing power). This formulation would require introducing binary variables to model the ON/OFF status of the generators. Then, the optimization problem becomes a mixed integer programming. The proposed algorithm in this chapter solves for continuous variables. Extending it to solve a mixed integer programming will be considered in the future work of the authors.

**Remark 3-2:** If the ramping constraints are also considered in the optimization, the local constraints would become:

$$ \forall i \in G \cup D : \tilde{P}_{i,\text{min}}(t) \leq P_i(t) \leq \tilde{P}_{i,\text{max}}(t), $$

(3-9)

where,
\[ \forall i \in G \cup D: \tilde{P}_{i, \text{max}}(t) = \min \left\{ P_{i, \text{max}}, P_i(t - 1) + \Delta P_{i, \text{max}}^{\text{up}} \right\}, \]

\[ \forall i \in G: \tilde{P}_{i, \text{min}}(t) = \max \left\{ 0, P_i(t - 1) - \Delta P_{i, \text{max}}^{\text{down}} \right\}, \quad (3-10) \]

\[ \forall i \in D: \tilde{P}_{i, \text{min}}(t) = \max \left\{ P_{i, \text{min}}, P_i(t - 1) - \Delta P_{i, \text{min}}^{\text{down}} \right\}. \]

Here, \( P_i(t - 1) \) is the allocation of power for the \( i \)-th unit in the previous optimization period, and \( \Delta P_{i, \text{max}}^{\text{up}}, \Delta P_{i, \text{min}}^{\text{down}} \) are the ramp up and ramp down limits for the \( i \)-th unit. In the rest of the chapter, the focus will be on optimization in one single period. So, for simplicity we will drop the time index \( t \) from the notations.

### 3.3 Incremental Welfare Consensus Algorithm

By substituting \( W_{i,D}(\cdot) \) and \( W_{i,G}(\cdot) \) for demand and generation units from (3-2) and (3-7) into (3-1) and presenting the optimization as a minimization problem, we can write (3-1) as:

\[
\text{Min } \sum_{i \in G} C_i(P_i) - \sum_{i \in D} U_i(P_i),
\]

\[
\text{s.t. } \sum_{i \in D} P_i = \sum_{i \in G} P_i, \forall i \in G: 0 \leq P_i(t) \leq P_{i, \text{max}}, \forall i \in D: P_{i, \text{min}} \leq P_i(t) \leq P_{i, \text{max}}. \quad (3-11)
\]

The optimization problem represented by (3-11) is a convex optimization problem with affine constraints. Therefore, satisfaction of Karush-Kuhn-Tucker (KKT) conditions would suffice for ensuring the global optimality [34].

#### A. Augmenting the Objective Function with KKT Multipliers

The first step in solving (3-11) is to augment the objective function with KKT multipliers:

\[
J = \sum_{i \in G} C_i(P_i) - \sum_{i \in D} U_i(P_i) + \lambda \left( \sum_{i \in D} P_i - \sum_{i \in G} P_i \right), \quad (3-12)
\]

46
where $\lambda$ represents the KKT multiplier or dual variable for the global equality constraint. Then $\mathbf{P} = [P_1, P_2, ..., P_{|D|+|G|}]^T$ is called the primal vector. As the inequality constraints are local, they do not need to be added in the augmented cost function because they can be treated as the boundaries of the domain of the problem.

### B. Dual Decomposition

The next step is to use dual decomposition [59] and develop a distributed iterative approach that moves in the direction of minimizing the augmented objective function $J$ in terms of the primal vector $\mathbf{P}$ and in the direction of maximizing $J$ in terms of the dual variable, $\lambda$:

$$
\forall i \in D: P_i^k = \arg \min_{P_{i,\min} \leq P_i \leq P_{i,\max}} \left( \lambda^k P_i - U_i(P_i) \right),
$$

$$
\forall i \in G: P_i^k = \arg \min_{0 \leq P_i \leq P_{i,max}} \left( C_i(P_i) - \lambda^k P_i \right),
$$

$$
\lambda^{k+1} = \lambda^k + \eta \Delta P^k, \quad \Delta P^k = \sum_{i \in D} P_i^k - \sum_{i \in G} P_i^k,
$$

where $P_i^k$ and $\lambda^k$ are the values of the primal and dual variables at the $k^{th}$ iteration, $\Delta P^k$ represents the power mismatch between generation and demand units at the $k^{th}$ iteration, and $\eta$ represents the updating step-size. If (3-11) is feasible, by choosing a small enough value for $\eta$, the iterative procedure (3-13) converges to the global optimum.

### C. Distributed Estimation of Global Information through Distributed Observers

The iterative procedure in (3-13) requires a central coordinator to gather the local information about power consumption and generation levels, update the dual variable, and broadcast the dual variable to all of the generation and demand units. However, we are seeking a center-free solution.
To eliminate the central coordinator, we take advantage of the consensus protocol [60] and design distributed observers to estimate the value of the global information only through local communications with their neighbors. Here, the global information is the power mismatch between the generation and demand units at the $k^{th}$ iteration, denoted by $\Delta P^k$.

The update equations for the $i^{th}$ distributed observer to estimate $\Delta P^k$ on the generation side would be:

$$\forall i \in G : $$

$$\Delta \hat{P}^{k+1}_i = \Delta \hat{P}^k_i + \sum_{j \in N_i} w_{ij} \left( \Delta \hat{P}^k_j - \Delta \hat{P}^k_i \right) + P^k_i - P^{k+1}_i,$$

(3-14)

where $\Delta \hat{P}^k_i$ is the estimate of the power imbalance at the $k^{th}$ iteration ($\Delta P^k$) by the $i^{th}$ unit, $N_i$ is the set of neighbors of node $i$, and $w_{ij} = w_{ji}$ is the connectivity strength between node $i$ and node $j$ and is chosen such that $0 \leq w_{ij} < \left( \max_{i=1,..,N} |N_i| \right)^{-1}$ to ensure the convergence [60].

A similar equation can be written for the demand side:

$$\forall i \in D : $$

$$\Delta \hat{p}^{k+1}_i = \Delta \hat{p}^k_i + \sum_{j \in N_i} w_{ij} \left( \Delta \hat{p}^k_j - \Delta \hat{p}^k_i \right) + P^{k+1}_i - P^k_i.$$

(3-15)

Summing $\Delta \hat{p}^{k+1}_i$ over all units,

$$\sum_j \Delta \hat{p}^{k+1}_i = \sum_j \Delta \hat{p}^k_j + \sum_i \sum_{j \in N_i} w_{ij} \left( \Delta \hat{p}^k_j - \Delta \hat{p}^k_i \right)$$

$$+ \left( \sum_{i \in D} P^{k+1}_i - \sum_{i \in G} P^k_i \right) - \left(3-16(242,713),(782,838)$

The second term equals to zero because for all $i$ and $j$, $w_{ij} = w_{ji}$. The third term and the fourth term are the power imbalances at iterations $k+1$ and $k$, respectively. So, (3-16) can be written as,
\[ \sum_i \Delta \hat{P}_i^{k+1} = \sum_i \Delta \hat{P}_i^k + \Delta P^{k+1} - \Delta P^k, \]  

Equation (3-17) suggests that the summation of \( \Delta \hat{P}_i^{k+1} \) for all \( i \) equals the power imbalance of the entire network. When the algorithm converges, the consensus variables converge toward each other,

\[ \lim_{k \to \infty} \Delta \hat{P}_1^k = \Delta \hat{P}_2^k = \ldots = \Delta \hat{P}_N^k. \]  

By continuing this process,

\[ \sum_i \Delta \hat{P}_i^{k+1} = \Delta P^{k+1}. \]  

Equation (3-19) suggests that the summation of \( \Delta \hat{P}_i^{k+1} \) for all \( i \) equals the power imbalance of the entire network. When the algorithm converges, the consensus variables converge toward each other,

\[ \forall i \in G \cup D : \lim_{k \to \infty} \Delta \hat{P}_i^k = \frac{\Delta P^k}{N}, \]

which validates that for all \( i \), \( \Delta \hat{P}_i^k \) is observing the average total power mismatch of the entire network.

Now, the estimated value of the power imbalance can be used at each unit to update the dual variable. However, to ensure that the dual variables at all the units converge to the same
value, they need to be coordinated among neighbors. Therefore, the update equation for the
dual variable would be written as:
\[
\forall i \in G \cup D: \lambda_i^{k+1} = \lambda_i^k + \sum_{j \in N_i} w_{ij} \left( \lambda_j^k - \lambda_i^k \right) + \eta \Delta P_i^k ,
\]
(3-22)
where $\lambda_i^k$ is the value of the dual variable in the $i$th unit at the $k$th iteration. By having the value
of the dual variable, each generation and consumer unit can update its generation/consumption
level as:
\[
\forall i \in D: P_i^k = \arg \min_{P_i \in [P_{i,\text{min}}^k, P_{i,\text{max}}^k]} \left( \lambda_i^k P_i - U_i(P_i) \right),
\]
\[
\forall i \in G: P_i^k = \arg \min_{0 \leq P_i \leq P_{i,\text{max}}} \left( C_i(P_i) - \lambda_i^k P_i \right).
\]
(3-23)

Here, we present two theorems. The first one ensures that under feasibility conditions, the fixed
point(s) of the algorithm presented by (3-14), (3-15), (3-22), and (3-23) satisfy the KKT
conditions of optimality for (3-11). As (3-11) is a convex problem with affine constraints,
satisfaction of KKT conditions translates into the global optimality [34]. The second theorem
ensures that by choosing a sufficiently small value for $\eta$, the fixed point of the algorithm is
stable.

**Theorem 3-1:** If the problem in (3-11) is feasible, and the cost/utility functions are continuous
differentiable convex/concave functions, then the fixed point(s) of the iterative algorithm
presented by (3-14), (3-15), (3-22) and (3-23) triggered from an initially balanced state (i.e. for
all $i$, $\Delta P_i^0 = 0$) satisfy the KKT conditions of optimality for the optimization problem of (3-11).

**Proof:**

Let $P^* = [P_i^*]_{i \in G \cup D}, \lambda^* = [\lambda_i^*]_{i \in G \cup D}$, and $\Delta P^* = [\Delta P_i^*]_{i \in G \cup D}$ be the fixed point of the iterative
procedure presented by (3-14), (3-15), (3-22), and (3-23). The KKT conditions of optimality
imply that $P^*_i = \{P^*_i\}_{i \in G \cup D}$ is the solution for (3.11) if there exist constants $\mu_i$, and $\xi_i$ for all $i \in G \cup D$ such that:

$$\Delta P^* = \sum_{i \in D} P^*_i - \sum_{i \in G} P^*_i = 0,$$

(3.24)

$$\forall i \in D: -\left. \frac{\partial U_i(P_i)}{\partial P_i} \right|_{P_i=p^*_i} + \lambda_i - \mu_i + \xi_i = 0,$$

(3.25)

$$\forall i \in G: \left. \frac{\partial C_i(P_i)}{\partial P_i} \right|_{P_i=p^*_i} - \lambda_i - \mu_i + \xi_i = 0,$$

(3.26)

$$\forall i \in D: \mu_i (P_{i,\min} - P_i^*) = 0, \forall i \in G: \mu_i (-P_i^*) = 0,$$

(3.27)

$$\forall i \in G \cup D: \xi_i (P_i^* - P_{i,\max}) = 0.$$

(3.28)

Summing $\lambda_i^*$ over all units using (3.22),

$$\forall i \in G \cup D: \sum_i \lambda_i^* = \sum_i \lambda_i^* + \sum_{i \in G \cup D} w_{ij} (\lambda_j^* - \lambda_i^*) + \eta \sum_i \Delta P_i^* ,$$

(3.29)

The second term equals to zero because for all $i$ and $j$, $w_{ij} = w_{ji}$. By substituting for the third term from (3.19),

$$\forall i \in G \cup D: \sum_i \lambda_i^* = \sum_i \lambda_i^* + \eta \Delta P^* ,$$

(3.30)

Therefore $\Delta P^* = 0$ and (3.24) is satisfied. On the other hand, from (3.22) and the fact that $\Delta P^* = 0$, we can verify that at the fixed point,

$$\lambda_1^* = \lambda_2^* = \ldots = \lambda_{|G|+|D|}^* = \lambda^* .$$

(3.31)

Now from (3.23), on the demand side at the fixed point:

$$\forall i \in D: P_i^* = \min_{P_i,\min \leq P_i \leq P_i,\max} \left( \lambda^* P_i - U_i(P_i) \right).$$

(3.32)
Let $P_i^{**}$ be the minimum of $\lambda^* P_i - U_i(P_i)$, for all $i \in D$, without considering the constraint $P_{i,\text{min}} < P_i < P_{i,\text{max}}$. Thus, as $\lambda^* P_i - U_i(P_i)$ is a convex differentiable function, its derivative at the minimum should be zero, i.e. $\lambda^* - \partial U_i / \partial P_i|_{P_i^*} = 0$. For any $i \in D$ three possibilities arise:

1. $P_{i,\text{min}} < P_i^* < P_{i,\text{max}}$: In this case, $P_i^* = P_i^{**}$ and $\lambda^* - \partial U_i / \partial P_i|_{P_i^*} = 0$. So by taking $\lambda = \lambda^*$ and $\mu_i = \xi_i = 0$, (3-25), (3-27), and (3-28) are satisfied.

2. $P_i^{**} > P_{i,\text{max}}$: In this case, $P_i^* = P_{i,\text{max}} < P_i^{**}$, so $\lambda^* - \partial U_i / \partial P_i|_{P_i^*} \leq 0$, as otherwise $P_i^{**}$ would be a maximum. Thus, by taking $\lambda = \lambda^*$, $\xi_i = -\lambda^* + \partial U_i / \partial P_i|_{P_i^*}$, and $\mu_i = 0$, (3-25), (3-27), and (3-28) are satisfied.

3. $P_i^{**} < P_{i,\text{min}}$: In this case, $P_i^* = P_{i,\text{min}} > P_i^{**}$, so $\lambda^* - \partial U_i / \partial P_i|_{P_i^*} \geq 0$, as otherwise $P_i^{**}$ would be a maximum. Thus, by taking $\lambda = \lambda^*$, $\xi_i = 0$, and $\mu_i = \lambda^* - \partial U_i / \partial P_i|_{P_i^*}$, (3-25), (3-27), and (3-28) are satisfied.

For the fixed point on the generation side, similar arguments can be made, and it can be shown that (3-26), (3-27), and (3-28) are satisfied. Thus, the theorem is proven.

**Corollary 3-1:** If the problem (3-11) is feasible and the cost/utility functions are strictly convex/concave, then the fixed point of the iterative algorithm presented by (3-14), (3-15), (3-22) and (3-23) is unique.

**Proof:** Based on Theorem 1, the fixed point of the algorithm satisfies KKT conditions of optimality for problem (11). If the cost/utility functions are strictly convex/concave, problem (11) is a strictly convex optimization problem and so it has a unique optimal point [34]. On the other hand, satisfaction of KKT conditions suffices for global optimality for any convex optimization problem with affine constraints [34]. If the algorithm has more than one fixed
point, then it means problem (11) has more than one global optimum which contradicts the strict convexity assumption. So, if the cost/utility functions are strictly convex/concave, the algorithm has a unique fixed point. ■

**Remark 3-3:**

The technical result of Theorem 3-1, is in agreement with the fundamental welfare theorems of micro-economics. The first fundamental theorem states that every competitive equilibrium is Pareto-optimal [61]. The second theorem states that every Pareto-optimal allocation is an equilibrium for a competitive economy [62], [63]. Basically, IWC algorithm provides a competitive market among the suppliers and consumers of energy which results in a Pareto-optimal equilibrium. However, cooperation among price regulators together with the convexity assumptions ensures that the Pareto-optimal point of this market is indeed the global optimum point. Moreover, the structure of the IWC algorithm allows this market to work in a fully distributed fashion without requiring any central price provider.

**Theorem 3-2:** If the problem in (3-11) is feasible and the cost/utility functions are strictly convex/concave and continuously differentiable, there exist a positive value \( \varepsilon \) such that for all values of \( 0 < \eta < \varepsilon \), the fixed point of the algorithm presented by (3-14), (3-15), (3-22) and (3-23) is attractive.

**Proof:**

Writing (3-14), (3-15), and (3-22) in vector form:

\[
\begin{bmatrix}
\lambda^{k+1} \\
\Delta \hat{P}^{k+1}
\end{bmatrix} =
\begin{bmatrix}
W & \eta I \\
0 & W
\end{bmatrix}
\begin{bmatrix}
\lambda^k \\
\Delta \hat{P}^k
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} u^k,
\]

(3-33)

where \( \lambda = [\lambda_i] \), \( \Delta \hat{P} = [\Delta \hat{P}_i] \), \( W \) is the consensus update matrix,
and \( u^k \) is a nonlinear feedback defined as:

\[
\forall i \in D: u^k(i) = \arg \min_{P_i, \min \leq P_i \leq P_i, \max} \left( \lambda_i^{k+1} P_i - U_i(P_i) \right) - \arg \min_{P_i, \min \leq P_i \leq P_i, \max} \left( \lambda_i^k P_i - U_i(P_i) \right),
\]

(3-35)

\[
\forall i \in G: u^k(i) = -\arg \min_{0 \leq P_i \leq P_i, \max} \left( C_i(P_i) - \lambda_i^{k+1} P_i \right) + \arg \min_{0 \leq P_i \leq P_i, \max} \left( C_i(P_i) - \lambda_i^k P_i \right).
\]

(3-36)

Denoting \( P_i^* = \arg \min_{P_i, \min \leq P_i \leq P_i, \max} (\lambda_i^k P_i - U_i(P_i)) \) for demand units:

\[
P_i^* + u^k(i) = \arg \min_{P_i, \min \leq P_i \leq P_i, \max} \left( (\lambda_i^k + \Delta \lambda_i) P_i - U_i(P_i) \right).
\]

(3-37)

where \( \Delta \lambda_i = \lambda_i^{k+1} - \lambda_i^k \). Without considering the local limits, the derivative at the minimum point would be zero. So:

\[
\lambda_i^k + \Delta \lambda_i - \partial U_i / \partial P_i |_{P_i^* + u^k(i)} = 0 .
\]

(3-38)

To analyze the dynamics around the fixed point, we write the first-order Taylor expansion for \( \partial U_i / \partial P_i \):

\[
\lambda_i^k + \Delta \lambda_i - \left( \partial U_i / \partial P_i |_{P_i} + \partial^2 U_i / \partial P_i^2 |_{P_i} \times u^k(i) \right) = 0 ,
\]

(3-39)

Noting that \( \lambda_i^k - (\partial U_i / \partial P_i)|_{P_i^*} = 0 \), (3-38) yields:

\[
u^k(i) = \Delta \lambda_i \left( \partial^2 U_i / \partial P_i^2 |_{P_i^*} \right)^{-1}.
\]
As we have local bounds, the absolute value of \( u^k(i) \) would be smaller than the one presented in (3-39). Then, considering that the utility functions are strictly concave (\( \frac{\partial^2 U_i}{\partial P_i^2} < 0 \)):

\[
\forall i \in D: u^k(i) = -K_{i,D}\Delta \lambda_i, \quad 0 \leq K_{i,D} \leq -\left( \frac{\partial^2 U_i}{\partial P_i^2} |_{P_i} \right)^{-1}, \tag{3-40}
\]

The same analysis can be done for the generation side. So:

\[
\forall i \in G: u^k(i) = -K_{i,G}\Delta \lambda_i, \quad 0 \leq K_{i,G} \leq \left( \frac{\partial^2 C_i}{\partial P_i^2} |_{P_i} \right)^{-1}. \tag{3-41}
\]

Considering that \( \Delta \lambda_i = \lambda_i^{k+1} - \lambda_i^k = w_i\lambda^k + \eta \Delta \tilde{P}^k - \lambda_i^k \), where \( w_i \) is the \( i \)-th row of the \( W \) matrix, we can combine (3-40) and (3-41) as:

\[
u^k = G \left( (I - W)\lambda^k - \eta \Delta \tilde{P}^k \right), \tag{3-42}\]

where \( G \) is a diagonal matrix defined as:

\[
G(i,i) = \begin{cases} K_{i,D} & i \in D \\ K_{i,G} & i \in G \end{cases}. \tag{3-43}
\]

Now (3-33) and (3-35) can be written as a closed-loop system:

\[
\begin{bmatrix} \lambda_i^{k+1} \\ \Delta \tilde{P}_i^{k+1} \end{bmatrix} = \begin{bmatrix} W & \eta I \\ G(I - W) & (W - \eta G) \end{bmatrix} \begin{bmatrix} \lambda_i^k \\ \Delta \tilde{P}_i^k \end{bmatrix}. \tag{3-44}
\]

System (3-44) represents a discrete-time linear system. To study the dynamics, we do eigenvalue analysis. If \( \eta \) is zero, the updating matrix becomes:

\[
\tilde{A} = A(\eta = 0) = \begin{bmatrix} W & 0 \\ G(I - W) & W \end{bmatrix}. \tag{3-45}
\]

This matrix is a triangular matrix and so its eigenvalues are the eigenvalues of the matrices in the diagonal i.e. \( W \). As \( W \) is the consensus update matrix, its eigenvalues are all inside the unit circle except one, which is located at 1 [60]. So, \( \tilde{A} \) has two eigenvalues at 1 (\( \sigma_1 = \sigma_2 = 1 \),
and all the other eigenvalues are inside the unit circle. The sensitivity of the $i$-th eigenvalue of $A$ w.r.t. $\eta$ at $\eta = 0$ from the theory of eigenvalue derivatives [64] is:

$$\frac{\partial \sigma_i}{\partial \eta} = \mathbf{u}_i^T \left. \frac{\partial A}{\partial \eta} \right|_{\eta=0} \mathbf{v}_i,$$

(3-46)

where $\mathbf{u}_i$ and $\mathbf{v}_i$ are left and right eigenvectors of $A(\eta = 0)$ such that:

$$\mathbf{u}_i^T \tilde{A} = \sigma_i \mathbf{u}_i^T, \quad \tilde{A} \mathbf{v}_i = \sigma_i \mathbf{v}_i, \quad \mathbf{u}_i^T \mathbf{v}_i = 1, \quad \mathbf{u}_i^T \mathbf{v}_j = 0,$$

(3-47)

where $i \neq j$. It can be verified that for $\sigma_1 = \sigma_2 = 1$ the following eigenvectors satisfy (3-47):

$$\mathbf{u}_1^T = \left[ (1/N) \mathbf{1}^T \mathbf{G} \quad (1/N) \mathbf{1}^T \right], \quad \mathbf{u}_2^T = \left[ (1/N) \mathbf{1}^T \quad \mathbf{0} \right],$$

$$\mathbf{v}_1^T = \left[ \mathbf{0} \quad \mathbf{1} \right], \quad \mathbf{v}_2^T = \left[ \mathbf{1} \quad (-1/N) \mathbf{1}^T \mathbf{G} \mathbf{1} \right],$$

(3-48)

where $N$ is the total number of units and $\mathbf{1}$ is a vector of ones with size $N$. By substituting for $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1,$ and $\mathbf{v}_2$ in (3-46), we get:

$$\frac{\partial \sigma_1}{\partial \eta} = \mathbf{u}_1^T \left. \left( \frac{\partial A}{\partial \eta} \right) \right|_{\eta=0} \mathbf{v}_1 = 0, \quad \frac{\partial \sigma_2}{\partial \eta} = \mathbf{u}_2^T \left. \left( \frac{\partial A}{\partial \eta} \right) \right|_{\eta=0} \mathbf{v}_2$$

$$= (-1/N) \left( \mathbf{1}^T \mathbf{G} \mathbf{1} \right) = (-1/N) \left( \sum_{i \in D} K_{i,D} + \sum_{i \in G} K_{i,G} \right).$$

(3-49)

Based on (3-40) and (3-41), $K_{i,D}, K_{i,G} \geq 0$ and at least one of them is non-zero, because otherwise it means at the $k$-th iteration the algorithm has already converged. So, $\partial \sigma_2 / \partial \eta < 0$.

This means by increasing $\eta$ from zero, $\sigma_2$ moves toward inside the unit circle. Therefore, there is a small enough positive value, $\varepsilon_I$ such that for $0 < \eta < \varepsilon_I$, $\sigma_2$ falls inside the unit circle. It can be verified that regardless of the value of $\eta$, $[1 \quad 0]^T$ is always an eigenvector of $A$ associated with eigenvalue 1. Thus, $A$ would always have an eigenvalue at 1. So, $\sigma_1 = 1$ remains at one for any value of $\eta$. To analyze what happens to other eigenvalues of the system, we use Bauer-
Fike theorem [65]. This theorem states that the following relationship holds between the eigenvalues of $\tilde{A}$ and $\tilde{A} + \Delta$:

$$\forall j : \min_i |\sigma_i(\tilde{A}) - \sigma_j(\tilde{A} + \Delta)| \leq \|V\|_p \|V^{-1}\|_p \|\Delta\|_p,$$

(3-50)

where $\sigma_i(\tilde{A})$ is the $i$-th eigenvalue of matrix $\tilde{A}$, $V$ is the matrix consisting of eigenvectors of $\tilde{A}$, and $\|\cdot\|_p$ is any matrix $p$ norm. To use this theorem, $A$ is written as:

$$A = \tilde{A} + \Delta, \Delta = \begin{bmatrix} 0 & \eta I \\ 0 & -\eta G \end{bmatrix}.$$  

(3-51)

Substituting for $\Delta$ and using infinite norm, (3-50) becomes:

$$\forall j : \min_i |\sigma_i(\tilde{A}) - \sigma_j(\tilde{A} + \Delta)| \leq \eta C \max \{1, \max_i G(i,i)\},$$

(3-52)

where $C = \|V\|_\infty \|V^{-1}\|_\infty$. Inequality (3-52) states that the perturbation of the eigenvalues of $A$ from the eigenvalues of $\tilde{A}$ is bounded by a multiplication of $\eta$. Therefore, we can always choose a sufficiently small value $\epsilon_2$ such that for all $0 < \eta < \epsilon_2$, those eigenvalues of $A$ which are inside the unit circle for $\eta = 0$, remain inside. Now, by choosing $\epsilon = \min \{\epsilon_1, \epsilon_2\}$, we can assure that for all $0 < \eta < \epsilon$, the eigenvalues of $A$ are inside the unit circle, except one of them which is always on the unit circle and results from the consensus dynamics. This eigenvalue leads the evolution of the consensus states to a common value. Let’s consider that the eigenvalues of $A$ are ordered such that $|\sigma_1| \geq |\sigma_2| \geq \cdots \geq |\sigma_n|$. Then, based on the difference equations theory [66], the response of the system can be written as:

$$\left[ \lambda^k \ D^k \right] = a_1 \sigma_1^k q_1 + a_2 \sigma_2^k q_2 + \cdots + a_n \sigma_n^k q_n,$$

(3-53)
where $a_1, a_2, \ldots, a_n$ are constant values and $q_1, q_2, \ldots, q_n$ are the eigenvectors of the system. Then, as $|\sigma_2|, \ldots, |\sigma_n|$ are smaller than one and $q_i = [1 \ 0]^T$ is an eigenvector of $A$ corresponding to the eigenvalue $\sigma_1 = 1$ (i.e. $Aq_i = q_i$), regardless of the value of $\eta$,

$$
\lim_{k \to \infty} \left[ \lambda^k \quad \Delta \hat{P}^k \right]^T = a_i [1 \ 0]^T
$$

This means that:

$$
\lim_{k \to \infty} \lambda_1^k = \lim_{k \to \infty} \lambda_2^k = \ldots = \lim_{k \to \infty} \lambda_N^k,
$$

$$
\lim_{k \to \infty} \Delta \hat{P}_1^k = \lim_{k \to \infty} \Delta \hat{P}_2^k = \ldots = \lim_{k \to \infty} \Delta \hat{P}_N^k = 0.
$$

According to Theorem 1, the fixed point with the properties of (3-55) satisfies KKT conditions of optimality for the convex optimization problem of (3-11) and therefore is an optimal point for the problem. If the value of the states at the optimal point is shown by $[\lambda^* \ 0]^T$, (3-55) can be rephrased as:

$$
\lim_{k \to \infty} \left[ \lambda^k \quad \Delta \hat{P}^k \right]^T = [\lambda^* \ 0]^T,
$$

which proves the theorem. ■

By comparing (3-23) with the welfare functions of demand and generation units given in (3-2) and (3-7), we can interpret the local dual variables $\lambda^k$ as the local prices of energy, which are updated in real-time based on the price of energy at neighbor units and the local estimated imbalance in supply and demand. Local prices of energy act as incremental welfares for the supply and demand sides, and once these incremental welfares equate with each other, the power imbalance goes to zero and the global optimum for social welfare is reached. Therefore,
we call this algorithm the Incremental Welfare Consensus (IWC) algorithm. In the IWC block diagram shown in Figure 3-1, local price regulators are comprised of two blocks: The distributed observer to observe the global power imbalance information; and the price update to adjust the local price; Also, each generation/demand unit has a power regulator to adjust the generation/consumption level based on the local price in a greedy manner. In this structure, the price regulators do not have access to the individual utility/cost function parameters and the individual generation/demand units do not have access to the information provided by the neighbors. As a result, the privacy of information in distributed units is assured.

**Remark 3-4:** In the proposed structure, there exists one local price regulator for each generation/demand unit. Without requiring any central price regulator by using the IWC algorithm, local prices converge to the optimal value throughout the entire network and the generation/demand levels converge to the global optimum point. The price regulators can act similar to an Independent System Operator (ISO) to manage the energy market. Their job is to ensure the balance of the power and maximize the social welfare by controlling the local prices of energy. Local prices of energy are used for the energy transactions. Therefore, in the case of generation units, these are the prices the generation units are paid for their provided energy. In the case of demand units, these are the prices the demand units pay for consuming the energy.

**Remark 3-5:** IWC algorithm requires two way communication capability among neighbor devices. The existing infrastructures proposed for Smart-Grid communications such as Zigbee [67], Wireless Mesh [68], Cellular Network [69], and Power Line Communication [70] allow two way communications among multiple devices. Therefore, with proper adjustments considering the specific requirements of each protocol, the IWC algorithm can be implemented within the existing communications infrastructures.
3.4 Numerical Results

In this section, we validate the performance of the IWC algorithm through numerical simulations and benchmarking it against centralized Quadratic Programming (QP). First, we test the algorithm on IEEE 39 bus system. Then, we use Monte Carlo simulations, and generate multiple random cases and check the convergence of the algorithm to the global optimum. Monte Carlo simulation also allows us to analyze the scalability of the system. Note that the current version of the algorithm does not consider the line losses.

Figure 3-2: The IEEE 39 bus power network with communications topology
IWC Algorithm on IEEE 39 Bus Test System

The IEEE 39 bus test system is shown in Figure 3-2. The communications network is chosen similar to the physical power network. This means that there is a communication link between two nodes, if there is a physical connection between them. There are 10 generation units.
units and 19 demand units. The generation and demand unit parameters are considered as shown in Table 3-1. The value for $\eta$ is chosen as 0.003. Considering that the maximum degree of the communications graph ($\max_{i=1,\ldots,N} |N_i|$) is 5, all the connectivity strengths are chosen as $w_{ij}=0.19$ to be in the feasible range for stability of the consensus network $0 \leq w_{ij} < \left( \max_{i=1,\ldots,N} |N_i| \right)^{-1} = 1/5$. 

Figure 3-3: Evolution of a) generation and demand powers, b) price, and c) social welfare in case study with IEEE 39 bus test system.
Figure 3-3 shows the evolution of the generation and demand curves, the evolution of price, and the evolution of the social welfare value using the IWC algorithm. As the algorithm proceeds, the summation of generation powers equates the summation of demand powers as shown in Figure 3-3 (a) and thus, the power balance condition is being satisfied.

Figure 3-3(b) shows the evolution of local prices. It consists of 29 curves, because the number of local price regulators is to the number of generation/demand units (10 generation units + 19 demand units). While the local prices differ at different units, the coordination among price regulators allows the local prices to converge to a unique value. As the price settles down, social welfare converges to its optimal value of 5.1 $/hour, drawn as a red line in Figure 3-3 (c).

3.4.2 Monte Carlo Simulations for Scalability Analysis

To study the scalability of the IWC algorithm, we use Monte Carlo simulations. We increase the number of nodes from 10 to 460, and for each value of the number of nodes, we perform 100 random simulations. Each simulation involves a random number of generation and demand units, random parameters, and a random network topology. For each experiment (which embraces 100 simulations), we record the average number of iterations, \( k \), required such that:

\[
\|P^k - P^*\|_2 \leq \varepsilon, \quad (3-57)
\]

where \( P^k \) is the vector of generation and consumption powers at the \( k \)th iteration of the IWC algorithm; \( P^* \) is the optimal vector from centralized QP; \( \| \cdot \|_2 \) is the Euclidean norm; and \( \varepsilon \) is a positive small scalar. For the experiments in this part, we use \( \varepsilon = 0.1 \). Also, the value of \( \eta \) for the algorithm is chosen as 0.001. Figure 3-4 shows the average number of iterations for convergence as a function of the number of nodes. Based on the drawn trend line, the number of iterations for convergence increases almost logarithmically as the number of nodes
increases. As the number of iterations required for convergence determines the extent of the computations being performed at each node, we conclude that the IWC algorithm is scalable in terms of its computational burden. Moreover, the number of times each agent is required to communicate with its neighbors, also directly depends on the required iterations for convergence. Therefore, we can conclude that the IWC algorithm is scalable in terms of the required communications resources as well. Also, the fact that in all the randomly generated scenarios the algorithm reaches the close vicinity of the optimal point (required by (3-57)) is a numerical validation for convergence of the algorithm.

3.5 Conclusion

In this chapter we proposed a cooperative distributed algorithm (IWC) to address the energy-management problem in the smart grid populated with distributed generators and responsive loads. The proposed IWC algorithm precludes the need for a central controller/price coordinator or leader so it is completely center-free. It converges to the global optimum of social welfare and is scalable. The properties of the algorithm are verified through theoretical analysis and numerical case studies. This chapter did not consider the line losses, voltage

Figure 3-4: Average number of iterations required for convergence as a function of the number of nodes for the IWC algorithm
deviations, and the power flow dynamics of physical power networks. These issues will be analyzed in future work.
CHAPTER 4. DISTRIBUTED REAL-TIME PRICING CONTROL FOR LARGE SCALE UNIDIRECTIONAL V2G WITH MULTIPLE ENERGY SUPPLIERS

Abstract:

With the increasing trend in adoption of Plug-in Hybrid and Plug-in Hybrid Electric Vehicles (PHEVs/PEVs), Vehicle to Grid (V2G) will play a prominent role in the future smart-grid and electric energy market. From the energy supplier side, PHEVs/PEVs can provide ancillary service to the grid and facilitate the integration of renewables by acting as energy buffers. From the vehicle owners’ side, the vehicle can be charged with low cost by providing demand response services to the grid. Conventionally, optimal management of V2G requires gathering and processing data from multiple vehicles in a center. However, with the increasing number of vehicles as well as energy suppliers, central V2G optimization would not be scalable and would be vulnerable to single node/link failures. This chapter introduces a novel distributed approach for optimal management of unidirectional V2G considering multiple energy suppliers. In this approach, each charging station as well as each individual energy supplier is equipped with a local price regulator. They control the price paid to the suppliers and the price paid by the vehicles. Price regulators coordinate with their neighbors and update the local energy price. In response to the updated prices, the vehicles adjust their charging rate to maximize their utility and energy suppliers adjust their production rate to maximize their benefit. It is shown that by iteratively repeating this process, the entire system converges to the global optimum. Therefore the main advantages of the proposed approach are: 1) solves V2G management problem in a fully distributed way considering multiple energy suppliers, 2) converges to the global optimum despite the greedy behavior of the individuals.
4.1 Introduction

The predictions indicate that by 2023, there would be 1.8-7 million Plug-in Hybrid/Plug-in Electric Vehicles (PHEVs/PEVs) on the road in United States [71]. The collective charging power of these vehicles, can be a threat as well as an opportunity for the grid. On one hand, if the charging process is not controlled, the peak electricity demand would be increased [15] which can overload the local infrastructures in certain areas.

On the other hand due to the flexible nature of the charging load of the electric vehicles, they can be used for ancillary services (such as frequency regulation, smoothing the load curve, etc.) to the benefit of the grid [72]. This is mainly because the charging process has a high power rating as well as large ramp up/ramp down rates. The role of PHEV/PEVs as ancillary service providers becomes especially important as larger number of renewables are integrated to the grid increasing the intermittency and fluctuations in the power profile [73].

The enabling technology to mitigate the possible threats of PHEVs/PEVs to the grid, and to use them as potential service providers, is called the Vehicle to Grid (V2G) technology which allows interaction between the vehicle and the grid. V2G can be unidirectional [17], where the power flow is only from the grid to the PHEVs/PEVs and vehicles can regulate their charging power, or bidirectional where the connected PHEVs/PEVs can also discharge to the grid [74]. The focus of this chapter is on optimal dispatch of PHEVs/PEVs using unidirectional V2G technology.

Most of the existing scheduling algorithms for charging of PHEVs/PEVs in the literature, are centralized. The charging stations/vehicles are required to transmit data to a control center called aggregator [18]. The control center performs the necessary computations and
determines the optimal power allocation for each unit and transmits it to the charging stations [22]–[24], [75], [76].

The centralized approach works well for small-scale problems. However, as the number of PHEVs/PEVs increases and they spread over a wide geographical area, the centralized approach would not scale well and becomes fragile to single points of failure. The situation is aggravated when multiple distributed energy suppliers are also added to the grid, and the control center has to get data from them as well. For this reason, many of researchers have proposed decentralized/distributed approaches to do the charge optimization of PHEVs/PEVs to relieve the computational burden from the control center [26]–[30], [77], [78], [51], [79].

In [26]–[28] the energy provider sends a central price/event signal to all the charging stations to manage the aggregate charging load; In [29] the distributed units respond to changes in the price signal adjusted through a center and in [30], [78] decentralized methods are introduced based on a multi-agent framework in which central aggregator broadcasts signals to lower level aggregators, which send pricing signals to individual vehicles. Although the aforementioned works are decentralized, they are still vulnerable to single points of failures, because higher level agents broadcast real-time control signals to lower level nodes. In [51], [79] fully distributed algorithms are introduced which do not require any control center, coordinator or aggregator, and therefore not only they are scalable, but also are robust against single point of failures. However, these two works consider a single energy provider.

In this chapter, we propose a Distributed Real-time Pricing Control (DRPC) technology for energy management of large-scale electrical vehicles considering multiple energy providers and renewables. Figure 4-1 shows the basic concept of the proposed approach: Each charging
station, each individual energy supplier and each demand is connected to a local price regulator with the communication capability. The price regulators control the price that each individual device is charged/paid for energy usage/production. There is one price regulator for each device. Price regulators coordinate with their neighbors and update the local energy price. In response to the updated prices, the vehicles adjust their charging rate to maximize their utility and energy suppliers adjust their production rate to maximize their own benefit. It is shown that by iteratively repeating this process, the entire system converges to the global optimum. The main advantages of this algorithm are:

1. While most of the existing distributed charge scheduling algorithms only assume a single energy provider, this technology considers multiple heterogeneous energy providers. This assumption is more suited to the future of the smart-grid.
2. This technology is fully distributed, meaning that it requires no coordinator/leader/aggregator.
3. It respects the privacy of individual units. No device has to share its preferences, energy generation/consumption, etc. with other units.

4. Although each unit selfishly tries to choose the optimal local power consumption/generation to maximize its own benefit, the local price regulators adjust the prices such that the behavior of the overall system moves toward the global optimum.

The chapter is organized as follows. Section II formulates the unidirectional V2G management with multiple energy providers, as a Social Welfare Maximization problem [80]. Section III introduces the DRPCG algorithm as a cooperative distributed solution to solve the energy management problem. Section IV brings numeric simulations to validate the proposed approach. The concluding remarks come in Section V.

4.2 Problem Formulation

Unidirectional V2G energy management with multiple energy suppliers can be formulated as a social welfare maximization problem, which is an optimization problem. It consists of an objective function, and constraints.

4.2.1 Objective Function

Social welfare maximization at time $t$ is defined as maximizing the summation of welfares of all the generation and consumer units [52], [55]:

$$
\text{Max}_{P(t),p} \left\{ \sum_{i \in EV} W_{i, EV}(P_{i, EV}(t), p(t)) + \sum_{i \in D} W_{i, D}(P_{i, D}(t), p(t)) + \sum_{i \in G} W_{i, G}(P_{i, G}(t), p(t)) + \sum_{i \in R} W_{i, R}(P_{i, R}(t), p(t)) \right\},
$$

(4-1)

where $EV$ is the set of indices of electric vehicles, $D$ is the set of indices of constant loads, $G$ is the set of indices of dispatchable generation units, $R$ is the set of indices of renewable
generation units, and $W_{i,j}$ is the welfare function of the device with index $i$ of type $j$. Here $\mathbf{P}(t)$ is the vector of powers of all generation and demand, and $p(t)$ is the price of energy at time $t$.

4.2.1.1 Welfare for Electric Vehicles

The welfare for electric vehicles is the utility they get for being charged minus the cost they pay for charging:

$$W_{i, EV}(t) = U_{i, EV}(t) - CC_{i, EV}(t),$$

where $U_{i, EV}(t)$ is the utility of the $i$-th vehicle at time $t$ and $CC_{i, EV}(t)$ is the charging cost for the $i$-th PHEV/PEV user at time $t$.

The utility function $U_{i, EV}(t)$, should represent the satisfaction of the customer for the amount of energy he/she buys. To define an appropriate utility function for PHEV/PEV users, we make the following assumptions:

1. At each time instant, the satisfaction of the user depends on the State of Charge (SoC) of the battery pack of the vehicle.
2. The satisfaction is non-decreasing in terms of SoC. By increasing the SoC, the user would be more satisfied, i.e.:

$$\frac{\partial U_{i, EV}(SoC_{i, EV})}{\partial SoC_{i, EV}} \geq 0$$

(4-3)

3. The level of satisfaction saturates by increasing the SoC. Equal amounts of increase in SoC results in higher increase in the user satisfaction at lower SoC levels than in higher SoC levels:

$$\frac{\partial^2 U_{i, EV}(SoC)}{\partial SoC^2} \leq 0$$

(4-4)
The satisfaction reaches its maximum at a certain SoC level (e.g. 0.9) and any increase in SoC beyond that limit does not change the user’s satisfaction level.

The following quadratic function defined over $0 \leq \text{SoC}_{i,\text{EV}}(t) \leq \text{SoC}_{i,\text{EV},d}$ satisfies these requirements and thus can be chosen as the PHEV/PEV user utility function:

$$U_{i,\text{EV}}(\text{SoC}_{i,\text{EV}}(t)) = \omega_{i,\text{EV}} \text{SoC}_{i,\text{EV},d} \left( \frac{\text{SoC}_{i,\text{EV}}(t)}{\text{SoC}_{i,\text{EV},d}} - \frac{1}{2} \left( \frac{\text{SoC}_{i,\text{EV}}(t)}{\text{SoC}_{i,\text{EV},d}} \right)^2 \right)$$  \hspace{1cm} (4-5)

4. The satisfaction reaches its maximum at a certain SoC level (e.g. 0.9) and any increase in SoC beyond that limit does not change the user’s satisfaction level.

The following quadratic function defined over $0 \leq \text{SoC}_{i,\text{EV}}(t) \leq \text{SoC}_{i,\text{EV},d}$ satisfies these requirements and thus can be chosen as the PHEV/PEV user utility function:

Here, $\text{SoC}_{i,\text{EV},d}$ is the SoC level that provides the $i$-th PHEV/PEV user with the maximum satisfaction level and $\omega_{i,\text{EV}}$ is a positive value to differentiate the various PHEV/PEV users.

Figure 4-1 shows the typical shape for this utility function.

The utility function defined in (4-5) is not yet in the form that can be used for decision making for charging power because of two main reasons:

1. The utility function in (4-5) is in terms of SoC, however we need a utility function in terms of the charging power.
2. The utility function in (4-5) has an undetermined parameter $\omega_{i,\text{EV}}(t)$.

To solve the first issue, we look into the dynamics of the SoC:

$$SoC_{i,\text{EV}}(t) = SoC_{i,\text{EV}}(t - \Delta t) + \frac{1}{Cap_{i,\text{EV}}} \eta_{i,\text{EV}} P_{i,\text{EV}}(t) \Delta t,$$

(4-6)

where $P_{i,\text{EV}}(t)$ is the charging power of the $i$-th vehicle at time $t$, $\eta_{i,\text{EV}}$ is the efficiency of the battery pack of the $i$-th vehicle, $Cap_{i,\text{EV}}$ is the capacity of the battery pack (kWh), and $\Delta t$ is the time-step. Note that equation (4-6) is a linearized equivalent for equation (2-5) which was used in chapter 2. Substituting for SoC from (4-6) into (4-5), we can write the utility function in terms of the charging power:

$$U_{i,\text{EV}}(t) = P_{i,\text{EV}}(t)^2 \left( -\eta_{i,\text{EV}}^2 \omega_{i,\text{EV}} \Delta t^2 \right) \left( \frac{-\eta_{i,\text{EV}}^2 \omega_{i,\text{EV}} \Delta t^2}{2Cap_{i,\text{EV}}^2 SoC_{i,\text{EV},d}^2} \right) + P_{i,\text{EV}}(t) \frac{\omega_{i,\text{EV}} \eta_{i,\text{EV}} \Delta t}{Cap_{i,\text{EV}}} \left( 1 - \frac{SoC_{i,\text{EV}}(t - \Delta t)}{SoC_{i,\text{EV},d}} \right) + \text{const},$$

(4-7)

where “const” denotes all the terms which are not a function of $P_{i,\text{EV}}(t)$. As the decision is to be made on $P_{i,\text{EV}}(t)$, the constant terms have no effect on the optimization.

To find a meaningful way to set the parameter $\omega_{i,\text{EV}}(t)$, we look into the other factor in welfare of the $i$-th PHEV/PEV user, i.e. the charging cost. The dynamic for the charging cost can be described by the following equation:

$$CC_{i,\text{EV}}(t) = CC_{i,\text{EV}}(t - \Delta t) + p(t) P_{i,\text{EV}}(t) \Delta t,$$

(4-8)

where $p(t)$ is the price of energy at time $t$. By substituting for $U_{i,\text{EV}}(t)$ and $CC_{i,\text{EV}}(t)$ from (4-7) and (4-8) into (4-2), and removing all the constant terms (as they have no effect on the
optimization) we get the following equation for the welfare of the electric vehicle with index $i$:

$$W_{i, EV}(t) = P_{i, EV}(t)^2 \left( -\eta_{i, EV}^2 \frac{\omega_{i, EV} \Delta t^2}{2 Cap_{i, EV}^2 SoC_{i, EV, d}} \right)$$

$$+ P_{i, EV}(t) \left( \frac{\omega_{i, EV} \eta_{i, EV} \Delta t}{Cap_{i, EV}} \left( 1 - \frac{SoC_{i, EV}(t - \Delta t)}{SoC_{i, EV, d}} \right) - p(t) \Delta t \right)^3,$$

(4-9)

By taking the derivative of $W_{i, EV}(t)$ with respect to $P_{i, EV}(t)$ and equating to zero, the optimal charging power to maximize the welfare of the vehicle with index $i$ would be:

$$P_{i, EV}^*(t) = \frac{\left( \frac{\omega_{i, EV} \eta_{i, EV} \Delta t}{Cap_{i, EV}} \left( 1 - \frac{SoC_{i, EV}(t - \Delta t)}{SoC_{i, EV, d}} \right) - p(t) \Delta t \right)^3}{\eta_{i, EV}^2 \frac{\omega_{i, EV} \Delta t^2}{2 Cap_{i, EV}^2 SoC_{i, EV, d}}}.$$

(4-10)

Based on (4-10), as the price of energy $p(t)$ goes up, the optimal charging rate for the EV drops down, until a specific price where it reaches zero. Let’s denote this price with $p_{i, EV}^{\text{max}}(t)$. Based on (4-10),

$$p_{i, EV}^{\text{max}}(t) = \frac{\omega_{i, EV} \eta_{i, EV}}{Cap_{i, EV}} \left( 1 - \frac{SoC_{i, EV}(t - \Delta t)}{SoC_{i, EV, d}} \right).$$

(4-11)

Equation (4-11) is important because it gives a relationship between the maximum price per kWh of energy that the user of the EV with index $i$ is willing to pay to increase its SoC, and $\omega_{i, EV}$. It is worthy to note that $p_{i, EV}^{\text{max}}(t)$ is a function of SOC of the vehicle as well. The higher the SoC of the vehicle, the lesser is $p_{i, EV}^{\text{max}}(t)$. Now, if we know the $p_{i, EV}^{\text{max}}(t)$ of the user at a certain SoC level, we can determine $\omega_{i, EV}$ as:
\[
\alpha_{i, EV} = \frac{\text{Cap}_{i, EV} p_{i, EV, x}^{\text{max}}}{\eta_{i, EV} \left(1 - \frac{x}{\text{SoC}_{i, EV, d}}\right)},
\]

where \( p_{i, EV, x}^{\text{max}} \) is \( p_{i, EV}^{\text{max}}(\cdot) \) of the user of the EV with index \( i \) at the SoC level of \( x \).

4.2.1.2 Welfare for Constant Loads

The utility for constant demands is constant. So the welfare would be:

\[
W_{i, D}(P_{i, D}(t), p(t)) = c_{i, D} - p(t)P_{i, D}(t)\Delta t,
\]

where \( c_{i, D} \) is the constant utility associated with the constant demand with index \( i \).

4.2.1.3 Welfare for Dispatchable Generation Units

The welfare for the dispatchable generation unit with index \( i \) can be defined as the net profit:

\[
W_{i, G}(P_{i, G}(t), p(t)) = p(t)P_{i, G}(t)\Delta t - C_{i, G}(P_{i, G}(t)),
\]

where \( C_{i, G}(P_{i, G}(t)) \) is the cost of the dispatchable generation unit with index \( i \) for providing \( P_{i, G}(t) \). We consider the cost function to be a differentiable convex function.

4.2.1.4 Welfare for Renewable Generation Units

The welfare for the renewable generation unit with index \( i \) is also the net profit. However, the cost is constant,

\[
W_{i, R}(P_{i, R}(t), p(t)) = p(t)P_{i, R}(t)\Delta t - c_{i, R},
\]

where \( c_{i, R} \) is the constant cost associated with the non-dispatchable generation unit with index \( i \).
4.2.2 Constraints

4.2.2.1 Power Balance Constraint

The summation of all the generated powers should equal the summation of all the consumed powers:

\[ \sum_{i \in G} P_{i,G}(t) + \sum_{i \in R} P_{i,R}(t) = \sum_{i \in EV} P_{i, EV}(t) + \sum_{i \in D} P_{i, D}(t), \]  

(4-16)

Note that in this chapter, we do not consider the transmission losses.

4.2.2.2 Local Device Constraints

Generators and vehicles have power rating and cannot produce or consume more power than a certain limit:

\[ \forall i \in EV \cup D \cup G \cup R, j \in \{EV, D, G, R\} : 0 \leq P_{i,j}(t) \leq P_{i,j,\text{max}}(t), \]  

(4-17)

where \( P_{i,j,\text{max}}(t) \) is the maximum power that can be generated/consumed by the device with index \( i \) of type \( j \). Also, as we are considering unidirectional V2G the lower limit for charging power of electric vehicles should be zero.

4.3 Distributed Real Time Pricing for Charge/Generation Control

In this section, the steps to derive the DRPCG algorithm based on KKT multipliers and consensus networks are explained.

4.3.1 Simplifying the Objective Function

By replacing for welfare functions from (4-2), (4-8), (4-13) - (4-15) in (4-1), applying the power balance constraint, dropping the constant terms (as they have no effect on the optimization), and representing the problem as a minimization, the optimization problem would be:
\[
\begin{align*}
\text{Min}_{\{P_i(t)\in EV\cup G\}} \left( - \sum_{i \in EV} U_{i, EV}(P_{i, EV}(t)) + \sum_{i \in G} C_{i, G}(P_{i, G}(t)) \right), \\
\text{s.t.} \sum_{i \in G} P_{i, G}(t) + \sum_{i \in R} P_{i, R}(t) = \sum_{i \in EV} P_{i, EV}(t) + \sum_{i \in D} P_{i, D}(t), \quad (4-18)
\end{align*}
\]

\( \forall i \in EV \cup D \cup G \cup R, j \in \{EV, D, G, R\} : 0 \leq P_{i, j}(t) \leq P_{i, j, \text{max}}(t). \)

### 4.3.2 Augment the Objective Function with KKT Multipliers

To solve (4-18), first, the objective function is augmented by the constraints and KKT multipliers. The Lagrangian would be:

\[
L = - \sum_{i \in EV} U_{i, EV}(P_{i, EV}(t)) + \sum_{i \in G} C_{i, G}(P_{i, G}(t)) + \lambda \left( \sum_{i \in EV} P_{i, EV}(t) + \sum_{i \in D} P_{i, D}(t) - \sum_{i \in G} P_{i, G}(t) - \sum_{i \in R} P_{i, R}(t) \right), \quad (4-19)
\]

where \( \lambda \) is the KKT multiplier for the power balance constraint. Note that we have not considered KKT multipliers for local device constraints, because those constraints can be handled by limiting the domain of each decision variable \( P_{i, j}(t) \) between its minimum and maximum.

### 4.3.3 Dual Decomposition

Rephrasing (4-19):

\[
L = \sum_{i \in EV} \left( -U_{i, EV}(P_{i, EV}(t)) + \lambda P_{i, EV}(t) \right) + \lambda \left( \sum_{i \in G} \left( -\lambda P_{i, G}(t) + C_{i, G}(P_{i, G}(t)) \right) \right) + \lambda \left( \sum_{i \in D} P_{i, D}(t) - \sum_{i \in R} P_{i, R}(t) \right). \quad (4-20)
\]

Using dual decomposition [59], an iterative approach that moves in the direction of minimizing \( L \) in terms of the primal vector \( \mathbf{P} \) and maximizing \( L \) in terms of the dual variable \( \lambda \) can converge to the optimal point:
\[ \forall i \in EV: \]
\[ P_{i,EV}^k(t) = \arg \max_{0 \leq P_{i,EV}(t) \leq P_{i,EV}^{\max}} \left( U_{i,EV}(P_{i,EV}(t)) - \lambda^k P_{i,EV}(t) \right), \quad (4-21) \]

\[ \forall i \in G: \]
\[ P_{i,G}^k(t) = \arg \max_{0 \leq P_{i,G}(t) \leq P_{i,G}^{\max}} \left( \lambda^k P_{i,G}(t) - C_{i,G}(P_{i,G}(t)) \right), \quad (4-22) \]

\[ \lambda^{k+1} = \lambda^k + \eta \Delta P^k, \quad (4-23) \]

where,
\[ \Delta P^k = \sum_{i \in EV} P_{i,EV}(t) + \sum_{i \in D} P_{i,D}(t) - \sum_{i \in G} P_{i,G}(t) - \sum_{i \in R} P_{i,R}(t), \quad (4-24) \]

and \( \eta > 0 \) is the updating step size for dual variable \( \lambda \). The convergence/stability of this iterative approach depends on the value of the updating step size \([59]\). As the optimization is over a single time step, in the remainder of the chapter, we drop notation \( t \) for simplicity.

### 4.3.4 Distributed Observers to Estimate Global Information

The iterative procedure represented by (4-21) - (4-23), consists of local optimizations that can be done at each device (eq. (4-21), and eq. (4-22)). However, these local optimizations require knowledge of global variable \( \lambda^k \). Also updating this global variable (eq. (4-23)) requires knowledge of generation and demand powers of all the devices at the \( k \)-th iteration. Therefore, in this section with the help of consensus networks, a procedure is designed so that each node can estimate the value of required global variables by collaboration with its neighbors. We introduce the following variables to serve as estimations of global variables:

\[ \hat{\lambda}_i^k : \text{Estimation of device with index } i \text{ of } \lambda^k, \]

\[ \Delta \hat{P}_i^k : \text{Estimation of device with index } i \text{ of average of } \Delta P^k \text{ over all devices.} \]

The update rules for estimated values are as follows:
\[ \forall i \in G \cup R, j \in \{G, R\} : \Delta \hat{P}_i^{k+1} = \Delta \hat{P}_i^k + \sum_{j \in N_j} w_j \left( \Delta \hat{P}_j^k - \Delta \hat{P}_i^k \right) + P_{i,j}^k - P_{i,j}^{k+1}, \]  \hfill (4-25)

\[ \forall i \in D \cup \text{EV}, j \in \{D, \text{EV}\} : \Delta \hat{P}_i^{k+1} = \Delta \hat{P}_i^k + \sum_{j \in N_j} w_j \left( \Delta \hat{P}_j^k - \Delta \hat{P}_i^k \right) + P_{i,j}^{k+1} - P_{i,j}^k, \]  \hfill (4-26)

\[ \forall i \in G \cup R \cup \text{EV} \cup D : \hat{\lambda}_i^{k+1} = \hat{\lambda}_i^k + \sum_{j \in N_i} w_j \left( \hat{\lambda}_j^k - \hat{\lambda}_i^k \right) + \eta_i \Delta \hat{P}_i^{k+1}, \]  \hfill (4-27)

where \( \eta_i > 0 \) is the updating step size for estimation of dual variable \( \hat{\lambda} \) by the device with index \( i \).

Then each EV, and each dispatchable generation unit will update its power consumption and generation as follows:

\[ \forall i \in \text{EV} : P_{i,\text{EV}}^k = \arg \min_{0 \leq P_{i,\text{EV}} \leq P_{i,\text{EV}}^{\max}} \left( -U_{i,\text{EV}}(P_{i,\text{EV}}) + \hat{\lambda}_i^k P_{i,\text{EV}} \right), \]  \hfill (4-28)

\[ \forall i \in G : P_{i,G}^k = \arg \min_{0 \leq P_{i,G} \leq P_{i,G}^{\max}} \left( -\hat{\lambda}_i^k P_{i,G} + C_{i,G}(P_{i,G}) \right). \]  \hfill (4-29)

It can be shown that if the entire communication network is connected and for all \( i, \eta_i > 0 \) is chosen small enough, the algorithm represented by (4-25)-(4-29) converges to the global optimum of the original optimization problem [80]. If we compare (4-28), (4-29) with welfare definitions in section III-A, we find that \( \hat{\lambda}_i^k \) (with inclusion of a factor of \( \Delta t \) ) can be interpreted as price of energy offered to device \( i \), at the \( k \)-th iteration.

4.3.5 Smart Transformers as Communication Nodes between Zones

Equations (4-25)-(4-29), assume that the entire communication network among all generation and load devices forms a connected network. This can be realized among the devices inside each zone. However, it is not likely that devices located in different zones of the...
power network, have the ability to exchange information with each other. One possible solution is to use smart transformers that beside their main functionality (stepping up/down the voltage), have the ability to exchange information among each other as well as with the devices inside their zone. It means in the communication network, the transformer nodes, should have neighbors from their own zone, as well as neighbors from other transformers in the network such that the entire communication network forms a connected graph. Therefore, we add two more equations for transformers:

\[ \forall i \in T: \Delta \hat{P}_i^{k+1} = \Delta \hat{P}_i^k + \sum_{j \in N_i} w_{ij} \left( \Delta \hat{P}_j - \Delta \hat{P}_i \right), \quad (4-30) \]

\[ \forall i \in T: \hat{\lambda}_i^{k+1} = \hat{\lambda}_i^k + \sum_{j \in N_i} w_{ij} \left( \hat{\lambda}_j^k - \hat{\lambda}_i^k \right) + \eta_i \Delta \hat{P}_i^{k+1}, \quad (4-31) \]

where \( T \) is the set of indices for transformers. Equations (4-25)-(4-31) form the DRPCG algorithm. The building blocks of this algorithm are shown in Figure 4-2. Each energy device is connected to a local price regulator. Local price regulators use distributed observer equations.
(4-25), (4-26), and (4-30) to estimate the imbalance of power. Based on the output of distributed observers and coordination with neighbors, the local price of energy offered to each device is updated. If the device is dispatchable resource or electrical vehicle, it will respond to changes in the price by regulating its generation/consumption power in a greedy way (eq.(4-28), (4-29)). If the device is non-dispatchable resource or a constant load, it will not respond to the changes in price, but its generation/consumption information would be used as a feedback in the system. If the device is transformer, its main role would be to propagate the pricing information among other zones. Next, we provide a theorem that shows the fixed point of the DRPCG algorithm satisfies KKT conditions of optimality. As the optimization problem is convex and all the constraints are affine, satisfaction of KKT conditions translates into global optimality.

**Theorem 4-1:** If the optimization problem in (4-18) is feasible, and the cost (utility) functions are convex (concave) and continuously differentiable, then the fixed points of the iterative algorithm presented by (4-25)-(4-31), triggered from an initially balanced state (i.e. for all \( i \), \( \Delta P_i^0 = 0 \)) satisfy the KKT conditions of optimality for the optimization problem. ■

**Proof:**

The proof is similar to Theorem 3-1, with slight difference because of the addition of smart transformers, constant demands, and renewables. For the sake of completeness, we bring the proof. Let \( P^* = [P^*_{i,j}]_{i \in G, j \in [G, EV]} \), \( \lambda^* = [\lambda^*_{i}]_{i \in GUEVUDURUT} \), and \( \Delta P^* = [\Delta P^*_{i}]_{i \in GUEVUDURUT} \) be a fixed point of the iterative procedure presented by (4-25)-(4-31). KKT conditions of optimality are satisfied if there exist constants \( \lambda, \mu_i \geq 0 \), and \( \xi_i \geq 0 \) for \( i \in G \cup E \cup U \cup R \cup D \cup T \) such that:
\[ \sum_{i \in G} P_{i,G}(t) + \sum_{i \in R} P_{i,R}(t) = \sum_{i \in EV} P_{i, EV}(t) + \sum_{i \in B} P_{i, D}(t), \quad (4-32) \]

\[ \forall i \in EV : \frac{\partial U_{i, EV}(P_{i, EV})}{\partial P_{i, EV}} |_{P_{i, EV} = P_{i, EV}^*} + \lambda - \mu_i + \xi_i = 0, \quad (4-33) \]

\[ \forall i \in G : \frac{\partial C_{i, G}(P_{i, G})}{\partial P_{i, G}} |_{P_{i, G} = P_{i, G}^*} - \lambda - \mu_i + \xi_i = 0, \quad (4-34) \]

\[ \forall i \in G \cup EV, j \in \{G, EV\} : \mu_i P_{i,j}^* = 0, \quad (4-35) \]

\[ \forall i \in G \cup EV, j \in \{G, EV\} : \xi_i \left(P_{i,j}^* - P_{i,j \text{ max}}\right) = 0. \quad (4-36) \]

**Checking satisfaction of KKT condition (4-32):**

Summing \( \lambda_i^* \) over all devices using (4-27), (4-31):

\[ \sum_i \lambda_i^* = \sum_i \lambda_i^* + \sum_{i,j \in N_i} w_{ij} \left( \lambda_j^* - \lambda_i^* \right) + \eta \sum_i \Delta \hat{P}_i^*, \quad (4-37) \]

The second term equals to zero because for all \( i \) and \( j \), \( w_{ij} = w_{ji} \):

\[ \sum_i \Delta \hat{P}_i^* = 0. \quad (4-38) \]

Summing \( \Delta \hat{P}_i^{k+1} \) over all units using (4-25), (4-26) and (4-30):

\[ \sum_i \Delta \hat{P}_i^{k+1} = \sum_i \Delta \hat{P}_i^k + \sum_{i,j \in N} w_{ij} \left( \Delta \hat{P}_j^k - \Delta \hat{P}_i^k \right) + \sum_{i,j \in N} P_{i,j}^{k+1} - \sum_{i,j \in G \cup R} P_{i,j}^k \quad (4-39) \]

The second term equals to zero because for all \( i \) and \( j \), \( w_{ij} = w_{ji} \). The third term and the fourth term are the power imbalances at iterations \( k+1 \) and \( k \), respectively. So:
\[
\sum_i \Delta \hat{P}_{i}^{k+1} = \sum_i \Delta \hat{P}_{i}^{k} + \Delta P^{k+1} - \Delta P^{k},
\]

(4-40)

Assuming the algorithm is triggered from an initially balanced state, for all \(i\), \(\Delta P_{i}^{0} = \Delta \hat{P}_{i}^{0} = 0\). So, based on (4-40),

\[
\sum_i \Delta \hat{P}_{i}^{1} = \sum_i \Delta \hat{P}_{i}^{0} + \Delta P^{1} - \Delta P^{0} = \Delta P^{1},
\]

(4-41)

\[
\sum_i \Delta \hat{P}_{i}^{2} = \sum_i \Delta \hat{P}_{i}^{1} + \Delta P^{2} - \Delta P^{1} = \Delta P^{2},
\]

(4-41)

\[
\sum_i \Delta \hat{P}_{i}^{3} = \sum_i \Delta \hat{P}_{i}^{2} + \Delta P^{3} - \Delta P^{2} = \Delta P^{3},
\]

By continuing this process,

\[
\sum_i \Delta \hat{P}_{i}^{k+1} = \Delta P^{k+1}.
\]

(4-42)

Now, using (4-38) and (4-42):

\[
\Delta P^{*} = \sum_{i \in EV} P_{i, EV}(t) + \sum_{i \in D} P_{i, D}(t) - \sum_{i \in G} P_{i, G}(t) - \sum_{i \in R} P_{i, R}(t) = 0.
\]

(4-43)

Therefore KKT condition (4-32) is satisfied.

Checking satisfaction of KKT conditions (4-33)-(4-36):

Based on (4-43), and considering the consensus property that the values \(\Delta P_{i}^{k+1}\) converge to their average,

\[
\forall i \in G_d \cup R \cup EV \cup D \cup T : \Delta \hat{P}_{i}^{*} = 0.
\]

(4-44)

Based on (4-27), (4-31), (4-44), and considering the consensus property:

\[
\forall i, j \in G \cup R \cup EV \cup D \cup T : \lambda_{i}^{*} = \lambda_{j}^{*}.
\]

(4-45)

We show this common value by \(\lambda^{*}\). Now for the set of PHEVs/PEVs at the fixed point:

\[
\forall i \in EV : P_{i, EV}^{*} = \arg \min_{0 \leq P_{i, EV} \leq P_{i, EV}^{\text{max}}} \left( -U_{i, EV} (P_{i, EV}) + \lambda^{*} P_{i, EV} \right)
\]

(4-46)
Let $P_{i, EV}^*$ be the minimum of $-U_{i, EV}(P_{i, EV}) + \lambda^* P_i$, without considering the local constraint $0 \leq P_{i, EV} \leq P_{i, EV}^{max}$. Thus, as $-U_{i, EV}(P_{i, EV}) + \lambda^* P_{i, EV}$ is a convex differentiable function, its derivative at the maximum should be zero, i.e. $\lambda^* - \partial U_{i, EV}/\partial P_{i, EV} |_{P_{i, EV}^*} = 0$. For any $i \in E \ V$ three possibilities arise:

1. $0 < P_{i, EV}^{**} < P_{i, EV}^{max}$: In this case, $P_i^* = P_{i, EV}^{**}$ and $\lambda^* - \partial U_{i, EV}/\partial P_{i, EV} |_{P_{i, EV}^*} = 0$. So by taking $\lambda = \lambda^*$ and $\mu_i = \xi_i = 0$, (4-33), (4-35), and (4-36) are satisfied.

2. $P_{i, EV}^{**} \geq P_{i, EV}^{max}$: In this case, $P_{i, EV}^* = P_{i, EV}^{max} < P_{i, EV}^{**}$, so $-\partial U_{i, EV}/\partial P_{i, EV} |_{P_{i, EV}^*} + \lambda^* \leq 0$, as otherwise $P_{i, EV}^{**}$ would be a maximum. Thus, by taking $\lambda = \lambda^*$, $\xi_i$ = $+\partial U_{i, EV}/\partial P_{i, EV} |_{P_{i, EV}^*} - \lambda^*$, and $\mu_i = 0$, (4-33), (4-35), and (4-36) are satisfied.

3. $P_{i, EV}^{**} < 0$: In this case, $P_{i, EV}^* = P_{i, EV}^{min} > P_{i, EV}^{**}$, so $-\partial U_{i, EV}/\partial P_{i, EV} |_{P_{i, EV}^*} + \lambda^* \geq 0$, as otherwise $P_{i, EV}^{**}$ would be a maximum. Thus, by taking $\lambda = \lambda^*$, $\xi_i = 0$, and $\lambda^* = \partial U_{i, EV}/\partial P_{i, EV} |_{P_{i, EV}^*}$, (4-33), (4-35), and (4-36) are satisfied.

For the fixed point for dispatchable generation units, similar arguments can be made, and it can be shown that (4-34), (4-35), and (4-36) are satisfied. Thus, the theorem is proven.

**Theorem 4-2**: If the optimization problem in (4-18) is feasible, and the utility functions for EVs are defined as (4-7) and cost functions for dispatchable generation units are strictly convex and continuously differentiable, there exist a positive value $\varepsilon$ such that for all values of $0 < \eta < \varepsilon$, the fixed points of the iterative algorithm presented by (4-25)-(4-31) is attractive.

**Proof:**

The proof is similar to Theorem 3-2, with slight difference because of the update equation for smart transformers, constant demands, and renewables. For the sake of completeness, we
bring the part of the proof that accounts for these differences. Writing (4-25)-(4-31) in vector form:

\[
\begin{bmatrix}
\lambda^{k+1} \\
\Delta \hat{\mathbf{p}}^{k+1}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{W} & \eta \mathbf{I} \\
0 & \mathbf{W}
\end{bmatrix}
\begin{bmatrix}
\lambda^k \\
\Delta \hat{\mathbf{p}}^k
\end{bmatrix} +
\begin{bmatrix}
0 \\
\mathbf{1}
\end{bmatrix} \mathbf{u}^k,
\]  
(4-47)

where \( \lambda = [\lambda_i] \), \( \Delta \hat{\mathbf{p}} = [\Delta \hat{\mathbf{p}}_i] \), \( \mathbf{W} \) is the consensus update matrix,

\[
W(i,j) = \begin{cases} 
  w_{ij} & i \neq j, j \in N_i \\
  0 & i \neq j, j \in N_i \\
  1 - \sum_{j \in N_i} w_{ij} & i = j
\end{cases},
\]  
(4-48)

and \( \mathbf{u}^k \) is a nonlinear input defined as:

\[
\forall i \in EV: \mathbf{u}^k(i) = \min_{0 \leq P_{i,EV} \leq P_{i,EV,\text{max}}} \left( \lambda_i^{k+1} P_{i,EV} - U_{i,EV}(P_{i,EV}) \right) \\
- \min_{0 \leq P_{i,EV} \leq P_{i,EV,\text{min}}} \left( \lambda_i^k P_{i,EV} - U_{i,EV}(P_{i,EV}) \right),
\]  

\[
\forall i \in G: \mathbf{u}^k(i) = - \min_{0 \leq P_{i,G} \leq P_{i,G,\text{max}}} \left( C_{i,G}(P_{i,G}) - \lambda_i^{k+1} P_{i,G} \right) \\
+ \min_{0 \leq P_{i,G} \leq P_{i,G,\text{min}}} \left( C_{i,G}(P_{i,G}) - \lambda_i^k P_{i,G} \right),
\]  
(4-49)

\[
\forall i \in D: \mathbf{u}^k(i) = P_{i,D}^{k+1} - P_{i,D}^k, \\
\forall i \in R: \mathbf{u}^k(i) = P_{i,R}^k - P_{i,R}^{k+1}, \\
\forall i \in T: \mathbf{u}^k(i) = 0.
\]

Denoting \( P_{i,EV}^* = \min_{P_{i,EV}} \left( \lambda_i^k P_{i,EV} - U_{i,EV}(P_{i,EV}) \right) \) for EVs:

\[
P_{i,EV}^* + \mathbf{u}^k(i) = \min_{0 \leq P_{i,EV} \leq P_{i,EV,\text{max}}} \left( \left( \lambda_i^k + \Delta \lambda_i \right) P_{i,EV} - U_{i,EV}(P_{i,EV}) \right),
\]  
(4-50)

where \( \Delta \lambda_i = \lambda_i^{k+1} - \lambda_i^k \). Without considering the local limits, the derivative at the minimum point would be zero. So:
\[ \lambda_i^k + \Delta \lambda_i - \frac{\partial U_{i, EV}}{\partial P_{i, EV}} \bigg|_{P_{i, EV}^* + \Delta P_{i, EV}} = 0. \] (4-51)

To analyze the dynamics around the fixed point, we write the first-order Taylor expansion for \( \partial U_{i, EV} / \partial P \):

\[
\lambda_i^k + \Delta \lambda_i - \left( \frac{\partial U_{i, EV}}{\partial P_{i, EV}} \bigg|_{P_{i, EV}^*} + \frac{\partial^2 U_{i, EV}}{\partial P_{i, EV}^2} \bigg|_{P_{i, EV}^*} \cdot \Delta P_{i, EV} \right) = 0, \tag{4-52}
\]

Noting that \( \lambda_i^k - \left( \frac{\partial U_{i, EV}}{\partial P_{i, EV}} \bigg|_{P_{i, EV}^*} \right) = 0 \), (4-52) yields:

\[
\Delta \lambda_i = \frac{2 \text{Cap}_{i, EV}^2 \text{SoC}_{i, EV, d}}{\eta_{i, EV}^2 \omega_{i, EV} \Delta t^2}.
\] (4-53)

As we have local bounds, the absolute value of \( \Delta \lambda_i \) would be smaller than the one presented in (4-53):

\[
\forall i \in EV: \Delta \lambda_i = \frac{2 \text{Cap}_{i, EV}^2 \text{SoC}_{i, EV, d}}{\eta_{i, EV}^2 \omega_{i, EV} \Delta t^2}, \tag{4-54}
\]

The same analysis can be done for the generation side. So:

\[
\forall i \in G: \Delta \lambda_i = \frac{2 \text{Cap}_{i, G} \Delta \lambda_i}{\eta_{i, G} \omega_{i, G} \Delta t^2}.
\] (4-55)

Also, note that when the algorithm converges, the inputs to the system from renewable generation, and constant demands do not change, i.e. \( \forall i \in D, R: \Delta \lambda_i = 0 \) around the fixed point. Then, considering \( \Delta \lambda_i = \lambda_i^{k+1} - \lambda_i^k = \mathbf{w}_i \lambda_i^k + \eta \Delta \hat{P}_i^k - \lambda_i^k \), where \( \mathbf{w}_i \) is the \( i \)-th row of the \( \mathbf{W} \) matrix, we can combine (4-54) and (4-55) as:

\[
\mathbf{u}_i^k = \mathbf{G} \left( (\mathbf{I} - \mathbf{W}) \lambda_i^k - \eta \Delta \hat{P}_i^k \right),
\] (4-56)

where \( \mathbf{G} \) is a diagonal matrix defined as:
\[ G(i,i) = \begin{cases} 
K_{i,\text{EV}} & i \in \text{EV} \\
K_{i,G} & i \in G \\
0 & i \in D \\
0 & i \in R \\
0 & i \in T 
\end{cases} \quad (4-57) \]

Now (4-47) and (4-49) can be written as a closed-loop system:

\[
\begin{bmatrix}
\lambda^{k+1} \\
\Delta \hat{P}^{k+1}
\end{bmatrix} = \begin{bmatrix} 
W & \eta \mathbf{I} \\
G (I - W) & (W - \eta G)
\end{bmatrix} \begin{bmatrix}
\lambda^{k} \\
\Delta \hat{P}^{k}
\end{bmatrix} \quad (4-58)
\]

The rest of the proof is the same as Theorem 3-2. Also, note that the upper bound of \( K_{i,\text{EV}} \) is independent of the SOC of the vehicle, and it is a function of static parameters for the EV.■

4.4 Numerical Simulations

In this section, we validate the performance of the DRPCG algorithm through numerical simulations and benchmarking it against centralized optimization. We test the algorithm on a 5 bus system (Garver Network).

Figure 4-3 shows the setup of the case study. It is a 5-bus system with multiple energy devices such as EVs, thermal generation units, renewables and constant loads. The energy devices are scattered in 5 different zones. The devices inside each zone have the ability to exchange information with each other as well as with the transformer that isolates the zone from the rest of the network. On a higher level, the transformers have the ability to exchange information within themselves. The parameters for dispatchable devices (i.e. Thermal generation units and EVs) is shown in Table 4-1. The time step \( \Delta t \) is taken as 10 min. For thermal generation units, we have used quadratic convex cost functions [58]:

\[ C_{i,G}(P_{i,G}) = a_{i,G}P_{i,G}^2 + b_{i,G}P_{i,G} + c_{i,G}, \quad (4-59) \]
where $a_{i,G}$, $b_{i,G}$, and $c_{i,G}$ are predetermined constants and $P_{i,G}$ is the amount of generated power. For EVs, we have used the utility functions with the form of (4-7). The maximum power for electric vehicles is considered to be 19.2 kW which is the maximum power that can be provided in level 2 AC charging. The generated power by the solar generation unit and the consumed power by the constant loads are as shown in Table 4-2. The optimization problem is how to regulate the generated power of the dispatchable generation units and charging power of the PHEV/PEVs so that the social welfare of the entire system is maximized. Applying the DRPCG algorithm to do the price negotiation among zones and inside each zone, the plots of Figure 4-4 are resulted. It can be seen that as the algorithm proceeds the prices inside zones and among zones converges to a unique value, and the objective value converges to the optimal
As the price evolves, EVs regulate their charging powers. The minimum charging power is allocated to the EV connected to node 3, which has the highest SoC. The maximum charging power is allocated to the EV connected to node 4, which has relatively small SoC, and pays more than the EV connected to node 9 which has smaller SoC than it. On the generation side, we can see that dispatchable generation units also regulate their production in response to the evolution in the price.

Now consider the same case study, with the difference that while the algorithm is still running, information regarding power production of the solar unit and some of the constant loads is updated. At iteration 800 of the algorithm, solar production changes from 40 kW to 20 kW,
and the load connected to node 2, changes from 15 kW to 40 kW. The results are shown in Figure 4-5. It can be seen that in response to these changes (decrease in the renewables and increase in constant loads), the price among zones increases (because demand is more than
supply) and as a result the generators increase their production and EVs decrease their consumption and the system converges to the new equilibrium point.

4.5 Conclusion

A novel distributed real-time pricing control for optimal management of unidirectional V2G was proposed which considers multiple energy suppliers. In this approach, each charging station as well as each individual energy supplier is equipped with a local price regulator which control the price of energy by coordinating with their neighbors and considering the balance

Figure 4-5. Response of DRPCG algorithm to changes in the loading condition of the system a) EV response, b) Generation response, c) Price response, d) Power imbalance
of supply and demand. In response to the updated prices, the vehicles adjust their charging rate
to maximize their utility while energy suppliers adjust their production rate to maximize their
benefit. The main advantages of the proposed approach are: 1) solves V2G management
problem in a fully distributed way considering the vehicle side and supplier side
simultaneously, 2) converges to the global optimum despite the greedy behavior of the
individuals.
CHAPTER 5. ASYNCHRONOUS INCREMENTAL WELFARE CONSENSUS ALGORITHM

Abstract:
A paradigm shift is occurring in the literature of the smart grid toward using distributed techniques to solve the energy management problem of the future power grid. To manage a smart grid environment populated with responsive demands and dispatchable distributed generators, the Incremental Welfare Consensus (IWC) algorithm was introduced in chapter 3. This algorithm can drive the behavior of the system toward the global optimum based solely on local computational and communications capabilities. However, one of the major requirements of this algorithm is synchronous updating of the distributed generators and responsive demands. If the units were to fail to update their states synchronously, IWC would not be able to converge to the global optimum and balance the generation and demand. This chapter introduces the Asynchronous IWC (AIWC) algorithm, which eliminates the synchronous updates requirement of the IWC algorithm. Thus, it is more suitable for real-world applications where maintaining synchronous behavior among a multitude of geographically distributed generation/demand units is a challenging task. The performance of the algorithm is validated through numerical case studies.

5.1 Introduction

Typically, energy management of the power grid is achieved through centralized Supervisory, Control, and Data Acquisition (SCADA) systems [1], [4]. However, in recent years, a paradigm shift is happening in the literature of the smart grid toward distributed methods [27], [30], [32], [43], [44], [46]–[50], [52]. The reason for this paradigm shift is the
introduction of a large number of heterogeneous controllable devices with local computational/communications capabilities and the physically distributed structure of the smart grid. This demands new technologies for management of the smart grid that are scalable and consider the local intelligence of the distributed devices, as well as their communications capability [5]. Distributed algorithms employ the local capabilities of the distributed devices and are scalable in computational and communications efforts. Moreover, they are more robust than centralized approaches when faced with single points of failures and, therefore, can better survive the inevitable device/link failures in the complex system of the smart grid.

The Incremental Welfare Consensus (IWC) algorithm introduced in chapter 3 is one of the distributed algorithms proposed for the energy management of the smart grid [55], [80]. This algorithm has two salient features:

1. IWC considers the management of responsive distributed loads and dynamic distributed generators simultaneously. Most of the existing distributed solutions focus on either the generation side or the demand side. For example, there are distributed algorithms, such as [32], [44], [46], which are proposed to regulate the distributed energy resources optimally. Also, there are algorithms such as distributed load-shedding [43], distributed demand response [27], [48], and distributed Plug-in Hybrid Electric Vehicle (PHEV)/Plug-in Electric Vehicle (PEV) demand management [30], which mainly focus on demand-side management.

2. IWC bypasses the need for any central controller/ coordinator/leader and requires only local communications capability among distributed devices. Nevertheless, most of the existing distributed energy management algorithms are not completely center-free. They either need a central coordinator [27], [50] or a leader [32] to
gather specific information from the distributed devices and update certain global information (such as price) or the distributed devices need to have the ability to communicate with all the other devices in the network [48], [49]. However, one of the major requirements of the IWC algorithm is synchronous updates of the distributed units. If the units were to fail to update their states synchronously, IWC would not be able to converge to the global optimum and would not even balance generation and demand. Maintaining synchronous behavior among a multitude of geographically distributed heterogeneous units is a challenging task.

In this chapter, the Asynchronous IWC (AIWC) algorithm is introduced. The AIWC possesses all the advantages of the IWC algorithm while it does not require synchronization among distributed units and therefore, is more robust against asynchronous behaviors. Thus, it is a better candidate for real-world applications where maintaining synchronous behavior among a multitude of geographically distributed units is difficult.

The chapter is organized as follows. Section II provides preliminary information and reviews the IWC algorithm. Section III demonstrates the sensitivity of the IWC algorithm to the asynchronous behavior of the distributed units. Section IV introduces the AIWC algorithm. Section V uses numerical analysis to show that the AIWC algorithm can drive the behavior of the system toward the global optimum using asynchronous updates and is robust to asynchronous behavior of the units. Finally, the concluding remarks come in Section VI.

5.2 Incremental Welfare Consensus Algorithm

The IWC [55], [80] is a distributed algorithm that deals with the energy management issue in a smart grid environment populated with distributed dispatchable generation units and
responsive demands. The IWC algorithm consists of three processes running on distributed controllers:

5.2.1 Distributed Estimation of Power Imbalance

Each distributed controller uses the information from its neighbors and its own information to update its estimate of the average power imbalance between generation and demand throughout the entire network:

\[ \forall i \in G : \]
\[
\Delta \hat{P}_{i}^{k+1} = \Delta \hat{P}_{i}^{k} + \sum_{j \in N_i} w_{ij} \left( \Delta \hat{P}_{j}^{k} - \Delta \hat{P}_{i}^{k} \right) + P_{i}^{k} - P_{i}^{k+1}, \tag{5-1}
\]

\[ \forall i \in D : \]
\[
\Delta \hat{P}_{i}^{k+1} = \Delta \hat{P}_{i}^{k} + \sum_{j \in N_i} w_{ij} \left( \Delta \hat{P}_{j}^{k} - \Delta \hat{P}_{i}^{k} \right) + P_{i}^{k+1} - P_{i}^{k}, \tag{5-2}
\]

where \( \Delta \hat{P}_{i}^{k} \) is the estimate of the \( i \)th unit from the value of the average power imbalance at the \( k \)th iteration, \( N_i \) is the set of neighbors of node \( i \), \( w_{ij} = w_{ji} \) is the connectivity strength between node \( i \) and node \( j \) and chosen so that \( 0 \leq w_{ij} < \left( \max_{i=1,...,N} |N_i| \right)^{-1} \). \( P_{i}^{k} \) is the generated/consumed power of the \( i \)th unit at the \( k \)th iteration. The set of indices of demand units is denoted by \( D \), and the set of indices of generation units is denoted by \( G \).

5.3 Local Price Update

Based on the estimation of power imbalance between supply and demand, the local price is updated at each distributed unit:

\[ \forall i \in G \cup D : \lambda_{i}^{k+1} = \lambda_{i}^{k} + \sum_{j \in N_i} w_{ij} \left( \lambda_{j}^{k} - \lambda_{i}^{k} \right) + \eta \Delta \hat{P}_{i}^{k}, \tag{5-3} \]

where \( \lambda_{i}^{k} \) is the price of energy at the \( i \)th unit at the \( k \)th iteration, and \( \eta \) is the updating step size.
5.4 Power Regulation

Based on the updated price of energy, each unit updates the amount of power consumption/generation to maximize its profit (or equivalently minimize its cost):

\[
\forall i \in D : P_{i}^{k+1} = \arg \min_{0 \leq P_i \leq P_{i,\text{max}}} \left( \lambda_i^{k} P_i - U_i(P_i) \right),
\]

\[
\forall i \in G : P_{i}^{k+1} = \arg \min_{P_{i,\text{min}} \leq P_i \leq P_{i,\text{max}}} \left( C_i(P_i) - \lambda_i^{k} P_i \right),
\]

(5-4)

where \( U_i(.) \) and \( C_i(.) \) are the utility and cost functions of the \( i \)th unit respectively and \( P_{i,\text{min}} \) and \( P_{i,\text{max}} \) represent the minimum and maximum bounds for power consumption/generation. The utility function for the \( i \)th consumer unit is a concave function defined as:

\[
U_i(P_i) = \begin{cases} 
\omega_i P_i - \alpha_i P_i^2 & P_i < \omega_i / 2 \alpha_i \\
\omega_i^2 / 4 \alpha_i & P_i \geq \omega_i / 2 \alpha_i
\end{cases}
\]

(5-5)
where $\omega$ and $\alpha$ are positive values differentiating the consumers.

The cost function for the $i^{th}$ generation unit is a convex function considered as:

$$C_i(P_i) = a_i P_i^2 + b_i P_i + c_i,$$

where $a_i \geq 0$, $b_i$ and $c_i$ are predetermined constants and differentiate the cost functions of different generators.

Equations (5-1) through (5-4) constitute the update equations of the IWC algorithm for distributed generation and demand devices. It was shown in chapter 3 that the fixed point of this iterative algorithm is the solution of the social welfare maximization problem [52], [55] represented by:

$$\text{Min } \sum_{i \in G} C_i(P_i) - \sum_{i \in D} U_i(P_i), \text{ subject to:}$$

$$\sum_{i \in D} P_i = \sum_{i \in G} P_i, \forall i \in G \cup D: P_{i,\min} \leq P_i \leq P_{i,\max}.$$  \hspace{1cm} (5-7)

where the objective is to minimize the cost on the generation side and maximize the utility on the demand side, while the power balance is satisfied.

**Example 5-1:**

The Garver power network [81] is shown in Figure 4-3. This system consists of five load (LD) units and two generation (G) units. Of the five demand units, three are responsive demands (LD2, LD4, and LD5) and two are constant demands (LD1 and LD3). A communications node is considered for each generation/demand unit and the communications topology is assumed to be similar to the power network topology.
The utility/cost function parameters, as well as the generation/consumption limits, are provided in Table I. The value for $\eta$ is chosen as 0.003, and all connectivity strengths are chosen as $w_{ij} = 0.3233$ to satisfy $0 \leq w_{ij} < \left( \max_{i=1,...,N} |N_i| \right)^{-1} = 1/3$ as the maximum degree of the communications graph ($\max_{i=1,...,N} |N_i|$) is 3.

Figure 5-2 shows the evolution of generation and consumption powers, as well as the social welfare value. It can be seen that around iteration 200, the generation and demand reach a balance, the price settles down, and the algorithm converges to the optimal social welfare value.

5.5 Sensitivity of IWC to Synchronous Updates

The IWC algorithm reviewed in Section II requires distributed agents to update their states synchronously. In this section we investigate the effect of asynchronous updates on IWC. Consider the IWC algorithm running on the Garver network with the parameters provided in Example 5-1. We consider a probability of $p$ for each agent to not update at the same instant that its neighbors are updating.

This simulates the asynchronous behavior of the agent. Thus, the update equation for each agent becomes:

Table 5-1. Generation/Demand Units Parameters Chosen For Garver Network

<table>
<thead>
<tr>
<th>Node</th>
<th>Demand</th>
<th>$\omega$ (cents/kW)</th>
<th>$a$ (cents/kWh$^2$)</th>
<th>$P_{min}$ (kW)</th>
<th>$P_{max}$ (kW)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>LD1</td>
<td>-</td>
<td>-</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>LD2</td>
<td>15</td>
<td>0.0750</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>4</td>
<td>LD3</td>
<td>-</td>
<td>-</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>6</td>
<td>LD4</td>
<td>12</td>
<td>0.1200</td>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td>7</td>
<td>LD5</td>
<td>20</td>
<td>0.2000</td>
<td>0</td>
<td>50</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Generation</th>
<th>$a$</th>
<th>$b$</th>
<th>$P_{min}$ (kW)</th>
<th>$P_{max}$ (kW)</th>
</tr>
</thead>
<tbody>
<tr>
<td>G1</td>
<td>0.0080</td>
<td>7</td>
<td>0</td>
<td>100</td>
</tr>
</tbody>
</table>

The utility/cost function parameters, as well as the generation/consumption limits, are provided in Table I. The value for $\eta$ is chosen as 0.003, and all connectivity strengths are chosen as $w_{ij} = 0.3233$ to satisfy $0 \leq w_{ij} < \left( \max_{i=1,...,N} |N_i| \right)^{-1} = 1/3$ as the maximum degree of the communications graph ($\max_{i=1,...,N} |N_i|$) is 3.
and $f_i$ is the update dynamics of the IWC algorithm represented in equations (5-1) through (5-4).

Figure 5-3 shows the evolution of social welfare and generation/demand for $p = 0.1, 0.2, 0.5$. The results indicate that the IWC algorithm fails if synchronous behavior is not maintained among the agents.

5.6 Asynchronous Incremental Welfare Consensus Algorithm

To eliminate the synchronization requirement for the IWC algorithm, we extend the IWC algorithm to an asynchronous version using the pairwise gossip algorithm [82]. We call this algorithm: AIWC. In the new algorithm, at each instant, each agent only communicates with
one of its neighbors. While the two communicating agents have not updated their states, they reject any communication attempts from other neighbors.

Therefore, the update equations (5-1), (5-2), and (5-3) of the IWC algorithm change in the AIWC algorithm. In AIWC, at each moment of time, we have two kinds of nodes:
5.6.1 Communicating nodes

If node \( i \) is communicating with node \( j \), the update equations for \( \lambda_i^k \) and \( \hat{\lambda}_i^k \) are:

\[
\forall i \in G:
\Delta \hat{\lambda}_i^{k+1} = \Delta \hat{\lambda}_i^k + w_{ij} \left( \Delta \hat{\lambda}_j^k - \Delta \hat{\lambda}_i^k \right) + P_i^k - P_i^{k+1},
\]

where \( 0 \leq w_{ij} < 1 \) and \( w_{ij} = w_{ji} \).

\[
\forall i \in D:
\Delta \hat{\lambda}_i^{k+1} = \Delta \hat{\lambda}_i^k + w_{ij} \left( \Delta \hat{\lambda}_j^k - \Delta \hat{\lambda}_i^k \right) + P_i^{k+1} - P_i^k,
\]

\[
\forall i \in G \cup D: \lambda_i^{k+1} = \lambda_i^k + w_{ij} \left( \lambda_j^k - \lambda_i^k \right) + \eta \Delta \hat{\lambda}_i^k,
\]

where \( 0 \leq w_{ij} < 1 \) and \( w_{ij} = w_{ji} \).

5.6.2 Non-communicating nodes

If node \( i \) has not been able to set up any communications session with one of its neighbors, its states would not change, i.e.,

\[
\left[ \begin{array}{c} P_i^{k+1} \\ \lambda_i^{k+1} \\ \Delta \hat{\lambda}_i^{k+1} \end{array} \right]^T = \left[ \begin{array}{c} P_i^k \\ \lambda_i^k \\ \Delta \hat{\lambda}_i^k \end{array} \right]^T.
\]

With this setup, each distributed unit has event-driven and time-driven dynamics. The time-driven dynamics are represented by update equations (5-4), (5-9)-(5-11). The event-driven dynamics is represented by the automaton in Figure 5-4.
Figure 5-5 shows a possible communications scenario at a time instant of the AIWC algorithm running on the Garver network. At this instant, unit 5 is a non-communicating node and other units are communicating nodes.

In the AIWC algorithm, different agents can update arbitrarily at different times and as long as they have not updated their states, they do not initiate any new communications with any of their neighbors.

5.7 Numerical Results

We consider the Garver network with the parameters provided in Table 5-1 and run the AIWC algorithm. Figure 5-6 shows the evolution of generation and consumption powers, the social welfare value and price. It can be seen that the values converge to the optimal values of
Example 5-1 in about 800 iterations. It should be noted that, as the updates are asynchronous, iteration does not represent time (there is no synchronous time). Rather, each iteration represents a pair-wise information exchange between two nodes. Therefore, the algorithm converges after total of 800 pair-wise information exchanges among nodes.

In the case shown in Figure 5-6, the nodes were not synchronized globally. However, it was assumed that in each pair-wise communication, the two nodes updated their states upon the information exchange. Now, we consider the case where even in pair-wise information
exchange, the nodes show asynchronous behavior. That is, after the information exchange is finished, one of them updates and looks to exchange information with another of its neighbors, while the other node does not update and, therefore, rejects all communications attempts from

Figure 5-7. Evolution of generation, demand and social welfare for AIWC algorithm under probability of asynchronous behavior for the pairwise updates a) $p = 0.1$, b) $p = 0.2$, c) $p = 0.5$
its neighbors. We consider a probability of \( p \) for each node to not update when its paired neighbor has updated. Figure 5-7 shows the social welfare and generation/demand curves for \( p = 0.1, 0.2, \) and 0.5. Here, each iteration represents a communication attempt between two non-communicating nodes. This communication attempt can be accepted or rejected depending on whether the nodes have updated their states in their previous information exchange session. In all the scenarios, the AIWC algorithm is able to converge to the global optimum. The only effect of this asynchronous behavior is increasing the number of required iterations (communication attempts) for the convergence. The results of this section validate that the AIWC algorithm does not require synchronization between nodes and is robust against asynchronous behaviors.

5.8 Conclusion

This chapter introduced the AIWC algorithm. The algorithm is an extension of the IWC algorithm for energy management of a smart grid environment populated with responsive demands and distributed dispatchable generators. The AIWC eliminates the requirement of synchronous updates for the distributed units. Therefore, it is more suitable for real-world applications where maintaining synchronization among distributed units is challenging.
CHAPTER 6. CONSENSUS BASED DISTRIBUTED SCHEDULING FOR COOPERATIVE OPERATION OF DISTRIBUTED ENERGY RESOURCES AND STORAGE DEVICES IN SMART GRIDS

Abstract:
Optimal dispatch of storage devices is crucial for the economic operation of smart grids with distributed energy resources. Through appropriate scheduling, storage devices can store the energy when the renewable production is high or electricity price is low, and support the demand when electricity is expensive. Conventionally, this scheduling requires a control center to gather information from the entire system and find the optimal schedule in the required horizon for the controllable devices. This chapter proposes a fully distributed scheduling methodology based on discrete-time optimal control, primal-dual gradient descent, and consensus networks. In the proposed approach, the requirement for the control center is eliminated and the optimal schedule for all the devices is found solely through iterative coordination of each device with its neighbors. The application of the algorithm is demonstrated in a 5-bus system and its convergence to the global optimum is validated through Monte Carlo simulations. Further, it is shown that the algorithm is robust against communication link failures provided that the communications topology remains connected or reconnects after being disconnected.

6.1 Introduction

Distributed Generation (DG) has the promise of reduced greenhouse emission, increased efficiency, and improved reliability [83]. Storage devices not only improve the reliability of DGs but also enable the increasing renewable penetrations [84]. However, optimal scheduling is essential for efficient cooperation of storage devices and DGs. This scheduling should
consider real-time operating conditions of the system as well as the forecasted information about the future renewable generation, demands [85], and electricity prices which can be available from weather/load forecasting models and utility pricing policies [86].

As storage devices can store the energy in one time step and release it in others, energy management decisions in different time steps are not independent of each other. Consequently, optimal scheduling of the system with storage devices is inherently a multi-step decision making problem and conventionally centralized technologies are utilized to solve it such as particle swarm algorithm in [87], mixed integer linear programming in [88], and genetic algorithms in [89] and [90].

With the increasing number of DGs and storage devices integrated into the distribution system, the centralized scheduling solutions would face certain challenges:

1. Vulnerability to central point of failure: In centralized approaches, the optimization is carried out in a central location. Failure of the center would affect the entire system performance.
2. Vulnerability to communication failures: Centralized solutions require the control center to have communications with all the controllable devices in the system. Failure of each communication link would mean that the respective device no longer can be monitored/controlled.
3. Challenge of privacy: Centralized technologies require the individual devices to reveal certain information regarding their characteristics and consumption/generation preferences to the center. This endangers the privacy of the devices which might have different owners.
Due to the aforementioned limitations of the centralized approaches, the literature of smart grids shows an increasing interest in distributed solutions. For instance, distributed load shedding [43], distributed economic dispatch for energy resources [46], [53], [91], distributed generation and demand response [80], and distributed electric vehicle charge optimization [51]. Nevertheless, most of the existing distributed approaches for optimal operation of smart grids are single step solvers and therefore are not appropriate to do the optimal scheduling. In [92], [93], and [94], distributed techniques are introduced for the scheduling problem, but the distributed controllers need to communicate with all the other controllers regarding their common binding constraints. In [95], a distributed multi-step approach is proposed but requires defining strict convex quadratic cost functions for all the controllable devices which is not a trivial process specially for storage units.

This chapter introduces a novel fully distributed algorithm for optimal scheduling of storage devices and DGs in smart grids. The theoretical basis of the algorithm is to solve a discrete time optimal control problem by combining primal-dual gradient descent and consensus networks. In the framework of this chapter, each energy device is equipped with a distributed controller with the ability to exchange information with its neighbors. The distributed controllers can find the optimal schedule by doing local calculations and sharing information with their neighbors in an iterative manner. The main features of the proposed algorithm are as follows:

1. The multi-step scheduling problem is solved in a fully distributed way without requiring any central controller/coordinator/leader and the convergence point of the algorithm is the global optimum of the system;

2. The nodes only require local communications with their neighbors;
3. The algorithm is robust against communication failures as long as the communications topology remains connected or reconnects after being disconnected;

4. The devices do not need to disclose their private information such as load profile, storage states or cost functions to other devices. Only estimations of global variables are exchanged among them.

The organization of this chapter is as follows. Section 2 formulates the optimal scheduling of the storage devices in the microgrid as a discrete time optimal control problem. Section 3 explains the derivation process for the proposed distributed scheduling algorithm. Section 4 provides numerical simulations to demonstrate the properties of the algorithm. Finally, section 5 summarizes the chapter and brings the concluding remarks.

6.2 Problem Formulation

As shown in Figure 6-1, the system of interest in this chapter is a microgrid consisting of distributed renewable resources, conventional generators, Distributed Energy Storage Devices (DESDs), and loads operating in the grid connected mode and monitored/controlled through distributed controllers. Optimal operation of distributed resources in this system in the requested horizon can be formulated as a finite horizon optimal control problem. Thus, we need to define objective function, inputs, states, and constraints. It is assumed that all the devices in the system are assigned with a unique index, which is also the index of the respective communications node. The set of indices of all devices is denoted by $I$.

6.2.1 Objective function

The objective is to minimize the operational cost of the system over a finite time horizon,

$$
\min_{u(t), t=1, \ldots, T} \left( J = \sum_{t=1}^{T} \gamma^{t-1} C(x(t), u(t), w(t)) \right),
$$

(6-1)
where $C(x(t), u(t), w(t))$ is the operational cost of the system at time-step $t$, $x(t)$ is the vector of system states, $u(t)$ is the vector of control inputs, and $w(t)$ is the vector of uncontrollable inputs. Here, $0 < \gamma \leq 1$ is a discount factor to put less weight on future values of the costs.

### 6.2.2 Controllable Inputs

The vector of controllable inputs, $u(t)$, consists of:

1) Power commands to storage devices: The power command to the storage device with index $i$ at time step $t$ is denoted by $P_{i,B}(t)$, where $i \in B$, and $B$ is the set of indices of storage units. A positive value denotes the discharging command to the storage device.

2) Power commands to conventional generators (e.g. micro turbine): The power command to conventional generator with index $i$ at time step $t$ is denoted by $P_{i,G}(t)$, $i \in G$, and $G$ is the set of indices of conventional generation units.
3) Import command to the grid interface: The power to be imported at time step \( t \) from the grid interface is denoted by \( P_{i,\text{grid}}(t) \), where \( i \) is the index of the respective communications node.

### 6.2.3 Uncontrollable Inputs

The vector of uncontrollable inputs, \( \mathbf{w}(t) \), consists of:

1) Renewable generation: The power produced by the renewable generation unit with index \( i \) at time step \( t \) is denoted by \( P_{i,R}(t) \), where \( i \in R \), and \( R \) is the set of indices of renewable units.

2) Demands: The power consumed by the load unit with index \( i \) at time step \( t \) is denoted by \( P_{i,D}(t) \), where \( i \in D \), and \( D \) is the set of indices of demand units.

3) Energy price: The price of energy at the grid at time step \( t \) is denoted by \( p(t) \).

### 6.2.4 System States

Neglecting the states resulting from fast dynamics of the individual devices, the amount of energy stored in storage devices would be the states of the system. The energy stored at the storage device with index \( i \) at time \( t \) is denoted by \( E_i(t) \). The vector of states is denoted by

\[
\mathbf{x}(t) = \left[ E_i(t) : i \in B \right].
\]

### 6.2.5 Operational Cost

The operational cost of the system consists of two parts:

1. *Fuel cost of the energy produced by conventional generators:* The operational cost of the conventional generation unit with index \( i \) at time step \( t \) is usually approximated by a quadratic convex function \([58]\) as:

\[
\forall 1 \leq t \leq T, \forall i \in G : C_{i,G}(P_{i,G}(t), \Delta t) = \left( a_i P_{i,G}(t)^2 + b_i P_{i,G}(t) + c_i \right) \Delta t ,
\]  

(6-2)
where $a_i$, $b_i$, and $c_i$ are predetermined constants and $\Delta t$ is the time interval between two time steps (for instance 1 hour).

2. **Cost of buying energy from grid**: The cost of buying energy from grid at time step $t$ is a linear function of the energy price:

$$\forall 1 \leq t \leq T : C_{grid}(P_{i,grid}(t), \Delta t) = p(t)P_{i,grid}(t)\Delta t .$$

(6-3)

### 6.2.6 Constraints

Three classes of constraints should be satisfied:

1. **State dynamics**: Dynamics of the energy stored in the storage devices is as follows:

$$\forall 1 \leq t \leq T, \forall i \in B : E_i(t+1) = E_i(t) - P_{i,B}(t)\Delta t .$$

(6-4)

Note that in this chapter we do not consider the efficiency of the storage devices as it will introduce a discrete mode to the optimization problem. We will consider it in our future work.

2. **State constraints**: The energy stored in the storage devices is to be bounded between the maximum capacity of the device and a minimum desired state of energy:

$$\forall 1 \leq t \leq T, i \in B : E_{i,\min} \leq E_i(t) \leq E_{i,\text{full}} ,$$

(6-5)

where $E_{i,\min}$ is the minimum desired energy level of the storage device and $E_{i,\text{full}}$ is the energy capacity of the storage device with index $i$. Combining (6-4) and (6-5) results:

$$\forall i \in B, \forall t = 1, ..., T : E_{i,0} - E_{i,\text{full}} \leq \sum_{k=1}^{t} P_{i,B}(k)\Delta t \leq E_{i,0} - E_{i,\min} .$$

(6-6)

3. **Input constraints**:

   a. **Power balance**: At all times, the amount of generation should equal to the amount of load:
\( \forall 1 \leq t \leq T: \sum_{i \in B} P_{i,B}(t) + \sum_{i \in R} P_{i,R}(t) + P_{i,\text{grid}} = \sum_{i \in D} P_{i,D}(t). \) (6-7)

Note that in this chapter, we do not consider losses.

b. **Power rating:** At all times, the power commands should be within the maximum and minimum limits defined by the device characteristics:

\[ \forall 1 \leq t \leq T, i \in B \cup G, j \in \{B, G\} : \]

\[ P_{i,j,\text{min}} \leq P_{i,j}(t) \leq P_{i,j,\text{max}}, \] (6-8)

\[ 0 \leq P_{i,\text{grid}}(t) \leq P_{\text{grid, max}}, \]

where \( P_{i,j,\text{min}} \) and \( P_{i,j,\text{max}} \) are the minimum and maximum power limits of the unit with index \( i \) of type \( j \) and \( P_{\text{grid, max}} \) is the maximum power that can be drawn from the grid interface.

Considering the above definitions, the optimal control problem can be defined as:

\[
\min_{u(t) \in \Omega} J = \sum_{t=1}^{T} \gamma^{t-1} \left( p(t) P_{i,\text{grid}}(t) \Delta t + \sum_{i \in G} \left( a_i P_{i,G}(t)^2 + b_i P_{i,G}(t) + c_i \right) \Delta t \right),
\]

subject to (6-4)-(6-8).

### 6.3 Consensus based distributed iterative algorithm

#### 6.3.1 Primal-dual gradient descent method

The optimal control problem in (6-9), can be solved using dual decomposition and applying the primal-dual gradient descent approach [96]. To do that, first we form the augmented Lagrangian by adding the constraints to the objective function:
\[
L = \sum_{t=1}^{T} \gamma^{-1} \left( p(t)P_{i,\text{grid}}(t)\Delta t + \sum_{i \in G} (a_i P_{i,G}(t)^2 + b_i P_{i,G}(t) + c_i)\Delta t \right) \\
+ \sum_{i=1}^{T} \lambda_i(t) \left( \sum_{i \in D} P_{i,D}(t) - \sum_{i \in B} P_{i,B}(t) - \sum_{i \in R} P_{i,R}(t) - P_{i,\text{grid}}(t) \right) \\
+ \sum_{i=1}^{T} \mu_{i_D}(t) \left( E_{i_D} - E_{i,\text{full}} - \sum_{s=1}^{T} P_{i,B}(s)\Delta t \right) + \sum_{i=1}^{T} \mu_{i_B}(t) \left( \sum_{s=1}^{T} P_{i,B}(s)\Delta t - E_{i_D} + E_{i,\text{min}} \right) \\
+ \frac{\rho}{2} \sum_{i=1}^{T} \left( \sum_{i \in D} P_{i,D}(t) - \sum_{i \in B} P_{i,B}(t) - \sum_{i \in R} P_{i,R}(t) - P_{i,\text{grid}}(t) \right)^2 \\
+ \frac{\rho}{2} \sum_{i=1}^{T} \sum_{i \in B} \left( \left[ E_{i_D} - E_{i,\text{full}} - \sum_{s=1}^{T} P_{i,B}(s)\Delta t \right]_{[0,\infty]} \right)^2 \\
+ \frac{\rho}{2} \sum_{i=1}^{T} \sum_{i \in B} \left( \left[ \sum_{s=1}^{T} P_{i,B}(s)\Delta t - E_{i_D} + E_{i,\text{min}} \right]_{[0,\infty]} \right)^2 ,
\]

where \( \lambda_i(t) \), \( \mu_{i_D}(t) \), and \( \mu_{i_B}(t) \) are the KKT multipliers for the constraints. The last three terms are penalty functions to convexify the Lagrangian, \( \rho \) is the penalty factor [97], and \( \left[ . \right]_{[x_{\text{min}}, x_{\text{max}}]} \) is the projection operator defined as:

\[
\left[ x \right]_{[x_{\text{min}}, x_{\text{max}}]} = \begin{cases} 
  x_{\text{min}} & x < x_{\text{min}} \\
  x & x_{\text{min}} \leq x \leq x_{\text{max}} \\
  x_{\text{max}} & x \geq x_{\text{max}}
\end{cases}
\]

Note that the local power rating constraints are not included in the Lagrangian as they can be treated as the domain of each control input.

As the cost function is convex and the constraints are affine, the optimal solution is the saddle point of the Lagrangian. Applying the primal-dual gradient descent method [96] to find the saddle point of the Lagrangian, the following iterative equations are resulted:
\[ \forall t \in \{1, ..., T\}: \]
\[
    p_{i,G}^{t+1}(t) = \left[ p_{i,G}^t(t) - \eta \left( y^{t-1} p(t) - \lambda^k(t) - \rho \Delta p^k(t) \right) \right]_{0, p_{i,G,\text{max}}} ,
\]
(6-12)

\[ \forall t \in \{1, ..., T\}: \]
\[
    p_{i,G}^{k+1}(t) = \left[ p_{i,G}^k(t) - \eta \left( 2 a_i p_{i,G}^k(t) \Delta t + b_i \Delta t - \lambda^k(t) - \rho \Delta p^k(t) \right) \right]_{p_{i,G,\text{min}}, p_{i,G,\text{max}}},
\]
(6-13)

\[ \forall t \in \{1, ..., T\}, i \in B: \]
\[
    p_{i,B}^{k+1}(t) = \left[ p_{i,B}^k(t) - \eta \left( -\lambda^k(t) - \rho \Delta p^k(t) - \sum_{l=1}^T \mu_{i}^k(l) \Delta t + \sum_{l=1}^T \mu_{2i}^k(l) \Delta t \right) \right. \\
    + \rho \Delta t \sum_{l=1}^T \sum_{s=1}^l p_{i,B}^k(s) \Delta t - E_{i0} + E_{i,\text{min}} \left. \right]_{0, \infty} \left[ \right]_{p_{i,B,\text{max}}, p_{i,B,\text{max}}},
\]
(6-14)

where \( \Delta p^k(t) \) is the power imbalance for time step \( t \) at the \( k \)-th iteration,

\[ \forall t \in \{1, ..., T\}: \]
\[
    \Delta p^k(t) = \sum_{i \in D} p_{i,D}^k(t) - \sum_{i \in B} p_{i,B}^k(t) - \sum_{i \in R} p_{i,R}^k(t) - \sum_{i \in G} p_{i,G}^k(t) - p_{i,G,\text{grid}}^k(t) ,
\]
(6-15)

and \( \lambda^k(t) \), \( \mu_{1i}^k(t) \), and \( \mu_{2i}^k(t) \) are the KKT multipliers [98] for the constraints for time step \( t \) updated as follows:

\[ \forall t \in \{1, ..., T\}, i \in G_j: \lambda^{k+1}(t) = \lambda^k(t) + \eta \Delta p^k(t) ,
\]
(6-16)

\[ \forall t \in \{1, ..., T\}, i \in B: \mu^{k+1}_{1i}(t) = \left[ \mu^k_{1i}(t) + \eta \left( E_{i0} - E_{i,\text{full}} - \sum_{s=1}^l p_{i,B}^k(s) \Delta t \right) \right]_{0, \infty} \right. \left. \right]_{0, \infty} ,
\]
(6-17)

\[ \forall t \in \{1, ..., T\}, i \in B: \mu^{k+1}_{2i}(t) = \left[ \mu^k_{2i}(t) + \eta \left( \sum_{s=1}^l p_{i,B}^k(s) \Delta t - E_{i0} + E_{i,\text{min}} \right) \right]_{0, \infty} ,
\]
(6-18)
By choosing small enough values for $\eta$ and $\rho$, the iterative algorithm presented by (6-12)-(6-18) converges to the optimal point. However, every iteration requires each device having access to the global information $\Delta P^k(t)$ and $\lambda^k(t)$ for all $t$. According to (6-15), this means each device will need to have access to the decisions made by all other devices at each iteration of the algorithm as well as the forecasted load of all the demand units and forecasted generation of all the renewable units. This may not be feasible nor desirable in a large system due to communications limitations and/or privacy concerns.

### 6.3.2 Distributed estimation of global information with consensus networks

To make the iterative algorithm presented by (6-12)-(6-18) fully distributed such that no device requires having access to the global information or revealing its private information to others, each unit will use an estimation of global variables instead of their true values. Similar to [80] for single step optimization, this estimation will be based on distributed observers that coordinate through consensus networks. Therefore the update equations (6-12) and (6-14) would become:

\[
\forall t \in \{1, \ldots, T\}:
\]

\[
P^{k+1}_{i, \text{grid}}(t) = \left[ P^k_{i, \text{grid}}(t) - \eta \left( \gamma^{t-1} P(t) - \hat{\lambda}^k(t) - \rho \Delta \hat{P}^k(t) \right) \right]_{\min \text{, max}, grid},
\]

\[
(6-19)
\]

\[
\forall t \in \{1, \ldots, T\}, i \in G:
\]

\[
P^{k+1}_{i, G}(t) = \left[ P^k_{i, G}(t) - \eta \left( \gamma^{t-1} \left( 2a_i P^k_{i, G}(t) \Delta t + b_i \Delta t \right) - \hat{\lambda}^k_i(t) - \rho \Delta \hat{P}^k_i(t) \right) \right]_{\min \text{, max}, G},
\]

\[
(6-20)
\]
where $\hat{\lambda}_i(t)$ and $\Delta P^k(t)$ are estimations of unit with index $i$ of $\lambda^k(t)$ and $\Delta P^k(t)$. Each unit will use the information it gets from neighbor units along with its own decisions to update its estimation of global variables as follows:

$$\forall t \in \{1, ..., T\}, i \in B :$$

$$P_{i,B}^{k+1}(t) = P_{i,B}^k(t) - \eta \begin{cases} 
-\hat{\lambda}_i(t) - \rho \Delta \hat{P}_i^k(t) - \frac{\sum_{j=1}^T}{\sum_{j=1}^T} \mu_{i,j}(l) \Delta t + \frac{\sum_{j=1}^T}{\sum_{j=1}^T} \mu_{i,j}(l) \Delta t \n \rho \Delta t \sum_{j=1}^T \left[ E_{i,j} - E_{i,j,full} - \frac{\sum_{x=1}^T}{\sum_{x=1}^T} \Delta P_{i,B}^k(s) \Delta t \right] \n + \rho \Delta t \sum_{j=1}^T \left[ \sum_{x=1}^T \Delta P_{i,j}(s) \Delta t - E_{i,j,0} + E_{i,j,\min} \right]
\end{cases},$$

where $\hat{\lambda}_i(t)$ and $\Delta \hat{P}_i^k(t)$ are estimations of unit with index $i$ of $\lambda^k(t)$ and $\Delta P^k(t)$. Each unit will use the information it gets from neighbor units along with its own decisions to update its estimation of global variables as follows:

$$\forall t \in \{1, ..., T\}, i \in B \cup G \cup R \cup \{\text{grid}\}, j \in \{B, G, R, \text{grid}\} :$$

$$\Delta \hat{P}_i^{k+1}(t) = \Delta \hat{P}_i^k(t) + \sum_{j \in N_i} \omega_{ij} \left( \Delta \hat{P}_j^k(t) - \Delta \hat{P}_i^k(t) \right) + \hat{P}_{i,j}^k(t) - \hat{P}_{i,j}^{k+1}(t).$$

$$\forall t \in \{1, ..., T\}, i \in D :$$

$$\Delta \hat{P}_i^{k+1}(t) = \Delta \hat{P}_i^k(t) + \sum_{j \in N_i} \omega_{ij} \left( \Delta \hat{P}_j^k(t) - \Delta \hat{P}_i^k(t) \right) + \hat{P}_{i,j}^k(t) - \hat{P}_{i,j}^{k+1}(t).$$
\[
\forall t \in \{1, \ldots, T\}, i \in I:
\hat{\lambda}_i^{k+1}(t) = \hat{\lambda}_i^k(t) + \sum_{j \in N_i} w_{ij} \left( \hat{\lambda}_j^k(t) - \hat{\lambda}_i^k(t) \right) + \eta \Delta P_i^k(t),
\]  

(6-24)

where \( N_i \) is the set of indices for neighbors of node \( i \), and \( w_{ij} = w_{ji} \) is the connectivity strength between node \( i \) and node \( j \) and is chosen such that \( 0 \leq w_{ij} < \left( \max_{l=1,\ldots,N} |N_l| \right)^{-1} \) to ensure the convergence of the consensus networks [60], and \( N \) is the total number of nodes.

Note that for renewable units and demand units, \( P^k_{i,R}(t) \) and \( P^k_{i,D}(t) \) are not controllable. They represent the information the unit with index \( i \) has from the predicted generation/demand of the unit at time \( t \) in the \( k \)-th iteration of the algorithm.

Update equations (6-17) and (6-18) remain unchanged as they do not need global information. Equations (6-17)-(6-24) represent the proposed cooperative distributed algorithm to solve the optimal control problem (6-9). The general structure for each distributed controller is shown in Figure 6-2. Each controller uses information from its neighbors’ estimations as well as its own local decisions to update its estimations (distributed observer) which are then used to update the decisions (local decision maker).

The following theorem, shows that under the feasibility conditions, the fixed points of the algorithm presented by (6-17)-(6-24) satisfy KKT conditions of optimality for the optimization problem in (6-9). As the optimization problem is a convex problem with affine constraints, satisfaction of KKT conditions translates into the global optimality [34].

**Theorem 6-1**: If the optimization problem in (6-9) is feasible, then the fixed points of the iterative algorithm presented by (6-17)-(6-24) starting from an initially balanced state (i.e. for all \( i, t \): \( \Delta P^0_i(t) = 0, \Delta P^0_i(t) = 0 \)), satisfy the KKT conditions of optimality for this optimization problem. ■
Proof: Let’s show the fixed point of the iterative algorithm with $P_{i,j}^*(t)$ for $1 \leq t \leq T$, $i \in G \cup B \cup grid$, and $j \in \{G,B,grid\}$. Now, we form the Lagrangian of the optimization problem in (6-9), by considering all the local constraints:

$$L = \sum_{t=1}^{T} \gamma^{t-1} \left( p(t)P_{i,grid}(t)\Delta t + \sum_{i \in G} (a_iP_{i,G}(t)^2 + b_iP_{i,G}(t) + c_i)\Delta t \right)$$

$$+ \sum_{t=1}^{T} \lambda(t) \left( \sum_{i \in D} P_{i,D}(t) - \sum_{i \in B} P_{i,B}(t) - \sum_{i \in R} P_{i,R}(t) - P_{grid}(t) \right)$$

$$+ \sum_{t=1}^{T} \sum_{i \in B} \mu_1(t) \left( E_{i,0} - E_{i,full} - \sum_{s=1}^{t} P_{i,B}(s)\Delta t \right)$$

$$+ \sum_{t=1}^{T} \sum_{i \in B} \mu_\ell(t) \left( \sum_{s=1}^{t} P_{i,B}(s)\Delta t - E_{i,0} + E_{i,min} \right)$$

$$+ \sum_{t=1}^{T} \sum_{i \in B,G} \xi_{i,t}(t) \left( P_{i,j,min} - P_{i,j}(t) \right) + \sum_{t=1}^{T} \sum_{i \in B,G} \xi_{\ell,t}(t) \left( P_{i,j}(t) - P_{i,j,max} \right)$$

$$+ \sum_{t=1}^{T} \sum_{i \in grid} \xi_{i,t}(t) \left( -P_{i,grid}(t) \right) + \sum_{t=1}^{T} \sum_{i \in grid} \xi_{\ell,t}(t) \left( P_{i,grid}(t) - P_{i,grid,max}(t) \right).$$

The KKT conditions of optimality imply that at the optimal point, for $1 \leq t \leq T$, there exist real values $\lambda^*(t)$, and positive values of $\mu_{1i}^*(t)$, $\mu_{\ell i}^*(t)$, $\xi_{i,t}^*(t)$, and $\xi_{\ell,t}^*(t)$ for all $i$ such that the following conditions hold:

1. Derivative of the Lagrangian w.r.t the primal variables is zero:

$$\forall i \in G,t \in \{1,\ldots,T\}: \gamma^{t-1} \left( 2a_iP_{i,G}(t)\Delta t + b_i\Delta t \right) - \hat{\lambda}^*(t) - \xi_{i,t}^*(t) + \xi_{\ell,t}^*(t) = 0,$$  

$$\forall i \in B,t \in \{1,\ldots,T\}:$$

$$-\hat{\lambda}^*(t) - \sum_{l=t}^{T} \mu_{1i}^*(l)\Delta t + \sum_{l=t}^{T} \mu_{\ell i}^*(l)\Delta t - \xi_{i,t}^*(t) + \xi_{\ell,t}^*(t) = 0.$$  

120
\[ \forall t \in \{1, \ldots, T\}: \quad \gamma^{-1} p(t) - \hat{\lambda}^* (t) - \xi^*_D(t) + \xi^*_G(t) = 0. \]  

(6-28)

2. All equality and inequality constraints should be satisfied:

\[ \forall t \in \{1, \ldots, T\}: \sum_{i \in D} P_{i,D}^* (t) - \sum_{i \in B} P_{i,B}^* (t) - \sum_{i \in R} P_{i,R}^* (t) - P_{i,grid}^* (t) = 0, \]  

(6-29)

\[ \forall t \in \{1, \ldots, T\}, i \in B: E_{i,0} - E_{i,full} - \sum_{s=1}^{t} P_{i,B}^* (s) \Delta t \leq 0, \]  

(6-30)

\[ \forall t \in \{1, \ldots, T\}, i \in B: \sum_{s=1}^{t} P_{i,B}^* (s) \Delta t - E_{i,0} + E_{i,\text{min}} \leq 0, \]  

(6-31)

\[ \forall t \in \{1, \ldots, T\}, i \in G: P_{i,G}^* (t) \geq P_{i,\text{min}}, \]  

(6-32)

\[ \forall t \in \{1, \ldots, T\}, i \in G: P_{i,G}^* (t) \leq P_{i,\text{max}}, \]  

(6-33)

\[ \forall t \in \{1, \ldots, T\}, i \in B: P_{i,B}^* (t) \geq P_{i,\text{min}}, \]  

(6-34)

\[ \forall t \in \{1, \ldots, T\}, i \in B: P_{i,B}^* (t) \leq P_{i,\text{max}}, \]  

(6-35)

\[ \forall t \in \{1, \ldots, T\}, i \in \text{grid}: P_{i,\text{grid}}^* (t) \geq 0, \]  

(6-36)

\[ \forall t \in \{1, \ldots, T\}, i \in \text{grid}: P_{i,\text{grid}}^* (t) \leq P_{i,\text{max}}, \]  

(6-37)

3. Complementary slackness holds for dual variables for the inequality constraints:

\[ \forall t \in \{1, \ldots, T\}, i \in B: \mu_{i1} (t) \left( E_{i,0} - E_{i,\text{full}} - \sum_{s=1}^{t} P_i (s) \Delta t \right) = 0, \]  

(6-38)

\[ \forall t \in \{1, \ldots, T\}, i \in B: \mu_{i2} (t) \left( \sum_{s=1}^{t} P_i (s) \Delta t - E_{i,0} + E_{i,\text{min}} \right) = 0, \]  

(6-39)

\[ \forall t \in \{1, \ldots, T\}, i \in G: \xi_{i1} (t) \left( P_{i,G,\text{min}} - P_{i,G}^* (t) \right) = 0, \]  

(6-40)
\[ \forall t \in \{1, \ldots, T\}, i \in G : \xi_{2i}(t) \left( P_{i,G}(t) - P_{i,G_{\max}} \right) = 0, \quad (6-41) \]

\[ \forall t \in \{1, \ldots, T\}, i \in B : \xi_{2i}(t) \left( P_{i,B_{\min}} - P_{i,b}(t) \right) = 0, \quad (6-42) \]

\[ \forall t \in \{1, \ldots, T\}, i \in B : \xi_{2i}(t) \left( P_{i,b}(t) - P_{i,B_{\max}} \right) = 0, \quad (6-43) \]

\[ \forall t \in \{1, \ldots, T\}, i \in \text{grid} : \xi_{1\text{grid}i}(t) \left( -P_{i,\text{grid}}(t) \right) = 0, \quad (6-44) \]

\[ \forall t \in \{1, \ldots, T\}, i \in \text{grid} : \xi_{2\text{grid}i}(t) \left( P_{i,\text{grid}}(t) - P_{i,i_{\max}} \right) = 0. \quad (6-45) \]

In the following we show that each of the conditions (6-26) to (6-45) are satisfied:

**Satisfaction of the global equality constraint (6-29):**

If we sum (6-24) over all devices at the fixed point:

\[ \forall t \in \{1, \ldots, T\} : \]

\[ \sum_{i \in I} \hat{\lambda}^*_i(t) = \sum_{i \in I} \hat{\lambda}^*_i(t) + \sum_{i \in I} \sum_{j \in N_i} w_j \left( \hat{\lambda}^*_j(t) - \hat{\lambda}^*_i(t) \right) + \eta \sum_{i \in I} \Delta \hat{P}^*_i(t). \quad (6-46) \]

The term \[ \sum_{i \in I} \sum_{j \in N_i} w_j \left( \hat{\lambda}^*_j(t) - \hat{\lambda}^*_i(t) \right) \] equals to zero because for all \( i, j \) \[ w_{ij} = w_{ji}. \] Thus:

\[ \forall t \in \{1, \ldots, T\}, i \in I : \sum_{i \in I} \Delta \hat{P}^*_i(t) = 0. \quad (6-47) \]

If we sum (6-22) and (6-23) over all units:

\[ \forall t \in \{1, \ldots, T\} : \]

\[ \sum_{i \in I} \Delta \hat{P}^{k+1}_i(t) = \sum_{i \in I} \Delta \hat{P}^k_i(t) + \sum_{i \in I} \sum_{j \in N_i} w_j \left( \Delta \hat{P}^k_j(t) - \Delta \hat{P}^k_i(t) \right) \]

\[ + \left( \sum_{i \in D} P^{k+1}_{i,D}(t) - \sum_{i \in B} P^{k+1}_{i,B}(t) - \sum_{i \in R} P^{k+1}_{i,R}(t) - P^{k+1}_{i,\text{grid}}(t) \right), \quad (6-48) \]

\[ - \left( \sum_{i \in D} P^k_{i,D}(t) - \sum_{i \in B} P^k_{i,B}(t) - \sum_{i \in R} P^k_{i,R}(t) - P^k_{i,\text{grid}}(t) \right) \]
The term $\sum_{i \in I} \sum_{j \in N_i} w_{ij} \left( \Delta \hat{P}_j^k(t) - \Delta \hat{P}_i^k(t) \right)$ equals to zero because for all $i, j$ $w_{ij} = w_{ji}$. Also based on (6-15), $\Delta P_i^k(t) = \sum_{i \in D} P_{i,D}^k(t) - \sum_{i \in B} P_{i,B}^k(t) - \sum_{i \in R} P_{i,R}^k(t) - P_{i,\text{grid}}^k(t)$. Thus:

$$\forall t \in \{1, \ldots, T\} : \sum_{i \in I} \Delta \hat{P}_i^{k+1}(t) = \sum_{i \in I} \Delta \hat{P}_i^{k}(t) + \Delta P_i^{k+1}(t) - \Delta P_i^{k}(t).$$

(6-49)

Considering the algorithm starts from an initially balanced state (i.e. for all $t$, $\Delta P_i^0(t) = 0$),

$$\forall t \in \{1, \ldots, T\} :$$

$$\sum_{i \in I} \Delta \hat{P}_i^{1}(t) = \sum_{i \in I} \Delta \hat{P}_i^{0}(t) + \Delta P_i^{1}(t) - \Delta P_i^{0}(t) = \Delta P_i^{1}(t),$$

$$\sum_{i \in I} \Delta \hat{P}_i^{2}(t) = \sum_{i \in I} \Delta \hat{P}_i^{1}(t) + \Delta P_i^{2}(t) - \Delta P_i^{1}(t) = \Delta P_i^{2}(t),$$

$$\sum_{i \in I} \Delta \hat{P}_i^{3}(t) = \sum_{i \in I} \Delta \hat{P}_i^{2}(t) + \Delta P_i^{3}(t) - \Delta P_i^{2}(t) = \Delta P_i^{3}(t),$$

$$\vdots$$

By continuing this process:

$$\forall t \in \{1, \ldots, T\} : \sum_{i \in I} \Delta \hat{P}_i^{k}(t) = \Delta P_i^{k}(t).$$

(6-51)

At the fixed point:

$$\forall t \in \{1, \ldots, T\} : \sum_{i \in I} \Delta \hat{P}_i^{*}(t) = \Delta P_i^{*}(t).$$

(6-52)

Based on (6-47), at the fixed point $\sum_{i \in I} \Delta \hat{P}_i^{*}(t) = 0$. Thus:

$$\forall t \in \{1, \ldots, T\} : \Delta P_i^{*}(t) = \sum_{i \in D} P_{i,D}^{*}(t) - \sum_{i \in B} P_{i,B}^{*}(t) - \sum_{i \in R} P_{i,R}^{*}(t) - P_{i,\text{grid}}^{*}(t) = 0.$$  

(6-53)

Therefore, at the fixed point, (6-29) is satisfied.

**Satisfaction of conditions involving $P_{i,\text{grid}}(t)$ (conditions (6-28), (6-36), (6-37), (6-44), and (6-45)):**

Based on the property of the consensus networks, the consensus variables converge to the average of their summation. Therefore:
\[
\forall i \in I, \forall t \in \{1, \ldots, T\} : \lim_{k \to \infty} \Delta \hat{P}_i^k (t) = \frac{\sum_{i=1}^{\infty} \Delta \hat{P}_i^k (t)}{|I|}.
\] 

(6-54)

Thus, based on (6-47), at the fixed point:

\[
\forall i \in I, \forall t \in \{1, \ldots, T\} : \Delta \hat{P}_i^* (t) = 0.
\] 

(6-55)

At the fixed point, update equation (6-19) can be written as:

\[
\forall t \in \{1, \ldots, T\} : P_{i, \text{grid}}^* (t) = \left[ P_{i, \text{grid}}^* (t) - \eta \left( \gamma^{i-1} p(t) - \hat{\lambda}_i^* (t) - \rho \Delta \hat{P}_i^* (t) \right) \right]_{[0, P_{i, \text{grid max}}^*]},
\] 

(6-56)

First of all, it is evident that because of the projection $0 \leq P_{i, \text{grid}}^* (t) \leq P_{i, \text{grid max}}^*$ conditions (6-36) and (6-37) are always satisfied. For each $t \in \{1, \ldots, T\}$, three possibilities arise:

1. $0 < P_{i, \text{grid}}^* (t) < P_{i, \text{grid max}}^*$: In this case, (6-56) can be written as:

\[
P_{i, \text{grid}}^* (t) = P_{i, \text{grid}}^* (t) - \eta \left( \gamma^{i-1} p(t) - \hat{\lambda}_i^* (t) - \rho \Delta \hat{P}_i^* (t) \right),
\]

(6-57)

Based on (6-55), $\Delta \hat{P}_i^* (t) = 0$. So:

\[
\gamma^{i-1} p(t) - \hat{\lambda}_i^* (t) = 0.
\]

(6-58)

Therefore, by choosing $\xi_{i, \text{grid}}^* (t) = \xi_{i, \text{grid}}^* (t) = 0$, conditions (6-28), (6-44) and (6-45) are satisfied.

2. $P_{i, \text{grid}}^* (t) = 0$: In this case based on (6-56),

\[
P_{i, \text{grid}}^* (t) - \eta \left( \gamma^{i-1} p(t) - \hat{\lambda}_i^* (t) - \rho \Delta \hat{P}_i^* (t) \right) \leq 0,
\]

(6-59)

As $P_{i, \text{grid}}^* (t) = 0$, and $\eta > 0$, and $\Delta \hat{P}_i^* (t) = 0$:

\[
\gamma^{i-1} p(t) - \hat{\lambda}_i^* (t) \geq 0,
\]

(6-60)
Therefore, by choosing \( \xi_{\text{grid}}(t) = \gamma^{t-1} p(t) - \hat{\lambda}^*(t) \), \( \xi_{\text{grid}}^*(t) = 0 \), conditions (6-28), (6-44) and (6-45) are satisfied.

3. \( P_{i,\text{grid}}^*(t) = P_{i,\text{grid max}}^* \): In this case based on (6-56),

\[
P_{i,\text{grid}}^*(t) - \eta \left( \gamma^{t-1} p(t) - \hat{\lambda}^*(t) - \rho \Delta \bar{P}(t) \right) \geq P_{i,\text{grid max}}^*,
\]

(6-61)

As \( P_{i,\text{grid}}^*(t) = P_{i,\text{grid max}}^* \), and \( \eta > 0 \),

\[
\gamma^{t-1} p(t) - \hat{\lambda}^*(t) \leq 0,
\]

(6-62)

Therefore, by choosing \( \xi_{\text{grid}}(t) = 0 \), \( \xi_{\text{grid}}^*(t) = -\gamma^{t-1} p(t) + \hat{\lambda}^*(t) \), conditions (6-28), (6-44) and (6-45) are satisfied.

**Satisfaction of conditions involving \( P_{i,G}(t) \) (conditions (6-26), (6-32), (6-33), (6-40), and (6-41)):**

The line of proof is very similar to the satisfaction of conditions involving \( P_{i,\text{grid}}(t) \) by looking at the update equation (6-20) at the fixed point.

**Satisfaction of conditions involving \( P_{i,B}(t) \) (conditions (6-27), (6-30), (6-31), (6-34), (6-35), (6-38), (6-39), (6-42), and (6-43)):**

At the fixed point, we can write the update equation (6-17) as:

\[
\forall t \in \{1, \ldots, T\}, i \in B: \mu_{i,i}^*(t) = \begin{bmatrix} \mu_{i,i}^*(t) + \eta \left( E_{i,0} - E_{i,\text{full}} - \sum_{s=1}^{t} P_{i,B}^*(s) \Delta t \right) \end{bmatrix}_{[0,\infty]}. \]

(6-63)

Two possibilities arise:

1. \( \mu_{i,i}^*(t) > 0 \): In this case, (6-63) can be written:
\[ \forall t \in \{1, \ldots, T\}, i \in B: \mu^*_i(t) = \mu^*_i(t) + \eta \left( E_{i0} - E_{i,full} - \sum_{s=1}^{t} P_{i,B}^*(s) \Delta t \right), \quad (6-64) \]

which results in:

\[ E_{i0} - E_{i,full} - \sum_{s=1}^{t} P_{i,B}^*(s) \Delta t = 0. \quad (6-65) \]

Therefore, conditions (6-30) and (6-38) are satisfied.

2. \( \mu^*_i(t) = 0 \): In this case, based on (6-63):

\[ \forall t \in \{1, \ldots, T\}, i \in B: \mu^*_i(t) + \eta \left( E_{i0} - E_{i,full} - \sum_{s=1}^{t} P_{i,B}^*(s) \Delta t \right) \leq 0, \quad (6-66) \]

As \( \mu^*_i(t) = 0 \), and \( \eta > 0 \):

\[ \forall t \in \{1, \ldots, T\}, i \in B: E_{i0} - E_{i,full} - \sum_{s=1}^{t} P_{i,B}^*(s) \Delta t \leq 0, \quad (6-67) \]

Therefore, at the fixed point, conditions (6-30) and (6-38) are satisfied.

Similar arguments can be made based on update equation (6-18) to show that at the fixed point conditions (6-31) and (6-39) are satisfied.

At the fixed point, we can write the update equation (6-21) as:

\[ \forall t \in \{1, \ldots, T\}, i \in B: \]

\[ P_{i,B}^*(t) = \begin{bmatrix} -\hat{d}_i^*(t) - \rho \Delta t \hat{P}_{i,B}^*(t) - \sum_{l=1}^{T} \mu^*_i(l) \Delta t + \sum_{l=1}^{T} \mu^*_{i,l}(l) \Delta t \\ -\rho \Delta t \sum_{l=1}^{T} \left[ E_{i0} - E_{i,full} - \sum_{s=1}^{t} P_{i,B}^*(s) \Delta t \right]_{[0, \infty]} \\ + \rho \Delta t \sum_{l=1}^{T} \left[ \sum_{s=1}^{t} P_{i,B}^*(s) \Delta t - E_{i0} + E_{i,\text{min}} \right]_{[0, \infty]} \end{bmatrix}_{P_{i,B}^*, P_{i,B}^*, P_{i,B}^*} \quad , (6-68) \]
First of all based on the projection, at the fixed point, conditions \((6-34)\), \((6-35)\) are always satisfied. We already showed that at the fixed point conditions \((6-30)\) and \((6-31)\) are satisfied. Therefore:

\[
\forall l \in \{1, \ldots, T\} : \left[ E_{i0} - E_{i, \text{full}} - \sum_{s=1}^{T} \mu_s^* (s) \Delta t \right] = 0 ,
\]

\[
\forall l \in \{1, \ldots, T\} : \left[ \sum_{s=1}^{T} P_{i,B}^* (s) \Delta t - E_{i0} + E_{i, \text{min}} \right] = 0 .
\]

(6-69)

Considering that \(\Delta P_i^* (t) = 0\), we can write \((6-68)\) as:

\[
\forall t \in \{1, \ldots, T\}, i \in B : \quad P_{i,B}^* (t) = \left[ P_{i,B}^* (t) - \eta \left( -\tilde{\lambda}_i^* (t) - \sum_{l=t}^{T} \mu_{i1}^* (l) \Delta t + \sum_{l=t}^{T} \mu_{i2}^* (l) \Delta t \right) \right]_{[P_{i,B_{\text{min}}}, P_{i,B_{\max}}]} .
\]

(6-70)

Now, three possibilities arise:

1. \(P_{i,B_{\text{min}}} < P_{i,B}^* (t) < P_{i,B_{\max}}\) : In this case \((6-70)\) can be written as:

\[
\forall t \in \{1, \ldots, T\}, i \in B : \quad P_{i,B}^* (t) = P_{i,B}^* (t) - \eta \left( -\tilde{\lambda}_i^* (t) - \rho \Delta \tilde{P}_i^* (t) - \sum_{l=t}^{T} \mu_{i1}^* (l) \Delta t + \sum_{l=t}^{T} \mu_{i2}^* (l) \Delta t \right) ,
\]

(6-71)

which results in:

\[
\forall t \in \{1, \ldots, T\}, i \in B : \quad -\tilde{\lambda}_i^* (t) - \rho \Delta \tilde{P}_i^* (t) - \sum_{l=t}^{T} \mu_{i1}^* (l) \Delta t + \sum_{l=t}^{T} \mu_{i2}^* (l) \Delta t = 0 .
\]

(6-72)

Now by choosing \(c_{i1} (t) = c_{i2} (t) = 0\), conditions \((6-27)\), \((6-42)\), and \((6-43)\) are satisfied.

2. \(P_{i,B}^* (t) = P_{i,B_{\max}}\) : In this case based on \((6-70)\):

\[
\forall t \in \{1, \ldots, T\}, i \in B : \quad P_{i,B}^* (t) - \eta \left( -\tilde{\lambda}_i^* (t) - \sum_{l=t}^{T} \mu_{i1}^* (l) \Delta t + \sum_{l=t}^{T} \mu_{i2}^* (l) \Delta t \right) \geq P_{i,B_{\max}} (t) ,
\]

(6-73)

Considering that \(P_{i,B}^* (t) = P_{i,B_{\max}} (t)\), and \(\eta > 0\) :
\begin{align}
\forall t \in \{1, \ldots, T\}, i \in B: & \quad -\lambda_i^* (t) - \sum_{l=1}^{T} \mu_{1i}^* (l) \Delta t + \sum_{l=1}^{T} \mu_{2i}^* (l) \Delta t \leq 0. 
\end{align}

(6-74)

Therefore, by choosing \( \xi_{i,j}^* (t) = 0 \) and \( \xi_{2,j}^* (t) = + \lambda_i^* (t) + \sum_{l=1}^{T} \mu_{1i}^* (l) \Delta t - \sum_{l=1}^{T} \mu_{2i}^* (l) \Delta t \geq 0 \), conditions (6-27), (6-42), and (6-43) are satisfied.

3. \( P_{i,B}(t) = P_{i,B_{\text{min}}} \): In this case based on (6-70):

\begin{align}
\forall t \in \{1, \ldots, T\}, i \in B: & \quad P_{i,B}^* (t) - \eta \left( -\lambda_i^* (t) - \sum_{l=1}^{T} \mu_{1i}^* (l) \Delta t + \sum_{l=1}^{T} \mu_{2i}^* (l) \Delta t \right) \leq P_{i,B_{\text{min}}} (t),
\end{align}

(6-75)

Considering that \( P_{i,B}^* (t) = P_{i,B_{\text{min}}} (t) \), and \( \eta > 0 \):

\begin{align}
\forall t \in \{1, \ldots, T\}, i \in B: & \quad -\lambda_i^* (t) - \sum_{l=1}^{T} \mu_{1i}^* (l) \Delta t + \sum_{l=1}^{T} \mu_{2i}^* (l) \Delta t \geq 0.
\end{align}

(6-76)

Therefore, by choosing \( \xi_{i,j}^* (t) = -\lambda_i^* (t) - \sum_{l=1}^{T} \mu_{1i}^* (l) \Delta t + \sum_{l=1}^{T} \mu_{2i}^* (l) \Delta t \) and \( \xi_{2,j}^* (t) = 0 \), conditions (6-27), (6-42), and (6-43) are satisfied.

We showed that at the fixed points of the proposed algorithm, all the KKT conditions are satisfied and therefore the theorem is proven. ■

6.4 Numerical Simulations

6.4.1 Demonstrate the application of the algorithm

To illustrate that the proposed approach finds the global optimal schedule in a fully distributed way, we apply it on the 5-bus Garver power network [81] shown in Figure 6-3 and benchmark the results against Quadratic Programming (QP) as a centralized method. Multiple energy devices are considered to be connected at each bus of the system. Each energy device as well as each bus in the system is equipped with a communications node. The communications network among the nodes is assumed to be similar to the physical power network. The network
is connected to the grid from bus 1 and can buy power from the utility with Real Time (RT) prices.

Table 6-1 shows the specifications for the type of device/load connected to each node of the network.

For the wind/photovoltaic (PV) generation online profiles\(^2\) are used and scaled to residential rating. For loads, we have used typical residential load provided by XCEL Energy\(^3\). For RT prices, PJM\(^4\) provided data is used. We want to optimally schedule the resources for the next two hours with 15 min intervals. Therefore based on the algorithm formulations \(\Delta t = 0.25 \text{ h} \) and \(T = 8\).

\(^3\) [http://www.xcelenergy.com/staticfiles/xe/Corporate/Corporate%20PDFs/AppendixD-Hourly_Load_Profiles.pdf](http://www.xcelenergy.com/staticfiles/xe/Corporate/Corporate%20PDFs/AppendixD-Hourly_Load_Profiles.pdf)
The initial energy stored in each storage device is considered to be 20% of the full capacity and it is required that the storage devices at all the times have at least 20% of charge. The profiles for the loads, renewable generation and prices are shown in Figure 6-4.

Table 6-1. Node Types in the Network

<table>
<thead>
<tr>
<th>#</th>
<th>Specification</th>
<th>#</th>
<th>Specification</th>
<th>#</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>12 kW peak</td>
<td>7</td>
<td>Wind Turbine: 15 kW</td>
<td>4</td>
<td>$P_{\text{max}} = 20 \text{ kW}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>16</td>
<td>PV Panel: 20 kW</td>
<td></td>
<td>$a_i = 0.0031 \text{ ¢/kW}^{2}\text{h}$</td>
</tr>
<tr>
<td>8</td>
<td>16 kW peak</td>
<td>6</td>
<td>Storage: 32 kWh/10 kW</td>
<td>10</td>
<td>$b_i = 6.60 \text{ ¢/kWh}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12</td>
<td>Storage: 15 kWh/4.5 kW</td>
<td></td>
<td>$P_{\text{max}} = 30 \text{ kW}$</td>
</tr>
<tr>
<td>11</td>
<td>24 kW peak</td>
<td>14</td>
<td>Storage: 32 kWh/10 kW</td>
<td></td>
<td>$a_i = 0.0074 \text{ ¢/kW}^{2}\text{h}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$b_i = 5.20 \text{ ¢/kWh}$</td>
</tr>
<tr>
<td>15</td>
<td>36 kW peak</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 6-4. Input profiles for the case study
The distributed algorithm is applied with sampling time of 1 ms for each distributed controller (i.e. each iteration of the algorithms takes 1 ms).

Figure 6-5 shows the progress in dual variable estimations as well as the discharging rate decisions for the eight time steps by the DESDs. It can be seen that as the dual estimations converge, the DESD decisions also converge.

The distributed algorithm is applied with sampling time of 1 ms for each distributed controller (i.e. each iteration of the algorithms takes 1 ms). Figure 6-5 shows the progress in dual variable estimations as well as the discharging rate decisions for the eight time steps by the DESDs. It can be seen that as the dual estimations converge, the DESD decisions also converge. Figure 6-6 shows the resultant schedule and the amount of energy stored in each storage device. Based on the resultant schedule, when the price at the grid side is low (time steps 1 to 4), the DESDs are charging and power is drawn from the grid. When the energy becomes expensive (time steps 5 to 8) the DESDs are discharging and no power is drawn from the grid. We can also see that at all time steps the energy stored is always greater than 20%
of the full capacity as required. Figure 6-7 shows the convergence of the objective value (blue line) to the global optimum (drawn as a red line) found by centralized quadratic programming.

### 6.4.2 Robustness to communication link failures

One of the advantages of the proposed approach is that unlike centralized approaches, if a communication link is lost, the algorithm is still able to operate and find the optimal schedule provided that the communications network remains connected. To illustrate this property, we analyze the behavior of the algorithm under different single link failure scenarios. We consider the scenario where one link is failed at \( t = 0.1 \) sec while the algorithm is running. When link \( (i, j) \) fails, the connectivity strength between node \( i \) and node \( j \) \((w_{ij} \) and \( w_{ji} \) in the formulations) becomes zero. Figure 6-8 shows the progress of the objective value without failure, under
The convergence time is measured as the time when the summation of L1 norms of constraint violations as well as changes in dispatch decisions becomes less than 0.01. We can see that in both cases the scheduling algorithm converges to the optimal value, but with different convergence rates. The same experiment is repeated for all the links. Figure 6-9 shows the objective value and the convergence time under the failure of each of the links. It can be seen that the algorithm converges to the optimal value in all the cases but with variable convergence rates.

Now we study cases where the communication link failures make the topology disconnected. We simulate two scenarios:

1. Scenario 1: Links (1, 2) and (9, 13) fail at $t = 0.1$ sec.
2. Scenario 2: Links (1, 2) and (9, 13) fail at $t = 0.1$ sec, and link (1, 2) recovers at $t = 0.5$ sec.

Figure 6-10 shows the progress of the objective value for each scenario. In scenario 1, that the communications topology disconnects and remains disconnected, the algorithm converges
to a suboptimal value that is away from the optimal value. However, in scenario 2, as the communication topology becomes connected again due to the recovery of link (1, 2), the algorithm is able to converge to the global optimum.
To validate the convergence of the algorithm to the global optimum in general, 100 random scenarios are generated and the objective values reached by the proposed algorithm are compared with the objective value reached by applying the centralized QP. For each scenario, the experiment is performed as follows:

1. Choose a random number smaller than 100 to be the total number of nodes.
2. Randomly assign one node to represent the grid.
3. Randomly assign the remaining nodes to be a load, PV panel, wind turbine, storage device or conventional generator.
4. Randomly choose the parameters of each device in the ranges shown in the following table.

Figure 6-10. Objective value under link failures that cause communications topology to be disconnected

6.4.3 Validate the convergence of the algorithm to global optimum using Monte Carlo simulations

(a) Links (1, 2) and (9, 13) fail at $t = 0.1$ sec

(b) Links (1, 2) and (9, 13) fail at $t = 0.1$ sec and link (1, 2) recovers at $t = 0.5$ sec
5. Use the load/PV and wind turbine profiles of section 4-1 and scale them for each load/PV and wind turbine based on the $P_{\text{max}}$ assigned to that device.

6. Generate a random communications network among devices.

7. Apply the proposed distributed algorithm as well as the centralized QP on this system and record the objective values. Stop the distributed scheduling algorithm when the summation of L1 norms of all the constraints as well as the relative change in charge/discharge decisions is less than 0.01

Figure 6-11 shows the histogram of the percentage difference in the objective value between the proposed distributed algorithm and the centralized QP in 100 random scenarios. The maximum percentage difference is less than 0.03%. The 98% confidence interval for the mean of the percentage differences is $[-0.012\%, 0.024\%]$ which includes zero. This result

<table>
<thead>
<tr>
<th>Table 6-2. Range of the random parameters for Monte Carlo simulations</th>
</tr>
</thead>
<tbody>
<tr>
<td>PV</td>
</tr>
<tr>
<td>Wind Turbine</td>
</tr>
<tr>
<td>Micro Turbine</td>
</tr>
<tr>
<td>Storage</td>
</tr>
<tr>
<td>Load</td>
</tr>
</tbody>
</table>
indicates that there is no significant difference between the objective values reached by proposed distributed algorithm and centralized QP.

6.5 Conclusion

A fully distributed algorithm is introduced for optimal scheduling of dispatchable distributed energy resources and storage devices in smart grids. The algorithm combines the notion of primal-dual gradient descent from optimization theory with consensus networks. Each dispatchable device in the system finds its optimal schedule solely through peer-to-peer coordination with its neighbors. The only information being exchanged among devices are the estimation of global variables and no device needs to disclose information regarding its generation/demand with others. The application of the algorithm is illustrated on a 5-bus network and shown that the algorithm is robust against communication link failures provided that the communication topology remains connected or reconnects after being disconnected. The convergence of the algorithm to the global optimum is validated through Monte Carlo simulations.
CHAPTER 7. CONCLUSION

This thesis presented our work in the area of cooperative distributed energy management for future smart grids. Five algorithms were developed. First algorithm (CDPDM) was developed for demand side management of large scale PHEV/PEV charging considering limits on the total available power. The second algorithm (IWC) was proposed for energy management of a grid populated with distributed generators and responsive demands. The third algorithm (DRPCG) was proposed to manage PHEV/PEV charging considering heterogeneity in the generation side based on the framework of the IWC algorithm. The fourth algorithm (AIWC) was proposed to relieve the synchronous update requirement from IWC algorithm. Finally, the fifth algorithm (CoDES) was proposed to do distributed scheduling in grids consisting of dispatchable generation units, renewables, and storage devices. While all the first four proposed algorithms were single step optimization solvers, the fifth algorithm is a multi-step optimization solver taking the future behavior of the system into account. All the proposed algorithms work solely based on local communications and computational capabilities of distributed devices and do not require a control center. The main advantages of the proposed solutions can be summarized as:

- Relieve the computational and communications burden from the control center,
- More robust to single point of failure compared to centralized solutions,
- More robust to communication failures,
- Scalable with increasing the number of distributed devices,
- Increased privacy as distributed units do not have to disclose their private information to other devices.
To employ these algorithms in real-world applications, practical issues such as power losses, energy efficiencies, line constraints, grid dynamics, and uncertainties due to inaccurate forecasts, noise, measurement errors, communication imperfections, and security issues of distributed algorithms [99]–[101] need to be considered. Therefore, following three tracks are proposed as the future research directions, based on the proposed work on this thesis:

1. Extend the proposed algorithms to consider more physical constraints such as the line losses, energy efficiencies, voltage constraints, and power flow dynamics.

2. Extend the proposed algorithms to consider uncertainties in the system parameters, in load forecast, and in renewable generations. The newly developed algorithms should be able to solve stochastic optimization problems.

3. Implement the algorithms in hardware platforms and analyze their real time behavior.
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