ABSTRACT

GAITHER, JEFFREY BEAU SELLERS. The Fixed Points of a Seasonal Model of Population Infectives. (Under the direction of John Franke.)

We model the spread of epidemics among insect populations. The mapping $F_t = \gamma (1 - e^{-\frac{\alpha I}{N_t}})(N_t - I) + \gamma \sigma I$ iterates on the current number of infectants to produce the number of infectants in the next time-period. The value $N_t$ is the current population, and it is known that population follows a globally attracting cycle $N_1 \ldots N_p$, which represents the population at various times of the year. Thus, the function $F = F_p \circ \cdots \circ F_1$ maps infectants to infecants on a month-to-month or season-to-season basis. We show that for $p = 2$, $F$ has only one attractor. We also show that for any $F$ there is $\alpha_0$ such that for any $\alpha > \alpha_0$, $F$ has only one attractor. We give an example of multiple attractors in the $p = 4$ case, and provide a means by which the composition $F$ can be represented as a composition of functions which are all scalar multiples of $F_1$. 
THE FIXED POINTS OF A SEASONAL MODEL OF POPULATION INFECTIVES

by

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1 Introduction

In this paper we consider a finite sequence of mappings \( f_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \),

\[
f_t(N, I) = \begin{pmatrix} G_t(N) \\ H(N, I) \end{pmatrix} = \begin{pmatrix} k_t(1 - \gamma) + \gamma N \\ \gamma(1 - e^{-\gamma N})(N - I) + \gamma \sigma I \end{pmatrix}
\]

which will generally be composed in the form \( f_p \circ \cdots \circ f_1 \). These mappings, discussed extensively by Franke and Yakubu in their paper "Discrete Time SIS Epidemic Model in a Seasonal Environment," model the spread of epidemics among insect populations. The function \( G_t \) operates on the current population \( N \) to produce the population in the next time-period, while \( H \) operates on both the current population \( N \) and the current number of infectants \( I \) to produce the number of infectants in the next time-period. (Naturally we require that \( I \leq N \).) The parameter \( k_t \) is \( p \)-periodic and is supposed to model seasonality. Thus, we might have a different mapping \( f_t \) for each month of the year, and \( f_{12} \circ f_{11} \circ \cdots \circ f_1 \) would map population and infectants from one year to the next on a month-to-month basis.

The parameters \( \gamma, \sigma \) and \( \alpha \) are constants which depend on the environment and the nature of the epidemic. \( \gamma \) is, quite simply, the probability that a given insect will survive until the next time-period. (The term "survive" is unrelated to the disease, which is assumed to be nonlethal.) The parameter \( \sigma \), on the other hand, represents the probability that an infected individual will remain infected, i.e. that in the next time-period the insect will still have the disease. Since both \( \gamma \) and \( \sigma \) are probabilities, we have \( \gamma, \sigma \in (0, 1) \). The final parameter \( \alpha \), called the "transmission constant," measures the communicability of the disease. It is positive but can be unboundedly large.

The ultimate focus of this paper is to consider the \( p \)-cycles of the functions \( f_t \). (We consider \( p \)-cycles for the simple reason that there are \( p \) values of \( k_t \) and therefore \( p \) functions \( f_t \).) A cycle is precisely a set of ordered pairs \((N_t, I_t)\) such that \( f_t(N_t, I_t) = (N_{t+1}, I_{t+1}) \) for all \( t \in [1, p - 1] \), and \( f_{p-1}(N_{p-1}, I_{p-1}) = (N_0, I_0) \).

Franke and Yakubu showed that the population-function \( G_t \) has a globally attracting \( p \)-cycle \( N_0 \ldots N_{p-1} \). Given this, we now present a theorem on the cycles of \( f_t \).

**Theorem 1** The set \( (N_0, I_0) \ldots (N_{p-1}, I_{p-1}) \) is a cycle of \( f_t \) iff \( N_0 \ldots N_{p-1} \) is the globally attracting cycle of \( G_t \) and \( I_0 \ldots I_{p-1} \) is a cycle of the functions \( H(N_t, I) \).

**PROOF:** Suppose \( (N_0, I_0) \ldots (N_{p-1}, I_{p-1}) \) is a cycle of \( f_t \). Then for every \( t \) we have \( G_t(N_t) = N_{t+1} \), so \( N_0 \ldots N_{p-1} \) is a \( p \)-cycle of \( G_t \), and therefore must be the global cycle. Similarly, we must have \( H(N_t, I_t) = I_{t+1} \) for every \( t \), so \( I_0 \ldots I_{p-1} \) is a \( p \)-cycle of \( H(N_t, I) \). Conversely, suppose we have \( N_0 \ldots N_{p-1} \) as the global cycle of \( G_t \), and that we have \( I_0 \ldots I_{p-1} \) as a cycle of \( H(N_t, I) \). Then for any \( t \) we have \( f_t(N_t, I_t) = (G_t(N_t), H(N_t, I_t)) = (N_{t+1}, I_{t+1}) \), so \( (N_0, I_0) \ldots (N_{p-1}, I_{p-1}) \) is a cycle of \( f_t \). QED

This theorem provides the basis for our entire paper. It tells us that, to find the cycles of \( f_t \), we need only find the cycles of the functions \( H(N_t, I) \) and tack them onto the global population.
cycle. Intuitively, this theorem says that the infection-map $H_t$ will eventually become, in effect, a function of one variable, because the variable $N$ will converge to some point $N_t$ in the globally attracting cycle.

The great majority of our paper, indeed, will deal with the functions $H(N_t, I)$. However, we first present a few results about the population-functions $G_t$.

2 The Population Mapping $G_t$

Franke and Yakubu showed that the population mapping $G_t$ lives on a globally attracting cycle whose first point is $N_0 = \frac{1 - \gamma}{1 - \gamma^p} \sum_{i=0}^{p-1} k_i \gamma^{p-1-i}$. We now present two results which expand on this conclusion.

**Theorem 2** For any point $N_t$ in the globally attracting $p$-cycle of $G_t$, we have $N_t = \frac{1 - \gamma}{1 - \gamma^p} \sum_{i=0}^{p-1} k_i + t \gamma^{p-1-i}$, where the index of $k$ is computed modulo $p$.

**Proof:**

We have $N_0 = \frac{1 - \gamma}{1 - \gamma^p} \sum_{i=0}^{p-1} k_i \gamma^{p-1-i}$. Suppose for induction that for some $t < p$ we have $N_t = \frac{1 - \gamma}{1 - \gamma^p} \sum_{i=0}^{p-1} k_i + \gamma^{p-1-i}$. Then $N_{t+1} = k_t (1 - \gamma) + \gamma N_t = k_t (1 - \gamma) + \frac{1 - \gamma}{1 - \gamma^p} \sum_{i=0}^{p-1} k_i + t \gamma^{p-1-i} = \frac{1 - \gamma}{1 - \gamma^p} (k_t - k_t \gamma^p + \sum_{i=0}^{p-1} k_i + t \gamma^{p-1-i}) = \frac{1 - \gamma}{1 - \gamma^p} (k_t + \sum_{i=1}^{p-1} k_i + t \gamma^{p-1-i}) = \frac{1 - \gamma}{1 - \gamma^p} \sum_{i=1}^{p} k_i + (t+1) \gamma^{p-1-i}$.

Then by induction we have $N_t = \frac{1 - \gamma}{1 - \gamma^p} \sum_{i=0}^{p-1} k_i + t \gamma^{p-1-i}$ for $0 \leq t \leq p$. QED

This theorem is significant; it gives us an easy way of calculating the value of any point $N_t$ in the globally attracting population cycle $N_0 \ldots N_{p-1}$. Next we present a theorem which bounds the possible ratios of the $N_t$-values.

**Theorem 3** Let $N_t$, $N_s$ be any points in the $p$-cycle of $G_t$. Then $\gamma^{p-1} \leq \frac{N_t}{N_s} \leq \frac{1}{\gamma^{p-1}}$. 

2
PROOF: We have
\[\frac{N_t}{N_s} = \frac{1 - \gamma}{1 - \gamma^p} \sum_{i=0}^{p-1} k_{i+s} \gamma^{p-1-i} \left( \sum_{i=0}^{p-1} k_{i+t} \gamma^{p-1-i} \right).\]

Now we make use of the following handy fact: given \(x_1 \ldots x_n\) and \(y_1 \ldots y_n\), if \(A \leq \frac{x_t}{y_t} \leq B\) for every \(t\), then we have \(A \leq \frac{x_1 + \ldots + x_n}{y_1 + \ldots + y_n} \leq B\). Note that for every \(i + t\) on \([0, k - 1]\) we have
\[
\gamma^{p-1} \leq \frac{k_{i+t} \gamma^{p-1-i}}{k_{(i+t-s)+s} \gamma^{p-1-(i+t-s)}} = \frac{\gamma^{p-1-i}}{\gamma^{p-1-(i+t-s)}} \leq \frac{1}{\gamma^{p-1}}
\]
(where the bottom exponent and subscript are computed modulo \(p\)), so
\[
\gamma^{p-1} \leq \frac{1}{\gamma^{p-1}} \frac{\sum_{i=0}^{p-1} k_{i+t} \gamma^{p-1-i}}{\sum_{i=0}^{p-1} k_{i+s} \gamma^{p-1-i}} = \frac{N_t}{N_s} \leq \frac{1}{\gamma^{p-1}}. \text{ QED}
\]

This is an extremely significant result. It tells us that the points on the population-cycle cannot vary by arbitrarily large gaps, but can differ by a factor of at most \(\gamma^{p-1}\). In other words, we cannot have 5 insects in the winter and 1,000,000,000 insects in the summer, unless our value of \(\gamma\) is exceptionally small.

This concludes our work on the population mapping \(G_t\). The remainder of our work will deal with the infection mapping \(H\).

### 3 The Infection Mapping \(F_t\)

As we mentioned in the introduction, the existence of a cycle in \(f_t\) is equivalent to the existence of a cycle in the functions \(H(N_t, I)\), where \(N_t\) is a point in the globally attracting cycle of \(G_t\). Our focus for the remainder of this paper will be the functions \(H(N_t, I)\), and we will use the label \(F_t\) to denote these functions. Thus

\[F_t(I) = H(N_t, I) = \gamma(1 - e^{-\alpha I / N_t})(N_t - I) + \gamma \sigma I.\]

We call this set of functions \(F_t\), identical except for the cycle-value \(N_t\), a set of infection functions. We also define the composition \(F = F_p \circ \cdots \circ F_1\). (Henceforth we index the population cycle \(N_1 \ldots N_p\), rather than \(N_0 \ldots N_{p-1}\).)
Our first step will be to prove a number of useful results which apply to any set of infection functions.

**Theorem 4** Let $F_1, \ldots, F_p$ be a set of infection functions. Then

1. $F_t([0, N_t]) \subset [0, N_{t+1}]$ for $t = 1 \cdots p - 1$, and $F_p([0, N_p]) \subset [0, N_1]$.
2. $F_t$ is continuous and differentiable throughout its domain.
3. $F_t(0) = 0$
4. $F_t(N_t) < N_t$
5. $F'_t(I) = \gamma e^{-\frac{\alpha I}{N_t}}(\alpha - \frac{\alpha I}{N_t} + 1) - \gamma + \gamma \sigma$
6. $F^{(n)}_t(I) = \gamma(-\frac{\alpha}{N_t})^{n-1} e^{-\frac{\alpha I}{N_t}}(\alpha - \frac{\alpha I}{N_t} + n)$ for all $n > 1$, where $F^{(n)}_t$ is the $n$th derivative of $F_t$.
7. $F'_t(N_t) > -1$
8. $F''_t(I) < 0$ for all $I$ in the domain.

**PROOF:**

1. Franke and Yakubu showed this.
2. $F_t(I) = \gamma(1 - e^{-\frac{\alpha I}{N_t}})(N_t - I) + \gamma \sigma I$ is clearly continuous and differentiable everywhere.
3. $F_t(0) = \gamma(0)(N_t - 0) + \gamma \sigma 0 = 0$.
4. $F_t(N_t) = \gamma \sigma N_t < N_t$.
5. This is just an application of the laws of differentiation.
6. Suppose for induction that $F^{(n)}_t(I) = \gamma(-\frac{\alpha}{N_t})^{n-1} e^{-\frac{\alpha I}{N_t}}(\alpha - \frac{\alpha I}{N_t} + n)$ for some $n$. Then

   $F^{(n+1)}_t(I) = \gamma(-\frac{\alpha}{N_t})^n e^{-\frac{\alpha I}{N_t}}(\alpha - \frac{\alpha I}{N_t} + n) + \gamma(-\frac{\alpha}{N_t})^{n-1} e^{-\frac{\alpha I}{N_t}}(-\frac{\alpha}{N_t}) = \gamma(-\frac{\alpha}{N_t})^n e^{-\frac{\alpha I}{N_t}}(\alpha - \frac{\alpha I}{N_t} + (n + 1))$.

   So our induction hypothesis is satisfied for $n + 1$. Some calculation yields

   $F''_t(I) = -\frac{\gamma \alpha}{N_t} e^{-\frac{\alpha I}{N_t}}(\alpha - \frac{\alpha I}{N_t} + 2)$

   so the condition is satisfied for $n = 2$ and therefore for all $n > 1$.
7. $F'_t(N_t) = \gamma(e^{-\alpha} + \sigma - 1) > -1$.
8. $F''_t(I) = \frac{\gamma \alpha}{N_t} e^{-\frac{\alpha I}{N_t}}(\alpha - \frac{\alpha I}{N_t} - 2) \leq \frac{\gamma \alpha}{N_t} e^{-\frac{\alpha I}{N_t}}(-2) < 0$. 

4
The most important of these conditions are (1), (3) and (8). (1) guarantees that each \( F_t \) maps its whole domain into the domain of \( F_{t+1} \), and thus allows us to construct the composition \( F = F_p \circ \cdots \circ F_1 : [0, N_1] \to [0, N_1] \), which maps a point through a whole year of \( p \) months or seasons. (Note that by letting \( p = 1 \), we can prove that \( F_t([0, N_t]) \subset [0, N_t] \) for any infection-function \( F_t \).) (1) also guarantees that \( F_t \) is nonnegative. (3) and (8), together, tell us that that \( F_t \) begins at 0 and has a constantly decreasing derivative. So \( F_t \) is roughly along the lines of a concave-downward bell curve. We provide a picture of a sample \( F_t \) below, as Figure 1. (We plot the line \( y = x \) on the same graph; it intersects \( F_t \) at a fixed point.)

![Figure 1: \( F_t(I) \) with \( N_t = 47.50, \gamma = .3, \sigma = .01, \alpha = 75 \)](image)

Now, the preceding results were fairly basic, and though they were derived independently, most of them appeared in Franke and Yakubu’s earlier work. We now present a new theorem which, while of no immediate utility, may yet prove to be the stepping-stone towards deeper results regarding these mappings. (The result of this theorem is equivalent to the existence of a topological conjugacy relation between any two functions \( F_t \) and \( F_s \) of the same infection-set.)

**Theorem 5** Let \( F_t, F_s \) be infection functions of the same infection-set. Then for all \( I_s, I_t \) such that \( \frac{I_s}{N_s} = \frac{I_t}{N_t} \), we have \( F_s'(I_s) = F_t'(I_t) \). We also have \( F_s'(I) = F_t'(\frac{N_t}{N_s} I) \) for any \( I \in [0, N_s] \). In addition, if \( I_0 \) is a fixed point of \( F_s \), then \( \frac{N_s}{N_t} I_0 \) is a fixed point of \( F_t \). Finally, for any \( I \in [0, N_s] \) we have \( F_s(I) = \frac{N_s}{N_t} F_t(\frac{N_t}{N_s} I) \).

**PROOF:** We have
\[ F'_s(I) = \gamma e^{-\frac{\alpha I}{N_s}}(\alpha - \frac{\alpha I}{N_s} + 1) + \gamma \sigma - \gamma \]

\[ F'_t(I) = \gamma e^{-\frac{\alpha I}{N_t}}(\alpha - \frac{\alpha I}{N_t} + 1) + \gamma \sigma - \gamma, \]

so clearly we have \( F'_s(I_s) = F'_t(I_t) \) for \( \frac{I_s}{N_s} = \frac{I_t}{N_t} \). Also, we have \( \frac{I_t}{N_t} = \frac{N_t I_s}{N_t} \), which gives us \( F'_s(I) = F'_t(\frac{N_t I_s}{N_t} I) \) for arbitrary \( I \in [0, N_s] \). Moreover, for \( I_0 \leq N_s \), we can integrate \( \int_0^{I_0} F'_s(I) dI = \int_0^{I_0} F'_t(\frac{N_t I_s}{N_t} I) dI \), which gives us \( F_s(I_0) - F_s(0) = \frac{N_t}{N_s} (F_t(\frac{N_t I_s}{N_t} I_0) - F_t(0)) \), or simply \( F_s(I_0) = \frac{N_t}{N_s} F_t(\frac{N_t I_s}{N_t} I_0) \).

Finally, if \( I_0 \) happens to be a fixed point of \( F_s \), we have \( F_s(I_0) = I_0 = \frac{N_t}{N_s} F_t(\frac{N_t I_s}{N_t} I_0) \), so \( \frac{N_t}{N_s} I_0 = F_t(\frac{N_t I_s}{N_t} I_0) \), so \( \frac{N_t}{N_s} I_0 \) is a fixed point of \( F_t \). QED

The power of Theorem 5 is apparent; it allows us to equate any two infection-functions by topological conjugation. This allows us to work with only one infection-function, rather than \( p \) of them. This topological conjugacy also sheds new light on the composition \( F_p \circ \cdots \circ F_1 \), and reveals it to be identical to a much simpler composition, as shown in our next theorem.

**Theorem 6** Let \( N_1 \ldots N_p \) be the globally attracting \( p \)-cycle of \( G_t \), and for \( t = 1 \ldots p - 1 \), define \( L_t = \frac{N_t}{N_{t+1}} F_1 \). Also, let \( L_p = \frac{N_p}{N_1} F_1 \). Then \( L_p \circ \cdots \circ L_1 = F_p \circ \cdots \circ F_1 \).

**PROOF:** We have \( L_1 = \frac{N_1}{N_2} F_1 \). Suppose for induction that for some \( t < p \), we have \( L_t \circ \cdots \circ L_1 = \frac{N_t}{N_{t+1}} F_t \circ \cdots \circ F_1 \). Then \( L_{t+1} \circ L_t \circ \cdots \circ L_1 = L_{t+1}(\frac{N_{t+1}}{N_1} F_t \circ \cdots \circ F_1) = \frac{N_{t+1}}{N_{t+2}} F_1(\frac{N_{t+1}}{N_{t+2}} F_t \circ \cdots \circ F_1) \).

Theorem 5 tells us that \( F_1(\frac{N_{t+1}}{N_1} F_t \circ \cdots \circ F_1) = \frac{N_{t+1}}{N_{t+2}} F_{t+1}(F_t \circ \cdots \circ F_1) \), so

\[ L_{t+1} \circ L_t \circ \cdots \circ L_1 = \frac{N_{t+1}}{N_{t+2}} \frac{N_{t+1}}{N_{t+2}} F_{t+1}(F_t \circ \cdots \circ F_1) = \frac{N_{t+1}}{N_{t+2}} F_{t+1} \circ F_t \circ \cdots \circ F_1. \]

This satisfies our induction hypothesis, so we have \( L_t \circ \cdots \circ L_1 = \frac{N_{t+1}}{N_{t+2}} F_t \circ \cdots \circ F_1 \) for any \( t \leq p \).

Then, observing that \( N_{p+1} = N_1 \), we have \( L_p \circ \cdots \circ L_1 = \frac{N_1}{N_{p+1}} F_p \circ \cdots \circ F_1 = F_p \circ \cdots \circ F_1 \). QED

This theorem has tremendous promise; it allows us to regard the composition \( F_p \circ \cdots \circ F_1 \) as a composition of scalar multiples of \( F_1 \), rather than a composition of \( p \) different functions. We discovered Theorem 6 late in our studies, and were not able to derive any further results from it, but we are confident that it will be of great use in future studies of these mappings.

This concludes our general background-work on the mapping \( F_t \).

### 4 The Fixed Points of \( F_t \) and Its Composition \( F \)

The ultimate goal of this paper is to consider the fixed points of the functions \( F \) and \( F_t \). We are almost ready to undertake this goal. Before we begin, however, we will prove a pair of theorems
about fixed points in general, which will make several of our later proofs much easier than they would be otherwise.

**Theorem 7** Let \( F : [0, N] \rightarrow [0, N] \) be continuous and differentiable, and let \( F(0) = 0 \). Suppose there are exactly \( m \) values \( y_1 \ldots y_m \in [0, N] \) such that \( F'(y_t) = 1 \). Then \( F \) has at most \( m \) nonzero fixed points.

**PROOF:** Suppose there are exactly \( m \) values \( y_t \in [0, N] \) such that \( F'(y_t) = 1 \), and let \( x_1, x_2 \) be any two fixed points of \( F \). The Mean Value Theorem guarantees there is \( c \in (x_1, x_2) \) with

\[
F'(c) = \frac{F(x_2) - F(x_1)}{x_2 - x_1} = \frac{x_2 - x_1}{x_2 - x_1} = 1.
\]

Then \( F \) cannot have an infinite number of fixed points, because if it did we could find fixed points \( x_1 < x_2 < \ldots < x_{m+2} \), and then for \( t = 1 \ldots m + 1 \) there would be \( y_t \in (x_t, x_{t+1}) \) with \( F'(y_t) = 1 \), and this would contract our hypothesis that there are only \( m \) such values \( y_t \). If \( F \) has finitely many fixed points, on the other hand, we can order them \( x_0, x_1, \ldots, x_n \), where \( x_1, \ldots, x_n \) are nonzero and \( x_0 = 1 \). Then for every nonzero fixed point \( x_n \) we can find \( y_n \in (x_{n-1}, x_n) \) with \( F'(y_n) = 1 \), so we cannot have more than \( m \) nonzero fixed points without contradicting our hypothesis. QED

The vital case of this theorem is that if \( F \) has only one point of derivative 1, then \( F \) has at most one nonzero fixed point.

The previous theorem dealt with the existence of fixed points. Our next theorem assumes there is a fixed point, and gives a sufficient condition for convergence to it. Before we state the theorem, however, we should clarify a certain point. We will often use the phrase "unique nonzero fixed point." By this, we do not mean to exclude 0 from the set of fixed points; 0 is a fixed point of any infection-function \( F \) and of any infection-composition \( F \). By "unique nonzero fixed point" we mean that a fixed point is the only nonzero fixed point, and is therefore unique in that respect. Having cleared up this ambiguity, we proceed.

**Theorem 8** Let continuous, differentiable \( F : [0, N] \rightarrow [0, N] \) have a unique nonzero fixed point \( x_0 \). Also, let there be \( a < x_0 \) such that \( |F'(x)| < 1 \) for all \( x \in (a, N) \). Then for all \( x \in [a, x_0], F^n(x) \rightarrow x_0 \).

**PROOF:** Take \( x \in [a, x_0) \). By the Mean Value Theorem, there is \( c_1 \in (a, x_0) \) with

\[
F'(c_1) = \frac{F(x_0) - F(x)}{x_0 - x} = \frac{x_0 - F(x)}{x_0 - x}.
\]

Then, if we let \( \epsilon = \max\{F'(x) : a \leq x \leq N\} \), we have

\[
|F'(c_1)| \leq \epsilon. \quad \text{Note that by our hypothesis,} \quad \epsilon < 1. \quad \text{Then we have} \quad |x_0 - F(x)| = |F'(c_1)||x - x_0| \leq \epsilon|x - x_0|.
\]

Now suppose for induction that for some \( n \) we have \( |x_0 - F^n(x)| \leq \epsilon^n|x_0 - x| \). Since \( x < x_0 \), this gives us \( F^n(x) > x \). Then we have \( F^n(x) \in (a, N) \). Now we apply the Mean Value Theorem and find \( c_n \) between \( F^n(x) \) and \( x_0 \) with \( |x_0 - F^{n+1}(x)| = |F'(c_n)||x_0 - F^n(x)| \leq \epsilon|x_0 - F^n(x)| \leq \epsilon^{n+1}|x_0 - x| \). Then by induction we have \( |x_0 - F^n(x)| \leq \epsilon^n|x_0 - x| \) for any \( n \), and therefore \( |x_0 - F^n(x)| \rightarrow 0 \) and \( F^n(x) \rightarrow x_0 \). QED
We will often consider functions with exactly one nonzero fixed point, and the above theorem will be extremely useful for showing convergence in these cases. We now begin to consider the mappings relevant to this paper, and begin with a theorem dealing with the fixed points of $F_t$.

**Theorem 9** The following are equivalent:

1. $F_t$ has a nonzero fixed point.
2. $F'_t(0) > 1$.
3. $\alpha > \frac{1}{\gamma} - \sigma$.

**Proof:**

1 $\Rightarrow$ 2 Suppose $F_t$ has a nonzero fixed point $I_0$. The Intermediate Value Theorem tells us there is $c \in (0, I_0)$ with $F'_t(c) = \frac{F_t(I_0) - F_t(0)}{I_0 - 0} = \frac{I_0}{I_0} = 1$. Theorem 4 tells us that $F'_t$ is always decreasing, so we must have $F'_t(0) > 1$.

2 $\Rightarrow$ 1. Suppose $F'_t(0) > 1$. Then there must be $\delta > 0$ such that $F_t(I) > I$ for all $I \in (0, \delta]$, so $F_t(\delta) > \delta$. Also, Theorem 4 tells us that $F_t(N_t) < N_t$. These two conditions, together, guarantee there is $I_0 \in (\delta, N_t)$ with $F_t(I_0) = I_0$.

2 $\Rightarrow$ 3. Suppose $F'_t(0) > 1$. We have $F'_t(0) = \gamma \alpha + \gamma \sigma$, so $\gamma \alpha + \gamma \sigma > 1$, which we can easily rearrange into $\alpha > \frac{1}{\gamma} - \sigma$.

3 $\Rightarrow$ 2. Suppose $\alpha > \frac{1}{\gamma} - \sigma$. Then $\gamma \alpha + \gamma \sigma = F'_t(0) > 1$. QED

This theorem is useful, since it gives us an easy way of determining whether $F_t$ has a fixed point; we need simply consider the derivative at 0. The next theorem makes a few statements about the nature of the fixed point of $F_t$.

**Theorem 10** If $F_t$ has a nonzero fixed point $I_0$, then that nonzero fixed point is unique and globally attracting. Moreover, $|F'_t(I_0)| < 1$.

**Proof:** Let $I_0$ be a nonzero fixed point of $F_t$. Since the derivative of $F_t$ is always decreasing, there can be at most one $a$ on $[0, N_t]$ with $F'_t(a) = 1$. Therefore, by Theorem 7, $I_0$ is unique as a nonzero fixed point, and $I_0 > a$. We know $F'_t(N_t) > -1$, so we have $|F'_t(I)| < 1$ for all $I \in (a, N_t)$. Then by Theorem 8 we have $F^n_t(I) \to I_0$ for all $I \in [a, I_0]$.

We now show that for any $I \notin [a, I_0]$, there is $n$ such that $F^n_t(I) \in [a, I_0]$.

**Case 1:** $I \in (I_0, N_t)$. In this case, suppose $F^n_t(I) > I_0$ for all $n$. Then $F^n_t(I)$ is a monotone decreasing sequence and bounded below by $I_0$, and therefore must converge. Now, Theorem 20 in the Appendix says that if $F^n_t(I)$ converges at all, it must converge to a fixed point, so we have
\( F^n(t) \rightarrow I_0 \). Otherwise we have \( F^n(I) < I_0 \) for some \( n \), so we either have \( F^n(I) \in [a, I_0] \) (which is our goal) or we have \( F^n(I) < a \), which is covered by the next case.

CASE 2: \( I \in (0, a) \). Suppose for contradiction that \( F^n(I) \leq a \) for all \( n \). Then \( F^n(I) < I_0 \) for all \( n \), and \( F^n(I) \) is a monotone increasing sequence bounded above by \( a \), so it must converge to some value \( \leq a \). But, based on Theorem 12 in our Appendix, \( F^n(I) \) must converge to a fixed point if it converges at all, and there are no fixed points on \([I, a]\), contradiction. Therefore there is \( n \) such that \( F^n(I) \leq a \), but \( F^{n+1}(I) > a \).

So we have \( F^n(I) \leq a < F^{n+1}(I) \). Suppose that in addition, \( F^{n+1}(I) < I_0 \). Then we have accomplished our purpose; we have \( F^{n+1}(I) \in (a, I_0) \). Suppose on the other hand that \( F^{n+1}(I) > I_0 \). Then, since we have \( F^n(I) \leq a \) and \( F'(a) = 1 \), we have \( F(I) > I_0 \). Now, the Mean Value Theorem guarantees there is \( d_1 \in (a, I_0) \) with \( F'(d_1) = \frac{I_0 - F(a)}{I_0 - a} \), and since we have \( a < I_0 < F(a) \), the numerator and denominator have different signs and we must have \( F'(d_1) < 0 \), which gives us \( F(I) < 0 \) since \( d_1 < I_0 \) and \( F'(a) \) is always decreasing.

We also have \( d_2 \in (I_0, F(I)) \) with \( F'(d_2) = \frac{I_0 - F^2(a)}{I_0 - F(a)} \). The fact that \( F'(I_0) < 0 \) gives us \( F'(d_2) < 0 \), which in turn gives us \( F^2(a) < I_0 \). Finally, we have \( |F^2(a) - I_0| = |F'(d_2)||F'(d_1)||a - I_0| < |a - I_0| \) since \( d_1, d_2 \in (a, N_t) \). Then we have \( F^2(a) > a \). Then, since we have \( I_0 < F^{n+1}(I) \leq F(I) \) and \( F'(I_0) < 0 \), we have \( F^{n+2}(I) > F^2(a) > a \). And since \( F'(d_2) < 0 \) and \( F^{n+1}(I) > I_0 \), we must have \( F^{n+2}(I) < I_0 \). Therefore we have \( F^{n+2}(I) \in (a, I_0) \).

Therefore, for any nonzero \( I \notin [a, I_0] \), there is \( n \) such that \( F^n(I) \in [a, I_0] \), therefore \( F^n(I) \rightarrow I_0 \) for all \( I \in [0, N_t] \). QED

The fact that \( F_t \) has a globally attracting fixed point is of some interest; it gives us insight into the function which is the building-block of our composition \( F \). Biologically, \( F_t \) represents the behavior of an epidemic over a single month/season, so we could interpret Theorem 10 to mean, "If there were only one season, if climate was always the same, then any epidemic outbreak among insects would eventually result in \( I_0 \) infected insects, regardless of how many insects were infected initially."

We now attention to that composition \( F = F_p \circ \cdots \circ F_1 \), and prove a number of theorems about \( F \) for both arbitrary and fixed cycle-length \( p \). First, we present a very basic theorem regarding the fixed points of \( F \).

**Theorem 11** Provided that \( \alpha > \frac{1}{\gamma} - \sigma \), \( F \) has a smallest nonzero fixed point \( I_0 \), and \( F'(I_0) \leq 1 \).

**Proof:** \( F \) is a composition of continuous differentiable functions, and therefore continuous and differentiable. By the chain rule, \( F'(0) = \prod_{1 \leq i \leq p} F_i'(F_{i-1} \circ \cdots \circ F_1(0)) = \prod_{1 \leq i \leq p} F_i'(0) \). Our hypothesis that \( \alpha > \frac{1}{\gamma} - \sigma \) guarantees that \( F_i'(0) > 1 \) for all \( i \), so we have \( F'(0) > 1 \). The fact that \( F'(0) > 1 \) allows us to conclude that there is an interval \([0, \delta]\) such that \( F(I) > I \) for all \( I \) in the interval.

We now consider the set \( X = \{ x : F(I) - I > 0 \text{ for all } I \in (0, x) \} \). This set is nonempty, since \( \delta \) is a member. And since \( F([0, N_1]) \subset (0, N_1) \) we must have \( F(N_1) \leq N_1 \), so \( N_1 \) is an upper bound for
We know that \( F(I_0) - I_0 > 0 \). Since the function \( F(I) - I \) is clearly continuous, there must then be \( I_1 > I_0 \) such that for every \( I \in (I_0, I_1) \), \( F(I) - I > 0 \). In this case \( I_1 \) would be a member of \( X \) and \( I_0 \) would not be an upper bound for \( X \), contradiction.

If, on the other hand, \( F(I_0) - I_0 < 0 \), there must be \( I_2 < I_0 \) with \( F(I_2) - I_2 < 0 \). In this case, \( I_2 \) will serve as an upper bound for \( X \), which contradicts the fact that \( I_0 \) is the supremum.

So, by exhaustion, \( F(I_0) - I_0 = 0 \), so \( F(I_0) = I_0 \) and \( I_0 \) is fixed. We know \( F(I) > I \) for all \( I \in (0, I_0) \), so \( I_0 \) is the unique smallest nonzero fixed point.

We now suppose for contradiction that \( F'(I_0) > 1 \). Then, since \( F' \) is continuous, there is an interval \([c, I_0] \) with \( F'(I) > 1 \) for all \( I \) on the interval.

Next, we employ the Mean Value Theorem on \((c, I_0)\) and find some \( d \) in the interval such that

\[
\frac{F(I_0) - F(c)}{I_0 - c} = \frac{I_0 - F(c)}{I_0 - c} = F'(d) > 1.
\]

We know that \( I_0 - c > 0 \), so \( I_0 - F(c) > 0 \), too; and because the quotient is \( > 1 \), we can conclude that \( c > F(c) \).

So \( F(c) < c \), but \( F(\delta) > \delta \), so there must be a fixed point on \((\delta, c)\), but \( I_0 \) is the smallest fixed point, contradiction. Therefore, our assumption that \( F'(I_0) > 1 \) was false, and \( F'(I_0) \leq 1 \). QED

This result is of great significance. It tells us that, provided we have \( \alpha > \frac{1}{\gamma} - \sigma \), \( F \) must have a smallest nonzero fixed point. This allows us, in most cases, to conjure a nonzero fixed point out of thin air, so to speak, in much the same way that, in a compact set, a finite subcover can be magically conjured out of any open cover.

Now we begin to set stringent conditions for the composition \( F \), and find results for particular cases. Our next theorem provides a result for the case when \( p = 2 \).

**Theorem 12** If \( p = 2 \), then \( F \) has at most one nonzero fixed point.

**PROOF:** Our goal is to show that there is at most one point \( a \in [0, N_1] \) with \( F'(a) = 1 \). To do this we handle two separate cases, and prove the uniqueness of such an \( a \) in each.

**CASE 1:** We first deal with the strange (but possible) case where \( F_1'(I) \geq 0 \) for all \( I \in [0, N_1] \). This gives us an \( F_1(I) \) that is always increasing, so we have \( F_2'(F_1(I)) \) always decreasing. And we know from Theorem 4 that \( F_1'(I) \) is always decreasing. So \( F'(I) = F_2'(F_1(I))F_1'(I) \) is the product of two decreasing terms, one of which is always positive. Therefore \( F' \) is always either decreasing or negative, and there can be at most one point \( a \in [0, N_1] \) with \( F'(a) = 1 \).

**CASE 2:** There is \( c_1 \in (0, N_1) \) with \( F_1'(c_1) = 0 \). Then we consider the interval \((0, c_1)\). On this interval, \( F_1 \) is increasing, so \( F_2'(F_1(I)) \) is decreasing. We also know that \( F_1'(I) \) is decreasing.
everywhere. So we can state that on the interval \((0, c_1)\), the derivative 
\(F'(I) = F_2'(F_1(1))F_1'(I)\) is the product of two decreasing terms, one of which, 
\(F_1'(I)\), is positive. This allows us to conclude that \(F'\) is either decreasing or negative on the interval \((0, c_1)\). So there is at most one point 
\(I \in (0, c_1)\) with 
\(F'(I) = 1\). And on the interval \([c_1, N_1]\), we have 
\(-1 < F_1'(I) \leq 0\), so we know that 
\(F'(I) = F_2'(F_1(1))F_1'(I) < 1\), since 
\(F_2'(F_1(I)) > -1\).

Therefore, in either case, we have at most one point \(a\) with 
\(F'(a) = 1\). Then by Theorem 7, 
\(F\) has at most one nonzero fixed point. QED

We see, then, that for a cycle of length 2, there can be only one fixed point. The biological implication of this is that if some environment (perhaps somewhere near the equator) has only two seasons, then the eventual number of infectants is predetermined, and completely independent of initial conditions.

The idea of infection-sets with multiple attractors provided the original motivation behind our project. We have shown that multiple attractors cannot exist in the \(p = 1\) or \(p = 2\) cases. However, numerical experiments revealed that a set of 4 infection functions can have multiple attractors. The next theorem provides a sufficient condition for the presence of multiple attractors in the \(p = 4\) case.

**Theorem 13** Let the cycle-length \(p = 4\), and for \(t = 1 \ldots 4\) let \(c_t\) be the unique value on \([0, N_t]\) such that \(F_t'(c_t) = 0\). Also, let the functions \(F_1 \ldots F_4\) and their composition \(F\) have the following properties:

1. \(F_t(c_t) > c_{t+1}\) for \(t = 0 \ldots 3\) and \(F_4(c_4) > c_1\).
2. \(F_1(N_1) < c_2\) and \(F_3(N_3) < c_4\)
3. \(F_3F_2(c_2) < c_4\)
4. \(F_1F_4(c_4) < c_2\)
5. \(F_2F_1(N_1) < c_3\) or \(F_3F_2F_1(N_1) > c_4\)
6. \(F_3F_2F_1F_4(c_4) > c_4\)
7. \(F_1F_4F_3F_2(c_2) > c_2\).
8. \(F_t'(N_t) < 0\) for all \(t\). (This condition guarantees the existence of each \(c_t\).

Then \(F\) has multiple nonzero fixed points.

**PROOF:** (1) tells us that \(F_3(c_3) > c_4\), and (2) tells us that \(F_3(N_3) < c_4\), so there is \(x \in (c_3, N_3)\) with \(F_3(x) = c_4\).

We now suppose for contradiction that \(x \geq F_2(c_2)\). (1) tells us that \(F_2(c_2) > c_3\), so we have \(x \geq F_2(c_2) > c_3\), which allows us to conclude that \(F_3(x) = c_4 \leq F_3F_2(c_2)\). This contradicts (3), so our assumption was wrong and we know \(x < F_2(c_2)\).
This, in turn, allows us to conclude there is \( y \in (0, c_2) \) with \( F_2(y) = x \).

Now we suppose for contradiction that \( y < F_1(N_1) \). Then (2) allows us to state that \( y < F_1(N_1) < c_2 \), which in turn implies that \( F_2(y) = x < F_2F_1(N_1) \). We know \( x > c_3 \), so we have \( F_2F_1(N_1) > c_3 \), which contradicts half of (5).

We then take \( F_3 \) of the inequality \( c_3 < x < F_2F_1(N_1) \) to conclude that \( F_3(x) = c_4 > F_3F_2F_1(N_1) \), which contradicts the other half of (5). So we have reached a contradiction, our assumption was wrong, and we know that \( y \geq F_1(N_1) \). Then, since we have \( F_1(N_1) \leq y < c_2 < F_1(c_1) \), there must then be \( z \in (c_1, N_1) \) with \( F_1(z) = y \).

Now we assume for contradiction that \( z \geq F_4(c_4) \). Then \( z \geq F_4(c_4) > c_1 \), by (1). From here we can conclude that \( F_1(z) = y \leq F_1F_4(c_4) \).

Then, by (4), we have \( y \leq F_1F_4(c_4) < c_2 \). Then \( F_2(y) = x \leq F_2F_1F_4(c_4) \). And we know \( x > c_3 \), so we have \( c_3 < x \leq F_2F_1F_4(c_4) \), which implies that \( F_3(x) = c_4 \geq F_3F_2F_1F_4(c_4) \), which contradicts (6).

Therefore our assumption that \( z \geq F_4(c_4) \) was incorrect, and \( z < F_4(c_4) = F(z) \). This completes one of our two major steps.

Now we start over. (1) and (2) tell us that \( F_1(c_1) > c_2 > F_1(N_1) \), so there is \( w \in (c_1, N_1) \) with \( F_1(w) = c_2 \).

From (7) we know that \( F_1F_4F_3F_2(c_2) > c_2 \), so, substituting \( F_1(w) \) for \( c_2 \), we find that \( F_1F_4F_3F_2(F_1(w)) = F_1(F(w)) > F_1(w) \). We know that \( w > c_1 \), so this allows us to conclude that \( F(w) < w \). This is our second major step.

We know that \( z, w > c_1 \), and \( F_1(w) = c_2 > F_1(z) = y \), so \( w < z \).

So we have \( w < z, F(w) < w, F(z) > z \). Then by the Intermediate Value Theorem there must be \( K_0 \in (w, z) \) with \( F(K_0) = K_0 \), i.e. there must be a fixed point \( K_0 \) on \( (w, z) \). We also know there is an interval \( (0, \delta) \) with \( F(I) > I \) for all \( I \) in the interval, so by identical reasoning we can deduce the existence of a fixed point \( K_1 \in (0, w) \). Finally, we know that \( F(N_1) \leq N_1 \), so there is a fixed point \( K_2 \in (z, N_1) \). Here are three nonzero fixed points. QED.

As a curiosity, we now present an instance of multiple attractors in the \( p = 4 \) case, in which the existence of said attractors could have been deduced by Theorem 13. The function appears on the following page as Figure 2.
Figure 2: A 4-periodic infection set with multiple attractors

Parameters and values:

\[
\begin{align*}
\gamma &= .4; & k_1 &= 1; & N_1 &= 60.32; & c_1 &= 1.33; \\
\sigma &= .02; & k_2 &= 200; & N_2 &= 144.13; & c_2 &= 3.18; \\
\alpha &= 200; & k_3 &= 1; & N_3 &= 58.25; & c_3 &= 1.29; \\
& & k_4 &= 210; & N_4 &= 149.30; & c_4 &= 3.30.
\end{align*}
\]

Conditions of Theorem 11:

1. \(F_1(c_1) - c_2 = 20.14; \quad F_2(c_2) - c_3 = 54.44; \quad F_3(c_3) - c_4 = 19.22; \quad F_4(c_4) - c_1 = 56.39\)
2. \(F_1(N_1) - c_2 = -2.70; \quad F_3(N_3) - c_4 = -2.83\)
3. \(F_3F_2(c_2) - c_4 = -1.84\)
4. \(F_1F_4(c_4) - c_2 = -1.68\)
5. \(F_2F_1(N_1) - c_3 = 26.76; \quad F_3F_2F_1(c_1) - c_4 = 9.01\)
6. \(F_3F_2F_1(c_1) - c_4 = .42\)
7. \(F_1F_4F_3F_2(c_2) - c_2 = 1.05\)

Now, our results up until this point raise an interesting question. If there cannot be multiple attractors in the \(p = 2\) case, but there can be multiple attractors in the \(p = 4\) case, can there exist multiple attractors in the \(p = 3\) case? Unfortunately, we were unable to answer this question.
However, we were able to uncover a result of sorts; that is, we found an example of a function which was very similar to $F$, and which had multiple attractors. We sum up this result in the following theorem.

**Theorem 14** The following conditions are NOT sufficient to conclude that $F$ does not have multiple attractors in the $p = 3$ case:

1. $F_t([0, N_t]) \subset [0, N_{t+1}]$ for $t = 1, 2$, and $F_3([0, N_3]) \subset [0, N_1]$.
2. $F_t(0) = 0$ for every $t$.
3. $F_t'(0) > 1$ for every $t$.
4. $F_t''(I) < 0$ for every $t$ and all $I \in [0, N_t]$.

**PROOF:** Consider the functions $F_t : [0, 2] \to [0, 2]$,

$$F_t(I) = \frac{-I^2}{1,000,000} + 1.001 \sin(k_1I) \quad k_1 = 2.53, \quad k_2 = 1, \quad k_3 = 3.$$

Then we have

1. $F_t(I) \in [0, 2]$ for every $t$, every $I \in [0, 2]$.
2. $F_t(0) = 0$ for every $t$.
3. $F_t'(I) = -\frac{I}{500,000} + 1.001k_1\cos(k_1I)$ and

$F_t'(0) = 1.001k_1 > 1$ for every $t$.
4. $F_t''(I) = -\frac{1}{500,000} - k_2^21.001\sin(k_2I) < 0$ for all $I \in [0, 2]$, every $t$.

However, $F$ has attractors at $I = .58$ and $I = .999$. QED

We present a graph of this function as Figure 3 on the next page. It is remarkable how nearly it approximates several of the $p = 3$ functions we encountered.
Another of our longstanding questions was, is there some infection set $F_1 \ldots F_p$ such that $F_p \circ \cdots \circ F_1$ has more than two attractors? The answer to this question is yes. For the $p = 8$ case, the parameters

\[
\alpha = 370; \quad \gamma = .504; \quad \sigma = .004093 \\
k_1 = 1; \quad k_2 = 402; \quad k_3 = 1; \quad k_4 = 400; \quad k_5 = 1; \quad k_6 = 402; \quad k_7 = 1; \quad k_8 = 270
\]

yielded attractors at $I_1 = 96.9742$, $I_2 = 98.2979$, $I_3 = 99.6540$. The graph of the $F$ obtained from these parameters appears as Figure 4 below.

Figure 4: Close-up of an $F$ with 3 attractors
The sinelike oscillation of $F$ towards the end of its interval suggests that for larger values of $p$, even more oscillations, and thus more attractors, could be obtained. We conjecture that for any $n$, there is a map $F$ with $n$ attractors, but were not able to prove this conjecture.

5 The Parameter $\alpha$

Our last few results deal with the effects of the parameter $\alpha$ on the general composition $F = F_p \circ \cdots \circ F_1$. First we present a case in which there are no nonzero fixed points.

**Theorem 15** If $\alpha < \frac{1}{\gamma} - \sigma$, 0 is a globally attracting fixed point of $F$.

**PROOF:** By the chain rule, $F'(0) = \prod_{1 \leq t \leq k} F'_t(F_{t-1} \circ \cdots \circ F_1(0))$. We know that $F_t(0) = 0$ for any $t$, so $F'(0) = \prod_{1 \leq t \leq k} F'_t(0)$.

We examine $F'_t(0)$, then. Referring to our derivatives, we see that for any $t$, $F'_t(0) = \gamma \alpha + \gamma \sigma$. We have assumed that $\alpha < \frac{1}{\gamma} - \sigma$, and we can easily transform this inequality into $\gamma \alpha + \gamma \sigma = F'_t(0) < 1$. Then, since the derivative of $F_t$ is always decreasing, we have $F'_t(I) < 1$ for all $I \in [0, N_t]$. We also have $F'_t(N_t) > -1$, so for $t = 1 \ldots p$ we have $|F'_t(I)| < 1$ for all $I \in [0, N_t]$. This gives us $F'(I) < 1$ for all $I \in [0, N_t]$. Then we have no points of derivative 1, and consequently, by Theorem 7, no nonzero fixed points.

Now we show that 0 is globally attracting. To do this, we first observe that $F'(0) = \prod_{1 \leq t \leq k} F'_t(0) < 1$, since every individual term is $< 1$ in absolute value. Since $F'(0) < 1$ and 0 is the only fixed point, we know that $F(I) < I$ for all $I > 0$. Then for arbitrary $I > 0$ the sequence $F^n(I)$ is strictly decreasing and bounded below by 0, so it must converge, therefore it must converge to a fixed point, therefore it must converge to 0, so we have $F^n(I) \rightarrow 0$. Therefore 0 is a globally attracting fixed point. QED

This result is highly significant. In conjunction with Theorem 9, it gives $\alpha = \frac{1}{\gamma} - \sigma$ as a ”threshold” for persistence or extinction of an epidemic. If $\alpha$ is larger than $\frac{1}{\gamma} - \sigma$, then the disease will never die out, since 0 will in this case be a repelling fixed point of $F$ (see Theorem 20 in the Appendix); if, on the other hand, $\alpha$ is smaller than $\frac{1}{\gamma} - \sigma$, the number of infectants will eventually converge to 0.

From a biological perspective, this idea of an $\alpha$-threshold makes perfect sense, when one considers that $\alpha$ is the ”transmission constant,” a measure of the communicability of the disease. Theorem 15 says, in effect, that if a disease is not contagious enough, it will die out.

It is also interesting that the $\alpha$-threshold varies inversely with $\gamma$ and $\sigma$ (although in the first case the variation is geometric, while in the second case it is merely arithmetic.) Biologically, $\gamma$ represents the probability of any individual’s surviving until the next time period, so we see that the shorter the lifespan of the organisms in question, the greater the degree of contagion required for the disease to establish itself permanently in the community. Analogously, $\sigma$ is the probability that a given
individual will NOT recover from the disease from one time-period to the next; so for a larger \( \sigma \), we will have more infected individuals, and thus will require a smaller degree of contagiousness to attain permanence.

We mentioned that \( \alpha = \frac{1}{\gamma} - \sigma \) was the “cutoff point” for disease persistence or disease extinction. We now put this idea into mathematical terms, and show that \( F \) undergoes a bifurcation at that value of \( \alpha \).

**Theorem 16** \( F \) undergoes a simple transcritical bifurcation (with respect to \( \alpha \)) at \( \bar{\alpha} = \frac{1}{\gamma} - \sigma \).

**PROOF:** We must show that four things are true.

1. \( F_\alpha(0) = 0 \) for all \( \alpha \).

\[
F_t(0) = 0 \text{ for any } t \text{ regardless of } \alpha, \text{ so the composition } F(0) = F_p \circ \cdots \circ F_1(0) = 0.
\]

2. \( F'_\alpha(0) = 1 \).

If we set \( \alpha = \bar{\alpha} = \frac{1}{\gamma} - \sigma \), then for any \( t \), \( F'_t(0) = \gamma(\frac{1}{\gamma} - \sigma) + \gamma \sigma = 1 - \gamma \sigma + \gamma \sigma = 1 \).

And since \( F'(0) = \prod_{1 \leq t \leq k} F'_t(0) \), we know that \( F'(0) = 1 \).

3. \( F''_\alpha(0) \neq 0 \).

We have \( F''(I) = \sum_{t=1}^{p} F''(F_{t-1} \circ \cdots \circ F_1(I)) \prod_{1 < i < t} F'_t(F_{i-1} \circ \cdots \circ F_1(I)) \prod_{t < i \leq p} F'_t(F_{i-1} \circ \cdots \circ F_1(I)) \).

When we substitute 0 for \( I \), this expression resolves to \( F''(0) = \sum_{t=1}^{p} F''_t(0) \).

We know \( F''_t(0) < 0 \) for all \( t \), and therefore \( F''(0) < 0 \) regardless of \( \alpha \), so \( F''_\alpha(0) \neq 0 \).

4. \( \frac{\partial F'_\alpha(0)}{\partial \alpha}(\bar{\alpha}) \neq 0 \).

We know that \( F'(0) = \gamma \alpha + \gamma \sigma \), so

\[
\frac{\partial F'_\alpha(0)}{\partial \alpha}(\bar{\alpha}) = \gamma \neq 0.
\]

Therefore \( F \) undergoes a simple transcritical bifurcation at \( \alpha = \frac{1}{\gamma} - \sigma \). QED
In our work with the parameter \( \alpha \), we have shown that below a certain \( \alpha \), there are no nonzero fixed points, and above that same \( \alpha \), there is at least one nonzero fixed point. A natural successor to these questions is, what are the fixed points of \( F \) for very large \( \alpha \)? The answer is revealed in our next theorem.

**Theorem 17** Given any \( \gamma, \sigma \in (0, 1) \), there is \( \alpha_0 \) such that \( \alpha > \alpha_0 \Rightarrow F \) has a globally attracting nonzero fixed point.

**PROOF:**
To show that there is a nonzero fixed point, we simply set \( \alpha > \frac{1}{\gamma} - \sigma \) and apply Theorem 9 to obtain the smallest fixed point \( I_0 \).

Now we show that for sufficiently large \( \alpha \), \( I_0 \) is a globally attracting fixed point. We begin by choosing \( \delta_1 \) such that

1. \( \delta_1 > 0 \).
2. \( F_{t-1} \circ \cdots \circ F_1(\delta_1) < F_t \circ F_{t-1} \circ \cdots \circ F_1(\delta_1) \) for \( t = 1 \ldots p \).
3. \( F_{t-1} \circ \cdots \circ F_1(\delta_1) < F_t(I_t) \) for \( t = 1 \ldots p \).

We must first show that such a \( \delta_1 \) exists. Since \( F_t \) is continuous and \( F_t(0) = 0 \) for every \( t \), we can get \( F_1(\delta_1) \) as small as we like by choosing small enough \( \delta_1 \). Suppose for induction that we can get \( F_t \circ \cdots \circ F_1(\delta_1) \) as small as we like by choosing small enough \( \delta_1 \). Then we can get \( F_{t+1} \circ F_t \circ \cdots \circ F_1(\delta_1) = F_{t+1}(F_t \circ \cdots \circ F_1(\delta_1)) \) as small as we like, also. Then by induction, we can get \( F_t \circ \cdots \circ F_1(\delta_1) \) as small we like for any \( t \). So we simply choose a finite number of \( \delta \)'s, each one satisfying satisfying one of the above conditions, and let \( \delta_1 \) be the minimum of all those \( \delta \)'s.

So there is a \( \delta_1 \) satisfying the above three conditions. Note these conditions imply that \( \delta_1 < F_t(I_t) \) for \( t = 1 \ldots p \), since \( \delta_1 < F_{t-1} \circ \cdots \circ F_1(\delta_1) \).

During the rest of this proof, we will do much work with the functions \( \bar{F}_t \), which are just like their counterparts \( F_t \), except that in place of \( \alpha \) they have \( \bar{\alpha} \), where \( \bar{\alpha} > \alpha \). Note that \( F_t(N_t) = \gamma \sigma N_t \) and is therefore totally independent of \( \alpha \), so for each \( t \) we have \( \bar{F}_t(N_t) = F_t(N_t) \), and we retain \( \delta_1 < \bar{F}_t(N_t) \) for \( t = 1 \ldots p \). We also let \( \bar{F} = \bar{F}_p \circ \cdots \circ \bar{F}_1 \).

We now show there is \( \alpha_0 \) such that \( \bar{\alpha} > \alpha_0 \Rightarrow \bar{F} \) has no fixed points on \((0, \delta_1)\). To do this, we first take the derivative of \( F_t \) with respect to \( \alpha \), and obtain

\[
\frac{\partial F_t}{\partial \alpha} = \frac{\gamma I}{N_t} e^{\alpha t} (N_t - I).
\]

The above expression is clearly \( > 0 \) for any \( I \in (0, N_t) \), so we can state that \( F_t \) is always increasing with respect to \( \alpha \) (except at 0 and \( N_t \)). Then for any \( I \in (0, N_t) \) we have \( \bar{F}_t(I) > F_t(I) \); and specifically, for the fixed point \( I_t \) of \( F_t \) we have \( \bar{F}_t(I_t) > F_t(I_t) = I_t \). This gives us \( I_t < \bar{I}_t \), where \( \bar{I}_t \) is the unique nonzero fixed point of \( \bar{F}_t \).
Next, we observe that
\[
\lim_{\alpha \to \infty} \mathcal{F}_t'(I_t) = \lim_{\alpha \to \infty} \gamma e^{-\frac{\alpha t}{N_t}} (\alpha - \frac{\bar{\alpha} I_t}{N_t} + 1) - \gamma + \gamma \sigma = \gamma \sigma - \gamma < 0.
\]

This tells us there is \( \alpha_0 \) such that \( \bar{\alpha} > \alpha_0 \Rightarrow \mathcal{F}_t'(I_t) < 0 \). Then for \( \alpha > \alpha_0 \), we also have \( \mathcal{F}_t'(I_t) < 0 \), since \( I_t > I_r \). That is to say, for \( \bar{\alpha} > \alpha_0 \) we have \( \bar{\alpha} t < I_t \), where \( \bar{\alpha} \) is the unique critical point of \( \mathcal{F}_t \).

We now fix \( \bar{\alpha} > \alpha_0 \), and suppose for contradiction that for some \( I < \delta \), \( \mathcal{F}(I) \leq I \). We then concoct an unusual induction scenario, in which the induction goes backwards from \( p \) to \( 0 \). Our opening hypothesis is that \( \mathcal{F}_p \circ \cdots \circ \mathcal{F}_1(I) < \delta_1 \). This is satisfied by our assumption that \( \mathcal{F}(I) \leq I \), since \( \mathcal{F}_p \circ \cdots \circ \mathcal{F}_1(I) = F(I) \leq I < \delta_1 \). We then suppose for induction that for some \( t \leq p \), we have \( \mathcal{F}_t \circ \mathcal{F}_{t-1} \circ \cdots \circ \mathcal{F}_1(I) < \delta_1 \). Then, since \( \delta_1 < \mathcal{F}_t(N_t) \), we must have \( \mathcal{F}_t \circ \mathcal{F}_{t-1} \circ \cdots \circ \mathcal{F}_1(I) < \mathcal{F}_t(N_t) \). This allows us to conclude that \( \mathcal{F}_{t-1} \circ \cdots \circ \mathcal{F}_1(I) < \bar{\alpha}_t \). (This is the key step. In English, we are arguing that we could not possibly have mapped to \( \mathcal{F}_t \circ \cdots \circ \mathcal{F}_1(I) \) from any point beyond the critical point \( \bar{\alpha}_t \) of \( \mathcal{F}_t \), so we must have mapped from some point \( < \bar{\alpha}_t \).) Then, since \( \bar{\alpha}_t < I_t \), we have \( \mathcal{F}_{t-1} \circ \cdots \circ \mathcal{F}_1(I) < I_t \). Then, since \( I_t \) is the fixed point of \( \mathcal{F}_t \), we have \( \mathcal{F}_{t-1} \circ \cdots \circ \mathcal{F}_1(I) < \mathcal{F}_t \circ \mathcal{F}_{t-1} \circ \cdots \circ \mathcal{F}_1(I) < \delta_1 \).

Our induction hypothesis is satisfied for \( t = p \), so we have \( I < \mathcal{F}_1(I) < \mathcal{F}_2 \circ \mathcal{F}_1(I) < \cdots < \mathcal{F}_p \circ \cdots \circ \mathcal{F}_1(I) = \mathcal{F}(I) \). But this is a contradiction, because we assumed that \( \mathcal{F}(I) \leq I \). Therefore, we have \( \mathcal{F}(I) > I \) for all \( I \in (0, \delta_1) \). So there are no fixed points in \( (0, \delta_1) \).

Now, throughout this proof, we have been working with \( \delta_t \). We now define \( \delta_2 \ldots \delta_p \) inductively as \( \delta_{t+1} = \inf \{ F_t([\delta_t, N_t]) \} \). Then for any \( I \in [\delta_t, N_t] \), we have \( F_t(I) \in [\delta_{t+1}, N_{t+1}] \), so \( F_t([\delta_t, N_t]) \subset [\delta_{t+1}, N_{t+1}] \) for \( t = 1 \ldots p - 1 \). Moreover, we have \( F_t(I) \geq F_t(I) \) for any \( I \in [0, N_t] \), so we have \( F_t([\delta_t, N_t]) \subset [\delta_{t+1}, N_{t+1}] \) for \( t = 1 \ldots p - 1 \). Finally, for any \( I \in [\delta_t, N_t] \), we have \( F_t(I) \leq F_t'(\delta_t) \), since \( F_t' \) is always decreasing.

We know that \( \lim_{\alpha \to \infty} \mathcal{F}_t'(\delta_t) = \gamma \sigma - \gamma < 0 \), so for each \( \delta_t \) there is \( \alpha_t \) such that \( \bar{\alpha} > \alpha_t \Rightarrow \mathcal{F}_t'(\delta_t) < 0 \).

We now fix \( \bar{\alpha} > \max \{ \alpha_t : 0 \leq t \leq p \} \), where \( \alpha_0 \) was the \( \alpha \) required for the earlier phase of the problem. Then for \( t = 1 \ldots p \) and for any \( I \in [\delta_t, N_t] \), we have \( F_t(I) \leq F_t'(\delta_t) < 0 \). From Theorem 4, we also have \( F_t'(I) > -1 \), so we have \( |F_t'(I)| < 1 \) for \( t = 1 \ldots p \) and for all \( I \in [\delta_t, N_t] \).

Now we take any \( I \in [\delta_t, N_t] \), and take the derivative of \( \mathcal{F} \) at \( I \). The Chain Rule gives us \( \mathcal{F}'(I) = \Pi_{1 \leq t \leq p} \mathcal{F}_t'(\mathcal{F}_{t-1} \circ \cdots \circ \mathcal{F}_1(I)) \). We always have \( \mathcal{F}_{t-1} \circ \cdots \circ \mathcal{F}_1(I) \in [\delta_t, N_t] \), so we have \( |\mathcal{F}_t'(\mathcal{F}_{t-1} \circ \cdots \circ \mathcal{F}_1(I))| < 1 \) for every \( t \). This gives us \( |\mathcal{F}'(I)| < 1 \). Our value \( I \) was arbitrary, so we have \( |\mathcal{F}'(I)| < 1 \) for all \( I \in [\delta_1, N_1] \).

From here we can easily conclude that the nonzero fixed point \( I_0 \), guaranteed by Theorem 6, is unique. All fixed points must reside in \( [\delta_1, N_1] \) and if we had two fixed points there, the Mean Value Theorem would guarantee the existence of some point of derivative 1 between them, which is impossible. Therefore \( I_0 \) is the unique nonzero fixed point of \( \mathcal{F} \).

Finally, we show that \( I_0 \) is globally attracting. To do this, we must first show that \( \mathcal{F}_p([\delta_p, N_p]) \subset [\delta_1, N_1] \). To do that, we first assume for induction that for some \( t, 1 < t < p \), we have \( \delta_t = \mathcal{F}_{t-1} \circ \cdots \circ \mathcal{F}_1(\delta_1) \). (Note that we are working with \( F \), not \( \mathcal{F} \).) Then \( [\delta_t, N_t] = [\mathcal{F}_{t-1} \circ \cdots \circ \mathcal{F}_1(\delta_1), N_t] \). By our
definition of $\delta_1$, we have guaranteed that $F_t \circ \cdots \circ F_1(\delta_1) < F_t(N_t)$, so we have $\delta_{t+1} = F_t \circ \cdots \circ F_1(\delta_1)$. We also have $F_1(\delta_1) < F_1(N_1)$, so $\delta_2 = F_1(\delta_1)$. Our induction hypotheses are satisfied, then, and we have $\delta_p = F_{p-1} \circ \cdots \circ F_1(\delta_1)$. This, in turn, gives us $F_p(\delta_p) = F(\delta_1)$, and we know $F(\delta_1) > \delta_1$ by our first condition for $\delta_1$, so we have $F_p(\delta_p) > \delta_1$. Then we have $F_p(\delta_p) > \delta_1$, also. Finally, we have $F_p(N_p) > \delta_1$, so we have $F_p(I) > \delta_1$ for all $I \in [\delta_p, N_p]$, and therefore $F_p(\delta_p, N_p) \subset [\delta_1, N_1]$. This allows us to draw the all-important conclusion that $\bar{F}(\delta_1, N_1) \subset [\delta_1, N_1]$. 

Very good. Our next goal is to show that for any $I \in (0, N_1]$, there is $n$ such that $\bar{F}^n(I) \in [\delta_1, I_0]$. We can then apply Theorem 8 and show that $\bar{F}^n(I) \rightarrow I_0$.

CASE 1: $I \in (0, \delta_1)$. Then $\bar{F}(I) > I$. If $\bar{F}^n(I) > I$ for all $n$ then $\bar{F}^n(I)$ is a monotone increasing sequence, bounded above by $I_0$, and therefore must converge to $I_0$. Otherwise there is $n$ such that $\bar{F}^n(I) \geq I_0$. We cover this circumstance in our next case.

CASE 2: $I \in (I_0, N_1)$. Then $I \in [\delta_1, N_1]$ and so we have $\bar{F}^n(I) \in [\delta_1, N_1]$ for all $n$. Suppose that for every $n$ we have $\bar{F}^{n+1}(I) < \bar{F}^n(I)$. Then $\bar{F}^n(I)$ is a monotone decreasing sequence, bounded below by $I_0$, and therefore must converge to $I_0$. Otherwise we have $\bar{F}^{n+1}(I) \geq \bar{F}^n(I)$ for some $n$ and therefore $\bar{F}^{n+1}(I) \in [\delta_1, I_0]$.

Therefore, for any $I \in (0, N_1)$, there is $n$ such that $\bar{F}^n(I) \in [\delta_1, I_0]$. We know that $|\bar{F}'(I)| < 1$ for all $I \in (\delta_1, I_0)$, so we can apply Theorem 8 and conclude that $\bar{F}^n(I) \rightarrow I_0$ for all $I \in (\delta_1, I_0)$. Then we have $\bar{F}^n(I) \rightarrow I_0$ for all $I \in (0, N_1)$, so $I_0$ is globally attracting. QED

The implications of this theorem, from a mathematical perspective, are simple. It states that for sufficiently large $\alpha$, $F$ has exactly one nonzero fixed point. The reason for this is that as we increase $\alpha$, $F$ converges to a function which is linear over $(0, N_1]$ and discontinuous at 0. We provide a graph of an $F$ with a very large $\alpha$ in Figure 5 below.

![Figure 5: A 4-periodic infection set with $\alpha = 500$](image)
From a biological standpoint, Theorem 15 tells us that if a disease is sufficiently infectious (i.e. if \( \alpha \) is sufficiently large), we can predict its long-term behavior without worrying about the initial conditions. A hypothetical application: a biologist concocts a super-infective insect virus, and asks his assistant to infect a certain number of insects with it. The assistant does so, but then forgets how many insects he infected. If the disease is sufficiently contagious, the assistant’s forgetfulness does not matter: the biologist can calculate the long-term dynamics of the epidemic without knowing how many insects were originally infected.

6 Conclusion

In this paper we reached a great many conclusions about the functions \( f_t \), the infection-maps \( F_t \), and the infection-composition \( F \). However, we also encountered several questions which we were not able to answer.

First, does every \( F \) with \( p = 3 \) have at most one nonzero fixed point? All our experiments point towards the conclusion that yes, only one fixed point is possible in the \( p = 3 \) case, but we were unable to prove this proposition. Perhaps Theorem 5, which equates \( F \) to a mapping \( H \) which is a composition of functions which are all scalar multiples of \( F_1 \), could be used to answer this question, but unfortunately we discovered Theorem 5 at a late point in our studies.

Secondly, is it possible to have three attractors in the \( p = 4 \) case? Again, all our research implies that this is not possible, but we were unable to come to a rigorous conclusion regarding this question.

Finally, can the number of attractors be made arbitrarily large by increasing the cycle-length \( p \) and tweaking the parameters? Overwhelmingly, we suspect that the answer is yes, but it is extremely difficult to mathematically deduce an attractor, particularly as the cycle-length \( p \) grows large. We suspect that Theorem 6 might be of some utility in answering this question.

At any rate, we are happy to have answered several unknown questions about this particular mapping, and to have propounded several more questions which we, or other mathematicians, might address in the future.
REFERENCES

Before beginning this project, we spent a considerable amount of time proving, "from scratch," the relationship between the derivative at a fixed point and that fixed point’s behavior. We did this because, not having a strong background in fixed-point real analysis, we wished to familiarize ourselves with the subject before undertaking any new research. The following results summarize our work.

We first consider a linear mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ with eigenvalues $\lambda_1, \lambda_2$, where $|\lambda_1| \geq |\lambda_2|$.

**Theorem 18** If $|\lambda_1| < 1$, then for any $\alpha > 0$ there is $N$ such that $n > N$ implies $||A^n x|| < \alpha ||x||$ for all $x \in \mathbb{R}^2, x \neq 0$.

**PROOF:** We have the Jordan decomposition $A = PDP^{-1}$ and $A^n = PD^n P^{-1}$.

If $\lambda_1 \neq \lambda_2$, then $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $D^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$. With the column sum norm, $||D^n|| = |\lambda_1^n|$, and since $|\lambda_1| < 1$ we know that $\lim_{n \to \infty} ||D^n|| = 0$.

On the other hand, if $\lambda_1 = \lambda_2$, then $D = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$. We suppose for induction that for some $k$,

$D^k = \begin{pmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{pmatrix}$. Then $D^{k+1} = DD^k = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{pmatrix} = \begin{pmatrix} \lambda_1^{k+1} & (k+1)\lambda_1^k \\ 0 & \lambda_1^{k+1} \end{pmatrix}$.

$D^n = \begin{pmatrix} \lambda_1^n & n\lambda_1^{n-1} \\ 0 & \lambda_1^n \end{pmatrix}$ so by induction we have $D^n = \begin{pmatrix} \lambda_1^n & n\lambda_1^{n-1} \\ 0 & \lambda_1^n \end{pmatrix}$ for all $n \in \mathbb{N}$.

Now we show that the upper-right term goes to zero as $n \to \infty$. By L’Hospital’s Rule,

$$\lim_{n \to \infty} n\lambda_1^{n-1} = \lim_{n \to \infty} \frac{n}{\lambda_1^{1-n}} = \lim_{n \to \infty} \frac{1}{-\ln(\lambda_1)\lambda_1^{1-n}} = \lim_{n \to \infty} \frac{\lambda_1^{n-1}}{-\ln(\lambda_1)} = 0.$$  

We have established that as $n \to \infty$, $n\lambda_1^{n-1} \to 0$, we know that $\lambda^n \to 0$, therefore with the column sum norm we have $||D^n|| \to 0$ as $n \to \infty$. We already proved this condition for the case where $\lambda_1 \neq \lambda_2$, so now we have it in both cases.

So, $\lim_{n \to \infty} ||D^n|| = 0$. Then for any $\alpha > 0$ there is $N$ such that $n > N \to ||D^n|| < \frac{\alpha}{2||P||||P^{-1}||}$.

Then for any $x \in \mathbb{R}^2$ we have $||A^n x|| \leq ||P|| ||D^n|| ||P^{-1}|| ||x|| < ||P|| \frac{\alpha}{2||P||||P^{-1}||} ||P^{-1}|| ||x|| = \alpha ||x||$. QED

Next, we consider the reverse case, where $|\lambda_1| \geq |\lambda_2| > 1$.

**Theorem 19** Let $|\lambda_2| > 1$. Then for any $\alpha > 0$ there is $N$ such that $n > N$ implies $||A^n x|| > \alpha ||x||$.
for all \( x \neq 0 \).

**PROOF:** We use the 2-norm throughout this proof. We know that \( \min_{x \in \mathbb{R}^2} \frac{|Cx|}{||x||} = \frac{1}{||C^{-1}||} \)
for any invertible matrix \( C \). We have \( |\lambda_1| \geq |\lambda_2| > 1 \), so \( A \) is invertible and we can say that \( \min_{x \in \mathbb{R}^2} \frac{|A^n x|}{||x||} = \frac{1}{||A^{-n}||} \). Since the eigenvalues of \( A \) are all greater than 1 in absolute value, the eigenvalues of \( A^{-1} \) are all less than 1 in absolute value, so, by Theorem 1, for arbitrary \( \alpha > 0 \) there is \( N \) such that \( n > N \) implies \( ||A^{-n}|| < \frac{1}{\alpha} ||x|| \) for any \( x \neq 0 \). This, in turn, implies that \( ||A^{-n}|| < \frac{1}{\alpha} \), so we have \( \alpha < \frac{1}{||A^{-1}||} = \min_{x \in \mathbb{R}^2} \frac{||A^n x||}{||x||} \). From here we can easily conclude that \( ||A^n x|| > \alpha ||x|| \) for any \( x \neq 0 \). QED

We now consider arbitrary \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), not necessarily linear, which is differentiable at some fixed point \( p \). We let \( A = Df(p) \), and \( \lambda_1, \lambda_2 \) be the eigenvalues of \( A \), \( |\lambda_1| \geq |\lambda_2| \).

**Theorem 20** \( \text{If } |\lambda_1| < 1, \text{ then } f \text{ is an attracting fixed point.} \)

**PROOF:** We take \( \epsilon \in (0, 1) \). Then, by Theorem 1, there is \( N \) such that \( n > N \Rightarrow ||A^n x|| < \frac{\epsilon}{2} ||x|| \) for all \( x \neq 0 \). We fix \( n \) at some value beyond \( N \).

Next, we consider the definition of the derivative of \( f^{n+t} \) at \( p \), for some integer \( t \in [0, n-1] \). This definition tells us there is \( \delta_t > 0 \) such that \( ||x - p|| < \delta_t \Rightarrow ||f^{n+t}(x) - f^{n+t}(p) - D(f^{n+t})(p)(x - p)|| < \frac{\epsilon}{2} ||x - p|| \). Then, if we let \( \delta = \max(\delta_t) \), we have

\[
||x - p|| < \delta \Rightarrow ||f^{n+t}(x) - f^{n+t}(p) - D(f^{n+t})(p)(x - p)|| < \frac{\epsilon}{2} ||x - p||
\]

for all integers \( t \in [0, n-1] \). Henceforth we fix \( t \) at 0, and work with an arbitrary \( x \) such that \( ||x - p|| < \delta ||x - p|| \).

Now we rewrite the above inequality. We know \( p \) is a fixed point, so \( f^n(p) = p \). Moreover, by the chain rule \( D(f^n)(p) = \Pi_{0 \leq i < n} Df(f^i(p)) = \Pi_{0 \leq t < n} Df(p) = A^n \). Also, using the stock norm-inequality \( ||x|| - ||y|| \leq ||x - y|| \), we can rewrite \( ||f^n(x) - f^n(p)|| \) as the difference of two absolute values, and still preserve the inequality. All this finagling yields

\[
||f^n(x) - p|| - ||A^n(x - p)|| < \frac{\epsilon}{2} ||x - p||.
\]

In the first line of this theorem we obtained \( ||A^n(x - p)|| < \frac{\epsilon}{2} ||x - p|| \), so we can add that to the above inequality and obtain \( ||f^n(x) - p|| < \epsilon ||x - p|| \).

We are basically done. We now suppose for induction that \( ||f^{kn+t}(x) - p|| < \epsilon^k ||x - p|| \) for some \( k \) and some \( t = 0 \cdots k - 1 \). Then, since \( \epsilon \in (0, 1) \), we know that \( ||f^{kn+t}(x) - p|| < ||x - p|| < \delta \) and
we can write \( \|f^{(k+1)n+t}(x) - p\| < \varepsilon \|f^{kn+t}(x) - p\| < \varepsilon^{k+1}\|x - p\| < \varepsilon\|x - p\| \). Then our induction condition is satisfied for \( k + 1 \). We know the induction condition is satisfied for \( k = 1 \) (that is, \( \|f^{n+t}(x) - p\| < \varepsilon\|x - p\| \) for any \( t < n \)), so we have \( \|f^{kn+t}(x) - p\| < \varepsilon^{k}\|x - p\| < \varepsilon \) for all \( k \geq 1 \), all integers \( t \in [0, n) \). This is equivalent to saying ”all integers greater than \( n \),” and since \( \varepsilon \) was arbitrary we can conclude that the sequence \( f^n(x) \) converges to \( p \), therefore \( p \) is an attracting fixed point. QED

Next, we prove an analogous result for the case where the eigenvalues of \( A \) are \( > 1 \).

**Theorem 21** If \( |\lambda_2| > 1 \), then \( p \) is a repelling fixed point.

**PROOF:** We choose \( \varepsilon \in (0, 1) \). Theorem 1 tells us there is \( N \) such that \( n > N \Rightarrow \|Ax\| > (2+\varepsilon)|x| \) for all \( x \neq 0 \). We fix \( n \) at some value greater than \( N \).

Next, we consider the definition of the derivative of \( f^n \) at \( p \). We know there is \( \delta > 0 \) such that

\[
\|x - p\| < \delta \Rightarrow \|A^n(x - p)\| - \|f^n(x) - p\| < \varepsilon\|x - p\| .
\]

Henceforth we fix \( x \) at an arbitrary value such that \( \|x - p\| < \delta \). We know that \( \|A^n(x - p)\| > (2+\varepsilon)|x - p| \), which is the same as \( -\|A^n(x - p)\| < (-2 - \varepsilon)|x - p| \), so, adding this inequality to the inequality near the beginning of this proof, we obtain \( -\|f^n(x) - p\| < -2|x - p| \) which, when multiplied by \( -1 \), yields \( \|f^n(x) - p\| > 2|x - p| \).

Now we suppose for contradiction that \( \|f^{kn}(x) - p\| < \delta \) for all positive integers \( k \). Then we suppose for induction that \( \|f^{kn}(x) - p\| > 2^k|x - p| \) for some \( k \).

Then, noting that \( f^n(f^{kn}(x)) = f^{(k+1)n}(x) \), we see that \( \|f^{(k+1)n}(x) - p\| > 2\|f^{kn}(x) - p\| > 2^{k+1}|x - p| \). We know that \( \|f^n(x) - p\| > 2|x - p| \) (i.e. the \( k = 1 \) case holds), so we have \( \|f^{kn}(x) - p\| > 2^k|x - p| \) for all \( k \). Clearly there must be \( k \) such that \( 2^k\|x - p\| > \delta \), here is our contradiction, so we do, in fact, have \( k \) such that \( \|f^{kn}(x) - p\| \geq \delta \).

Therefore, for any \( x \) such that \( \|x - p\| < \delta \), there is \( n \) such that \( \|f^n(x) - p\| \geq \delta \), so \( x \) is a repelling fixed point. QED

Finally, we prove a very well-known general result which we will cite numerous times in the body of this paper.

**Theorem 22** Let \( f : A \subset \mathbb{R} \rightarrow \mathbb{R} \) be continuous and differentiable throughout \( A \), and let the derivative of \( f \) be bounded on \( A \). Then, given the sequence \( F^n(x) = \{x, F(x), F^2(x), \ldots \} \), if \( F^n(x) \) converges, it must converge to a fixed point.

**PROOF:**

Suppose for contradiction that there is nonzero \( x \) such that \( F^n(x) \) converges to some non-fixed \( y \). Then for any \( \varepsilon > 0 \), there is \( N \) such that \( n > N \Rightarrow |F^n(x) - y| < \varepsilon \).
Now we do some absolute value fiddling, and journey from the standard Mean-Value-Theorem numerator to our desired destination. 

\[ |F^{n+1}(x) - F(y)| = |F(y) - F^{n+1}(x)| = |F(y) - y + y - F^{n+1}(x)| = |F(y) - y - (F^{n+1}(x) - y)| \geq ||F(y) - y| - |F^{n+1}(x) - y|| \geq |F(y) - y| - \epsilon. \]

So we have

\[ |F^{n+1}(x) - F(y)| \geq |F(y) - y| - \epsilon \text{ and } |F^n(x) - y| < \epsilon \]

which allows us to conclude that

\[ \frac{|F^{n+1}(x) - F(y)|}{|F^n(x) - y|} > \frac{|F(y) - y| - \epsilon}{\epsilon}. \]

Now, the Mean Value Theorem tells us that there must be \( c \) between \( F^n(x) \) and \( y \) such that

\[ |F'(c)| = \frac{|F^{n+1}(x) - F(y)|}{|F^n(x) - y|} > \frac{|F(y) - y| - \epsilon}{\epsilon}. \]

But clearly

\[ \lim_{\epsilon \to 0} \frac{|F(y) - y| - \epsilon}{\epsilon} = \infty, \]

and since for any \( \epsilon > 0 \) we can find \( c \) such that the above inequalities hold, we can find \( c \) with \( |F'(c)| \) unboundedly large. This contradicts our assumption that \( F' \) is bounded on \( A \). Therefore, the sequence \( F^n(x) \) cannot converge to nonfixed \( y \), and must converge, if it converges at all, to some fixed point. QED