Abstract

RAHMOELLER, MARGARET LYNN . On Demazure Crystals for the Quantum Affine Algebra $U_q\left(\hat{sl}(n)\right)$. (Under the direction of Dr. Kailash Misra.)

Lie algebras and their representations have been an important area of mathematical research due to their relevance in several areas of mathematics and physics. It is known that the representation of a Lie algebra can be studied by considering the representations of its universal enveloping algebra, which is an associative algebra. In 1968, Victor Kac and Robert Moody defined a class of infinite dimensional Lie algebras called affine Lie algebras. An affine Lie algebra can be viewed as the universal central extension of the Lie algebra of polynomial maps from the unity circle to a finite dimensional simple Lie algebra. In this thesis, we consider the affine Lie algebra $\hat{sl}(n)$ associated with the finite dimensional simple Lie algebra $sl(n)$ of $n \times n$ trace zero matrices over the field of complex numbers. We consider certain representations of the quantum affine Lie algebra $U_q(\hat{sl}(n))$, which is a q-deformation of the universal enveloping algebra $U(\hat{sl}(n))$ of $\hat{sl}(n)$ introduced by Michio Jimbo and Vladimir Drinfeld in 1985. In 1988, it was shown by George Lusztig that the q-deformation of a representation of $U_q(\hat{sl}(n))$ parallels the representation of $\hat{sl}(n)$ for generic q.

It is known that for any dominant integral weight λ there is a unique (up to isomorphism) irreducible $U_q(\hat{sl}(n))$- module $V(\lambda)$. It was shown by Kashiwara that this irreducible module $V(\lambda)$ admits a crystal base $(L(\lambda), B(\lambda))$. The set $B(\lambda)$ is called a crystal, and it provides a nice combinatorial tool to study the combinatorial properties of the representation space $V(\lambda)$.

Let $\mathcal{W}$ be the Weyl group of the affine Lie algebra $\hat{sl}(n)$. For $w \in \mathcal{W}$ we consider the subspace $V_w(\lambda)$ generated by the extremal weight vector $v_{w\lambda}$ and the positive half $U_q^+(\hat{sl}(n))$ of the quantum affine algebra $U_q(\hat{sl}(n))$. We note that $\bigcup_{w\in W} V_w(\lambda) = V(\lambda)$. These subspaces $V_w(\lambda)$ are called Demazure modules and are used for inductive arguments. In 1993, Kashiwara showed that $V_w(\lambda)$ admits a crystal base and that a certain subset $B_w(\lambda)$ of the crystal $B(\lambda)$ is the crystal for $V_w(\lambda)$. Kang, Kashiwara, Misra, Miwa, Nakashima and Nakayashiki gave the path realizations of affine crystals as a semi-infinite tensor product of some finite crystals called perfect crystals in 1991.

In this thesis, we use this path realization and study the combinatorial properties of certain Demazure crystals $B_w(\lambda)$ for the quantum affine algebra $U_q(\hat{sl}(n))$. In particular, it is shown that the intersection and union of a certain family of Demazure crystals for $U_q(\hat{sl}(n))$ can be realized as tensor products of finitely many perfect crystals.

On Demazure Crystals for the Quantum Affine Algebra $U_q\left(\hat{sl}(n)\right)$

by
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Dedication

I dedicate this to my Mom. She is my biggest role model, and has supported me wholeheartedly throughout my entire life. Thank you, Mom, for all of your encouragement when life threw me curveballs, your patience when I was struggling, your comfort when I was sad or upset, and your unconditional love.

Biography

Margaret Rahmoeller was born in June of 1988 in the wonderful Webster Groves, a suburb of St. Louis, Missouri. Nicknamed Maggie, she was the third child of Ken and Diane Rahmoeller, their first being Katie and second being Brad. Not too long after, Maggie was the proud older sister of Karen and Brenda. Maggie grew up playing outside, climbing trees, collecting bugs and slugs for her teachers, and being an all-around adorable child.

As a child, she dreamed of being an astronaut or an actress. Little did she know that she was destined to be a teacher. In sixth grade, she began playing the oboe, much to the horror of her family. A unique woodwind instrument, the oboe is typically called the "duck" instrument. However, after starting lessons and practicing very hard, she began to sound more like an oboist. She already had a love for music, since she had taken piano lessons for several years; however, playing the oboe increased her appreciation for classical music.

At the end of high school, she had a very important decision to make- what to study in college. She chose to study mathematics and music, since she loved them both dearly. She spent an amazing four years studying both at McKendree University, located in Lebanon, Illinois. Along the way, she also picked up a computer science minor. Although most of her time was filled with her studies and tutoring, she managed to spend quality time with stupendous new friends, performing with the St. Louis Symphony Youth Orchestra as principal oboist, and helping out with math club. At the end of her time at good 'ole McK, she had another important decision to make- what to do after graduation.

Maggie spent a summer doing an internship in Maryland before heading off to graduate school to study mathematics at North Carolina State University in Raleigh, North Carolina. During her first semester as a teaching assistant for a Calculus III class, she had the opportunity to teach one lecture. She fell in love with teaching that day. By participating in pedagogical programs such as Certificate of Accomplishment in Teaching and Preparing the Professoriate, she learned as much as she could about different aspects of teaching.

She also performed as principal oboist with both the Raleigh Civic Symphony and the Raleigh Civic Chamber Orchestra, ensembles at NCSU consisting of both students and community members. In her fourth year at NCSU, she joined the Raleigh Civic Symphony Association, a nonprofit organization, which works in partnership with the music department to promote the orchestras at NCSU.

But most of her time was spent studying mathematics. She was introduced to Lie algebras in her third year, and quickly became intrigued. She decided to work with Dr. Kailash Misra in the field of Lie algebra representation theory. After several years of hard work, she will be defending her thesis in front of a live audience in the summer of 2015. Then, in August of 2015, she will begin a position at Roanoke College in Salem, Virginia as a Visiting Assistant Professor. She is super excited for the next chapter of her life to begin!

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I would also like to thank my Grandpa for his words of wisdom and encouragement. I've always looked up to him, both literally and figuratively, and have enjoyed all of the times we have shared. Thank you to Mr. and Mrs. Nuelle, who I have always considered grandparents, for the many memorable lunches and for many of my wonderful childhood memories. Mr. Nuelle, thank you for pushing me to finish my degree in five years and for all the math documentaries we watched together. And I of course want to thank one Mr. Russ for all of the fun times in orchestra at State and the delightful duets we played. You are an incredibly talented bassoonist and a wonderful friend.

I had other mentors at NCSU who I'd also like to thank. Professor Stitz, you have always been immensely helpful, whether by having an answer to all of my questions, or helping me make important decisions, or by always having a smile and words of encouragement. You never minded when I stopped by to ask a question or just to have a friendly conversation. Thank you! I would also like to thank Dr. Molly Fenn for all of her help with my teaching dilemmas and questions. You are an amazing teacher, and I learned so much from you. And thank you to my committee members, Dr. Lada, Dr. Stitz, and Dr. Martin, for all of your help, your time, and your advice.

Last, but not least, I'd like to thank all of my friends. Abby, I'm so glad we became friends- I couldn't have made it through graduate school without you. We survived the initial shock of graduate school together and had so many awesome times together. I still can't believe we made ravioli! I'd also like to thank Kristin for being the best roommate ever, and for all the fun times we had. Thank you, Alex, for all of your helpful lecture notes, your wisdom (you are still one of the smartest people I know), and all of our adventures. Will, thanks for your help with the LaTex technicalities, the many trips to different restaurants, and your friendship. TJ, thanks for all of the long walks and good conversations! And, Alyssa, thank you for all of your help, patience when answering all my questions, and words of encouragement.

There are many others who I'd like to thanks, but I have to keep this decently short. So, thank you to all of my other friends who have supported me and who have helped create wonderful memories for the past five years. I could not have made it without you.

Table of Contents

LIST OF FIGURES

Chapter

1

Introduction

In the late 1800's, Sophus Lie was studying contact transformations in geometry and differential equations. This led to his discovery of continuous transformation groups and infinitesimal groups, which today are what we call Lie groups and Lie algebras, respectively. Wilhelm Killing also discovered Lie algebras, independently of Lie. His main contribution in the late 1880's was the classification of the complex finite-dimensional simple Lie algebras. Along the way, he made several important conjectures and introduced the concepts behind Cartan subalgebras, the Cartan matrix, and root systems. He also discovered the exceptional Lie algebras, though he tried hard to get rid of them. Élie Cartan continued the work of Lie and Killing, rewriting a lot of Killing's work on the classification of semisimple Lie algebras and proving the existence of the exceptional Lie algebras (cf. [7]). He also introduced representations of semisimple Lie algebras, tying together geometry, differential geometry, and topology. In differential geometry, he developed a theory of moving frames, leading to the idea of a fibre bundle [3]. The Cartan matrix $A=(a_{ij})$ associated with a finite dimensional semisimple Lie algebra $\mathfrak{g}$ is a $n\times n$ positive definite integral matrix with $a_{ii}=2, a_{ij}\leq 0$ and $a_{ii}=0$ if and only if $a_{ji}=0$. The algebraist Jean-Pierre Serre [29] gave the construction of any finite dimensional semisimple Lie algebra with Cartan matrix A via generators and relations.

An $n\times n$ integral matrix A is a generalized Cartan matrix (GCM) if $a_{ii}=2, a_{ij}\leq 0$ and $a_{ii}=0$ if and only if $a_{ji} = 0$. A GCM A is said to be an affine GCM if it has corank 1 and if it has a positive null vector. An affine GCM is indecomposable if it is not equivalent to a matrix in block form. If we delete the first row and column of an indecomposable GCM A the remaining matrix is a Cartan matrix for a finite dimensional simple Lie algebra. Motivated by Serre's result, Victor Kac [10] and Robert Moody [28] defined Lie algebras associated with any GCM A via generators and relations that we now call Kac-Moody Lie algebras. When the GCM A is not positive definite the associated Kac-Moody Lie algebra is infinite dimensional. The Kac-Moody Lie algebra associated with an affine GCM is called an affine Lie algebra.

Consider the finite-dimensional simple special linear Lie algebra $sl(n,\mathbb{C})$, consisting of the set of all $n \times n$ trace zero matrices with entries in the field of complex numbers $\mathbb{C}$. In this thesis, we focus on the corresponding affine Kac-Moody Lie algebra $\hat{sl}(n,\mathbb{C}) = sl(n,\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where c is a central element and d is a derivation. In Chapter 2 of this thesis, we review some basic definitions about the structure and representation theory of Lie algebras from ([7], [12]).

In 1985, Michio Jimbo [8] and Vladimir Drinfeld [4] introduced the notion of the quantum group $U_q(\mathfrak{g})$, as a q-deformation of the universal enveloping algebra of a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. It is known [23] that for generic q, the integrable representation of $U_q(\mathfrak{g})$ parallels that of $\mathfrak{g}$. The quantum group associated with an affine Lie algebra is called a quantum affine algebra. In this thesis we focus on the quantum affine algebra $U_q(\hat{sl}(n))$. In Chapter 3, we recall some necessary definitions and properties about the representation theory of quantum groups from [6].

For each dominant integral weight λ, there exists a unique (up to isomorphism) irreducible integrable highest weight $U_q(\mathfrak{g})$-module $V^q(\lambda)$. Let $\mathcal{W}$ denote the Weyl group of $\mathfrak{g}$. For each $w \in \mathcal{W}$, the $w\lambda$ weight space of $V^q(\lambda)$ is one-dimensional and spanned by the extremal weight vector $v_{w\lambda}$. Let $U_q^+(\mathfrak{g})$ be the upper half of the quantum group $U_q(\mathfrak{g})$. The subspace $V_w(\lambda) = U_q^+(\mathfrak{g})v_{w\lambda}$ is called a Demazure module.

The work by Jimbo and Drinfeld led to the development of crystal base theory by Masaki Kashiwara [16] and Lusztig [23], independently. Crystal base theory provides a combinatorial tool to study Lie algebra representation theory. Each irreducible integrable highest weight $U_q(\mathfrak{g})$-module $V^q(\lambda)$ has an associated crystal basis [16]. In Chapter 3 we recall some basic definitions and properties about crystal bases from [6]. We also recall the path realization of affine crystals as semi-infinite tensor products of certain finite crystals called "perfect crystals" [14].

In 1993, Kashiwara proved the existence of crystal bases for Demazure modules [18] and showed that the associated crystals are subsets of the crystal for $V^q(\lambda)$. He showed that the Demazure crystals have certain a recursive property. Later, Kuniba, Misra, Okada, and Uchiyama gave path realizations of Demazure crystals and showed that under certain criterion the Demazure crystals have tensor product-like structures [21].

In Chapter 4, we define Demazure modules and their crystals. For certain Weyl group elements $w(L,i)$, we give explicit descriptions for Demazure crystals $B_{w(L,i)}(\ell\Lambda_j)$ for $U_q(\hat{sl}(n))$. We then prove that the union (respectively, the intersection) over i of these Demazure crystals can be realized as the tensor product of L (respectively, $L-1$) copies of the associated perfect crystal. In Chapter 5, we give explicit realizations of Demazure crystals $B_{w(L,i)}(\lambda)$ for $U_q(\hat{sl}(3))$, where λ is any dominant integrable weight. In Chapter 6, we use our algorithms to generalize the previous results, giving realizations of Demazure crystals $B_{w(L,i)}(\lambda)$ for $U_q(\hat{sl}(n))$ and showing that the union and the intersection over i of these crystals are tensor products of finitely many perfect crystals.

Chapter

2

Lie Algebra Representations

In this chapter, we recall the basic definitions and properties of Lie algebras and their representations from [7], [11], and [25]. We use the Lie algebra $sl(n,\mathbb{C})$ and its associated affine Lie algebra $\hat{sl}(n,\mathbb{C})$ as the running example.

2.1 Lie Algebras

We begin this section with the definition of a Lie algebra, and then recall background information about Lie algebra representation theory.

Definition 2.1.1. [25] A vector space $\mathfrak{g}$ over the field $\mathbb{C}$ is a *Lie algebra* if there is a product, which is called the *bracket*, $[\cdot,\cdot]: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, such that

1. $[ax+by,z]=a[x,z]+b[y,z]$ and $[x,ay+bz]=a[x,y]+b[x,z]$ (bilinearity),

2. $[x,x]=0$,

3. $[x,[y,z]]=[[x,y],z]+[y,[x,z]]$ (Jacobi identity),

for all $x,y,z\in\mathfrak{g}$ and $a,b\in\mathbb{C}$.

Note that the first two axioms in Definition 2.1.1 imply that $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is *abelian* if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

Any associative algebra $\mathfrak{g}$ is a Lie algebra under the *commutator bracket*, defined by $[x, y] = xy - yx$ for all $x, y \in \mathfrak{g}$. One of the simplest Lie algebra is the vector space of all 2×2, trace-zero matrices with the commutator bracket.

Example 2.1.2. Let $\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a, b, c \in \mathbb{C} \right\}$ be the vector space of all 2×2, trace-zero matrices. Then the set

$$\left\{ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

is a basis for $\mathfrak{g}$. Using the commutator bracket, $[A, B] = AB - BA$ for all $A, B \in \mathfrak{g}$, we have

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

Then $\mathfrak{g}$ is a Lie algebra, which we denote by both A_1 and $sl(2, \mathbb{C})$.

Example 2.1.3. [25] The Lie algebra $sl(2,\mathbb{C})$ from the above example can be extended to the Lie algebra $sl(n+1,\mathbb{C})$ of $(n+1)\times(n+1)$ trace-zero matrices, called the *special linear Lie algebra*. We can also denote this Lie algebra by A_n. The set $\left\{ H_i = E_{ii} - E_{i+1,i+1}, E_{jk} \middle| 1 \le i \le n, 1 \le j \ne k \le n+1 \right\}$, where E_{ij} denotes the $(n+1)\times(n+1)$ matrix with a one in the (i, j)-entry and zeros elsewhere, is a basis for the Lie algebra $sl(n+1,\mathbb{C})$. Note that, under the commutator bracket,

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}, \text{ where } \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \ne j. \end{cases}$$

A subspace $\mathfrak{s}$ of a Lie algebra $\mathfrak{g}$ is a *subalgebra* if $[x, y] \in \mathfrak{s}$ for all $x, y \in \mathfrak{s}$. Under the bracket in $\mathfrak{g}$, a subalgebra $\mathfrak{s}$ is a Lie algebra.

Example 2.1.4. [7] Let $\mathfrak{g} = sl(n+1,\mathbb{C})$. Then the set $\mathfrak{h} = \operatorname{span}\left\{ H_i = E_{ii} - E_{i+1,i+1} \mid 1 \le i \le n \right\}$ is a subalgebra of $\mathfrak{g}$. We call this subalgebra the *Cartan subalgebra*.

Let $\mathfrak{g}$ be a Lie algebra. Then a subspace $\mathfrak{s}$ of $\mathfrak{g}$ is an *ideal* of $\mathfrak{g}$ if $[x, y] \in \mathfrak{s}$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{s}$. If $\mathfrak{g}$ is nonabelian, i.e. $[\mathfrak{g}, \mathfrak{g}] \ne \{0\}$, and if $\{0\}$ and $\mathfrak{g}$ are the only ideals of $\mathfrak{g}$, then $\mathfrak{g}$ is *simple*. For example, $sl(n+1,\mathbb{C})$ is a simple Lie algebra. A Lie algebra $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^{(m)} = \{0\}$ for some $m \in \mathbb{Z}_{>0}$, where

$$\mathfrak{g}^{(0)} = \mathfrak{g},$$

$$\mathfrak{g}^{(m)} = \left[\mathfrak{g}^{(m-1)}, \mathfrak{g}^{(m-1)}\right],$$

for $m \in \mathbb{Z}_{>0}$. The $\mathfrak{g}^{(m)}$ are ideals of $\mathfrak{g}$, and the series

$$\mathfrak{g} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \mathfrak{g}^{(3)} \supseteq \cdots$$

is called the *derived series* of $\mathfrak{g}$. If $\mathfrak{g}$ contains no nonzero proper solvable ideals, then we say $\mathfrak{g}$ is *semisimple.* In other words, if we can write $\mathfrak{g}$ as a direct sum of simple ideals, then $\mathfrak{g}$ is semisimple. Note that a simple Lie algebra is also a *semisimple* Lie algebra, but the converse is not necessarily true.

2.2 Lie Algebra Representation Theory

Let $(A, +, \cdot, 0, 1)$ be a ring over $\mathbb{C}$, where A is a vector space over $\mathbb{C}$ with the same addition rule and zero element. Suppose $a(x \cdot y) = (ax) \cdot y = x \cdot (ay)$ for all $a \in \mathbb{C}$ and $x, y \in A$. Then we say that A is an *associative algebra.* Under the commutator bracket, with an associative product, A is a Lie algebra.

Example 2.2.1. [25] Let V be a vector space over $\mathbb{C}$, and let $gl(V)$ denote the set of all linear transformations from V to V. Then $gl(V)$ is an associative algebra, where the associative product is the composition of maps. Hence, $gl(V)$ is a Lie algebra under the commutator bracket.

Not every Lie algebra is an associative algebra. But, if we are given a Lie algebra, we can construct an associative algebra called a *universal enveloping algebra.* Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. Suppose we have an associative algebra $U(\mathfrak{g})$ over $\mathbb{C}$ and a linear map $j : \mathfrak{g} \to U(\mathfrak{g})$ such that $j([x, y]) = j(x)j(y) - j(y)j(x)$ for all $x, y \in \mathfrak{g}$. Then the pair $(U(\mathfrak{g}), j)$ is called a *universal enveloping algebra* of $\mathfrak{g}$ if it satisfies the following universal property [6]:

> For any pair (A, ϕ), where A is an associative algebra and $\phi : \mathfrak{g} \to A$ is a linear map satisfying $\phi([x, y]) = \phi(x)\phi(y) - \phi(y)\phi(x)$ for all $x, y \in \mathfrak{g}$, there exists a unique homomorphism of associative algebras $\psi : U(\mathfrak{g}) \to A$ such that $\phi = \psi \circ j$.

The following commutative diagram depicts the universal property of $U(\mathfrak{g})$ [6]:

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "j"] \arrow[dr, "\phi"'] & & U(\mathfrak{g}) \arrow[dl, dashed, "\exists!\psi"] \\ & A & \end{tikzcd}$$

Figure 2.1 Universal Property for the Universal Enveloping Algebra

Let $T(\mathfrak{g})$ be the tensor algebra of $\mathfrak{g}$ so that

$$T(\mathfrak{g}) = \bigoplus_{k \geq 0} T^k \mathfrak{g},$$

and let I be the two-sided ideal generated by elements of the form $x \otimes y - y \otimes x - [x, y]$, for $x, y \in \mathfrak{g}$. Then the universal enveloping algebra can be constructed as $U(\mathfrak{g}) = T(\mathfrak{g})/I$. The linear map $j : \mathfrak{g} \to U(\mathfrak{g})$ is constructed by composing the natural maps $\mathfrak{g} \hookrightarrow T(\mathfrak{g})$ and $\pi : T(\mathfrak{g}) \to U(\mathfrak{g})$. The Poincaré-Birkhoff-Witt Theorem, which is known as the PBW Theorem, proves that j is injective and gives a basis for $U(\mathfrak{g})$ [7].

Theorem 2.2.2. *[7] [Poincaré-Birkhoff-Witt]*

1. *The map $j : \mathfrak{g} \to U(\mathfrak{g})$ is injective.*
2. *Let $\{x_i | i \in I\}$ be an ordered basis for $\mathfrak{g}$, where I is an index set. Then the set $\{x_{i_1} x_{i_2} \cdots x_{i_k} | i_1 \leq i_2 \leq \cdots \leq i_k, k \geq 0\}$, along with 1, forms a basis for $U(\mathfrak{g})$.*

The PBW Theorem allows us to identify the Lie algebra $\mathfrak{g}$ with the image $j(\mathfrak{g})$.

We now discuss special maps on Lie algebras called *homomorphisms*. Suppose we have two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. We say that a linear transformation $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a *Lie algebra homomorphism* if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}_1$. A Lie algebra homomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is an *epimorphism* if φ is onto and is a *monomorphism* if φ is one-to-one. If φ is both one-to-one and onto, we say that φ is a *Lie algebra isomorphism.*

Let V be a vector space over $\mathbb{C}$. Recall that $gl(V)$, the set of all linear operators on V, is a Lie algebra under the commutator bracket. A *representation* of the Lie algebra $\mathfrak{g}$ on the vector space V is a homorphism $\varphi : \mathfrak{g} \to gl(V)$. We say that V is a $\mathfrak{g}$-*module* if there is an operation $\mathfrak{g} \times V \to \mathfrak{g}$, given by $(x, v) \mapsto x \cdot v$, such that

1. $x \cdot (au + bv) = a(x \cdot u) + b(x \cdot v)$,
2. $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$,
3. $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$

for all $x, y \in \mathfrak{g}$, $u, v \in V$, and $a, b \in \mathbb{C}$ [7].

Suppose V is a $\mathfrak{g}$-module. Then the map $\varphi : \mathfrak{g} \to gl(V)$ defined by $\varphi(x)v = x \cdot v$ for all $x \in \mathfrak{g}$ and $v \in V$ is a representation. Conversely, if $\varphi : \mathfrak{g} \to gl(V)$ is a representation, then by defining $x \cdot v = \varphi(x)v$ for all $x \in \mathfrak{g}$ and $v \in V$, we see that V is a $\mathfrak{g}$-module. Thus, representations and $\mathfrak{g}$-modules can be used interchangeably.

Example 2.2.3. [25] Let $\mathfrak{g} = sl(2, \mathbb{C})$ and $V = \mathbb{C}^2$. Define the map $\varphi : sl(2, \mathbb{C}) \to gl(\mathbb{C}^2)$ by $\varphi(x)v = xv$ (matrix multiplication) for $x \in sl(2, \mathbb{C})$ and $v \in \mathbb{C}^2$. The map φ is a representation, and hence, $\mathbb{C}^2$ is an $sl(2, \mathbb{C})$-module by $x \cdot v = xv$.

Example 2.2.4. [25] Let $\mathfrak{g}$ be a Lie algebra. Define the map ad:$\mathfrak{g} \to gl(\mathfrak{g})$ by $\mathrm{ad}(x)y = \mathrm{ad}_x y = [x, y]$ for $x, y \in \mathfrak{g}$. This map is a representation of $\mathfrak{g}$ on itself and is called the *adjoint representation* [7].

Let V be a $\mathfrak{g}$-module, and let S be a subspace of V. We say that S is a *submodule* of V if $x \cdot u \in S$ for all $x \in \mathfrak{g}$ and $u \in S$ [7]. Suppose that $V \neq \{0\}$ and that V has no proper submodules. Then V is *irreducible*.

Example 2.2.5. [25] Let $\mathfrak{g} = sl(n+1, \mathbb{C})$ and $V = \mathbb{C}^n$. Under the action $x \cdot v = xv$ (matrix multiplication), V is an irreducible $\mathfrak{g}$-module.

Now let $\mathfrak{g}$ be a Lie algebra, and let V and W be two $\mathfrak{g}$-modules. Suppose we have a linear transformation $\varphi : V \longrightarrow W$. If $\varphi(x \cdot v) = x \cdot \varphi(v)$ for all $x \in \mathfrak{g}$ and $v \in V$, then φ is a $\mathfrak{g}$-*module homomorphism*. As with Lie algebra homomorphisms, if φ is one-to-one and onto, then the $\mathfrak{g}$-module homomorphism φ is an *isomorphism*, i.e. $V \cong W$.

In [6], we see that the representation theory of a Lie algebra $\mathfrak{g}$ parallels the representation theory of its universal enveloping algebra $U(\mathfrak{g})$. Hence we can extend all of the aforementioned definitions for the Lie algebra $\mathfrak{g}$ to $U(\mathfrak{g})$.

2.3 Kac-Moody Lie algebras

In this section, we define a class of infinite-dimensional Lie algebras called Kac-Moody Lie algebras. These algebras can be constructed from a special matrix, called a generalized Cartan matrix, and are a generalization of finite-dimensional semisimple Lie algebras.

But first, we will define the *special affine Lie algebra*, denoted $A_n^{(1)}$ or $\hat{sl}(n+1,\mathbb{C})$, by construction, starting from the finite-dimensional simple Lie algebra $sl(n+1,\mathbb{C})$. This affine Lie algebra is infinite-dimensional and is the base of our main focus in this thesis. First recall that a linear transformation $\partial : \mathfrak{g} \to \mathfrak{g}$ is a *derivation* of $\mathfrak{g}$ if $\partial([x,y]) = [\partial(x), y] + [x, \partial(y)]$ for all $x, y \in \mathfrak{g}$.

The *special affine Lie algebra* can be constructed as follows [12]. Let

$$A_n^{(1)} = \hat{sl}(n+1,\mathbb{C}) = sl(n+1,\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where $\mathbb{C}[t,t^{-1}]$ is the Laurent polynomial ring, c is central, and d is the derivation $1 \otimes t\frac{d}{dt}$, with the following bracket structure:

$$\begin{aligned}
[x \otimes t^i, y \otimes t^j] &= [x,y] \otimes t^{i+j} + \mathrm{tr}(x,y) i \delta_{i+j,0} c, \\
[d, x \otimes t^i] &= i(x \otimes t^i), \\
[d, c] &= 0,
\end{aligned}$$

for all $x, y \in sl(n+1,\mathbb{C})$ and $i, j \in \mathbb{Z}$.

The following elements generate this algebra:

$$\begin{aligned}
e_0 &= F_0 \otimes t, & e_i &= E_i \otimes 1, \\
f_0 &= E_0 \otimes t^{-1}, & f_i &= F_i \otimes 1, \\
h_0 &= -H_0 \otimes 1 + c, & h_i &= H_i \otimes 1,
\end{aligned}$$

and $c = h_0 + h_1 + \cdots + h_n$ and the E_i, F_i, and H_i generate $sl(n+1,\mathbb{C})$.

An $n \times n$ integer matrix $A = (a_{ij})$ is called a *generalized Cartan matrix* (GCM) if the following conditions hold:

1. $a_{ii} = 2$ for all $i = 1, \ldots, n$,

2. $a_{ij} \leq 0$ if $i \neq j$,

3. $a_{ij} = 0$ if and only if $a_{ji} = 0$.

A matrix A is a *Cartan matrix* if it satisfies the above conditions and is positive-definite. If there exists an $n \times n$ diagonal matrix $D = \operatorname{diag}(s_i)$, such that $s_i \in \mathbb{Z}_{>0}$ and DA is symmetric, then A is *symmetrizable*. The GCM A is *indecomposable* if for every pair of nonempty subsets $I_1, I_2 \subset I$ with $I_1 \cup I_2 = I$, there exists some $i \in I_1$ and $j \in I_2$ such that $a_{ij} \neq 0$.

Theorem 2.3.1. *[12] Let $A = (a_{ij})$ be an indecomposable $n \times n$ GCM, and let $u, v \in \mathbb{R}^n$. We say that $u > 0$ if $u_i > 0$ and that $u \geq 0$ if $u_i \geq 0$ for all $i = 1, 2, \ldots, n$. Then one and only one of the following three possibilities hold for both A and A^T:*

(Finite) $\det A \neq 0$*; there exists $u > 0$ such that $Au > 0$; $Av \geq 0$ implies $v > 0$ or $v = 0$.*

(Affine) $\operatorname{corank} A = 1$*; there exists $u > 0$ such that $Au = 0$; $Av \geq 0$ implies $Av = 0$.*

(Indefinite) There exists $u > 0$ such that $Au < 0$; $Av \geq 0$ and $v \geq 0$ imply $v = 0$.

We say that the GCM A is of finite*,* affine*, or* indefinite type *if A satisfies the corresponding condition.*

Let $A = (a_{ij})$ be an $n \times n$ GCM. Then there exists an oriented graph, which we call the *Dynkin diagram*, which has n vertices. The directed edges are defined by the following [12]:

- If $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, then the vertices i and j are connected with $|a_{ij}|$ edges, with the arrow pointing toward i if $|a_{ij}| > 1$.

- If $a_{ij}a_{ji} > 4$, then the vertices i and j are connected with a bold edge, labeled with the ordered pair $(|a_{ij}|, |a_{ji}|)$.

Hence, given a GCM we can create a Dynkin diagram, and we can recover the GCM if given the Dynkin diagram, up to the order of indices.

Example 2.3.2. Here is one example of a GCM and its corresponding Dynkin diagram:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Figure 2.2 Generalized Cartan matrix and Dynkin diagram

Let $A = (a_{ij})$ be an $n \times n$ symmetrizable GCM. The *Cartan datum* [12] of A is the quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$, where Π, $\Pi^\vee$, P, and $P^\vee$ are defined as follows:

(dual-weight lattice) $P^\vee$ is a free abelian group of rank $2n - \text{rank}A$ with $\mathbb{Z}$-basis

$$\{h_i | i = 1, \dots, n\} \cup \{d_s | s = 1, 2, \dots, n - \text{rank}A\}.$$

(Cartan subalgebra) $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$.

(weight lattice) $P = \{\lambda \in \mathfrak{h}^* | \lambda(P^\vee) \subset \mathbb{Z}\}$.

(simple coroots) $\Pi^\vee = \{h_i | i = 1, \dots, n\}$.

(simple roots) $\Pi = \{\alpha_i | i = 1, \dots, n\} \subset \mathfrak{h}^*$, a linearly independent subset satisfying

$$\alpha_j(h_i) = \alpha_{ij} \quad \text{and} \quad \alpha_j(d_s) = \delta_{sj}.$$

The *fundamental weights* $\Lambda_i \in \mathfrak{h}^*$ are linear functionals on $\mathfrak{h}$ and defined as follows:

$$\Lambda_i(h_j) = \delta_{ij} \qquad \text{and} \qquad \Lambda_i(d_s) = 0$$

for $i, j \in I$ and $s = 1, 2, \dots, |I| - \text{rank}A$.

Definition 2.3.3. [12] The *Kac-Moody Lie algebra* $\mathfrak{g}$ associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the Lie algebra generated by the elements e_i, f_i $(i \in I)$ and $h \in P^\vee$ such that the following relations hold:

1. $[h, h'] = 0$ for $h, h' \in P^\vee$,
2. $[e_i, f_i] = \delta_{ij} h_i$,
3. $[h, e_i] = \alpha_i(h) e_i$ for $h \in P^\vee$,
4. $[h, f_i] = -\alpha_i(h) f_i$ for $h \in P^\vee$,
5. $(\text{ad} e_i)^{1-a_{ij}} e_j = 0$ for $i \neq j$,
6. $(\text{ad} f_i)^{1-a_{ij}} f_j = 0$ for $i \neq j$.

Example 2.3.4. [12] The special linear Lie algebra $A_n = sl(n+1, \mathbb{C})$ is a Kac-Moody Lie algebra and has the following GCM and corresponding Dynkin diagram:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \ddots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & -1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

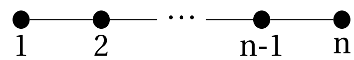

Figure 2.3 Generalized Cartan matrix and Dynkin diagram for $A_n = sl(n+1,\mathbb{C})$

If A is a GCM of affine type, then we call the quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$ an *affine Cartan datum* and to each we can associate the *affine Kac-Moody Lie algebra* $\mathfrak{g}$.

Example 2.3.5. [12] The special affine Lie algebra $A_n^{(1)} = \hat{sl}(n+1,\mathbb{C})$ is also a Kac-Moody Lie algebra, with GCM and Dynkin diagram as follows:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & \ddots & 0 \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ 0 & 0 & -1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

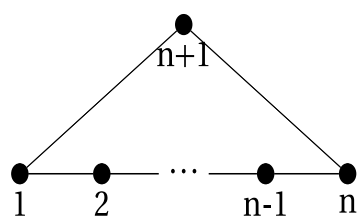

Figure 2.4 Generalized Cartan matrix and Dynkin diagram for $A_n^{(1)} = \hat{sl}(n+1,\mathbb{C})$

The affine Kac-Moody Lie algebra $A_n^{(1)}$ has root basis $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, coroot basis $\Pi^\vee = \{h_0, h_1, \ldots, h_n\} \subset \mathfrak{h}$, and the imaginary roots of $\mathfrak{g}$ are nonzero integral multiples of the *null root* $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_n$.

The free abelian group $A = \bigoplus_i \mathbb{Z}\alpha_i$, for $i = 1, \ldots, n$, is called the *root lattice*, which can be separated into two parts: the *positive root lattice* $Q_+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ and the *negative root lattice* $Q_- = -Q_+$ [12].

For each $\alpha \in Q$, let

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$, then α is a *root* of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is called the *root space*. The dimension of $\mathfrak{g}_\alpha$ is called the *root multiplicity* of α. Let Δ represent the set of all roots, $\Delta_+ = \Delta \cap Q_+$ be the set of all positive roots, and $\Delta_- = \Delta \cap Q_-$ be the set of all negative roots.

Proposition 2.3.6. *[12] The Kac-Moody Lie algebra $\mathfrak{g}$ has the following triangular decomposition:*

$$\mathfrak{g} = \left(\bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \right) = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+,$$

where $\dim \mathfrak{g}_\alpha < \infty$ *for all* $\alpha \in \Delta$.

We also have the *Chevalley involution*, which leads to special properties of the triangular decomposition.

Proposition 2.3.7. *[12] There exists an involution $\omega : \mathfrak{g} \to \mathfrak{g}$, called the Chevalley involution, such that $e_i \mapsto -f_i$, $f_i \mapsto -e_i$, and $h \mapsto -h$.*

The triangular decomposition given in Proposition 2.3.6 implies that if α is a positive root, then we have $\mathfrak{g}_\alpha \in \mathfrak{g}_+$, and if α is negative, then $\mathfrak{g}_\alpha \in \mathfrak{g}_-$. Proposition 2.3.7 implies that $\mathrm{mult}(\alpha) = \mathrm{mult}(-\alpha)$ [12].

Now we introduce the *Weyl group*, which plays a key role in this paper. The *simple reflection* r_i, i=1,...,n, on $\mathfrak{h}^*$ is the linear functional given by $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$ [7]. The simple reflections generate a subgroup $\mathcal{W}$ of End($\mathfrak{h}^*$), which we call the *Weyl group*. Let $w \in \mathcal{W}$. Then w can be written in terms of the simple reflections, $w = r_{i_1} r_{i_2} \cdots r_{i_t}$. If t is minimal, then we say that w is *reduced* and t is called the *length* of w, denoted $l(w)$.

Example 2.3.8. [25] The Weyl group for $A_n = sl(n+1, \mathbb{C})$ is isomorphic to the symmetric group S_{n+1}. The Weyl group for $A_n^{(1)}$ is $\mathcal{W} = \langle r_0, r_1, \ldots, r_n \rangle$. In particular, the Weyl group $\mathcal{W} = \langle r_0, r_1 \rangle$ for $A_1^{(1)}$ is the infinite Dihedral group $\{(r_0 r_1)^m, r_1(r_0 r_1)^m | m \in \mathbb{Z}\}$.

Let $\mathfrak{g}$ be a Kac-Moody Lie algebra. We can construct an associative algebra for $\mathfrak{g}$ using the following proposition.

Proposition 2.3.9. *[6] The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is the associative algebra over $\mathbb{C}$ with unity generated by e_i, f_i $(i \in I)$ and $\mathfrak{h}$ satisfying the following relations:*

1. $hh' = h'h$ *for* $h, h' \in \mathfrak{h}$,

2. $e_i f_j - f_j e_i = \delta_{ij} h_i$, *for* $i, j \in I$,

3. $he_i - e_i h = \alpha_i(h) e_i$ *for* $h \in \mathfrak{h}$, $i \in I$,

4. $hf_i - f_ih = -\alpha_i(h)f_i$ *for* $h \in \mathfrak{h}$, $i \in I$,

5. $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ *for* $i \neq j$,

6. $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ *for* $i \neq j$.

Let U^+ (respectively, U^0 and U^-) be the subalgebra of $U(\mathfrak{g})$ generated by the elements e_i (respectively, $\mathfrak{h}$ and f_i), for $i = 1, \ldots, n$. We define the root spaces of $U(\mathfrak{g})$ by the following equations:

$$U_\beta = \{u \in U(\mathfrak{g}) | hu - uh = \beta(h)u \text{ for all } h \in \mathfrak{h}\} \text{ for } \beta \in Q,$$
$$U_\beta^\pm = \{u \in U^\pm | hu - uh = \beta(h)u \text{ for all } h \in \mathfrak{h}\} \text{ for } \beta \in Q_\pm.$$

Proposition 2.3.10. *[6] Let* $\mathfrak{g}$ *be a Kac-Moody Lie algebra and let* $U(\mathfrak{g})$ *be the associated universal enveloping algebra. Then we have the following:*

1. $U(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+$.

2. $U(\mathfrak{g}) = \displaystyle\bigoplus_{\beta \in Q} U_\beta$.

3. $U^\pm = \displaystyle\bigoplus_{\beta \in Q_\pm} U_\beta^\pm$.

2.4 Representation Theory

For this section, let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra, and let V be a $\mathfrak{g}$-module. Recall that, by the PBW Theorem, the representation theory of $\mathfrak{g}$ parallels the representation theory of its universal enveloping algebra $U(\mathfrak{g})$. Hence, the definitions and properties in this section can be extended for $U(\mathfrak{g})$.

For any $\mu \in \mathfrak{h}^*$, the μ-*weight space* is defined as

$$V_\mu = \{v \in V | hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}.$$

If $V_\mu \neq 0$, then μ is called a *weight* and vectors $v \in V_\mu$ are called *weight vectors* of weight μ. The dimension of V_μ is called the *weight multiplicity* of μ [25]. A $\mathfrak{g}$-module V is called a *weight module* if it admits a *weight space decomposition*

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu.$$

Recall the positive root lattice $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. Define a partial ordering on $\mathfrak{h}^*$ by

$$\lambda \geq \mu \text{ if and only if } \lambda - \mu \in Q_+$$

for $\lambda, \mu \in \mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*$, let $D(\lambda) = \{\mu \in \mathfrak{h}^* | \mu \leq \lambda\}$, which is called the λ-cone [25]. The *category* $\mathcal{O}$ [25] is the set of weight modules V over $\mathfrak{g}$ with finite-dimensional weight spaces for which there exists a finite number of elements $\lambda_1, \lambda_2, \ldots, \lambda_s \in \mathfrak{h}^*$ such that

$$\mathrm{wt}(V) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s).$$

A weight module V is a *highest weight module* of *highest weight* $\lambda \in \mathfrak{h}^*$ if there exists a nonzero vector $v_\lambda \in V$, called a *highest weight vector*, such that

$$\begin{aligned} e_i v_\lambda &= 0 \text{ for all } i \in I, \\ h v_\lambda &= \lambda(h) v_\lambda \text{ for all } h \in \mathfrak{h}, \\ V &= U(\mathfrak{g}) v_\lambda. \end{aligned}$$

For example, if we fix $\lambda \in \mathfrak{h}^*$ and let $J(\lambda)$ be the ideal of $U(\mathfrak{g})$ generated by all e_i and $h - \lambda(h)1$, for $i = 1, \ldots, n$ and $h \in \mathfrak{h}$, then $M(\lambda) = U(\mathfrak{g})/J(\lambda)$ is called the *Verma module* if it has a $U(\mathfrak{g})$-module structure by left multiplication [25].

Proposition 2.4.1. *[12]*

1. *$M(\lambda)$ is a highest weight $\mathfrak{g}$-module with highest weight λ and highest weight vector $v_\lambda = 1 + J(\lambda)$.*

2. *Every highest weight $\mathfrak{g}$-module with highest weight λ is a homomorphic image of $M(\lambda)$.*

3. *As a U^--module, $M(\lambda)$ is free of rank 1, generated by the highest weight vector $v_\lambda = 1 + J(\lambda)$.*

4. *$M(\lambda)$ has a unique maximal submodule.*

Let $N(\lambda)$ be the unique maximal submodule of $M(\lambda)$. Then $V(\lambda) = M(\lambda)/N(\lambda)$ is the *irreducible highest weight module* [25].

Proposition 2.4.2. *[12] Every irreducible $\mathfrak{g}$-module in the category $\mathcal{O}$ is isomorphic to $V(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.*

Suppose we have a Kac-Moody algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module V. If for any $v \in V$, there exists a positive integer N such that $x^N \cdot v = 0$, then $x \in \mathfrak{g}$ is *locally nilpotent* on V. We say that a weight module V over a Kac-Moody Lie algebra $\mathfrak{g}$ is *integrable* if all e_i and f_i, $i \in I$, are locally nilpotent on V [25]. We call the elements in the weight lattice $P = \{\lambda \in \mathfrak{h}^* | \lambda(h_i) \in \mathbb{Z} \text{ for all } i = 1, \ldots, n\}$ *integral weights*, and if we restrict this set to the positive part $P^+ = \{\lambda \in P | \lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i = 1, \ldots, n\}$, we call the resulting set the *dominant integral weights*.

We define *category $\mathcal{O}_{int}$* to be the category containing all integrable $\mathfrak{g}$-modules in the category $\mathcal{O}$ such that $\mathrm{wt}(V) \subset P$. Any $\mathfrak{g}$-module in category $\mathcal{O}_{int}$ is completely reducible, with weight space decomposition $V = \bigoplus_{\lambda \in P} V_\lambda$, where $V_\lambda = \{v \in V | h_i v = \lambda(h_i) v \text{ for all } i \in I\}$ [25].
We have the following results:

Proposition 2.4.3. *[12] Let $V(\lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$. Then $V(\lambda)$ is in category $\mathcal{O}_{int}$. Namely, $V(\lambda)$ is integrable if and only if $\lambda \in P^+$.*

Theorem 2.4.4. *[12] Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. Then every irreducible $\mathfrak{g}$-module in the category $\mathcal{O}_{int}$ is isomorphic to $V(\lambda)$, where $\lambda \in P^+$.*

Chapter

3

Quantum Groups and Crystal Bases

In this chapter, we recall the definitions of quantum groups, their representation theory, and the associated crystal bases from [6]. We'll start with the definition of the quantum group $U_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$, focusing on the special case $\mathfrak{g} = A_n^{(1)}$. Then we'll discuss the integrable representations of $U_q(\mathfrak{g})$ and their crystal bases.

3.1 Quantum Groups

In this section, we introduce some definitions and facts about quantum groups and their representations.

Let $m, n \in \mathbb{Z}$ and let q be any indeterminate. A *q-integer* is defined as $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$. We also define *$q$-factorials* by the following equations:

$$[0]_q! = 1 \qquad \text{and} \qquad [n]_q! = [n]_q [n-1]_q \cdots [1]_q,$$

for $n > 0$. Let $m \geq n \geq 0$. Then we define the *q-binomial coefficient* by the equation [6]:

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}.$$

Let $(A, \Pi, \Pi^\vee, P, P^\vee)$ be a Cartan datum associated with $n \times n$ symmetrizable GCM $A = (a_{ij})$. Let $D = \mathrm{diag}(s_i \in \mathbb{Z}_{>0} | i = 1, \ldots, n)$ be the symmetrizing matrix for A. The *quantum group* or the *quantized universal enveloping algebra* $U_q(\mathfrak{g})$ associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the associative algebra over $\mathbb{C}(q)$ with unity generated by the elements e_i, f_i, and q^h, where $i = 1, \ldots, n$ and $q \in P^\vee$, with the following relations [6]:

1. $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^\vee$,
2. $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ for $h \in P^\vee$,
3. $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ for $h \in P^\vee$,
4. $e_i f_j - f_j e_i = \delta_{ij} \dfrac{q^{s_i h_i} - q^{-s_i h_i}}{q^{s_i} - q^{-s_i}}$ for $i, j \in I$,
5. $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{s_i}} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$,
6. $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{s_i}} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$.

As $q \to 1$, notice that $U_q(\mathfrak{g}) \to U(\mathfrak{g})$.

Example 3.1.1. [25] Let $\mathfrak{g} = sl(2, \mathbb{C})$. Then the quantum group $U_q(sl(2, \mathbb{C}))$ is the associative algebra generated by $\{e, f, q^h\}$ satisfying the following relations:

1. $q^h e q^{-h} = q^2 e$,
2. $q^h f q^{-h} = q^{-2} f$,
3. $ef - fe = \dfrac{q^h - q^{-h}}{q - q^{-1}}$.

As with $U(\mathfrak{g})$, the quantum group $U_q(\mathfrak{g})$ has the following triangular decomposition [6]:

$$U_q(\mathfrak{g}) \cong U_q(\mathfrak{g})^+ \otimes U_q(\mathfrak{g})^0 \otimes U_q(\mathfrak{g})^-,$$

where $U_q(\mathfrak{g})^+$ (respectively, $U_q(\mathfrak{g})^-$ and $U_q(\mathfrak{g})^0$) is the subalgebra generated by the elements e_i (respectively, f_i and q^h for $h \in P^\vee$).

Now let V^q be a $U_q(\mathfrak{g})$-module. The following definitions are recalled from [6]. For any $\mu \in P$, the *μ-weight space* is

$$V_\mu^q = \{v \in V^q | q^h v = q^{\mu(h)} v \text{ for all } h \in P^\vee.$$

If $V_\mu^q \neq 0$, then μ is called a *weight*, and all vectors $v \in V_\mu^q$ are *weight vectors* of weight μ. The *weight multiplicity* of μ is defined as the dimension of V_μ^q. If a $U_q(\mathfrak{g})$-module V^q admits a *weight space decomposition*

$$V^q = \bigoplus_{\mu \in P} V_\mu^q,$$

then V^q is a *weight module*. And V^q is a *highest weight module* with *highest weight* $\lambda \in P$ if there exists $0 \neq v_\lambda \in V^q$ such that

$$\begin{aligned} e_i v_\lambda &= 0 \text{ for all } i \in I, \\ q^h v_\lambda &= q^{\lambda(h)} v_\lambda \text{ for all } h \in P^\vee, \\ V^q &= U_q(\mathfrak{g}) v_\lambda. \end{aligned}$$

As with $U(\mathfrak{g})$, we can define the *Verma module* associated with $U_q(\mathfrak{g})$ by

$$M^q(\lambda) = U_q(\mathfrak{g}) / J^q(\lambda),$$

where $\lambda \in P$ is fixed and $J^q(\lambda)$ is the left ideal of $U_q(\mathfrak{g})$ generated by e_i and $q^h - q^{\lambda(h)}1$, for $i = 1, \ldots, n$ and $h \in P^\vee$ [6]. The Verma module $M^q(\lambda)$ is a $U_q(\mathfrak{g})$-module if we use left multiplication. By [6], $M^q(\lambda)$ is a highest weight module with highest weight λ and highest weight vector $v_\lambda = 1 + J^q(\lambda)$, and $M^q(\lambda)$ has a unique maximal submodule, denoted $N^q(\lambda)$. The module $V^q(\lambda) = M^q(\lambda)/N^q(\lambda)$ is the *irreducible highest weight module* with highest weight λ [6]. $V^q(\lambda)$ is *integrable* if all e_i and f_i are locally nilpotent on V^q. We also recall the definition of the *category* $\mathcal{O}^q$, which consists of weight modules V^q with finite-dimensional weight spaces such that

$$\mathrm{wt}(V^q) \subset D(\lambda_1) \cup D(\lambda_2) \cup \cdots \cup D(\lambda_s),$$

where $\mathrm{wt}(V^q)$ is the set of weights of V^q, $s < \infty$, and $D(\lambda) = \{\mu \in P | \mu \leq \lambda\}$ is the λ-cone [6]. If we restrict category $\mathcal{O}$ to integrable $U_q(\mathfrak{g})$-modules, then we have the *category* $\mathcal{O}_{int}^q$, in which all modules are completely reducible and have weight space decompositions.

Proposition 3.1.2. *[6] Let $V^q(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda \in P$. Then $V^q(\lambda)$ is in category $\mathcal{O}^q_{int}$. Namely, $V^q(\lambda)$ is integrable if and only if $\lambda \in P^+$.*

3.2 Crystal Base Theory

We recall necessary definitions and properties of crystal base theory from [6]. We begin by recalling the definitions of Kashiwara operators upon which we will rely heavily.

Lemma 3.2.1. *[6] Let $V^q = \bigoplus_{\lambda \in P} V^q_\lambda$ be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$ and let λ be a weight of V^q such that $V^q_\lambda \neq 0$. For each $i \in I$, every weight vector $v \in V^q_\lambda$ may be written in the form*

$$v = v_0 + f_i v_1 + \cdots + f_i^{(n)} v_n,$$

where $n \in \mathbb{Z}_{\geq 0}$, $v_k \in V^q_{\lambda + k\alpha_i} \cap \ker e_i$, and $f_i^{(k)} = \dfrac{f_i^k}{[k]_q!}$. Each v_k in the above expression is uniquely determined by v, and $v_k \neq 0$ only if $\lambda(h_i) + k \geq 0$.

We can now define the *Kashiwara operators* $\tilde{e}_i, \tilde{f}_i : V^q \to V^q$, for $i \in I$, which are endomorphisms on V^q such that for $v \in V^q$,

$$\tilde{e}_i v = \sum_{k=1}^{n} f_i^{(k-1)} v_k \qquad \text{and} \qquad \tilde{f}_i v = \sum_{k=1}^{n} f_i^{(k+1)} v_k.$$

Note that for $v \in V^q_\lambda$, $\tilde{e}_i v \in V^q_{\lambda+\alpha_i}$ and $\tilde{f}_i v \in V^q_{\lambda-\alpha_i}$. Let

$$\mathbb{A} = \left\{ \frac{g(q)}{h(q)} \middle| g(q), h(q) \in \mathbb{C}[q], h(0) \neq 0 \right\},$$

which is a principal ideal domain with $\mathbb{C}(q)$ as its field of quotients. And let $V^q \in \mathcal{O}^q_{int}$. A free $\mathbb{A}$-submodule $\mathcal{L}$ of V^q is a *crystal lattice* if

1. $\mathcal{L}$ generates V^q as a vector space over $\mathbb{C}(q)$, or equivalently, $V^q_\mu \cong \mathbb{C}(q) \otimes_{\mathbb{A}} \mathcal{L}_\mu$ for each $\mu \in \mathrm{wt}(V^q)$,
2. $\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda$, where $\mathcal{L}_\lambda = \mathcal{L} \cap V^q_\lambda$ for all $\lambda \in P$,
3. $\tilde{e}_i \mathcal{L} \subset \mathcal{L}$, $\tilde{f}_i \mathcal{L} \subset \mathcal{L}$ for all $i \in I$.

Let $\mathbb{J} = \langle q \rangle$ be the unique maximal ideal of $\mathbb{A}$. Then there exists an isomorphism of fields $\mathbb{A}/\mathbb{J} \to \mathbb{C}$ given by $f(q) + \mathbb{J} \mapsto f(0)$. Hence, $\mathbb{C} \otimes_{\mathbb{A}} \mathcal{L} \cong \mathcal{L}/q\mathcal{L}$. Since the operators $\tilde{e}_i$ and $\tilde{f}_i$ preserve the lattice

$\mathcal{L}$, we can also define the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathcal{L}/q\mathcal{L}$ [6].

Recall from [6] that a *crystal base* for V^q is a pair $(\mathcal{L}, \mathcal{B})$ such that

1. $\mathcal{L}$ is a crystal lattice of V^q,
2. $\mathcal{B}$ is a $\mathbb{C}$-basis of $\mathcal{L}/q\mathcal{L} \cong \mathbb{C} \otimes_{\mathbb{A}} \mathcal{L}$,
3. $\mathcal{B} = \bigsqcup_{\lambda \in P} \mathcal{B}_\lambda$, where $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathcal{L}_\lambda / q\mathcal{L}_\lambda)$,
4. $\tilde{e}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$, $\tilde{f}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$ for all $i \in I$,
5. for any $b, b' \in \mathcal{B}$ and $i \in I$, we have $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$.

Each crystal base $(\mathcal{L}, \mathcal{B})$ for $V^q \in \mathcal{O}^q_{int}$ has an associated *crystal graph*, which is a directed graph. The vertex set consists of all elements of $\mathcal{B}$, and the edge set consists of i-colored arrows. Two nodes $b, b' \in \mathcal{B}$ are joined by an i-colored arrow, $b \xrightarrow{i} b'$, if and only if $\tilde{f}_i b = b'$ for $i \in I$. We denote the crystal graph of V^q by $\mathcal{B}$. We define the maps $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z}$ for $i \in I$ by

$$\varepsilon_i(b) = \max\left\{k \geq 0 | \tilde{e}_i^{\,k} b \in \mathcal{B}\right\},$$
$$\varphi_i(b) = \max\left\{k \geq 0 | \tilde{f}_i^{\,k} b \in \mathcal{B}\right\}.$$

Hence, ε_i denotes the number of i-colored arrows coming into the vertex b, and φ_i denotes the number of i-colored arrows coming out of the vertex b. Therefore, $\varphi_i(b) + \varepsilon_i(b)$ is the length of the i-string through b and $\varphi_i(b) - \varepsilon_i(b) = \lambda(h_i)$ [6].

Theorem 3.2.2. *(cf. [6]) Let $\lambda \in P^+$ be a dominant integral weight, and let $V^q(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight λ and highest weight vector v_λ. Let*

$$\mathcal{L}(\lambda) = \sum_{r \geq 0, i_k \in I} \mathbb{A} \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_r} v_\lambda,$$

and

$$\mathcal{B}(\lambda) = \left\{ \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_r} v_\lambda + q\mathcal{L}(\lambda) \in \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \middle| \, r \geq 0, i_k \in I \right\} \setminus \{0\}.$$

Then the pair $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ is a crystal base of $V^q(\lambda)$.

We have the following result for crystal bases:

Theorem 3.2.3. *[6] Let V_j^q be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_{int}^q$, and let $(\mathcal{L}_j, \mathcal{B}_j)$ be a crystal base of V_j^q, for $j = 1, 2$. Set $\mathcal{L} = \mathcal{L}_1 \otimes_{\mathbb{A}} \mathcal{L}_2$ and $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$. Then $(\mathcal{L}, \mathcal{B})$ is a crystal base of $V_1^q \otimes_{\mathbb{C}(q)} V_2^q$ where the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathcal{B}$ are defined as follows:*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

Hence, we have the following:

$$\begin{aligned} wt(b_1 \otimes b_2) &= wt(b_1) + wt(b_2), \\ \varepsilon_i(b_1 \otimes b_2) &= max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1)\rangle), \\ \varphi_i(b_1 \otimes b_2) &= max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, wt(b_2)\rangle). \end{aligned}$$

Note that $b_1 \otimes 0 = 0 \otimes b_2 = 0$, and that we write $b_1 \otimes b_2$ instead of $(b_1, b_2) \in \mathcal{B} \times \mathcal{B}_2$. Also, we denote the crystal graph of $V_1^q \otimes V_2^q$ as $\mathcal{B}_1 \otimes \mathcal{B}_2$.

Let $A = (a_{ij})$ be an $n \times n$ GCM with Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. Define the maps wt: $\mathcal{B} \to P$, $\tilde{e}_i, \tilde{f}_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$, and $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$, for $i \in I$, a finite index set, such that the following conditions hold:

1. $\varphi_i(b) = \varepsilon(b) + \langle h_i, \mathrm{wt}(b)\rangle$ for all $i \in I$,
2. $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\tilde{e}_i b \in \mathcal{B}$,
3. $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $\tilde{f}_i b \in \mathcal{B}$,
4. $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $\tilde{e}_i b \in \mathcal{B}$,
5. $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $\tilde{f}_i b \in \mathcal{B}$,
6. $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in \mathcal{B}$, $i \in I$,
7. if $\varphi_i(b) = -\infty$ for $b \in \mathcal{B}$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

A set $\mathcal{B}$, along with the above maps and conditions, is called a *crystal* that is associated with the Cartan datum and hence, with $U_q(\mathfrak{g})$ [6]. Recall from [6] that Theorem 3.2.3 also holds for these crystals. Now suppose we have two crystals $\mathcal{B}_1, \mathcal{B}_2$ associated with the same Cartan datum. A *crystal morphism* $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ is a map $\Psi : \mathcal{B}_1 \cup \{0\} \to \mathcal{B}_2 \cup \{0\}$ such that

1. $\Psi(0) = 0$,
2. if $b \in \mathcal{B}_1$ and $\Psi(b) \in \mathcal{B}_2$, then $\mathrm{wt}\,(\Psi(b)) = \mathrm{wt}\,(b)$, $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\Psi(b)) = \varphi_i(b)$ for all $i \in I$,
3. if $b, b' \in \mathcal{B}_1$, $\Psi(b), \Psi(b') \in \mathcal{B}_2$, and $\tilde{f}_i b = b'$, then $\tilde{f}_i \Psi(b) = \Psi(b')$ and $\Psi(b) = \tilde{e}_i \Psi(b')$ for all $i \in I$.

A crystal morphism $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ is an *isomorphism* if it is both one-to-one and onto from $\mathcal{B}_1 \cup \{0\}$ to $\mathcal{B}_2 \cup \{0\}$ [6].

3.3 Perfect Crystals

In this section, we introduce the notion of a perfect crystal for a quantum affine algebra. Again, let $A = (a_{ij})$ be an $(n+1) \times (n+1)$ affine GCM, with indices $i, j = 0, 1, \ldots, n$. Let $(A, \Pi, \Pi^\vee, P, P^\vee)$ be the Cartan datum for the quantum affine algebra $U_q(\mathfrak{g})$.

Let $U_q'(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{e_i, f_i, q^{\pm s_i h_i} | i \in I\}$. This subalgebra is also called the *quantum affine algebra* [6]. Let

$$\bar{P}^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \qquad \text{and} \qquad \bar{\mathfrak{h}} = \mathbb{C} \otimes_{\mathbb{Z}} \bar{P}^\vee.$$

If we consider α_i and Λ_i as linear functionals on $\bar{\mathfrak{h}}$, we can define the *classical weights* as elements of

$$\bar{P} = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n.$$

We call the quintuple $(A, \Pi, \Pi^\vee, \bar{P}, \bar{P}^\vee)$ a *classical Cartan datum.* Hence, the quantum affine algebra $U_q'(\mathfrak{g})$ is the quantum group associated with the classical Cartan datum [6]. The crystal associated with $(A, \Pi, \Pi^\vee, \bar{P}, \bar{P}^\vee)$ is called a *classical crystal*, or a $U_q'(\mathfrak{g})$-*crystal* [6].

Example 3.3.1. [25] Let $\mathfrak{g} = A_1^{(1)}$ and consider $U_q(\mathfrak{g})$ and $V^q = \mathbb{C}(q)v_0 \oplus \mathbb{C}(q)v_1$. Then V^q is a $U_q'(\mathfrak{g})$-module where

$$\begin{aligned} e_1 v_0 &= 0, & e_1 v_1 &= v_0, \\ f_1 v_0 &= v_1, & f_1 v_1 &= 0, \\ q^{s_1 \lambda(h_1)} v_0 &= q\, v_0, & q^{s_1 \lambda(h_1)} v_1 &= q^{-1} v_1, \end{aligned}$$

and the affine action is given by:

$$\begin{aligned} e_0 v_0 &= v_1, & e_0 v_1 &= 0, \\ f_0 v_0 &= 0, & f_0 v_1 &= v_0, \\ q^{s_0\lambda(h_0)} v_0 &= q^{-1} v_0, & q^{s_0\lambda(h_0)} v_1 &= q\, v_1. \end{aligned}$$

Hence, defining $\mathcal{L} = \mathbb{A} v_0 \oplus \mathbb{A} v_1$ and $\mathcal{B} = \{v_0, v_1\}$, we have that $(\mathcal{L}, \mathcal{B})$ is a crystal base for V^q as a $U_q'(\mathfrak{g})$-module. The following diagram depicts the crystal graph for V^q:

$$\begin{tikzcd} v_0 \arrow[r, "1"'] & v_1 \arrow[l, bend right, "0"'] \end{tikzcd}$$

Figure 3.1 Crystal graph for $U_q'(\mathfrak{g})$-module V^q

Now, let $\mathcal{B}$ be a classical crystal. For $b \in \mathcal{B}$, define

$$\varepsilon(b) = \sum_{i\in I} \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i\in I} \varphi_i(b)\Lambda_i.$$

Note that $\mathrm{wt}\,(b) = \varphi(b) - \varepsilon(b)$. For a positive integer $\ell > 0$, set

$$\bar{P}^+_\ell = \left\{ \lambda \in \bar{P}^+ \middle| \langle c, \lambda \rangle = \ell \right\}.$$

Definition 3.3.2. [6] For a positive integer $\ell > 0$, we say that a finite classical crystal $\mathcal{B}_\ell$ is a *perfect crystal of level* ℓ if it satisfies the following conditions:

1. There exists a finite-dimensional $U_q'(\mathfrak{g})$-module with a crystal basis whose crystal graph is isomorphic to $\mathcal{B}_\ell$.

2. $\mathcal{B}_\ell \otimes \mathcal{B}_\ell$ is connected.

3. There exists a classical weight $\lambda_0 \in \bar{P}$ such that

$$\mathrm{wt}\,(\mathcal{B}) \subset \lambda_0 + \sum_{i\neq 0} \mathbb{Z}_{\leq 0}\alpha_i, \qquad \#(\mathcal{B}_{\lambda_0}) = 1.$$

4. For any $b \in \mathcal{B}$, we have $\langle c, \varepsilon(b) \rangle \geq \ell$.

5. For each $\lambda \in \bar{P}_\ell^+$, there exist unique vectors $b^\lambda, b_\lambda \in \mathcal{B}$ such that $\varepsilon(b^\lambda) = \lambda$ and $\varphi(b_\lambda) = \lambda$.

For $\lambda \in \bar{P}_\ell^+$, we say that b_λ is λ-*minimal* if

$$\lambda = \varphi(b_\lambda) = \sum_{i \in I} \varphi_i(b_\lambda)\Lambda_i.$$

Example 3.3.3. [6] Let $\mathfrak{g} = A_2^{(1)}$ and let $\ell = 2$. Define

$$\mathcal{B}_2 = \left\{ (m_1, m_2, m_0) \in \mathbb{Z}_{\geq 0}^3 \;\middle|\; \sum_{i=1}^{3} m_i = \ell \right\}.$$

Define the following for $b \in \mathcal{B}_2$:

$$\begin{aligned}
&\tilde{e}_0(b) = (m_1 - 1, m_2, m_0 + 1), && \tilde{e}_1(b) = (m_1 + 1, m_2 - 1, m_0), && \tilde{e}_2(b) = (m_1, m_2 + 1, m_0 - 1),\\
&\tilde{f}_0(b) = (m_1 + 1, m_2, m_0 - 1), && \tilde{f}_1(b) = (m_1 - 1, m_2 + 1, m_0), && \tilde{f}_2(b) = (m_1, m_2 - 1, m_0 + 1),\\
&\varphi_0(b) = m_0, && \varphi_1(b) = m_1, && \varphi_2(b) = m_2,\\
&\varepsilon_0(b) = m_1, && \varepsilon_1(b) = m_2, && \varepsilon_2(b) = m_0,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{wt}(b) &= (m_n - m_0)\Lambda_n + \sum_{i=0}^{n-1}(m_i - m_{i+1})\Lambda_i\\
&= \sum_{i=0}^{n}(\varphi_i(b) - \varepsilon_i(b))\Lambda_i,\\
\varphi(b) &= \sum_{i=0}^{n}\varphi_i(b)\Lambda_i,\\
\varepsilon(b) &= \sum_{i=0}^{n}\varepsilon_i(b)\Lambda_i.
\end{aligned}$$

By the axioms listed in Definition 3.3.2, $\mathcal{B}_2$ is a perfect crystal of level 2 for $U_q\left(A_2^{(1)}\right)$, and is depicted in Figure 3.2. In this figure, the labels on the edges represent the Kashiwara operator action used. Notice that for $\lambda = 2\Lambda_0$, the λ-minimal element is $b_{2\Lambda_0} = (0,0,2)$, since $\varphi(0,0,2) = \lambda$.

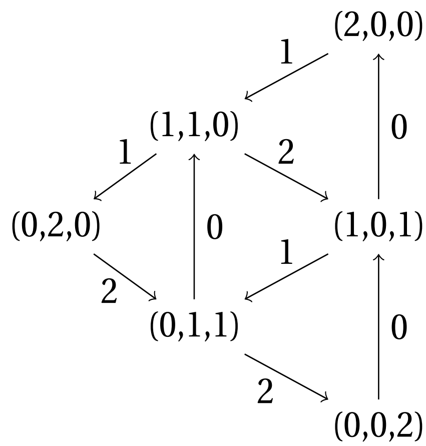

Figure 3.2 Perfect crystal of level 2 for $U_q\left(A_2^{(1)}\right)$

We end this section with a result that is essential for understanding *path realization*, which we discuss in the next section.

Theorem 3.3.4. *[6] The following map*

$$\Psi : \mathcal{B}(\lambda) \longrightarrow \mathcal{B}(\varepsilon(b_\lambda)) \otimes \mathcal{B}_\ell$$

given by $u_\lambda \mapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda$, *where* u_λ *is the highest weight vector of* $\mathcal{B}(\lambda)$ *and* $u_{\varepsilon(b_\lambda)}$ *is the highest weight vector of* $\mathcal{B}(\varepsilon(b_\lambda))$, *is a strict isomorphism of crystals.*

3.4 Path Realizations

At the end of the previous section, we gave Theorem 3.3.4. This theorem contributed to the development of path realizations of crystal graphs, which we recall from [6].

Let

$$\begin{aligned} \lambda_1 &= \lambda, & \lambda_{k+1} &= \varepsilon(b_{\lambda_k}), \\ b_1 &= b_\lambda, & b_{k+1} &= b_{\lambda_{k+1}}. \end{aligned}$$

These definitions give us a crystal isomorphism

$$\Psi : \mathcal{B}(\lambda_j) \to \mathcal{B}(\lambda_{j+1}) \otimes \mathcal{B}_\ell,$$

which is defined by $u_{\lambda_j} \mapsto u_{\lambda_{j+1}} \otimes b_j$. We can then construct a sequence of crystal isomorphisms

$$\mathcal{B}(\lambda) \to \mathcal{B}(\lambda_2) \otimes \mathcal{B}_\ell \to \mathcal{B}(\lambda_3) \otimes \mathcal{B}_\ell \otimes \mathcal{B}_\ell \to \cdots \to \mathcal{B}(\lambda_{k+1}) \otimes \mathcal{B}_\ell^{\otimes k} \to \cdots,$$

which is defined by

$$u_\lambda \mapsto u_{\lambda_2} \otimes b_1 \mapsto u_{\lambda_3} \otimes b_2 \otimes b_1 \mapsto \cdots \mapsto u_{\lambda_{k+1}} \otimes b_k \otimes \cdots \otimes b_2 \otimes b_1 \mapsto \cdots.$$

Now we have two infinite sequences, which we denote by

$$\mathbf{w}_\lambda = (\lambda_k)_{k=1}^\infty = (\ldots, \lambda_{k+1}, \lambda_k, \ldots, \lambda_2, \lambda_1) \in (\bar{P}_\ell^+)^\infty,$$
$$\mathbf{p}_\lambda = (b_k)_{k=1}^\infty = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_2 \otimes b_1 \in \mathcal{B}^{\otimes\infty}.$$

Hence, for each $k \geq 1$, we have the crystal isomorphism

$$\Psi_k : \mathcal{B}(\lambda) \to \mathcal{B}(\lambda_{k+1}) \otimes \mathcal{B}^{\otimes k}$$

defined by

$$u_\lambda \mapsto u_{\lambda_{k+1}} \otimes b_k \otimes \cdots \otimes b_2 \otimes b_1.$$

Because $\bar{P}_\ell^+$ and in $\mathcal{B}_\ell$ each have a finite number of elements, there exist $N > 0$ and $k \geq 1$ such that $\lambda_{k+N} = \lambda_k$, and hence, $b_{k+N} = b_k$ since φ is bijective. Since ε is also bijective, we have the following equations:

$$\begin{aligned}
b_1 &= \varphi^{-1}(\lambda_1) = \varphi^{-1}(\lambda_{N+1}) = b_{N+1}, \\
\lambda_2 &= \varepsilon(b_1) = \varepsilon(b_{N+1}) = \lambda_{N+2}, \\
b_2 &= \varphi^{-1}(\lambda_2) = \varphi^{-1}(\lambda_{N+2}) = b_{N+2}, \\
&\vdots \\
\lambda_j &= \varepsilon(b_{j-1}) = \varepsilon(b_{N+j-1}) = \lambda_{N+j}, \\
b_j &= \varphi^{-1}(\lambda_j) = \varphi^{-1}(\lambda_{N+j}) = b_{N+j}, \\
&\vdots
\end{aligned}$$

$$\begin{aligned}\lambda_N &= \varepsilon(b_{N-1}) = \varepsilon(b_{2N-1}) = \lambda_{2N},\\ b_N &= \varphi^{-1}(\lambda_N) = \varphi^{-1}(\lambda_{2N}) = b_{2N}.\end{aligned}$$

Thus, $\mathbf{w}_\lambda$ and $\mathbf{p}_\lambda$ are periodic sequences, each with period $N > 0$. We call $\mathbf{p}_\lambda = (b_k)_{k=1}^{\infty} = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_2 \otimes b_1$ the *ground-state path* of weight λ, and $\mathbf{p} = (\mathbf{p}_k)_{k=1}^{\infty} = \cdots \mathbf{p}_{k+1} \otimes \mathbf{p}_k \otimes \cdots \otimes \mathbf{p}_2 \otimes \mathbf{p}_1$ a λ*-path*, where $\mathbf{p}_k \in \mathcal{B}_\ell$ such that $\mathbf{p}_k = b_k$ for all $k \gg 1$. We use $\mathcal{P}(\lambda)$ to denote the set of all λ-paths in $\mathcal{B}_\ell$ [6].

Example 3.4.1. We refer back to Example 3.3.3, in which $\mathfrak{g} = A_2^{(1)}$, level $\ell = 2$, and $\lambda = 2\Lambda_0$. We construct the ground-state path for this example- see Figure 3.2 for a visual of the perfect crystal $\mathcal{B}_2$. From Example 3.3.3, we know that the λ-minimal element is $(0,0,2)$; hence, we set $b_{\lambda_1} = b_1 = (0,0,2)$ and $\lambda_1 = 2\Lambda_0 = \lambda$. We compute $\varepsilon(b_1) = 2\Lambda_2$, and set $\lambda_2 = 2\Lambda_2$ and $b_{\lambda_2} = (0,2,0)$. Next we compute $\varepsilon(b_2) = 2\Lambda_1$, and set $\lambda_3 = 2\Lambda_1$ and $b_3 = (2,0,0)$. If we compute $\varepsilon(b_3)$, we obtain $2\Lambda_0$, which is equivalent to λ_1.

Hence, the ground-state path is

$$\begin{aligned}\mathbf{p}_\lambda &= \ldots \otimes b_3 \otimes b_2 \otimes b_1 \otimes b_3 \otimes b_2 \otimes b_1\\ &= \ldots \otimes (2,0,0) \otimes (0,2,0) \otimes (0,0,2) \otimes (2,0,0) \otimes (0,2,0) \otimes (0,0,2).\end{aligned}$$

An example of a λ-path is

$$\mathbf{p} = \ldots \otimes (2,0,0) \otimes (0,2,0) \otimes (0,0,2) \otimes (2,0,0) \otimes (1,1,0) \otimes (0,1,1).$$

Theorem 3.4.2. *[6] Let $\boldsymbol{p} = (\boldsymbol{p}_k)_{k=1}^{\infty}$ be a λ-path in $\mathcal{B}_\ell$ and let $N > 0$ be the smallest positive integer such that $\boldsymbol{p}_k = b_k$ for all $k \geq N$. For each $i \in I$, define*

$$\begin{aligned}\overline{wt}\boldsymbol{p} &= \lambda_N + \sum_{k=1}^{N-1} \overline{wt}\boldsymbol{p}_k,\\ \tilde{e}_i \boldsymbol{p} &= \cdots \otimes \boldsymbol{p}_{N+1} \otimes \tilde{e}_i(\boldsymbol{p}_N \otimes \cdots \otimes \boldsymbol{p}_1),\\ \tilde{f}_i \boldsymbol{p} &= \cdots \otimes \boldsymbol{p}_{N+1} \otimes \tilde{f}_i(\boldsymbol{p}_N \otimes \cdots \otimes \boldsymbol{p}_1),\\ \varepsilon_i(\boldsymbol{p}) &= max(\varepsilon_i(\boldsymbol{p}') - \varphi_i(b_N), 0),\\ \varphi_i(\boldsymbol{p}) &= \varphi_i(\boldsymbol{p}') + max(\varphi_i(b_N) - \varepsilon_i(\boldsymbol{p}'), 0),\end{aligned}$$

where $\boldsymbol{p}' = \boldsymbol{p}_{N-1} \otimes \cdots \otimes \boldsymbol{p}_1 \otimes \boldsymbol{p}_1$ and $\overline{wt}$ denotes the classical weights. Then the maps $\overline{wt} : \mathcal{P}(\lambda) \to \bar{P}$, $\tilde{e}_i, \tilde{f}_i : \mathcal{P}(\lambda) \to \mathcal{P}(\lambda) \sqcup \{0\}$, and $\varepsilon_i, \varphi_i : \mathcal{P}(\lambda) \to \mathbb{Z}$ define a $U_q'(\mathfrak{g})$-crystal structure on $\mathcal{P}(\lambda)$.

Theorem 3.4.3. *[14] There exists an isomorphism of $U'_q(\mathfrak{g})$-crystals*

$$\Psi : \mathcal{B}(\lambda) \longrightarrow \mathcal{P}(\lambda)$$

given by

$$u_\lambda \mapsto \boldsymbol{p}_\lambda.$$

The previous theorem defines the *path realization* of the classical crystal $\mathcal{B}(\lambda)$. Now we are ready to provide the reader with the definitions of Demazure modules and crystals.

Chapter

4

Demazure Modules and Crystals

A Demazure module is a certain finite-dimensional subspace of an integrable module for a Kac-Moody algebra parameterized by elements in the Weyl group and dominant integral weights. Kashiwara showed that the crystal for a Demazure module is a subset of the crystal for the corresponding integrable highest weight module. Subsequently, path realizations of these Demazure crystals were given in [21]. They also showed that these path realizations have tensor product-like structures. We focus on Demazure crystals for $U_q\left(A_n^{(1)}\right)$. We recall definitions and main results from ([20], [21], [14]).

4.1 Demazure Modules and Crystals

Let $V^q(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module in $\mathcal{O}^q_{int}$. Then $V^q(\lambda)$ has highest weight λ and highest weight vector u_λ [6]. We recall that $\dim V^q(\lambda)_\lambda = \dim V^q(\lambda)_{w\lambda} = 1$ for any $w \in \mathcal{W}$, where $\mathcal{W}$ is the Weyl group [12]. We call the basis vector $u_{w\lambda}$ of $V^q(\lambda)_{w\lambda}$ the *extremal vector*.

Definition 4.1.1. Fix $w \in \mathcal{W}$ and $\lambda \in \bar{P}^+$. Then $V_w(\lambda) = U_q(\mathfrak{g})^+ u_{w\lambda}$ is called the *Demazure module* associated with w.

Demazure modules are finite-dimensional subspaces of $V^q(\lambda)$ such that

1. $V^q(\lambda) = \bigcup_{w \in \mathcal{W}} V_w(\lambda)$,

2. For $w, w' \in \mathcal{W}$ with $w \preceq w'$ (the Bruhat order), $V_w(\lambda) \subset V_{w'}(\lambda)$.

Now let $\mathcal{B}(\lambda)$ and $\mathcal{L}(\lambda)$ be the crystal and crystal lattice for $V^q(\lambda)$, respectively. Kashiwara [18] showed that for each $w \in \mathcal{W}$, there exists a subset $B_w(\lambda)$ of $B(\lambda)$ such that

$$\frac{V_w(\lambda) \cap \mathcal{L}(\lambda)}{V_w(\lambda) \cap q\mathcal{L}(\lambda)} = \bigoplus_{b \in B_w(\lambda)} \mathbb{Q}b.$$

The subset $B_w(\lambda)$ of $B(\lambda)$ defined above is the crystal for the Demazure module $V_w(\lambda)$, and it is called the *Demazure crystal* [18]. Kashiwara also proved that the Demazure crystal $B_w(\lambda)$ has the following recursive property:

$$\text{If } r_i w \succ w, \text{ then } B_{r_i w}(\lambda) = \bigcup_{n \geq 0} \tilde{f}_i^n B_w(\lambda) \backslash \{0\}. \tag{4.1}$$

Let $\mathcal{B}_\ell$ be a perfect crystal of level ℓ for $U_q(\mathfrak{g})$, where $\lambda(c) = \ell \geq 1$. Recall that by Theorem 3.3.4, there exists an isomorphism between the crystal $\mathcal{B}(\lambda)$ and the set of λ-paths $\mathcal{P} = \mathcal{P}(\lambda, \mathcal{B}_\ell)$, in which the highest weight vector $u_\lambda \in \mathcal{B}(\lambda)$ identifies with the ground-state path $\mathbf{p}_\lambda = \cdots \otimes b_3 \otimes b_2 \otimes b_1$. The path realizations for the Demazure crystals $B_w(\lambda)$ are defined in the following way [21]:

First, choose $d, \kappa \in \mathbb{Z}_{>0}$. For a sequence of elements $\{i_a^{(j)} | j \geq 1, 1 \leq a \leq d\} \subset \{0, 1, \ldots, n\}$, define the subsets $\{B_a^{(j)} | j \geq 1, 0 \leq a \leq d\}$ as follows:

$$B_0^{(j)} = \{b_j\}, \qquad B_a^{(j)} = \bigcup_{k \geq 0} \tilde{f}_{i_a^{(j)}}^k B_{a-1}^{(j)} \backslash \{0\}.$$

Now define $B_a^{(j+1,j)}$ for $j \geq 1, 0 \leq a \leq d$, by

$$B_0^{(j+1,j)} = B_0^{(j+1)} \otimes B_d^{(j)}, \qquad B_a^{(j+1,j)} = \bigcup_{k \geq 0} \tilde{f}_{i_a^{(j+1)}}^k B_{a-1}^{(j+1,j)} \backslash \{0\}.$$

Continue in this way, until we define:

$$B_0^{(j+\kappa-1,\ldots,j)} = B_0^{(j+\kappa-1)} \otimes B_d^{(j+\kappa-2,\ldots,j)},$$
$$B_a^{(j+\kappa-1,\ldots,j)} = \bigcup_{k \geq 0} \tilde{f}_{i_a^{(j+\kappa-1)}}^k B_{a-1}^{(j+\kappa-1,\ldots,j)} \backslash \{0\}.$$

Let

$$w^{(0)} = 1, \qquad w^{(k)} = r_{i_a}^{(j)} w^{(k-1)}$$

define a sequence of Weyl group elements $\{w^{(k)} | w^{(k)} \in \mathcal{W}, k > 0\}$, where j and a are fixed by $k = (j-1)d + a$, for $j \geq 1$, $1 \leq a \leq d$. Also, define the subsets $\mathcal{P}^{(k)}(\lambda, \mathcal{B}_\ell)$ of $\mathcal{P}(\lambda, \mathcal{B}_\ell)$, for $k \geq 0$, by the following:

$$\mathcal{P}^{(0)}(\lambda, \mathcal{B}_\ell) = \{\mathbf{p}_\lambda\},$$
$$\mathcal{P}^{(k)}(\lambda, \mathcal{B}_\ell) = \begin{cases} \cdots \otimes B_0^{(j+2)} \otimes B_0^{(j+1)} \otimes B_a^{(j,\ldots,1)} & \text{if } j < \kappa, \\ \cdots \otimes B_0^{(j+2)} \otimes B_0^{(j+1)} \otimes B_a^{(j,\ldots,j-\kappa+1)} \otimes B^{\otimes(j-\kappa)} & \text{if } j \geq \kappa. \end{cases}$$

We end this section with a theorem that describes the combinatorial structure of Demazure crystals $B_{w^{(k)}}(\lambda)$.

Theorem 4.1.2. *[21] Let $\lambda \in \bar{P}^+$ with $\lambda(c) = \ell$, and let $\mathcal{B}_\ell$ be a perfect crystal of level ℓ for the quantum affine algebra $U_q(\mathfrak{g})$. For fixed $d, \kappa \in \mathbb{Z}_{>0}$, suppose we have a sequence of integers $\{i_a^{(j)} \mid j \geq 1, 1 \leq a \leq d\} \subset \{0, 1, \ldots, n\}$ satisfying the following conditions:*

1. *for any $j \geq 1$, $B_d^{(j+\kappa-1,\ldots,j)} = B_d^{(j+\kappa-1,\ldots,j+1)} \otimes \mathcal{B}_\ell$,*
2. *for any $j \geq 1$ and $1 \leq a \leq d$, $\langle \lambda_j, h_{i_a^{(j)}} \rangle \leq \varepsilon_{i_a^{(j)}}(b)$, $b \in B_{a-1}^{(j)}$, and*
3. *the sequence of elements $\{w^{(k)}\}_{k \geq 0}$ is an increasing sequence of Weyl group elements with respect to the Bruhat order.*

Then we have that $B_{w^{(k)}}(\lambda) \cong \mathcal{P}^{(k)}(\lambda, \mathcal{B}_\ell)$.

4.2 $\lambda = \ell\Lambda_j$ Case

In this section, we give explicit descriptions corresponding to Demazure crystals $B_w(\lambda)$ of $U_q(A_n^{(1)})$, for which $\lambda = \ell\Lambda_j$. For the rest of this thesis, assume that $\mathfrak{g} = A_n^{(1)}$.

First, we define the set

$$\mathcal{B}_\ell = \left\{ (m_1, m_2, \ldots, m_n, m_0) \in \mathbb{Z}_{\geq 0}^{n+1} \,\middle|\, \sum_{i=0}^{n} m_i = \ell \right\}$$

and give the actions of $\tilde{e}_i$ and $\tilde{f}_i$ as follows:

$$\tilde{e}_0(m_1, m_2, \ldots, m_n, m_0) = (m_1 - 1, m_2, \ldots, m_n, m_0 + 1),$$
$$\tilde{e}_i(m_1, m_2, \ldots, m_n, m_0) = (m_1, \ldots, m_i + 1, m_{i+1} - 1, m_{i+2}, \ldots, m_n, m_0),$$
$$\tilde{f}_0(m_1, m_2, \ldots, m_n, m_0) = (m_1 + 1, m_2, \ldots, m_n, m_0 - 1),$$

$$\tilde{f}_i(m_1, m_2, \ldots, m_n, m_0) = (m_1, \ldots, m_i - 1, m_{i+1} + 1, m_{i+2}, \ldots, m_n, m_0),$$

for $1 \le i \le n$. For any $b \in \mathcal{B}_\ell$, if $\tilde{e}_i(b)$ or $\tilde{f}_i(b) \notin \mathcal{B}_\ell$, e.g. if m_i becomes negative, then we understand it to be 0. We also define

$$\begin{aligned}
\varepsilon_i(b) &= m_{i+1}, \\
\varphi_i(b) &= m_i, \\
\mathrm{wt}(b) &= \sum_{i=0}^{n} (\varphi_i(b) - \varepsilon_i(b))\Lambda_i,
\end{aligned}$$

for $0 \le i \le n$, where $n+1 \equiv 0$.

By [15], $\mathcal{B}_\ell$ is a perfect crystal of level ℓ for $U_q(A_n^{(1)})$. We note that $\mathcal{B}_\ell$, with φ_i, ε_i, $\tilde{e}_i$, $\tilde{f}_i$, and $\mathrm{wt}(b)$ defined as above for $1 \le i \le n$, is the crystal for the irreducible $sl(n+1)$-module $V(\ell\Lambda_1)$. By [7], we then have

$$|\mathcal{B}_\ell| = \frac{\prod_{\alpha \succ 0}(\ell\Lambda_1 + \delta, \alpha)}{\prod_{\alpha \succ 0}(\delta, \alpha)}.$$

Note that

$$\begin{aligned}
\delta &= \frac{1}{2}\sum_{\alpha \succ 0} \alpha \\
&= \frac{1}{2}\sum_{1 \le i < j \le n+1} (\varepsilon_i - \varepsilon_j) \\
&= \frac{1}{2}\sum_{m=1}^{n+1} (n - 2(m-1))\varepsilon_m.
\end{aligned}$$

Hence, the numerator

$$\begin{aligned}
\prod_{\alpha \succ 0}(\ell\Lambda_1 + \delta, \alpha) &= \prod_{1 \le i < j \le n+1} \left(\ell\varepsilon_1 + \frac{1}{2}\sum_{m=1}^{n+1}(n - 2(m-1))\varepsilon_m, \varepsilon_i - \varepsilon_j \right) \\
&= \prod_{2 \le j \le n+1} \left(\frac{1}{2}\left[(2\ell + n)\varepsilon_1 + \sum_{m=2}^{n+1}(n - 2(m-1))\varepsilon_m \right], \varepsilon_1 - \varepsilon_j \right) \\
&\quad \cdot \prod_{2 \le i < j \le n+1} \left(\frac{1}{2}\left[(2\ell + n)\varepsilon_1 + \sum_{m=2}^{n+1}(n - 2(m-1))\varepsilon_m \right], \varepsilon_i - \varepsilon_j \right) \\
&= \prod_{2 \le j \le n+1} \frac{1}{2}\big(2\ell + n - (n - 2(j-1))\big) \prod_{2 \le i < j \le n+1} \frac{1}{2}\big(n - 2(i-1) - (n - 2(j-1))\big),
\end{aligned}$$

and the denominator

$$\begin{aligned}\prod_{\alpha \succ 0}(\delta, \alpha) &= \prod_{1\le i<j\le n+1}\left(\frac{1}{2}\sum_{m=1}^{n+1}(n-2(m-1))\varepsilon_m, \varepsilon_i - \varepsilon_j\right)\\ &= \prod_{1\le i<j\le n+1}\left(\frac{1}{2}\big[n-2(i-1)-(n-2(j-1))\big]\right)\\ &= \prod_{1\le i<j\le n+1}(j-i).\end{aligned}$$

Thus,

$$|\mathcal{B}_\ell| = \frac{(k+1)(k+2)\cdots(k+n)(1)^{n-1}(2)^{n-2}\cdots(n-2)^2(n-1)}{(1)^n(2)^{n-1}(3)^{n-2}\cdots(n-2)^3(n-1)^2(n)^1} = \binom{n+\ell}{\ell}.$$

Now we construct the sequence of integers

$$\{i_a^{(j)} | j \ge 1, 1 \le a \le d\} \subset \{0, 1, \dots, n\}.$$

In this thesis, we study $A_n^{(1)}$-perfect crystals $\mathcal{B}_\ell = \mathcal{B}^{1,\ell}$. So, as in [20], set $d = n$, $r = a-1-ng$, and $g = \left\lfloor \frac{a-1}{n} \right\rfloor$, where $1 \le a \le n$. Hence, $g = 0$ for all a and n. Then we define

$$i_a^{(j)} = 1 - j - g + r = 1 - j + (a-1) = a - j.$$

Thus, we have the sequence of integers $\{i_a^{(j)} = a - j | j \ge 1, 1 \le a \le n\}$. Using this sequence of integers, we can define

$$w^{(0)} = 1 \qquad \text{and} \qquad w^{(k)} = r_{i_a^{(j)}} w^{(k-1)},$$

where j and a are fixed by $k = (j-1)d + a = (j-1)n + a$.

We define

$$\begin{aligned} w^{(n)} &= r_{i_n^{(1)}} r_{i_{n-1}^{(1)}} \cdots r_{i_2^{(1)}} r_{i_1^{(1)}}\\ &= r_{n-1} r_{n-2} \cdots r_1 r_0,\\ w^{(2n)} &= \left(r_{i_n^{(2)}} r_{i_{n-1}^{(2)}} \cdots r_{i_2^{(2)}} r_{i_1^{(2)}}\right)\left(r_{i_n^{(1)}} r_{i_{n-1}^{(1)}} \cdots r_{i_2^{(1)}} r_{i_1^{(1)}}\right)\\ &= (r_{n-2} \cdots r_0 r_n)(r_{n-1} \cdots r_1 r_0),\\ &\vdots\\ w^{(Ln)} &= \left(r_{i_n^{(L)}} r_{i_{n-1}^{(L)}} \cdots r_{i_2^{(L)}} r_{i_1^{(L)}}\right)\cdots\left(r_{i_n^{(2)}} r_{i_{n-1}^{(2)}} \cdots r_{i_2^{(2)}} r_{i_1^{(2)}}\right)\left(r_{i_n^{(1)}} r_{i_{n-1}^{(1)}} \cdots r_{i_2^{(1)}} r_{i_1^{(1)}}\right)\end{aligned}$$

$$= (r_{n-L} \cdots r_{n-L+3} r_{n-L+2}) \cdots (r_{n-2} \cdots r_0 r_n)(r_{n-1} \cdots r_1 r_0).$$

We denote $w(L,0) = w^{(Ln)}$. Note that $w(L,0)$ has length Ln.

Remark 1. Each sequence of simple reflections $r_{i_s} \cdots r_{i_2} r_{i_1}$, for $i_t \in \{0,1,\ldots,n\}$, corresponds to a sequence of Kashiwara operators $\tilde{f}_{i_s} \cdots \tilde{f}_{i_2} \tilde{f}_{i_1}$, in which each Kashiwara operator is applied as many times as possible to elements in $\mathcal{B}_\ell$.

We have the automorphism σ such that $\sigma(r_i) = r_{i+1}$ for $0 \le i \le n$, where r_i is a simple reflection in $\mathcal{W}$ and $\sigma(r_n) = r_0$. We define the notation

$$w(L,j) = \sigma^j(w^{(Ln)}) = \left(r_{j+n-L} \cdots r_{j+2-L} r_{j+1-L}\right) \cdots \left(r_{j+n-2} \cdots r_j r_{j+n}\right)\left(r_{j+n-1} \cdots r_{j+1} r_j\right).$$

Let $\lambda = \ell\Lambda_j$. Then the λ-minimal element is $(0,\ldots,0,\ell,0,\ldots,0)$, where the ℓ is in the $(j)^{th}$ coordinate, and the ground-state path is:

$$\begin{aligned}
\mathbf{p}_\lambda &= \ldots \otimes b_1 \otimes b_{n+1} \otimes b_n \otimes \ldots \otimes b_2 \otimes b_1 \\
&= \ldots \otimes (0,\ldots,0,\ell,0,\ldots,0) \otimes \ldots \otimes (0,\ldots,\ell,0) \otimes (0,\ldots,0,\ell) \otimes (\ell,0,\ldots,0) \otimes \ldots \\
&\qquad \ldots \otimes (0,\ldots,\ell,0,0,\ldots,0) \otimes (0,\ldots,0,\ell,0,\ldots,0),
\end{aligned}$$

where the ℓ in the rightmost component of the tensor product is in the $(j)^{th}$ component, and as we move left along the tensor product, the ℓ shifts left in the $(n+1)$-tuple.

We wish to construct the Demazure crystals $B_{w(L,j)}(\ell\Lambda_j)$. Recall Equation 4.1, which gives a recursive property of these Demazure crystals. In order to use this definition, we need to define the (i)-signature of a path [21].

Let $p \in \mathcal{P}(\lambda, B)$ correspond to $u_{\lambda_k} \otimes p(k) \otimes \cdots \otimes p(1)$. For each $p(t)$, we associate the following:

$$\begin{aligned}
\epsilon^{(t)} &= \left(\epsilon_1^{(t)}, \epsilon_2^{(t)}, \ldots, \epsilon_m^{(t)}\right), \\
m &= \varepsilon_i(p(t)) + \varphi_i(p(t)), \\
\epsilon_a^{(t)} &= \begin{cases} -, & \text{if } 1 \le a \le \varepsilon_i(p(t)) \\ +, & \text{if } \varepsilon_i(p(t)) < a \le m. \end{cases}
\end{aligned}$$

For the highest weight vector u_{λ_k}, we have $\epsilon^{(k+1)} = (+,\ldots,+)$, where there are $\langle \lambda_k, h_i \rangle$ pluses. Then we append the $\epsilon^{(j)}$'s, forming $\epsilon = \left(\epsilon^{(k+1)}, \epsilon^{(k)}, \ldots, \epsilon^{(1)}\right)$. We call this the (i)-*signature* of p truncated at the k^{th} position. We can also form a sequence of signatures, $\epsilon = \eta_0, \eta_1, \ldots, \eta_{\max}$, where η_{t+1} is obtained from η_t by removing the leftmost adjacent $(+,-)$ pair of η_t and $\eta_{\max}$ is of the form:

$$\eta_{\max} = (-, \ldots, -, +, \ldots, +)$$

for $n_- \geq 0$ minuses and $n_+ \geq 0$ pluses. We call $\eta_{\max}$ the *reduced* (i)*-signature* and denote it by $\bar{\epsilon}$.

We can use these (i)-signatures to determine on which component of the path p $\tilde{e}_i$ or $\tilde{f}_i$ acts. If $n_- = 0$ (respectively, $n_+ = 0$), then we set $\tilde{e}_i p = 0$ (respectively, $\tilde{f}_i p = 0$). Otherwise, take the rightmost $-$ (respectively, leftmost $+$) and find the component $\epsilon^{(t)}$ to which it belongs. Then

$$\tilde{e}_i p = \cdots \otimes \tilde{e}_i p(t) \otimes \cdots \otimes p(2) \otimes p(1),$$

or respectively,

$$\tilde{f}_i p = \cdots \otimes \tilde{f}_i p(t) \otimes \cdots \otimes p(2) \otimes p(1).$$

Note that if k is large enough, then t doesn't depend on the choice of k.

Example 4.2.1. Referring back to Example 3.4.1, we have $\mathfrak{g} = A_2^{(1)}$, level $\ell = 2$, and $\lambda = 2\Lambda_0$. Our example of a λ-path was

$$\mathbf{p} = \ldots \otimes (2,0,0) \otimes (0,2,0) \otimes (0,0,2) \otimes (2,0,0) \otimes (1,1,0) \otimes (0,1,1).$$

The (2)-signature for the above λ-path is $(\ldots, \bullet, +_2, -_2, \bullet, +, -+)$, where we place $\bullet$'s as placeholders for components in the tensor product that don't have any $+$'s or $-$'s in the (2)-signature. The corresponding reduced (2)-signature is $(+)$, since the sequence $(\bullet, +_2, -_2)$ is repeated infinitely many times and all $(+,-)$ pairs cancel. The (0)-signature for the above λ-path is $(\ldots, -_2, \bullet, +_2, -_2, -, +)$, which reduces to $(-,+)$, and the (1)-signature is $(\ldots, +_2, -_2, \bullet, +_2, -+, -)$, which reduces to $(+, \bullet, \bullet)$.

In general, for $b = (m_1, m_2, \ldots, m_n, m_0) \in \mathcal{B}_\ell$, the (i)-signature is $\left(-_{m_{i+1}} +_{m_i}\right)$.

We write the ground-state path as

$$\begin{aligned} u_\lambda &= \ldots \otimes b_{n+2} \otimes b_{n+1} \otimes \ldots \otimes b_3 \otimes b_2 \otimes b_1 \\ &= \ldots \otimes b_{\ell\Lambda_j} \otimes b_{\ell\Lambda_{j+1}} \otimes \ldots \otimes b_{\ell\Lambda_{j+n-1}} \otimes b_{\ell\Lambda_{j+n}} \otimes b_{\ell\Lambda_j}, \end{aligned}$$

and we denote

$$\begin{aligned} u_{\ell\Lambda_{j-L}} \otimes \mathcal{B}_\ell^L &= u_{\ell\Lambda_{j-L}} \otimes \underbrace{\mathcal{B}_\ell \otimes \ldots \otimes \mathcal{B}_\ell}_{L} \\ &= \left\{ u_{\ell\Lambda_{j-L}} \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_L} \,\middle|\, b_{\rho_t} \in \mathcal{B}_\ell \text{ for } 1 \le t \le L \right\}. \end{aligned}$$

Remark 2. The (i)-signature for any path in the set $u_{\ell\Lambda_{j-L}} \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$ is of the form

$$\begin{cases} (-_\ell, \bullet, *_\mu, *_1, \ldots, *_{L-1}), & \text{if } i = j-L-1, \\ (+_\ell, *_\mu, *_1, \ldots, *_{L-1}), & \text{if } i = j-L, \\ (*_\mu, *_1, \ldots, *_{L-1}), & \text{if } i \neq j-L-1, j-L, \end{cases}$$

where $*_t$ is a sequence of the form $(-_{a_t} +_{s_t})$, $a_t, s_t \in \mathbb{Z}_{\ge 0}$, for each $1 \le t \le L-1$. Hence, the $(L+1)^{st}$ component from the right for any path in the set $u_{\ell\Lambda_{j-L}} \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$ is only affected if $i = j-L$.

Lemma 4.2.2. *Let $\lambda = \ell\Lambda_j$. Then $B_{w(L,j)}(\lambda) \subset u_{\ell\Lambda_{j-L}} \otimes \mathcal{B}_\ell^L$.*

Proof. Let $L = 1$. Then we have $w(1, j) = (r_{j+n-1} \cdots r_{j+1} r_j)$. By Remark 2, the second component from the right in $u_{\ell\Lambda_j}$ can only be affected if r_{j-1} is in the sequence $w(1, j)$. Hence, the only component affected in $u_{k\Lambda_j}$ is the first component from the right. Thus, $B_{w(1,j)}(\ell\Lambda_j) \subset u_{\ell\Lambda_{j-1}} \otimes \mathcal{B}_\ell$.

Now, suppose $B_{w(L-1,j)}(\ell\Lambda_j) \subset u_{\ell\Lambda_{j-L+1}} \otimes B^{L-1}$. We want to prove the claim for L. Note that

$$\begin{aligned} w(L, j) &= (r_{j+n-L} \cdots r_{j+2-L} r_{j+1-L}) \cdots (r_{j+n-2} \cdots r_j r_{j+n})(r_{j+n-1} \cdots r_{j+1} r_j) \\ &= (r_{j+n-L} \cdots r_{j+2-L} r_{j+1-L}) w(L-1, j). \end{aligned}$$

By our induction hypothesis, we know that $B_{w(L-1,j)}(\ell\Lambda_j) \subset u_{\ell\Lambda_{j-L+1}} \otimes \mathcal{B}_\ell^{L-1}$. By definition, $B_{w(L,j)}(\ell\Lambda_j)$ is formed by applying each $\tilde{f}_t$, for $j+2-L \le t \le j+n-L$, in the given order as many times as possible to every path in $B_{r_{t-1} \cdots r_{j+1-L} w(L-1,j)}(\ell\Lambda_j)$. By Remark 2, we need to apply $\tilde{f}_{j-L}$ to paths in the set $u_{\ell\Lambda_{j-L}} \otimes b_{\ell\Lambda_{j-L+1}} \otimes \mathcal{B}_\ell^{L-1}$ in order to affect the $(L+1)^{st}$ component from the right. But the Weyl group sequence $w(L, j)$ does not contain the simple reflection r_{j-L} in the last n simple reflections. Hence, only the first L components in the tensor product $u_{\ell\Lambda_{j-L}} \otimes b_{\ell\Lambda_{j-L+1}} \otimes \mathcal{B}_\ell^{L-1}$ can be affected. Thus, $B_{w(L,j)}(k\Lambda_j) \subset u_{k\Lambda_{j-L}} \otimes B^L$. $\square$

We now give explicit descriptions for the Demazure crystals $B_{w(L,j)}(\ell\Lambda_j)$.

Theorem 4.2.3. *We have the following:*

$$\begin{aligned}
B_{w(L,j+1)}(\ell\Lambda_j) &= u_{\ell\Lambda_{j-L+1}} \otimes \mathcal{B}_\ell^{L-1},\\
B_{w(L,j+s)}(\ell\Lambda_j) &= \left\{u_{\ell\Lambda_{j-L}} \otimes b_{\mu_{s-1}} \otimes \mathcal{B}_\ell^{L-1}\right\},\\
B_{w(L,j)}(\ell\Lambda_j) &= u_{\ell\Lambda_{j-L}} \otimes \mathcal{B}_\ell^{L},
\end{aligned}$$

where

$$b_{\mu_{s-1}} = \left(m_1, m_2, \dots, m_{j+s-L}, 0, \dots, 0, m_{j+1-L}, m_{j+2-L}, \dots, m_n, m_0\right) \in \mathcal{B}_\ell,\ s = 2, 3, \dots, n.$$

Proof. Let $L = 1$. Then we have the sequence of Weyl group elements $w(1, j) = r_{j+n-1} \cdots r_{j+1} r_j$. By definition and by Remark 2, the only Kashiwara operator that will affect $u_{\ell\Lambda_j}$ is $\tilde{f}_j$. Hence, we apply $\tilde{f}_j$ exactly (m_{j+1}) times to $u_{\ell\Lambda_j}$ to produce the paths $u_{\ell\Lambda_{j-1}} \otimes b_{\mu_1}$, where

$$b_{\mu_1} = (0, \dots, 0, m_j, m_{j+1}, 0, \dots, 0) \in \mathcal{B}_\ell.$$

Then these paths have $(j+1)$-signature $(+_{m_{j+1}})$. Hence, we can apply $\tilde{f}_{j+1}$ a total of (m_{j+2}) times to these new paths to produce the following paths $u_{\ell\Lambda_{j-1}} \otimes b_{\mu_2}$, where

$$b_{\mu_2} = (0, \dots, 0, m_j, m_{j+1}, m_{j+2}, 0, \dots, 0) \in \mathcal{B}_\ell.$$

The $(j+2)$-signature for these paths is $\left(+_{m_{j+2}}\right)$, and hence, we can apply $\tilde{f}_{j+3}$ a sufficient number of times to produce the paths $u_{\ell\Lambda_{j-1}} \otimes b_{\mu_3}$, where $b_{\mu_3} = (0, \dots, 0, m_j, m_{j+1}, m_{j+2}, m_{j+3}, 0, \dots, 0) \in \mathcal{B}_\ell$. If we continue this pattern, we form the paths $u_{\ell\Lambda_{j-1}} \otimes b_{\mu_s}$, where

$$b_{\mu_s} = (m_1, \dots, m_{j+s}, 0, \dots, 0, m_j, m_{j+1}, \dots, m_n, m_0) \in \mathcal{B}_\ell,$$

with $(j+s)$-signature $(+_{m_{j+s}})$, for $s = 1, 2, \dots, n-1$.

If $s = n-1$, then we can apply $\tilde{f}_{j+n-1}$ a total of (m_{j+n}) times to produce the paths $u_{\ell\Lambda_{j-1}} \otimes b_{\mu_n}$, where

$$b_{\mu_n} = \left(m_1, \dots, m_{j+n-1}, m_{j+n}, m_j, \dots, m_0\right) \in \mathcal{B}_\ell.$$

Note that the set

$$\left\{u_{\ell\Lambda_{j-1}} \otimes b_{\mu_n}\right\} = u_{\ell\Lambda_{j-1}} \otimes \mathcal{B}_\ell = B_{w(1,j)}(\ell\Lambda_j),$$

by construction. Also, notice that

$$\begin{aligned} w(1, j+1) &= \left(r_{j+n} r_{j+n-1} \cdots r_{j+2} r_{j+1}\right), \text{ which doesn't contain } r_j, \text{ and} \\ w(1, j+s) &= \left(r_{j+s+n-1} \cdots r_{j+s+1} r_{j+s}\right) \\ &= \left(r_{j+s+n-1} \cdots r_{j+1} r_j \cdots r_{j+s+1} r_{j+s}\right). \end{aligned}$$

Since only r_j affects $u_{\ell\Lambda_j}$, we have

$$\begin{aligned} B_{w(1,j+1)}(\ell\Lambda_j) &= u_{\ell\Lambda_j}, \\ B_{w(1,j+s)}(\ell\Lambda_j) &= B_{\left(r_{j+s+n-1} \cdots r_{j+1} r_j\right)}(\ell\Lambda_j) \\ &= \left\{ u_{\ell\Lambda_{j-1}} \otimes b_{\mu_{s-1}} \right\}, \end{aligned}$$

where $b_{\mu_{s-1}} = (m_1, \ldots, m_{j+s-1}, 0, \ldots, 0, m_j, m_{j+1}, \ldots, m_n, m_0) \in \mathcal{B}_\ell$ and $s = 2, \ldots, n$.

Now, let $L = 2$. Then we have the sequence of Weyl group elements

$$\begin{aligned} w(2, j) &= \left(r_{j+n-2} \cdots r_j r_{j+n}\right)\left(r_{j+n-1} \cdots r_{j+1} r_j\right) \\ &= \left(r_{j+n-2} \cdots r_j r_{j+n}\right) w(1, j). \end{aligned}$$

By Lemma 4.2.2, we know that $B_{w(2,j)}(\ell\Lambda_j) \subset u_{\ell\Lambda_{j-2}} \otimes \mathcal{B}_\ell^2$. We want to show that $u_{\ell\Lambda_{j-2}} \otimes \mathcal{B}_\ell^2 \subset B_{w(2,j)}(\ell\Lambda_j)$. Note that by applying the Kashiwara operators corresponding to the Weyl group sequence $w(1, j)$, we obtain all paths in the set $u_{\ell\Lambda_{j-1}} \otimes \mathcal{B}_\ell$. Suppose we want to obtain all paths of the form

$$u_{\ell\Lambda_{j-2}} \otimes b_{\mu_1} \otimes \mathcal{B}_\ell,$$

where $b_{\mu_1} = (0, \ldots, 0, m_{j-1}, m_j, 0, \ldots, 0) \in \mathcal{B}_\ell$. Note that the $(j-1)$-signature for paths of the form $u_{\ell\Lambda_{j-1}} \otimes b_\nu$, where $b_\nu \in \mathcal{B}_\ell$, is of the form $(+_\ell, \ast_1)$. Suppose $\ast_1$ contains no more than $(\ell - m_j)$ negatives. Then we can apply $\tilde{f}_{j-1}$ exactly m_j times to form the paths $u_{\ell\Lambda_{j-2}} \otimes b_{\mu_1} \otimes b_\nu$. If $\ast_1$ contains more than $(\ell - m_j)$ negatives, then we can form all other paths of the form $u_{\ell\Lambda_{j-2}} \otimes b_{\mu_1} \otimes b_\nu$ by letting $\tilde{f}_{j-1}$ act a total of $(m_j + v_j - (\ell - m_j))$ times on the paths

$$u_{\ell\Lambda_{j-1}} \otimes \left(v_1, \ldots, v_{j-1} + v_j - (\ell - m_j), \ell - m_j, v_{j+1}, \ldots, v_n, v_0\right),$$

which have $(j-1)$-signatures $\left(+_{m_j}, +_{v_{j-1}+v_j-(\ell-m_j)}\right)$. By only using the Weyl reflections $r_{j-1} w(1, j)$, we have produced all paths of the form $u_{\ell\Lambda_{j-2}} \otimes b_{\mu_1} \otimes \mathcal{B}_\ell$.

Now suppose we want to obtain all paths of the form

$$u_{\ell\Lambda_{j-2}} \otimes b_{\mu_2} \otimes \mathcal{B}_\ell,$$

where $b_{\mu_2} = (0, \ldots, 0, m_{j-1}, m_j, m_{j+1}, 0, \ldots, 0) \in \mathcal{B}_\ell$. Note that the (j)-signature for paths $u_{\ell\Lambda_{j-2}} \otimes b_{\mu_1} \otimes b_\nu$, where $b_\nu \in \mathcal{B}_\ell$, is of the form $\left(+_{m_j}, \ast_1\right)$. Suppose $\ast_1$ contains no more than $(m_j - m_{j+1})$ negatives. Then we can apply $\tilde{f}_j$ exactly m_{j+1} times to form the paths $u_{\ell\Lambda_{j-2}} \otimes b_{\mu_2} \otimes b_\nu$. If $\ast_1$ contains more than $(m_j - m_{j+1})$ negatives, then we can construct all other paths of the form $u_{\ell\Lambda_{j-2}} \otimes b_{\mu_2} \otimes b_\nu$ by letting $\tilde{f}_j$ act on the paths

$$u_{\ell\Lambda_{j-2}} \otimes b_{\mu_1} \otimes \left(v_1, \ldots, v_j + v_{j+1} - (m_j - m_{j+1}), m_j - m_{j+1}, v_{j+2}, \ldots, v_n, v_0\right),$$

which have (j)-signatures $\left(+_{m_{j+1}}, +_{v_j + v_{j+1} - (m_j - m_{j+1})}\right)$. Hence, applying $\tilde{f}_j$ a total number of $\left(m_{j+1} + v_{j+1} - (m_j - m_{j+1})\right)$ times will produce the desired paths.

Suppose we now have paths of the form $u_{\ell\Lambda_{j-2}} \otimes b_{\mu_s} \otimes \mathcal{B}_\ell$, where

$$b_{\mu_s} = \left(m_1, \ldots, m_{j-1+s}, 0, \ldots, 0, m_{j01}, \ldots, m_n, m_0\right) \in \mathcal{B}_\ell,$$

and that we constructed these paths using the Weyl reflections $\left(r_{j-2+s} \cdots r_j r_{j-1}\right) w(1, j)$. We want to construct all paths of the form

$$u_{\ell\Lambda_{j-2}} \otimes b_{\mu_{s+1}} \otimes \mathcal{B}_\ell,$$

where $b_{\mu_{s+1}} = \left(m_1, \ldots, m_{j+s}, 0, \ldots, 0, m_{j-1}, \ldots, m_n, m_0\right) \in \mathcal{B}_\ell$. Note that the $(j-1+s)$-signature for paths $u_{\ell\Lambda_{j-2}} \otimes b_{\mu_s} \otimes b_\nu$, where $b_\nu \in \mathcal{B}_\ell$, is of the form $\left(+_{m_{j-1+s}}, \ast_1\right)$. Suppose $\ast_1$ has no more than $(m_{j-1+s} - m_{j+s})$ negatives. Then we can apply $\tilde{f}_{j-1+s}$ exactly m_{j+s} times to form the paths $u_{\ell\Lambda_{j-2}} \otimes b_{\mu_{s+1}} \otimes b_\nu$. If $\ast_1$ contains more than $(m_{j-1+s} - m_{j+s})$ negatives, then we can construct these paths by letting $\tilde{f}_{j-1+s}$ act on the paths

$$u_{\ell\Lambda_{j-2}} \otimes b_{\mu_s} \otimes \left(v_1, \ldots, v_{j-1+s} + v_{j+s} - (m_{j-1+s} - m_{j+s}), m_{j-1+s} - m_{j+s}, v_{j+s+1}, \ldots, v_n, v_0\right),$$

which have $(j-1+s)$-signatures $\left(+_{m_{j+s}}, +_{v_{j+s} - (m_{j-1+s} - m_{j+s})}\right)$, a total number of $(m_{j+s} + v_{j+s} - (m_{j-1+s} - m_{j+s}))$ times.

Note that using the Weyl reflections $w(2, j)$ will thus produce all paths of the form $u_{\ell\Lambda_{j-2}} \otimes b_{\mu_n} \otimes \mathcal{B}_\ell$, where

$$b_{\mu_n} = \left(m_1, \ldots, m_{j-2}, m_{j-1}, \ldots, m_n, m_0\right) \in \mathcal{B}_\ell,$$

and hence $B_{w(2,j)}(\ell\Lambda_j) = u_{\ell\Lambda_{j-2}} \otimes \mathcal{B}_\ell^2$. We also have that

$$\begin{aligned}
B_{w(2,j+1)}(\ell\Lambda_j) &= B_{w(1,j)w(1,j+1)}(\ell\Lambda_j)\\
&= B_{w(1,j)}(\ell\Lambda_j)\\
&= u_{\ell\Lambda_{j-1}} \otimes \mathcal{B}_\ell,\\
B_{w(2,j+s)}(\ell\Lambda_j) &= B_{\left(r_{j+s+n-2}\cdots r_{j+s} r_{j+s+n}\right)w(1,j+s)}(\ell\Lambda_j)\\
&= B_{\left(r_{j+s+n-2}\cdots r_j r_{j-1}\right)w(1,j)}(\ell\Lambda_j)\\
&= \left\{ u_{\ell\Lambda_{j-2}} \otimes b_{\mu_{s-1}} \otimes b_\nu \,\middle|\, b_\nu \in \mathcal{B}_\ell \right\},
\end{aligned}$$

where $b_{\mu_{s-1}} = \left(m_1, \ldots, m_{j+s-2}, 0, \ldots, 0, m_{j-1}, m_j, \ldots, m_n, m_0\right)$ and $s = 2, 3, \ldots, n$.

Now suppose the claim holds for $L-1$. Mainly, suppose that $B_{w(L-1,j)}(\ell\Lambda_j) = u_{\ell\Lambda_{j+1-L}} \otimes \mathcal{B}_\ell^{L-1}$. We want to prove the the L case. The sequence of Weyl group elements

$$\begin{aligned}
w(L,j) &= \left(r_{j+n-L}\cdots r_{j+2-L} r_{j+1-L}\right)\cdots\left(r_{j+n-2}\cdots r_j r_{j+n}\right)\left(r_{j+n-1}\cdots r_{j+1} r_j\right)\\
&= \left(r_{j+n-L}\cdots r_{j+2-L} r_{j+1-L}\right) w(L-1,j).
\end{aligned}$$

By Lemma 4.2.2, we know that $B_{w(L,j)}(\ell\Lambda_j) \subset u_{\ell\Lambda_{j-L}} \otimes \mathcal{B}_\ell^L$. We want to show that $u_{\ell\Lambda_{j-L}} \otimes \mathcal{B}_\ell^L \subset B_{w(L,j)}(\ell\Lambda_j)$. Note that by applying the Kashiwara operators corresponding to the Weyl group sequence $w(L-1,j)$, we obtain all paths in the set $u_{\ell\Lambda_{j+1-L}} \otimes \mathcal{B}_\ell^{L-1}$. Suppose we want to obtain all paths of the form

$$u_{\ell\Lambda_{j-L}} \otimes b_{\mu_1} \otimes \mathcal{B}_\ell^{L-1},$$

where $b_{\mu_1} = \left(0, \ldots, 0, m_{j+1-L}, m_{j+2-L}, 0, \ldots, 0\right) \in \mathcal{B}_\ell$. Note that the $(j+1-L)$-signature for paths of the form

$$u_{\ell\Lambda_{j+1-L}} \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes \cdots \otimes b_{\rho_{L-1}},$$

where $b_{\rho_i} = \left(p_{i,1}, p_{i,2}, \ldots, p_{i,n}, p_{i,0}\right) \in \mathcal{B}_\ell$, is of the form $(+_\ell, \ast_1, \ast_2, \ldots, \ast_{L-1})$. Problems arise when there are negatives in the rightmost $(L-1)$ components of this signature that cancel too many pluses in the $(L)^{th}$ component from the right, leaving fewer than m_{j+2-L} pluses.

Suppose a_1 is the smallest positive integer such that

$$\sum_{i=1}^{a_1} p_{i,j+2-L} > \sum_{i=1}^{a_1} p_{i-1,j+1-L},$$

where we define $p_{0,j+1-L} = 0$. In other words, $b_{\rho_{a_1}}$ is the first b_{ρ_t} component of the tensor product from the left in which the total number of negatives in the $(j+1-L)$-signature is more than the total number of positives. Let

$$c_1 = \sum_{i=1}^{a_1} p_{i,j+2-L} - \sum_{i=1}^{a_1} p_{i-1,j+1-L}$$

represent how many more negatives there are than positives in these components of the $(j+1-L)$-signature. If $\ell - c_1 < m_{j+2-L}$, let $k_1 = a_1$. However, if $\ell - c_1 \geq m_{j+2-L}$, then let a_2 be the smallest positive integer such that

$$\sum_{i=a_1+1}^{a_2} p_{i,j+2-L} > \sum_{i=a_1+1}^{a_2} p_{i-1,j+1-L},$$

and define

$$c_2 = \sum_{i=a_1+1}^{a_2} p_{i,j+2-L} - \sum_{i=a_1+1}^{a_2} p_{i-1,j+1-L}.$$

If $\ell - c_1 - c_2 < m_{j+2-L}$, then let $k_1 = a_2$. Otherwise, continue this process until we have a_t, the smallest positive integer such that

$$\sum_{i=a_{t-1}+1}^{a_t} p_{i,j+2-L} > \sum_{i=a_{t-1}+1}^{a_t} p_{i-1,j+1-L},$$

and define

$$c_t = \sum_{i=a_{t-1}+1}^{a_t} p_{i,j+2-L} - \sum_{i=a_{t-1}+1}^{a_t} p_{i-1,j+1-L},$$

where $\ell - \sum_{s=1}^{t} c_s < m_{j+2-L}$. If no such a_t exists, then there are no problem spots. Let $k_1 = a_t$.

Case 4.2.1. Suppose that

$$\sum_{i=k_1+1}^{z} p_{i,j+2-L} \leq \sum_{i=k_1+1}^{z} p_{i-1,j+1-L} \tag{4.2}$$

for all $z = k_1+1, \ldots, L-1$. Let $d_1 = \left(\sum_{s=1}^{t} c_s\right) - \left(\ell - m_{j+2-L}\right)$. Define

$$b_{\rho^*_{k_1}} = \left(p_{k_1,1}, p_{k_1,2}, \ldots, p_{k_1,j-L}, p_{k_1,j+1-L} + d_1, p_{k_1,j+2-L} - d_1, p_{k_1,j+3-L}, \ldots, p_{k_1,n}, p_{k_1,0}\right) \in \mathcal{B}_\ell.$$

The path

$$u_{\ell\Lambda_{j-L+1}} \otimes b_{\rho_1} \otimes \cdots b_{\rho^*_{k_1}} \otimes b_{\rho_{k_1+1}} \otimes \cdots \otimes b_{\rho_{L-1}}$$

has $(j+1-L)$-signature

$$\left(+_{\ell}, \left(-_{c_1+c_2+\cdots+c_{t-1}+c_t-d_1}+_{p_{k_1,j+1-L}+d_1}\right), (*)\right),$$

which reduces to

$$\left(+_{\ell}, \left(-_{\ell-m_{j+2-L}}+_{p_{k_1,j+1-L}+d_1}\right), (*)\right) = \left(+_{m_{j+2-L}}, (+_{p_{k_1,j+1-L}+d_1}), (*)\right).$$

By applying $\tilde{f}_{j+1-L}$ to these paths $(m_{j+2-L}+d_1)$ times, we produce the paths

$$u_{\ell\Lambda_{j-L}} \otimes \left(0, \ldots, 0, m_{j+1-L}, m_{j+2-L}, 0, \ldots, 0\right) \otimes b_{\rho_1} \otimes \cdots \otimes b_{\rho_{L-1}},$$

which were the desired paths.

Case 4.2.2. Now suppose that Equation 4.2 were not true. Then, suppose k_2 is the smallest positive integer such that

$$\sum_{i=k_1+1}^{k_2} p_{i,j+2-L} > \sum_{i=k_1+1}^{k_2} p_{i-1,j+1-L}.$$

If

$$\sum_{i=k_2+1}^{z} p_{i,j+2-L} \leq \sum_{i=k_2+1}^{z} p_{i-1,j+1-L}$$

for all $z = k_1+1, \ldots, L-1$, then we define

$$d_2 = \sum_{i=k_1+1}^{k_2} p_{i,j+2-L} - \sum_{i=k_1+1}^{k_2-1} p_{i-1,j+1-L} \in \mathbb{Z}_{\geq 0}.$$

Also, define the element

$$b_{\rho^*_{k_2}} = \left(p_{k_2,1}, p_{k_2,2}, \ldots, p_{k_2,j-L}, p_{k_2,j+1-L}+d_2, p_{k_2,j+2-L}-d_2, p_{k_2,j+3-L}, \ldots, p_{k_2,n}, p_{k_2,0}\right) \in \mathcal{B}_{\ell}.$$

Since the path

$$u_{\ell\Lambda_{j-L+1}} \otimes b_{\rho_1} \otimes \cdots \otimes b_{\rho^*_{k_1}} \otimes b_{\rho_{k_1+1}} \otimes \cdots \otimes b_{\rho^*_{k_2}} \otimes b_{\rho_{k_2+1}} \otimes \cdots \otimes b_{\rho_{L-1}}$$

has $(j-L+1)$-signature

$$\left(+_{m_{j+2-L}}, (+_{p_{k_1,j+1-L}+d_1}), (-_{p_{k_2,j+1-L}}, \bullet, +_{p_{k_2,j+2-L}-d_2}), (-_{p_{k_2,j+2-L}-d_2}+_{p_{k_2,j+1-L}+d_2}), (*)\right)$$

$$= \left(+_{m_{j+2-L}}, (+_{d_1}), (+_{p_{k_2,j+1-L}+d_2}), (*)\right),$$

we can apply $\tilde{f}_{j+1-L}$ to this path $(m_{j+2-L} + d_1 + d_2)$ times to produce the desired path

$$u_{\ell\Lambda_{j-L}} \otimes b_{\mu_1} \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes \cdots \otimes b_{\rho_{L-1}}.$$

Note that the $-p_{k_2,j+1-L}$ is part of the signature because $p_{k_2,j+1-L}$ is included in the definition of d_2; so, we know those pluses must cancel with minuses. Also, note that with the original paths, there should be an extra d_2 negatives; hence the $+_{p_{k_2,j+2-L}-d_2}$ would have canceled with most of $-_{p_{k_2,j+2-L}}$, leaving behind d_2 negatives.

If k_2 is not the last time there are more negatives than positives, then we continue with this process. Suppose we have a solution for the k_p case, i.e. let

$$\begin{aligned}
b_{\rho^*_{k_s}} &= \left(p_{k_s,1}, \ldots, p_{k_s,j-L}, p_{k_s,j+1-L} + d_s, p_{k_s,j+2-L} - d_s, p_{k_s,j+3-L}, \ldots, p_{k_s,n}, p_{k_s,0}\right),\\
d_1 &= \sum_{s=1}^{t} c_s - \left(\ell - m_{j+2-L}\right),\\
d_s &= \sum_{i=k_{s-1}+1}^{k_s} p_{i,j+2-L} - \sum_{i=k_{s-1}+1}^{k_s-1} p_{i-1,j+1-L} \in \mathbb{Z}_{\geq 0},
\end{aligned}$$

for $s = 1, 2, \ldots, p$. Then applying $\tilde{f}_{j+1-L}$ the correct number of times to the path $u_{\ell\Lambda_{j+1-L}} \otimes b'_{\rho_1} \otimes \cdots \otimes b_{\rho'_{L-1}}$, where

$$b'_{\rho_t} = \begin{cases} b_{\rho^*_t}, & \text{if } t = k_s,\ s = 1, 2, \ldots, p,\\ b_{\rho_t}, & \text{else,}\end{cases}$$

produces the desired path $u_{\ell\Lambda_{j-L}} \otimes b_{\mu_1} \otimes b_{\rho_1} \otimes \ldots \otimes b_{\rho_{L-1}}$.

We want to show this is true for the k_{p+1} case. Define d_{p+1} in the same way. Then the path $u_{\ell\Lambda_{j+1-L}} \otimes b'_{\rho_1} \otimes \cdots \otimes b_{\rho'_{L-1}}$, where

$$b'_{\rho_t} = \begin{cases} b_{\rho^*_t}, & \text{if } t = k_s,\ s = 1, 2, \ldots, p+1,\\ b_{\rho_t}, & \text{else,}\end{cases}$$

has $(j+1-L)$-signature

$$\left(+_{m_{j+2-L}}, (+_{d_1+\cdots+d_{p-1}+p_{k_p,j+1-L}+d_p}), (-_{p_{k_p,j+1-L}}, +_{p_{k_{p+1},j+2-L}-d_{p+1}}), (-_{p_{k_{p+1},j+2-L}-d_{p+1}} +_{p_{k_{p+1},j+1-L}+d_{p+1}}), (*)\right)$$

$$= \left(+_{m_{j+2-L}}, (+_{d_1+\cdots+d_p}), (+_{p_{k_{p+1},j+1-L}+d_{p+1}}), (*)\right).$$

Hence, we can apply $\tilde{f}_{j+1-L}$ to this path $(m_{j+2-L} + d_1 + \cdots + d_{p+1})$ times to obtain the desired path $u_{\ell\Lambda_{j-L}} \otimes b_{\mu_1} \otimes b_{\rho_1} \otimes \cdots \otimes b_{\rho_{L-1}}$.

Now, suppose we have all paths of the form $u_{\ell\Lambda_{j-L}} \otimes b_{\mu_s} \otimes \mathcal{B}_\ell^{L-1}$, where

$$b_{\mu_s} = \left(m_1, \ldots, m_{j+1+s-L}, 0, \ldots, 0, m_{j+1-L}, \ldots, m_n, m_0\right) \in \mathcal{B}_\ell,$$

and that we constructed these paths using the Weyl reflections $\left(r_{j+s-L} \cdots r_{j+2-L} r_{j+1-L}\right) w(L-1, j)$. We want to construct all paths of the form

$$u_{\ell\Lambda_{j-L}} \otimes b_{\mu_{s+1}} \otimes \mathcal{B}_\ell^{L-1},$$

where $b_{\mu_{s+1}} = \left(m_1, \ldots, m_{j+s+2-L}, 0, \ldots, 0, m_{j+1-L}, \ldots, m_n, m_0\right) \in \mathcal{B}_\ell$.

The paths

$$u_{\ell\Lambda_{j-L}} \otimes b_{\mu_s} \otimes b_{\rho_1} \otimes \cdots \otimes b_{\rho_{L-1}},$$

where $b_{\rho_t} \in \mathcal{B}_\ell$, have $(j+s+1-L)$-signatures of the form $\left(\ldots, +_{m_{j+1+s-L}}, *_1, *_2, \ldots, *_{L-1}\right)$. So, again, problems arise when there are negatives in the last $(L-1)$ components of the signature that cancel with the $(m_{j+1+s-L})$ pluses in the L^{th} component from the right. Using the same approach as when forming the paths $u_{\ell\Lambda_{j-L}} \otimes b_{\mu_1} \otimes b_{\rho_1} \otimes \cdots \otimes b_{\rho_{L-1}}$, we can obtain the desired paths. So, if we define a_t, c_t, d_t, k_t, and $b_{\rho_t^*}$ as before, applying $\tilde{f}_{j+s+1-L}$ a certain number of times to the path

$$u_{\ell\Lambda_{j-L}} \otimes b_{\mu_s} \otimes b'_{\rho_1} \otimes \cdots \otimes b'_{\rho_{L-1}},$$

where

$$b'_{\rho_t} = \begin{cases} b_{\rho_t^*}, & \text{if } t = k_s,\ s = 1, 2, \ldots, p, \\ b_{\rho_t}, & \text{else,} \end{cases}$$

will produce the desired paths.

Now, suppose that $s = n$. In other words, using the Weyl reflections $(r_{j+n-L} \cdots r_{j+2-L} r_{j+1-L}) w(L-1, j)$, we constructed all paths of the form

$$u_{\ell\Lambda_{j-L}} \otimes b_{\mu_n} \otimes b_{\rho_1} \otimes \cdots \otimes b_{\rho_{L-1}},$$

where $b_{\mu_n} = (m_1, \ldots, m_n, m_0)$, $b_{\rho_t} \in \mathcal{B}_\ell$. Thus, by definition, we formed all paths in the set $u_{\ell\Lambda_{j-L}} \otimes \mathcal{B}_\ell^L$.

In conclusion, we have the following:

$$\begin{aligned}
B_{w(L,j)}(\ell\Lambda_j) &= u_{\ell\Lambda_{j-L}} \otimes \mathcal{B}_\ell^L, \\
B_{w(L,j+1)}(\ell\Lambda_j) &= B_{w(L-1,j)}(\ell\Lambda_j) \\
&= u_{\ell\Lambda_{j+1-L}} \otimes \mathcal{B}_\ell^{L-1}, \\
B_{w(L,j+s)}(\ell\Lambda_j) &= B_{(r_{j+s+n-L}\cdots r_{j+2-L}r_{j+1-L})w(L-1,j)}(\ell\Lambda_j) \\
&= \left\{ u_{\ell\Lambda_{j-L}} \otimes b_{\mu_{s-1}} \otimes b_{\rho_1} \otimes \ldots \otimes b_{\rho_{L-1}} \,\middle|\, b_{\rho_t} \in \mathcal{B}_\ell \right\},
\end{aligned}$$

for $s = 2, 3, \ldots, n$, which is what we wanted to show. □

Corollary 4.2.4. *We have the following:*

1. $\bigcup_{s\geq 0} B_{w(L,s)}(\ell\Lambda_j) = u_{\ell\Lambda_{j-L}} \otimes \mathcal{B}_\ell^L$.
2. $\bigcap_{s\geq 0} B_{w(L,s)}(\ell\Lambda_j) = u_{\ell\Lambda_{j+1-L}} \otimes \mathcal{B}_\ell^{L-1}$.

Proof. First, note that $w(L,s) = w(L, s+n+1)$. Hence,

$$\bigcup_{s\geq 0} B_{w(L,s)}(\ell\Lambda_j) = \bigcup_{s=0}^{n} B_{w(L,s)}(\ell\Lambda_j),$$
$$\bigcap_{s\geq 0} B_{w(L,s)}(\ell\Lambda_j) = \bigcap_{s=0}^{n} B_{w(L,s)}(\ell\Lambda_j).$$

By Theorem 4.2.3, we have that

$$u_{\ell\Lambda_{j+1-L}} \otimes \mathcal{B}_\ell^{L-1} = B_{w(L,j+1)}(\ell\Lambda_j) \subset B_{w(L,j+2)}(\ell\Lambda_j) \subset \cdots \subset B_{w(L,j)}(\ell\Lambda_j) = u_{\ell\Lambda_{j-L}} \otimes \mathcal{B}_\ell^L. \tag{4.3}$$

The statement of the corollary follows directly from Equation 4.3. □

Example 4.2.5. We explicitly describe the Demazure crystals for $U_q\left(A_2^{(1)}\right)$ for which $\lambda = 2\Lambda_2$. We have depicted the perfect crystal $\mathcal{B}_2$ in Figure 3.2. Notice that there are

$$\binom{n+\ell}{\ell} = \binom{4}{2} = 6$$

elements in B_2.

Let $L = 1$. Then we have the following:

$$w(1,0) = (r_1 r_0),$$
$$w(1,1) = (r_2 r_1),$$
$$w(1,2) = (r_0 r_2).$$

($B_{w(1,0)}(2\Lambda_2)$-case): The (0)-signature on $u_{2\Lambda_2}$ is $(\ldots, \bullet, +_2, -_2, \bullet) = (\bullet)$. Hence, we cannot apply $\tilde{f}_0$ to $u_{2\Lambda_2}$. The (1)-signature on $u_{2\Lambda_2}$ is $(\ldots, \bullet, +_2, -_2) = (\bullet)$. So, we cannot apply the $\tilde{f}_1$ action either. Thus, $B_{w(1,0)}(2\Lambda_2) = \{u_{2\Lambda_2}\}$.

($B_{w(1,1)}(2\Lambda_2)$-case): The $\tilde{f}_1$ operator produces no paths. The (2)-signature on $u_{2\Lambda_2}$ is $(\ldots, \bullet, +_2, -_2, \bullet, +_2) = (+_2)$. Hence, we can apply $\tilde{f}_2$ to $u_{2\Lambda_2}$ twice. We have

$$\tilde{f}_2(u_{2\Lambda_2}) = u_{2\Lambda_1} \otimes \tilde{f}_2(0,2,0) = u_{2\Lambda_1} \otimes (0,1,1),$$
$$\tilde{f}_2^2(u_{2\Lambda_2}) = u_{2\Lambda_1} \otimes \tilde{f}_2(0,1,1) = u_{2\Lambda_1} \otimes (0,0,2).$$

Thus $B_{w(1,1)}(2\Lambda_2) = \left\{u_{2\Lambda_2}, u_{2\Lambda_1} \otimes (0,1,1), u_{2\Lambda_1} \otimes (0,0,2)\right\}$.

($B_{w(1,2)}(2\Lambda_2)$-case): Applying $\tilde{f}_2$ produces the paths $u_{2\Lambda_1} \otimes (0,1,1)$ and $u_{2\Lambda_1} \otimes (0,0,2)$. Now we apply the $\tilde{f}_0$ operator to all paths obtained thusfar. Again, $\tilde{f}_0$ acting on $u_{2\Lambda_2}$ produces no new paths. However, the (0)-signature on $u_{2\Lambda_1} \otimes (0,1,1)$ is $(\ldots, \bullet, ++, --, +) = (+)$. So, we can apply $\tilde{f}_0$ once:

$$\tilde{f}_0(u_{2\Lambda_1} \otimes (0,1,1)) = u_{2\Lambda_1} \otimes \tilde{f}_0(0,0,1) = u_{2\Lambda_1} \otimes (1,1,0).$$

The (0)-signature on $u_{2\Lambda_1} \otimes (0,0,2)$ is $(\ldots, \bullet, ++, --, ++) = (+_2)$. Thus, we can apply $\tilde{f}_0$ twice:

$$\tilde{f}_0(u_{2\Lambda_1} \otimes (0,0,2)) = u_{2\Lambda_1} \otimes \tilde{f}_0(0,0,2) = u_{2\Lambda_1} \otimes (1,0,1),$$
$$\tilde{f}_0^2(u_{2\Lambda_1} \otimes (0,0,2)) = u_{2\Lambda_1} \otimes \tilde{f}_0(1,0,1) = u_{2\Lambda_1} \otimes (0,2,0).$$

Hence, $B_{w(1,2)}(\lambda) = u_{2\Lambda_1} \otimes \mathcal{B}_2$.

Now let $L = 2$. Then we have the following:

$$w(2,0) = (r_0 r_2)(r_1 r_0),$$
$$w(2,1) = (r_1 r_0)(r_2 r_1),$$

$$w(2,2) = (r_2 r_1)(r_0 r_2).$$

($B_{w(2,0)}(\lambda)$-case): Notice that $B_{w(2,0)}(2\Lambda_2) = B_{(r_0 r_2)w(1,0)}(2\Lambda_2) = B_{w(1,2)}(2\Lambda_2) = u_{2\Lambda_1} \otimes \mathcal{B}_\ell$.

($B_{w(2,1)}(\lambda)$-case): Note that $B_{w(2,1)}(2\Lambda_2) = B_{(r_1 r_0)(r_2)}(2\Lambda_2) = B_{r_1 w(1,2)}(2\Lambda_2)$. So, we need only apply $\tilde{f}_1$ to the paths $u_{2\Lambda_1} \otimes \mathcal{B}_\ell$. Let $b_\mu \in \mathcal{B}_\ell$. Then the (1)-signature on $u_{2\Lambda_1} \otimes b_\mu$ is $(\ldots, \bullet, +_2, -_2, \bullet, +_2, \ast) = (+_2, \ast)$. We have the following cases:

1. If $b_\mu = (0,2,0)$, then $\ast = (-_2)$. Hence, we produce no new paths.

2. If $b_\mu = (0,1,1)$, then $\ast = (-)$, and we can produce one new path:

$$\tilde{f}_1(u_{2\Lambda_1} \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_1(2,0,0) \otimes (0,1,1) = u_{2\Lambda_0} \otimes (1,1,0) \otimes (0,1,1).$$

3. If $b_\mu = (0,0,2)$, then $\ast = (\bullet)$, and we can produce two new paths:

$$\tilde{f}_1(u_{2\Lambda_1} \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_1(2,0,0) \otimes (0,0,2) = u_{2\Lambda_0} \otimes (1,1,0) \otimes (0,0,2),$$
$$\tilde{f}_1^2(u_{2\Lambda_1} \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_1(1,1,0) \otimes (0,0,2) = u_{2\Lambda_0} \otimes (0,2,0) \otimes (0,0,2).$$

4. If $b_\mu = (1,1,0)$, then $\ast = (-+)$ and we can apply $\tilde{f}_1$ twice:

$$\tilde{f}_1(u_{2\Lambda_1} \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_1(2,0,0) \otimes (1,1,0) = u_{2\Lambda_0} \otimes (1,1,0) \otimes b(1,1,0),$$
$$\tilde{f}_1^2(u_{2\Lambda_1} \otimes b_\mu) = u_{2\Lambda_0} \otimes (1,1,0) \otimes \tilde{f}_1(1,1,0) = u_{2\Lambda_0} \otimes (1,1,0) \otimes (0,2,0).$$

5. If $b_\mu = (1,0,1)$, then $\ast = (+)$, and so we can apply $\tilde{f}_1$ three times:

$$\tilde{f}_1(u_{2\Lambda_1} \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_1(2,0,0) \otimes (1,0,1) = u_{2\Lambda_0} \otimes (1,1,0) \otimes (1,0,1),$$
$$\tilde{f}_1^2(u_{2\Lambda_1} \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_1(1,1,0) \otimes (1,0,1) = u_{2\Lambda_0} \otimes (0,2,0) \otimes (1,0,1),$$
$$\tilde{f}_1^3(u_{2\Lambda_1} \otimes b_\mu) = u_{2\Lambda_0} \otimes (0,2,0) \otimes \tilde{f}_1(1,0,1) = u_{2\Lambda_0} \otimes (0,2,0) \otimes (0,1,1).$$

6. If $b_\mu = (2,0,0)$, then $\ast = (+_2)$, and so we can produce four new paths:

$$\tilde{f}_1(u_{2\Lambda_1} \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_1(2,0,0) \otimes (2,0,0) = u_{2\Lambda_0} \otimes (1,1,0) \otimes (2,0,0),$$
$$\tilde{f}_1^2(u_{2\Lambda_1} \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_1(1,1,0) \otimes (2,0,0) = u_{2\Lambda_0} \otimes (0,2,0) \otimes (2,0,0),$$
$$\tilde{f}_1^3(u_{2\Lambda_1} \otimes b_\mu) = u_{2\Lambda_0} \otimes (0,2,0) \otimes \tilde{f}_1(2,0,0) = u_{2\Lambda_0} \otimes (0,2,0) \otimes (1,1,0),$$

$$\tilde{f}_1^4(u_{2\Lambda_1} \otimes b_\mu) = u_{2\Lambda_0} \otimes (0,2,0) \otimes \tilde{f}_1(1,1,0) = u_{2\Lambda_0} \otimes (0,2,0) \otimes (0,2,0).$$

Hence, $B_{w(2,1)}(2\Lambda_2) = \left\{u_{2\Lambda_1} \otimes \mathcal{B}_\ell,\, u_{2\Lambda_0} \otimes (1,1,0) \otimes \mathcal{B}_\ell,\, u_{2\Lambda_0} \otimes (0,2,0) \otimes \mathcal{B}_\ell\right\}$.

($B_{w(2,2)}(2\Lambda_2)$-case): Notice that $B_{w(2,2)}(2\Lambda_2) = B_{(r_2)(r_1 w(1,2))}(2\Lambda_2)$. So, we need only apply $\tilde{f}_2$ to all paths in $B_{w(2,1)}(2\Lambda_2)$. Let $b_\mu \in \mathcal{B}_\ell$. Then the (2)-signature on $u_{2\Lambda_1} \otimes b_\mu$ is $(\ldots, \bullet, +_2, -_2, \bullet, \ast) = (\ast)$. Hence, we will not produce any new paths. The (2)-signature on $u_{2\Lambda_0} \otimes (1,1,0) \otimes b_\mu$ is $(\ldots, \bullet, +_2, -_2, +, \ast) = (+, \ast)$. We have the following cases:

1. If $b_\mu = (0,0,2)$, $(1,0,1)$, or $(0,1,1)$, then $\ast$ has at least one negative, resulting in the reduced (2)-signature $(\ast)$. Thus, no new paths are produced.

2. If $b_\mu = (1,1,0)$, then $\ast = (+)$, and so, two new paths are produced:

$$\begin{aligned}
\tilde{f}_2(u_{2\Lambda_0} \otimes (1,1,0) \otimes b_\mu) &= u_{2\Lambda_0} \otimes \tilde{f}_2(1,1,0) \otimes (1,1,0) = u_{2\Lambda_0} \otimes (1,0,1) \otimes (1,1,0), \\
\tilde{f}_2^2(u_{2\Lambda_0} \otimes (1,1,0) \otimes b_\mu) &= u_{2\Lambda_0} \otimes (1,0,1) \otimes \tilde{f}_2(1,1,0) = u_{2\Lambda_0} \otimes (1,0,1) \otimes (1,0,1).
\end{aligned}$$

3. If $b_\mu = (0,2,0)$, then $\ast = (+_2)$. Thus, three new paths are produced:

$$\begin{aligned}
\tilde{f}_2(u_{2\Lambda_0} \otimes (1,1,0) \otimes b_\mu) &= u_{2\Lambda_0} \otimes \tilde{f}_2(1,1,0) \otimes (0,2,0) = u_{2\Lambda_0} \otimes (1,0,1) \otimes (0,2,0), \\
\tilde{f}_2^2(u_{2\Lambda_0} \otimes (1,1,0) \otimes b_\mu) &= u_{2\Lambda_0} \otimes (1,0,1) \otimes \tilde{f}_2(0,2,0) = u_{2\Lambda_0} \otimes (1,0,1) \otimes (0,1,1), \\
\tilde{f}_2^3(u_{2\Lambda_0} \otimes (1,1,0) \otimes b_\mu) &= u_{2\Lambda_0} \otimes (1,0,1) \otimes \tilde{f}_2(0,1,1) = u_{2\Lambda_0} \otimes (1,0,1) \otimes (0,0,2).
\end{aligned}$$

4. If $b_\mu = (2,0,0)$, then $\ast = (\bullet)$. So, one new path is produced:

$$\tilde{f}_2(u_{2\Lambda_0} \otimes (1,1,0) \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_2(1,1,0) \otimes (2,0,0) = u_{2\Lambda_0} \otimes (1,0,1) \otimes (2,0,0)$$

Finally, the (2)-signature on $u_{2\Lambda_0} \otimes (0,2,0) \otimes b_\mu$ is $(\ldots, \bullet, +_2, -_2, +_2, \ast)$. We have the following cases:

1. If $b_\mu = (0,0,2)$, then $\ast = (-_2)$. So, no new paths are produced.

2. If $\mu = (0,1,1)$, then $\ast = (-+)$. Thus, we can apply $\tilde{f}_2$ twice:

$$\begin{aligned}
\tilde{f}_2(u_{2\Lambda_0} \otimes (0,2,0) \otimes b_\mu) &= u_{2\Lambda_0} \otimes \tilde{f}_2(0,2,0) \otimes (0,1,1) = u_{2\Lambda_0} \otimes (0,1,1) \otimes (0,1,1), \\
\tilde{f}_2^2(u_{2\Lambda_0} \otimes b_{2\Lambda_2} \otimes b_\mu) &= u_{2\Lambda_0} \otimes (0,1,1) \otimes \tilde{f}_2(0,1,1) = u_{2\Lambda_0} \otimes (0,1,1) \otimes (0,0,2).
\end{aligned}$$

3. If $b_\mu = (1,0,1)$, then $\ast = (-)$, and so one new path can be produced:

$$\tilde{f}_2(u_{2\Lambda_0} \otimes (0,2,0) \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_2(0,2,0) \otimes (1,0,1) = u_{2\Lambda_0} \otimes (0,1,1) \otimes (1,0,1).$$

4. If $b_\mu = (2,0,0)$, then $\ast = (\bullet)$. Thus two new paths can be produced:

$$\tilde{f}_2(u_{2\Lambda_0} \otimes (0,2,0) \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_2(0,2,0) \otimes (2,0,0) = u_{2\Lambda_0} \otimes (0,1,1) \otimes (2,0,0),$$
$$\tilde{f}_2^2(u_{2\Lambda_0} \otimes (0,2,0) \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_2(0,1,1) \otimes (2,0,0) = u_{2\Lambda_0} \otimes (0,0,2) \otimes (2,0,0).$$

5. If $b_\mu = (1,1,0)$, then $\ast = (+)$. So, we can apply $\tilde{f}_2$ three times:

$$\tilde{f}_2(u_{2\Lambda_0} \otimes (0,2,0) \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_2(0,2,0) \otimes (1,1,0) = u_{2\Lambda_0} \otimes (0,1,1) \otimes (1,1,0),$$
$$\tilde{f}_2^2(u_{2\Lambda_0} \otimes (0,2,0) \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_2(0,1,1) \otimes (1,1,0) = u_{2\Lambda_0} \otimes (0,0,2) \otimes (1,1,0),$$
$$\tilde{f}_2^3(u_{2\Lambda_0} \otimes (0,2,0) \otimes b_\mu) = u_{2\Lambda_0} \otimes (0,0,2) \otimes \tilde{f}_2(1,1,0) = u_{2\Lambda_0} \otimes (0,0,2) \otimes (1,0,1).$$

6. If $b_\mu = (0,2,0)$, then $\ast = (+_2)$. So, we can produce four new paths:

$$\tilde{f}_2(u_{2\Lambda_0} \otimes (0,2,0) \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_2(0,2,0) \otimes (0,2,0) = u_{2\Lambda_0} \otimes (0,1,1) \otimes (0,2,0),$$
$$\tilde{f}_2^2(u_{2\Lambda_0} \otimes (0,2,0) \otimes b_\mu) = u_{2\Lambda_0} \otimes \tilde{f}_2(0,1,1) \otimes (0,2,0) = u_{2\Lambda_0} \otimes (0,0,2) \otimes (0,2,0),$$
$$\tilde{f}_2^3(u_{2\Lambda_0} \otimes (0,2,0) \otimes b_\mu) = u_{2\Lambda_0} \otimes (0,0,2) \otimes \tilde{f}_2(0,2,0) = u_{2\Lambda_0} \otimes (0,0,2) \otimes (0,1,1),$$
$$\tilde{f}_2^4(u_{2\Lambda_0} \otimes (0,2,0) \otimes b_\mu) = u_{2\Lambda_0} \otimes (0,0,2) \otimes \tilde{f}_2(0,1,1) = u_{2\Lambda_0} \otimes (0,0,2) \otimes (0,0,2).$$

Hence, $B_{w(2,2)}(\lambda) = u_{2\Lambda_0} \otimes \mathcal{B}_\ell^2$.

Chapter

5

$U_q\left(\hat{sl}(3,\mathbb{C})\right)$-Demazure Crystals

In this chapter, we give explicit descriptions corresponding to Demazure crystals $B_w(\lambda)$ of $U_q(A_2^{(1)})$, for $\lambda \in \bar{P}^+$.

We first give an identity for $|\mathcal{B}_\ell|$:

Lemma 5.1. *We have the following identity:*

$$\sum_{j=0}^{\ell}(j+1)\binom{n-2+\ell-j}{\ell-j}=\binom{n+\ell}{\ell}.$$

Proof. Let $n=2$. Then

$$\begin{aligned}\sum_{j=0}^{\ell}(j+1)\binom{n-2+\ell-j}{\ell-j} &= \sum_{j=0}^{\ell}(j+1)\binom{\ell-j}{\ell-j}\\ &= \sum_{j=0}^{\ell}(j+1)\\ &= \left(\frac{1}{2}\ell(\ell+1)\right)+(\ell-0+1)\end{aligned}$$

$$
\begin{aligned}
&= (\ell+1)\left(\frac{1}{2}\ell+1\right) \\
&= \frac{(\ell+1)(\ell+2)}{2} \\
&= \binom{2+\ell}{\ell}.
\end{aligned}
$$

Now let $n = 3$. Then

$$
\begin{aligned}
\sum_{j=0}^{\ell}\left(j+1\right)\binom{n-2+\ell-j}{\ell-j} &= \sum_{j=0}^{\ell}\left(j+1\right)\binom{1+\ell-j}{\ell-j} \\
&= \sum_{j=0}^{\ell}\left(j+1\right)(1+\ell-j) \\
&= \sum_{j=0}^{\ell} j\ell - j^2 + \ell + 1 \\
&= (\ell+1)(\ell+1) + \frac{1}{2}\ell^2(\ell+1) - \frac{1}{6}\ell(\ell+1)(2\ell+1) \\
&= (\ell+1)\left(\ell+1+\frac{1}{2}\ell^2 - \frac{1}{6}(2\ell^2) - \frac{1}{6}\ell\right) \\
&= (\ell+1)\left(\frac{1}{6}\ell^2 + \frac{5}{6}\ell + 1\right) \\
&= \frac{1}{6}(\ell+1)(\ell+2)(\ell+3) \\
&= \binom{3+\ell}{\ell}.
\end{aligned}
$$

Now, suppose we have the following identity for n:

$$
\sum_{j=0}^{\ell}\left(j+1\right)\binom{n-2+\ell-j}{\ell-j} = \binom{n+\ell}{\ell}.
$$

We want to show this is true for $n+1$, i.e. that

$$
\sum_{j=0}^{\ell}\left(j+1\right)\binom{(n+1)-2+\ell-j}{\ell-j} = \binom{(n+1)+\ell}{\ell}.
$$

We will use the following recursive identity:

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

We have:

$$\begin{aligned}
\sum_{j=0}^{\ell}(j+1)\binom{(n+1)-2+\ell-j}{\ell-j} &= \sum_{j=0}^{\ell}(j+1)\left(\binom{n-2+\ell-j}{\ell-j} + \binom{n-2+\ell-j}{\ell-j-1}\right) \\
&= \sum_{j=0}^{\ell}(j+1)\binom{n-2+\ell-j}{\ell-j} + \sum_{j=0}^{\ell}(j+1)\binom{n-2+\ell-j}{\ell-j-1} \\
&= \binom{n+\ell}{\ell} + \sum_{j=0}^{\ell}(j+1)\binom{n-2+\ell-j}{\ell-j-1},
\end{aligned}$$

where the last step uses the inductive hypothesis. If we look at the second term, we have:

$$\begin{aligned}
\sum_{j=0}^{\ell}(j+1)\binom{n-2+\ell-j}{\ell-j-1} &= \sum_{j=0}^{\ell-1}(j+1)\binom{n-2+\ell-j}{\ell-j-1} \\
&= \sum_{j=0}^{\ell-1}(j+1)\binom{(n+1)-2+(\ell-1)-j}{(\ell-1)-j} \\
&= \binom{(n+1)+(\ell-1)}{(\ell-1)} \\
&= \binom{n+\ell}{\ell-1},
\end{aligned}$$

again using the inductive hypothesis. Hence, we have that

$$\sum_{j=0}^{\ell}(j+1)\binom{(n+1)-2+\ell-j}{k-j} = \binom{n+\ell}{\ell} + \binom{n+\ell}{\ell-1} = \binom{(n+1)+\ell}{\ell}.$$

□

In general, if $\lambda \in \bar{P}^+$ such that $\lambda(c) = \ell$, where c is the central element, we can write $\lambda = \sum_{i=0}^{n} m_i \Lambda_i$, where $\sum_{i=0}^{n} m_i = \ell$ and $m_i \geq 0$ for all i. Recall that $m_0 = m_{n+1}$ and that $b_\lambda = (m_1, m_2, \ldots, m_n, m_0)$, since

$$\varphi(b_\lambda) = \varphi(m_1, m_2, ..., m_n, m_0)$$

$$
\begin{aligned}
&= \sum_{i=0}^{n} \varphi_i(m_1, m_2, \ldots, m_n, m_0)\Lambda_i \\
&= \sum_{i=0}^{n} m_i \Lambda_i \\
&= \lambda.
\end{aligned}
$$

Hence, $b(\mu) = b_\mu \in \mathcal{B}_\ell$ for all $\mu \in \bar{P}^+$. Thus, $u_\lambda = \ldots \otimes b_{\sigma^2(\lambda)} \otimes b_{\sigma(\lambda)} \otimes b_\lambda$, where $\sigma(\lambda) = \varepsilon(b_\lambda)$. Note, for example, that $b_{\sigma(\lambda)} = (m_2, m_3, \ldots, m_n, m_0, m_1)$. This leads to the following lemma.

Lemma 5.2. *Given $b_\lambda = (m_1, m_2, \ldots, m_n, m_0)$, we have*

$$
b_{\sigma^L(\lambda)} = (m_{L+1}, m_{L+2}, \ldots, m_{L+n}, m_L),
$$

where $m_p = m_{p \bmod (n+1)}$.

Proof. We have $b_\lambda = (m_1, m_2, \ldots, m_n, m_0)$. Let $L = 1$. Then

$$
\begin{aligned}
\sigma(\lambda) &= \varepsilon(b_\lambda) \\
&= \varepsilon(m_1, m_2, \ldots, m_n, m_0) \\
&= \sum_{i=0}^{n} \varepsilon_i(m_1, m_2, \ldots, m_n, m_0)\Lambda_i \\
&= \sum_{i=0}^{n} m_{i+1}\Lambda_i,
\end{aligned}
$$

where $m_{n+1} = m_0$. Hence, $b(\sigma(\lambda)) = b_{\sigma(\lambda)} = (m_2, m_3, \ldots, m_n, m_0, m_1)$. Suppose that $b(\sigma^{L-1}(\lambda)) = b_{\sigma^{L-1}(\lambda)} = (m_L, m_{L+1}, \ldots, m_{L+n})$. Then

$$
\begin{aligned}
\sigma^L(\lambda) &= \sigma(\sigma^{L-1}(\lambda)) \\
&= \varepsilon\left(b_{\sigma^{L-1}(\lambda)}\right) \\
&= \varepsilon(m_L, m_{L+1}, \ldots, m_{L+n}) \\
&= \sum_{i=0}^{n} \varepsilon_i(m_L, m_{L+1}, \ldots, m_{L+n})\Lambda_i \\
&= \sum_{i=0}^{n} m_{i+L}\Lambda_i.
\end{aligned}
$$

Thus, $b_{\sigma^L(\lambda)} = (m_{L+1}, m_{L+2}, \ldots, m_{L+n}, m_L)$. $\square$

The following statement is an immediate consequence of Lemma 5.2.

Corollary 5.3. *For any $\lambda \in \bar{P}^+$, we have $u_{\sigma^L(\lambda)} = u_{\sigma^{L+1}(\lambda)} \otimes b_{\sigma^L(\lambda)}$.*

Corollary 5.4. *Given the set of λ-paths $u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L$, where $\lambda \in \bar{P}^+$, we have*

$$u_{\sigma^L(\lambda)} \otimes B_\ell^L = u_{\sigma^{L+1}(\lambda)} \otimes (m_{L+1}, m_{L+2}, \ldots, m_{L+n}, m_L) \otimes \mathcal{B}_\ell^L.$$

Proof. By Corollary 5.3, we know that $u_{\sigma^L(\lambda)} = u_{\sigma^{L+1}(\lambda)} \otimes (m_{L+1}, m_{L+2}, \ldots, m_{L+n}, m_L)$. Hence, we have $u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L = u_{\sigma^{L+1}(\lambda)} \otimes (m_{L+1}, m_{L+2}, \ldots, m_{L+n}, m_L) \otimes \mathcal{B}_\ell^L$. □

Now let $n = 2$. Then we have $u_\lambda = \ldots \otimes b_\lambda \otimes b_{\sigma^2(\lambda)} \otimes b_{\sigma(\lambda)} \otimes b_\lambda$, where

$$\begin{aligned}
b_\lambda &= (m_1, m_2, m_0),\\
b_{\sigma(\lambda)} &= b(\varepsilon(b_\lambda)) = b(m_2\Lambda_1 + m_0\Lambda_2 + m_1\Lambda_0) = (m_2, m_0, m_1),\\
b_{\sigma^2(\lambda)} &= b(\varepsilon(b_{\sigma(\lambda)})) = b(m_0\Lambda_1 + m_1\Lambda_2 + m_2\Lambda_0) = (m_0, m_1, m_2),\\
b_{\sigma^3(\lambda)} &= b(\varepsilon(b_{\sigma^2(\lambda)})) = b(m_1\Lambda_1 + m_2\Lambda_2 + m_0\Lambda_0) = (m_1, m_2, m_0) = b_\lambda.
\end{aligned}$$

Hence, we have $u_\lambda = \ldots \otimes (m_1, m_2, m_0) \otimes (m_0, m_1, m_2) \otimes (m_2, m_0, m_1) \otimes (m_1, m_2, m_0)$. This implies that

$$\begin{aligned}
u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L = \ldots &\otimes (m_{L+1}, m_{L+2}, m_L) \otimes (m_L, m_{L+1}, m_{L+2})\\
&\otimes (m_{L+2}, m_L, m_{L+1}) \otimes (m_{L+1}, m_{L+2}, m_L) \otimes \mathcal{B}_\ell^L.
\end{aligned}$$

Recall that the sequence $w(L, j)$ of simple reflections is defined by

$$w(L,j) = \left(r_{j+n-L} \cdots r_{j+2-L}\, r_{j+1-L}\right) \cdots \left(r_{j+n-2} \cdots r_j\, r_{j+n}\right)\left(r_{j+n-1} \cdots r_{j+1}\, r_j\right), \tag{5.1}$$

and so for $n = 2$, we have $w(L, j) = (r_{j+2-L}, r_{j+1-L}) \cdots (r_j\, r_{j+2})(r_{j+1}\, r_j)$.

Also, in general, we have the ground state path for $\lambda = m_0\Lambda_0 + m_1\Lambda_1 + \cdots + m_n\Lambda_n$:

$$u_\lambda = \ldots \otimes (m_3, m_4, \ldots, m_1, m_2) \otimes (m_2, m_3, \ldots, m_0, m_1) \otimes (m_1, m_2, \ldots, m_n, m_0), \tag{5.2}$$

which has (i)-signature

$$\left(\ldots, -_{m_{i+3}} +_{m_{i+2}}, -_{m_{i+2}} +_{m_{i+1}}, -_{m_{i+1}} +_{m_i}\right) = \left(+_{m_i}\right). \tag{5.3}$$

The (i)-signature for the λ-path $u_{\sigma(\lambda)} \otimes b_\nu$ where $b_\nu = (v_1, v_2, \ldots, v_n, v_0)$ is

$$\left(\ldots, -_{m_{i+3}} +_{m_{i+2}}, -_{m_{i+2}} +_{m_{i+1}}, -_{v_{i+1}} +_{v_i}\right) = \left(+_{m_{i+1}}, -_{v_{i+1}} +_{m_i}\right), \tag{5.4}$$

and the set of λ-paths $u_{\sigma^{L-1}(\lambda)} \otimes \mathcal{B}_\ell^{L-1} = u_{\sigma^L(\lambda)} \otimes (m_L, m_{L+1}, \ldots, m_{L+n}) \otimes \mathcal{B}_\ell^{L-1}$ has (i)-signature of the form

$$\left(\ldots, -_{m_{i+L+1}} +_{m_{i+L}}, -_{m_{i+L}} +_{m_{i+L-1}}, *_1, *_2, \ldots, *_{L-1}\right) = \left(+_{m_{i+L-1}}, *_1, *_2, \ldots, *_{L-1}\right). \tag{5.5}$$

The set of λ-paths $u_{\sigma^L(\lambda)} \otimes b_\nu \otimes \mathcal{B}_\ell^{L-1}$ where $b_\nu = (v_1, v_2, \ldots, v_n, v_0)$ has (i)-signature

$$\left(+_{m_{i+L}}, -_{v_{i+L}} +_{v_{i+L-1}}, *_1, \ldots, *_{L-1}\right). \tag{5.6}$$

We use these equations in the proof of the following lemma.

Lemma 5.5. *For each $0 \le i \le n$, choose s sufficiently large so that the reduced (i)-signature of*

$$u_p \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s}$$

is $(-_g +_r)$, where the r pluses correspond to $b_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s}$. Starting with the set of λ-paths

$$u_p \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes \mathcal{B}_\ell^L,$$

where $\mathcal{B}_\ell^L = \{a_1 \otimes a_2 \otimes \ldots \otimes a_L \,|\, a_t \in \mathcal{B}_\ell,\ t = 1, 2, \ldots, L\}$, we can obtain

$$u_p \otimes b_{\rho_1, j} \otimes b_{\rho_2, j} \otimes \ldots \otimes b_{\rho_s, j} \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_L$$

for all $\{a_1 \otimes a_2 \otimes \ldots \otimes a_L\} \in \mathcal{B}_\ell^L$ and $1 \le j \le r$, by applying $\tilde{f}_i$ a sufficient number of times, where

$$b_{\rho_1, j} \otimes b_{\rho_2, j} \otimes \ldots \otimes b_{\rho_s, j} = \tilde{f}_i^{\,j}\left(b_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s}\right).$$

Proof. First, note that by Equation 5.6, such an s exists. We will prove this lemma by induction on L and by induction on r. Fix $i \in \{0, 1, \ldots, n\}$.

Case 5.1 (L=1)**.** Given $u_p \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes \mathcal{B}_\ell$ with (i)-signature of the form $\left((-_g +_r), *\right)$.

Subcase 5.1.1. If $r = 0$, then there's nothing to prove. So, let $r = 1$. Then we have the (i)-signature of the form $\left((-_g +), *\right)$.

If $* = +_\ell$, then the λ-path is $u_p \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes b_\mu$, where

$$b_\mu = (m_1, m_2, \ldots, m_i = \ell, m_{i+1}, \ldots, m_n, m_0),$$

with $m_t = 0$ for $t \neq i$ and reduced (i)-signature $\left((-_g+), +_\ell\right)$. Hence, we will obtain $\ell + 1$ new λ-paths of the form $u_p \otimes b_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\nu$, where $b_\nu = (v_1, v_2, \ldots, v_n, v_0)$ such that $v_t = 0$ for $t \neq i, i+1$ and $v_i + v_{i+1} = \ell$.

If $* = \bullet_1 +_{\ell-1}$, then the λ-path is $u_p \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes b_\mu$, where

$$b_\mu = (m_1, m_2, \ldots, m_i = \ell - 1, m_{i+1} = 0, \ldots, m_n, m_0),$$

with $\sum_{t \neq i, i+1} m_t = 1$ and reduced (i)-signature $\left((-_g+), +_{\ell-1}\right)$. Hence, we will obtain ℓ new λ-paths of the form $u_p \otimes b_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\nu$, for each $b_\nu = (v_1, v_2, \ldots, v_n, v_0)$ such that $\sum_{t \neq i, i+1} v_t = 1$ and $v_i + v_{i+1} = \ell - 1$. So, in total we have $\ell(n-1)$ new λ-paths of the above form, since the number of solutions to $\sum_{t \neq i, i+1} v_t = 1$ is equal to the number of ways to distribute $(\ell - 1)$ 1's among $(n-1)$ coordinates in an $(n+1)$-tuple, i.e. the "balls in bins" combinatorics problem. Hence, there are

$$\ell \binom{(n-1)+(1)-1}{1} = \ell(n-1)$$

new λ-paths.

In general, if $* = \bullet_{\ell-j} +_j$ for $j = 1, 2, \ldots, \ell$, then the λ-path is $u_p \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes b_\mu$, where $b_\mu = \left(m_1, m_2, \ldots, m_i = j, m_{i+1} = 0, \ldots, m_n, m_0\right)$, with $\sum_{t \neq i, i+1} m_t = \ell - j$ and reduced (i)-signature $\left((-_g+), +_j\right)$. Hence, we will obtain $j+1$ new λ-paths of the form $u_p \otimes b_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\nu$, for each $b_\nu = (v_1, v_2, \ldots, v_n, v_0)$ such that $\sum_{t \neq i, i+1} v_t = \ell - j$ and $v_i + v_{i+1} = j$. Again, this is the "balls in bins" problem, so there are

$$(j+1)\binom{(n-1)+(\ell-j)-1}{\ell-j}$$

new λ-paths for $j = 1, 2, \ldots, \ell$.

Now, suppose $* = \bullet_\ell$. Then the λ-path is $u_p \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes b_\mu$, where $b_\mu = (m_1, m_2, \ldots, m_n, m_0)$ such that $\sum_{t \neq i, i+1} m_t = \ell$ and $m_i = 0 = m_{i+1}$, with (i)-signature is $\left((-_g+), \bullet_\ell\right)$. Then there is one new λ-path of the form $u_p \otimes b_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\nu$, where $b_\nu = (v_1, v_2, \ldots, v_n, v_0)$ such that $v_i = 0 = v_{i+1}$

for each solution of $\sum_{t\neq i,i+1} v_t = \ell$. Again, by the same combinatorial problem, there are

$$\binom{(n-1)+(\ell)-1}{\ell}$$

solutions to this equation.

Note that if $*$ contains any negatives, then no new λ-paths are produced. So, by Lemma 5.1, altogether there are

$$\left(\sum_{j=1}^{\ell}(j+1)\binom{n-1+\ell-j-1}{\ell-j}\right)+\binom{n-1+\ell-1}{\ell}=\sum_{j=0}^{\ell}(j+1)\binom{n-2+\ell-j}{\ell-j}=\binom{n+\ell}{\ell}$$

unique elements $b_\mu \in \mathcal{B}_\ell$ in the λ-paths of the form $u_p \otimes b_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\mu$, i.e. we've obtained the set of λ-paths $u_p \otimes b_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes \mathcal{B}_\ell$.

Subcase 5.1.2. Now, suppose that if we have the set of λ-paths $u_p \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes \mathcal{B}_\ell$ with (i)-signature of the form $\left((-_g+_r),*\right)$, we can apply $\tilde{f}_i$ a sufficient number of times and produce the paths $u_p \otimes b_{\rho_1^j} \otimes b_{\rho_2^j} \otimes \ldots \otimes b_{\rho_s^j} \otimes \mathcal{B}_\ell$ for $j = 1,2,\ldots,r$. We want to show that this is true for $r+1$. So, suppose we have the λ-paths $u_p \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes \mathcal{B}_\ell$ with (i)-signature of the form $\left((-_g+_{r+1}),*\right)$.

If $r+1>\ell$, then $*$ won't affect at least the first $r+1-\ell$ pluses in the (i)-signature. For each $*$, let $\tilde{f}_i$ act on the component with the leftmost plus in the (i)-signature; then we have $u_p \otimes b_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots b_{\rho_s^1} \otimes \mathcal{B}_\ell$ with (i)-signature of the form $\left((-_g+_r),*\right)$. Hence, by the inductive hypothesis, we can apply $\tilde{f}_i$ a sufficient number of times to certain λ-paths in the above set to produce the set of λ-paths $u_p \otimes b_{\rho_1^j} \otimes b_{\rho_2^j} \otimes \ldots \otimes b_{\rho_s^j} \otimes \mathcal{B}_\ell$, for $j = 1,2,\ldots,r+1-\ell$.

If $r+1 \leq \ell$ and $*$ has less than $r+1$ negatives, then we have the λ-paths $u_p \otimes u_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes b_\mu$, where $b_\mu = (m_1, m_2, \ldots, m_n, m_0)$ such that $m_{i+1} < r+1$ and $\sum_{t=0}^{n} m_t = \ell$, with (i)-signature $\left((-_g+_{r+1}),-_{m_{i+1}}+_{m_i}\right)$. Then we can apply $\tilde{f}_i$ at least once to these λ-paths to produce the λ-paths $u_p \otimes u_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\mu$. If $r+1 \leq \ell$ and $*$ has at least $r+1$ negatives, then we have the λ-paths $u_p \otimes u_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes b_\mu$, where $b_\mu = (m_1, m_2, \ldots, m_n, m_0)$ such that $m_{i+1} \geq r+1$ and $\sum_{t=0}^{n} m_t = \ell$, with (i)-signature $\left((-_g+_{r+1}),-_{m_{i+1}}+_{m_i}\right)$. Then how do we obtain the λ-paths of the form $u_p \otimes u_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\mu$? We have λ-paths of the form $u_p \otimes u_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes b_\nu$, where $b_\nu = (v_1, v_2, \ldots, v_n, v_0)$ such that $v_{i+1} = r$, $v_i = m_i + m_{i+1} - r$, and $\sum_{t=0}^{n} v_t = \ell$. The (i)-signature for these λ-paths is $\left((-_g+_{r+1}),-_r+_{m_i+m_{i+1}-r}\right) = \left((-_g+),+_{m_i+m_{i+1}-r}\right)$. By applying $\tilde{f}_i$ a sufficient number of times, we obtain the λ-paths $u_p \otimes u_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\gamma$, for $b_\gamma = \left(y_1, y_2, \ldots, y_n, y_0\right)$ such that

$y_{i+1} = r, r+1, \ldots, m_i + m_{i+1}$; $y_i + y_{i+1} = m_i + m_{i+1}$; and $\sum_{t=0}^{n} y_t = \ell$, which contains the $(n+1)$-tuples b_ν that we wanted. Hence, altogether we have obtained the set of λ-paths $u_p \otimes u_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes \mathcal{B}_\ell$, with (i)-signature of the form $\left((-_g+_r), *\right)$. Thus, by the inductive hypothesis, we have $u_p \otimes u_{\rho_1^j} \otimes b_{\rho_2^j} \otimes \ldots \otimes b_{\rho_s^j} \otimes \mathcal{B}_\ell$ for $j = 2, 3, \ldots, r+1$.

Case 5.2 (Induction)**.** Now suppose the lemma is true for L and for any r, i.e. given $u_p \otimes u_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes \mathcal{B}_\ell^L$ with (i)-signature of the form $\left(\left(-_g+_r\right), *_1, \ldots, *_L\right)$, we can obtain the set of λ-paths $u_p \otimes u_{\rho_1^j} \otimes b_{\rho_2^j} \otimes \ldots \otimes b_{\rho_s^j} \otimes \mathcal{B}_\ell^L$ for $j = 1, 2, \ldots, r$ by applying $\tilde{f}_i$ a sufficient number of times. We want to show that the lemma is true for $L+1$. Suppose we are given the set of λ-paths $u_p \otimes u_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes \mathcal{B}_\ell^{L+1}$ with (i)-signature of the form $\left((-_g+_r), *_1, \ldots, *_{L+1}\right)$.

Subcase 5.2.1. If $r = 0$, then there's nothing to prove. So, suppose $r = 1$. Then we have the (i)-signature of the form $\left((-_g+), *_1, \ldots, *_{L+1}\right)$.

If $*_1 = +_\ell$, then we have the set of λ-paths

$$u_p \otimes u_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes b_\mu \otimes \mathcal{B}_\ell^L,$$

where $b_\mu = (m_1, m_2, \ldots, m_n, m_0)$ such that $m_i = \ell$ and $m_t = 0$ for $t \neq i$. And this set of λ-paths has (i)-signature of the form $\left((-_g+), +_\ell, *_2, \ldots, *_{L+1}\right) = \left(\left(-_g+_{\ell+1}\right), *_2, \ldots, *_{L+1}\right)$. By the inductive hypothesis on L, if we let $\tilde{f}_i$ act a sufficient number of times on certain λ-paths in this set, then we obtain all λ-paths in the set $u_p \otimes u_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\nu \otimes \mathcal{B}_\ell^L$, where $b_\nu = (v_1, v_2, \ldots, v_n, v_0)$ such that $v_i + v_{i+1} = \ell$ and $v_t = 0$ for $t \neq i, i+1$. Hence, we have obtained $\ell + 1$ new b_ν.

In general, if $* = \bullet_{\ell-j}+_j$ for $j = 1, 2, \ldots, \ell$, then we have the set of λ-paths $u_p \otimes u_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes b_\mu \otimes \mathcal{B}_\ell^L$ where $b_\mu = (m_1, m_2, \ldots, m_n, m_0)$ such that $m_i = j$, $m_{i+1} = 0$, and $\sum_{t \neq i, i+1} m_t = \ell - j$ and with (i)-signature of the form $\left((-_g+), +_j, *_2, \ldots, *_{L+1}\right)$. If we apply $\tilde{f}_i$ to these paths a sufficient number of times, then by the inductive hypothesis, we obtain all λ-paths in the set $u_p \otimes u_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\nu \otimes \mathcal{B}_\ell^L$ where $b_\nu = (v_1, v_2, \ldots, v_n, v_0)$ such that $v_i + v_{i+1} = j$ and $\sum_{t \neq i, i+1} v_t = \ell - j$. So, by the "balls in bins" combinatorial problem, we obtain

$$\sum_{j=1}^{\ell} (j+1) \binom{n-1+\ell-j-1}{\ell-j}$$

new b_ν in the $(L+1)^{st}$ component from the right.

If $* = \bullet_\ell$, then we have the set of λ-paths $u_p \otimes u_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes b_\mu \otimes \mathcal{B}_\ell^L$ where $b_\mu = (m_1, m_2, \ldots, m_n, m_0)$

such that $m_i = 0 = m_{i+1}$ and $\sum_{t\neq i,i+1} m_t = \ell$. This set of λ-paths has (i)-signature of the form $\left((-_g+),\bullet,\ast_2,\ldots,\ast_{L+1}\right) = \left((-_g+),\ast_2,\ldots,\ast_{L+1}\right)$. Hence, by the inductive hypothesis on L, we obtain the set of λ-paths $u_p \otimes u_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\mu \otimes \mathcal{B}_\ell^L$, of which there are

$$\binom{n-1+\ell-1}{\ell}$$

b_μ in the $(L+1)^{st}$ component from the right in the tensor product. Thus, we have

$$\left(\sum_{j=1}^{\ell}(j+1)\binom{n-1+\ell-j-1}{\ell-j}\right)+\binom{n-1+\ell-1}{\ell}=\binom{n+\ell}{\ell}$$

unique b_μ in the set $u_p \otimes u_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\mu \otimes \mathcal{B}_\ell^L$, and hence, have $u_p \otimes u_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes \mathcal{B}_\ell^{L+1}$.

Subcase 5.2.2. Now, suppose that if we have the set of λ-paths $u_p \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes \mathcal{B}_\ell^{L+1}$ with (i)-signature of the form $\left((-_g+_r),\ast_1,\ldots,\ast_{L+1}\right)$, we can apply $\tilde{f}_i$ a sufficient number of times and produce the set of λ-paths $u_p \otimes b_{\rho_1^j} \otimes b_{\rho_2^j} \otimes \ldots \otimes b_{\rho_s^j} \otimes \mathcal{B}_\ell^{L+1}$ for $j = 1,2,\ldots,r$. We want to show it's true for $r+1$. Suppose we have the λ-path $u_p \otimes u_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes \mathcal{B}_\ell^{L+1}$ with (i)-signature of the form $\left((-_g+_{r+1}),\ast_1,\ldots,\ast_{L+1}\right)$.

If $r+1 > \ell$, then $\ast_1$ won't affect the leftmost plus in the (i)-signature. If we fix $\ast_1 = -_a +_b \bullet_c$, then for each such $\ast_1$, we have the set of λ-paths $u_p \otimes u_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes b_\mu \otimes \mathcal{B}_\ell^L$, with (i)-signature of the form $\left((-_g+_{r+1-a+b}),\ast_2,\ldots,\ast_{L+1}\right)$. By the inductive hypothesis on L, we can apply $\tilde{f}_i$ a sufficient number of times to get the set $u_p \otimes u_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\mu \otimes \mathcal{B}_\ell^L$, with (i)-signature of the form $\left((-_g+_{r-a+b}),\ast_2,\ldots,\ast_{L+1}\right) = \left((-_g+_r),\ast_1,\ast_2,\ldots,\ast_{L+1}\right)$. Then, by the inductive hypothesis on r, we obtain $u_p \otimes u_{\rho_1^j} \otimes b_{\rho_2^j} \otimes \ldots \otimes b_{\rho_s^j} \otimes \mathcal{B}_\ell^{L+1}$ for $j = 2,3,\ldots,r+1$.

Now suppose that $r+1 \le \ell$. If $\ast_1$ has less than $r+1$ negatives, then we have the set of λ-paths

$$u_p \otimes u_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes b_\mu \otimes \mathcal{B}_\ell^L, \tag{5.7}$$

where $b_\mu = (m_1, m_2, \ldots, m_n, m_0)$ such that $m_{i+1} < r+1$ and $\sum_{t=0}^{n} m_t = \ell$. Then the leftmost plus in the associated (i)-signature of the form $\left((-_g+_{r+1}),-_{m_{i+1}}+_{m_i},\ast_2,\ldots,\ast_{L+1}\right)$ won't be affected. So, we can apply $\tilde{f}_i$ a sufficient number of times by the inductive hypothesis on L to obtain the set of λ-paths $u_p \otimes u_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\mu \otimes \mathcal{B}_\ell^L$, with (i)-signature of the form $\left((-_g+_r),\ast_1,\ldots,\ast_{L+1}\right)$, where b_μ is such that $m_{i+1} < r+1$. On the other hand, if $\ast_1$ has at least $r+1$ negatives, then we have $u_p \otimes u_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes b_\mu \otimes \mathcal{B}_\ell^L$ where $b_\mu = (m_1, m_2, \ldots, m_n, m_0)$ such that $m_{i+1} \ge r+1$

and $\sum_{t=0}^{n} m_t = \ell$. This set of λ-paths has (i)-signature of the form $\left((-_g +_{r+1}), -_{m_{i+1}} +_{m_i}, *_2, \ldots, *_{L+1}\right)$. Then how do we obtain λ-paths of the form $u_p \otimes u_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\mu \otimes \mathcal{B}_\ell^L$? From Equation 5.7, we have λ-paths of the form $u_p \otimes u_{\rho_1} \otimes b_{\rho_2} \otimes \ldots \otimes b_{\rho_s} \otimes b_\nu \otimes \mathcal{B}_\ell^L$ where $b_\nu = (v_1, v_2, \ldots, v_n, v_0)$ such that $v_i = m_{i+1} + m_i - r$, $v_{i+1} = r$, and $\sum_{t=0}^{n} m_t = \ell$ and with (i)-signature $\left(\ldots, (+_{r+1}), -_r +_{m_{i+1}+m_i-r}, *_2, \ldots, *_L\right) = \left(\ldots, (+), +_{m_{i+1}+m_i-r}, *_2, \ldots, *_L\right)$. By the inductive hypothesis on L, a certain number of applications of $\tilde{f}_i$ produces λ-paths of the form

$$u_p \otimes u_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes b_\gamma \otimes \mathcal{B}_\ell^L, \tag{5.8}$$

where $b_\gamma = \left(y_1, y_2, \ldots, y_n, y_0\right)$ such that $y_{i+1} = r, r+1, \ldots, m_i + m_{i+1}$; $y_i + y_{i+1} = m_i + m_{i+1}$; and $\sum_{t=0}^{n} y_t = \ell$, which contains the $(n+1)$-tuples b_ν that we wanted. Hence, combining Equations 5.7 and 5.8, we have obtained $u_p \otimes u_{\rho_1^1} \otimes b_{\rho_2^1} \otimes \ldots \otimes b_{\rho_s^1} \otimes \mathcal{B}_\ell^{L+1}$, with (i)-signature $(\ldots, (+_r), *_1, *_2, \ldots, *_{L+1})$. Thus, by the inductive hypothesis on r, we have $u_p \otimes u_{\rho_1^j} \otimes b_{\rho_2^j} \otimes \ldots \otimes b_{\rho_s^j} \otimes \mathcal{B}_\ell^{L+1}$ for $j = 2, 3, \ldots, r+1$.

$\square$

Suppose that we have the set of λ-paths $u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L = u_{\sigma^{L+1}(\lambda)} \otimes b_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L$, where $\tilde{f}_i\left(b_{\sigma^L(\lambda)}\right) \neq 0$. Equation 5.6 and Lemma 5.5 tell us that if we apply $\tilde{f}_i$ a sufficient number of times to certain λ-paths in the set $u_{\sigma^L(\lambda)} \otimes B_\ell^L$, that we will obtain the set of λ-paths $u_{\sigma^{L+1}(\lambda)} \otimes \tilde{f}_i\left(b_{\sigma^L(\lambda)}\right) \otimes \mathcal{B}_\ell^L$.

Similarly, suppose that $\tilde{f}_i^j\left(b_{\sigma^L(\lambda)}\right) \neq 0$ for a positive integer j. Then Lemma 5.5 implies that we can obtain the set of λ-paths $u_{\sigma^{L+1}(\lambda)} \otimes \tilde{f}_i^j\left(b_{\sigma^L(\lambda)}\right) \otimes \mathcal{B}_\ell^L$ by applying $\tilde{f}_i$ a sufficient number of times to certain λ-paths in the set $u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L$.

We can also extend this result to sequences containing more than one Kashiwara operator. In other words, suppose that we have the set of λ-paths $u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L$. Also, suppose that $\tilde{f}_{i_t}^{j_t} \cdots \tilde{f}_{i_2}^{j_2} \tilde{f}_{i_1}^{j_1}\left(u_{\sigma^L(\lambda)}\right) \neq 0$. Then, by applying each $\tilde{f}_{i_p}$ a sufficient number of times to certain λ-paths in the set $u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L$, we can obtain the set $\tilde{f}_{i_t}^{j_t} \cdots \tilde{f}_{i_2}^{j_2} \tilde{f}_{i_1}^{j_1}\left(u_{\sigma^L(\lambda)}\right) \otimes \mathcal{B}_\ell^L$.

Now, we'll work through an example to help understand Lemma 5.5.

Example 5.6. Suppose that $L = 2$, $n = 3$, and $\ell = 1$. Then we have that

$$|\mathcal{B}_\ell| = \binom{n+\ell}{\ell} = \binom{3+1}{1} = 4.$$

Suppose that $b_\lambda = (1,0,0,0)$ and that we have the set of λ-paths

$$u_{\sigma^5(\lambda)} \otimes (0,0,0,1) \otimes (0,0,1,0) \otimes (1,0,0,0) \otimes \mathcal{B}_1^2. \tag{5.9}$$

The (0)-signature for the set of λ-paths in Equation 5.9 is of the form

$$(\ldots,+,-,\bullet,\bullet,+,+,\bullet,-,\ast_1,\ast_2) = ((+),\ast_1,\ast_2)$$

where the $(+)$ corresponds to

$$b_{\sigma^5(\lambda)} \otimes (0,0,0,1) \otimes (0,0,1,0) \otimes (1,0,0,0) = b_{\rho_1} \otimes b_{\rho_2} \otimes b_{\rho_3} \otimes b_{\rho_4}.$$

Note that $r = 1$ and

$$\begin{aligned} & \tilde{f}_0\left(b_{\sigma^5(\lambda)} \otimes (0,0,0,1) \otimes (0,0,1,0) \otimes (1,0,0,0)\right) \\ = \;& (1,0,0,0) \otimes (0,0,0,1) \otimes (0,0,1,0) \otimes (1,0,0,0) \\ = \;& b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1}. \end{aligned}$$

We want to produce the set of λ-paths $u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes \mathcal{B}_1^2$ by applying $\tilde{f}_0$ a sufficient number of times to specific λ-paths in the set $u_{\sigma^6(\lambda)} \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes b_{\rho_3} \otimes b_{\rho_4} \otimes \mathcal{B}_1^2$. We'll denote these specific λ-paths by $u_{\sigma^6(\lambda)} \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes b_{\rho_3} \otimes b_{\rho_4} \otimes a_1 \otimes a_2$.

Case 5.3. Suppose that $\ast_1 = +$. Then this implies that $a_1 = (0,0,0,1)$, and that we have the (0)-signature of the form $((+),+,\ast_2)$.

Subcase 5.3.1. Suppose that $\ast_2 = +$. Then $a_2 = (0,0,0,1)$ and the (0)-signature becomes $((+),+,+)$. Hence, we can produce three new λ-paths:

$$\begin{aligned} & u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,0,0,1) \otimes (0,0,0,1), \\ & u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (1,0,0,0) \otimes (0,0,0,1), \\ & u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (1,0,0,0) \otimes (1,0,0,0). \end{aligned}$$

Subcase 5.3.2. Suppose that $\ast_2 = \bullet$. Then $a_2 = (0,1,0,0)$ or $(0,0,1,0)$, and the (0)-signature becomes $((+),+,\bullet)$. Hence, we can produce four new λ-paths:

$$\begin{aligned} & u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,0,0,1) \otimes (0,1,0,0), \\ & u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (1,0,0,0) \otimes (0,1,0,0), \end{aligned}$$

$$u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,0,0,1) \otimes (0,0,1,0),$$
$$u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (1,0,0,0) \otimes (0,0,1,0).$$

Subcase 5.3.3. Suppose that $\ast_2 = -$. Then $a_2 = (1,0,0,0)$, and the (0)-signature becomes $((+),+,-) = ((+))$. Hence, we can only produce one new λ-path:

$$u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,0,0,1) \otimes (1,0,0,0).$$

Notice that so far, applying $\tilde{f}_0$ a sufficient number of times to these specific λ-paths has produced the sets

$$u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,0,0,1) \otimes \mathcal{B}_1,$$
$$u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (1,0,0,0) \otimes \mathcal{B}_1.$$

Case 5.4. Suppose that $\ast_1 = \bullet$. Then $a_1 = (0,1,0,0)$ or $(0,0,1,0)$, and we have the following set

$$u_{\sigma^6(\lambda)} \otimes b_{\rho_1} \otimes b_{\rho_2} \otimes b_{\rho_3} \otimes b_{\rho_4} \otimes a_1 \otimes \mathcal{B}_1$$

with (0)-signature of the form $((+),\bullet,\ast_2)$.

Subcase 5.4.1. Suppose that $\ast_2 = +$. Then $a_2 = (0,0,0,1)$, and the (0)-signature becomes $((+),\bullet,+)$. Hence, we can produce four new λ-paths:

$$u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,1,0,0) \otimes (0,0,0,1),$$
$$u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,1,0,0) \otimes (1,0,0,0),$$
$$u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,0,1,0) \otimes (0,0,0,1),$$
$$u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,0,1,0) \otimes (1,0,0,0).$$

Subcase 5.4.2. Suppose that $\ast_2 = \bullet$. Then $a_2 = (0,1,0,0)$ or $(0,0,1,0)$, and the (0)-signature becomes $((+),\bullet,\bullet)$. Hence, we can produce four new λ-paths:

$$u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,1,0,0) \otimes (0,1,0,0),$$
$$u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,1,0,0) \otimes (0,0,1,0),$$
$$u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,0,1,0) \otimes (0,1,0,0),$$
$$u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,0,1,0) \otimes (0,0,1,0).$$

Subcase 5.4.3. Suppose that $*_2 = -$. Then $a_2 = (1,0,0,0)$, and the (0)-signature becomes $((+), \bullet, -) = (\bullet)$. So, we cannot produce any new λ-paths.

Notice that by applying $\tilde{f}_0$ a sufficient number of times to these specific λ-paths we have produced the sets

$$\begin{aligned}
&u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,0,0,1) \otimes \mathcal{B}_1,\\
&u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (1,0,0,0) \otimes \mathcal{B}_1,\\
&u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,1,0,0) \otimes \mathcal{B}_1,\\
&u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes (0,0,1,0) \otimes \mathcal{B}_1,
\end{aligned}$$

and thus we have produced the set of paths

$$u_{\sigma^6(\lambda)} \otimes b_{\rho_1,1} \otimes b_{\rho_2,1} \otimes b_{\rho_3,1} \otimes b_{\rho_4,1} \otimes \mathcal{B}_1^2.$$

Now we give an explicit description of the Demazure crystal $B_{w(L,0)}(\lambda)$ for $U_q(A_2^{(1)})$.

Lemma 5.7. *Let $\lambda = m_0\Lambda_0 + m_1\Lambda_1 + m_2\Lambda_2 \in \bar{P}^+$, where $\lambda(c) = \ell$. Then the Demazure crystal $B_{w(L,0)}(\lambda)$ for $U_q(A_2^{(1)})$ can be explicitly defined by one of the following three descriptions:*

1. *If $L \equiv 1 \bmod 3$, then*

$$\begin{aligned}
B_{w(L,0)}(\lambda) = \big\{ & u_\lambda \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\sigma(\lambda)} \otimes (m_1 + k_0, m_2, m_0 - k_0) \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\sigma(\lambda)} \otimes (m_1 - k_1, m_2 + k_1, m_0) \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\sigma(\lambda)} \otimes \big(m_1 + k_0 - c_{1,0}, m_2 + c_{1,0}, m_0 - k_0\big) \otimes \mathcal{B}_\ell^{L-1} \mid\\
& k_t = 1,2,\ldots,m_t,\quad t = 0,1,\quad c_{1,0} = 1,2,\ldots,m_1 + k_0 \big\}.
\end{aligned}$$

2. *If $L \equiv 2 \bmod 3$, then*

$$\begin{aligned}
B_{w(L,0)}(\lambda) = \big\{ & u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\sigma^2(\lambda)} \otimes (m_2 + k_1, m_0, m_1 - k_1) \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\sigma^2(\lambda)} \otimes \big(m_2 + c_{1,0}, m_0 - k_0, m_1 + k_0 - c_{1,0}\big) \otimes \mathcal{B}_\ell^{L-1} \mid\\
& k_t = 1,2,\ldots,m_t,\quad t = 0,1,\quad c_{1,0} = 1,2,\ldots,m_1 + k_0 \big\}.
\end{aligned}$$

3. *If* $L \equiv 0 \bmod 3$*, then*

$$\begin{aligned}B_{w(L,0)}(\lambda) = \{ & u_{\sigma^2(\lambda)} \otimes \mathcal{B}_\ell^{L-1}, \\ & u_\lambda \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes \mathcal{B}_\ell^{L-1}, \\ & u_\lambda \otimes (m_0, m_1 - k_1, m_2 + k_1) \otimes \mathcal{B}_\ell^{L-1}, \\ & u_\lambda \otimes \left(m_0 - k_0, m_1 + k_0 - c_{1,0}, m_2 + c_{1,0}\right) \otimes \mathcal{B}_\ell^{L-1} \mid \\ & k_t = 1,2,\ldots,m_t, \quad t = 0,1, \quad c_{1,0} = 1,2,\ldots,m_1 + k_0 \}.\end{aligned}$$

Proof. Let $L = 1$. We can construct $B_{w(1,0)}(\lambda)$ by applying the sequence $w(1,0) = r_1 r_0$ to the ground-state path $u_\lambda = \ldots \otimes (m_1, m_2, m_0) \otimes (m_0, m_1, m_2) \otimes (m_2, m_0, m_1) \otimes (m_1, m_2, m_0)$. The (0)-signature on u_λ is $\left(+_{m_0}\right)$. Hence, we can apply $\tilde{f}_0$ a total of m_0 times, and thus we obtain the set of paths $\left\{u_{\sigma(\lambda)} \otimes (m_1 + k_0, m_2, m_0 - k_0) \mid k_0 = 1,2,\ldots,m_0\right\}$. Similarly, the (1)-signature on u_λ is $\left(+_{m_1}\right)$. So we can apply $\tilde{f}_1$ a total of m_1 times, and thus we obtain the set of paths

$$\left\{u_{\sigma(\lambda)} \otimes (m_1 - k_1, m_2 + k_1, m_0) \mid k_1 = 1,2,\ldots,m_1\right\}.$$

Now we need to apply the $\tilde{f}_1$ operator to the paths we obtained by applying the $\tilde{f}_0$ operator to u_λ. The (1)-signature on these paths is $\left(+_{m_1+k_0}\right)$. Hence we obtain the set of paths

$$\left\{u_{\sigma(\lambda)} \otimes \left(m_1 + k_0 - c_{1,0}, m_2 + c_{1,0}, m_0 - k_0\right) \mid k_0 = 1,2,\ldots,m_0, \quad c_{1,0} = 1,2,\ldots,m_1 + k_0\right\}.$$

Therefore we have

$$\begin{aligned}B_{w(1,0)}(\lambda) = \{ & u_\lambda, \\ & u_{\sigma(\lambda)} \otimes (m_1 + k_0, m_2, m_0 - k_0), \\ & u_{\sigma(\lambda)} \otimes (m_1 - k_1, m_2 + k_1, m_0), \\ & u_{\sigma(\lambda)} \otimes \left(m_1 + k_0 - c_{1,0}, m_2 + c_{1,0}, m_0 - k_0\right) \mid \\ & k_t = 1,2,\ldots,m_t, \quad t = 0,1, \quad c_{1,0} = 1,2,\ldots,m_1 + k_0 \}.\end{aligned}$$

Notice that we now have

$$1 + m_0 + m_1 + \sum_{k_0=1}^{m_0} (m_1 + k_0) = 1 + \frac{3}{2} m_0 + m_1 + m_0 m_1 + \frac{1}{2} m_0^2$$

$b_\mu \in \mathcal{B}_\ell$ in the form $u_{\sigma(\lambda)} \otimes b_\mu$. In order to form the set $u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell$, we'll need to obtain a total of $\binom{2+\ell}{\ell}$

unique b_μ in the form $u_{\sigma(\lambda)} \otimes b_\mu$.

Next let $L = 2$. Then we have $w(2,0) = (r_0 r_2)(r_1 r_0) = (r_0 r_2) w(1,0)$. So, we need only apply $\tilde{f}_2$ and $\tilde{f}_0$ to all paths in the set $B_{w(1,0)}(\lambda)$. First, we'll apply $\tilde{f}_2$ to the path u_λ. The (2)-signature on u_λ is $(+_{m_2})$. Hence, we obtain the following set of paths:

$$\left\{u_{\sigma(\lambda)} \otimes (m_1, m_2 - k_2, m_0 + k_2) | k_2 = 1, 2, \ldots, m_2\right\}. \tag{5.10}$$

We've already applied $\tilde{f}_0$ to the path u_λ, so we don't need to repeat this step. Next we'll apply $\tilde{f}_2$ to the set of paths $\left\{u_{\sigma(\lambda)} \otimes (m_1 + k_0, m_2, m_0 - k_0)\right\}$. The (2)-signature on these paths is $\left(+_{k_0}, +_{m_2}\right)$. Thus, we obtain the following set of paths:

$$\begin{aligned}
&\Big\{u_{\sigma^2(\lambda)} \otimes \left(m_2, m_0 - k_0', m_1 + k_0'\right) \otimes (m_1 + k_0, m_2, m_0 - k_0), \\
&u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes (m_1 + k_0, m_2 - k_2, m_0 - k_0 + k_2) \Big| \\
&k_2 = 1, 2, \ldots, m_2, \quad k_0' = 1, 2, \ldots, k_0\Big\}.
\end{aligned} \tag{5.11}$$

If we were to apply $\tilde{f}_0$ to the same set of paths, we wouldn't produce any new paths. So, let's apply $\tilde{f}_2$ to the paths $u_{\sigma(\lambda)} \otimes (m_1 - k_1, m_2 + k_1, m_0)$. We have the following (2)-signature: $\left(+_{m_2+k_1}\right)$. This action produces the set of paths

$$\left\{u_{\sigma(\lambda)} \otimes \left(m_1 - k_1, m_2 + k_1 - c_{2,1}, m_0 + c_{2,1}\right)\right\}, \tag{5.12}$$

where $c_{2,1} = 1, 2, \ldots, m_2 + k_1$.

Next let's apply $\tilde{f}_0$ to this same set of paths. We have the (0)-signature $\left(+_{k_1}, +_{m_0}\right)$. Hence, we produce the following set of paths:

$$\begin{aligned}
&\Big\{u_{\sigma^2(\lambda)} \otimes \left(m_2 + k_1', m_0, m_1 - k_1'\right) \otimes (m_1 - k_1, m_2 + k_1, m_0), \\
&u_{\sigma^2(\lambda)} \otimes (m_2 + k_1, m_0, m_1 - k_1) \otimes (m_1 - k_1 + k_0, m_2 + k_1, m_0 - k_0) \Big| \\
&k_0' = 1, 2, \ldots, k_0\Big\}.
\end{aligned} \tag{5.13}$$

Now, let's apply $\tilde{f}_2$ to the set $\left\{u_{\sigma(\lambda)} \otimes \left(m_1 + k_0 - c_{1,0}, m_2 + c_{1,0}, m_0 - k_0\right)\right\}$. The (2)-signature is

$$\left(+_{k_0}, +_{m_2+c_{1,0}}\right).$$

This results in the following set of paths:

$$\begin{aligned}&\{u_{\sigma^2(\lambda)}\otimes\left(m_2, m_0-k_0', m_1+k_0'\right)\otimes\left(m_1+k_0-c_{1,0}, m_2+c_{1,0}, m_0-k_0\right),\\&u_{\sigma^2(\lambda)}\otimes(m_2, m_0-k_0, m_1+k_0)\otimes\left(m_1+k_0-c_{1,0}, m_2+c_{1,0}-c'_{2;1,0}, m_0-k_0+c'_{2;1,0}\right)\Big|\\&c'_{2;1,0}=1,2,\ldots,m_2+c_{1,0}\}.\end{aligned} \tag{5.14}$$

Next, let's apply $\tilde{f}_0$ to the same paths. The (0)-signature is $\left(\ldots,-_{m_2}+_{m_1},-_{m_1+k_0-c_{1,0}}+_{m_0-k_0}\right)$, which can be rewritten as:

$$\begin{aligned}&\begin{cases}\left(+_{c_{1,0}-k_0},+_{m_0-k_0}\right) & \text{if } c_{1,0}>k_0\\ \left(+_{m_0-k_0}\right) & \text{if } c_{1,0}\le k_0\end{cases}\\ =&\begin{cases}\left(+_{k_1},+_{m_0-k_0}\right) & \text{if } c_{1,0}>k_0\\ \left(+_{m_0-k_0}\right) & \text{if } c_{1,0}\le k_0.\end{cases}\end{aligned}$$

Let's break this section into cases:

Case 5.1. Suppose that $c_{1,0}>k_0$ and let $k_1=c_{1,0}-k_0$. Then we obtain the paths

$$\begin{aligned}&\{u_{\sigma^2(\lambda)}\otimes\left(m_2+k_1', m_0, m_1-k_1'\right)\otimes\left(m_1+k_0-c_{1,0}, m_2+c_{1,0}, m_0-k_0\right),\\&u_{\sigma^2(\lambda)}\otimes(m_2+k_1, m_0, m_1-k_1)\otimes\left(m_1+k_0-c_{1,0}+c_{0,-0}, m_2+c_{1,0}, m_0-k_0-c_{0,-0}\right)\Big|\\&c_{0,-0}=1,2,\ldots,m_0-k_0\}.\end{aligned} \tag{5.15}$$

Case 5.2. Suppose that $c_{1,0}\le k_0$. Then we obtain the paths

$$\left\{u_{\sigma(\lambda)}\otimes\left(m_1+k_0-c_{1,0}+c_{0,-0}, m_2+c_{1,0}, m_0-k_0-c_{0,-0}\right)\right\}.$$

Note that we will have some repeated paths here. We have that

$$\tilde{f}_0^{t_3}\tilde{f}_1^{t_2}\tilde{f}_0^{t_1}\left(b_\mu\right)=\tilde{f}_1^{t_2}\tilde{f}_0^{t_1+t_3}\left(b_\mu\right) \tag{5.16}$$

if $t_2\le t_1$ and $t_1+t_3\le\mu_0,\;\; t_1,t_2,t_3\in\mathbb{Z}_{\ge 0}, \mu=\mu_0\Lambda_0+\mu_1\Lambda_1+\mu_2\Lambda_2$. This follows logically from the definition of $\tilde{f}_i\left(b_\mu\right)$. Hence, all paths from Case 5.2 have already been obtained; i.e. the only new paths produced are from Case 5.1.

Next we'll let $\tilde{f}_0$ act on the paths $\left\{u_{\sigma(\lambda)}\otimes(m_1, m_2-k_2, m_0+k_2)|k_2=1,2,\ldots,m_2\right\}$. We have the (0)-

signature $\left(+_{m_0+k_2}\right)$. Thus, we have the paths

$$\left\{u_{\sigma(\lambda)} \otimes \left(m_1 + c_{0,2}, m_2 - k_2, m_0 + k_2 - c_{0,2}\right)\right\}, \tag{5.17}$$

where $c_{0,2} = 1, 2, \ldots, m_0 + k_2$.

Now let $\tilde{f}_0$ act on the set:

$$\begin{aligned}&\left\{u_{\sigma^2(\lambda)} \otimes \left(m_2, m_0 - k_0', m_1 + k_0'\right) \otimes (m_1 + k_0, m_2, m_0 - k_0),\right.\\ &u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes (m_1 + k_0, m_2 - k_2, m_0 - k_0 + k_2)\Big|\\ &\left. k_2 = 1, 2, \ldots, m_2,\ \ k_0' = 1, 2, \ldots, k_0\right\}.\end{aligned}$$

Rewriting these paths, we have:

$$\begin{cases} u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes (m_1 + k_0, m_2, m_0 - k_0) & [1] \\ u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes (m_1 + k_0, m_2 - k_2, m_0 - k_0 + k_2) & [2] \\ u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes \left(m_1 + k_0 + c_{0,-0}, m_2, m_0 - k_0 - c_{0,-0}\right) & [3] \end{cases}$$

where $k_2 = 1, 2, \ldots, m_2$, $c_{0,-0} = 1, 2, \ldots, m_0 - k_0$ for each $k_0 = 1, 2, \ldots, m_0$. Let's break this into three sections:

Case 5.3. Looking at paths [1] displayed above, we have the (0)-signature $\left(+_{m_0-k_0}\right)$. This gives the following paths: $\left\{u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes \left(m_1 + k_0 + c_{0,-0}, m_2, m_0 - k_0 - c_{0,-0}\right)\right\}$. But these are exactly the same as paths [3] above.

Case 5.4. Now let's look at the paths [2]. We have the (0)-signature $\left(+_{m_0-k_0+k_2}\right)$. Thus, we obtain the following set of paths:

$$\begin{aligned}&\left\{u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes (m_1 + k_0 + 1, m_2 - k_2, m_0 - k_0 + k_2 - 1),\right.\\ &u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes (m_1 + k_0 + 2, m_2 - k_2, m_0 - k_0 + k_2 - 2), \ldots,\\ &\left.u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes (m_1 + m_0 + k_2, m_2 - k_2, 0)\right\}.\end{aligned} \tag{5.18}$$

Case 5.5. Finally, let's look at the paths [3]. We have the (0)-signature $\left(+_{m_0-k_0-c_{0,-0}}\right)$. Again, this will only produce paths like [3].

Next, let $\tilde{f}_0$ act on the set $\left\{u_{\sigma(\lambda)} \otimes \left(m_1 - k_1, m_2 + k_1 - c_{2,1}, m_0 + c_{2,1}\right)\right\}$, where $c_{2,1} = 1, 2, \ldots, m_2 + k_1$.

This gives the (0)-signature $\left(+_{k_1}, +_{m_0+c_{2,1}}\right)$. Hence we obtain the following paths:

$$\begin{aligned}
&\big\{ u_{\sigma^2(\lambda)} \otimes \left(m_2+k_1', m_0, m_1-k_1'\right) \otimes \left(m_1-k_1, m_2+k_1-c_{2,1}, m_0+c_{2,1}\right),\\
&u_{\sigma^2(\lambda)} \otimes (m_2+k_1, m_0, m_1-k_1) \otimes \left(m_1-k_1+c_{0;2,1}', m_2+k_1-c_{2,1}, m_0+c_{2,1}-c_{0;2,1}'\right) \Big|\\
&k_1'=1,2,\ldots,k_1,\quad c_{0;2,1}'=1,2,\ldots,m_0+c_{2,1}\big\}.
\end{aligned} \tag{5.19}$$

Finally, let $\tilde{f}_0$ act on the following set of paths:

$$\begin{aligned}
&\big\{ u_{\sigma^2(\lambda)} \otimes \left(m_2, m_0-k_0', m_1+k_0'\right) \otimes \left(m_1+k_0-c_{1,0}, m_2+c_{1,0}, m_0-k_0\right),\\
&u_{\sigma^2(\lambda)} \otimes (m_2, m_0-k_0, m_1+k_0) \otimes \left(m_1+k_0-c_{1,0}, m_2+c_{1,0}-c_{2;1,0}', m_0-k_0+c_{2;1,0}'\right) \Big|\\
&c_{2;1,0}'=1,2,\ldots,m_2+c_{1,0}\big\},
\end{aligned}$$

which we rewrite as:

$$\begin{cases}
u_{\sigma^2(\lambda)} \otimes (m_2, m_0-k_0, m_1+k_0) \otimes \left(c_{1,0}', \ell-i, i-c_{1,0}'\right) & [1]\\
u_{\sigma^2(\lambda)} \otimes (m_2, m_0-k_0, m_1+k_0) \otimes \left(\ell-m_2-c_{0,-0}, m_2+t, c_{0,-0}-t\right) & [2]
\end{cases}$$

where $c_{1,0}'=0,1,\ldots,m_1+k_0-1$, $i=c_{1,0}', c_{1,0}'+1,\ldots,\ell$, $c_{0,-0}=1,2,\ldots,m_0-k_0$, and $t=1,2,\ldots,c_{0,-0}$. Again, let's break this into cases:

Case 5.6. The (0)-signature on the paths [1] is $\left(\ldots, -_{m_2} +_{m_1+k_0}, -_{c_{1,0}'} +_{i-c_{1,0}'}\right)$, which reduces to

$$\left(+_{m_1+k_0}, -_{c_{1,0}'} +_{i-c_{1,0}'}\right) = \left(+_{c_{1,0}}, +_{i-c_{1,0}'}\right) = \left(+_{c_{1,0}}, +_z\right),$$

where $z=0,1,\ldots,\ell-c_{1,0}'$. Hence, we have obtained the following set of paths:

$$\left\{ u_{\sigma^2(\lambda)} \otimes \left(m_2+c_{1,0}', m_0-k_0, m_1+k_0-c_{1,0}'\right) \otimes b_\mu \,\middle|\, c_{1,0}'=1,2,\ldots,c_{1,0} \right\} \tag{5.20}$$

for specific $b_\mu \in \mathcal{B}_\ell$.

Case 5.7. The (0)-signature on the paths [2] is

$$\begin{aligned}
\left(\ldots, +_{m_1+k_0}, -_{\ell-m_2-c_{0,-0}} +_{c_{0,-0}-t}\right) &= \left(+_{m_1+k_0}, -_{\ell-m_2-c_{0,-0}} +_{c_{0,-0}-t}\right)\\
&= \left(+_{m_1+k_0}, -_{m_1+m_0-c_{0,-0}} +_{c_{0,-0}-t}\right)\\
&= \left(+_{c_{0,-0}-t}\right).
\end{aligned}$$

Hence, this application of $\tilde{f}_0$ produces no new paths, since we already have all paths of the form

$u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes b_\mu$, which we'll show below.

The following is a summary of all the new elements we have found. From Equation 5.10, we have m_2 new b_μ in the form $u_{\sigma(\lambda)} \otimes b_\mu$. From Equation 5.11, we have $(m_0 - (k_0 - 1)) + m_2$ new b_μ in the form $u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes b_\mu$. Equation 5.12 produces $\sum_{k_1=1}^{m_1} m_2 + k_1 = m_1 m_2 + \frac{1}{2} m_1(m_1+1)$ unique b_μ in the form $u_{\sigma(\lambda)} \otimes b_\mu$. Equation 5.13 gives $(m_1 - (k_1 - 1)) + m_0$ new b_μ in the form $u_{\sigma^2(\lambda)} \otimes (m_2 + k_1, m_0, m_1 - k_1) \otimes b_\mu$. Equation 5.14 produces $\sum_{i=1}^{m_1+k_0} (1 + m_2 + i) + \sum_{i=k_0+1}^{m_0} (m_1 + i)$ new b_μ in the form $u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes b_\mu$. Equation 5.15 gives

$$\sum_{k_0=1}^{m_0} [(1 + m_0 - k_0) + (m_1 + k_0 - (k_0 + k_1 + 1) + 1)]$$

new b_μ of the form $u_{\sigma^2(\lambda)} \otimes (m_2 + k_1, m_0, m_1 - k_1) \otimes b_\mu$. Equation 5.17 has $\sum_{k_2=1}^{m_2} (m_0 + k_2)$ new b_μ in the form $u_{\sigma(\lambda)} \otimes b_\mu$. Equation 5.18 produces $\sum_{k_2=1}^{m_2} (m_0 - k_0 + k_2)$ new b_μ in the form $u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes b_\mu$. Equation 5.19 has $\sum_{c_{2,1}=1}^{m_2+k_1} \left(1 + m_0 + c_{2,1}\right) + \sum_{i=k_1+1}^{m_1} (m_2 + i)$ new b_μ in the form $u_{\sigma^2(\lambda)} \otimes (m_2 + k_1, m_0, m_1 - k_1) \otimes b_\mu$. Finally, Equation 5.20 gives

$$\sum_{t=0}^{\ell - c'_{1,0}} (t+1) + \sum_{c'_{1,0}=0}^{m_1+k_0-t-1} \sum_{t=0}^{\ell - c'_{1,0}} (1)$$

new b_μ in the form $u_{\sigma^2(\lambda)} \otimes (m_2 + t, m_0 - k_0, m_1 + k_0 - t) \otimes b_\mu$.

From this summary, we see that we have:

- Total b_μ in the form $u_{\sigma(\lambda)} \otimes b_\mu$:

$$1 + \frac{3}{2} m_0 + \frac{3}{2} m_1 + m_0 m_1 + \frac{1}{2} m_0^2 + m_2 + m_1 m_2 + \frac{1}{2} m_1^2 + m_2 m_0 + \frac{1}{2} m_2 (m_2 + 1) = \binom{2+\ell}{\ell}.$$

- Total b_μ in the form $u_{\sigma^2(\lambda)} \otimes (m_2 + k_1, m_0, m_1 - k_1) \otimes b_\mu$:

$$m_1 - k_1 + 1 + m_0 + \sum_{k_0=1}^{m_0} (1 + m_0 + m_1 - k_0 - k_1) + \sum_{c_{2,1}=1}^{m_2+k_1} \left(1 + m_0 + c_{2,1}\right) + \sum_{i=k_1+1}^{m_1} (m_2 + i) = \binom{2+\ell}{\ell}.$$

- Total b_μ in the form $u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes b_\mu$:

$$m_0 - (k_0 - 1) + m_2 + \sum_{i=1}^{m_1+k_0} (1 + m_2 + i) + \sum_{i=k_0+1}^{m_0} (m_1 + i) + \sum_{k_2=1}^{m_2} (m_0 - k_0 + k_2) = \binom{2+\ell}{\ell}.$$

- Total b_μ in the form $u_{\sigma^2(\lambda)} \otimes (m_2 + i, m_0 - k_0, m_1 + k_0 - i) \otimes b_\mu$:

$$\sum_{t=0}^{\ell - c'_{1,0}} (t+1) + \sum_{c'_{1,0}=0}^{m_1+k_0-i-1} \sum_{t=0}^{\ell - c'_{1,0}} (1) = \binom{2+\ell}{\ell}.$$

Therefore, we see that

$$\begin{aligned} B_{w(2,0)}(\lambda) = \big\{ & u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell, \\ & u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes \mathcal{B}_\ell, \\ & u_{\sigma^2(\lambda)} \otimes (m_2 + k_1, m_0, m_1 - k_1) \otimes \mathcal{B}_\ell, \\ & u_{\sigma^2(\lambda)} \otimes \left(m_2 + c_{1,0}, m_0 - k_0, m_1 + k_0 - c_{1,0}\right) \otimes \mathcal{B}_\ell \mid \\ & k_t = 1, 2, \ldots, m_t, \quad t = 0, 1, \quad c_{1,0} = 1, 2, \ldots, m_1 + k_0 \big\}. \end{aligned}$$

Next let $L = 3$. Then we have $w(3,0) = (r_2 r_1)(r_0 r_2)(r_1 r_0) = (r_2 r_1) w(2,0)$. So, we need only apply $\tilde{f}_1$ and $\tilde{f}_2$ to all paths in the set $B_{w(2,0)}(\lambda)$. First, let's apply $\tilde{f}_1$ to certain paths in the set $u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell$. The (1)-signature is of the form $\left(+_{m_2}, *_1\right)$, where $*_1$ is the (1)-signature for any $b_\mu \in \mathcal{B}_\ell$. Hence, by Lemma 5.5, we have the following set of paths:

$$\left\{ u_{\sigma^2(\lambda)} \otimes (m_2 - k_2, m_0 + k_2, m_1) \otimes \mathcal{B}_\ell \middle| k_2 = 1, 2, \ldots, m_2 \right\}. \tag{5.21}$$

Note that we don't need to apply $\tilde{f}_2$ to the paths in the set $u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell$ because the paths that would be produced are already paths in $B_{w(2,0)}(\lambda)$. Next, let's apply $\tilde{f}_1$ to paths in the set $u_{\sigma^2(\lambda)} \otimes (m_2 + k_1, m_0, m_1 - k_1) \otimes \mathcal{B}_\ell$. The (1)-signature is of the form $\left(+_{m_2+k_1}, *_1\right)$. By Lemma 5.5, we have the set

$$\left\{ u_{\sigma^2(\lambda)} \otimes \left(m_2 + k_1 - c_{2,1}, m_0 + c_{2,1}, m_1 - k_1\right) \otimes \mathcal{B}_\ell \mid c_{2,1} = 1, 2, \ldots, m_2 + k_1 \right\}. \tag{5.22}$$

Now we can apply $\tilde{f}_2$ to the same set of paths. The (2)-signature is of the form $\left(+_{k_1}, +_{m_0}, *_1\right)$. Again, by Lemma 5.5, we have the following paths

$$\big\{ u_{\sigma^3(\lambda)} \otimes \left(m_0, m_1 - k'_1, m_2 + k'_1\right) \otimes (m_2 + k_1, m_0, m_1 - k_1) \otimes \mathcal{B}_\ell,$$

$$u_{\sigma^3(\lambda)} \otimes (m_0, m_1 - k_1, m_2 + k_1) \otimes (m_2 + k_1, m_0 - k_0, m_1 - k_1 + k_0) \otimes \mathcal{B}_\ell \Big| \\ k_1 = 1, 2, \ldots, m_1, \;\; k_1' = 1, 2, \ldots, k_1 \Big\}. \tag{5.23}$$

Then we can apply $\tilde{f}_1$ to paths in the set $u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes \mathcal{B}_\ell$. The (1)-signature is of the form $\left(+_{k_0}, +_{m_2}, *_1\right)$. Hence, by Lemma 5.5, we have the following paths:

$$\Big\{ u_{\sigma^3(\lambda)} \otimes \left(m_0 - k_0', m_1 + k_0', m_2\right) \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes \mathcal{B}_\ell, \\ u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes (m_2 - k_2, m_0 - k_0 + k_2, m_1 + k_0) \otimes \mathcal{B}_\ell \Big| \\ k_2 = 1, 2, \ldots, m_2, \;\; k_0' = 1, 2, \ldots, k_0 \Big\}. \tag{5.24}$$

But if we apply $\tilde{f}_2$ to the same paths, we won't produce any new paths. So, let's apply $\tilde{f}_1$ to the paths in the set $u_{\sigma^2(\lambda)} \otimes \left(m_2 + c_{1,0}, m_0 - k_0, m_1 + k_0 - c_{1,0}\right) \otimes \mathcal{B}_\ell$. The (1)-signature is of the form $\left(+_{k_0}, +_{m_2 + c_{1,0}}, *_1\right)$. By Lemma 5.5, we obtain the set

$$\Big\{ u_{\sigma^3(\lambda)} \otimes \left(m_0 - k_0', m_1 + k_0', m_2\right) \otimes \left(m_2 + c_{1,0}, m_0 - k_0, m_1 + k_0 - c_{1,0}\right) \otimes \mathcal{B}_\ell, \\ u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes \left(m_2 + c_{1,0} - c_{2;1,0}', m_0 - k_0 + c_{2;1,0}', m_1 + k_0 - c_{1,0}\right) \otimes \mathcal{B}_\ell \Big| \\ c_{2;1,0}' = 1, 2, \ldots, m_2 + c_1, 0 \Big\}.$$

Note that we can rewrite these paths as

$$\begin{cases} u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes \left(\ell - i, i - c_{1,0}', c_{1,0}'\right) \otimes \mathcal{B}_\ell \\ u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes \left(m_2 + t, c_{0,-0} - t, \ell - m_2 - c_{0,-0}\right) \otimes \mathcal{B}_\ell \end{cases} \tag{5.25}$$

where $c_{1,0}' = 0, 1, \ldots, m_1 + k_0 - 1$, $i = c_{1,0}', c_{1,0}' + 1, \ldots, \ell$, $c_{0,-0} = 1, 2, \ldots, m_0 - k_0$, and $t = 1, 2, \ldots, c_{0,-0}$.

Next we will apply $\tilde{f}_2$ to the certain paths from the same sets. The (2)-signature is of the form

$$\left(\ldots, -_{m_0} +_{m_2}, -_{m_2} +_{m_1}, -_{m_1 + k_0 - c_{1,0}} +_{m_0 - k_0}, *_1\right),$$

which reduces to the following:

$$\begin{cases} \left(+_{c_{1,0} - k_0}, +_{m_0 - k_0}, *_1\right) & \text{if } c_{1,0} > k_0 \\ \left(+_{m_0 - k_0}, *_1\right) & \text{if } c_{1,0} \le k_0 \end{cases}$$

$$= \begin{cases} \left(+_{k_1}, +_{m_0-k_0}, *_1\right) & \text{if } c_{1,0} > k_0 \\ \left(+_{m_0-k_0}, *_1\right) & \text{if } c_{1,0} \leq k_0. \end{cases}$$

Let's break this section into cases:

Case 5.8. Suppose that $c_{1,0} > k_0$ and let $k_1 = c_{1,0} - k_0 \in \{1, 2, \ldots, m_1\}$. Then, by Lemma 5.5, we obtain the paths

$$\begin{aligned} &\left\{ u_{\sigma^3(\lambda)} \otimes \left(m_0, m_1 - k_1', m_2 + k_1'\right) \otimes \left(m_2 + c_{1,0}, m_0 - k_0, m_1 + k_0 - c_{1,0}\right) \otimes \mathcal{B}_\ell\ , \right. \\ &u_{\sigma^3(\lambda)} \otimes (m_0, m_1 - k_1, m_2 + k_1) \otimes \left(m_2 + c_{1,0}, m_0 - k_0 - c_{0,-0}, m_1 + k_0 - c_{1,0} + c_{0,-0}\right) \otimes \mathcal{B}_\ell \Big| \\ &\left. c_{0,-0} = 1, 2, \ldots, m_0 - k_0 \right\}. \end{aligned} \tag{5.26}$$

Case 5.9. Suppose that $c_{1,0} \leq k_0$. Then, by Lemma 5.5, we obtain the paths

$$\left\{ u_{\sigma^2(\lambda)} \otimes \left(m_2 + c_{1,0}, m_0 - k_0 - c_{0,-0}, m_1 + k_0 - c_{1,0} + c_{0,-0}\right) \otimes \mathcal{B}_\ell \right\}.$$

But by a variation of Equation 5.16, these paths are a repeat of the paths we started with in this case. Hence, the only new paths produced are from Case 5.8.

Next, let's apply $\tilde{f}_2$ to the paths in Equation 5.21. The (2)-signature is of the form $\left(+_{k_1}, +_{m_0+c_{2,1}}, *_1\right)$. Hence, by Lemma 5.5, we produce the new paths

$$\left\{ u_{\sigma^2(\lambda)} \otimes \left(m_2 - k_2, m_0 + k_2 - c_{0,2}, m_1 + c_{0,2}\right) \otimes \mathcal{B}_\ell \right\} \tag{5.27}$$

where $c_{0,2} = 1, 2, \ldots, m_0 + k_2$.

Applying $\tilde{f}_2$ to the paths in Equation 5.22, we have the (2)-signature of the form $\left(+_{k_1}, +_{m_0+c_{2,1}}, *_1\right)$. By Lemma 5.5, this application produces the new paths

$$\begin{aligned} &\left\{ u_{\sigma^3(\lambda)} \otimes \left(m_0, m_1 - k_1', m_2 + k_1'\right) \otimes \left(m_2 + k_1 - c_{2,1}, m_0 + c_{2,1}, m_1 - k_1\right) \otimes \mathcal{B}_\ell, \right. \\ &u_{\sigma^3(\lambda)} \otimes (m_0, m_1 - k_1, m_2 + k_1) \otimes \left(m_0 + c_{2,1} - c'_{0;2,1}, m_1 - k_1 + c'_{0;2,1}, m_2 + k_1 - c_{2,1}\right) \otimes \mathcal{B}_\ell \Big| \\ &\left. k_1' = 1, 2, \ldots, k_1, \quad c'_{0;2,1} = 1, 2, \ldots, m_0 + c_{2,1} \right\}. \end{aligned} \tag{5.28}$$

Now, let's apply $\tilde{f}_2$ to the paths in Equation 5.24. If we rewrite Equation 5.24 as

$$\begin{cases} u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2,) \otimes (m_1, m_0 - k_0, m_1 + k_0) \otimes \mathcal{B}_\ell & [1] \\ u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2,) \otimes (m_2 - k_2, m_0 - k_0 + k_2, m_1 + k_0) \otimes \mathcal{B}_\ell & [2] \\ u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2,) \otimes \left(m_2, m_0 - k_0 - c_{0,-0}, m_1 + k_0 + c_{0,-0}\right) \otimes \mathcal{B}_\ell & [3] \end{cases}$$

then we can break this application of $\tilde{f}_2$ into three cases.

Case 5.10. First, note that the (2)-signature on [1] is of the form $\left(+_{m_0-k_0}, *_1\right)$. This action will produce the same paths as in [3]; hence, no new paths are produced.

Case 5.11. The (2)-signature on [2] is $\left(+_{m_0-k_0+k_2}, *_1\right)$. So, by Lemma 5.5, we have the paths

$$\begin{cases} u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes (m_2 - k_2, m_0 - k_0 + k_2 - 1, m_1 + k_0 + 1) \otimes \mathcal{B}_\ell \\ u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes (m_2 - k_2, m_0 - k_0 + k_2 - 2, m_1 + k_0 + 2) \otimes \mathcal{B}_\ell \\ \qquad \vdots \\ u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes (m_2 - k_2, 0, m_1 + m_0 + k_2) \otimes \mathcal{B}_\ell. \end{cases} \tag{5.29}$$

Case 5.12. Finally, the (2)-signature on [3] is $\left(+_{m_0-k_0-c_{0,-0}}, *_1\right)$. Hence, by Lemma 5.5, this action produces the same paths as in [3].

Next, let's apply $\tilde{f}_2$ to the paths in Equation 5.25, which are of the form

$$\begin{cases} u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes \left(\ell - i, i - c'_{1,0}, c'_{1,0}\right) \otimes \mathcal{B}_\ell & [1] \\ u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes \left(m_2 + t, c_{0,-0} - t, \ell - m_2 - c_{0,-0}\right) \otimes \mathcal{B}_\ell & [2] \end{cases}$$

where $c'_{1,0} = 0, 1, \ldots, m_1 + k_0 - 1$, $i = c'_{1,0}, c'_{1,0} + 1, \ldots, \ell$, $c_{0,-0} = 1, 2, \ldots, m_0 - k_0$, and $t = 1, 2, \ldots, c_{0,-0}$. Let's also break this into cases:

Case 5.13. For [1], we have the (2)-signature of the form $\left(\ldots, -_{m_0} +_{m_2}, -_{m_2} +_{m_1+k_0}, -_{c'_{1,0}} +_{i-c'_{1,0}}, *_1\right)$, which reduces to

$$\left(+_{m_1+k_0}, -_{c'_{1,0}} +_{i-c'_{1,0}}, *_1\right) = \left(+_{c_{1,0}}, +_{i-c'_{1,0}}, *_1\right) = \left(+_{c_{1,0}}, +_z, *_1\right),$$

where $z = 0, 1, \ldots, \ell - c'_{1,0}$. Hence, by Lemma 5.5, we have the paths

$$u_{\sigma^3(\lambda)} \otimes \left(m_0 - k_0, m_1 + k_0 - c'_{1,0}, m_2 + c'_{1,0}\right) \otimes b_\mu \otimes \mathcal{B}_\ell \tag{5.30}$$

for specific $b_\mu \in \mathcal{B}_\ell$.

Case 5.14. For [2], we have the (2)-signature of the form

$$\left(\ldots, -_{m_0} +_{m_2}, -_{m_2} +_{m_1+k_0}, -_{\ell-m_2-c_{0,-0}} +_{c_{0,-0}-t}, *_1\right),$$

which reduces to

$$\left(+_{m_1+k_0}, -_{\ell-m_2-c_{0,-0}} +_{c_{0,-0}-t}, *_1\right) = \left(+_{m_1+k_0}, -_{m_1+m_0-c_{0,-0}} +_{c_{0,-0}-t}, *_1\right) = \left(+_{c_{0,-0}-t}, *_1\right),$$

since $m_0 - c_{0,-0} \geq k_0$. Thus, no new paths are produced in this case, because we already have all paths of the form $u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes b_\mu \otimes \mathcal{B}_\ell$, which we'll show below.

Let's summarize the paths that the operators $\tilde{f}_1$ and $\tilde{f}_2$ have produced. From Equation 5.21, we have an additional m_2 new b_μ in the form $u_{\sigma 2(\lambda)} \otimes b_\mu \otimes \mathcal{B}_\ell$. We see from Equation 5.24 that we have an additional $(m_0-(k_0-1)) + m_2$ new b_μ in the form $u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes b_\mu \otimes \mathcal{B}_\ell$. Equation 5.22 produced an additional $\sum_{k_1=1}^{m_1} m_2 + k_1 = m_1 m_2 + \frac{1}{2} m_1(m_1+1)$ unique b_μ in the form $u_{\sigma^2(\lambda)} \otimes b_\mu \otimes \mathcal{B}_\ell$. From Equation 5.23, we have an additional $(m_1-(k_1-1)) + m_0$ new b_μ in the form $u_{\sigma^3(\lambda)} \otimes (m_0, m_1 - k_1, m_2 + k_1) \otimes b_\mu \otimes \mathcal{B}_\ell$. We see from Equation 5.25 that we added $\sum_{i=1}^{m_1+k_0} (1 + m_2 + i) + \sum_{i=k_0+1}^{m_0} (m_1 + i)$ new b_μ in the form $u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes b_\mu \otimes \mathcal{B}_\ell$. In Equation 5.26, we have $\sum_{k_0=1}^{m_0} [(1 + m_0 - k_0) + (m_1 + k_0 - (k_0 + k_1 + 1) + 1)]$ new b_μ of the form $u_{\sigma^3(\lambda)} \otimes (m_0, m_1 - k_1, m_2 + k_1) \otimes b_\mu \otimes \mathcal{B}_\ell$. From Equation 5.27, we produced $\sum_{k_2=1}^{m_2} (m_0 + k_2)$ new b_μ in the form $u_{\sigma^2(\lambda)} \otimes b_\mu \otimes \mathcal{B}_\ell$. Equation 5.28 produced $\sum_{k_2=1}^{m_2} (m_0 - k_0 + k_2)$ new b_μ in the form $u_{\sigma^3(\lambda)} \otimes (m_0, m_1 - k_1, m_2 + k_1) \otimes b_\mu \otimes \mathcal{B}_\ell$. From Equation 5.29, we have $\sum_{c_{2,1}=1}^{m_2+k_1} \left(1 + m_0 + c_{2,1}\right) + \sum_{i=k_1+1}^{m_1} (m_2 + i)$ new b_μ in the form $u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes b_\mu \otimes \mathcal{B}_\ell$. Finally, we see from Equation 5.30 that we have $\sum_{t=0}^{\ell - c'_{1,0}} (t+1) + \sum_{c'_{1,0}=0}^{m_1+k_0-t-1} \sum_{t=0}^{\ell - c'_{1,0}} (1)$ new b_μ in the form $u_{\sigma^3(\lambda)} \otimes \left(m_0 - k_0, m_1 + k_0 - c_{1,0}, m_2 + c_{1,0}\right) \otimes b_\mu \otimes \mathcal{B}_\ell$.

From the above list, we see that we have:

- Total b_μ in the form $u_{\sigma^2(\lambda)} \otimes b_\mu \otimes \mathcal{B}_\ell$:

$$1 + \frac{3}{2} m_0 + \frac{3}{2} m_1 + m_0 m_1 + \frac{1}{2} m_0^2 + m_2 + m_1 m_2 + \frac{1}{2} m_1^2 + m_2 m_0 + \frac{1}{2} m_2 (m_2 + 1) = \binom{2+\ell}{\ell}.$$

- Total b_μ in the form $u_{\sigma^3(\lambda)} \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes b_\mu \otimes \mathcal{B}_\ell$:

$$m_0 - (k_0 - 1) + m_2 + \sum_{i=1}^{m_1+k_0} (1 + m_2 + i) + \sum_{i=k_0+1}^{m_0} (m_1 + i) + \sum_{k_2=1}^{m_2} (m_0 - k_0 + k_2) = \binom{2+\ell}{\ell}.$$

- Total b_μ in the form $u_{\sigma^3(\lambda)} \otimes (m_0, m_1-k_1, m_2+k_1) \otimes b_\mu \otimes \mathcal{B}_\ell$:

$$m_1-(k_1-1)+m_0+\sum_{k_0=1}^{m_0}(1+m_0+m_1-k_0-k_1)+\sum_{c_{2,1}=1}^{m_2+k_1}\left(1+m_0+c_{2,1}\right)+\sum_{i=k_1+1}^{m_1}(m_2+i)=\binom{2+\ell}{\ell}.$$

- Total b_μ in the form $u_{\sigma^3(\lambda)} \otimes \left(m_0-k_0, m_1+k_0-c_{1,0}, m_2+c_{1,0}\right) \otimes b_\mu \otimes \mathcal{B}_\ell$:

$$\sum_{t=0}^{\ell-c'_{1,0}}(t+1)+\sum_{c'_{1,0}=0}^{m_1+k_0-i-1}\sum_{t=0}^{\ell-c'_{1,0}}(1)=\binom{2+\ell}{\ell}.$$

Therefore, we see that

$$\begin{aligned}B_{w(3,0)}(\lambda)=\big\{&u_{\sigma^2(\lambda)}\otimes\mathcal{B}_\ell,\\ &u_{\sigma^3(\lambda)}\otimes(m_0-k_0,m_1+k_0,m_2)\otimes\mathcal{B}_\ell,\\ &u_{\sigma^3(\lambda)}\otimes(m_0,m_1-k_1,m_2+k_1)\otimes\mathcal{B}_\ell,\\ &u_{\sigma^3(\lambda)}\otimes\left(m_0-k_0,m_1+k_0-c_{1,0},m_2+c_{1,0}\right)\otimes\mathcal{B}_\ell \mid\\ &k_t=1,2,\ldots,m_t,\ \ t=0,1,\ \ c_{1,0}=1,2,\ldots,m_1+k_0\big\}.\end{aligned}$$

Now, suppose we have explicit descriptions of $B_{w(L,0)}(\lambda)$, for $L\equiv 1 \bmod 3$, $L\equiv 2 \bmod 3$, or $L\equiv 0 \bmod 3$. We want to show that our explicit descriptions hold for $B_{w(L+1,0)}(\lambda)$.

First, let's suppose $L\equiv 1 \bmod 3$. Then $L+1=2 \bmod 3$. Note that

$$w(L+1,0)=(r_{1-L}\,r_{-L})\,w(L,0)=(r_0\,r_2)\,w(L,0).$$

Hence we only need to apply $\tilde{f}_2$ and $\tilde{f}_0$ to certain paths in the set

$$\begin{aligned}B_{w(L,0)}(\lambda)=\big\{&u_\lambda\otimes\mathcal{B}_\ell^{L-1},\\ &u_{\sigma(\lambda)}\otimes(m_1+k_0,m_2,m_0-k_0)\otimes\mathcal{B}_\ell^{L-1},\\ &u_{\sigma(\lambda)}\otimes(m_1-k_1,m_2+k_1,m_0)\otimes\mathcal{B}_\ell^{L-1},\\ &u_{\sigma(\lambda)}\otimes\left(m_1+k_0-c_{1,0},m_2+c_{1,0},m_0-k_0\right)\otimes\mathcal{B}_\ell^{L-1}\big\}\end{aligned}$$

The only difference between this case and the case when $L=2$ is the rightmost $L-1$ components, $\mathcal{B}_\ell^{L-1}$, in the tensor product. So, by the same process as in the case when $L=2$ and by Lemma 5.5, we have

$$\begin{aligned}B_{w(L+1,0)}(\lambda) = \{ & u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell^L, \\ & u_{\sigma^2(\lambda)} \otimes (m_2, m_0 - k_0, m_1 + k_0) \otimes \mathcal{B}_\ell^L, \\ & u_{\sigma^2(\lambda)} \otimes (m_2 + k_1, m_0, m_1 - k_1) \otimes \mathcal{B}_\ell^L, \\ & u_{\sigma^2(\lambda)} \otimes \left(m_2 + c_{1,0}, m_0 - k_0, m_1 + k_0 - c_{1,0}\right) \otimes \mathcal{B}_\ell^L \}\end{aligned}$$

Next, let's suppose that $L = 2 \bmod 3$. Then $L+1 = 0 \bmod 3$. Note that

$$w(L+1,0) = (r_{1-L}\, r_{-L})\, w(L,0) = (r_2\, r_1)\, w(L,0).$$

Thus, we only need to apply $\tilde{f}_1$ and $\tilde{f}_2$ to certain paths in the set $B_{w(L,0)}(\lambda)$. But this case is exactly the same as when $L = 3$, except for the rightmost $L-1$ components, $\mathcal{B}_\ell^{L-1}$, in the tensor product. So, by Lemma 5.5 and by using the same process as in the case when $L = 3$, we have

$$\begin{aligned}B_{w(L+1,0)}(\lambda) = \{ & u_{\sigma^2(\lambda)} \otimes \mathcal{B}_\ell^L, \\ & u_\lambda \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes \mathcal{B}_\ell^L, \\ & u_\lambda \otimes (m_0, m_1 - k_1, m_2 + k_1) \otimes \mathcal{B}_\ell^L, \\ & u_\lambda \otimes \left(m_0 - k_0, m_1 + k_0 - c_{1,0}, m_2 + c_{1,0},\right) \otimes \mathcal{B}_\ell^L \}\end{aligned}$$

Finally, let's suppose that $L = 0 \bmod 3$. Then $L+1 = 1 \bmod 3$. Note that

$$w(L+1,0) = (r_{1-L}\, r_{-L})\, w(L,0) = (r_1\, r_0)\, w(L,0).$$

and hence we need only apply $\tilde{f}_0$ and $\tilde{f}_1$ to certain paths in the set $B_{w(L,0)}(\lambda)$. Let's apply $\tilde{f}_0$ to certain paths in each of these sets:

Case 5.15. Let $\tilde{f}_0$ act on certain paths in the set $u_{\sigma^2(\lambda)} \otimes \mathcal{B}_\ell^{L-1}$. The (0)-signature is of the form $\left(+_{m_2}, *_1, \ldots, *_{L-1}\right)$. By Lemma 5.5, this produces the set $\left\{u_\lambda \otimes (m_0 + k_2, m_1, m_2 - k_2) \otimes \mathcal{B}_\ell^{L-1}\right\}$.

Case 5.16. Now let $\tilde{f}_0$ act on certain paths in the set $u_\lambda \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes \mathcal{B}_\ell^{L-1}$. We have the (0)-signature of the form $\left(+_{k_0}, +_{m_2}, *_1, \ldots, *_{L-1}\right)$. By Lemma 5.5, this produces the paths

$$\begin{aligned}\{ & u_{\sigma(\lambda)} \otimes \left(m_1 + k_0', m_2, m_0 - k_0'\right) \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes \mathcal{B}_\ell^{L-1}, \\ & u_{\sigma(\lambda)} \otimes (m_1 + k_0, m_2, m_0 - k_0) \otimes (m_0 - k_0 + k_2, m_1 + k_0, m_2 - k_2) \otimes \mathcal{B}_\ell^{L-1}, \}\end{aligned}$$

where $k_0' = 1, 2, \ldots, k_0$.

Case 5.17. Next, let $\tilde{f}_0$ act on certain paths in the set $u_\lambda \otimes (m_0, m_1 - k_1, m_2 + k_1) \otimes \mathcal{B}_\ell^{L-1}$. The (0)-signature is of the form $\left(+_{m_2+k_1}, \ast_1, \ldots, \ast_{L-1}\right)$. Lemma 5.5 implies that we have the set

$$u_\lambda \otimes \left(m_0 + c_{2,1}, m_1 - k_1, m_2 + k_1 - c_{2,1}\right) \otimes \mathcal{B}_\ell^{L-1},$$

where $c_{2,1} = 1, 2, \ldots, m_2 + k_1$.

Case 5.18. Finally, let $\tilde{f}_0$ act on certain paths in the set $u_\lambda \otimes \left(m_0 - k_0, m_1 + k_1 - c_{1,0}, m_2 + c_{1,0}\right) \otimes \mathcal{B}_\ell^{L-1}$. The (0)-signature is of the form $\left(+_{k_0}, +_{m_2+c_{1,0}}, \ast_1, \ldots, \ast_{L-1}\right)$. By Lemma 5.5, we have

$$\begin{aligned}&\left\{u_{\sigma(\lambda)} \otimes \left(m_1 + k_0', m_2, m_0 - k_0'\right) \otimes \left(m_0 - k_0, m_1 + k_0 - c_{1,0}, m_2 + c_{1,0}\right) \otimes \mathcal{B}_\ell^{L-1},\right.\\ &\left.u_{\sigma(\lambda)} \otimes (m_1 + k_0, m_2, m_0 - k_0) \otimes \left(m_0 - k_0 + c_{2;1,0}', m_1 + k_0 - c_{1,0}, m_2 + c_{1,0} - c_{2;1,0}'\right) \otimes \mathcal{B}_\ell^{L-1}\right\},\end{aligned}$$

where $c_{2;1,0}' = 1, 2, \ldots, m_2 + c_{1,0}$.

Currently, we have the following:

- Total number of b_μ in the form $u_\lambda \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$:

$$m_2 + \sum_{k_1=1}^{m_1} (m_2 + k_1) + 1 + m_0 + m_1 + \sum_{i=1}^{m_0} (i + m_1).$$

- Total number of b_μ in the form $u_{\sigma(\lambda)} \otimes (m_1 + k_0, m_2, m_0 - k_0) \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$:

$$(m_0 - (k_0 - 1)) + m_2 + \sum_{i=1}^{m_1+k_0} (1 + m_2 + i) + \sum_{i=k_0+1}^{m_0} (m_1 + i).$$

Next, let's apply $\tilde{f}_1$ to certain paths in $B_{w(L,0)}(\lambda)$:

Case 5.19. Let $\tilde{f}_1$ act on certain paths in the set $u_{\sigma^2(\lambda)} \otimes \mathcal{B}_\ell^{L-1}$. We have the (1)-signature of the form $\left(+_{m_0}, \ast_1, \ldots, \ast_{L-1}\right)$. By Lemma 5.5, this action produces the set $u_\lambda \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes \mathcal{B}_\ell^{L-1}$, which we already have.

Case 5.20. Letting $\tilde{f}_1$ act on any path in the set $u_\lambda \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes \mathcal{B}_\ell^{L-1}$ produces no new paths.

Case 5.21. Let $\tilde{f}_1$ act on certain paths in the set $u_\lambda \otimes (m_0, m_1 - k_1, m_2 + k_1) \otimes \mathcal{B}_\ell^{L-1}$. The (1)-signature is of the form $\left(+_{k_1}, +_{m_0}, \ast_1, \ldots, \ast_{L-1}\right)$. By Lemma 5.5, we have the following paths:

$$\left\{u_{\sigma(\lambda)} \otimes \left(m_1 - k_1', m_2 + k_1', m_0\right) \otimes (m_0, m_1 - k_1, m_2 + k_1) \otimes \mathcal{B}_\ell^{L-1},\right.$$

$$u_{\sigma(\lambda)} \otimes (m_1+k_1, m_2+k_1, m_0) \otimes (m_0-k_0, m_1-k_1+k_0, m_2+k_1) \otimes \mathcal{B}_\ell^{L-1}\big\}.$$

Case 5.22. Let $\tilde{f}_1$ act on certain paths in the set $u_\lambda \otimes \left(m_0-k_0, m_1+k_0-c_{1,0}, m_2+c_{1,0}\right) \otimes \mathcal{B}_\ell^{L-1}$. The (1)-signature is of the form

$$\left(\ldots, -_{m_0} +_{m_2}, -_{m_2} +_{m_1}, -_{m_1+k_0-c_{1,0}} +_{m_0-k_0}, *_1, \ldots, *_{L-1}\right),$$

which reduces to

$$\begin{aligned} &\begin{cases} \left(+_{c_{1,0}-k_0}, +_{m_0-k_0}, *_1, \ldots, *_{L-1}\right) & \text{if } c_{1,0} > k_0 \\ \left(+_{m_0-k_0}, *_1, \ldots, *_{L-1}\right) & \text{if } c_{1,0} \le k_0 \end{cases} \\ =&\begin{cases} \left(+_{k_1}, +_{m_0-k_0}, *_1, \ldots, *_{L-1}\right) & \text{if } c_{1,0} > k_0 \\ \left(+_{m_0-k_0}, *_1, \ldots, *_{L-1}\right) & \text{if } c_{1,0} \le k_0. \end{cases} \end{aligned}$$

Subcase 5.22.1. Suppose $c_{1,0} > k_0$ and let $k_1 = c_{1,0} - k_0$. Then, by Lemma 5.5, this action produces the following paths:

$$\begin{aligned} &\big\{u_{\sigma(\lambda)} \otimes \left(m_1-k_1', m_2+k_1', m_0\right) \otimes \left(m_0-k_0, m_1+k_0-c_{1,0}, m_2+c_{1,0}\right) \otimes \mathcal{B}_\ell^{L-1}, \\ &u_{\sigma(\lambda)} \otimes (m_1-k_1, m_2+k_1, m_0) \otimes \left(m_0-k_0-c_{0,-0}, m_1+k_0-c_{1,0}+c_{0,-0}, m_2+c_{1,0}\right) \otimes \mathcal{B}_\ell^{L-1}\big\}, \end{aligned}$$

where $c_{0,-0} = 1, 2, \ldots, m_0-k_0$.

Subcase 5.22.2. Suppose $c_{1,0} \le k_0$. Then the (1)-signature is $\left(+_{m_0-k_0}\right)$. Hence, by Lemma 5.5, we have the paths $u_\lambda \otimes \left(m_0-k_0-c_{0,-0}, m_1+k_0-c_{1,0}+c_{0,-0}, m_2+c_{1,0}\right) \otimes \mathcal{B}_\ell^{L-1}$, which are the same as $u_\lambda \otimes \left(m_0-k_0, m_1+k_0-c_{1,0}, m_2+c_{1,0}\right) \otimes \mathcal{B}_\ell^{L-1}$. But we've already obtained this paths.

Now, we currently have the following paths:

- Total number of b_μ in the form $u_\lambda \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$:

$$m_2 + \sum_{k_1=1}^{m_1} (m_2+k_1) + 1 + m_0 + m_1 + \sum_{i=1}^{m_0} (i+m_1).$$

- Total number of b_μ in the form $u_{\sigma(\lambda)} \otimes (m_1+k_0, m_2, m_0-k_0) \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$:

$$(m_0-(k_0-1)) + m_2 + \sum_{i=1}^{m_1+k_0} (1+m_2+i) + \sum_{i=k_0+1}^{m_0} (m_1+i).$$

- Total number of b_μ in the form $u_{\sigma(\lambda)} \otimes (m_1 - k_1, m_2 + k_1, m_0) \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$:

$$(m_1 - (k_1 - 1)) + m_0 + \sum_{k_0=1}^{m_0} (1 + m_0 - k_0 + m_1 - k_1).$$

Finally, let's apply $\tilde{f}_1$ to certain paths produced when applying $\tilde{f}_0$ to $B_{w(L,0)}(\lambda)$:

Case 5.23. Let $\tilde{f}_1$ act on certain paths in the set $u_\lambda \otimes (m_0 + k_2, m_1, m_2 - k_2) \otimes \mathcal{B}_\ell^{L-1}$. The (1)-signature is of the form

$$\left(+_{m_0+k_2}, *_1, \ldots, *_{L-1}\right).$$

By Lemma 5.5, we have $u_\lambda \otimes \left(m_0 + k_2 - c_{0,2}, m_1 + c_{0,2}, m_2 - k_2\right) \otimes \mathcal{B}_\ell^{L-1}$.

Case 5.24. Let $\tilde{f}_1$ act on certain paths in the following sets:

$$\begin{cases} u_{\sigma(\lambda)} \otimes (m_1 + k_0, m_2, m_0 - k_0) \otimes (m_0 - k_0, m_1 + k_0, m_2) \otimes \mathcal{B}_\ell^{L-1} & [1] \\ u_{\sigma(\lambda)} \otimes (m_1 + k_0, m_2, m_0 - k_0) \otimes (m_0 - k_0 + k_2, m_1 + k_0, m_2 - k_2) \otimes \mathcal{B}_\ell^{L-2} & [2] \\ u_{\sigma(\lambda)} \otimes (m_1 + k_0, m_2, m_0 - k_0) \otimes \left(m_0 - k_0 - c_{0,-0}, m_0 + k_0 + c_{0,-0}, m_2\right) \otimes \mathcal{B}_\ell^{L-2}. & [3] \end{cases}$$

Subcase 5.24.1. Let $\tilde{f}_1$ act on certain paths in [1]. The (1)-signature is of the form

$$\left(+_{m_0-k_0}, *_1, \ldots, *_{L-1}\right).$$

By Lemma 5.5, we have

$$u_{\sigma(\lambda)} \otimes (m_1 + k_0, m_2, m_0 - k_0) \otimes \left(m_0 - k_0 - c_{0,-0}, m_1 + k_0 + c_{0,-0}, m_2\right) \otimes \mathcal{B}_\ell^{L-1},$$

which is the same set of paths as in [3].

Subcase 5.24.2. Next let $\tilde{f}_1$ act on certain paths in [2]. The (1)-signature is of the form

$$\left(+_{m_0-k_0+k_2}, *_1, \ldots, *_{L-1}\right).$$

By Lemma 5.5, we have

$$\begin{aligned}&\left\{u_{\sigma(\lambda)} \otimes (m_1 + k_0, m_2, m_0 - k_0) \otimes (m_0 - k_0 + k_2 - 1, m_1 + k_0 + 1, m_2 - k_2) \otimes \mathcal{B}_\ell^{L-1},\right.\\ &u_{\sigma(\lambda)} \otimes (m_1 + k_0, m_2, m_0 - k_0) \otimes (m_0 - k_0 + k_2 - 2, m_1 + k_0 + 2, m_2 - k_2) \otimes \mathcal{B}_\ell^{L-1}, \ldots,\\ &\left.u_{\sigma(\lambda)} \otimes (m_1 + k_0, m_2, m_0 - k_0) \otimes (0, m_1 + m_0 + k_2, m_2 - k_2) \otimes \mathcal{B}_\ell^{L-1}\right\}.\end{aligned}$$

Subcase 5.24.3. Finally let $\tilde{f}_1$ act on certain paths in [3]. The (1)-signature is of the form

$$\left(+_{m_0-k_0-c_{0,-0}}, *_1, \ldots, *_{L-1}\right).$$

But by Lemma 5.5, this action will also produce the same paths as in [3].

Case 5.25. Next let $\tilde{f}_1$ act on certain paths in the set $u_\lambda \otimes \left(m_0+c_{2,1}, m_1-k_1, m_2+k_1-c_{2,1}\right) \otimes \mathcal{B}_\ell^{L-1}$. The (1)-signature is of the form $\left(+_{k_1}, +_{m_0+c_{2,1}}, *_1, \ldots, *_{L-1}\right)$. By Lemma 5.5, this action produces the following paths:

$$\begin{aligned}&\left\{u_{\sigma(\lambda)} \otimes \left(m_1-k_1', m_2+k_1', m_0\right) \otimes \left(m_0+c_{2,1}, m_1-k_1, m_2+k_1-c_{2,1}\right) \otimes \mathcal{B}_\ell^{L-1},\right.\\ &\left. u_{\sigma(\lambda)} \otimes (m_1-k_1, m_2+k_1, m_0) \otimes \left(m_0+c_{2,1}-c_{0;2,1}', m_1-k_1+c_{0;2,1}', m_2+k_1-c_{2,1}\right) \otimes \mathcal{B}_\ell^{L-1}\right\},\end{aligned}$$

where $c_{0;2,1}' = 1, 2, \ldots, m_0+c_{2,1}$.

Case 5.26. Finally, let $\tilde{f}_1$ act on certain paths in the following sets:

$$\begin{cases} u_{\sigma(\lambda)} \otimes (m_1+k_0, m_2, m_0-k_0) \otimes \left(i-c_{1,0}', c_{1,0}', \ell-i\right) \otimes \mathcal{B}_\ell^{L-1} & [1] \\ u_{\sigma(\lambda)} \otimes (m_1+k_0, m_2, m_0-k_0) \otimes \left(c_{0,-0}-t, \ell-m_2-c_{0,-0}, m_2+t\right) \otimes \mathcal{B}_\ell^{L-2} & [2] \end{cases}$$

where $c_{1,0}' = 0, 1, \ldots, m_1+k_0-1$, $i = c_{1,0}', c_{1,0}'+1, \ldots, \ell$, $c_{0,-0} = 1, 2, \ldots, m_0-k_0$, and $t = 1, 2, \ldots, c_{0,-0}$.

Subcase 5.26.1. Let $\tilde{f}_1$ act on certain paths in [1]. The (1)-signature is of the form

$$\begin{aligned}&\left(\ldots, -_{m_0} +_{m_2}, -_{m_2} +_{m_1+k_0}, -_{c_{1,0}'} +_{i-c_{1,0}'}, *_1, \ldots, *_{L-1}\right)\\ =&\left(+_{m_1+k_0}, -_{c_{1,0}'} +_{i-c_{1,0}'}, *_1, \ldots, *_{L-1}\right)\\ =&\left(+_{c_{1,0}}, +_{i-c_{1,0}'}, *_1, \ldots, *_{L-1}\right)\\ =&\left(+_{c_{1,0}}, +_z, *_1, \ldots, *_{L-1}\right),\end{aligned}$$

where $z = 0, 1, \ldots, \ell - c_{1,0}'$. By Lemma 5.5, we have the set of paths

$$u_{\sigma(\lambda)} \otimes \left(m_1+k_0-c_{1,0}', m_2+c_{1,0}', m_0-k_0\right) \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$$

for specific $b_\mu \in \mathcal{B}_\ell$.

Subcase 5.26.2. Letting $\tilde{f}_1$ act on certain paths in [2], we have (1)-signature of the form

$$\left(\ldots, -_{m_0} +_{m_2}, -_{m_2} +_{m_1+k_0}, -_{\ell-m_2-c_{0,-0}} +_{c_{0,-0}-t}, *_1, \ldots, *_{L-1}\right)$$

$$\begin{aligned}
&=\left(+_{m_1+k_0}, -_{\ell-m_2-c_{0,-0}} +_{c_{0,-0}-t}, *_1,\dots,*_{L-1}\right)\\
&=\left(+_{m_1+k_0}, -_{m_1+m_0-c_{0,-0}}, +_{c_{0,-0}-t} *_1,\dots,*_{L-1}\right)\\
&=\left(+_{c_{0,-0}-t}, *_1,\dots,*_{L-1}\right).
\end{aligned}$$

Hence, no new paths are produced in this case.

Thus, we have obtained the following paths:

- Total number of b_μ in the form $u_\lambda \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$:

$$m_2+\sum_{k_1=1}^{m_1}(m_2+k_1)+1+m_0+m_1+\sum_{i=1}^{m_0}(i+m_1)+\sum_{k_2=1}^{m_2}(m_0+k_2)=\binom{2+\ell}{\ell}.$$

- Total number of b_μ in the form $u_{\sigma(\lambda)} \otimes (m_1+k_0, m_2, m_0-k_0) \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$:

$$(m_0-(k_0-1))+m_2+\sum_{i=1}^{m_1+k_0}(1+m_2+i)+\sum_{i=k_0+1}^{m_0}(m_1+i)+\sum_{k_2=1}^{m_2}(m_0-k_0+k_2)=\binom{2+\ell}{\ell}.$$

- Total number of b_μ in the form $u_{\sigma(\lambda)} \otimes (m_1-k_1, m_2+k_1, m_0) \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$:

$$(m_1-(k_1-1))+m_0+\sum_{k_0=1}^{m_0}(1+m_0-k_0+m_1-k_1)+\sum_{c_{2,1}=1}^{m_2+k_1}\left(1+m_0+c_{2,1}\right)+\sum_{i=k_1+1}^{m_1}(m_2+i)=\binom{2+\ell}{\ell}.$$

- Total number of b_μ in the form $u_{\sigma(\lambda)} \otimes \left(m_1+k_0-c_{1,0}, m_2+c_{1,0}, m_0-k_0\right) \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$:

$$\sum_{t=0}^{\ell-c'_{1,0}}(t+1)+\sum_{c'_{1,0}=0}^{m_1+k_0-t-1}\sum_{t=0}^{\ell-c'_{1,0}}1=\binom{2+\ell}{\ell}.$$

□

The next Corollary follows directly from Lemma 5.7.

Corollary 5.8. *Let $\lambda = m_0\Lambda_0 + m_1\Lambda_1 + m_2\Lambda_2 \in \bar{P}^+$, where $\lambda(c) = \ell$. Then the Demazure crystal $B_{w(L,j)}(\lambda)$ for $U_q(A_2^{(1)})$, where $j = 0,1,2$, can be explicitly defined by one of the following descriptions:*

1. *If $L \equiv 1 \bmod 3$, then*

$$B_{w(L,0)}(\lambda)=\left\{u_\lambda \otimes \mathcal{B}_\ell^{L-1},\right.$$

$$
\begin{aligned}
&u_{\sigma(\lambda)} \otimes (m_1+k_0, m_2, m_0-k_0) \otimes \mathcal{B}_\ell^{L-1},\\
&u_{\sigma(\lambda)} \otimes (m_1-k_1, m_2+k_1, m_0) \otimes \mathcal{B}_\ell^{L-1},\\
&u_{\sigma(\lambda)} \otimes \left(m_1+k_0-c_{1,0}, m_2+c_{1,0}, m_0-k_0\right) \otimes \mathcal{B}_\ell^{L-1} \mid\\
&k_t = 1,2,\ldots,m_t, \quad t=0,1, \quad c_{1,0}=1,2,\ldots,m_1+k_0\big\},
\end{aligned}
$$

$$
\begin{aligned}
B_{w(L,1)}(\lambda) = \big\{ &u_\lambda \otimes \mathcal{B}_\ell^{L-1},\\
&u_{\sigma(\lambda)} \otimes (m_1-k_1, m_2+k_1, m_0) \otimes \mathcal{B}_\ell^{L-1},\\
&u_{\sigma(\lambda)} \otimes (m_1, m_2-k_2, m_0+k_2) \otimes \mathcal{B}_\ell^{L-1},\\
&u_{\sigma(\lambda)} \otimes \left(m_1-k_1, m_2+k_1-c_{2,1}, m_0+c_{2,1}\right) \otimes \mathcal{B}_\ell^{L-1} \mid\\
&k_t = 1,2,\ldots,m_t, \quad t=1,2, \quad c_{2,1}=1,2,\ldots,m_2+k_1\big\},
\end{aligned}
$$

$$
\begin{aligned}
B_{w(L,2)}(\lambda) = \big\{ &u_\lambda \otimes \mathcal{B}_\ell^{L-1},\\
&u_{\sigma(\lambda)} \otimes (m_1, m_2-k_2, m_0+k_2) \otimes \mathcal{B}_\ell^{L-1},\\
&u_{\sigma(\lambda)} \otimes (m_1+k_0, m_2, m_0-k_0) \otimes \mathcal{B}_\ell^{L-1},\\
&u_{\sigma(\lambda)} \otimes \left(m_1+c_{0,2}, m_2-k_2, m_0+k_2-c_{0,2}\right) \otimes \mathcal{B}_\ell^{L-1} \mid\\
&k_t = 1,2,\ldots,m_t, \quad t=0,2, \quad c_{0,2}=1,2,\ldots,m_0+k_2\big\}.
\end{aligned}
$$

2. *If* $L \equiv 2 \bmod 3$, *then*

$$
\begin{aligned}
B_{w(L,0)}(\lambda) = \big\{ &u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell^{L-1},\\
&u_{\sigma^2(\lambda)} \otimes (m_2, m_0-k_0, m_1+k_0) \otimes \mathcal{B}_\ell^{L-1},\\
&u_{\sigma^2(\lambda)} \otimes (m_2+k_1, m_0, m_1-k_1) \otimes \mathcal{B}_\ell^{L-1},\\
&u_{\sigma^2(\lambda)} \otimes \left(m_2+c_{1,0}, m_0-k_0, m_1+k_0-c_{1,0}\right) \otimes \mathcal{B}_\ell^{L-1} \mid\\
&k_t = 1,2,\ldots,m_t, \quad t=0,1, \quad c_{1,0}=1,2,\ldots,m_1+k_0\big\},
\end{aligned}
$$

$$
\begin{aligned}
B_{w(L,1)}(\lambda) = \big\{ &u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell^{L-1},\\
&u_{\sigma^2(\lambda)} \otimes (m_2+k_1, m_0, m_1-k_1) \otimes \mathcal{B}_\ell^{L-1},\\
&u_{\sigma^2(\lambda)} \otimes (m_2-k_2, m_0+k_2, m_1) \otimes \mathcal{B}_\ell^{L-1},\\
&u_{\sigma^2(\lambda)} \otimes \left(m_2+k_1-c_{2,1}, m_0+c_{2,1}, m_1-k_1\right) \otimes \mathcal{B}_\ell^{L-1} \mid
\end{aligned}
$$

$$k_t = 1,2,\ldots,m_t,\quad t=1,2,\quad c_{2,1}=1,2,\ldots,m_2+k_1\big\},$$

$$\begin{aligned}
B_{w(L,2)}(\lambda) = \big\{ & u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\sigma^2(\lambda)} \otimes (m_2-k_2, m_0+k_2, m_1) \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\sigma^2(\lambda)} \otimes (m_2, m_0-k_0, m_1+k_0) \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\sigma^2(\lambda)} \otimes \big(m_2-k_2, m_0+k_2-c_{0,2}, m_1+c_{0,2}\big) \otimes \mathcal{B}_\ell^{L-1} \mid\\
& k_t = 1,2,\ldots,m_t,\quad t=0,2,\quad c_{0,2}=1,2,\ldots,m_0+k_2\big\}.
\end{aligned}$$

3. If $L \equiv 0 \bmod 3$, then

$$\begin{aligned}
B_{w(L,0)}(\lambda) = \big\{ & u_{\sigma^2(\lambda)} \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\lambda} \otimes (m_0-k_0, m_1+k_0, m_2) \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\lambda} \otimes (m_0, m_1-k_1, m_2+k_1) \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\lambda} \otimes \big(m_0-k_0, m_1+k_0-c_{1,0}, m_2+c_{1,0}\big) \otimes \mathcal{B}_\ell^{L-1} \mid\\
& k_t = 1,2,\ldots,m_t,\quad t=0,1,\quad c_{1,0}=1,2,\ldots,m_1+k_0\big\},
\end{aligned}$$

$$\begin{aligned}
B_{w(L,1)}(\lambda) = \big\{ & u_{\sigma^2(\lambda)} \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\lambda} \otimes (m_0, m_1-k_1, m_2+k_1) \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\lambda} \otimes (m_0+k_2, m_1, m_2-k_2) \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\lambda} \otimes \big(m_0+c_{2,1}, m_1-k_1, m_2+k_1-c_{2,1}\big) \otimes \mathcal{B}_\ell^{L-1} \mid\\
& k_t = 1,2,\ldots,m_t,\quad t=1,2,\quad c_{2,1}=1,2,\ldots,m_2+k_1\big\},
\end{aligned}$$

$$\begin{aligned}
B_{w(L,2)}(\lambda) = \big\{ & u_{\sigma^2(\lambda)} \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\lambda} \otimes (m_0+k_2, m_1, m_2-k_2) \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\lambda} \otimes (m_0-k_0, m_1+k_0, m_2) \otimes \mathcal{B}_\ell^{L-1},\\
& u_{\lambda} \otimes \big(m_0+k_2-c_{0,2}, m_1+c_{0,2}, m_2-k_2\big) \otimes \mathcal{B}_\ell^{L-1} \mid\\
& k_t = 1,2,\ldots,m_t,\quad t=0,2,\quad c_{0,2}=1,2,\ldots,m_0+k_2\big\}.
\end{aligned}$$

Corollary 5.9. *For $\lambda = m_0\Lambda_0 + m_1\Lambda_1 + m_2\Lambda_2 \in \bar{P}^+$, where $\lambda(c)=\ell$, we have the following property for the Demazure crystals $B_{w(L,j)}(\lambda)$ for $U_q(A_2^{(1)})$:*

1. $\bigcup_{j\geq 0} B_{w(L,j)}(\lambda) = u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell^L$.
2. $\bigcap_{j\geq 0} B_{w(L,j)}(\lambda) = u_\lambda \otimes \mathcal{B}_\ell^{L-1}$.

Proof. As in the $\lambda = \ell\Lambda_j$ case, we note that $w(L,j) = w(L,j+n+1)$. Hence, we have:

$$\bigcup_{j\geq 0} B_{w(L,j)}(\lambda) = \bigcup_{j=0}^{n} B_{w(L,j)}(\lambda),$$
$$\bigcap_{j\geq 0} B_{w(L,j)}(\lambda) = \bigcap_{j=0}^{n} B_{w(L,j)}(\lambda).$$

The proof follows precisely from Corollary 5.8. Suppose $L \equiv 1 \bmod 3$. Then

$$\bigcap_{j=0}^{2} B_{w(L,j)}(\lambda) = B_{w(L,0)}(\lambda) \cap B_{w(L,1)}(\lambda) \cap B_{w(L,2)}(\lambda).$$

But the only sets of paths that $B_{w(L,0)}(\lambda)$ and $B_{w(L,1)}(\lambda)$ have in common are

$$\left\{u_{\sigma(\lambda)} \otimes (m_1 - k_1, m_2 + k_1, m_0) \otimes \mathcal{B}_\ell^{L-1}\right\}$$

and

$$u_\lambda \otimes \mathcal{B}_\ell^{L-1}.$$

Since $B_{w(L,2)}(\lambda)$ only contains $u_\lambda \otimes \mathcal{B}_\ell^{L-1}$ out of these two sets, the intersection

$$\bigcap_{j=0}^{2} B_{w(L,j)}(\lambda) = u_\lambda \otimes \mathcal{B}_\ell^{L-1}$$

for $L \equiv 1 \bmod 3$.

For the union, we have

$$\bigcup_{j=0}^{2} B_{w(L,j)}(\lambda) = B_{w(L,0)}(\lambda) \cup B_{w(L,1)}(\lambda) \cup B_{w(L,2)}(\lambda).$$

From $B_{w(L,0)}(\lambda)$, we have

$$1 + m_0 + m_1 + \sum_{k_0=1}^{m_0} (m_1 + k_0) = 1 + m_0 + m_1 + m_0 m_1 + \frac{1}{2} m_0 (m_0 + 1)$$

unique b_μ in the form $u_{\sigma(\lambda)} \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$. From $B_{w(L,1)}(\lambda)$, we add

$$m_2 + \sum_{k_1=1}^{m_1} (m_2 + k_1) = m_2 + m_1 m_2 + \frac{1}{2} m_1 (m_1 + 1)$$

new b_μ in the form $u_{\sigma(\lambda)} \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$. From $B_{w(L,2)}(\lambda)$, we add

$$\sum_{k_2=1}^{m_2} (m_0 + k_2) = m_2 m_0 + \frac{1}{2} m_2 (m_2 + 1)$$

new b_μ in the form $u_{\sigma(\lambda)} \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$. So, altogether we have a total of

$$\begin{aligned}
&= 1 + m_0 + m_1 + m_0 m_1 + \frac{1}{2} m_0 (m_0 + 1) + m_2 + m_1 m_2 + \frac{1}{2} m_1 (m_1 + 1) + m_2 m_0 + \frac{1}{2} m_2 (m_2 + 1) \\
&= 1 + \frac{3}{2} m_0 + \frac{3}{2} m_1 + \frac{3}{2} m_2 + m_0 m_1 + m_1 m_2 + m_2 m_0 + \frac{1}{2} m_0^2 + \frac{1}{2} m_1^2 + \frac{1}{2} m_2^2 \\
&= \binom{2+\ell}{\ell}
\end{aligned}$$

unique b_μ in the form $u_{\sigma(\lambda)} \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$. Thus,

$$\bigcup_{j=0}^{2} B_{w(L,j)}(\lambda) = u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell^{L}$$

for $L \equiv 1 \bmod 3$. Note that for the other two cases of L, we have a similar argument. $\square$

Chapter

6

$U_q\left(\hat{sl}(n+1,\mathbb{C})\right)$-Demazure Crystals

In this chapter, we describe a specific property for Demazure crystals $B_w(\lambda)$ of $U_q(A_n^{(1)})$, and give algorithms that help us find a sufficient Weyl group sequence for obtaining certain paths in this crystal.

Let $\mathfrak{g} = A_n^{(1)}$, let $\lambda = m_0\Lambda_0 + m_1\Lambda_1 + \cdots + m_n\Lambda_n \in \bar{P}^+$, where $m_0 + m_1 + \cdots + m_n = \ell$, and let $\mathcal{B}_\ell$ be the perfect crystal of level ℓ for $U_q(A_n^{(1)})$. Also recall that $m_0 = m_{n+1}$, and recall Equations 5.1-5.6 from Chapter 5.

Our goal is to build an algorithm that will provide a sequence of Kashiwara operators such that when applied to the ground-state path u_λ it produces $u_{\sigma(\lambda)} \otimes b_\nu$, for some $b_\nu \in \mathcal{B}_\ell$. We use the following two lemmas to help build such an algorithm.

Lemma 6.1. *We can obtain any λ-path of the form $u_{\sigma^{L+1}(\lambda)} \otimes b_\nu \otimes b_\mu \otimes p_\ell$, where*

$$b_\nu = (m_{L+1}, m_{L+2}, \ldots, m_{L+i-1}, v_i, v_{i+1}, m_{L+i+2}, \ldots, m_L)$$

such that $v_i < m_{L+i}$, b_μ is some element in $\mathcal{B}_\ell$, and $p_\ell \in \mathcal{B}_\ell^{L-1}$, by applying $\tilde{f}_i^{k_i}\tilde{f}_{i+1}^{k_{i+1}}$ to a path in $u_{\sigma^{L-1}(\lambda)} \otimes \mathcal{B}_\ell^{L-1}$ for sufficiently large k_i and k_{i+1}.

Proof. According to Equation 5.5, the set $u_{\sigma^{L-1}(\lambda)} \otimes \mathcal{B}_\ell^{L-1}$ has (i)-signature of the form

$$\left(+_{m_{i+L-1}}, *_1, *_2, \ldots, *_{L-1}\right).$$

Hence, by Lemma 5.5, letting $\tilde{f}_i$ act on certain λ-paths in the set $u_{\sigma^{L-1}(\lambda)} \otimes \mathcal{B}_\ell^{L-1}$ a sufficient number of times will produce the set $u_{\sigma^L(\lambda)} \otimes \tilde{f}_i\left(b_{\sigma^{L-1}(\lambda)}\right) \otimes \mathcal{B}_\ell^{L-1}$. However, we want $\tilde{f}_i$ to act on the $(L+1)^{st}$ component from the right, not the L^{th} component. If we look at the form of the (i)-signature for $u_{\sigma^{L-1}(\lambda)} \otimes \mathcal{B}_\ell^{L-1}$ more carefully, we see that we have

$$\left(\ldots, -_{m_{i+L+1}} +_{m_{i+L}}, -_{m_{i+L}} +_{m_{i+L-1}}, *_1, *_2, \ldots, *_{L-1}\right).$$

Now, in order to keep pluses in the $(L+1)^{st}$ component of this (i)-signature, we need there to be fewer than (m_{i+L}) minuses in the L^{th} component. The only way to do this is to let $\tilde{f}_{i+1}$ act on certain λ-paths in the set $u_{\sigma^{L-1}(\lambda)} \otimes \mathcal{B}_\ell^{L-1}$ a sufficient number of times to produce the set of λ-paths

$$u_{\sigma^L(\lambda)} \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}. \tag{6.1}$$

By Lemma 5.5, this is possible, since the $(i+1)$-signature on $u_{\sigma^{L-1}(\lambda)} \otimes \mathcal{B}_\ell^{L-1}$ is of the form

$$\left(+_{m_{i+L}}, *_1, *_2, \ldots, *_{L-1}\right).$$

Now, suppose that $b_\mu = \tilde{f}_{i+1}^{z_{i+1}}\left(b_{\sigma^{L-1}(\lambda)}\right)$. Then the (i)-signature on the set in Equation 6.1 is of the form

$$\begin{aligned}&\left(\ldots, -_{m_{i+L+2}} +_{m_{i+L+1}}, -_{m_{i+L+1}} +_{m_{i+L}}, -_{m_{i+L}-z_{i+1}} +_{m_{i+L-1}}, *_1, \ldots, *_{L-1}\right)\\ =&\left(+_{z_{i+1}}, +_{m_{i+L-1}}, *_1, \ldots, *_{L-1}\right).\end{aligned}$$

Hence, by Lemma 5.5, we can apply $\tilde{f}_i$ a sufficient number of times to certain λ-paths in the set in Equation 6.1 to obtain the desired set of λ-paths $u_{\sigma^{L+1}(\lambda)} \otimes b_\nu \otimes b_\mu \otimes \mathcal{B}_\ell^{L-1}$, where $b_\nu = (m_{L+1}, m_{L+2}, \ldots, m_{L+i-1}, v_i, v_{i+1}, m_{L+i+2}, \ldots, m_L)$, $v_i < m_{i+L}$. □

Lemma 6.2. *Choose $b_\nu = (m_{L+1}, m_{L+2}, \ldots, m_{L+i-1}, v_i, v_{i+1}, m_{L+i+2}, \ldots, m_L) \in \mathcal{B}_\ell$, where $v_i < m_{i+L-1}$ for some $i = 0, 1, \ldots, n$. We can obtain the λ-path $u_{\sigma^L(\lambda)} \otimes b_\nu \otimes b_{\mu_{L-1}} \otimes b_{\mu_{L-2}} \otimes \ldots \otimes b_{\mu_1}$, for $b_{\mu_p} = \tilde{f}_{i+L-p}^{z_{i+L-p}}\left(b_{\sigma^{p-1}(\lambda)}\right) \in \mathcal{B}_\ell$, for $p = 1, 2, \ldots, L-1$, by applying $\tilde{f}_i^{z_i} \tilde{f}_{i+1}^{z_{i+1}} \cdots \tilde{f}_{i+L-1}^{z_{i+L-1}}$, for some $z_i, z_{i+1}, \ldots, z_{i+L-1} \in \mathbb{Z}_{>0}$ such that $z_i \le z_{i+p}$, to the ground-state path $u_\lambda = u_{\sigma(\lambda)} \otimes (m_1, m_2, \ldots, m_0)$.*

Proof. We'll prove the lemma using induction on L. Let $L=2$. By Lemma 6.1, we know that

$$\tilde{f}_i^{z_i}\tilde{f}_{i+1}^{z_{i+1}}(u_\lambda)=u_{\sigma^2(\lambda)}\otimes b_\nu\otimes b_{\mu_{L-1}},$$

where $b_\nu=(m_3,m_4,\ldots,m_{i+1},v_i,v_{i+1},m_{i+4},\ldots,m_2)$, $v_i<m_{i+2}$, and $z_i\leq z_{i+1}$. Hence, we've proved the base case.

Note that $\tilde{f}_{i+1}^{z_{i+1}}(u_\lambda)=u_{\sigma(\lambda)}\otimes(m_1,m_2,\ldots,m_{i+1}-z_{i+1},m_{i+2}+z_{i+2},m_{i+3},\ldots,m_0)$, which has (i)-signature

$$(\ldots,-_{m_{i+3}}+_{m_{i+2}},-_{m_{i+2}}+_{m_{i+1}},-_{m_{i+1}-z_{i+1}}+_{m_i})=(+_{z_{i+1}},+_{m_i}).$$

And so, applying $\tilde{f}_i^{z_i}$ to the λ-path $\tilde{f}_{i+1}^{z_{i+1}}(u_\lambda)$ will only affect the second component from the right, which implies that $b_\nu=\tilde{f}_i^{z_i}(b_\lambda)$ and $b_{\mu_{L-1}}=\tilde{f}_{i+1}^{z_{i+1}}(b_{\sigma(\lambda)})$.

Now, let $z_i\leq z_{i+p}$, for $p=1,2,\ldots,L-2$. And suppose that

$$\tilde{f}_i^{z_i}\tilde{f}_{i+1}^{z_{i+1}}\cdots\tilde{f}_{i+L-2}^{z_{i+L-2}}(u_\lambda)=u_{\sigma^{L-1}(\lambda)}\otimes b_\nu\otimes b_{\mu_{L-2}}\otimes b_{\mu_{L-3}}\otimes\ldots\otimes b_{\mu_1},$$

where $b_{\mu_p}=\tilde{f}_{i+L-1-p}^{z_{i+L-1-p}}\big(b_{\sigma^{p-1}(\lambda)}\big)\in\mathcal{B}_\ell$, for $p=1,2,\ldots,L-2$. We want to show that

$$\tilde{f}_i^{z_i}\tilde{f}_{i+1}^{z_{i+1}}\cdots\tilde{f}_{i+L-1}^{z_{i+L-1}}(u_\lambda)=u_{\sigma^{L}(\lambda)}\otimes b_\nu\otimes b_{\mu_{L-1}}\otimes b_{\mu_{L-2}}\otimes\ldots\otimes b_{\mu_1},$$

where $b_{\mu_p}=\tilde{f}_{i+L-p}^{z_{i+L-p}}\big(b_{\sigma^{p-1}(\lambda)}\big)\in\mathcal{B}_\ell$, for $p=1,2,\ldots,L-1$. By the inductive hypothesis, we have

$$\tilde{f}_{i+1}^{z_{i+1}}\tilde{f}_{i+2}^{z_{i+2}}\cdots\tilde{f}_{i+L-1}^{z_{i+L-1}}(u_\lambda)=u_{\sigma^{L-1}(\lambda)}\otimes b_{\mu_{L-1}}\otimes b_{\mu_{L-2}}\otimes b_{\mu_{L-3}}\otimes\ldots\otimes b_{\mu_1},$$

where $b_{\mu_p}=\tilde{f}_{i+L-p}^{z_{i+L-p}}\big(b_{\sigma^{p-1}(\lambda)}\big)\in\mathcal{B}_\ell$, for $p=1,2,\ldots,L-1$.

Note that

$$\begin{aligned}b_{\mu_p}&=\tilde{f}_{i+L-p}^{z_{i+L-p}}\big(b_{\sigma^{p-1}(\lambda)}\big)\\&=\big(m_p,m_{p+1},\ldots,m_{i+L-1}-z_{i+L-p},m_{i+L}+z_{i+L-p},m_{i+L+1},\ldots,m_{p+n}\big).\end{aligned}$$

Hence, the (i)-signature for b_{μ_p} is $\big(-_{m_{p+i}}+_{m_{p-1+i}}\big)$, unless $p+i=i+L-1$ or $p+i=i+L$. Note that if $p+i=i+L-1$, then $p=L-1$. In this case, we have the element $b_{\mu_{L-1}}$, which has (i)-signature $\big(-_{m_{p+i}-z_{i+1}}+_{m_{p-1+i}}\big)$. If $p+i=i+L$, then $p=L$ and b_{μ_p} is not defined.

Thus, the λ-path $u_{\sigma^{L-1}(\lambda)} \otimes b_{\mu_{L-1}} \otimes b_{\mu_{L-2}} \otimes \ldots \otimes b_{\mu_1}$ has (i)-signature

$$\left(\ldots, -_{m_{i+L}} +_{m_{i+L-1}}, -_{m_{i+L-1}-z_{i+1}} +_{m_{i+L-2}}, -_{m_{i+L-2}} +_{m_{i+L-3}}, -_{m_{i+L-3}} +_{m_{i+L-4}}, \ldots, -_{m_{i+1}} +_{m_i}\right)$$

$$= \left(+_{z_{i+1}}, \bullet_{L-1}, \bullet_{L-2}, \ldots, \bullet_2, +_{m_i}\right).$$

So, we can apply $\tilde{f}_i^{z_i}$ to the λ-path $u_{\sigma^{L-1}(\lambda)} \otimes b_{\mu_{L-1}} \otimes b_{\mu_{L-2}} \otimes \ldots \otimes b_{\mu_1}$ to produce the λ-path

$$\begin{aligned} & u_{\sigma^L(\lambda)} \otimes \tilde{f}_i^{z_i}\left(b_{\sigma^{L-1}(\lambda)}\right) \otimes b_{\mu_{L-1}} \otimes b_{\mu_{L-2}} \otimes \ldots \otimes b_{\mu_1} \\ = \quad & u_{\sigma^L(\lambda)} \otimes b_\nu \otimes b_{\mu_{L-1}} \otimes b_{\mu_{L-2}} \otimes \ldots \otimes b_{\mu_1}. \end{aligned}$$

Therefore, $\tilde{f}_i^{z_i} \tilde{f}_{i+1}^{z_{i+1}} \cdots \tilde{f}_{i+L-1}^{z_{i+L-1}}(u_\lambda) = u_{\sigma^L(\lambda)} \otimes b_\nu \otimes b_{\mu_{L-1}} \otimes b_{\mu_{L-2}} \otimes \ldots \otimes b_{\mu_1}$, which is what we wanted to show. $\square$

Given the ground-state path $u_\lambda = u_{\sigma(\lambda)} \otimes b_\lambda = u_{\sigma(\lambda)} \otimes (m_1, m_2, \ldots, m_n, m_0)$, we want to obtain $u_{\sigma(\lambda)} \otimes b_\nu = u_{\sigma(\lambda)} \otimes (v_1, v_2, \ldots, v_n, v_0)$. The following algorithm provides a sequence of Kashiwara operators such that when applied to u_λ it produces $u_{\sigma(\lambda)} \otimes b_\nu$.

Algorithm 1: To construct the necessary sequence of Kashiwara operators, do the following:

1. Find all v_i such that $v_i > m_i$, $i = 0, 1, 2, \ldots, n$. Label them (in order of appearance in the $(n+1)$-tuple) as $v_{s_1}, v_{s_2}, \ldots, v_{s_r}$, where $1 < r < n+1$. Define $z_0 = 0$.

2. If $s_1 = 1$, then skip this step.
   Otherwise, we know that $v_t < m_t$ for all $t = 1, 2, \ldots, s_1 - 1$. So, recursively define $z_1 = m_1 - v_1$ and $z_t = z_{t-1} + m_t - v_t$ for $t = 2, 3, \ldots, s_1 - 1$. Hence, we'll need $\tilde{f}_{s_1-1}^{z_{s_1-1}} \cdots \tilde{f}_2^{z_2} \tilde{f}_1^{z_1}$ as part of the sequence.

3. Compute $d_1 = v_{s_1} - m_{s_1} - z_{s_1-1}$. Define

$$(d_1)_+ = \max\left\{v_{s_1} - m_{s_1} - z_{s_1-1}, 0\right\}$$

   and

$$(-d_1)_+ = \max\left\{m_{s_1} + z_{s_1-1} - v_{s_1}, 0\right\}.$$

   Let $z_{s_1} = (-d_1)_+$. The sequence becomes

$$\tilde{f}_{s_1}^{(-d_1)_+} \tilde{f}_{s_1-1}^{z_{s_1-1}+(d_1)_+} \cdots \tilde{f}_1^{z_1+(d_1)_+} \tilde{f}_0^{(d_1)_+}.$$

4. Do the following for $g = 2, \dots, r$.
   Let $z_t = z_{t-1} + m_t - v_t$ for $t = s_{g-1}+1, s_{g-1}+2, \dots, s_g - 1$. Compute $d_g = v_{s_g} - m_{s_g} - z_{s_g - 1}$. However, if $s_r = 0$, then set $d_r = 0$ and $z_{s_r} = 0$. Define

$$(d_g)_+ = \max\left\{v_{s_g} - m_{s_g} - z_{s_g-1}, 0\right\} \tag{6.2}$$

   and

$$(-d_g)_+ = \max\left\{m_{s_g} + z_{s_g-1} - v_{s_g}, 0\right\}. \tag{6.3}$$

   Let $z_{s_g} = (-d_g)_+$.

   Note that after repeating step 4 for all g, the sequence becomes

$$\begin{aligned}\tilde{f}_{s_r}^{(-d_r)_+}\tilde{f}_{s_r-1}^{z_{s_r-1}+(d_r)_+}\cdots\tilde{f}_{s_{r-1}+1}^{z_{s_{r-1}+1}+(d_r)_+}\tilde{f}_{s_{r-1}}^{(-d_{r-1})_++(d_r)_+}\tilde{f}_{s_{r-1}-1}^{z_{s_{r-1}-1}+(d_r)_++(d_{r-1})_+}\\ \cdots\tilde{f}_{s_{r-2}+1}^{z_{s_{r-2}+1}+(d_r)_++(d_{r-1})_+}\tilde{f}_{s_{r-2}}^{(-d_{r-2})_++(d_{r-1})_++(d_r)_+}\\ \cdots\\ \cdots\tilde{f}_{s_1+1}^{z_{s_1+1}+\sum_{i=2}^{r}(d_i)_+}\tilde{f}_{s_1}^{(-d_1)_++\sum_{i=2}^{r}(d_i)_+}\tilde{f}_{s_1-1}^{z_{s_1-1}+\sum_{i=1}^{r}(d_i)_+}\\ \cdots\tilde{f}_1^{z_1+\sum_{i=1}^{r}(d_i)_+}\tilde{f}_0^{\sum_{i=1}^{r}(d_i)_+}.\end{aligned}$$

   Let's rewrite the sequence as $\tilde{f}_{s_r}^{z'_{s_r}}\tilde{f}_{s_r-1}^{z'_{s_r-1}}\cdots\tilde{f}_1^{z'_1}\tilde{f}_0^{z'_0}$.

5. Next, let $z_t = z_{t-1} + m_t - v_t$ for $t = s_r+1, s_r+2, \dots, n$.
   Then we also need the following in the sequence:

$$\tilde{f}_n^{z_n}\tilde{f}_{n-1}^{z_n-1}\cdots\tilde{f}_{s_r+1}^{z_{s_r+1}}.$$

   Then the entire sequence becomes

$$\tilde{f}_n^{z_n}\tilde{f}_{n-1}^{z_n-1}\cdots\tilde{f}_{s_r+1}^{z_{s_r+1}}\tilde{f}_{s_r}^{z'_{s_r}}\tilde{f}_{s_r-1}^{z'_{s_r-1}}\cdots\tilde{f}_1^{z'_1}\tilde{f}_0^{z'_0},$$

   which we'll rewrite as

$$\tilde{f}_n^{z'_n}\tilde{f}_{n-1}^{z'_{n-1}}\cdots\tilde{f}_1^{z'_1}\tilde{f}_0^{z'_0},$$

   where

$$z'_t = z_t \qquad \text{for } t = s_r+1, s_r+2, \dots, n, \tag{6.4}$$

$$z'_t = z_t + \sum_{i=g+1}^{r} (d_i)_+ \qquad \text{for } t = s_g, \text{ where } g = 1,2,\dots,r, \tag{6.5}$$

$$z'_t = z_t + \sum_{i=g}^{r} (d_i)_+ \qquad \text{for } t = s_{g-1}+1, s_{g-1}+2,\dots, s_g - 1, \text{ where } g = 2,3,\dots,r, \tag{6.6}$$

$$z'_t = z_t + \sum_{i=1}^{r} (d_i)_+ \qquad \text{for } t = 0,1,\dots,s_1 - 1 \text{ and } z_0 = 0. \tag{6.7}$$

6. Finally, we want to rearrange the current sequence so that no $\tilde{f}_i$ occurs after $\tilde{f}_{i+1}$.
First, note that there exists some $t^* \in \{0,1,\dots,n\}$ such that $z'_{t^*} = 0$. How do we know this? Let's prove this by contradiction. Suppose that $z'_i \neq 0$ for all i, i.e. $z'_i > 0$ for all $i \in \{0,1,\dots,n\}$. Then since
$$z'_{s_g} = (-d_g)_+ = \max\left\{m_{s_g} + z_{s_g-1} - v_{s_g}, 0\right\}$$
we have that $m_{s_g} + z_{s_g-1} - v_{s_g} > 0$, which implies that $(d_g)_+ = 0$ for all g. This implies that
$$z'_0 = z_0 + \sum_{i=1}^{r} (d_i)_+ = 0 + \sum_{i=1}^{r} 0 = 0,$$
which is a contradiction. Hence, there exists a t^* such that $z'_{t^*} = 0$.
Then, the sequence becomes
$$\tilde{f}_n^{z'_n} \tilde{f}_{n-1}^{z'_{n-1}} \cdots \tilde{f}_{t^*+1}^{z'_{t^*+1}} \tilde{f}_{t^*-1}^{z'_{t^*-1}} \cdots \tilde{f}_1^{z'_1} \tilde{f}_0^{z'_0}$$
which we'll rearrange to
$$F_{t^*} = \tilde{f}_{t^*-1}^{z'_{t^*-1}} \cdots \tilde{f}_1^{z'_1} \tilde{f}_0^{z'_0} \tilde{f}_n^{z'_n} \tilde{f}_{n-1}^{z'_{n-1}} \cdots \tilde{f}_{t^*+1}^{z'_{t^*+1}}, \tag{6.8}$$
where $z'_{t^*+1} \neq 0$. When we apply this sequence of Kashiwara operators to u_λ, we will obtain $u_{\sigma(\lambda)} \otimes b_\nu$.

Example 6.3. Suppose $b_\lambda = (5,7,2,1,3,2)$ and $b_\nu = (2,8,1,2,1,6)$. We'll use the algorithm to construct a sequence that when acting on u_λ, produces $u_{\sigma(\lambda)} \otimes b_\nu$.

1. We first find all v_i such that $v_i > m_i$. Hence, we have
$$v_{s_1} = v_2 = 8 > 7 = m_2,$$
$$v_{s_2} = v_4 = 2 > 1 = m_4,$$

$$v_{s_3} = v_0 = 6 > 2 = m_0.$$

Also, let $z_0 = 0$.

2. Since $s_1 \neq 1$, we will complete this step. We have

$$z_1 = m_1 - v_1 = 5 - 2 = 3.$$

Hence, we currently need $\tilde{f}_1^3 \tilde{f}_0^0$ in the sequence.

3. Next we compute

$$d_1 = v_{s_1} - m_{s_1} - z_{s_1 - 1} = v_2 - m_2 - z_1 = -2 < 0.$$

This tells us that $(d_1)_+ = 0$ and $(-d_1)_+ = 2$. Thus, $z_2 = (-d_1)_+ = 2$ and our current sequence becomes $\tilde{f}_2^2 \tilde{f}_1^3 \tilde{f}_0^0$.

4. For $g = 2$, we compute

$$z_3 = z_2 + m_3 - v_3 = 3$$
$$d_2 = v_{s_2} - m_{s_2} - z_{s_2 - 1} = v_4 - m_4 - z_3 = -2 < 0.$$

Since $d_2 < 0$, we have $(d_2)_+ = 0$ and $(-d_2)_+ = 2$, which implies that $z_4 = (-d_2)_+ = 2$. The current sequence is now $\tilde{f}_4^2 \tilde{f}_3^3 \tilde{f}_2^2 \tilde{f}_1^3 \tilde{f}_0^0$.
For $g = 3$, we have

$$z_5 = z_4 + m_5 - v_5 = 4,$$
$$d_3 = 0 \text{ since } s_3 = 0,$$
$$z_0 = 0.$$

Now the needed sequence is $\tilde{f}_5^4 \tilde{f}_4^2 \tilde{f}_3^3 \tilde{f}_2^2 \tilde{f}_1^3 \tilde{f}_0^0$.

5. We can ignore step 5 because $z_{s_r} = 0$.

6. We just need to remove the $\tilde{f}_0^0$ in our sequence to have it be in the correct order. Thus, the needed sequence is $\tilde{f}_5^4 \tilde{f}_4^2 \tilde{f}_3^3 \tilde{f}_2^2 \tilde{f}_1^3$.

7. Let's check the sequence:

$$\tilde{f}_5^4 \tilde{f}_4^2 \tilde{f}_3^3 \tilde{f}_2^2 \tilde{f}_1^3(u_\lambda) = \tilde{f}_5^4 \tilde{f}_4^2 \tilde{f}_3^3 \tilde{f}_2^2 \left(u_{\sigma(\lambda)} \otimes (2,10,2,1,3,2)\right)$$

$$
\begin{aligned}
&= \tilde{f}_5^4 \tilde{f}_4^2 \tilde{f}_3^3 \left(u_{\sigma(\lambda)} \otimes (2,8,4,1,3,2)\right) \\
&= \tilde{f}_5^4 \tilde{f}_4^2 \left(u_{\sigma(\lambda)} \otimes (2,8,1,4,3,2)\right) \\
&= \tilde{f}_5^4 \left(u_{\sigma(\lambda)} \otimes (2,8,1,2,5,2)\right) \\
&= u_{\sigma(\lambda)} \otimes (2,8,1,2,1,6) \\
&= u_{\sigma(\lambda)} \otimes b_\nu.
\end{aligned}
$$

Example 6.4. Suppose $b_\lambda = (3,4,5,1,7,6,4)$ and $b_\nu = (6,2,2,3,8,7,2)$.

1. We first find all v_i such that $v_i > m_i$. Hence, we have

$$
\begin{aligned}
v_{s_1} &= v_1 = 6 > 3 = m_1, \\
v_{s_2} &= v_4 = 3 > 1 = m_4, \\
v_{s_3} &= v_5 = 8 > 7 = m_5, \\
v_{s_4} &= v_6 = 7 > 6 = m_6.
\end{aligned}
$$

   Also, let $z_0 = 0$.

2. Since $s_1 = 1$, we will skip this step.

3. Next we compute

$$d_1 = v_{s_1} - m_{s_1} = v_1 - m_1 = 3 > 0.$$

   This tells us that $(d_1)_+ = 3$ and $(-d_1)_+ = 0$. Thus, $z_1 = 0$ and our current sequence becomes $\tilde{f}_1^0 \tilde{f}_0^3$.

4. For $g = 2$, we compute

$$
\begin{aligned}
z_2 &= z_1 + m_2 - v_2 = 2, \\
z_3 &= z_2 + m_3 - v_3 = 5, \\
d_2 &= v_{s_2} - m_{s_2} - z_{s_2 - 1} = v_4 - m_4 - z_3 = -3 < 0.
\end{aligned}
$$

   Since $d_2 < 0$, we have $(d_2)_+ = 0$ and $(-d_2)_+ = 3$, which implies that $z_4 = (-d_2)_+ = 3$. The current sequence is $\tilde{f}_4^3 \tilde{f}_3^5 \tilde{f}_2^2 \tilde{f}_1^0 \tilde{f}_0^3$.
   For $g = 3$, we have

$$d_3 = v_{s_3} - m_{s_3} - z_{s_3 - 1} = v_5 - m_5 - z_4 = -2 < 0,$$

   telling us that $(d_3)_+ = 0$ and $(-d_3)_+ = 2$. Thus, $z_5 = 2$ and the sequence becomes $\tilde{f}_5^2 \tilde{f}_4^3 \tilde{f}_3^5 \tilde{f}_2^2 \tilde{f}_1^0 \tilde{f}_0^3$.

For $g = 4$, we have

$$d_4 = v_{s_4} - m_{s_4} - z_{s_4-1} = v_6 - m_6 - z_5 = -1 < 0.$$

This means that $(d_4)_+ = 0$ and $(-d_4)_+ = 1$, and so $z_6 = 1$.
Now the needed sequence is $\tilde{f}_6\tilde{f}_5^2\tilde{f}_4^3\tilde{f}_3^5\tilde{f}_2^2\tilde{f}_1^0\tilde{f}_0^3$.

5. We can ignore step 5 because $s_r = 6 = n$.

6. We have the sequence $\tilde{f}_6\tilde{f}_5^2\tilde{f}_4^3\tilde{f}_3^5\tilde{f}_2^2\tilde{f}_1^0\tilde{f}_0^3$, which is not in the needed order. We need to rearrange it to

$$\tilde{f}_0^3\tilde{f}_6\tilde{f}_5^2\tilde{f}_4^3\tilde{f}_3^5\tilde{f}_2^2,$$

since $z_1' = 0$.

7. Let's check the sequence:

$$\begin{aligned}
\tilde{f}_0^3\tilde{f}_6\tilde{f}_5^2\tilde{f}_4^3\tilde{f}_3^5\tilde{f}_2^2(u_\lambda) &= \tilde{f}_0^3\tilde{f}_6\tilde{f}_5^2\tilde{f}_4^3\tilde{f}_3^5\left(u_{\sigma(\lambda)}\otimes(3,2,7,1,7,6,4)\right)\\
&= \tilde{f}_0^3\tilde{f}_6\tilde{f}_5^2\tilde{f}_4^3\left(u_{\sigma(\lambda)}\otimes(3,2,2,6,7,6,4)\right)\\
&= \tilde{f}_0^3\tilde{f}_6\tilde{f}_5^2\left(u_{\sigma(\lambda)}\otimes(3,2,2,3,10,6,4)\right)\\
&= \tilde{f}_0^3\tilde{f}_6\left(u_{\sigma(\lambda)}\otimes(3,2,2,3,8,8,4)\right)\\
&= \tilde{f}_0^3\left(u_{\sigma(\lambda)}\otimes(3,2,2,3,8,7,5)\right)\\
&= \left(u_{\sigma(\lambda)}\otimes(6,2,2,3,8,7,2)\right)\\
&= u_{\sigma(\lambda)}\otimes b_\nu.
\end{aligned}$$

Example 6.5. Suppose $b_\lambda = (3,3,3,3,0)$ and $b_\nu = (4,4,2,1,1)$.

1. We first find all v_i such that $v_i > m_i$. Hence, we have

$$\begin{aligned}
v_{s_1} &= v_1 = 4 > 3 = m_1,\\
v_{s_2} &= v_2 = 4 > 3 = m_2,\\
v_{s_3} &= v_0 = 1 > 0 = m_0.
\end{aligned}$$

Also, let $z_0 = 0$.

2. Since $s_1 = 1$, we will skip this step.

3. Next we compute

$$d_1 = v_{s_1} - m_{s_1} = v_1 - m_1 = 1 > 0.$$

This tells us that $(d_1)_+ = 1$ and $(-d_1)_+ = 0$. Thus, $z_1 = 0$ and our current sequence becomes $\tilde{f}_1^0 \tilde{f}_0$.

4. For $g = 2$, we compute

$$d_2 = v_{s_2} - m_{s_2} - z_{s_2 - 1} = v_2 - m_2 - z_1 = 1 > 0.$$

Since $d_2 > 0$, we have $(d_2)_+ = 1$ and $(-d_2)_+ = 0$, which implies that $z_2 = 0$ and the current sequence is $\tilde{f}_2^0 \tilde{f}_1 \tilde{f}_0^2$.
For $g = 3$, we have

$$\begin{aligned} z_3 &= z_2 + m_3 - v_3 = 1, \\ z_4 &= z_3 + m_4 - v_4 = 3, \\ d_3 &= 0 \text{ since } s_3 = 0. \end{aligned}$$

Hence, the sequence becomes $\tilde{f}_4^3 \tilde{f}_3 \tilde{f}_2^0 \tilde{f}_1 \tilde{f}_0^2$.

5. We can ignore step 5 because $s_r = 0$.

6. We have the sequence $\tilde{f}_4^3 \tilde{f}_3 \tilde{f}_2^0 \tilde{f}_1 \tilde{f}_0^2$, which is not in the needed order. We need to rearrange it to

$$\tilde{f}_1 \tilde{f}_0^2 \tilde{f}_4^3 \tilde{f}_3,$$

since $z_2' = 0$.

7. Let's check the sequence:

$$\begin{aligned} \tilde{f}_1 \tilde{f}_0^2 \tilde{f}_4^3 \tilde{f}_3(u_\lambda) &= \tilde{f}_1 \tilde{f}_0^2 \tilde{f}_4^3 \left(u_{\sigma(\lambda)} \otimes (3,3,2,4,0)\right) \\ &= \tilde{f}_1 \tilde{f}_0^2 \left(u_{\sigma(\lambda)} \otimes (3,3,2,1,3)\right) \\ &= \tilde{f}_1 \left(u_{\sigma(\lambda)} \otimes (5,3,2,1,1)\right) \\ &= u_{\sigma(\lambda)} \otimes (4,4,2,1,1) \\ &= u_{\sigma(\lambda)} \otimes b_\nu. \end{aligned}$$

Example 6.6. Suppose $b_\lambda = (4,1,3,2,4)$ and $b_\nu = (1,5,5,3,0)$.

1. We have

$$
\begin{aligned}
z_0 &= 0,\\
v_{s_1} &= v_2 = 5 > 1 = m_2,\\
v_{s_2} &= v_3 = 5 > 3 = m_3,\\
v_{s_3} &= v_4 = 3 > 2 = m_4.
\end{aligned}
$$

2. Now, $z_1 = m_1 - v_1 = 4-1 = 3$. Hence, we have $\tilde{f}_1^3 \tilde{f}_0^0$.

3. Next, $d_1 = v_{s_1} - m_{s_1} - z_{s_1-1} = v_2 - m_2 - z_1 = 5-1-3 = 1 > 0$. Hence, $(d_1)_+ = 1$ and $z_2 = (-d_1)_+ = 0$. The current sequence is $\tilde{f}_2^0 \tilde{f}_1^4 \tilde{f}_0^1$.

4. For $g = 2$, we compute

$$d_2 = v_{s_2} - m_{s_2} - z_{s_2-1} = v_3 - m_3 - z_2 = 5-3-0 = 2 > 0.$$

   Thus, $(d_2)_+ = 2$, $z_3 = (-d_2)_+ = 0$, and the sequence becomes $\tilde{f}_3^0 \tilde{f}_2^2 \tilde{f}_1^6 \tilde{f}_0^3$.
   For $g = 3$, we have

$$d_3 = v_{s_3} - m_{s_3} - z_{s_3-1} = v_4 - m_4 - z_3 = 3-2-0 = 1 > 0.$$

   Thus, $(d_3)_+ = 1$, $z_4 = (-d_3)_+ = 0$, and the sequence changes to $\tilde{f}_4^0 \tilde{f}_3^1 \tilde{f}_2^3 \tilde{f}_1^7 \tilde{f}_0^4$.

5. We can ignore this step because $s_r = 4 = n$.

6. We need only remove the last Kashiwara operator from the sequence to obtain $\tilde{f}_3^1 \tilde{f}_2^3 \tilde{f}_1^7 \tilde{f}_0^4$, the needed sequence.

7. Let's check the sequence:

$$
\begin{aligned}
\tilde{f}_3^1 \tilde{f}_2^3 \tilde{f}_1^7 \tilde{f}_0^4(u_\lambda) &= \tilde{f}_3^1 \tilde{f}_2^3 \tilde{f}_1^7 \left(u_{\sigma(\lambda)} \otimes (8,1,3,2,0)\right)\\
&= \tilde{f}_3^1 \tilde{f}_2^3 \left(u_{\sigma(\lambda)} \otimes (1,8,3,2,0)\right)\\
&= \tilde{f}_3^1 \left(u_{\sigma(\lambda)} \otimes (1,5,6,2,0)\right)\\
&= u_{\sigma(\lambda)} \otimes (1,5,5,3,0)\\
&= u_{\sigma(\lambda)} \otimes b_\nu.
\end{aligned}
$$

Lemma 6.7. *Suppose we have the ground-state path $u_\lambda = u_{\sigma(\lambda)} \otimes b_\lambda$ and the λ-path $u_{\sigma(\lambda)} \otimes b_\nu$, for some $b_\nu \in \mathcal{B}_\ell$. Then the sequence F_{t^*} of Kashiwara operators obtained using Algorithm 1 is such that $F_{t^*}(u_\lambda) = u_{\sigma(\lambda)} \otimes b_\nu$.*

Proof. Suppose we are given $u_\lambda = u_{\sigma(\lambda)} \otimes (m_1, m_2, \ldots, m_n, m_0)$ and $u_{\sigma(\lambda)} \otimes b_\nu = u_{\sigma(\lambda)} \otimes (v_1, v_2, \ldots, v_n, v_0)$. And suppose we construct the sequence F_{t^*} using Algorithm 1. In order to prove that $F_{t^*}(u_\lambda) = u_{\sigma(\lambda)} \otimes b_\nu$, we need to make sure that the exponent of each Kashiwara operator $\tilde{f}_i$ does not exceed the corresponding m_i in the $(n+1)$-tuple b_λ; otherwise, $\tilde{f}_i(b_\lambda) = 0$ and our algorithm fails. So, we only need to show that $z'_{t^*+1} \leq m_{t^*+1}$ and that $z'_{t+1} \leq m_{t+1} + z'_t$ for $t = t^*+2, t^*+3, \ldots, t^*-1$.

We recall the following notation from Algorithm 1:

$$z_t = \begin{cases} 0 & \text{if } t = 0, \\ (-d_t)_+, & \text{if } t = s_g, g = 1, 2, \ldots, r, \\ z_{t-1} + m_t - v_t, & \text{else,} \end{cases} \tag{6.9}$$

$$z'_t = \begin{cases} z_t + \sum_{i=1}^r (d_i)_+, & \text{if } t = 0, 1, \ldots, s_1 - 1, \\ z_t + \sum_{i=j}^r (d_i)_+ & \text{if } t = s_{j-1}, s_{j-1}+1, \ldots, s_j - 1, j = 2, 3, \ldots, r, \text{ or } t = s_r, \\ z_t, & \text{else.} \end{cases} \tag{6.10}$$

Now, we also recall that $z'_{t^*} = 0$, by definition.

Remark 1. By Equation 6.10, if $t = t^* = 0, 1, \ldots, s_r$, then $(d_i)_+ = 0$ for corresponding values of i.

We first want to show that $z'_{t^*+1} \leq m_{t^*+1}$. For $t^*+1 \neq 0$, this follows directly from Equations 6.9 and 6.10 and Remark 1. So, suppose $t^*+1 = 0$. Then $z'_{t^*} = z'_n = z_n = 0$. Now, shift both the ground state path and the desired path by applying σ, i.e. we have

$$\begin{aligned} u_{\sigma^2(\lambda)} \otimes b_{\sigma(\lambda)} &= u_{\sigma^2(\lambda)} \otimes (m_2, m_3, \ldots, m_1), \\ u_{\sigma^2(\lambda)} \otimes b_{\sigma(\nu)} &= u_{\sigma^2(\lambda)} \otimes (v_2, v_3, \ldots, v_1). \end{aligned}$$

Let $t^*_\nu = t^*$. If we use the algorithm on these paths, we'll have the sequence

$$\tilde{f}_{t^*_\nu-2}^{z'_{t^*_\nu-1}} \cdots \tilde{f}_{t^*_\nu+1}^{z'_{t^*_\nu+2}} \tilde{f}_{t^*_\nu}^{z'_{t^*_\nu+1}} = \tilde{f}_{t^*_{\sigma(\nu)-2}}^{z'_{t^*_\nu-1}} \cdots \tilde{f}_{t^*_{\sigma(\nu)+2}}^{z'_{t^*_\nu+2}} \tilde{f}_{t^*_{\sigma(\nu)+1}}^{z'_{t^*_\nu+1}}.$$

Hence, $t^*_{\sigma(\nu)+1} = n$, which implies by definition that $z'_{t^*_\nu+1} \leq m_{t^*_{(\sigma(\nu)+1)+1}} = m_{t^*_\nu+1}$, i.e. $z'_0 \leq m_0$, since

$$\tilde{f}_i(m_2, m_3, \ldots, m_1) = (m_2, \ldots, m_{i+1} - 1, m_{i+2} + 1, \ldots, m_1)$$

(see Example 6.8).

Now let's show that $z'_{t+1} \leq m_{t+1} + z'_t$ for $t = t^*+2, t^*+3, \ldots, t^*-1$. Suppose that $t+1 = s_r+1, s_r+2, \ldots, n$. Then $z'_{t+1} = z_{t+1} = z_t + m_{t+1} - v_{t+1}$. Since $z'_t \geq z_t$, we have that $z'_{t+1} \leq z'_t + m_{t+1} - v_{t+1} \leq z'_t + m_{t+1}$.

Now suppose that $t+1 = s_g$ for $g = 1, 2, \ldots, r$. Then

$$z'_{t+1} = z_{t+1} + \sum_{i=g+1}^{r} (d_i)_+ = (-d_g)_+ + \sum_{i=g+1}^{r} (d_i)_+.$$

We want to show that $m_{t+1} + z'_t \geq z'_{t+1}$, i.e. that $m_{t+1} + z'_t - z'_{t+1} \geq 0$. So,

$$\begin{aligned} m_{t+1} + z'_t - z'_{t+1} &= \left(m_{s_g} + z_{s_g-1} + \sum_{i=g}^{r} (d_i)_+ \right) - \left((-d_g)_+ + \sum_{i=g+1}^{r} (d_i)_+ \right) \\ &= m_{s_g} + z_{s_g-1} + (d_g)_+ - (-d_g)_+. \end{aligned}$$

If $(d_g)_+ \neq 0$, then $(-d_g)_+ = 0$. Hence, $m_{t+1} + z'_t - z'_{t+1} = m_{s_g} + z_{s_g-1} + (d_g)_+ \geq 0$.
If $(d_g)_+ = 0$ and $(-d_g)_+ = 0$, then $m_{t+1} + z'_t - z'_{t+1} = m_{s_g} + z_{s_g-1} \geq 0$.
If $(d_g)_+ = 0$ and $(-d_g)_+ = m_{s_g} + z_{s_g-1} - v_{s_g} > 0$, then

$$m_{t+1} + z'_t - z'_{t+1} = m_{s_g} + z_{s_g-1} + 0 - \left(m_{s_g} + z_{s_g-1} - v_{s_g} \right) = v_{s_g} \geq 0.$$

Hence, $z'_{t+1} \leq z'_t + m_{t+1}$.

Next suppose that $t+1 = s_{g-1}+1, s_{g-1}+2, \ldots, s_g - 1$, for $g = 2, \ldots, r$. Then

$$\begin{aligned} z'_{t+1} &= z_{t+1} + \sum_{i=g}^{r} (d_i)_+ \\ &= (z_t + m_{t+1} - v_{t+1}) + \sum_{i=g}^{r} (d_i)_+. \end{aligned}$$

If $t+1 = s_{g-1}+1$, then

$$\begin{aligned} z'_{t+1} &= z_{s_{g-1}} + m_{s_{g-1}+1} - v_{s_{g-1}+1} + \sum_{i=g}^{r}(d_i)_+ \\ &= (z_t + m_{t+1} - v_{t+1}) + \sum_{i=g}^{r}(d_i)_+. \end{aligned}$$

Since

$$z'_t = z'_{s_{g-1}} = z_{s_{g-1}} + \sum_{i=g}^{r}(d_i)_+ = z_t + \sum_{i=g}^{r}(d_i)_+,$$

we have $z'_{t+1} = z'_{s_{g-1}} + m_{s_{g-1}+1} - v_{s_{g-1}+1} \leq z'_{s_{g-1}} + m_{s_{g-1}+1} = z'_t + m_{t+1}$. If instead, $t+1 \neq s_{g-1}+1$, then

$$\begin{aligned} z'_{t+1} &= z_t + m_{t+1} - v_{t+1} + \sum_{i=g}^{r}(d_i)_+ \\ &= z_t + \sum_{i=g}^{r}(d_i)_+ + m_{t+1} - v_{t+1} \\ &= z'_t + m_{t+1} - v_{t+1} \\ &\leq z'_t + m_{t+1}. \end{aligned}$$

Now suppose $t+1 = 1,2,\ldots,s_1-1$. Then

$$\begin{aligned} z'_{t+1} &= z_{t+1} + \sum_{i=1}^{r}(d_i)_+ \\ &= z_t + m_{t+1} - v_{t+1} + \sum_{i=1}^{r}(d_i)_+ \\ &= z_t + \sum_{i=1}^{r}(d_i)_+ + m_{t+1} - v_{t+1} \\ &= z'_t + m_{t+1} - v_{t+1} \\ &\leq z'_t + m_{t+1}. \end{aligned}$$

Finally, suppose that $t+1=0$. Then we want to show that $z'_0 \leq z'_n + m_0$. Well, by a similar argument to when we proved that $z'_{t^*+1} \leq m_{t^*+1}$ for $t^*+1=0$, if we shift b_λ and b_ν by σ and use the algorithm to find a sequence that will take us from $b_{\sigma(\lambda)}$ to $b_{\sigma(\nu)}$, we will obtain the sequence

$$\tilde{f}_{t_\nu^*-2}^{z'_{t_\nu^*-1}} \cdots \tilde{f}_{t_\nu^*+1}^{z'_{t_\nu^*+2}} \tilde{f}_{t_\nu^*}^{z'_{t_\nu^*+1}}.$$

Thus, we have $z'_0 \le z'_n + m_0$, since

$$\tilde{f}_i(m_2, m_3, \ldots, m_1) = (m_2, m_3, \ldots, m_{i+1}-1, m_{i+2}+1, m_{i+3}, \ldots, m_1).$$

Hence, $z'_{t+1} \le z'_t + m_{t+1}$.

□

Remark 2. Algorithm 1 constructs a sequence F_{t^*}, such that $F_{t^*}(u_\lambda) = u_{\sigma(\lambda)} \otimes F_{t^*}(b_\lambda) = u_{\sigma(\lambda)} \otimes b_\nu$ for a chosen $b_\nu \in \mathcal{B}_\ell$. Since

$$\begin{aligned} F_{t^*}(b_\lambda) &= \tilde{f}_{t^*-1}^{z'_{t^*-1}} \cdots \tilde{f}_0^{z'_0} \tilde{f}_n^{z'_n} \cdots \tilde{f}_{t^*+1}^{z'_{t^*+1}}(m_1, m_2, \ldots, m_n, m_0) \\ &= \big(m_1 + z'_0 - z'_0, \ldots, m_{t^*-1} + z'_{t^*-2} - z'_{t^*-1}, m_{t^*} + z'_{t^*-1}, \\ &\qquad m_{t^*+1} - z'_{t^*+1}, m_{t^*+2} + z'_{t^*+1} - z'_{t^*+2}, \ldots, m_0 + z'_n - z'_0\big), \end{aligned}$$

we have the following equations:

$$\begin{aligned} \nu_{t^*} &= m_{t^*} + z'_{t^*-1}, \\ \nu_{t^*+1} &= m_{t^*+1} - z'_{t^*+1}, \\ \nu_i &= m_i + z'_{i+1} - z'_i, \text{for all } i \ne t^*, t^*+1. \end{aligned} \tag{6.11}$$

Example 6.8. For further clarification on proving $z'_{t^*+1} \le m_{t^*+1}$ for $t^*+1 = 0$ in Lemma 6.7, let's return to Example 6.6. We had that $b_\lambda = (4,1,3,2,4)$ and $b_\nu = (1,5,5,3,0)$. Using the algorithm, we produced the sequence $F_{t_\nu^*} = \tilde{f}_3 \tilde{f}_2^3 \tilde{f}_1^7 \tilde{f}_0^4$. If instead we had produced the sequence that when acting on $b_{\sigma(\lambda)} = (1,3,2,4,4)$ would give $b_{\sigma(\nu)} = (5,5,3,0,1)$, we would have $\tilde{f}_2 \tilde{f}_1^3 \tilde{f}_0^7 \tilde{f}_4^4$. Notice that $z'_{t_\nu^*+1} = z'_0 = 4 \le m_{t^*_{\sigma(\nu)+2}} = m_0$.

In order to apply the sequence F_{t^*} of Kashiwara operators obtained from Algorithm 1 to the ground-state path u_λ, we need the following sequence of simple reflections:

$$r_{t^*-1} \cdots r_1 r_0 r_n \cdots r_{t^*+1}.$$

Lemma 6.9. *The sequence*

$$r_{t^*-1} \cdots r_1 r_0 r_n \cdots r_{t^*+1}$$

is a subsequence of $w(2, j)$, for all $j = 0, 1, \ldots, n$.

Proof. Recall that $w(2, j) = (r_{j+n-2} \cdots r_j r_{j+n})(r_{j+n-1} \cdots r_{j+1} r_j)$. Notice that each simple reflection r_i appears within the first $n+1$ components from the right in $w(2, j)$. If we start at any of those r_i, we can form a sequence of simple reflections $(r_{i+n-1} \cdots r_{i+1} r_i)$ of length n that is a subsequence of $w(2, j)$. Let $i = t^* + 1$. Then we have that $r_{t^*-1} \cdots r_1 r_0 r_n \cdots r_{t^*+1}$ is a subsequence of $w(2, j)$, for all $j = 0, 1, \ldots, n$. $\square$

Corollary 6.10. *We have that $u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell \subset B_{w(2,j)}(\lambda)$ for all $j = 0, 1, \ldots, n$.*

Proof. For each λ-path $u_{\sigma(\lambda)} \otimes b_\nu$, Algorithm 1 produces a sequence $F_{t_\nu^*}$ that is associated with a sequence of simple reflections $r_{t^*-1} \cdots r_1 r_0 r_n \cdots r_{t^*+1}$ of length n. By Lemma 6.9, we know each of these sequences of simple reflections is a subsequence of $w(2, j)$ for all $j = 0, 1, \ldots, n$. Hence, by the definition of $B_{w(2,j)}(\lambda)$, it follows that $u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell \subset B_{w(2,j)}(\lambda)$ for all j. $\square$

Lemma 6.11. *We have that*

$$\bigcup_{j=0}^{n} B_{w(1,j)}(\lambda) = u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell.$$

Proof. By Lemma 6.7, we proved that there exists a sequence $F_{t_\nu^*}$ such that $F_{t_\nu^*}(u_\lambda) = u_{\sigma(\lambda)} \otimes b_\nu$ for $b_\nu \in \mathcal{B}_\ell$. Each sequence $F_{t_\nu^*}$ is associated with a sequence of simple reflections $w_{t_\nu^*} = r_{t_\nu^*-1} \cdots r_{t_\nu^*+2} r_{t_\nu^*+1}$ of length n. Note that $w_{t_\nu^*} = w(1, t_\nu^* + 1)$. Hence, $u_{\sigma(\lambda)} \otimes b_\nu \in B_{w(1,t_\nu^*+1)}(\lambda)$ for each $b_\nu \in \mathcal{B}_\ell$. Thus,

$$\bigcup_{j=0}^{n} B_{w(1,j)}(\lambda) \supseteq u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell.$$

Now we just need to show that

$$\bigcup_{j=0}^{n} B_{w(1,j)}(\lambda) \subseteq u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell.$$

In other words, we need to show that $u_{\sigma^2(\lambda)} \otimes b_\nu \otimes \mathcal{B}_\ell \not\subseteq B_{w(1,j)}(\lambda)$ for all $j = 0, 1, \ldots, n$. Well, by Lemma 6.1, we have that

$$\begin{aligned} \tilde{f}_i^{z_i} \tilde{f}_{i+1}^{z_{i+1}}(u_\lambda) &= \tilde{f}_i^{z_i} \tilde{f}_{i+1}^{z_{i+1}}\left(u_{\sigma^2(\lambda)} \otimes b_{\sigma(\lambda)} \otimes b_\lambda\right) \\ &= u_{\sigma^2(\lambda)} \otimes b_{\rho_1} \otimes b_{\rho_2}. \end{aligned}$$

This last equation implies that if any of our sequences contain $\tilde{f}_{i+1}$ first and then $\tilde{f}_i$, that the second component from the right in the tensor product will be affected. But $w(1, j) = (r_{j+n-1} \cdots r_{j+1} r_j)$ for $j = 0, 1, \ldots, n$, which means this situation never occurs. Hence, the second component from the right in the tensor product will never be affected and so

$$\bigcup_{j=0}^{n} B_{w(1,j)}(\lambda) \subseteq u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell.$$

Therefore, $\bigcup_{j=0}^{n} B_{w(1,j)}(\lambda) = u_{\sigma(\lambda)} \otimes \mathcal{B}_\ell.$ □

We want to extend this result to the L^{th} case.

Algorithm 2: Algorithm 1 can be rewritten for the general case.

Suppose we are given

$$u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L = u_{\sigma^{L+1}(\lambda)} \otimes (m_{L+1}, m_{L+2}, \ldots, m_{L+n}, m_L) \otimes \mathcal{B}_\ell^L.$$

We want to obtain

$$u_{\sigma^{L+1}(\lambda)} \otimes b_\nu \otimes \mathcal{B}_\ell^L = u_{\sigma^{L+1}(\lambda)} \otimes (v_1, v_2, \ldots, v_n, v_0) \otimes \mathcal{B}_\ell^L.$$

This algorithm provides a sequence of Kashiwara operators such that when applied to specific λ-paths in the set $u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L$, we produce the set $u_{\sigma^{L+1}(\lambda)} \otimes b_\nu \otimes \mathcal{B}_\ell^L$.

Recall that by Lemma 5.5, we can apply $\tilde{f}_i$ a sufficient number of times to specific λ-paths in the set $u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L$ to produce the set $u_{\sigma^{L+1}(\lambda)} \otimes \tilde{f}_i(b_{\sigma^L(\lambda)}) \otimes \mathcal{B}_\ell^L$. Hence with a few notation changes, Algorithm 2 is the same as the Algorithm 1. Note that $m_t = m_{t \bmod (L+1)}$.

1. Find all v_i such that $v_i > m_{L+i}$, $i = 0, 1, 2, \ldots, n$. Label them (in order of appearance in the $(n+1)$-tuple) as $v_{s_1}, v_{s_2}, \ldots, v_{s_r}$, where $1 < r < n+1$. Define $z_0 = 0$.

2. If $s_1 = 1$, then skip this step.
   Otherwise, we know that $v_t < m_{L+t}$ for all $t = 1, 2, \ldots, s_1 - 1$. So, recursively define $z_1 = m_{L+1} - v_1$ and $z_t = z_{t-1} + m_{L+t} - v_t$ for $t = 2, 3, \ldots, s_1 - 1$. Hence, we'll need $\tilde{f}_{s_1-1}^{z_{s_1-1}} \cdots \tilde{f}_2^{z_2} \tilde{f}_1^{z_1}$ as part of the sequence.

3. Compute $d_1 = v_{s_1} - m_{L+s_1} - z_{s_1-1}$. Define

   $$(d_1)_+ = \max\left\{v_{s_1} - m_{L+s_1} - z_{s_1-1}, 0\right\}$$

   and

   $$(-d_1)_+ = \max\left\{m_{L+s_1} + z_{s_1-1} - v_{s_1}, 0\right\}.$$

Let $z_{s_1} = (-d_1)_+$. The sequence becomes

$$\tilde{f}_{s_1}^{(-d_1)_+}\tilde{f}_{s_1-1}^{z_{s_1-1}+(d_1)_+}\cdots\tilde{f}_1^{z_1+(d_1)_+}\tilde{f}_0^{(d_1)_+}.$$

4. Do the following for $g = 2,\ldots,r$.
   Let $z_t = z_{t-1} + m_{L+t} - v_t$ for $t = s_{g-1}+1, s_{g-1}+2,\ldots,s_g-1$.
   Compute $d_g = v_{s_g} - m_{L+s_g} - z_{s_g-1}$. However, if $s_r = 0$, then set $d_r = 0$ and $z_{s_r} = 0$. Define

$$(d_g)_+ = \max\left\{v_{s_g} - m_{L+s_g} - z_{s_g-1}, 0\right\} \tag{6.12}$$

   and

$$(-d_g)_+ = \max\left\{m_{L+s_g} + z_{s_g-1} - v_{s_g}, 0\right\}. \tag{6.13}$$

   Let $z_{s_g} = (-d_g)_+$.

   Note that after repeating step 4 for all g, the sequence becomes

$$\begin{aligned}\tilde{f}_{s_r}^{(-d_r)_+}\tilde{f}_{s_r-1}^{z_{s_r-1}+(d_r)_+}&\cdots\tilde{f}_{s_{r-1}+1}^{z_{s_{r-1}+1}+(d_r)_+}\tilde{f}_{s_{r-1}}^{(-d_{r-1})_++(d_r)_+}\tilde{f}_{s_{r-1}-1}^{z_{s_{r-1}-1}+(d_r)_++(d_{r-1})_+}\\&\cdots\tilde{f}_{s_{r-2}+1}^{z_{s_{r-2}+1}+(d_r)_++(d_{r-1})_+}\tilde{f}_{s_{r-2}}^{(-d_{r-2})_++(d_{r-1})_++(d_r)_+}\\&\cdots\\&\cdots\tilde{f}_{s_1+1}^{z_{s_1+1}+\sum_{i=2}^r(d_i)_+}\tilde{f}_{s_1}^{(-d_1)_++\sum_{i=2}^r(d_i)_+}\tilde{f}_{s_1-1}^{z_{s_1-1}+\sum_{i=1}^r(d_i)_+}\\&\cdots\tilde{f}_1^{z_1+\sum_{i=1}^r(d_i)_+}\tilde{f}_0^{\sum_{i=1}^r(d_i)_+}.\end{aligned}$$

   Let's rewrite the sequence as

$$\tilde{f}_{s_r}^{z'_{s_r}}\tilde{f}_{s_r-1}^{z'_{s_r-1}}\cdots\tilde{f}_1^{z'_1}\tilde{f}_0^{z'_0}.$$

5. Next, let $z_t = z_{t-1} + m_{L+t} - v_t$ for $t = s_r+1, s_r+2,\ldots,n$.
   Then we also need the following in the sequence:

$$\tilde{f}_n^{z_n}\tilde{f}_{n-1}^{z_n-1}\cdots\tilde{f}_{s_r+1}^{z_{s_r+1}}.$$

   Then the entire sequence becomes

$$\tilde{f}_n^{z_n}\tilde{f}_{n-1}^{z_n-1}\cdots\tilde{f}_{s_r+1}^{z_{s_r+1}}\tilde{f}_{s_r}^{z'_{s_r}}\tilde{f}_{s_r-1}^{z'_{s_r-1}}\cdots\tilde{f}_1^{z'_1}\tilde{f}_0^{z'_0},$$

   which we'll rewrite as

$$\tilde{f}_n^{z'_n}\tilde{f}_{n-1}^{z'_{n-1}}\cdots\tilde{f}_1^{z'_1}\tilde{f}_0^{z'_0},$$

where

$$z'_t = z_t \qquad \text{for } t = s_r+1, s_r+2, \ldots, n \quad , \tag{6.14}$$

$$z'_t = z_t + \sum_{i=g+1}^{r}(d_i)_+ \qquad \text{for } t = s_g, \text{ where } g = 1,2,\ldots,r, \tag{6.15}$$

$$z'_t = z_t + \sum_{i=g}^{r}(d_i)_+ \qquad \text{for } t = s_{g-1}+1, s_{g-1}+2, \ldots, s_g-1, \text{ where } g = 2,3,\ldots,r, \tag{6.16}$$

$$z'_t = z_t + \sum_{i=1}^{r}(d_i)_+ \qquad \text{for } t = 0,1,\ldots,s_1-1 \text{ and } z_0 = 0. \tag{6.17}$$

6. Finally, we want to rearrange the current sequence so that no $\tilde{f}_i$ occurs after $\tilde{f}_{i+1}$.
First, note that there exists some $t^* \in \{0,1,\ldots,n\}$ such that $z'_{t^*} = 0$. How do we know this? Let's prove this by contradiction. So, suppose that $z'_i \neq 0$ for all i, i.e. $z'_i > 0$ for all $i \in \{0,1,\ldots,n\}$. Then since

$$z'_{s_g} = (-d_g)_+ = \max\left\{m_{L+s_g} + z_{s_g-1} - v_{s_g}, 0\right\}$$

we have that $m_{L+s_g} + z_{s_g-1} - v_{s_g} > 0$, which implies that $(d_g)_+ = 0$ for all g. This implies that

$$z'_0 = z_0 + \sum_{i=1}^{r}(d_i)_+ = 0 + \sum_{i=1}^{r} 0 = 0,$$

which is a contradiction. Hence, there exists a t^* such that $z'_{t^*} = 0$.
Then, the sequence becomes

$$\tilde{f}_n^{z'_n}\tilde{f}_{n-1}^{z'_{n-1}}\cdots\tilde{f}_{t^*+1}^{z'_{t^*+1}}\tilde{f}_{t^*-1}^{z'_{t^*-1}}\cdots\tilde{f}_1^{z'_1}\tilde{f}_0^{z'_0}$$

which we'll rearrange to

$$F_{t^*} = \tilde{f}_{t^*-1}^{z'_{t^*-1}}\cdots\tilde{f}_1^{z'_1}\tilde{f}_0^{z'_0}\tilde{f}_n^{z'_n}\tilde{f}_{n-1}^{z'_{n-1}}\cdots\tilde{f}_{t^*+1}^{z'_{t^*+1}}. \tag{6.18}$$

Hence, if we apply each of the following Kashiwara operators

$$\tilde{f}_{t^*+1}, \tilde{f}_{t^*+2}, \ldots, \tilde{f}_{n-1}, \tilde{f}_n, \tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_{t^*-1}$$

a sufficient number of times in the given order to specific λ-paths in the set $u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L$, we will obtain

$$u_{\sigma^{L+1}(\lambda)} \otimes F_{t^*}\left(b_{\sigma^L(\lambda)}\right) \otimes \mathcal{B}_\ell^L = u_{\sigma^{L+1}(\lambda)} \otimes b_\nu \otimes \mathcal{B}_\ell^L.$$

We also can adjust Equation 6.11 to the following:

$$\begin{aligned} v_{t^*} &= m_{L+t^*} + z'_{t^*-1}, \\ v_{t^*+1} &= m_{L+t^*+1} - z'_{t^*+1}, \\ v_i &= m_{L+i} + z'_{i-1} - z'_i, \text{ for all } i \neq t^*, t^*+1. \end{aligned} \tag{6.19}$$

Example 6.12. To demonstrate Algorithm 2, suppose we are given $u_{\sigma^5(\lambda)} \otimes \mathcal{B}^5_{12}$, where

$$b_\lambda = (2,1,0,4,2,3) = (m_1, m_2, \ldots, m_0).$$

Hence, $b_{\sigma^5(\lambda)} = (3,2,1,0,4,2) = (m_0, m_1, \ldots, m_5)$; so L = 5. Suppose we want to obtain the paths

$$u_{\sigma^6(\lambda)} \otimes (5,3,0,0,4,0) \otimes \mathcal{B}^5_{12} = u_{\sigma^6(\lambda)} \otimes b_\nu \otimes \mathcal{B}^5_{12}.$$

1. Let $z_0 = 0$. Find all v_i such that $v_i > m_{L+i} = m_{5+i}$:

$$\begin{aligned} v_{s_1} &= v_1 = 5 > 3 = m_{5+1} = m_0, \\ v_{s_2} &= v_2 = 3 > 2 = m_{5+2} = m_1. \end{aligned}$$

2. We'll skip this step because $s_1 = 1$.

3. Compute
$$d_1 = v_{s_1} - m_{5+s_1} - z_{s_1-1} = 5 - 3 - 0 = 2 > 0.$$
Hence, $(d_1)_+ = 2$, $z_1 = (-d_1)_+ = 0$, and we need the sequence $\tilde{f}_1^0 \tilde{f}_0^2$.

4. Let $g = 2$. Then
$$d_2 = v_{s_2} - m_{5+s_2} - z_{s_2-1} = v_2 - m_1 - z_1 = 3 - 2 - 0 = 1 > 0.$$
So, $(d_2)_+ = 1$, $z_2 = (-d_2)_+ = 0$, and the current sequence is $\tilde{f}_2^0 \tilde{f}_1 \tilde{f}_0^3$.

5. Now compute
$$\begin{aligned} z_3 &= z_2 + m_{5+3} - v_3 = 0 + m_2 - 0 = 1, \\ z_4 &= z_3 + m_{5+4} - v_4 = 1 + m_3 - 0 = 1 + 0 - 0 = 1, \\ z_5 &= z_4 + m_{5+5} - v_5 = 1 + m_4 - 4 = 1 + 4 - 4 = 1. \end{aligned}$$

The sequence becomes $\tilde{f}_5\tilde{f}_4\tilde{f}_3\tilde{f}_2^0\tilde{f}_1\tilde{f}_0^3$.

6. So, rearrange the sequence to $\tilde{f}_1\tilde{f}_0^3\tilde{f}_5\tilde{f}_4\tilde{f}_3$.

7. Let's check the sequence. Recall that by Lemma 5.5, we can apply the Kashiwara operators $\tilde{f}_3, \tilde{f}_4, \tilde{f}_5, \tilde{f}_0, \tilde{f}_1$, in that order, a sufficient number of times to certain λ-paths in the set $u_{\sigma^5(\lambda)}\otimes\mathcal{B}_{12}^5$ to produce the set

$$\begin{aligned}
& u_{\sigma^6(\lambda)}\otimes\tilde{f}_1\tilde{f}_0^3\tilde{f}_5\tilde{f}_4\tilde{f}_3(3,2,1,0,4,2)\otimes\mathcal{B}_{12}^5 \\
= \;& u_{\sigma^6(\lambda)}\otimes\tilde{f}_1\tilde{f}_0^3\tilde{f}_5\tilde{f}_4(3,2,0,1,4,2)\otimes\mathcal{B}_{12}^5 \\
= \;& u_{\sigma^6(\lambda)}\otimes\tilde{f}_1\tilde{f}_0^3\tilde{f}_5(3,2,0,0,5,2)\otimes\mathcal{B}_{12}^5 \\
= \;& u_{\sigma^6(\lambda)}\otimes\tilde{f}_1\tilde{f}_0^3(3,2,0,0,4,3)\otimes\mathcal{B}_{12}^5 \\
= \;& u_{\sigma^6(\lambda)}\otimes\tilde{f}_1(6,2,0,0,4,0)\otimes\mathcal{B}_{12}^5 \\
= \;& u_{\sigma^6(\lambda)}\otimes(5,3,0,0,4,0)\otimes\mathcal{B}_{12}^5 \\
= \;& u_{\sigma^6(\lambda)}\otimes b_\nu\otimes\mathcal{B}_{12}^5.
\end{aligned}$$

Example 6.13. Suppose we are given $u_{\sigma^{10}(\lambda)}\otimes\mathcal{B}_{14}^{10}$, where

$$b_\lambda=(3,1,3,0,1,2,4)=(m_1,m_2,\ldots,m_0),$$

and so $L=10$, $n=6$, and $b_{\sigma^{10}(\lambda)}=(0,1,2,4,3,1,3)$. And suppose we want to obtain the paths

$$u_{\sigma^{11}(\lambda)}\otimes(0,3,5,1,1,2,2)\otimes\mathcal{B}_{14}^{10}=u_{\sigma^{11}(\lambda)}\otimes b_\nu\otimes\mathcal{B}_{14}^{10}.$$

1. Let $z_0=0$. Find all v_i such that $v_i>m_{L+i}=m_{10+i}$:

$$\begin{aligned}
v_{s_1}&=v_2=3>1=m_{10+2}=m_5,\\
v_{s_2}&=v_3=5>2=m_{10+3}=m_6,\\
v_{s_3}&=v_6=2>1=m_{10+6}=m_2.
\end{aligned}$$

2. Compute

$$z_1=m_{10+1}-v_1=m_4-v_1=0-0=0.$$

Hence the sequence is $\tilde{f}_1^0\tilde{f}_0^0$.

3. Now compute

$$d_1 = v_{s_1} - m_{10+s_1} - z_{s_1-1} = v_2 - m_5 - z_1 = 3 - 1 - 0 = 2 > 0.$$

So, $(d_1)_+ = 2$, $z_2 = (-d_1)_+ = 0$, and the sequence becomes $\tilde{f}_2^0 \tilde{f}_1^2 \tilde{f}_0^2$.

4. Let $g = 2$. Then

$$d_2 = v_{s_2} - m_{10+s_2} - z_{s_2-1} = v_3 - m_6 - z_2 = 5 - 2 - 0 = 3 > 0.$$

Hence, $(d_2)_+ = 3$ and $z_3 = (-d_2)_+ = 0$, changing the sequence to $\tilde{f}_3^0 \tilde{f}_2^3 \tilde{f}_1^5 \tilde{f}_0^5$.
Next let $g = 3$. Then

$$\begin{aligned} z_4 &= z_3 + m_{10+4} - v_4 = 0 + m_0 - 1 = 4 - 1 = 3, \\ z_5 &= z_4 + m_{10+5} - v_5 = 3 + m_1 - 1 = 3 + 3 - 1 = 5, \\ d_3 &= v_{s_3} - m_{10+s_3} - z_{s_3-1} = v_6 - m_2 - z_5 = 2 - 1 - 5 = -4 < 0. \end{aligned}$$

So, $(d_3)_+ = 0$ and $z_6 = (-d_3)_+ = 4$, which makes the sequence $\tilde{f}_6^4 \tilde{f}_5^5 \tilde{f}_4^3 \tilde{f}_3^0 \tilde{f}_2^3 \tilde{f}_1^5 \tilde{f}_0^5$.

5. We don't need this step because $s_r = 6 = n$.

6. Rearrange the sequence to $\tilde{f}_2^3 \tilde{f}_1^5 \tilde{f}_0^5 \tilde{f}_6^4 \tilde{f}_5^5 \tilde{f}_4^3$.

7. Let's check the sequence. Recall that by Lemma 5.5, we can apply the Kashiwara operators $\tilde{f}_4, \tilde{f}_5, \tilde{f}_6, \tilde{f}_0, \tilde{f}_1, \tilde{f}_2$, in that order, a sufficient number of times to certain λ-paths in the set $u_{\sigma^{10}(\lambda)} \otimes \mathcal{B}_{14}^{10}$ to produce the set

$$\begin{aligned} & u_{\sigma^{11}(\lambda)} \otimes \tilde{f}_2^3 \tilde{f}_1^5 \tilde{f}_0^5 \tilde{f}_6^4 \tilde{f}_5^5 \tilde{f}_4^3 \left(b_{\sigma^{10}(\lambda)}\right) \otimes \mathcal{B}_{14}^{10} \\ = \; & u_{\sigma^{11}(\lambda)} \otimes \tilde{f}_2^3 \tilde{f}_1^5 \tilde{f}_0^5 \tilde{f}_6^4 \tilde{f}_5^5 \tilde{f}_4^3 (0,1,2,4,3,1,3) \otimes \mathcal{B}_{14}^{10} \\ = \; & u_{\sigma^{11}(\lambda)} \otimes \tilde{f}_2^3 \tilde{f}_1^5 \tilde{f}_0^5 \tilde{f}_6^4 \tilde{f}_5^5 (0,1,2,1,6,1,3) \otimes \mathcal{B}_{14}^{10} \\ = \; & u_{\sigma^{11}(\lambda)} \otimes \tilde{f}_2^3 \tilde{f}_1^5 \tilde{f}_0^5 \tilde{f}_6^4 (0,1,2,1,1,6,3) \otimes \mathcal{B}_{14}^{10} \\ = \; & u_{\sigma^{11}(\lambda)} \otimes \tilde{f}_2^3 \tilde{f}_1^5 \tilde{f}_0^5 (0,1,2,1,1,2,7) \otimes \mathcal{B}_{14}^{10} \\ = \; & u_{\sigma^{11}(\lambda)} \otimes \tilde{f}_2^3 \tilde{f}_1^5 (5,1,2,1,1,2,2) \otimes \mathcal{B}_{14}^{10} \\ = \; & u_{\sigma^{11}(\lambda)} \otimes \tilde{f}_2^3 (0,6,2,1,1,2,2) \otimes \mathcal{B}_{14}^{10} \\ = \; & u_{\sigma^{11}(\lambda)} \otimes (0,3,5,1,1,2,2) \otimes \mathcal{B}_{14}^{10} \\ = \; & u_{\sigma^{11}(\lambda)} \otimes b_\nu \otimes \mathcal{B}_{14}^{10}. \end{aligned}$$

Lemma 6.14. *Suppose we have the ground state path u_λ. Choose $b_\nu = (\nu_1, \nu_2, \ldots, \nu_n, \nu_0) \in \mathcal{B}_\ell$. Then*

$$F_{t^*} F'_{t^*+1} F'_{t^*+2} \cdots F'_{t^*+L-1}(u_\lambda) = u_{\sigma^L(\lambda)} \otimes b_\nu \otimes b_{\mu_{L-1}} \otimes b_{\mu_{L-2}} \otimes \ldots \otimes b_{\mu_1},$$

for some $b_{\mu_1}, b_{\mu_2}, \ldots, b_{\mu_{L-1}} \in \mathcal{B}_\ell$, where the sequence

$$F_{t^*} = \tilde{f}_{t^*-1}^{z'_{t^*-1}} \tilde{f}_{t^*-2}^{z'_{t^*-2}} \cdots \tilde{f}_{t^*+1}^{z'_{t^*+1}}$$

is constructed using Algorithm 2, and we define

$$F'_{t^*+p} = \tilde{f}_{t^*-1+p}^{z'_{t^*-1}} \tilde{f}_{t^*-2+p}^{z'_{t^*-2}} \cdots \tilde{f}_{t^*+1+p}^{z'_{t^*+1}},$$

for $p = 1, 2, \ldots, L-1$.

Proof. Suppose $L = 2$. Let

$$u_\lambda = \ldots \otimes (m_3, m_4, \ldots, m_2) \otimes (m_2, m_3, \ldots, m_1) \otimes (m_1, m_2, \ldots, m_0).$$

Then we want to show that

$$F_{t^*} F'_{t^*+1}(u_\lambda) = u_{\sigma^2(\lambda)} \otimes (\nu_1, \nu_2, \ldots, \nu_0) \otimes (m'_1, m'_2, \ldots, m'_0).$$

We first find F_{t^*} using Algorithm 2 and then form F'_{t^*+1}. This gives us the sequence

$$\left(\tilde{f}_{t^*-1}^{z'_{t^*-1}} \tilde{f}_{t^*-2}^{z'_{t^*-2}} \cdots \tilde{f}_{t^*+1}^{z'_{t^*+1}}\right)\left(\tilde{f}_{t^*}^{z'_{t^*-1}} \tilde{f}_{t^*-1}^{z'_{t^*-2}} \cdots \tilde{f}_{t^*+2}^{z'_{t^*+1}}\right).$$

Then we have

$$\begin{aligned}
&\left(\tilde{f}_{t^*-1}^{z'_{t^*-1}} \tilde{f}_{t^*-2}^{z'_{t^*-2}} \cdots \tilde{f}_{t^*+1}^{z'_{t^*+1}}\right)\left(\tilde{f}_{t^*}^{z'_{t^*-1}} \tilde{f}_{t^*-1}^{z'_{t^*-2}} \cdots \tilde{f}_{t^*+2}^{z'_{t^*+1}}\right)(u_\lambda) \\
=&\left(\tilde{f}_{t^*-1}^{z'_{t^*-1}} \tilde{f}_{t^*-2}^{z'_{t^*-2}} \cdots \tilde{f}_{t^*+1}^{z'_{t^*+1}}\right)\left(u_{\sigma(\lambda)} \otimes \left(m'_1, m'_2, \ldots, m'_n, m'_0\right)\right)
\end{aligned}$$

where

$$m'_\tau = \begin{cases} m_\tau + z'_{\tau-2}, & \text{for } \tau = t^*+1 \\ m_\tau - z'_{\tau-1}, & \text{for } \tau = t^*+2 \\ m_\tau + z'_{\tau-2} - z'_{\tau-1}, & \text{otherwise.} \end{cases}$$

Now, the (t^*+1)-signature on this λ-path is

$$\begin{aligned}&\left(\ldots,-_{m_{t^*+4}}+_{m_{t^*+3}},-_{m_{t^*+3}}+_{m_{t^*+2}},-_{m_{t^*+2}-z'_{t^*+1}}+_{m_{t^*+1}+z'_{t^*-1}}\right)\\ =&\left(+_{z'_{t^*+1}},+_{m_{t^*+1}+z'_{t^*-1}}\right).\end{aligned}$$

Hence, we can apply the next Kashiwara operator in the sequence to this λ-path:

$$\begin{aligned}\left(\tilde{f}_{t^*-1}^{z'_{t^*-1}}\tilde{f}_{t^*-2}^{z'_{t^*-2}}\cdots\tilde{f}_{t^*+2}^{z'_{t^*+2}}\right)\big(u_{\sigma^2(\lambda)}\otimes\big(m_2,\ldots,m_{t^*+2}-z'_{t^*+1},m_{t^*+3}+z'_{t^*+1},m_{t^*+4},\ldots,m_1\big)\\ \otimes\big(m'_1,m'_2,\ldots,m'_n,m'_0\big)\big).\end{aligned}$$

The (t^*+2)-signature on this λ-path is

$$\begin{aligned}&\left(\ldots,-_{m_{t^*+5}}+_{m_{t^*+4}},-_{m_{t^*+4}}+_{m_{t^*+3}+z'_{t^*+1}},-_{m_{t^*+3}+z'_{t^*+1}-z'_{t^*+2}}+_{m_{t^*+2}-z'_{t^*+1}}\right)\\ =&\left(+_{z'_{t^*+2}},+_{m_{t^*+2}-z'_{t^*+1}}\right).\end{aligned}$$

Thus, we can apply the next Kashiwara operator in the sequence to this λ-path:

$$\begin{aligned}\left(\tilde{f}_{t^*-1}^{z'_{t^*-1}}\cdots\tilde{f}_{t^*+3}^{z'_{t^*+3}}\right)\big(u_{\sigma^2(\lambda)}\otimes\big(m_2,\ldots,m_{t^*+2}-z'_{t^*+1},m_{t^*+3}+z'_{t^*+1}-z'_{t^*+2},m_{t^*+4}+z'_{t^*+2},m_{t^*+5},\ldots,m_1\big)\\ \otimes\big(m'_1,m'_2,\ldots,m'_n,m'_0\big)\big).\end{aligned}$$

We can continue with this process, since the (i)-signature for each new λ-path will be

$$\begin{aligned}&\left(\ldots,-_{m_{i+3}}+_{m_{i+2}},-_{m_{i+2}}+_{m_{i+1}+z'_{i-1}},-_{m'_{i+1}}+_{m'_i}\right)\\ =&\left(+_{m_{i+1}+z'_{i-1}-(m_{i+1}+z'_{i-1}-z'_i)},+_{m'_i}\right)\\ =&\left(+_{z'_i},+_{m'_i}\right)\end{aligned}$$

for $i=t^*+3,t^*+4,\ldots,t^*-1$. The key factor here is that each of the last n Kashiwara operators in the sequence $F_{t^*}F'_{t^*+1}$ will only affect the second component from the right in our tensor product. Hence,

$$\begin{aligned}F_{t^*}F'_{t^*+1}(u_\lambda)&=F_{t^*}\left(u_{\sigma(\lambda)}\otimes\left(m'_1,m'_2,\ldots,m'_n,m'_0\right)\right)\\ &=u_{\sigma^2(\lambda)}\otimes F_{t^*}\left(b_{\sigma(\lambda)}\right)\otimes\left(m'_1,m'_2,\ldots,m'_n,m'_0\right)\\ &=u_{\sigma^2(\lambda)}\otimes b_\nu\otimes\left(m'_1,m'_2,\ldots,m'_n,m'_0\right).\end{aligned}$$

Now let's suppose that

$$F_{g^*}F'_{g^*+1}F'_{g^*+2}\cdots F'_{g^*+L-1}(u_\lambda)=u_{\sigma^L(\lambda)}\otimes b_\nu\otimes b_{\mu_{L-1}}\otimes b_{\mu_{L-2}}\otimes\ldots\otimes b_{\mu_1},$$

where

$$F_{g^*}\left(u_{\sigma^{L-1}(\lambda)}\right)\otimes\mathcal{B}_\ell^{L-1}=u_{\sigma^L(\lambda)}\otimes b_\nu\otimes\mathcal{B}_\ell^{L-1},$$

$$b_{\mu_\tau}=\left(m'_\tau,m'_{\tau+1},\ldots,m'_{\tau+n}\right),$$

and each F_{g^*+p}, for $p=0,1,\ldots,L-1$, only affects the $(L-p)^{th}$ component from the right in the tensor product.

We want to show that there exists such a sequence such that when acting on u_λ it produces the λ-path

$$u_{\sigma^{L+1}(\lambda)}\otimes b_\gamma\otimes b_{\mu_L}\otimes b_{\mu_{L-1}}\otimes\ldots\otimes b_{\mu_1}.$$

By Algorithm 2, we have the sequence F_{t^*} where we can apply each of the Kashiwara operators in this sequence a sufficient number of times to certain λ-paths in the set $u_{\sigma^L(\lambda)}\otimes\mathcal{B}_\ell^L$ to obtain the set

$$\begin{aligned}F_{t^*}\left(u_{\sigma^L(\lambda)}\right)\otimes\mathcal{B}_\ell^L&=\tilde{f}_{t^*-1}^{z'_{t^*-1}}\tilde{f}_{t^*-2}^{z'_{t^*-2}}\cdots\tilde{f}_{t^*+1}^{z'_{t^*+1}}\left(u_{\sigma^L(\lambda)}\right)\otimes\mathcal{B}_\ell^L\\&=u_{\sigma^{L+1}(\lambda)}\otimes b_\gamma\otimes\mathcal{B}_\ell^L.\end{aligned}$$

If we choose b_{μ_L} to be the element such that

$$\begin{aligned}F'_{t^*+1}\left(u_{\sigma^{L-1}(\lambda)}\right)\otimes\mathcal{B}_\ell^{L-1}&=\tilde{f}_{t^*}^{z'_{t^*-1}}\tilde{f}_{t^*-1}^{z'_{t^*-2}}\cdots\tilde{f}_{t^*+2}^{z'_{t^*+1}}\left(u_{\sigma^{L-1}(\lambda)}\right)\otimes\mathcal{B}_\ell^{L-1}\\&=u_{\sigma^L(\lambda)}\otimes b_{\mu_L}\otimes\mathcal{B}_\ell^{L-1},\end{aligned}$$

then by our inductive hypothesis the sequence $F'_{t^*+1}F'_{t^*+2}\cdots F_{t^*+L}$ acting on u_λ produces the λ-path $u_{\sigma^L(\lambda)}\otimes b_{\mu_L}\otimes b_{\mu_{L-1}}\otimes\ldots\otimes b_{\mu_1}$, where

$$\begin{aligned}b_{\mu_L}&=F'_{t^*+1}\left(b_{\sigma^{L-1}(\lambda)}\right)\\&=\tilde{f}_{t^*}^{z'_{t^*-1}}\tilde{f}_{t^*-1}^{z'_{t^*-2}}\cdots\tilde{f}_{t^*+2}^{z'_{t^*+1}}(m_L,m_{L+1},\ldots,m_{L-2},m_{L-1})\\&=\big(m_L+z'_n-z'_0,\ldots,m_{t^*+L-1}+z_{t^*-2}-z_{t^*-1},m_{t^*+L}+z_{t^*-1},\\&\qquad m_{t^*+L+1}-z'_{t^*+1},m_{t^*+L+2}+z'_{t^*+1}-z'_{t^*+2},\ldots,m_{L-1}+z'_{n-1}-z'_n\big).\end{aligned}$$

Hence, the (t^*+1)-signature on the λ-path $u_{\sigma^L(\lambda)} \otimes b_{\mu_L} \otimes b_{\mu L-1} \otimes \ldots \otimes b_{\mu_1}$ is of the form

$$\begin{aligned}&\left(\ldots, -_{m_{t^*+L+2}} +_{m_{t^*+L+1}}, -_{m_{t^*+L+1}-z_{t^*+1}} +_{m_{t^*+L}+z_{t^*-1}}, *_{L-1}, *_{L-2}, \ldots, *_1\right)\\ =&\left(+_{z_{t^*+1}}, +_{m_{t^*+L}+z_{t^*-1}}, *_{L-1}, *_{L-2}, \ldots, *_1\right),\end{aligned}$$

where the $*_j$ represent sequences of minuses then pluses that don't affect the leftmost z_{t^*+1} pluses, by the inductive hypothesis. So, we can apply $\tilde{f}_{t^*+1}^{z'_{t^*+1}}$ and it will only affect the $(L+1)^{st}$ component from the right in the tensor product.

We thus obtain the λ-path:

$$u_{\sigma^{L+1}(\lambda)} \otimes b'_\nu \otimes b_{\mu_L} \otimes b_{\mu_{L-1}} \otimes \ldots \otimes b_{\mu_1},$$

where $b'_\nu = \left(m_{L+1}, m_{L+2}, \ldots, m_{t^*+L+1} - z'_{t^*+1}, m_{t^*+L+2} + z'_{t^*+1}, m_{t^*+L+3}, \ldots, m_L\right)$.

Now we want to apply $\tilde{f}_i^{z'_i}$. If we look at the (i)-signature for each of the λ-paths obtained after applying up to the Kashiwara operator $\tilde{f}_{i-1}^{z'_{i-1}}$, we'll notice that the only component in the tensor product that is affected is the one in the $(L+1)^{st}$ component, for $i = t^*+2, t^*+3, \ldots, t^*$:

$$\left(\ldots, -_{m_{L+i+2}} +_{m_{L+i+1}}, -_{m_{L+i+1}} +_{m_{L+i}+z'_{i-1}}, -_{m_{L+i}+z'_{i-1}-z'_i} +_{m_{L+i-1}-z'_{i-1}}, *_{L-1}, *_{L-2}, \ldots, *_1\right)$$

$$= \left(+_{z'_i}, +_{m_{L+i}-z'_{i-1}}, *_{L-1}, *_{L-2}, \ldots, *_1\right).$$

Hence,

$$\begin{aligned}F_{t^*}\left(u_{\sigma^L(\lambda)} \otimes b_{\mu_L} \otimes b_{\mu_{L-1}} \otimes \ldots \otimes b_{\mu_1}\right) &= u_{\sigma^{L+1}(\lambda)} \otimes F_{t^*}\left(b_{\sigma^L(\lambda)}\right) \otimes b_{\mu_L} \otimes \ldots \otimes b_{\mu_1}\\ &= u_{\sigma^{L+1}(\lambda)} \otimes b_\gamma \otimes b_{\mu_L} \otimes \ldots \otimes b_{\mu_1},\end{aligned}$$

which is what we wanted to show. □

Given the ground-state path

$$u_\lambda = u_{\sigma^2(\lambda)} \otimes (m_2, m_3, \ldots, m_1) \otimes (m_1, m_2, \ldots, m_0)$$

and elements b_ν, $b_\omega \in \mathcal{B}_\ell$, we want to produce a sequence G such that

$$G(u_\lambda) = u_{\sigma^2(\lambda)} \otimes b_\nu \otimes b_\omega$$

$$= u_{\sigma^2(\lambda)} \otimes (v_1, v_2, \ldots, v_0) \otimes (w_1, w_2, \ldots, w_0).$$

We'll start with an example, and then generalize the process for any λ-path.

Example 6.15. Suppose the ground-state path is given as

$$\begin{aligned} u_\lambda &= \ldots \otimes (4,7,3,1,0,2) \otimes (2,4,7,3,1,0) \\ &= \ldots \otimes (m_2, m_3, \ldots, m_1) \otimes (m_1, m_2, \ldots, m_0) \end{aligned}$$

and we want to find a sequence that produces the λ-path

$$u_{\sigma^2(\lambda)} \otimes (2,1,0,9,2,3) \otimes (5,0,7,1,0,4).$$

Hence, $b_\nu = (2,1,0,9,2,3)$ and $b_\omega = (5,0,7,1,0,4)$. By Algorithm 2, we construct the sequence

$$\begin{aligned} F_{t_\nu^*} &= \tilde{f}_5^{z_5'} \tilde{f}_4^{z_4'} \tilde{f}_3^{z_3'} \tilde{f}_2^{z_2'} \tilde{f}_1^{z_1'} \\ &= \tilde{f}_5 \tilde{f}_4^3 \tilde{f}_3^{11} \tilde{f}_2^8 \tilde{f}_1^2 \end{aligned}$$

where $F_{t_\nu^*}\left(u_{\sigma(\lambda)}\right) \otimes \mathcal{B}_{17} = u_{\sigma^2(\lambda)} \otimes b_\nu \otimes \mathcal{B}_{17}$. Hence $t_\nu^* = 0$ for this example. We will work backwards on the λ-path

$$u_{\sigma^2(\lambda)} \otimes (2,1,0,9,2,3) \otimes (5,0,7,1,0,4)$$

to determine whether we need to alter the exponents in the sequence $F_{t_\nu^*}$. The λ-path

$$\begin{aligned} & u_{\sigma^2(\lambda)} \otimes (v_1, v_2, v_3, v_4, v_5 + z_5', v_0 - z_5') \otimes (w_1, w_2, w_3, w_4, w_5, w_0) \\ = \quad & u_{\sigma^2(\lambda)} \otimes (2,1,0,9,3,2) \otimes (5,0,7,1,0,4) \end{aligned}$$

has (5)-signature

$$(\ldots, -_4 +_2, -_2 +_3, -_4 +_0) = (+_3, -_4 +_0)$$

which means $\tilde{f}_5$ acting on this λ-path won't produce our desired path $u_{\sigma^2(\lambda)} \otimes b_\nu \otimes b_\omega$. We need there to be at least 1 plus in the second component from right in order to change the second component. Hence, we need the $-_4$ to be a $-_2$ instead. Make the following definitions and changes:

$$\begin{aligned} a_5 &:= w_0 - v_5 = 4 - 2 = 2, \\ c_0 &:= v_5 = 2, \\ w_5 &\to w_5 + a_5 = 2, \end{aligned}$$

$$\tilde{f}_5 \to \tilde{f}_5^{1+a_5} = \tilde{f}_5^3.$$

Then we have

$$\begin{aligned}
& u_{\sigma^2(\lambda)} \otimes (v_1, v_2, v_3, v_4, v_5 + z_5', m_1) \otimes (w_1, w_2, w_3, w_4, w_5 + a_5, c_0) \\
= \quad & u_{\sigma^2(\lambda)} \otimes (2,1,0,9,3,2) \otimes (5,0,7,1,2,2) \\
\xrightarrow{\tilde{f}_5^3} \quad & u_{\sigma^2(\lambda)} \otimes b_\nu \otimes b_\omega.
\end{aligned}$$

Now, let's determine if we need to change the exponent of the Kashiwara operator $\tilde{f}_4^3$. The λ-path

$$\begin{aligned}
& u_{\sigma^2(\lambda)} \otimes (v_1, v_2, v_3, v_4 + z_4', v_5 + z_5' - z_4', m_1) \otimes (w_1, w_2, w_3, w_4, w_5 + a_5, c_0) \\
= \quad & u_{\sigma^2(\lambda)} \otimes (2,1,0,12,0,2) \otimes (5,0,7,1,2,2)
\end{aligned}$$

has (4)-signature

$$(\ldots, -_2 +_0, -_0 +_{12}, -_2 +_1) = (+_{10}, +_1)$$

which means $\tilde{f}_4^3$ acting on this λ-path will produce our desired λ-path

$$u_{\sigma^2(\lambda)} \otimes (2,1,0,9,3,2) \otimes (5,0,7,1,2,2).$$

Make the following definitions:

$$\begin{aligned}
a_4 &:= 0, \\
c_5 &:= w_5 + a_5 = 2.
\end{aligned}$$

Then we have

$$\begin{aligned}
& u_{\sigma^2(\lambda)} \otimes (v_1, v_2, v_3, v_4 + z_4', m_0, m_1) \otimes (w_1, w_2, w_3, w_4, c_5, c_0) \\
= \quad & u_{\sigma^2(\lambda)} \otimes (2,1,0,12,0,2) \otimes (5,0,7,1,2,2) \\
\xrightarrow{\tilde{f}_4^3} \quad & u_{\sigma^2(\lambda)} \otimes (2,1,0,9,3,2) \otimes (5,0,7,1,2,2).
\end{aligned}$$

Now, let's determine if we need to change the exponent of the Kashiwara operator $\tilde{f}_3^{11}$. The λ-path

$$\begin{aligned}
& u_{\sigma^2(\lambda)} \otimes (v_1, v_2, v_3 + z_3', v_4 + z_4' - z_3', m_0, m_1) \otimes (w_1, w_2, w_3, w_4, c_5, c_0) \\
= \quad & u_{\sigma^2(\lambda)} \otimes (2,1,11,1,0,2) \otimes (5,0,7,1,2,2)
\end{aligned}$$

has (3)-signature

$$(\ldots,-_0+_1,-_1+_{11},-_1+_7)=(+_{10},+_7)$$

which means $\tilde{f}_3^{11}$ acting on this λ-path won't produce our desired λ-path

$$u_{\sigma^2(\lambda)}\otimes(2,1,0,12,0,2)\otimes(5,0,7,1,2,2).$$

Make the following definitions and changes:

$$\begin{aligned}
a_3 &:= w_4-v_3=1-0=1,\\
c_4 &:= v_3=0,\\
w_3 &\longrightarrow w_3+a_3,\\
\tilde{f}_3^{11} &\longrightarrow \tilde{f}_3^{11+a_3}=\tilde{f}_3^{12}.
\end{aligned}$$

Then we have

$$\begin{aligned}
& u_{\sigma^2(\lambda)}\otimes(v_1,v_2,v_3+z_3',m_5,m_0,m_1)\otimes(w_1,w_2,w_3+a_3,c_4,c_5,c_0)\\
= \quad & u_{\sigma^2(\lambda)}\otimes(2,1,11,1,0,2)\otimes(5,0,8,0,2,2)\\
\xrightarrow{\tilde{f}_3^{12}} \quad & u_{\sigma^2(\lambda)}\otimes(2,1,0,12,0,2)\otimes(5,0,7,1,2,2).
\end{aligned}$$

Now, let's determine if we need to change the exponent of the Kashiwara operator $\tilde{f}_2^8$. The λ-path

$$\begin{aligned}
& u_{\sigma^2(\lambda)}\otimes(v_1,v_2+z_2',v_3+z_3'-z_2',m_5,m_0,m_1)\otimes(w_1,w_2,w_3+a_3,c_4,c_5,c_0)\\
= \quad & u_{\sigma^2(\lambda)}\otimes(2,9,3,1,0,2)\otimes(5,0,8,0,2,2)
\end{aligned}$$

has (2)-signature

$$(\ldots,-_1+_3,-_3+_9,-_8+_0)=(+_1,+_0)$$

which means $\tilde{f}_2^8$ acting on this λ-path won't produce our desired λ-path

$$u_{\sigma^2(\lambda)}\otimes(2,1,11,1,0,2)\otimes(5,0,8,0,2,2).$$

Make the following definitions and changes:

$$a_2 := w_3+a_3-v_2=7+1-1=7,$$

$$\begin{aligned}
c_3 &:= v_2 = 1,\\
w_2 &\to w_2 + a_2 = 0+7 = 7,\\
\tilde{f}_2^8 &\to \tilde{f}_2^{8+a_2} = \tilde{f}_2^{15}.
\end{aligned}$$

Then we have

$$\begin{aligned}
& u_{\sigma^2(\lambda)} \otimes (v_1, v_2 + z_2', m_4, m_5, m_0, m_1) \otimes (w_1, w_2 + a_2, c_3, c_4, c_5, c_0)\\
= \; & u_{\sigma^2(\lambda)} \otimes (2,9,3,1,0,2) \otimes (5,7,1,0,2,2)\\
\xrightarrow{\tilde{f}_2^{15}} \; & u_{\sigma^2(\lambda)} \otimes (2,1,11,1,0,2) \otimes (5,0,8,0,2,2).
\end{aligned}$$

Now, let's determine if we need to change the exponent of the Kashiwara operator $\tilde{f}_1^2$. The λ-path

$$\begin{aligned}
& u_{\sigma^2(\lambda)} \otimes (v_1 + z_1', v_2 + z_2' - z_1', m_4, m_5, m_0, m_1) \otimes (w_1, w_2 + a_2, c_3, c_4, c_5, c_0)\\
= \; & u_{\sigma^2(\lambda)} \otimes (4,7,3,1,0,2) \otimes (5,7,1,0,2,2)
\end{aligned}$$

has (1)-signature

$$(\ldots, -_3 +_7, -_7 +_4, -_7 +_5) = (+_4, -_7 +_5)$$

which means $\tilde{f}_1^2$ acting on this λ-path won't produce our desired λ-path

$$u_{\sigma^2(\lambda)} \otimes (2,9,3,1,0,2) \otimes (5,7,1,0,2,2).$$

Make the following definitions and changes:

$$\begin{aligned}
a_1 &:= w_2 + a_2 - v_1 = 0+7-2 = 5,\\
c_2 &:= v_1 = 2,\\
c_1 &:= w_1 + a_1,\\
w_1 &\to w_1 + a_1 = 5+5 = 10,\\
\tilde{f}_1^2 &\to \tilde{f}_1^{2+a_1} = \tilde{f}_1^{7}.
\end{aligned}$$

Then we have

$$\begin{aligned}
& u_{\sigma^2(\lambda)} \otimes (v_1 + z_1, m_3, m_4, m_5, m_0, m_1) \otimes (c_1, c_2, c_3, c_4, c_5, c_0)\\
= \; & u_{\sigma^2(\lambda)} \otimes (m_2, m_3, \ldots, m_1) \otimes c\\
= \; & u_{\sigma^2(\lambda)} \otimes (4,7,3,1,0,2) \otimes (10,2,1,0,2,2)
\end{aligned}$$

$$
\begin{aligned}
&= \quad u_{\sigma(\lambda)} \otimes (10,2,1,0,2,2) \\
&\xrightarrow{\tilde{f}_2^{15}} \quad u_{\sigma^2(\lambda)} \otimes (2,9,3,1,0,2) \otimes (5,7,1,0,2,2).
\end{aligned}
$$

To summarize, we have that

$$\tilde{f}_5^3 \tilde{f}_4^3 \tilde{f}_3^{12} \tilde{f}_2^{15} \tilde{f}_1^7 \left(u_{\sigma(\lambda)} \otimes c\right) = u_{\sigma^2(\lambda)} \otimes b_\nu \otimes b_\omega.$$

Now we need to construct a sequence of the form

$$\hat{F}'_{t^*_\nu+1} = \tilde{f}_0^{q_0} \tilde{f}_5^{q_5} \cdots \tilde{f}_2^{q_2}$$

such that $\hat{F}'_{t^*_\nu+1}(u_\lambda) = u_{\sigma(\lambda)} \otimes c$. Note that if we use Algorithm 1, we obtain the sequence

$$\hat{F}'_{t^*_\nu+1} = \tilde{f}_0^8 \tilde{f}_5^{10} \tilde{f}_4^{11} \tilde{f}_3^8 \tilde{f}_2^2.$$

This is the sequence we want; however, in Algorithm 3, we'll use a different method to construct this sequence. We need to make sure that we don't need to apply $\tilde{f}_0$ at all, and this new method will do just that. Thus, we have $\left(\tilde{f}_5^3 \tilde{f}_4^3 \tilde{f}_3^{12} \tilde{f}_2^{15} \tilde{f}_1^7\right)\left(\tilde{f}_0^8 \tilde{f}_5^{10} \tilde{f}_4^{11} \tilde{f}_3^8 \tilde{f}_2^2\right)(u_\lambda) = u_{\sigma^2(\lambda)} \otimes b_\nu \otimes b_\omega$.

Now we generalize the process explained in Example 6.15.
Algorithm 3: Given the λ-path

$$u_\lambda = u_{\sigma^2(\lambda)} \otimes (m_2, m_1, \dots, m_1) \otimes (m_1, m_2, \dots, m_0)$$

we want to produce a sequence G of length at most $2n$ such that

$$
\begin{aligned}
G(u_\lambda) &= u_{\sigma^2(\lambda)} \otimes b_\nu \otimes b_\omega \\
&= u_{\sigma^2(\lambda)} \otimes (v_1, v_2, \dots, v_0) \otimes (w_1, w_2, \dots, w_0).
\end{aligned}
$$

By Algorithm 2, we know that there exists a sequence

$$F_{t^*_\nu} = \tilde{f}_{t^*_\nu-1}^{z'_{t^*_\nu-1}} \tilde{f}_{t^*_\nu-2}^{z'_{t^*_\nu-2}} \cdots \tilde{f}_{t^*_\nu+1}^{z'_{t^*_\nu+1}}$$

such that

$$F_{t^*_\nu}\left(u_{\sigma(\lambda)}\right) \otimes \mathcal{B}_\ell = u_{\sigma^2(\lambda)} \otimes b_\nu \otimes \mathcal{B}_\ell.$$

So, by Lemma 6.14, we know we're going to need to produce a sequence of Kashiwara operators of the form

$$\left(\tilde{f}_{t_\nu^*-1}^{z'_{t_\nu^*-1}}\tilde{f}_{t_\nu^*-2}^{z'_{t_\nu^*-2}}\cdots\tilde{f}_{t_\nu^*+1}^{z'_{t_\nu^*+1}}\right)\left(\tilde{f}_{t_\nu^*}^{z'_{t_\nu^*-1}}\tilde{f}_{t_\nu^*-1}^{z'_{t_\nu^*-2}}\cdots\tilde{f}_{t_\nu^*+2}^{z'_{t_\nu^*+1}}\right)$$

with possibly different exponents, such that when acting on u_λ it produces the λ-path $u_{\sigma^2(\lambda)}\otimes b_\nu\otimes b_\omega$.

Define, as in Algorithm 1, the notation

$$(\rho)_+ = \max\{\rho, 0\}.$$

Just as in Example 6.15, we'll start by working backwards. The λ-path

$$u_{\sigma^2(\lambda)}\otimes(v_1, v_2, \dots, v_{t_\nu^*-1}+z'_{t_\nu^*-1}, v_{t_\nu^*}-z'_{t_\nu^*}, v_{t_\nu^*+1}, \dots, v_0)\otimes(w_1, w_2, \dots, w_0)$$

has (t_ν^*-1)-signature

$$\begin{aligned}
&\left(\dots, -_{m_{t_\nu^*+2}}+_{m_{t_\nu^*+1}}, -_{v_{t_\nu^*}-z'_{t_\nu^*-1}}+_{v_{t_\nu^*-1}+z'_{t_\nu^*-1}}, -_{w_{t_\nu^*}}+_{w_{t_\nu^*-1}}\right)\\
=\ &\left(\dots, -_{m_{t_\nu^*+2}}+_{m_{t_\nu^*+1}}, -_{m_{t_\nu^*+1}}+_{v_{t_\nu^*-1}+z'_{t_\nu^*-1}}, -_{w_{t_\nu^*}}+_{w_{t_\nu^*-1}}\right)\\
=\ &\left(+_{v_{t_\nu^*-1}+z'_{t_\nu^*-1}}, -_{w_{t_\nu^*}}+_{w_{t_\nu^*-1}}\right)\\
=\ &\begin{cases}\left(+_{v_{t_\nu^*-1}+z'_{t_\nu^*-1}-w_{t^*}}, +_{w_{t_\nu^*-1}}\right), & \text{if } v_{t_\nu^*-1}+z_{t_\nu^*-1} > w_{t_\nu^*},\\ \left(+_{w_{t_\nu^*-1}}\right), & \text{else.}\end{cases}
\end{aligned}$$

If $z'_{t_\nu^*-1}=0$, then we don't care what the (t_ν^*-1)-signature is, because we won't be applying the corresponding Kashiwara operator. If this were true, we'd set $a_{t_\nu^*-1}=0$. However, if $z'_{t_\nu^*-1}\neq 0$, then we need to ensure that we have at least $z'_{t_\nu^*-1}$ pluses in the second component from right in the (t_ν^*-1)-signature. This will only happen if $w_{t_\nu^*}\leq v_{t_\nu^*}$. If, however, this situation is not true, then we need to change the t_ν^*-1 and t_ν^* components to enforce this inequality.

Define the following:

$$a_{t_\nu^*-1} := \begin{cases}\left(w_{t^*\nu}-v_{t_\nu^*-1}\right)_+, & \text{if } z_{t_\nu^*-1}\neq 0\\ 0, & \text{else}\end{cases}$$

$$c_{t_\nu^*} := \begin{cases} v_{t_\nu^*-1}, & \text{if } a_{t_\nu^*-1}>0\\ w_{t_\nu^*}, & \text{else}\end{cases}$$

and make the following changes:

$$\begin{aligned}
w_{t_\nu^*} &\to c_{t_\nu^*},\\
w_{t_\nu^*-1} &\to w_{t_\nu^*-1} + a_{t_\nu^*-1},\\
\tilde{f}_{t_\nu^*-1}^{z'_{t_\nu^*-1}} &\to \tilde{f}_{t_\nu^*-1}^{z'_{t_\nu^*-1}+a_{t_\nu^*-1}}.
\end{aligned}$$

Also, recall that by Equation 6.19, we know the following:

$$\begin{aligned}
v_{t_\nu^*+1} &= m_{t_\nu^*+2} - z'_{t_\nu^*+1},\\
v_{t_\nu^*} &= m_{t_\nu^*+1} + z'_{t_\nu^*-1},\\
v_i &= m_{i+1} + z'_{i-1} - z'_i, \text{ for } i \neq t_\nu^*, t_\nu^*+1,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
m_{t_\nu^*+2} &= v_{t_\nu^*+1} + z'_{t_\nu^*+1},\\
m_{t_\nu^*+1} &= v_{t_\nu^*} - z'_{t_\nu^*-1},\\
m_{i+1} &= v_i - z'_{i-1} + z'_i, \text{ for } i \neq t_\nu^*, t_\nu^*+1.
\end{aligned}$$

Then we have

$$\begin{aligned}
& u_{\sigma^2(\lambda)} \otimes (v_1, v_2, \dots, v_{t_\nu^*-1} + z'_{t_\nu^*-1}, m_{t_\nu^*+1}, v_{t_\nu^*+1}, \dots, v_0)\\
&\qquad \otimes (w_1, w_2, \dots, w_{t_\nu^*-1} + a_{t_\nu^*-1}, c_{t_\nu^*}, w_{t_\nu^*+1}, \dots, w_0)\\
\xrightarrow{\tilde{f}_{t_\nu^*-1}^{z'_{t_\nu^*-1}+a_{t_\nu^*-1}}} \quad & u_{\sigma^2(\lambda)} \otimes b_\nu \otimes b_\omega.
\end{aligned}$$

Now, let's determine if we need to change the exponent of the Kashiwara operator $\tilde{f}_{t_\nu^*-2}^{z'_{t_\nu^*-2}}$. The λ-path

$$\begin{aligned}
& u_{\sigma^2(\lambda)} \otimes (v_1, v_2, \dots, v_{t_\nu^*-2} + z'_{t_\nu^*-2}, v_{t_\nu^*-1} + z'_{t_\nu^*-1} - z'_{t_\nu^*-2}, m_{t_\nu^*+1}, v_{t_\nu^*+1}, \dots, v_0)\\
&\qquad \otimes (w_1, w_2, \dots, w_{t_\nu^*-1} + a_{t_\nu^*-1}, c_{t_\nu^*}, w_{t_\nu^*+1}, \dots, w_0)\\
= \quad & u_{\sigma^2(\lambda)} \otimes (v_1, v_2, \dots, v_{t_\nu^*-2} + z'_{t_\nu^*-2}, m_{t_\nu^*}, m_{t_\nu^*+1}, v_{t_\nu^*+1}, \dots, v_0)\\
&\qquad \otimes (w_1, w_2, \dots, w_{t_\nu^*-1} + a_{t_\nu^*-1}, c_{t_\nu^*}, w_{t_\nu^*+1}, \dots, w_0).
\end{aligned}$$

has $(t^*_\nu-2)$-signature

$$\begin{aligned}
& \left(\ldots, -_{m_{t^*_\nu+1}} +_{m_{t^*_\nu}}, -_{m_{t^*_\nu}} +_{v_{t^*_\nu-2}+z'_{t^*_\nu-2}}, -_{w_{t^*_\nu-1}+a_{t^*_\nu-1}} +_{w_{t^*_\nu-2}}\right) \\
= & \left(+_{v_{t^*_\nu-2}+z'_{t^*_\nu-2}}, -_{w_{t^*_\nu-1}+a_{t^*_\nu-1}} +_{w_{t^*_\nu-2}}\right) \\
= & \begin{cases} \left(+_{v_{t^*_\nu-2}+z'_{t^*_\nu-2}-w_{t^*_\nu-1}-a_{t^*_\nu-1}}, +_{w_{t^*_\nu-2}}\right), & \text{if } v_{t^*_\nu-2}+z'_{t^*_\nu-2}-w_{t^*_\nu-1}-a_{t^*_\nu-1}>0, \\ \left(+_{w_{t^*_\nu-2}}\right), & \text{else.} \end{cases}
\end{aligned}$$

Similarly to the $t^*_\nu-1$ case, if $z'_{t^*_\nu-2}=0$, then we don't care what the $(t^*_\nu-2)$-signature is, because we won't be applying the corresponding Kashiwara operator. If this were true, we'd set $a_{t^*_\nu-2}=0$. However, if $z'_{t^*_\nu-2}\neq 0$, then we need to ensure that we have at least $z'_{t^*_\nu-2}$ pluses in the second component from right in the $(t^*_\nu-2)$-signature. This will only happen if $w_{t^*_\nu-1}+a_{t^*\nu}\leq v_{t^*_\nu-2}$. If, however, this situation is not true, then we need to change the $t^*_\nu-2$ and $t^*_\nu-1$ components to enforce this inequality.

Define the following:

$$a_{t^*_\nu-2} := \begin{cases} \left(w_{t^*\nu-1}+a_{t^*_\nu-1}-v_{t^*_\nu-2}\right)_+, & \text{if } z_{t^*_\nu-2}\neq 0 \\ 0, & \text{else} \end{cases}$$

$$c_{t^*_\nu-1} := \begin{cases} v_{t^*_\nu-2}, & \text{if } a_{t^*_\nu-2}>0 \\ w_{t^*_\nu-1}+a_{t^*_\nu-1}, & \text{else} \end{cases}$$

and make the following changes:

$$\begin{aligned}
w_{t^*_\nu-1}+a_{t^*_\nu-1} &\to c_{t^*_\nu-1}, \\
w_{t^*_\nu-2} &\to w_{t^*_\nu-2}+a_{t^*_\nu-2}, \\
\tilde{f}^{z'_{t^*_\nu-2}}_{t^*_\nu-2} &\to \tilde{f}^{z'_{t^*_\nu-2}+a_{t^*_\nu-2}}_{t^*_\nu-2}.
\end{aligned}$$

Then we have

$$\begin{aligned}
& u_{\sigma^2(\lambda)}\otimes(v_1,v_2,\ldots,v_{t^*_\nu-2}+z'_{t^*_\nu-2},m_{t^*_\nu},m_{t^*_\nu+1},v_{t^*_\nu+1},\ldots,v_0) \\
& \qquad \otimes(w_1,w_2,\ldots,w_{t^*_\nu-2}+a_{t^*_\nu-2},c_{t^*_\nu-1},c_{t^*_\nu},w_{t^*_\nu+1},\ldots,w_0) \\
\xrightarrow{\tilde{f}^{z'_{t^*_\nu-2}+a_{t^*_\nu-2}}_{t^*_\nu-2}} \quad & u_{\sigma^2(\lambda)}\otimes(v_1,v_2,\ldots,v_{t^*_\nu-1}+z'_{t^*_\nu-1},m_{t^*_\nu+1},v_{t^*_\nu+1},\ldots,v_0)
\end{aligned}$$

$$\otimes(w_1, w_2, \dots, w_{t^*_\nu -1} + a_{t^*_\nu -1}, c_{t^*_\nu}, w_{t^*_\nu +1}, \dots, w_0).$$

In general, if we define

$$a_i := \begin{cases} (w_{i+1} + a_{i+1} - v_i)_+, & \text{if } z'_i \neq 0 \\ 0, & \text{else} \end{cases}$$
$$c_{i+1} := \begin{cases} v_i, & \text{if } a_i > 0 \\ w_{i+1} + a_{i+1}, & \text{else} \end{cases},$$

then we have the following

$$\begin{aligned} & u_{\sigma^2(\lambda)} \otimes \left(m_2, m_3, \dots, m_{t^*_\nu +1}, v_{t^*_\nu +1}, \dots, v_i + z'_i, m_{i+2}, \dots, m_1\right) \\ & \qquad \otimes \left(c_1, c_2, \dots, c_{t^*_\nu}, w_{t^*_\nu +1}, \dots, w_i + a_i, c_{i+1}, \dots, c_0\right) \\ \xrightarrow{\tilde{f}_i^{z'_i + a_i}} \quad & u_{\sigma^2(\lambda)} \otimes \left(m_2, m_3, \dots, m_{t^*_\nu +1}, v_{t^*_\nu +1}, \dots, v_{i+1} + z'_{i+1}, m_{i+3}, \dots, m_1\right) \\ & \qquad \otimes \left(c_1, c_2, \dots, c_{t^*_\nu}, w_{t^*_\nu +1}, \dots, w_{i+1} + a_{i+1}, c_{i+2}, \dots, c_0\right). \end{aligned}$$

Thus, we end with

$$\begin{aligned} & u_{\sigma^2(\lambda)} \otimes \left(m_2, m_3, \dots, m_{t^*_\nu +1}, v_{t^*_\nu +1} + z'_{t^*_\nu +1}, m_{t^*_\nu +3}, \dots, m_1\right) \\ & \qquad \otimes \left(c_1, c_2, \dots, c_{t^*_\nu}, w_{t^*_\nu +1} + a_{t^*_\nu +1}, c_{t^*_\nu +2}, \dots, c_0\right) \\ = \quad & u_{\sigma^2(\lambda)} \otimes (m_2, m_3, \dots, m_1) \\ & \qquad \otimes (c_1, c_2, \dots, c_0) \\ = \quad & u_{\sigma(\lambda)} \otimes c \\ \xrightarrow{\tilde{f}_{t^*_\nu +1}^{z'_{t^*_\nu +1} + a_{t^*_\nu +1}}} \quad & u_{\sigma^2(\lambda)} \otimes \left(m_2, m_3, \dots, m_{t^*_\nu +1}, v_{t^*_\nu +1}, v_{t^*_\nu +1} + z'_{t^*_\nu +2}, m_{t^*_\nu +4}, \dots, m_1\right) \\ & \qquad \otimes \left(c_1, c_2, \dots, c_{t^*_\nu}, w_{t^*_\nu +1}, w_{t^*_\nu +2} + a_{t^*_\nu +2}, c_{t^*_\nu +3}, \dots, c_0\right). \end{aligned}$$

To summarize, we have that

$$\hat{F}_{t^*_\nu} = \tilde{f}_{t^*_\nu -1}^{z'_{t^*_\nu -1} + a_{t^*_\nu -1}} \tilde{f}_{t^*_\nu -2}^{z'_{t^*_\nu -2} + a_{t^*_\nu -2}} \cdots \tilde{f}_{t^*_\nu +1}^{z'_{t^*_\nu +1} + a_{t^*_\nu +1}} \tag{6.20}$$

where

$$\hat{F}_{t^*_\nu}\left(u_{\sigma(\lambda)} \otimes c\right) = u_{\sigma^2(\lambda)} \otimes b_\nu \otimes b_\omega.$$

And we define:

$$\begin{aligned}
a_{t_\nu^*} &:= 0\\
a_i &:= \begin{cases} (w_{i+1}+a_{i+1}-v_i)_+, & \text{if } z_i' \neq 0\\ 0, & \text{else}\end{cases}\\
c_{t_\nu^*+1} &:= w_{t_\nu^*+1}+a_{t_\nu^*+1}\\
c_{i+1} &:= \begin{cases} v_i, & \text{if } a_i > 0\\ w_{i+1}+a_{i+1}, & \text{else}\end{cases}
\end{aligned}$$

for $i \neq t_\nu^*$.

Now we need to show that there exists a sequence

$$\hat{F}'_{t_\nu^*+1} = \tilde{f}_{t_\nu^*}^{q_{t_\nu^*}} \tilde{f}_{t_\nu^*-1}^{q_{t_\nu^*-1}} \cdots \tilde{f}_{t_\nu^*+2}^{q_{t_\nu^*+2}}$$

such that

$$\hat{F}'_{t_\nu^*+1}(u_\lambda) = u_{\sigma(\lambda)} \otimes c.$$

If we let this sequence act on u_λ, we have

$$\begin{aligned}
& \tilde{f}_{t_\nu^*}^{q_{t_\nu^*}} \tilde{f}_{t_\nu^*-1}^{q_{t_\nu^*-1}} \cdots \tilde{f}_{t_\nu^*+3}^{q_{t_\nu^*+3}} \big(u_{\sigma(\lambda)} \otimes \big(m_1,\dots,m_{t_\nu^*+2}-q_{t_\nu^*+2}, m_{t_\nu^*+3}+q_{t_\nu^*+2}, m_{t_\nu^*+4},\dots,m_0\big)\big)\\
= \;& \tilde{f}_{t_\nu^*}^{q_{t_\nu^*}} \cdots \tilde{f}_{t_\nu^*+4}^{q_{t_\nu^*+4}} \big(u_{\sigma(\lambda)} \otimes \big(m_1,\dots,m_{t_\nu^*+2}-q_{t_\nu^*+2}, m_{t_\nu^*+3}+q_{t_\nu^*+2}-q_{t_\nu^*+3}, m_{t_\nu^*+4}+q_{t_\nu^*+3},\dots,m_0\big)\big)\\
= \;& \dots\\
= \;& u_{\sigma(\lambda)} \otimes \big(m_1+q_0-q_1,\dots,m_{t_\nu^*}+q_{t_\nu^*-1}-q_{t_\nu^*}, m_{t_\nu^*+1}+q_{t_\nu^*},\\
& \qquad m_{t_\nu^*+2}-q_{t_\nu^*+2}, m_{t_\nu^*+3}+q_{t_\nu^*+2}-q_{t_\nu^*+3},\dots,m_0+q_n-q_0\big).
\end{aligned}$$

Setting this equal to the right-hand side $u_{\sigma(\lambda)} \otimes c$, we obtain the following equations:

$$\begin{aligned} q_1 &= m_1 + q_0 - c_1, \\ q_2 &= m_2 + q_1 - c_2, \\ &\vdots \\ q_{t_\nu^*-1} &= m_{t_\nu^*-1} + q_{t_\nu^*-2} - c_{t_\nu^*-1}, \\ q_{t_\nu^*} &= m_{t_\nu^*} + q_{t_\nu^*-1} - c_{t_\nu^*}, \\ &= w_{t_\nu^*+1} + a_{t_\nu^*+1} - m_{t_\nu^*+1}, \end{aligned} \qquad \begin{aligned} q_{t_\nu^*+2} &= m_{t_\nu^*+2} - c_{t_\nu^*+2}, \\ q_{t_\nu^*+3} &= m_{t_\nu^*+3} + q_{t_\nu^*+2} - c_{t_\nu^*+3}, \\ q_{t_\nu^*+4} &= m_{t_\nu^*+4} + q_{t_\nu^*+3} - c_{t_\nu^*+4}, \\ &\vdots \\ q_0 &= m_0 + q_n - c_0. \end{aligned}$$

If we start with the equation

$$q_{t_\nu^*} = w_{t_\nu^*+1} + a_{t_\nu^*+1} - m_{t_\nu^*+1},$$

then we can find a formula for $q_{t_\nu^*-1}$:

$$\begin{aligned} q_{t_\nu^*-1} &= q_{t_\nu^*} - m_{t_\nu^*} + c_{t_\nu^*} \\ &= w_{t_\nu^*+1} + a_{t_\nu^*+1} - m_{t_\nu^*+1} - m_{t_\nu^*} + c_{t_\nu^*}. \end{aligned}$$

Next we can solve for $q_{t_\nu^*-2}$:

$$\begin{aligned} q_{t_\nu^*-2} &= q_{t_\nu^*-1} - m_{t_\nu^*-1} + c_{t_\nu^*-1} \\ &= \left(w_{t_\nu^*+1} + a_{t_\nu^*+1} - (m_{t_\nu^*} + m_{t_\nu^*+1}) + (c_{t_\nu^*})\right) - m_{t_\nu^*-1} + c_{t_\nu^*-1} \\ &= w_{t_\nu^*+1} + a_{t_\nu^*+1} - (m_{t_\nu^*-1} + m_{t_\nu^*} + m_{t_\nu^*+1}) + (c_{t_\nu^*-1} + c_{t_\nu^*}). \end{aligned}$$

In general, for $i \neq t_\nu^* + 1$, we have

$$q_i = w_{t_\nu^*+1} + a_{t_\nu^*+1} - (m_{i+1} + m_{i+2} + \cdots + m_{t_\nu^*} + m_{t_\nu^*+1}) + (c_{i+1} + c_{i+2} + \cdots + c_{t_\nu^*-1} + c_{t_\nu^*}).$$

Note that we'll have two equations for $q_{t_\nu^*+2}$:

$$q_{t_\nu^*+2} = m_{t_\nu^*+2} - c_{t_\nu^*+2}$$

and

$$q_{t_\nu^*+2} = w_{t_\nu^*+1} + a_{t_\nu^*+1} - (m_{t_\nu^*+3} + m_{t_\nu^*+4} + \cdots + m_{t_\nu^*+1}) + (c_{t_\nu^*+3} + c_{t_\nu^*+4} + \cdots + c_{t_\nu^*}).$$

Are these equivalent?

$$\begin{aligned}
q_{t_\nu^*+2} &= w_{t_\nu^*+1} + a_{t_\nu^*+1} - (m_{t_\nu^*+3} + m_{t_\nu^*+4} + \cdots + m_{t_\nu^*+1}) + (c_{t_\nu^*+3} + c_{t_\nu^*+4} + \cdots + c_{t_\nu^*}) \\
&= w_{t_\nu^*+1} + a_{t_\nu^*+1} - \left(\ell - m_{t_\nu^*+2}\right) + \left(\ell - c_{t_\nu^*+2} - c_{t_\nu^*+1}\right) \\
&= w_{t_\nu^*+1} + a_{t_\nu^*+1} + m_{t_\nu^*+2} - c_{t_\nu^*+2} - c_{t_\nu^*+1} \\
&= m_{t_\nu^*+2} - c_{t_\nu^*+2},
\end{aligned}$$

since $c_{t_\nu^*+1} = w_{t_\nu^*+1} + a_{t_\nu^*+1}$ by our definition. So, yes, they are equivalent.

Note that by construction, $q_{t_\nu^*+2} \le m_{t_\nu^*+2}$ and $q_i \le m_i + q_{i-1}$, so we can apply each Kashiwara operator the necessary number of times. Hence, the sequence

$$\hat{F}'_{t_\nu^*+1} = \tilde{f}_{t_\nu^*}^{q_{t_\nu^*}} \tilde{f}_{t_\nu^*-1}^{q_{t_\nu^*-1}} \cdots \tilde{f}_{t_\nu^*+2}^{q_{t_\nu^*+2}} \tag{6.21}$$

is such that

$$\hat{F}'_{t_\nu^*+1}(u_\lambda) = u_{\sigma(\lambda)} \otimes c.$$

Therefore, we have produced the sequence G such that

$$\begin{aligned}
G(u_\lambda) &= \hat{F}_{t_\nu^*} \hat{F}'_{t_\nu^*+1}(u_\lambda) \\
&= u_{\sigma^2(\lambda)} \otimes b_\nu \otimes b_\omega,
\end{aligned} \tag{6.22}$$

for any $b_\nu, b_\omega \in \mathcal{B}_\ell$.

Back to Example 6.15: We were given the ground-state path

$$u_\lambda = \ldots \otimes (4,7,3,1,0,2) \otimes (2,4,7,3,1,0)$$

where $b_\lambda = (m_1, m_2, \ldots, m_0) = (2,4,7,3,1,0)$. And we wanted to produce the λ-path

$$u_{\sigma^2(\lambda)} \otimes (2,1,0,9,2,3) \otimes (5,0,7,1,0,4)$$

where $b_\nu = (2,1,0,9,2,3)$ and $b_\omega = (5,0,7,1,0,4)$. In order to compute the sequence needed to obtain this λ-path, we need to do the following:

1. Use Algorithm 2 to find the sequence

$$F_{t_\nu^*} = \tilde{f}_5 \tilde{f}_4^3 \tilde{f}_3^{11} \tilde{f}_2^8 \tilde{f}_1^2$$

   where $t_\nu^* = 0$.

2. Compute the a_i's:

$$\begin{aligned}
a_0 &= 0,\\
a_5 &= (w_0 + a_0 - v_5)_+ = (4+0-2)_+ = 2,\\
a_4 &= (w_5 + a_5 - v_4)_+ = (0+2-9)_+ = 0,\\
a_3 &= (w_4 + a_4 - v_3)_+ = (1+0-0)_+ = 1,\\
a_2 &= (w_3 + a_3 - v_2)_+ = (7+1-1)_+ = 7,\\
a_1 &= (w_2 + a_2 - v_1)_+ = (0+7-2)_+ = 5.
\end{aligned}$$

3. Then compute the c_{i+1}'s:

$$\begin{aligned}
c_1 &= w_1 + a_1 = 5+5 = 10,\\
c_0 &= v_5 = 2,\\
c_5 &= w_5 + a_5 = 0+2 = 2,\\
c_4 &= v_3 = 0,\\
c_3 &= v_2 = 1,\\
c_2 &= v_1 = 2.
\end{aligned}$$

4. Form

$$\begin{aligned}
\hat{F}_{t_\nu^*} &= \tilde{f}_5^3 \tilde{f}_4^3 \tilde{f}_3^{12} \tilde{f}_2^{15} \tilde{f}_1^7,\\
c &= (10,2,1,0,2,2).
\end{aligned}$$

5. Compute the q_i's:

$$\begin{aligned}
q_1 &= 0,\\
q_2 &= w_1 + a_1 - (m_3 + m_4 + m_5 + m_0 + m_1) + (c_3 + c_4 + c_5 + c_0)\\
&= 5+5-(7+3+1+0+2)+(1+0+2+2) = 2,
\end{aligned}$$

$$\begin{aligned}
q_3 &= w_1 + a_1 - (m_4 + m_5 + m_0 + m_1) + (c_4 + c_5 + c_0) \\
&= 5 + 5 - (3 + 1 + 0 + 2) + (0 + 2 + 2) = 8, \\
q_4 &= w_1 + a_1 - (m_5 + m_0 + m_1) + (c_5 + c_0) \\
&= 5 + 5 - (1 + 0 + 2) + (2 + 2) = 11, \\
q_5 &= w_1 + a_1 - (m_0 + m_1) + (c_0) = 5 + 5 - (0 + 2) + (2) = 10, \\
q_0 &= w_1 + a_1 - (m_1) + (0) = 5 + 5 - (2) + 0 = 8.
\end{aligned}$$

6. Form $\hat{F}'_{t^*_\nu+1} = \tilde{f}_0^8 \tilde{f}_5^{10} \tilde{f}_4^{11} \tilde{f}_3^8 \tilde{f}_2^2$.

Hence, the sequence $\left(\tilde{f}_5 \tilde{f}_4^3 \tilde{f}_3^{11} \tilde{f}_2^8 \tilde{f}_1^2\right)\left(\tilde{f}_0^8 \tilde{f}_5^{10} \tilde{f}_4^{11} \tilde{f}_3^8 \tilde{f}_2^2\right)$ is such that

$$\left(\tilde{f}_5 \tilde{f}_4^3 \tilde{f}_3^{11} \tilde{f}_2^8 \tilde{f}_1^2\right)\left(\tilde{f}_0^8 \tilde{f}_5^{10} \tilde{f}_4^{11} \tilde{f}_3^8 \tilde{f}_2^2\right)(u_\lambda) = u_{\sigma^2(\lambda)} \otimes (2,1,0,9,2,3) \otimes (5,0,7,1,0,4).$$

The sequence $G = \hat{F}_{t^*_\nu} \hat{F}'_{t^*_\nu+1}$ corresponds to the sequence of Weyl group simple reflections

$$\mathcal{S} = \left(r_{t^*_\nu-1} r_{t^*_\nu-2} \cdots r_{t^*_\nu+1}\right)\left(r_{t^*_\nu} r_{t^*_\nu-1} \cdots r_{t^*_\nu+2}\right).$$

Lemma 6.16. *The sequence $\mathcal{S} = \left(r_{t^*_\nu-1} r_{t^*_\nu-2} \cdots r_{t^*_\nu+1}\right)\left(r_{t^*_\nu} r_{t^*_\nu-1} \cdots r_{t^*_\nu+2}\right)$ is a subsequence of $w(3,j)$ for all $j = 0,1,\ldots,n$.*

Proof. Recall that

$$w(3,j) = (r_{j+n-3} \cdots r_{j+n} r_{j+n-1})(r_{j+n-2} \cdots r_j r_{j+n})(r_{j+n-1} \cdots r_{j+1} r_j).$$

Notice that each simple reflection r_i appears within the first $n+1$ components from the right in $w(3,j)$. If we start at any of those r_i, we can form a sequence of simple reflections

$$(r_{i+n-2} \cdots r_i r_{i+n})(r_{i+n-1} \cdots r_{i+1} r_i)$$

of length $2n$ that is a subsequence of $w(3,j)$. Let $i = t^* + 2$. Then we have that

$$\mathcal{S} = \left(r_{t^*_\nu-1} r_{t^*_\nu-2} \cdots r_{t^*_\nu+1}\right)\left(r_{t^*_\nu} r_{t^*_\nu-1} \cdots r_{t^*_\nu+2}\right)$$

is a subsequence of $w(3,j)$ for all $j = 0,1,\ldots,n$. □

Corollary 6.17. *We have that $u_{\sigma^2(\lambda)} \otimes \mathcal{B}_\ell^2 \subset B_{w(3,j)}(\lambda)$ for all $j = 0,1,\ldots,n$.*

Proof. For each λ-path $u_{\sigma(\lambda)} \otimes b_\gamma \otimes b_\omega$, Algorithm 3 produces a sequence $\hat{F}_{t_\gamma^*}\hat{F}'_{t_\gamma^*+1}$ that is associated with a sequence of simple reflections $\mathcal{S} = \left(r_{t_\gamma^*-1} r_{t_\gamma^*-2} \cdots r_{t_\gamma^*+1}\right)\left(r_{t_\gamma^*} r_{t_\gamma^*-1} \cdots r_{t_\gamma^*+2}\right)$ of length $2n$. By Lemma 6.16, we know each of these sequences $\mathcal{S}$ of simple reflections is a subsequence of $w(3,j)$ for all j. Hence, by the definition of $B_{w(3,j)}(\lambda)$, it follows that $u_{\sigma^2(\lambda)} \otimes \mathcal{B}_\ell^2 \subset B_{w(3,j)}(\lambda)$ for all j. □

Lemma 6.18. *We have that*

$$\bigcup_{j=0}^{n} B_{w(2,j)} = u_{\sigma^2(\lambda)} \otimes \mathcal{B}_\ell^2.$$

Proof. By Algorithm 3, we proved that there exists a sequence $G = \hat{F}_{t_\gamma^*}\hat{F}'_{t_\gamma^*+1}$ such that

$$G(u_\lambda) = u_{\sigma(\lambda)} \otimes b_\gamma \otimes b_\omega$$

for all $b_\gamma, b_\omega \in \mathcal{B}_\ell$. Each sequence G is associated with a sequence of simple reflections

$$\mathcal{S} = \left(r_{t_\gamma^*-1} r_{t_\gamma^*-2} \cdots r_{t_\gamma^*+1}\right)\left(r_{t_\gamma^*} r_{t_\gamma^*-1} \cdots r_{t_\gamma^*+2}\right)$$

of length $2n$. Well, note that $\mathcal{S} = w(2, t_\gamma^*+1)$. Hence,

$$u_{\sigma^2(\lambda)} \otimes b_\gamma \otimes b_\omega \in B_{w(1,t_\gamma^*+1)}(\lambda)$$

for each $b_\gamma, b_\omega \in \mathcal{B}_\ell$. Thus,

$$\bigcup_{j=0}^{n} B_{w(2,j)} \supseteq u_{\sigma^2(\lambda)} \otimes \mathcal{B}_\ell^2.$$

Now we just need to show that

$$\bigcup_{j=0}^{n} B_{w(2,j)} \subseteq u_{\sigma^2(\lambda)} \otimes \mathcal{B}_\ell^2.$$

In other words, we need to show that $u_{\sigma^3(\lambda)} \otimes b_\gamma \otimes \mathcal{B}_\ell^2 \not\subseteq B_{w(2,j)}(\lambda)$ for all $j = 0, 1, \ldots, n$. Well, recall that

$$\begin{aligned}\tilde{f}_i^{z_i}\tilde{f}_{i+1}^{z_i}\tilde{f}_{i+2}^{z_i}(u_\lambda) &= \tilde{f}_i^{z_i}\tilde{f}_{i+1}^{z_i}\tilde{f}_{i+2}^{z_i}\left(u_{\sigma^3(\lambda)} \otimes b_{\sigma^2(\lambda)} \otimes b_{\sigma(\lambda)} \otimes b_\lambda\right)\\ &= u_{\sigma^3(\lambda)} \otimes b_{\rho_3} \otimes b_{\rho_2} \otimes b_{\rho_1}.\end{aligned}$$

This last equation implies that if any sequence of simple reflections contains $\tilde{f}_{i+2}$, then $\tilde{f}_{i+1}$, and then $\tilde{f}_i$, then the third component from the right in the tensor product will be affected. And such a sequence is needed in order for the third component to be affected, by Lemma 6.2. But $w(2,j) = (r_{j+n-2} \cdots r_j r_{j+n})(r_{j+n-1} \cdots r_{j+1} r_j)$ for $j = 0, 1, \ldots, n$, which means that this situation never occurs.

Hence, the third component from the right in the tensor product will never be affected, and so

$$\bigcup_{j=0}^{n} B_{w(2,j)} \subseteq u_{\sigma^2(\lambda)} \otimes \mathcal{B}_\ell^2.$$

Therefore,

$$\bigcup_{j=0}^{n} B_{w(2,j)} = u_{\sigma^2(\lambda)} \otimes \mathcal{B}_\ell^2.$$

□

Remark 3. Recall that with some notation changes, Algorithm 2 was a generalization of Algorithm 1. In a similar manner, with some notation changes, Algorithm 3 generalizes to produce a sequence G of Kashiwara operators such that $G(u_{\sigma^{L-1}(\lambda)}) \otimes \mathcal{B}_\ell^{L-1} = u_{\sigma^{L+1}(\lambda)} \otimes b_\nu \otimes b_\omega \otimes \mathcal{B}_\ell^{L-1}$.

Lemma 6.19. *Choose $b_{\gamma_1}, b_{\gamma_2}, \ldots, b_{\gamma_L} \in \mathcal{B}_\ell$. Then, given the ground-state path*

$$u_\lambda = \ldots \otimes (m_2, m_3, \ldots, m_1) \otimes (m_1, m_2, \ldots, m_0)$$

there exists a sequence

$$G = \left(\hat{F}_{t_\nu^*}\right)\left(\hat{F}'_{t_\nu^*+1}\right) \cdots \left(\hat{F}'_{t_\nu^*+L-1}\right)$$

of at most Ln Kashiwara operators such that

$$G(u_\lambda) = u_{\sigma^L(\lambda)} \otimes b_{\gamma_1} \otimes b_{\gamma_2} \ldots \otimes b_{\gamma_L}.$$

Proof. Let $L=1$. By Algorithm 1, we have the sequence $G = F_{t^*}$ of Kashiwara operators of length at most n such that $G(u_\lambda) = u_{\sigma(\lambda)} \otimes b_{\gamma_1}$, where F_{t^*} is defined as in Equation 6.8.

Let $L=2$. By Algorithm 3, we have the sequence $G = \hat{F}_{t_{\gamma_1}^*} \hat{F}'_{t_{\gamma_1}^*+1}$ of Kashiwara operators of length at most $2n$ such that $G(u_\lambda) = u_{\sigma^2(\lambda)} \otimes b_{\gamma_1} \otimes b_{\gamma_2}$, where $\hat{F}_{t_{\gamma_1}^*}$ is defined as in Equation 6.20 and $\hat{F}'_{t_{\gamma_1}^*+1}$ is defined as in Equation 6.21. Notice that except for possibly different exponents, the sequence $\hat{F}_{t_{\gamma_1}^*}$ of Kashiwara operators is the same as the sequence $F_{t_{\gamma_1}^*}$ of Kashiwara operators, which is obtained using Algorithm 2, where

$$F_{t_{\gamma_1}^*}\left(u_{\sigma(\lambda)}\right) \otimes \mathcal{B}_\ell = u_{\sigma^2(\lambda)} \otimes b_{\gamma_1} \otimes \mathcal{B}_\ell.$$

Now, suppose that there exists a sequence $\hat{G}$ of Kashiwara operators of length at most $(L-1)n$ such that

$$\hat{G}(u_\lambda) = u_{\sigma^{L-1}(\lambda)} \otimes b_{\gamma_2} \otimes b_{\gamma_3} \otimes \ldots b_{\gamma_L}$$

$$= u_{\sigma^L(\lambda)} \otimes b_{\sigma^{L-1}(\lambda)} \otimes b_{\gamma_2} \otimes b_{\gamma_3} \otimes \ldots b_{\gamma_L}.$$

for $b_{\gamma_j} \in \mathcal{B}_\ell$, and suppose that, except for possibly different exponents, the last n Kashiwara operators in the sequence $\hat{G}$ form the same sequence of Kashiwara operators as $F_{t^*_{\gamma_2}}$, which is obtained using Algorithm 2 for the element b_{γ_2}.

We want to show that we can find a sequence G for the L case so that when acting on the ground-state path u_λ it will produce the λ-path

$$u_{\sigma^L(\lambda)} \otimes b_{\gamma_1} \otimes b_{\gamma_2} \otimes \ldots \otimes b_{\gamma_L},$$

for $b_{\gamma_j} \in \mathcal{B}_\ell$.

Using Algorithm 2, we can construct a sequence $F_{t^*_{\gamma_1}}$ (defined in Equation 6.18) such that

$$F_{t^*_{\gamma_1}}\left(u_{\sigma^{L-1}(\lambda)}\right) \otimes \mathcal{B}_\ell^{L-1} = u_{\sigma^L(\lambda)} \otimes b_{\gamma_1} \otimes \mathcal{B}_\ell^{L-1}.$$

Then, by Remark 3, we can construct a sequence $\hat{F}_{t^*_{\gamma_1}}$ (defined in Equation 6.20) such that

$$\hat{F}_{t^*_{\gamma_1}}\left(u_{\sigma^{L-1}(\lambda)} \otimes c\right) \otimes \mathcal{B}_\ell^{L-1} = u_{\sigma^L(\lambda)} \otimes b_{\gamma_1} \otimes b_{\gamma_2} \otimes \mathcal{B}_\ell^{L-1},$$

and the sequence $\hat{F}'_{t^*_{\gamma_1}+1}$ (defined in Equation 6.21) such that

$$\hat{F}_{t^*_{\gamma_1}+1}\left(u_{\sigma^{L-2}(\lambda)}\right) \otimes \mathcal{B}_\ell^{L-2} = u_{\sigma^{L-1}(\lambda)} \otimes c \otimes \mathcal{B}_\ell^{L-2},$$

so that

$$\hat{F}_{t^*_{\gamma_1}} \hat{F}'_{t^*_{\gamma_1}+1}\left(u_{\sigma^{L-2}(\lambda)}\right) \otimes \mathcal{B}_\ell^{L-2} = u_{\sigma^L(\lambda)} \otimes b_{\gamma_1} \otimes b_{\gamma_2} \otimes \mathcal{B}_\ell^{L-1}.$$

Then, by our induction hypothesis, there exists a sequence $\hat{G}$ such that

$$\hat{G}\left(u_\lambda\right) = u_{\sigma^{L-1}(\lambda)} \otimes c \otimes b_{\gamma_3} \otimes b_{\gamma_4} \otimes \ldots \otimes b_{\gamma_L},$$

where, except for possibly different exponents, the last n Kashiwara operators in $\hat{G}$ form the same sequence of Kashiwara operators as $\hat{F}'_{t^*_\gamma+1}$. If we define the sequence $\hat{F}'_{t^*_{\gamma_1}}$ to have the same sequence of Kashiwara operators as $\hat{F}_{t^*_{\gamma_1}}$ but possibly different exponents, and we define $\hat{G}'$ to have the same sequence of Kashiwara operators as $\hat{G}$ but possibly different exponents, then we can form the sequence

$$G = \hat{F}'_{t^*_\nu}\hat{G}',$$

where

$$G(u_\lambda) = u_{\sigma^L(\lambda)} = u_{\sigma^L(\lambda)} \otimes b_{\gamma_1} \otimes b_{\gamma_2} \otimes \ldots \otimes b_{\gamma_L}.$$

□

The sequence G corresponds to the sequence of Weyl group reflections

$$\mathcal{S} = \left(r_{t^*_\nu-1} r_{t^*_\nu-2} \cdots r_{t^*_\nu+1}\right)\left(r_{t^*_\nu} r_{t^*_\nu-1} \cdots r_{t^*_\nu+2}\right)\cdots\left(r_{t^*_\nu+L-2} r_{t^*_\nu+L-3} \cdots r_{t^*_\nu+L}\right).$$

Lemma 6.20. *The sequence*

$$\mathcal{S} = \left(r_{t^*_\nu-1} r_{t^*_\nu-2} \cdots r_{t^*_\nu+1}\right)\left(r_{t^*_\nu} r_{t^*_\nu-1} \cdots r_{t^*_\nu+2}\right)\cdots\left(r_{t^*_\nu+L-2} r_{t^*_\nu+L-3} \cdots r_{t^*_\nu+L}\right)$$

is a subsequence of $w(L+1, j)$ *for all* $j = 0, 1, \ldots, n$.

Proof. Recall that

$$w(L+1, j) = (r_{j+n-L-1} \cdots r_{j+1-L} r_{j-L}) \cdots (r_{j+n-2} \cdots r_j r_{j+n})(r_{j+n-1} \cdots r_{j+1} r_j).$$

Notice that each simple reflection r_i appears within the first $n+1$ components from the right in $w(L+1, j)$. If we start at any of those r_i, we can form a sequence of simple reflections

$$(r_{i+L+n-2} \cdots r_{i-L} r_{i-L-1}) \cdots (r_{i+n-2} \cdots r_i r_{i+n})(r_{i+n-1} \cdots r_{i+1} r_i)$$

of length Ln that is a subsequence of $w(L+1, j)$ for all j. Let $i = t^*_\nu + L$. Then, we have that

$$\mathcal{S} = \left(r_{t^*_\nu-1} r_{t^*_\nu-2} \cdots r_{t^*_\nu+1}\right)\left(r_{t^*_\nu} r_{t^*_\nu-1} \cdots r_{t^*_\nu+2}\right)\cdots\left(r_{t^*_\nu+L-2} r_{t^*_\nu+L-3} \cdots r_{t^*_\nu+L}\right)$$

is a subsequence of $w(L+1, j)$ for all $j = 0, 1, \ldots, n$. □

Corollary 6.21. *We have that* $u_{\sigma^L(\lambda)} \otimes \mathcal{B}^L_\ell \subset B_{w(L+1,j)}(\lambda)$ *for all* $j = 0, 1, \ldots, n$.

Proof. For each λ-path

$$u_{\sigma^L(\lambda)} \otimes b_\nu \otimes b_\omega \otimes b_{\gamma_1} \otimes b_{\gamma_2} \otimes \ldots \otimes b_{\gamma_{L-2}},$$

by Lemma 6.19 there exists a sequence of Kashiwara operators

$$G = \left(\hat{F}_{t^*_\nu}\right)\left(\hat{F}'_{t^*_\nu+1}\right)\cdots\left(\hat{F}'_{t^*_\nu+L-1}\right)$$

such that $G(u_\lambda) = u_{\sigma^L(\lambda)} \otimes b_\nu \otimes b_\omega \otimes b_{\gamma_1} \otimes \ldots \otimes b_{\gamma_{L-2}}$. This sequence G is associated with a sequence of simple reflections

$$\mathcal{S} = \left(r_{t_\nu^*-1} r_{t_\nu^*-2} \cdots r_{t_\nu^*+1}\right)\left(r_{t_\nu^*} r_{t_\nu^*-1} \cdots r_{t_\nu^*+2}\right)\cdots\left(r_{t_\nu^*+L-2} r_{t_\nu^*+L-3} \cdots r_{t_\nu^*+L}\right)$$

of length Ln. By Lemma 6.20, we know each of these sequences of simple reflections W is a subsequence of $w(L+1, j)$ for all j. Hence, by the definition of $B_{w(L+1,j)}(\lambda)$, it follows that

$$u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L \subset B_{w(L+1,j)}(\lambda)$$

for all j. □

Corollary 6.22. *Choose $b_\nu, b_{\gamma_1}, b_{\gamma_2}, \ldots, b_{\gamma_{L-1}} \in \mathcal{B}_\ell$ such that $b_\nu \neq b_{\sigma^{L-1}(\lambda)}$. Then, for some j,*

$$u_{\sigma^L(\lambda)} \otimes b_\nu \otimes b_{\gamma_1} \otimes b_{\gamma_2} \otimes \ldots \otimes b_{\gamma_{L-1}} \notin B_{w(L,j)}(\lambda).$$

Proof. By Lemma 6.19, we know there exists a sequence G of Kashiwara operators such that

$$G(u_\lambda) = u_{\sigma^L(\lambda)} \otimes b_\nu \otimes b_{\gamma_1} \otimes b_{\gamma_2} \otimes \ldots \otimes b_{\gamma_{L-1}}.$$

This sequence G_ν is associated with a sequence of simple reflections

$$\mathcal{S} = \left(r_{t_\nu^*-1} r_{t_\nu^*-2} \cdots r_{t_\nu^*+1}\right)\left(r_{t_\nu^*} r_{t_\nu^*-1} \cdots r_{t_\nu^*+2}\right)\cdots\left(r_{t_\nu^*+L-2} r_{t_\nu^*+L-3} \cdots r_{t_\nu^*+L}\right)$$

of length Ln. Note that $\mathcal{S}$ is not a subsequence of $w(L, t_\nu^* + L + 1)$. Hence, by the definition of $B_{w(L,t_\nu^*+L+1)}(\lambda)$,

$$u_{\sigma^L(\lambda)} \otimes b_\nu \otimes b_{\gamma_1} \otimes b_{\gamma_2} \otimes \ldots \otimes b_{\gamma_{L-1}} \notin B_{w(L,t_\nu^*+L+1)}(\lambda).$$

□

Theorem 6.23. *(**Main Theorem**) We have that*

1. $\displaystyle\bigcup_{j\geq 0} B_{w(L,j)}(\lambda) = u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L,$
2. $\displaystyle\bigcap_{j\geq 0} B_{w(L,j)}(\lambda) = u_{\sigma^{L-1}}(\lambda) \otimes \mathcal{B}_\ell^{L-1}.$

Proof. First, note that $w(L, j) = w(L, j+n+1)$. Hence,

$$\bigcup_{j\geq 0} B_{w(L,j)}(\lambda) = \bigcup_{j=0}^{n} B_{w(L,j)}(\lambda),$$

$$\bigcap_{j\geq 0} B_{w(L,j)}(\lambda) = \bigcap_{j=0}^{n} B_{w(L,j)}(\lambda).$$

We'll first prove the union. By Lemma 6.19, we proved that there exists a sequence G such that

$$G(u_\lambda) = u_{\sigma(\lambda)} \otimes b_\nu \otimes b_{\gamma_1} \otimes b_{\gamma_2} \otimes \ldots \otimes b_{\gamma_{L-1}}$$

for $b_\nu, b_{\gamma_i} \in \mathcal{B}_\ell$. Each sequence G is associated with a sequence of simple reflections

$$\mathcal{S} = \left(r_{t_\nu^*-1} r_{t_\nu^*-2} \cdots r_{t_\nu^*+1}\right)\left(r_{t_\nu^*} r_{t_\nu^*-1} \cdots r_{t_\nu^*+2}\right) \cdots \left(r_{t_\nu^*+L-2} r_{t_\nu^*+L-3} \cdots r_{t_\nu^*+L}\right)$$

of length Ln. Note that $\mathcal{S} = w(L, t_\nu^* + L)$. Hence, by the definition of $B_{w(L,t_\nu^*+L)}(\lambda)$,

$$u_{\sigma^L(\lambda)} \otimes b_\nu \otimes b_{\gamma_1} \otimes \ldots \otimes b_{\gamma_{L-1}} \in B_{w(L,t_\nu^*+L)}(\lambda)$$

for $b_\nu, b_{\gamma_i} \in \mathcal{B}_\ell$. Thus,

$$\bigcup_{j=0}^{n} B_{w(L,j)}(\lambda) \supseteq u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L.$$

Now we just need to show that

$$\bigcup_{j=0}^{n} B_{w(L,j)}(\lambda) \subseteq u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L.$$

In other words, we need to show that $u_{\sigma^{L+1}(\lambda)} \otimes b_\zeta \otimes \mathcal{B}_\ell^L \not\subseteq B_{w(L,j)}(\lambda)$ for all $j = 0, 1, \ldots, n$. Well, recall that by Lemma 6.2

$$\begin{aligned}\tilde{f}_i^{z_i} \tilde{f}_{i+1}^{z_i} \cdots \tilde{f}_{i+L+n}^{z_i} \tilde{f}_{i+L}^{z_i}(u_\lambda) &= \tilde{f}_i^{z_i} \tilde{f}_{i+1}^{z_i} \cdots \tilde{f}_{i+L}^{z_i}\left(u_{\sigma^{L+1}(\lambda)} \otimes b_{\sigma^L(\lambda)} \otimes b_{\sigma^{L-1}(\lambda)} \otimes \ldots \otimes b_\lambda\right) \\ &= u_{\sigma^{L+1}(\lambda)} \otimes b_{\rho_{L+1}} \otimes b_{\rho_L} \otimes \ldots \otimes b_{\rho_1}.\end{aligned}$$

This last equation implies that if any sequence contains the simple reflections in that order, then the $(L+1)^{st}$ component from the right in the tensor product will be affected. And such a sequence is needed in order for the third component to be affected. This sequence would be associated with a sequence of simple reflections:

$$(r_i)(r_{i+n} \cdots r_{i+2} r_{i+1}) \cdots (r_{i+n-2+L} \cdots r_{i+L} r_{i+n+L})(r_{i+n-1+L} \cdots r_{i+1+L} r_{i+L})$$

of length $Ln + 1$. But

$$w(L, j) = (r_{j+n-L} \cdots r_{j+2-L} r_{j+1-L}) \cdots (r_{j+n-2} \cdots r_j r_{j+n})(r_{j+n-1} \cdots r_{j+1} r_j)$$

for $j=0,1,\ldots,n$, is of length Ln, which means that this situation never occurs. Hence, the $(L+1)^{st}$ component from the right in the tensor product will never be affected, and so

$$\bigcup_{j=0}^{n} B_{w(L,j)}(\lambda) \subseteq u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L.$$

Therefore,

$$\bigcup_{j=0}^{n} B_{w(L,j)}(\lambda) = u_{\sigma^L(\lambda)} \otimes \mathcal{B}_\ell^L.$$

Now, let's prove the intersection. By Corollary 6.21, we know that

$$u_{\sigma^{L-1}(\lambda)} \otimes b^{L-1} \subset B_{w(L,j)}(\lambda)$$

for all $j=0,1,\ldots,n$. Hence,

$$\bigcap_{j=0}^{n} B_{w(L,j)}(\lambda) \supseteq u_{\sigma^{L-1}}(\lambda) \otimes \mathcal{B}_\ell^{L-1}.$$

By Corollary 6.22, each λ-path

$$u_{\sigma^L(\lambda)} \otimes b_\nu \otimes b_{\gamma_1} \otimes b_{\gamma_2} \otimes \ldots \otimes b_{\gamma_{L-1}} \notin B_{w(L,j)}(\lambda)$$

for some j, which implies that

$$\bigcap_{j=0}^{n} B_{w(L,j)}(\lambda) \subseteq u_{\sigma^{L-1}}(\lambda) \otimes \mathcal{B}_\ell^{L-1}.$$

Therefore,

$$\bigcap_{j=0}^{n} B_{w(L,j)}(\lambda) = u_{\sigma^{L-1}}(\lambda) \otimes \mathcal{B}_\ell^{L-1}.$$

□

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