ABSTRACT

ARMSTRONG, ALYSSA MARIE. Demazure Crystals for the Quantum Affine Algebra $U_q(D_4^{(3)})$. (Under the direction of Dr. Kailash Misra.)

Kac-Moody algebras, independently discovered in 1968 by Victor Kac and Robert Moody, are an infinite dimensional analog of finite dimensional semisimple Lie algebras. An important class of Kac-Moody algebras are the affine Lie algebras due to their many applications in physics and mathematics. One area of study involves determining the structure of irreducible integrable highest weight modules of affine Kac-Moody algebras. After Vladimir Drinfeld and Michio Jimbo developed quantum group theory in 1985, George Lusztig showed that the representation theory of a Kac-Moody algebra is parallel to its quantum group in the generic case. In 1990, Masaki Kashiwara introduced the crystal basis as a nice combinatorial tool to study the irreducible highest weight modules over a quantum group. Demazure modules are certain finite dimensional subspaces of integrable highest weight modules of quantum affine algebras. Crystal base theory is used in studying the structure of a Demazure module and its corresponding crystal.

In this thesis, we study the Demazure crystals for a particular affine Lie algebra, $D_4^{(3)}$. In 1993, Kashiwara showed that the Demazure crystals have a recursive definition. In 1998, Kuniba, Misra, Okado, and Uchiyama gave a criterion for the Demazure crystals to have tensor product-like structures. In this study, a certain parameter, κ, called the mixing index enters the picture. For the quantum affine algebra $U_q(D_4^{(3)})$, we show $\kappa = 1$ when $\lambda = \ell\Lambda_2$ and conjecture that $\kappa = 2$ for $\lambda = \ell\Lambda_1$ and $\lambda = \ell(\Lambda_0 + \Lambda_1)$. We prove our conjecture for $\ell = 1$ and 2.

Demazure Crystals for the Quantum Affine Algebra $U_q(D_4^{(3)})$

by
Alyssa Marie Armstrong

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APPROVED BY:

Dr. Naihuan Jing	Dr. Ernest Stitzinger
Dr. Kimberly Weems	Dr. Kailash Misra Chair of Advisory Committee

BIOGRAPHY

Alyssa Armstrong was born on August 26, 1987 near Chicago, IL to Kevin and Diane Armstrong. She grew up just outside Cincinnati, OH where she learned how much she loved being in school. Math was one of her favorite subjects in middle and high school, and she was encouraged by her teachers and parents to pursue mathematics in college. Alyssa graduated from Lakota West High School in the top 25 students (of a class of near 600) in 2005, well prepared and excited to start college. She began her undergraduate career at Wittenberg University in Springfield, OH as a math education major, hoping to become a high school math teacher. Seeking as many opportunities as possible, Alyssa began tutoring math in the school's Math Workshop, and also became close with the math department faculty.

In her freshman year, she learned about a study abroad experience in Budapest, Hungary specifically for math majors. Enraptured by the idea of learning advanced mathematics and the chance to travel the world, Alyssa had a tough decision on her hands: she could either continue her math education major and complete her student-teaching semester on time, or she could go to Budapest and work toward a career as a mathematics professor at the college level. After weighing her options, she decided to apply for the Budapest Semesters in Mathematics program and pursue becoming a professor. Alyssa was accepted into the program, and had the opportunity to travel to Budapest in the spring of 2008. She studied abstract algebra, combinatorics, and number theory from some of the best Hungarian mathematics professors in the country. She also had a chance to delve into Hungarian culture and see much of Europe, which strengthened her desire to travel in the future.

After returning from Budapest, Alyssa continued to take as many math courses as she could, and began to focus on attending graduate school. She was intrigued by the beauty and simplicity of combinatorics, and continued studying the subject in an independent study her senior year. Alyssa graduated Summa cum Laude with Honors in 2009, ready to continue her studies in graduate school. She decided to attend North Carolina State University in Raleigh, NC in pursuit of a Ph.D. in mathematics. While she initially planned on studying combinatorics at NCSU, a desire to learn more abstract algebra slowly developed during her first two years. Alyssa found it was possible to combine her interests in combinatorics and algebra working with Dr. Kailash Misra in the field of combinatorial Lie algebra representation theory. During her time at NCSU, she obtained her M.S. in mathematics in 2011 and will defend her Ph.D. dissertation in June 2014 with a graduate statistics minor. She will then move to Springfield, OH to begin a position as a visiting assistant professor of mathematics at Wittenberg University. Alyssa is extremely excited to be back at her alma mater to "Pass the Light Onto Others!"

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My graduate career has had its ups and downs, but I have been extremely fortunate to have so many people supporting me over these past five years. I wish to thank each and every one of you for seeing me through to the end.

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I will never be able to thank my parents, Diane and Kevin, enough for all their support in the past twenty-six years. They have both encouraged me from the very beginning to pursue my dreams, no matter what. Their dedication to my success in graduate school has been overwhelming and I am so excited to make them proud as I continue to achieve my greatest dreams. Thank you both to listening to my good and bad days, for trying to understand my research, and for always telling me what I needed to hear to keep making progress. Also, I want to thank my brother, Steven, for his support and encouragement and the rest of my family for always being there with words of praise and motivation.

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I do not have enough room to list all my non-mathematical friends who have supported me, challenged me, and made the past five years a blast. I give my thanks for all of you (you know who you are), and I know I would not be where I am today without each and every one of you. I want to especially mention my church families - those at Holy Trinity Lutheran Church, 724 Church, BSF, and the math graduate Bible study - who have helped mold me and continue to look toward God every single day.

Finally and most importantly, I know I could never have accomplished writing a dissertation and surviving graduate school without God. This is all for You and I hope to continue to do Your will in all my days to come.

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Chapter 1

Introduction

In the nineteenth century, Sophus Lie developed the notion of a Lie group and a Lie algebra while studying the geometric symmetries of differential equations. Shortly after Lie defined the Lie group and Lie algebra, Wilhelm Killing and Élie Cartan showed the necessity to classify all finite dimensional simple Lie algebras. By the end of the nineteenth century, Cartan had completed this classification of finite dimensional simple Lie algebras over the complex numbers, $\mathbb{C}$ (cf. [3]). Then in 1968, Victor Kac [8] and Robert Moody [23] independently discovered Kac-Moody algebras, an infinite dimensional analog of the finite dimensional semisimple Lie algebras. An important class of Kac-Moody algebras are called affine Lie algebras, and they have been widely studied due to their numerous applications in physics, including conformal field theory and statistical physics, as well as in various fields of mathematics, including number theory, combinatorics, and algebraic geometry.

Each Kac-Moody algebra is constructed from an $n \times n$ integer matrix, called a generalized Cartan matrix, and a corresponding diagram, called a Dynkin diagram. Using the generalized Cartan matrix and Dynkin diagram, the Kac-Moody algebra can be realized in terms of generators and relations. Indecomposable, symmetrizable generalized Cartan matrices are classified into one of three types: finite, affine, or indefinite, and the corresponding Kac-Moody algebras are classified in the same way. In this thesis, we will focus on the affine Kac-Moody algebra $D_4^{(3)}$, which has the generalized Cartan matrix,

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$$

The representation theory of affine Kac-Moody algebras has been developed in the past forty years, when Kac introduced the concept of an integrable highest weight module [7]. It has been shown that for each dominant integral weight λ and Kac-Moody algebra, $\mathfrak{g}$, there is

a unique, up to isomorphism, irreducible integrable highest weight module, denoted by $V(\lambda)$ [2]. In Chapter 2, we review some of the fundamental definitions and theorems of Kac-Moody algebras and their representation theory.

In 1985, Drinfeld [1] and Jimbo [5] introduced the quantum group $U_q(\mathfrak{g})$ in their study of the quantum Yang-Baxter equation arising from two-dimensional solvable lattice models. They discovered the quantum group as a deformation of the universal enveloping algebra of a Kac-Moody algebra, $U(\mathfrak{g})$. Then in 1988, Lusztig showed that the representation theory of a Kac-Moody algebra, $\mathfrak{g}$, is parallel to the representation theory of the quantum group, $U_q(\mathfrak{g})$ for generic q. In particular, we can consider the unique irreducible integrable highest weight module over $U_q(\mathfrak{g})$, $V^q(\lambda)$. Then, for each element w in the Weyl group, we can construct a module from $V^q(\lambda)$ called the Demazure module associated with w. For each w, it is known that the extremal weight space, $V^q(\lambda)_{w\lambda}$ of $V^q(\lambda)$ is one-dimensional. If we denote $U_q^+(\mathfrak{g})$ to be the positive part of $U_q(\mathfrak{g})$, then the Demazure module, $V_w(\lambda)$, is the $U_q^+(\mathfrak{g})$-module generated by $V^q(\lambda)_{w\lambda}$. We are particularly interested in the structure of the Demazure module and their corresponding crystals in this thesis.

Associated with each irreducible integrable highest weight $U_q(\mathfrak{g})$-module, $V^q(\lambda)$, is a basis at $q = 0$ called a crystal basis, developed by Kashiwara in 1990 [12]. Crystal bases provide a nice combinatorial tool to study the structure of irreducible highest weight modules over $U_q(\mathfrak{g})$. In particular, a crystal base for a quantum affine algebra can be constructed using perfect crystals [10], whose crystal graphs are useful in determining the structure of Demazure modules. While studying Demazure modules, Kashiwara showed the existence of a crystal for the Demazure module in 1993 [15], which is a subset of the crystal for $V^q(\lambda)$. He also showed that the Demazure crystals have a certain recursive property. Using this property, Kuniba, Misra, Okado and Uchiyama [16] gave a criterion for the Demazure crystal to have a tenor product-like structure. In this realization, a parameter κ, called the mixing index, is defined. It has been shown that for $\lambda = \ell\Lambda$, where Λ is a dominant integral weight of level one, the mixing index $\kappa = 1$ for the affine Kac-Moody algebras, $\mathfrak{g} = A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$ [17], $D_4^{(3)}$ [21], and $G_2^{(1)}$ [4]. Also, its was shown in [16] that for any classical dominant integral weight, the mixing index $\kappa = 2$ for $\mathfrak{g} = A_n^{(1)}$. From these results, it is conjectured that $\kappa \leq 2$ for all quantum affine algebras. While the mixing index has been determined in some cases, there is still much work left in proving this conjecture in general.

In this thesis, we consider the particular affine Kac-Moody algebra, $\mathfrak{g} = D_4^{(3)}$. In 2007, Kashiwara, Misra, Okado, and Yamada [13] gave an explicit description of a perfect crystal for the quantum affine algebra $U_q(D_4^{(3)})$. Using this description along with Kashiwara's recursive property and path realizations of affine crystals [9], we construct $U_q(D_4^{(3)})$-Demazure crystals for $\lambda = \ell\Lambda_1$, $\lambda = \ell\Lambda_2$, and $\lambda = \ell(\Lambda_0 + \Lambda_1)$ in this thesis. We first provide the details of crystal base theory and perfect crystals in Chapter 3, and give the main theorem proved in [16] to show

path realizations of the Demazure crystals have tensor product-like structures. In Chapter 4, we give the necessary details regarding the quantum affine algebra $U_q(D_4^{(3)})$ as well as the construction of the $U_q(D_4^{(3)})$-perfect crystal as described in [13]. Then in Chapter 5, we prove that the main theorem holds with $\kappa = 1$ in the case when $\lambda = \ell\Lambda_2$, and conjecture that the main theorem holds with $\kappa = 2$ when $\lambda = \ell\Lambda_1$ and $\lambda = \ell(\Lambda_0 + \Lambda_1)$. We prove our conjecture when $\ell = 1$ and 2 in both cases. Along with the previous work in showing $\kappa = 1$ for $\lambda = \ell\Lambda_0$ as shown in [21], this work provides evidence that the mixing index $\kappa \leq 2$ for dominant integral weights $\lambda = \ell\Lambda$ where the level of $\Lambda \leq 3$ for $U_q(D_4^{(3)})$. These results continue to support the main conjecture in [16] that $\kappa \leq 2$ for all quantum affine algebras.

Chapter 2

Kac-Moody Algebras and Quantum Groups

We begin by providing an introduction to Lie algebras and a particular class of Lie algebras called Kac-Moody algebras. We will also introduce the representation theory and quantum groups associated with Kac-Moody algebras. Unless otherwise noted, we assume the underlying field to be $\mathbb{C}$, the field of complex numbers.

2.1 Lie Algebras and Representations

Definition 2.1.1. A *Lie algebra* $\mathfrak{g}$ is a vector space over the field $\mathbb{C}$ together with a product (called the *bracket*), $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that for all $x, y, z \in \mathfrak{g}$ and $a, b \in \mathbb{C}$,

(1) $[ax + by, z] = a[x, z] + b[y, z]$ and $[x, ay + bz] = a[x, y] + b[x, z]$,

(2) $[x, x] = 0$,

(3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$ (Jacobi identity)

Remark 2.1.2. By axioms (1) and (2) in the definition of a Lie algebra, the following property also holds: $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$.

Definition 2.1.3. A Lie algebra $\mathfrak{g}$ is *abelian* if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

Vector spaces are oftentimes constructed by a basis, so we have an equivalent definition of a Lie algebra based on the basis elements of the vector space:

Theorem 2.1.4. *Let $\{x_i \mid i \in I\}$ be a basis of a Lie algebra $\mathfrak{g}$ and $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be a bilinear map. Then $\mathfrak{g}$ is a* Lie algebra *if and only if*

(1) $[x_i, x_i] = 0$ *for all* $i \in I$,

(2) $[x_i, x_j] = -[x_j, x_i]$ *for all* $i \neq j \in I$,

(3) $[x_i, [x_j, x_k]] + [x_j, [x_k, x_i]] + [x_k, [x_i, x_j]] = 0$ *for all district triples* $i, j, k \in I$.

Many examples of Lie algebras involve matrices, in which we define the bracket, called the *commutator bracket*, $[A, B] = AB - BA$ for $A, B \in \mathfrak{g}$. An example of such a Lie algebra is the set of 2×2 traceless matrices.

Example 2.1.5. Consider the set of 2×2 traceless matrices, whose basis is as follows:

$$\left\{ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

With the bracket defined as $[A, B] = AB - BA$ for all $A, B \in \mathfrak{g}$, we have

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Then $\mathfrak{g}$ is a Lie algebra, denoted $s\ell(2, \mathbb{C})$.

Example 2.1.6. We can extend the example above to $n\times n$ traceless matrices, denoted $s\ell(n, \mathbb{C})$. A basis for $s\ell(n, \mathbb{C})$ is $\{h_i = E_{ii} - E_{i+1,i+1},\ E_{jk} \mid 1 \leq i \leq n-1,\ 1 \leq j \neq k \leq n\}$, where E_{ij} denote the $n \times n$ matrix with a one in the (i, j)-entry and zeros elsewhere. Note that, under the commutator bracket,

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}, \text{ where } \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Definition 2.1.7. An *associative algebra* $\mathcal{A}$ over $\mathbb{C}$ is a ring $\mathcal{A}$ which can also be viewed as a vector space over $\mathbb{C}$, such that the underlying addition and the zero element 0 are the the same in the ring and vector space, and $a(x \cdot y) = (ax) \cdot y = x \cdot (ay)$ for all $x, y \in A$, $a \in \mathbb{C}$.

For any associative algebra $\mathcal{A}$, we can define $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by $[x, y] = x \cdot y - y \cdot x$, where $\cdot$ denotes the associative product and $x, y \in \mathcal{A}$. With this bracket structure, $\mathcal{A}$ becomes a Lie algebra.

Definition 2.1.8. A subspace $\mathfrak{g}'$ of a Lie algebra $\mathfrak{g}$ is a *subalgebra* if $[x, y] \in \mathfrak{g}'$ for all $x, y \in \mathfrak{g}'$.

Example 2.1.9. For $s\ell(n, \mathbb{C})$, the set $\mathfrak{h} = \text{span}\{h_i = E_{ii} - E_{i+1,i+1}\}$ is a subalgebra. This subalgebra is very important and called the *Cartan subalgebra.*

Definition 2.1.10. A subspace $\mathfrak{g}'$ of a Lie algebra $\mathfrak{g}$ is an *ideal* if $[x, y] \in \mathfrak{g}'$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{g}'$. Then a Lie algebra is *simple* if it is nonabelian and its only ideals are $\{0\}$ and itself.

Simple Lie algebras give rise to a special class of Lie algebras, called *semisimple Lie algebras.*

Definition 2.1.11. For a Lie algebra, $\mathfrak{g}$, define the *derived series* of $\mathfrak{g}$ as

$$\mathfrak{g} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \mathfrak{g}^{(3)} \supseteq \cdots,$$

where $\mathfrak{g}^{(m)} = [\mathfrak{g}^{(m-1)}, \mathfrak{g}^{(m-1)}]$. Then, $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^{(m)} = \{0\}$ for some m.

Definition 2.1.12. A Lie algebra, $\mathfrak{g}$, is *semisimple* if it contains no nonzero solvable ideals. Equivalently, $\mathfrak{g}$ is semisimple if it can be written as a direct sum of simple ideals.

Not every Lie algebra is an associative algebra. However, we can construct an associative algebra from a Lie algebra, called a universal enveloping algebra.

Definition 2.1.13. Let $\mathfrak{g}$ be a Lie algebra. A *universal enveloping algebra* of $\mathfrak{g}$ is a pair $(U(\mathfrak{g}), \iota)$ such that $U(\mathfrak{g})$ is an associative algebra over $\mathbb{C}$ and $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is a linear map such that $\iota([x, y]) = \iota(x)\iota(y) - \iota(y)\iota(x)$ for all $x, y \in \mathfrak{g}$ and satisfying the universal property:
For any associative algebra $\mathcal{A}$ and any linear map $j : \mathfrak{g} \to \mathcal{A}$ satisfying $j([x, y]) = j(x)j(y) - j(y)j(x)$ for all $x, y \in \mathfrak{g}$, there exists a unique homomorphism of associate algebras $\phi : U(\mathfrak{g}) \to \mathcal{A}$ such that $j = \phi \circ \iota$. The universal property of $U(\mathfrak{g})$ can be shown in the commutative diagram below:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "j"'] & U(\mathfrak{g}) \arrow[d, dashed, "\exists!\ \phi"] \\
& \mathcal{A}
\end{tikzcd}$$

Figure 2.1: Universal property for the universal enveloping algebra

The universal enveloping algebra can be constructed as $U(\mathfrak{g}) = T(\mathfrak{g})/I$ where $T(\mathfrak{g}) = \bigoplus_{k \geq 0} \mathfrak{g}^{\otimes k}$ is the tensor algebra of $\mathfrak{g}$ and I is the two-sided ideal generated by elements of the form $x \otimes y - y \otimes x - [x, y]$, $x, y \in \mathfrak{g}$. The linear map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is constructed by composing the natural maps $\mathfrak{g} \hookrightarrow T(\mathfrak{g})$ and $\pi : T(\mathfrak{g}) \to U(\mathfrak{g})$. From this construction, the map ι may not be injective. However, the Poincaré-Birkhoff-Witt Theorem, known as the PBW Theorem, shows that ι is injective and gives a basis for $U(\mathfrak{g})$.

Theorem 2.1.14. *[2] [Poincaré-Birkhoff-Witt]*

(1) The map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is injective.

(2) Let $\{x_i \mid i \in I\}$ be an ordered basis for $\mathfrak{g}$, where I is an index set. Then, the set $\{x_{i_1} x_{i_2} \cdots x_{i_k} \mid i_1 \leq i_2 \leq \cdots \leq i_k,\ k \geq 0\}$ forms a basis for $U(\mathfrak{g})$.

In the study of abstract objects, we consider homomorphisms and their impact on the structure of the objects. An important homomorphism of Lie algebras is called a representation and leads to a field of study called representation theory.

Definition 2.1.15. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras. A linear transformation $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a *Lie algebra homomorphism* if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}_1$.

Definition 2.1.16. Let $\mathfrak{g}$ be a Lie algebra and V be a vector space over $\mathbb{C}$. A Lie algebra homomorphism $\varphi : \mathfrak{g} \to g\ell(V)$ is a *Lie algebra representation*, where $g\ell(V)$ denotes the set of all linear transformations from V to V. $g\ell(V)$ is an associative algebra, hence a Lie algebra under the commutator bracket.

Definition 2.1.17. Let $\mathfrak{g}$ be a Lie algebra and V be a vector space over $\mathbb{C}$. Then V is an $\mathfrak{g}$-*module* if there is a binary operation $\cdot : \mathfrak{g} \times V \to V$ such that for all $x, y \in \mathfrak{g}$, $u, v \in V$, and $a, b \in \mathbb{C}$,

(1) $x \cdot (au + bv) = a(x \cdot u) + b(x \cdot v)$,

(2) $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$,

(3) $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$.

Remark 2.1.18. Representations and $\mathfrak{g}$ modules are equivalent: If V is a $\mathfrak{g}$-module, then we can define the representation $\varphi : L \to g\ell(V)$ by $\varphi(x)v = x \cdot v$. Conversely, given a representation, $\varphi : L \to g\ell(V)$, V is a $\mathfrak{g}$-module given by $x \cdot v = \varphi(x)v$. We will use the terms "representations" and "modules" interchangeably due to this equivalence, noting that the representation is the homomorphism and the module is the vector space.

Example 2.1.19. Let $\mathfrak{g} = s\ell(2, \mathbb{C})$ and $V = \mathbb{C}^2$. Define the map $\varphi : s\ell(2, \mathbb{C}) \to g\ell(\mathbb{C}^2)$ by $\varphi(x)v = xv$ for $x \in s\ell(2, \mathbb{C})$ and $v \in \mathbb{C}^2$. This map is a representation, and thus $V = \mathbb{C}^2$ is an $s\ell(2, \mathbb{C})$-module by $x \cdot v = xv$.

Example 2.1.20. Let $\mathfrak{g}$ be a Lie algebra. Define the map $\mathrm{ad} : \mathfrak{g} \to g\ell(\mathfrak{g})$ by $\mathrm{ad}(x)y = [x, y]$ for $x, y \in \mathfrak{g}$. This map is a representation, called the adjoint representation. We usually denote $\mathrm{ad}(x) = \mathrm{ad}_x$.

Definition 2.1.21. Let V be a $\mathfrak{g}$-module. A subspace W of V is a *submodule* if $x \cdot w \in W$ for all $x \in \mathfrak{g}$ and $w \in W$.

Definition 2.1.22. A $\mathfrak{g}$-module V is *irreducible* if $V \neq 0$ and V has no proper submodules.

Definition 2.1.23. Let $\mathfrak{g}$ be a Lie algebra and V and W be two $\mathfrak{g}$-modules. A linear transformation $\varphi : V \to W$ is a $\mathfrak{g}$-*module homomorphism* if $\varphi(x \cdot v) = x \cdot \varphi(v)$ for all $x \in \mathfrak{g}$, $v \in V$. If φ is one-to-one and onto, then it is an isomorphism, and we say that V and W are isomorphic as $\mathfrak{g}$-modules, denoted by $V \cong W$.

Let $\mathfrak{g}$ be a Lie algebra and V be a $\mathfrak{g}$-module. By the universal property of $U(\mathfrak{g})$, the $\mathfrak{g}$-module structure on V extends naturally to the $U(\mathfrak{g})$-module structure on V: Since $U(\mathfrak{g})$ is generated by $\mathfrak{g}$, we define the module action of $U(\mathfrak{g})$ on V by

$$(x_{i_1} \cdots x_{i_k}) \cdot v = x_{i_1} \cdot (x_{i_2} \cdots (x_{i_k} \cdot v))$$

Because modules and representations are equivalent, we can extend representations of $\mathfrak{g}$ to representations of $U(\mathfrak{g})$. Conversely, the PBW Theorem states that $\mathfrak{g}$ is a subspace of $U(\mathfrak{g})$, so a representation of $U(\mathfrak{g})$ is also a representation of $\mathfrak{g}$. Therefore, the representation theory of a Lie algebra is parallel to that of its universal enveloping algebra. [2]

2.2 Kac-Moody Algebras

We restrict our study of Lie algebras to a class of infinite-dimensional Lie algebras called Kac-Moody algebras. These algebras are a generalization of finite-dimensional semisimple Lie algebras, and are constructed from a special matrix, called a generalized Cartan matrix.

Definition 2.2.1. Let I be a finite index set. A square matrix, $A = (a_{ij})_{i,j \in I}$, with integer entries is a *generalized Cartan matrix (GCM)* if it satisfies the following:

(1) $a_{ii} = 2$ for all $i \in I$,

(2) $a_{ij} < 0$ if $i \neq j$,

(3) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

If A is also positive-definite, then A is called a *Cartan matrix.*

Definition 2.2.2. A GCM A is *symmetrizable* if there exists a diagonal matrix, $S = \mathrm{diag}(s_i \mid i \in I,\ s_i \in \mathbb{Z}_{>0})$ such that SA is a symmetric matrix. Also, A is *indecomposable* if for every pair of nonempty subsets $I_1, I_2 \subseteq I$ with $I_1 \cup I_2 = I$, there exists some $i \in I_1$ and $j \in I_2$ such that $a_{ij} \neq 0$.

We will consider only indecomposable symmetrizable GCM's.

Theorem 2.2.3. *[6] Let $A = (a_{ij})$ be an indecomposable GCM. Then one and only one of the following three possibilities hold for both A and A^t.*

(Finite) $\det A \neq 0$*; there exists $u > 0$ such that $Au > 0$; $Av \geq 0$ implies $v > 0$ or $v = 0$.*

(Affine) corank $A = 1$; there exists $u > 0$ such that $Au = 0$; $Av \geq 0$ implies $Av = 0$.

(Indefinite) There exists $u > 0$ such that $Au < 0$; $Av \geq 0$ and $v \geq 0$ imply $v = 0$.

Here, u, v are column vectors in $\mathbb{R}^n$ and we say that $u > 0$ if $u_i > 0$ for all $i = 1, \ldots, n$. Also, A is said to be of finite, affine, *or* indefinite type *if A satisfies the corresponding condition.*

Definition 2.2.4. [6] For each GCM, A, we can associate an oriented graph called a *Dynkin diagram.* This diagram consists of vertices indexed by I and edges with arrows such that

- If $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, then the vertices i and j are connected with $|a_{ij}|$ edges and has an arrow pointing toward i if $|a_{ij}| > 1$.
- If $a_{ij}a_{ji} > 4$, then the vertices i and j are connected with a bold edge labeled with $(|a_{ij}|, |a_{ji}|)$.

We can also recover the corresponding GCM from a Dynkin diagram, up to the order of indices.

Example 2.2.5. The following GCM and its corresponding Dynkin diagram are given below:

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

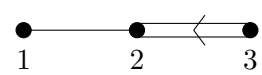

Figure 2.2: Generalized Cartan matrix and Dynkin diagram

Definition 2.2.6. Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable GCM. The *Cartan datum* of A is the quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$, where Π, $\Pi^\vee$, P, and $P^\vee$ are defined as follows:

$P^\vee$ is a free abelian group of rank $2|I| - \text{rank } A$ with $\mathbb{Z}$-basis $\{h_i \mid i \in I\} \cup \{d_s \mid s = 1, \ldots, |I| - \text{rank } A\}$, called the *dual weight lattice.*

Define $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$ to the be *Cartan subalgebra.*

$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\}$ is called the *weight lattice.*

$\Pi^\vee = \{h_i \mid i \in I\}$ is called the set of *simple coroots*.

$\Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ is a linearly independent subset satisfying $\alpha_j(h_i) = a_{ij}$ and $\alpha_j(d_s) = \delta_{sj}$, called the set of *simple roots*.

Definition 2.2.7. The *fundamental weights* $\Lambda_i \in \mathfrak{h}^*$ are linear functionals on $\mathfrak{h}$ given by,

$$\Lambda_i(h_j) = \delta_{ij}, \qquad \lambda_i(d_s) = 0 \qquad \text{for } i, j \in I, s \in 1, \ldots, |I| - \text{rank } A$$

With the Cartan datum, we can construct a Kac-Moody algebra as follows.

Definition 2.2.8. The *Kac-Moody algebra* $\mathfrak{g}$ associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the Lie algebra generated by the elements e_i, f_i $(i \in I)$ and $h \in P^\vee$ subject to the following relations:

(1) $[h, h'] = 0$ for $h, h' \in P^\vee$,

(2) $[e_i, f_j] = \delta_{ij} h_i$,

(3) $[h, e_i] = \alpha_i(h) e_i$ for $h \in P^\vee$,

(4) $[h, f_i] = -\alpha_i(h) f_i$ for $h \in P^\vee$,

(5) $(\text{ad } e_i)^{1-a_{ij}} e_j = 0$ for $i \neq j$,

(6) $(\text{ad } f_i)^{1-a_{ij}} f_j = 0$ for $i \neq j$.

The generators e_i and f_i $(i \in I)$ the *Chevalley generators*. Also, relations (1)-(4) are called the *Weyl relations*, while relations (5) and (6) are called the *Serre relations*.

Example 2.2.9. Consider the following GCM and corresponding Dynkin diagram.

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

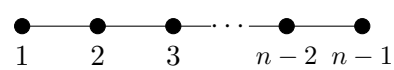

Figure 2.3: Generalized Cartan matrix and Dynkin diagram for $s\ell(n, \mathbb{C})$

This GCM is associated with the Kac-Moody algebra $\mathfrak{g} = s\ell(n, \mathbb{C})$. The Chevalley generators are $\{e_i = E_{i,i+1},\ f_i = E_{i+1,i} \mid 1 \leq i \leq n-1\}$ and the Cartan subalgebra is $\mathfrak{h} = \text{span}\{h_i = E_{ii} - E_{i+1,i+1} \mid 1 \leq i \leq n-1\}$.

Definition 2.2.10. For each $i \in I$, the *simple reflection* r_i on $\mathfrak{h}^*$ is the linear functional given by $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$. The subgroup W of $\mathrm{End}(\mathfrak{h}^*)$ generated by all simple reflections is called the *Weyl group*. Any element $w \in W$ can be written as a product of simple reflections, ie. $w = r_{i_1} r_{i_2} \cdots r_{i_t}$. If t is minimal among all such expressions, then w is a *reduced expression* and the minimal number t is called the *length*, denoted $l(w)$.

Recall, for a Kac-Moody algebra $\mathfrak{g}$, we have the set of simple roots, $\Pi = \{\alpha_i \mid i \in I\}$.

Definition 2.2.11. The free abelian group $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is called the *root lattice*, $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ is the *positive root lattice*, and $Q_- = -Q_+$ is the *negative root lattice*. For each $\alpha \in Q$, let

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, then α is called a *root* of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is called the *root space* attached to α. The dimension of $\mathfrak{g}_\alpha$ is called the *root multiplicity* of α.

We denote the set of all roots by Δ and denote the set of positive (respectively, negative) roots by $\Delta_+ = \Delta \cap Q_+$ (respectively, $\Delta_i = \Delta \cap Q_-$). We have the following decompositions of the Kac-Moody algebra.

Proposition 2.2.12. [6]

(1) The Kac-Moody algebra $\mathfrak{g}$ has the following root space decomposition:

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha,$$

with $\dim \mathfrak{g}_\alpha < \infty$ for all $\alpha \in Q$.

(2) Let $\mathfrak{g}_+ = \bigoplus \mathbb{C}e_i$ and $\mathfrak{g}_- = \bigoplus \mathbb{C}f_i$ $(i \in I)$ be subalgebras of the Kac-Moody algebra $\mathfrak{g}$. Then $\mathfrak{g}$ has the following triangular decomposition:

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+.$$

(3) There exists an involution $\omega : \mathfrak{g} \to \mathfrak{g}$, called the *Chevalley involution*, such that $e_i \mapsto -f_i$, $f_i \mapsto -e_i$, and $h \mapsto -h$.

Given the triangular decomposition, $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$, if α is a positive root, then $\mathfrak{g}_\alpha \in \mathfrak{g}_+$ and if α is a negative root, then $\mathfrak{g}_\alpha \in \mathfrak{g}_-$. Also, by the Chevalley involution, if α is a positive root, then $\omega(g_\alpha) = g_{-\alpha}$, implying $\mathrm{mult}(\alpha) = \mathrm{mult}(-\alpha)$.

Corresponding to a Kac-Moody algebra, $\mathfrak{g}$, we can construct its universal enveloping algebra as follows.

Proposition 2.2.13. [2] The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is the associative algebra over $\mathbb{C}$ with unity generated by e_i, f_i $(i \in I)$ and $\mathfrak{h}$ such that the following relations hold:

(1) $hh' = h'h$ for $h, h' \in \mathfrak{h}$,

(2) $e_i f_j - f_j e_i = \delta_{ij} h_i$ for $i, j \in I$,

(3) $he_i - e_i h = \alpha_i(h) e_i$ for $h \in \mathfrak{h},\ i \in I$,

(4) $hf_i - f_i h = -\alpha_i(h) f_i$ for $h \in \mathfrak{h},\ i \in I$,

(5) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1-a_{ij} \\ k \end{pmatrix} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$,

(6) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1-a_{ij} \\ k \end{pmatrix} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$.

We can extend the root space decomposition and triangular decomposition to the universal enveloping algebra of $\mathfrak{g}$ using the PBW Theorem. First, let U^+ (respectively, U^0 and U^-) denote the subalgebra of $U(\mathfrak{g})$ generated by the elements e_i (respectively, $\mathfrak{h}$ and f_i) for $i \in I$. Also, define the root spaces of $U(\mathfrak{g})$ as follows:

$$U_\beta = \{u \in U(\mathfrak{g}) \mid hu - uh = \beta(h)u \text{ for all } h \in \mathfrak{h}\} \quad \text{for } \beta \in Q;$$

$$U_\beta^\pm = \{u \in U^\pm \mid hu - uh = \beta(h)u \text{ for all } h \in \mathfrak{h}\} \quad \text{for } \beta \in Q_\pm.$$

Proposition 2.2.14. [2] For Kac-Moody algebra, $\mathfrak{g}$, and its corresponding universal enveloping algebra, $U(\mathfrak{g})$, we have the follow decompositions:

(1) $U(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+$

(2) $U(\mathfrak{g}) = \displaystyle\bigoplus_{\beta \in Q} U_\beta$

(3) $U^\pm = \displaystyle\bigoplus_{\beta \in Q_\pm} U_\beta^\pm$

2.3 Representation Theory

Given a Kac-Moody algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module V, we define the notion of weights and associated weight spaces of V. This leads to the study of integrable highest weight $\mathfrak{g}$-modules. We assume $\mathfrak{g}$ is a Kac-Moody algebra in this section.

Definition 2.3.1. Let V be a $\mathfrak{g}$-module. For any $\mu \in \mathfrak{h}^*$, the *μ-weight space* is

$$V_\mu = \{v \in V \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}.$$

If $V_\mu \neq 0$, then μ is called a *weight* and the dimension of V_μ is called the *weight multiplicity* of μ. A vector $v \in V_\mu$ is called a *weight vector* of weight μ.

Definition 2.3.2. A $\mathfrak{g}$-module V is a *weight module* if it admits a *weight space decomposition*:

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$$

Definition 2.3.3. Given a $\mathfrak{g}$-module V, if $\dim V_\mu < \infty$ for all weights μ, the *character* of V is

$$\text{ch } V = \sum_\mu \dim V_\mu e^\mu,$$

where e^μ are formal basis elements of the group algebra $\mathbb{C}[\mathfrak{h}^*]$ with multiplication defined as $e^\lambda e^\mu = e^{\lambda+\mu}$.

An important class of weight modules are called highest weight modules.

Definition 2.3.4. A weight module V is a *highest weight module* of *highest weight* $\lambda \in \mathfrak{h}^*$ if there exists nonzero vector $v_\lambda \in V$, called a *highest weight vector*, such that

$e_i v_\lambda = 0$ for all $i \in I$,

$hv_\lambda = \lambda(h)v_\lambda$ for all $h \in \mathfrak{h}$,

$V = U(\mathfrak{g})v_\lambda$.

We now consider a specific type of highest weight module called a Verma module.

Definition 2.3.5. Fix $\lambda \in \mathfrak{h}^*$ and let $J(\lambda)$ be the left ideal of $U(\mathfrak{g})$ generated by all e_i and $h - \lambda(h)1$ ($i \in I,\ h \in \mathfrak{h}$). Set $M(\lambda) = U(\mathfrak{g})/J(\lambda)$ and give $M(\lambda)$ a $U(\mathfrak{g})$-module structure by left multiplication. Then $M(\lambda)$ is called the *Verma module*.

We have the following properties regarding Verma modules.

Proposition 2.3.6. [6]

(1) $M(\lambda)$ is a highest weight $\mathfrak{g}$-module with highest weight λ and highest weight vector $v_\lambda = 1 + J(\lambda)$.

(2) Every highest weight $\mathfrak{g}$-module with highest weight λ is a homomorphic image of $M(\lambda)$.

(3) As a U^--module, $M(\lambda)$ is free of rank 1, generated by the highest weight vector $v_\lambda = 1 + J(\lambda)$.

(4) $M(\lambda)$ has a unique maximal submodule, $N(\lambda)$.

Definition 2.3.7. Given Verma module $M(\lambda)$ and its unique maximal submodule, $N(\lambda)$, define $V(\lambda) = M(\lambda)/N(\lambda)$. Then $V(\lambda)$ is the *irreducible highest weight module* of $U_q(\mathfrak{g})$.

Definition 2.3.8. Define a partial ordering on $\mathfrak{h}^*$ by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ for $\lambda, \mu \in \mathfrak{h}^*$. Then, for $\lambda \in \mathfrak{h}^*$, let $D(\lambda) = \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda\}$ called the λ-cone. The *category* $\mathcal{O}$ is the set of weight modules V over $\mathfrak{g}$ with finite dimensional weight spaces for which there exists a finite number of elements $\lambda_1, \lambda_2, \ldots, \lambda_s \in \mathfrak{h}^*$ such that

$$wt(V) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s).$$

The morphisms in the category $\mathcal{O}$ are $\mathfrak{g}$-module homomorphisms.

Proposition 2.3.9. [6] Every irreducible $\mathfrak{g}$-module in the category $\mathcal{O}$ is isomorphic to $V(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.

Definition 2.3.10. A weight module V over a Kac-Moody algebra $\mathfrak{g}$ is *integrable* if all e_i and f_i $(i \in I)$ are locally nilpotent on V, ie. for any $v \in V$, there exists $N \in \mathbb{Z}_{>0}$ such that $x^N \cdot v = 0$ for $x \in \{e_i, f_i \mid i \in I\}$.

Definition 2.3.11. Recall the weight lattice $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}, \text{ for all } i \in I\}$. Elements in this set are called *integral weights*. Also, define $P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0}, \text{ for all } i \in I\}$ to be the set of *dominant integral weights*.

We can now define a subcategory consisting of all integrable $\mathfrak{g}$-modules.

Definition 2.3.12. The *category* $\mathcal{O}_{int}$ consists of integrable $\mathfrak{g}$-modules in the category $\mathcal{O}$ such that $wt(V) \subset P$.

Any $\mathfrak{g}$-module V in category $\mathcal{O}_{\text{int}}$ is completely reducible and has a weight space decomposition,

$$V = \bigoplus_{\lambda \in P} V_\lambda,$$

where $V_\lambda = \{v \in V \mid h_i v = \lambda(h_i)v \text{ for all } i \in I\}$.

Proposition 2.3.13. [6] Let $V(\lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$. Then $V(\lambda)$ is in category $\mathcal{O}_{\text{int}}$, ie. $V(\lambda)$ is integrable, if and only if $\lambda \in P^+$.

This proposition leads to our main result regarding integrable highest weight $\mathfrak{g}$-modules.

Theorem 2.3.14. *[6] Let $\mathfrak{g}$ be a Kac-Moody algebra associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. Then every $\mathfrak{g}$-module in the category $\mathcal{O}_{int}$ is isomorphic to a direct sum of irreducible highest weight modules $V(\lambda)$ with $\lambda \in P^+$.*

2.4 Quantum Groups

In this section, we will introduce quantum deformations of the universal enveloping algebra of a Kac-Moody algebra $\mathfrak{g}$. These deformations are called quantum groups and denoted $U_q(\mathfrak{g})$. In particular, if $\mathfrak{g}$ is an affine Kac-Moody algebra, then the quantum deformation is called a quantum affine algebra. The quantum affine algebra $U_q(D_4^{(3)})$ is our main focus in future chapters, but we first develop the representation theory of quantum groups.

For $m, n \in \mathbb{Z}$ and q any indeterminate, define the following:

- $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$ is a *q-integer*;
- $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ $(n > 0)$ and $[0]_q! = 1$ are *q-factorials*;
- $\begin{bmatrix} m \\ n \end{bmatrix}_q = \dfrac{[m]_q!}{[n]_q![m-n]_q!}$ $(m \geq n \geq 0)$ is the *q-binomial coefficient.*

Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable GCM with symmetrizing matrix $S = \mathrm{diag}(s_i \mid i \in I,\ s_i \in \mathbb{Z}_{>0})$.

Definition 2.4.1. [2] The *quantum group* or the *quantized enveloping algebra* $U_q(\mathfrak{g})$ associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the associative algebra over $\mathbb{C}(q)$ with unity generated by the elements e_i, f_i $(i \in I)$ and q^h $(h \in P^\vee)$ with the following relations.

(1) $q^0 = 1$; $q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^\vee$,

(2) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ for $h \in P^\vee$, $i \in I$,

(3) $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ for $h \in P^\vee$, $i \in I$,

(4) $e_i f_j - f_j e_i = \delta_{ij} \dfrac{q^{s_i h_i} - q^{-s_i h_i}}{q^{s_i} - q^{-s_i}}$ for $i, j \in I$,

(5) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q^{s_i}} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$,

(6) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q^{s_i}} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$.

Note, that as $q \to 1$, $U_q(\mathfrak{g}) \to U(\mathfrak{g})$.

Example 2.4.2. Given $\mathfrak{g} = s\ell(2, \mathbb{C})$, the quantum groups $U_q(s\ell(2, \mathbb{C}))$ is the associative algebra generated by $\{e, f, q^h\}$ such that the following relations hold:

(1) $q^h e q^{-h} = q^2 e$,

(2) $q^h f q^{-h} = q^{-2} f$,

(3) $ef - fe = \dfrac{q^h - q^{-h}}{q - q^{-1}}$.

Since the defining relations for a quantum group $U_q(\mathfrak{g})$ are analogous to those for the universal enveloping algebra, $U(\mathfrak{g})$, $U_q(\mathfrak{g})$ has both a root space decomposition and a triangular decomposition.

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} U_q(\mathfrak{g})_\alpha,$$

where $U_q(\mathfrak{g})_\alpha = \{u \in U_q(\mathfrak{g}) \mid q^h u q^{-h} = q^{\alpha(h)} u \text{ for all } h \in P^\vee\}$.

Also, define $U_q^+(\mathfrak{g})$ (respectively, $U_q^-(\mathfrak{g})$ and $U_q^0(\mathfrak{g})$) to be the subalgebra generated by the elements e_i (respectively, the elements f_i and q^h for $h \in P^\vee$). Then, the triangular decomposition of $U_q(\mathfrak{g})$ is,

$$U_q(\mathfrak{g}) \cong U_q^+(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^-(\mathfrak{g}).$$

The representation theory of $\mathfrak{g}$-modules and $U(\mathfrak{g})$-modules discussed in the previous section is parallel to the representation theory of modules over the quantum group, $U_q(\mathfrak{g})$, denoted V^q.

Definition 2.4.3. Let V^q be a $U_q(\mathfrak{g})$-module. For any $\mu \in P$, the *μ-weight space* is

$$V_\mu^q = \{v \in V^q \mid q^h v = q^{\mu(h)} v \text{ for all } h \in P^\vee\}.$$

If $V_\mu^q \neq 0$, then μ is called a *weight* and the dimension of V_μ^q is the *weight multiplicity* of μ. A vector $v \in V_\mu^q$ is called a *weight vector* of weight μ.

Definition 2.4.4. A $U_q(\mathfrak{g})$-module V^q is a *weight module* if it admits a *weight space decomposition*,

$$V^q = \bigoplus_{\mu \in P} V_\mu^q.$$

Denote by $\mathrm{wt}(V^q)$ the set of weights of the $U_q(\mathfrak{g})$-module V^q.

Definition 2.4.5. Given a $U_q(\mathfrak{g})$-module V^q, if $\dim V_\mu^q < \infty$ for all weights $\mu \in \mathrm{wt}(V^q)$, the *character* of V^q is

$$\mathrm{ch}\ V^q = \sum_\mu \dim V_\mu^q e^\mu,$$

where e^μ are formal basis elements of the group algebra $\mathbb{C}[P]$ with multiplication defined as $e^\lambda e^\mu = e^{\lambda+\mu}$.

Definition 2.4.6. A weight module V^q is a *highest weight module* of *highest weight* $\lambda \in P$ if there exists a nonzero vector $v_\lambda \in V^q$, called a *highest weight vector*, such that

$$e_i v_\lambda = 0 \text{ for all } i \in I,$$

$$q^h v_\lambda = q^{\lambda(h)} v_\lambda \text{ for all } h \in P^\vee,$$

$$V^q = U_q(\mathfrak{g}) v_\lambda.$$

Just as we discussed in the representation theory for Kac-Moody algebras, we aim to study unique irreducible, integrable highest weight modules. The construction of such modules follows similarly to those of Kac-Moody algebras.

Definition 2.4.7. Fix $\lambda \in P$ are let $J^q(\lambda)$ be the left ideal of $U_q(\mathfrak{g})$ generated by all e_i and $q^h - q^{\lambda(h)}1$ ($i \in I$, $h \in P^\vee$). Set $M^q(\lambda) = U_q(\mathfrak{g})/J^q(\lambda)$ and give $M^q(\lambda)$ a $U_q(\mathfrak{g})$-module structure by left multiplication. Then $M^q(\lambda)$ is the *Verma module.*

Set $v_\lambda = 1 + J^q(\lambda)$. Then, we have the following,

$$q^h v_\lambda = q^h + J^q(\lambda) = q^{\lambda(h)}1 + J^q(\lambda) = q^{\lambda(h)} v_\lambda,$$

$$e_i v_\lambda = e_i + J^q(\lambda) = J^q(\lambda) = 0,$$

$$U_q(\mathfrak{g}) v_\lambda = U_q(\mathfrak{g})/J^q(\lambda) = M^q(\lambda).$$

Thus, $M^q(\lambda)$ is a highest weight module with highest weight λ and highest weight vector $v_\lambda = 1 + J^q(\lambda)$. Also, we know that $M^q(\lambda)$ has a unique maximal submodule, $N^q(\lambda)$.

Definition 2.4.8. Given Verma module $M^q(\lambda)$ and its unique maximal submodule, $N^q(\lambda)$, define $V^q(\lambda) = M^q(\lambda)/N^q(\lambda)$. Then $V^q(\lambda)$ is the *irreducible highest weight module* with highest weight λ.

Definition 2.4.9. A weight module $V^q(\lambda)$ over the quantum group $U_q(\mathfrak{g})$ is *integrable* if all e_i and f_i ($i \in I$) are locally nilpotent on V^q.

Just as before, for $\lambda \in P$, consider the λ-cone, $D(\lambda) = \{\mu \in P \mid \mu \leq \lambda\}$. The *category* $\mathcal{O}^q$ is the set of weight modules V^q over $U_q(\mathfrak{g})$ with finite dimensional weight spaces for which there exists a finite number of elements $\lambda_1, \lambda_2, \ldots, \lambda_s \in P$ such that

$$wt(V^q) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s).$$

We can then define the subcategory of category $\mathcal{O}^q$ of integrable $U_q(\mathfrak{g})$-modules.

Definition 2.4.10. The *category* $\mathcal{O}^q_{int}$ consists of integrable $U_q(\mathfrak{g})$-modules in the category $\mathcal{O}^q$.

Similar to category $\mathcal{O}_{\text{int}}$, any $U_q(\mathfrak{g})$-module in category $\mathcal{O}^q_{\text{int}}$ is completely reducible and has a weight space decomposition.

Proposition 2.4.11. [2] Let $V^q(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda \in P$. Then $V^q(\lambda)$ is in category $\mathcal{O}^q_{\text{int}}$, ie. $V^q(\lambda)$ is integrable, if and only if $\lambda \in P^+$.

This result shows that we can construct a unique irreducible, integrable highest weight $U_q(\mathfrak{g})$-module for each $\lambda \in P^+$, denoted $V^q(\lambda)$.

Chapter 3

Crystal Base Theory

Crystal base theory provides combinatorial tools to study the structure of integrable $U_q(\mathfrak{g})$-modules V^q in the category $\mathcal{O}^q_{\text{int}}$. Crystal bases can be viewed as bases at $q = 0$ and behave nicely with respect to the tensor product. Also, the crystal bases correspond to perfect crystals, whose crystal graphs are useful in determining the structure of the Demazure modules for $U_q(D_4^{(3)})$ described in the next chapter.

3.1 Crystal Base

Let $V^q = \bigoplus_{\lambda \in P} V^q_\lambda$ be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\text{int}}$ and let λ be a weight of V^q such that $V^q_\lambda \neq 0$. For each $i \in I$, every weight vector $v \in V^q_\lambda$ may be written as

$$v = v_0 + f_i v_1 + \cdots + f_i^{(N)} v_N,$$

where $N \in \mathbb{Z}_{\geq 0}$, $v_k \in V^q_{\lambda + k\alpha_i} \cap \ker e_i$, and $f_i^{(k)} = \dfrac{f_i^k}{[k]_q!}$. Each v_k in the above expression is determined uniquely by v and $v_k \neq 0$ only if $\lambda(h_i) + k \geq 0$.

Definition 3.1.1. The *Kashiwara operators* $\tilde{e}_i,\ \tilde{f}_i : V^q \to V^q$ $(i \in I)$ are endomorphisms such that for $v \in V^q$,

$$\tilde{e}_i v = \sum_{k=1}^{N} f_i^{(k-1)} v_k, \qquad \tilde{f}_i v = \sum_{k=1}^{N} f_i^{(k+1)} v_k.$$

We note that for $v \in V^q_\lambda$, $\tilde{e}_i v \in V^q_{\lambda+\alpha_i}$ and $\tilde{f}_i v \in V^q_{\lambda-\alpha_i}$.

Now, consider the principal ideal domain, $\mathbb{A}$, with $\mathbb{C}(q)$ as its field of quotients defined as follows,

$$\mathbb{A} = \left\{ \frac{g(q)}{h(q)} \mid g(q), h(q) \in \mathbb{C}[q],\ h(0) \neq 0 \right\}.$$

Definition 3.1.2. [2] A free $\mathbb{A}$-submodule $\mathcal{L}$ of $V^q \in \mathcal{O}^q_{\text{int}}$ is a *crystal lattice* if

(1) $\mathcal{L}$ generates V^q as a vector space over $\mathbb{C}(q)$, ie. $V^q \cong \mathcal{L}_\mu \otimes_{\mathbb{A}} \mathbb{C}(q)$ for each $\mu \in wt(V^q)$,

(2) $\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda$, where $\mathcal{L}_\lambda = \mathcal{L} \cap V^q_\lambda$ for all $\lambda \in P$,

(3) $\tilde{e}_i \mathcal{L} \subset \mathcal{L}$ and $\tilde{f}_i \mathcal{L} \subset \mathcal{L}$ for all $i \in I$.

Remark 3.1.3. Let $\mathbb{J} = \langle q \rangle$ be the unique maximal ideal of $\mathbb{A}$. Then, there exists an isomorphism of fields from $\mathbb{A}/\mathbb{J}$ onto $\mathbb{C}$ given by $f(q) + \mathbb{J} \mapsto f(0)$. This implies that $\mathbb{C} \otimes_{\mathbb{A}} \mathcal{L} \cong \mathcal{L}/q\mathcal{L}$. This passage from $\mathcal{L}$ to $\mathcal{L}/q\mathcal{L}$ is called taking the *crystal limit*. Since $\mathcal{L}$ is preserved by $\tilde{e}_i$ and $\tilde{f}_i$, we can also define the Kashiwara operators on $\mathcal{L}/q\mathcal{L}$ and use the same symbols.

We now define a crystal base for the $U_q(\mathfrak{g})$-module $V^q \in \mathcal{O}^q_{\text{int}}$:

Definition 3.1.4. [2] A *crystal base* for V^q is a pair $(\mathcal{L}, \mathcal{B})$ such that

(1) $\mathcal{L}$ is a crystal lattice of V^q,

(2) $\mathcal{B}$ is a $\mathbb{C}$-basis of $\mathcal{L}/q\mathcal{L} \cong \mathbb{C} \otimes_{\mathbb{A}} \mathcal{L}$,

(3) $\mathcal{B} = \bigsqcup_{\lambda \in P} \mathcal{B}_\lambda$, where $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathcal{L}_\lambda / q\mathcal{L}_\lambda)$,

(4) $\tilde{e}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$ and $\tilde{f}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$ for all $i \in I$,

(5) for any $b, b' \in \mathcal{B}$ and $i \in I$, we have $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$.

Definition 3.1.5. Given a crystal base, $(\mathcal{L}, \mathcal{B})$ for $V^q \in \mathcal{O}^q_{\text{int}}$, we can define a *crystal graph* of V^q in the following way: Let the elements of $\mathcal{B}$ be the set of vertices and for each $i \in I$, we join $b \in \mathcal{B}$ to $b' \in \mathcal{B}$ with an i-colored arrow $(b \xrightarrow{i} b')$ if and only if $\tilde{f}_i b = b'$.

Let $\mathcal{B}$ be the crystal graph for the $U_q(\mathfrak{g})$-module in $\mathcal{O}^q_{\text{int}}$. For $i \in I$ and $b \in \mathcal{B}_\lambda$, $\lambda \in P$, we define the maps $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z}$ by,

$$\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k b \in \mathcal{B}\},$$

$$\varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k b \in \mathcal{B}\}.$$

Then, by the crystal graph, ε_i denotes the number of i-colored arrows coming into the vertex b, and φ_i denotes the number of i-colored arrows coming out of the vertex b. Thus, $\varphi_i(b) + \varepsilon_i(b)$ denotes the length of the i-string through b, and we have $\varphi_i(b) - \varepsilon_i(b) = \lambda(h_i)$.

The existence and uniqueness of crystal bases for integrable $U_q(\mathfrak{g})$-modules in category $\mathcal{O}^q_{\text{int}}$ follows from the following theorem of Kashiwara:

Theorem 3.1.6. *[2] Let $\lambda \in P^+$ be a dominant integral weight and let $V^q(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight λ and highest weight vector v_λ. Let*

$$\mathcal{L}(\lambda) = \sum_{r \geq 0,\ i_k \in I} \mathbb{A} \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_r} v_\lambda,$$

$$\mathcal{B}(\lambda) = \{\tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_r} v_\lambda + q\mathcal{L}(\lambda) \in \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \mid r \geq 0,\ i_k \in I\} \setminus \{0\}.$$

Then the pair $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ is a crystal base of $V^q(\lambda)$.

Crystal bases behave nicely under the tensor product, providing one of the nicest combinatorial features of a crystal base.

Theorem 3.1.7. *[2] Let V_j^q be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_{int}^q$ and let $(\mathcal{L}_j, \mathcal{B}_j)$ be a crystal base of V_j^q $(j = 1, 2)$. Set $\mathcal{L} = \mathcal{L}_1 \otimes_{\mathbb{A}} \mathcal{L}_2$ and $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$. Then $(\mathcal{L}, \mathcal{B})$ is a crystal base of $V_1^q \otimes_{\mathbb{C}(q)} V_2^q$ where the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathcal{B}$ are given as follows:*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

Hence, we have:

$$\begin{aligned} wt(b_1 \otimes b_2) &= wt(b_1) + wt(b_2), \\ \varepsilon_i(b_1 \otimes b_2) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1) \rangle), \\ \varphi_i(b_1 \otimes b_2) &= \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, wt(b_2) \rangle). \end{aligned}$$

Note that we understand $b_1 \otimes 0 = 0 \otimes b_2 = 0$. Also, we write $b_1 \otimes b_2$ instead of $(b_1, b_2) \in \mathcal{B}_1 \times \mathcal{B}_2$, and denote the crystal graph of $V_1^q \otimes V_2^q$ as $\mathcal{B}_1 \otimes \mathcal{B}_2$.

By considering the abstract notion of a crystal base, we obtain a purely combinatorial structure.

Definition 3.1.8. [2] Let I be a finite index set and $A = (a_{ij})_{i,j \in I}$ be a GCM with Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. A *crystal* associated with the Cartan datum (and hence, $U_q(\mathfrak{g})$) is a set $\mathcal{B}$ together with the maps, wt: $\mathcal{B} \to P$, $\tilde{e}_i, \tilde{f}_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$, and $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$, satisfying the following properties:

(1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle$ for all $i \in I$,

(2) $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\tilde{e}_i b \in \mathcal{B}$,

(3) $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $\tilde{f}_i b \in \mathcal{B}$,

(4) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi(b) + 1$ if $\tilde{e}_i b \in \mathcal{B}$,

(5) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi(b) - 1$ if $\tilde{f}_i b \in \mathcal{B}$,

(6) $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in \mathcal{B}$ and $i \in I$,

(7) if $\varphi_i(b) = -\infty$ for $b \in \mathcal{B}$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

The tensor product rule defined above also holds for abstract crystals. Also, we can define maps between crystals in the following way:

Definition 3.1.9. Let $\mathcal{B}_1, \mathcal{B}_2$ be crystals associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. A *crystal morphism* $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ is a map $\Psi : \mathcal{B}_1 \cup \{0\} \to \mathcal{B}_2 \cup \{0\}$ such that,

(1) $\Psi(0) = 0$,

(2) if $b \in \mathcal{B}_1$ and $\Psi(b) \in \mathcal{B}_2$, then $\mathrm{wt}(\Psi(b)) = \mathrm{wt}(b)$, $\varepsilon(\Psi(b)) = \varepsilon(b)$, and $\varphi(\Psi(b)) = \varphi(b)$ for all $i \in I$,

(3) if $b, b' \in \mathcal{B}_1$, $\Psi(b), \Psi(b') \in \mathcal{B}_2$ and $\tilde{f}_i b = b'$, then $\tilde{f}_i \Psi(b) = \Psi(b')$ and $\Psi(b) = \tilde{e}_i \Psi(b')$ for all $i \in I$.

A crystal morphism $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ is a *crystal isomorphism* if Ψ is a bijection from $\mathcal{B}_1 \cup \{0\}$ to $\mathcal{B}_2 \cup \{0\}$.

3.2 Quantum Affine Algebras and Perfect Crystals

Using the crystal base theory defined above, we develop the notion of a perfect crystal for quantum affine algebras. Perfect crystals allow us to realize the crystal graph of the irreducible highest weight $U_q(\mathfrak{g})$-module in terms of certain paths. We begin by recalling basic definitions related to quantum affine algebras.

Let $I = \{0, 1, \ldots, n\}$ and $(A, \Pi, \Pi^\vee, P, P^\vee)$ be the Cartan datum of an affine Kac-Moody algebra $\mathfrak{g}$, where $A = (a_{ij})_{i,j \in I}$ is an affine GCM, $\Pi = \{\alpha_i \mid i \in I\}$ is the set of simple roots, and $\Pi^\vee = \{h_i \mid i \in I\}$ is the set of simple coroots. Recall that the dual weight lattice and affine weight lattice are,

$$P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d,$$

$$P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\frac{1}{d_0}\delta,$$

where $\{\Lambda_i \mid i \in I\}$ is the set of fundamental weights, $\delta = d_0\alpha_0 + d_1\alpha_1 + \cdots + d_n\alpha_n$ is the null root, and d is the scaling element such that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathbb{C}d$. Elements of P are called *affine weights*. Also, the Cartan subalgebra is $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$.

The linear functionals, α_i and Λ_i, act on $\mathfrak{h}$ in the following way:

$$\alpha_i(h_j) = a_{ji}, \qquad \alpha_i(d) = \delta_{0,i},$$

$$\Lambda_i(h_j) = \delta_{ij}, \qquad \Lambda_i(d) = 0.$$

Note that the center of $\mathfrak{g}$ is one-dimensional, spanned by the *canonical central element*, $c = c_0h_0 + c_1h_1 + \cdots + c_nh_n$.

Then, the quantum group associated with the affine Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ above is called a *quantum affine algebra*, denoted $U_q(\mathfrak{g})$.

Definition 3.2.1. Let $U_q'(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{e_i, f_i, q^{\pm s_i h_i} \mid i \in I\}$. This subalgebra is also called a *quantum affine algebra*. Let

$$\bar{P}^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n, \qquad \bar{\mathfrak{h}} = \mathbb{C} \otimes_{\mathbb{Z}} \bar{P}^\vee.$$

If we consider α_i and Λ_i as linear functionals on $\bar{\mathfrak{h}}$, we can similarly define the *classical weights* as elements of

$$\bar{P} = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n.$$

Then the quintuple $(A, \Pi, \Pi^\vee, \bar{P}, \bar{P}^\vee)$ is called the *classical Cartan datum* and $U_q'(\mathfrak{g})$ is the quantum group associated with the classical Cartan datum.

We distinguish between $U_q(\mathfrak{g})$ and $U_q'(\mathfrak{g})$ because the "classical" quantum affine algebra, $U_q'(\mathfrak{g})$, can have finite dimensional irreducible modules while $U_q(\mathfrak{g})$ has no nontrivial finite dimensional irreducible modules.

Definition 3.2.2. Let $P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}$ be the set of *affine dominant integral weights*. For an affine dominant integral weight, $\lambda \in P^+$, the *level* of λ is the nonnegative integer $\ell = \lambda(c)$, where c is the canonical central element defined above.

Definition 3.2.3. A *classical crystal* is a crystal associated with the classical Cartan datum $(A, \Pi, \Pi^\vee, \bar{P}, \bar{P}^\vee)$. This crystal is also known as a $U_q'(\mathfrak{g})$-*crystal*.

Let $\mathcal{B}$ be a classical crystal, and for $b \in \mathcal{B}$, define

$$\varepsilon(b) = \sum_i \varepsilon_i(b)\Lambda_i, \qquad \varphi(b) = \sum_i \varphi_i(b)\Lambda_i.$$

For a positive integer, $\ell > 0$, set $\bar{P}_\ell^+ = \{\lambda \in \bar{P} \mid \lambda(c) = \ell\}$.

Definition 3.2.4. [2] For a positive integer, $\ell > 0$, a finite classical crystal $\mathcal{B}_\ell$ is a *perfect crystal of level* ℓ if it satisfies the following properties:

(1) There exists a finite dimensional $U_q'(\mathfrak{g})$-module with a crystal base whose crystal graph is isomorphic to $\mathcal{B}_\ell$,

(2) $\mathcal{B}_\ell \otimes \mathcal{B}_\ell$ is connected,

(3) There exists a classical weight $\lambda_0 \in \bar{P}$ such that $\mathrm{wt}(\mathcal{B}_\ell) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\geq 0}\alpha_i$, $\#\mathcal{B}_{\lambda_0} = 1$,

(4) For any $b \in \mathcal{B}_\ell$, we have $\langle c, \varepsilon(b)\rangle \geq \ell$,

(5) For each $\lambda \in \bar{P}_\ell^+$, there exists unique vectors $b^\lambda, b_\lambda \in \mathcal{B}_\ell$ such that $\varepsilon(b^\lambda) = \lambda$ and $\varphi(b_\lambda) = \lambda$.

We now present a crystal isomorphism theorem that allows us to obtain what is called the *path realization* of a crystal graph. For the rest of the chapter, assume that $\mathfrak{g}$ is an affine Kac-Moody algebra, $\ell \in \mathbb{Z}_{\geq 0}$, $\mathcal{B}_\ell$ is a perfect crystal of level ℓ, $\lambda \in \bar{P}_\ell^+$ is a classical dominant integral weight, b_λ is the unique vector in $\mathcal{B}_\ell$ such that $\varphi(b_\lambda) = \lambda$, $L(\lambda)$ is the highest weight $U_q(\mathfrak{g})$-module with highest weight λ, and $\mathcal{B}(\lambda)$ is the crystal graph of $L(\lambda)$.

Theorem 3.2.5. *[2] The following map*

$$\Psi : \mathcal{B}(\lambda) \to \mathcal{B}(\varepsilon(b_\lambda)) \otimes \mathcal{B}_\ell \text{ given by } u_\lambda \mapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda,$$

where u_λ is the highest weight vector of $\mathcal{B}(\lambda)$ and $u_{\varepsilon(b_\lambda)}$ is the highest weight vector of $\mathcal{B}(\varepsilon(b_\lambda))$, is a strict isomorphism of crystals.

3.3 Path Realizations

We will use Theorem 3.2.5 to develop path realizations of crystal graphs. The path realization gives the combinatorial structure of a crystal, which we will use when studying Demazure crystals.

We first set

$$\lambda_0 = \lambda, \qquad \lambda_{k+1} = \varepsilon(b_{\lambda_k});$$

$$b_0 = b_\lambda, \qquad b_{k+1} = b_{\lambda_{k+1}}.$$

By Theorem 3.2.5, we have a crystal isomorphism,

$$\Psi : \mathcal{B}(\lambda_j) \to \mathcal{B}(\lambda_{j+1}) \otimes \mathcal{B}_\ell \text{ given by } u_{\lambda_j} \mapsto u_{\lambda_{j+1}} \otimes b_j.$$

Composing the Ψ's yields a sequence of crystal isomorphisms,

$$\mathcal{B}(\lambda) \to \mathcal{B}(\lambda_1) \otimes \mathcal{B}_\ell \to \mathcal{B}(\lambda_2) \otimes \mathcal{B}_\ell \otimes \mathcal{B}_\ell \to \cdots \mathcal{B}(\lambda_k) \otimes \mathcal{B}_\ell^{\otimes k} \to \cdots,$$

given by

$$u_\lambda \mapsto u_{\lambda_1} \otimes b_0 \mapsto u_{\lambda_2} \otimes b_1 \otimes b_0 \mapsto \cdots u_{\lambda_k} \otimes b_{k-1} \otimes \cdots \otimes b_1 \otimes b_0 \mapsto \cdots.$$

Thus, for each $k \geq 1$, we have the following crystal isomorphism,

$$\Psi_k : \mathcal{B}(\lambda) \to \mathcal{B}(\lambda_k) \otimes \mathcal{B}_\ell^{\otimes k} \text{ given by } u_\lambda \mapsto u_{\lambda_k} \otimes b_{k-1} \otimes \cdots \otimes b_1 \otimes b_0.$$

There also exists two infinite sequences,

$$\mathbf{w}_\lambda = (\lambda_k)_{k=0}^\infty = (\ldots, \lambda_{k+1}, \lambda_k, \ldots, \lambda_1, \lambda_0) \in (\bar{P}_\ell^+)^\infty,$$

$$\mathbf{p}_\lambda = (b_k)_{k=1}^\infty = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0 \in (\mathcal{B}_\ell)^\infty.$$

Since $\bar{P}_\ell^+$ and $\mathcal{B}_\ell$ contain only finitely many elements, there exists $N > 0$ such that $\lambda_N = \lambda_0$ and hence, $b_N = b_0 = b_\lambda$. Then, ε and φ are bijective, yielding the following equalities,

$$\begin{aligned}
&b_{\lambda_0} = \varphi^{-1}(\lambda_0) = \varphi^{-1}(\lambda_N) = b_{\lambda_N} \\
&\lambda_1 = \varepsilon(b_{\lambda_0}) = \varepsilon(b_{\lambda_N}) = \lambda_{N+1} \\
&b_{\lambda_1} = \varphi^{-1}(\lambda_1) = \varphi^{-1}(\lambda_{N+1}) = b_{\lambda_{N+1}} \\
&\quad\vdots \\
&\lambda_j = \varepsilon(b_{\lambda_{j-1}}) = \varepsilon(b_{\lambda_{N+j-1}}) = \lambda_{N+j} \\
&b_{\lambda_j} = \varphi^{-1}(\lambda_j) = \varphi^{-1}(\lambda_{N+j}) = b_{\lambda_{N+j}} \\
&\quad\vdots \\
&\lambda_{N-1} = \varepsilon(b_{\lambda_{N-2}}) = \varepsilon(b_{\lambda_{2N-2}}) = \lambda_{2N-1} \\
&b_{\lambda_{N-1}} = \varphi^{-1}(\lambda_{N-1}) = \varphi^{-1}(\lambda_{2N-1}) = b_{\lambda_{2N-1}}
\end{aligned}$$

Hence the sequences $\mathbf{w}_\lambda$ and $\mathbf{p}_\lambda$ are periodic with the same period $N > 0$.

Definition 3.3.1. The sequence $\mathbf{p}_\lambda = (b_k)_{k=0}^\infty = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0$ is called the *ground-state path* of weight λ.

Definition 3.3.2. A λ-*path* in $\mathcal{B}_\ell$ is a sequence $\mathbf{p} = (\mathbf{p}_k)_{k=0}^\infty = \cdots \mathbf{p}_{k+1} \otimes \mathbf{p}_k \otimes \cdots \otimes \mathbf{p}_1 \otimes \mathbf{p}_0$

with $\mathbf{p}_k \in \mathcal{B}_\ell$ such that $\mathbf{p}_k = b_k$ for all $k \gg 0$. We let $\mathcal{P}(\lambda)$ denote the set of all λ-paths in $\mathcal{B}_\ell$.

We can define a crystal structure on $\mathcal{P}(\lambda)$ as shown in the following theorem.

Theorem 3.3.3. *[2] Let $\mathbf{p} = (\mathbf{p}_k)_{k=0}^{\infty}$ be a λ-path in $\mathcal{B}_\ell$ and let $N > 0$ be the smallest positive integer such that $\mathbf{p}_k = b_k$ for all $k \geq N$. For each $i \in I$, define*

$$\begin{aligned}
&\overline{wt}\,\mathbf{p} = \lambda_N + \sum_{k=0}^{N-1} \overline{wt}\,\mathbf{p}_k,\\
&\tilde{e}_i\mathbf{p} = \cdots \otimes \mathbf{p}_{N+1} \otimes \tilde{e}_i(\mathbf{p}_N \otimes \cdots \otimes \mathbf{p}_0),\\
&\tilde{f}_i\mathbf{p} = \cdots \otimes \mathbf{p}_{N+1} \otimes \tilde{f}_i(\mathbf{p}_N \otimes \cdots \otimes \mathbf{p}_0),\\
&\varepsilon_i(\mathbf{p}) = \max\{\varepsilon_i(\mathbf{p}') - \varphi(b_N), 0\},\\
&\varphi_i(\mathbf{p}) = \varphi_i(\mathbf{p}') + \max\{\varphi_i(b_N) - \varepsilon_i(\mathbf{p}'), 0\},
\end{aligned}$$

where $\mathbf{p}' = \mathbf{p}_{N-1} \otimes \cdots \mathbf{p}_1 \otimes \mathbf{p}_0$ and $\overline{wt}$ denotes the classical weight. Then the maps $\overline{wt} : \mathcal{P}(\lambda) \to \bar{P}$, $\tilde{e}_i, \tilde{f}_i : \mathcal{P}(\lambda) \to \mathcal{P}(\lambda) \sqcup \{0\}$, $\varepsilon_i, \varphi_i : \mathcal{P}(\lambda) \to \mathbb{Z}$ define a $U_q'(\mathfrak{g})$-crystal structure on $\mathcal{P}(\lambda)$.

Since $\mathcal{P}(\lambda)$ has a $U_q'(\mathfrak{g})$-crystal structure, we can now define the *path realization* of the $U_q'(\mathfrak{g})$-crystal $\mathcal{B}(\lambda)$ associated with the highest weight $U_q'(\mathfrak{g})$-module, $L(\lambda)$.

Theorem 3.3.4. *[9] There exists an isomorphism of $U_q'(\mathfrak{g})$-crystals,*

$$\Psi : \mathcal{B}(\lambda) \to \mathcal{P}(\lambda) \text{ given by } u_\lambda \mapsto \mathbf{p}_\lambda.$$

Thus, $\mathcal{B}(\lambda)$ and $\mathcal{P}(\lambda)$ are isomorphic as $U_q'(\mathfrak{g})$-crystals.

3.4 Demazure Modules and Demazure Crystals

In this section, we define a Demzaure module and its corresponding Demazure crystal, developed by Kashiwara. We then give the main theorem, which states that under certain conditions, the path realizations of Demazure crystals have tensor product-like structures. We will use this theorem in Chapter 5.

Recall that an affine quantum algebra, $U_q(\mathfrak{g})$, has a triangular decomposition, $U_q(\mathfrak{g}) = U_q^+(\mathfrak{g}) \oplus U_q^0(\mathfrak{g}) \oplus U_q^-(\mathfrak{g})$. Also, let $V^q(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module in $\mathcal{O}^q_{\text{int}}$ with highest weight λ and highest weight vector u_λ.

Let W denote the Weyl group for an affine Kac-Moody algebra, $\mathfrak{g}$, generated by simple reflections, $\{r_i \mid i \in I\}$. For any $w \in W$, it is known that the weight space, $V^q(\lambda)_{w\lambda}$, is one-dimensional, with basis vector $u_{w\lambda}$ [6]. This vector is called the *extremal vector*.

Definition 3.4.1. For $w \in W$ and weight space $V^q(\lambda)_{w\lambda} = \mathbb{C}u_{w\lambda}$, the *Demazure module* associated with w is $V_w(\lambda) = U_q^+(\mathfrak{g})u_{w\lambda}$.

The Demazure modules are finite subspaces of $V^q(\lambda)$ and satisfy the following properties:

- $V^q(\lambda) = \bigcup_{w \in W} V_w(\lambda)$,
- For $w, w' \in W$ with $w \preceq w'$ (the Bruhat order), $V_w(\lambda) \subset V_{w'}(\lambda)$.

Now, let $\mathcal{B}(\lambda)$ and $\mathcal{L}(\lambda)$ be the crystal and the crystal lattice for $V^q(\lambda)$, respectively. Kashiwara showed in 1993 [15] that for each $w \in W$, there exists a subset $B_w(\lambda) \subseteq \mathcal{B}(\lambda)$ such that

$$\frac{V_w(\lambda) \cap L(\lambda)}{V_w(\lambda) \cap qL(\lambda)} = \bigoplus_{b \in B_w(\lambda)} \mathbb{Q}b.$$

Definition 3.4.2. For $w \in W$, the subset $B_w(\lambda)$ of $\mathcal{B}(\lambda)$ defined above is the crystal for the Demazure module, $V_w(\lambda)$. It is called the *Demazure crystal.*

The Demazure crystal has the following recursive property [15]:

$$\text{If } w \prec r_i w, \text{ then } B_{r_i w}(\lambda) = \bigcup_{n \geq 0} \tilde{f}_i^n B_w(\lambda) \setminus \{0\}.$$

Using this recursive property, we begin to establish the necessary notation used in the main theorem below.

Suppose $\lambda(c) = \ell \geq 1$ and $\mathcal{B}_\ell$ is a perfect crystal of level ℓ for $U_q(\mathfrak{g})$. By Theorem 3.3.4, the crystal $\mathcal{B}(\lambda)$ is isomorphic to the set of paths, $\mathcal{P}(\lambda) = \mathcal{P}(\lambda, \mathcal{B}_\ell)$. Under this isomorphism, we identify the highest weight vector $u_\lambda \in \mathcal{B}(\lambda)$ with the ground-state path $\mathbf{p}_\lambda = \cdots \otimes b_3 \otimes b_2 \otimes b_1$.

We can define the path realizations for the Demazure crystals, $B_w(\lambda)$, as shown in [16]. First, set $d, \kappa \in \mathbb{Z}_{>0}$. For a sequence of integers, $\{i_a^{(j)} \mid j \geq 1,\ 1 \leq a \leq d\} \subset \{0, 1, \ldots, n\}$, define the subsets $\{B_a^{(j)} \mid j \geq 1,\ 0 \leq a \leq d\}$ as follows,

$$B_0^{(j)} = \{b_j\}, \qquad B_a^{(j)} = \bigcup_{k \geq 0} \tilde{f}_{i_a^{(j)}}^k B_{a-1}^{(j)} \setminus \{0\}.$$

Using these subsets, we define $B_a^{(j+1,j)}$ for $j \geq 1$, $1 \leq a \leq d$ as,

$$B_0^{(j+1,j)} = B_0^{(j+1)} \otimes B_d^{(j)}, \qquad B_a^{(j+1,j)} = \bigcup_{k \geq 0} \tilde{f}_{i_a^{(j+1)}}^k B_{a-1}^{(j+1,j)} \setminus \{0\}.$$

We continue in this way until we define,

$$\begin{aligned} B_0^{(j+\kappa-1,\ldots,j)} &= B_0^{(j+\kappa-1)} \otimes B_d^{(j+\kappa-2,\ldots,j)} \\ B_a^{(j+\kappa-1,\ldots,j)} &= \bigcup_{k \geq 0} \tilde{f}_{i_a^{(j+\kappa-1)}}^k B_{a-1}^{(j+\kappa-1,\ldots,j)} \setminus \{0\}. \end{aligned}$$

Also, define a sequence of Weyl group elements, $\{w^{(k)} \mid w^{(k)} \in W\}$ as,

$$w^{(0)} = 1, \qquad w^{(k)} = r_{i_a^{(j)}} w^{(k-1)}, \quad (k > 0).$$

Here j and a are determined uniquely from k by the relation $k = (j-1)d + a$ for $j \geq 1$, $1 \leq a \leq d$.

Lastly, for $k \geq 0$, define the subsets, $\mathcal{P}^{(k)}(\lambda, \mathcal{B}_\ell)$ of $\mathcal{P}(\lambda, \mathcal{B}_\ell)$ as,

$$\mathcal{P}^{(0)}(\lambda, \mathcal{B}_\ell) = \{\mathbf{p}_\lambda\},$$
$$\mathcal{P}^{(k)}(\lambda, \mathcal{B}_\ell) = \begin{cases} \cdots \otimes B_0^{(j+2)} \otimes B_0^{(j+1)} \otimes B_a^{(j,\ldots,1)} & \text{if } j < \kappa \\ \cdots \otimes B_0^{(j+2)} \otimes B_0^{(j+1)} \otimes B_a^{(j,\ldots,j-\kappa+1)} \otimes B^{\otimes(j-\kappa)} & \text{if } j \geq \kappa. \end{cases}$$

The following theorem by Kuniba, Misra, Okado, and Uchiyama, gives a tensor product-like structure to path realizations of Demazure crystals, $B_{w^{(k)}}(\lambda)$.

Theorem 3.4.3. *[16] Let $\lambda \in \bar{P}^+$ with $\lambda(c) = \ell$ and $\mathcal{B}_\ell$ be a perfect crystal of level ℓ for the quantum affine algebra $U_q(\mathfrak{g})$. For fixed $d, \kappa \in \mathbb{Z}_{>0}$, suppose we have a sequence of integers $\{i_a^{(j)} \mid j \geq 1, 1 \leq a \leq d\} \subset \{0, 1, \ldots, n\}$ satisfying the following conditions:*

(1) for any $j \geq 1$, $B_d^{(j+\kappa-1,\ldots,j)} = B_d^{(j+\kappa-1,\ldots,j+1)} \otimes \mathcal{B}_\ell$,

(2) for any $j \geq 1$ and $1 \leq a \leq d$, $\langle \lambda_j, h_{i_a^{(j)}} \rangle \leq \varepsilon_{i_a^{(j)}}(b)$, $b \in B_{a-1}^{(j)}$, and

(3) the sequence of elements $\{w^{(k)}\}_{k \geq 0}$ is an increasing sequence of Weyl group elements with respect to the Bruhat order.

Then, we have $B_{w^{(k)}}(\lambda) \cong \mathcal{P}^{(k)}(\lambda, \mathcal{B}_\ell)$.

The positive integer κ in Theorem 3.4.3 is called the *mixing index* and is dependent on the choice of perfect crystal $\mathcal{B}_\ell$ [22]. There is a conjecture that for any affine quantum algebra $U_q(\mathfrak{g})$, the mixing index $\kappa \leq 2$. It has been shown that for $\lambda = \ell\Lambda$, where $\Lambda \in \bar{P}^+$, $\Lambda(c) = 1$, and the perfect crystal $\mathcal{B}_\ell$, there exists a sequence of Weyl group elements $\{w^{(k)}\}$ which satisfy the conditions in Theorem 3.4.3 with $\kappa = 1$ for any classical quantum affine algebra [17], $U_q(D_4^{(3)})$ [21], and $U_q(G_2^{(1)})$ [4].

We use the following proposition to check the validity of condition (3) in Theorem 3.4.3.

Proposition 3.4.4. [16] For $w \in W$, if $\langle w\mu, h_j \rangle > 0$ for some $\mu \in \bar{P}^+$, then $w \prec r_j w$.

Finally, for $b \in B$, let $\tilde{f}_i^{\max}(b)$ denote $\tilde{f}_i^{\varphi_i(b)}(b)$. Then, for $j \geq 1$ and $1 \leq a \leq d$, set $b_0^{(j)} = b_j$ and $b_a^{(j)} = \tilde{f}_{i_a^{(j)}}^{\max}(b_{a-1}^{(j)})$.

We will define the quantum affine algebra, $U_q(D_4^{(3)})$, the perfect crystal associated with $U_q(D_4^{(3)})$, and its path realization in the next chapter. Then, we will use Theorem 3.4.3 to show path realizations of Demazure crystals for $U_q(D_4^{(3)})$ have tensor product-like structure for one dominant integral weight and give evidence of this for two other dominant integral weights.

Chapter 4

The Quantum Affine Algebra $U_q(D_4^{(3)})$

In this chapter, we define the affine Kac-Moody algebra $D_4^{(3)}$ and the quantum affine algebra $U_q(D_4^{(3)})$. We then give a description of the $U_q(D_4^{(3)})$-perfect crystal, $\mathcal{B}_\ell$ of level $\ell > 0$ as shown in [13], and the corresponding path realization.

4.1 Defining the Quantum Affine Algebra $U_q(D_4^{(3)})$

We begin by constructing the affine Kac-Moody algebra, $D_4^{(3)}$. First, fix $I = \{0, 1, 2\}$.

Consider the affine Kac-Moody algebra, $\mathfrak{g} = D_4^{(3)}$, associated with the generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix},$$

obtained from the Dynkin diagram,

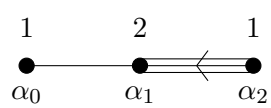

Figure 4.1: Dynkin Diagram for $D_4^{(3)}$

The numerical labels on the Dynkin diagram are the coordinates of the standard null root, $\delta = (d_0, d_1, d_2)^T$, such that $A\delta = 0$ and d_0, d_1, d_2 are relatively prime integers.

Let $\{\alpha_0, \alpha_1, \alpha_2\}$, $\{h_0, h_1, h_2\}$, and $\{\Lambda_0, \Lambda_1, \Lambda_2\}$ denote the set of simple roots, simple coroots, and fundamental weights, respectively.

The center of $\mathfrak{g} = D_4^{(3)}$ is one-dimensional, spanned by the canonical central element, $c = h_0 + 2h_1 + 3h_2 \in \mathfrak{h}$.

Let $d \in \mathfrak{h}$ such that

$$\alpha_0(d) = 1,\ \alpha_1(d) = 0,\ \alpha_2(d) = 0.$$

Then $\{h_0, h_1, h_2, d\}$ form a basis for the Cartan subalgebra, $\mathfrak{h}$. Therefore, the affine Kac-Moody algebra $D_4^{(3)}$ is generated by $\{e_i, f_i \mid i = 0, 1, 2\} \cup \mathfrak{h}$, satisfying the following relations:

(1) $[h, h'] = 0$ for $h, h' \in \mathfrak{h}$,

(2) $[e_i, f_j] = \delta_{ij} h_i$,

(3) $[h, e_i] = \alpha_i(h) e_i$ for $h \in \mathfrak{h}$,

(4) $[h, f_i] = -\alpha_i(h) f_i$ for $h \in \mathfrak{h}$,

(5) $(\text{ad } e_i)^{1-a_{ij}} e_j = 0$ for $i \neq j$,

(6) $(\text{ad } f_i)^{1-a_{ij}} f_j = 0$ for $i \neq j$.

Let $\bar{P} = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\Lambda_2$ be the set of classical weights for $\mathfrak{g} = D_4^{(3)}$, and let $\bar{P}^+ = \{\lambda \in \bar{P} \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for } i = 0, 1, 2\}$ be the set of classical dominant integral weights. Recall, the level for each classical dominant integral weight, $\lambda \in \bar{P}^+$, is $\ell = \lambda(c)$, where c is the canonical central element.

Example 4.1.1. For $\mathfrak{g} = D_4^{(3)}$, the level of each fundamental weight is

$$\Lambda_0(c) = 1, \qquad \Lambda_1(c) = 2, \qquad , \Lambda_2(c) = 3.$$

Let W be the Weyl group for $\mathfrak{g} = D_4^{(3)}$ generated by the simple reflections, $\{r_0, r_1, r_2\}$, where $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$, $i = 0, 1, 2$.

Recall that the affine generalized Cartan matrix associated with $\mathfrak{g} = D_4^{(3)}$ is symmetrizable, with symmetrizing matrix $S = \text{diag}\{s_1, s_2, s_3\} = \text{diag}\{1, 1, 3\}$.

Then, the quantum affine algebra $U_q(D_4^{(3)})$ is generated by the elements $\{e_i, f_i, q^{\pm h} \mid i = 0, 1, 2, h \in \mathfrak{h}\}$, satisfying the following relations:

(1) $q^0 = 1$; $q^h q^{h'} = q^{h+h'}$ for $h, h' \in \mathfrak{h}$,

(2) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ for $h \in \mathfrak{h}$, $i = 0, 1, 2$,

(3) $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ for $h \in \mathfrak{h}$, $i = 0, 1, 2$,

(4) $e_i f_j - f_j e_i = \delta_{ij} \dfrac{q^{s_i h_i} - q^{-s_i h_i}}{q^{s_i} - q^{-s_i}}$ for $i, j = 0, 1, 2$,

(5) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q^{s_i}} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$,

(6) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q^{s_i}} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$.

The representation theory for $\mathfrak{g} = D_4^{(3)}$ and $U_q(D_4^{(3)})$ remains the same as in the previous chapters and requires no modification. We now only consider the case when $\mathfrak{g} = D_4^{(3)}$ and $U_q(\mathfrak{g}) = U_q(D_4^{(3)})$.

4.2 $U_q(D_4^{(3)})$-Perfect Crystals and Path Realizations

We will use the quantum affine algebra $U_q(D_4^{(3)})$ to construct the perfect crystal $\mathcal{B}_\ell$ of level $\ell \geq 1$ as defined in [13].

Define the set,

$$\mathcal{B}_\ell = \left\{ b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in \mathbb{Z}_{\geq 0}^6 \;\middle|\; \begin{array}{l} x_3 \equiv \bar{x}_3 \pmod 2, \\ x_1 + x_2 + \frac{x_3 + \bar{x}_3}{2} + \bar{x}_2 + \bar{x}_1 \leq \ell \end{array} \right\}.$$

For $b = (x_1, x_2, x_3 \bar{x}_3, \bar{x}_2, \bar{x}_1) \in \mathcal{B}_\ell$, we denote

$$s(b) = x_1 + x_2 + \frac{x_3 + \bar{x}_3}{2} + \bar{x}_2 + \bar{x}_1, \qquad t(b) = x_2 + \frac{x_3 + \bar{x}_3}{2},$$

$$z_1 = \bar{x}_1 - x_1, \qquad z_2 = \bar{x}_2 - \bar{x}_3, \qquad z_3 = x_3 - x_2, \qquad z_4 = \frac{\bar{x}_3 - x_3}{2},$$

and

$$\mathcal{A} = (0, z_1, z_1 + z_2, z_1 + z_2 + 3z_4, z_1 + z_2 + z_3 + 3z_4, 2z_1 + z_2 + z_3 + 3z_4).$$

The Kashiwara operators, $\tilde{e}_i$ and $\tilde{f}_i$, for $i = 0, 1, 2$ are given as follows for $b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in \mathcal{B}_\ell$. We denote $(a)_+ = \max\{a, 0\}$.

$$\tilde{e}_1(b) = \begin{cases} (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2 + 1, \bar{x}_1 - 1) & \text{if } z_2 \geq (-z_3)_+, \\ (x_1, x_2, x_3 + 1, \bar{x}_3 - 1, \bar{x}_2, \bar{x}_1) & \text{if } z_2 < 0 \leq z_3, \\ (x_1 + 1, x_2 - 1, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \text{if } (z_2)_+ < -z_3, \end{cases}$$

$$\tilde{f}_1(b) = \begin{cases} (x_1 - 1, x_2 + 1, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \text{if } (z_2)_+ \leq -z_3, \\ (x_1, x_2, x_3 - 1, \bar{x}_3 + 1, \bar{x}_2, \bar{x}_1) & \text{if } z_2 \leq 0 < z_3, \\ (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2 - 1, \bar{x}_1 + 1) & \text{if } z_2 > (-z_3)_+, \end{cases}$$

$$\tilde{e}_2(b) = \begin{cases} (x_1, x_2, x_3, \bar{x}_3 + 2, \bar{x}_2 - 1, \bar{x}_1) & \text{if } z_4 \geq 0, \\ (x_1, x_2 + 1, x_3 - 2, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \text{if } z_4 < 0, \end{cases}$$

$$\tilde{f}_2(b) = \begin{cases} (x_1, x_2 - 1, x_3 + 2, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \text{if } z_4 \leq 0, \\ (x_1, x_2, x_3, \bar{x}_3 - 2, \bar{x}_2 + 1, \bar{x}_1) & \text{if } z_4 > 0, \end{cases}$$

$$\tilde{e}_0(b) = \begin{cases} (x_1 - 1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \text{if } (\mathcal{E}_1), \\ (x_1, x_2, x_3 - 1, \bar{x}_3 - 1, \bar{x}_2, \bar{x}_1 + 1) & \text{if } (\mathcal{E}_2), \\ (x_1, x_2, x_3 - 2, \bar{x}_3, \bar{x}_2 + 1, \bar{x}_1) & \text{if } (\mathcal{E}_3), \\ (x_1, x_2 - 1, x_3, \bar{x}_3 + 2, \bar{x}_2, \bar{x}_1) & \text{if } (\mathcal{E}_4), \\ (x_1 - 1, x_2, x_3 + 1, \bar{x}_3 + 1, \bar{x}_2, \bar{x}_1) & \text{if } (\mathcal{E}_5), \\ (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1 + 1) & \text{if } (\mathcal{E}_6), \end{cases}$$

$$\tilde{f}_0(b) = \begin{cases} (x_1 + 1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \text{if } (\mathcal{F}_1), \\ (x_1, x_2, x_3 + 1, \bar{x}_3 + 1, \bar{x}_2, \bar{x}_1 - 1) & \text{if } (\mathcal{F}_2), \\ (x_1, x_2, x_3 + 2, \bar{x}_3, \bar{x}_2 - 1, \bar{x}_1) & \text{if } (\mathcal{F}_3), \\ (x_1, x_2 + 1, x_3, \bar{x}_3 - 2, \bar{x}_2, \bar{x}_1) & \text{if } (\mathcal{F}_4), \\ (x_1 + 1, x_2, x_3 - 1, \bar{x}_3 - 1, \bar{x}_2, \bar{x}_1) & \text{if } (\mathcal{F}_5), \\ (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1 - 1) & \text{if } (\mathcal{F}_6), \end{cases}$$

where the conditions $(\mathcal{F}_1)-(\mathcal{F}_6)$ are given below. The conditions for $(\mathcal{E}_1)-(\mathcal{E}_6)$ are obtained from $(\mathcal{F}_1)-(\mathcal{F}_6)$ by replacing the inequality $>$ with $\geq$ and the inequality $\leq$ with $<$.

$$
\begin{aligned}
&(\mathcal{F}_1)\ z_1+z_2+z_3+3z_4\leq 0,\ z_1+z_3+3z_4\leq 0,\ z_1+z_2\leq 0,\ z_1\leq 0,\\
&(\mathcal{F}_2)\ z_1+z_2+z_3+3z_4\leq 0,\ z_2+3z_4\leq 0,\ z_2\leq 0,\ z_1>0,\\
&(\mathcal{F}_3)\ z_1+z_3+3z_4\leq 0,\ z_3+3z_4\leq 0,\ z_4\leq 0,\ z_2>0,\ z_1+z_2>0,\\
&(\mathcal{F}_4)\ z_1+z_2+3z_4>0,\ z_2+3z_4>0,\ z_4>0,\ z_3\leq 0,\ z_1+z_3\leq 0,\\
&(\mathcal{F}_5)\ z_1+z_2+z_3+3z_4>0,\ z_3+3z_4>0,\ z_3>0,\ z_1\leq 0,\\
&(\mathcal{F}_6)\ z_1+z_2+z_3+3z_4>0,\ z_1+z_3+3z_4>0,\ z_1+z_3>0,\ z_1>0.
\end{aligned}
$$

We then define the maps, ε_i and φ_i, $i=0,1,2$ for $b=(x_1,x_2,x_3,\bar{x}_3,\bar{x}_2,\bar{x}_1)\in\mathcal{B}_\ell$ as follows:

$$
\begin{aligned}
\varepsilon_1(b)&=\bar{x}_1+(\bar{x}_3-\bar{x}_2+(x_2-x_3)_+)_+,\\
\varepsilon_2(b)&=\bar{x}_2+\frac{1}{2}(x_3-\bar{x}_3)_+,\\
\varepsilon_0(b)&=\ell-s(b)+\max A-(2z_1+z_2+z_3+3z_4),\\
\varphi_1(b)&=x_1+(x_3-x_2+(\bar{x}_2-\bar{x}_3)_+)_+,\\
\varphi_2(b)&=x_2+\frac{1}{2}(\bar{x}_3-x_3)_+,\\
\varphi_0(b)&=\ell-s(b)+\max A.
\end{aligned}
$$

We note that for $b\in\mathcal{B}_\ell$, if $\tilde{e}_i(b)$ or $\tilde{f}_i(b)$ does not belong to $\mathcal{B}_\ell$, ie. if x_j or $\bar{x}_j$ becomes negative or if $s(b)>\ell$, we understand it to be 0.

Then, as shown in [13], for the quantum affine algebra $U_q(D_4^{(3)})$, the set $\mathcal{B}_\ell$ with the maps $\tilde{e}_i,\tilde{f}_i,\varepsilon_i,\varphi_i$, $i=0,1,2$, is a perfect crystal of level ℓ. Also, the minimal elements are given by [13],

$$(\mathcal{B})_{\min}=\{(\alpha,\beta,\beta,\beta,\beta,\alpha)\mid \alpha,\beta\in\mathbb{Z}_{\geq 0},\ 2\alpha+3\beta\leq\ell\}.$$

Example 4.2.1. For $\ell=1$, we have $\mathcal{B}_1=\{b_0,b_1,b_2,b_3,b_4,b_5,b_6,b_7\}$, where

$$
\begin{aligned}
b_0&=(0,0,0,0,0,0) & b_4&=(0,0,1,1,0,0)\\
b_1&=(1,0,0,0,0,0) & b_5&=(0,0,0,2,0,0)\\
b_2&=(0,1,0,0,0,0) & b_6&=(0,0,0,0,1,0)\\
b_3&=(0,0,2,0,0,0) & b_7&=(0,0,0,0,0,1)
\end{aligned}
$$

The perfect crystal graph for $\mathcal{B}_1$ is,

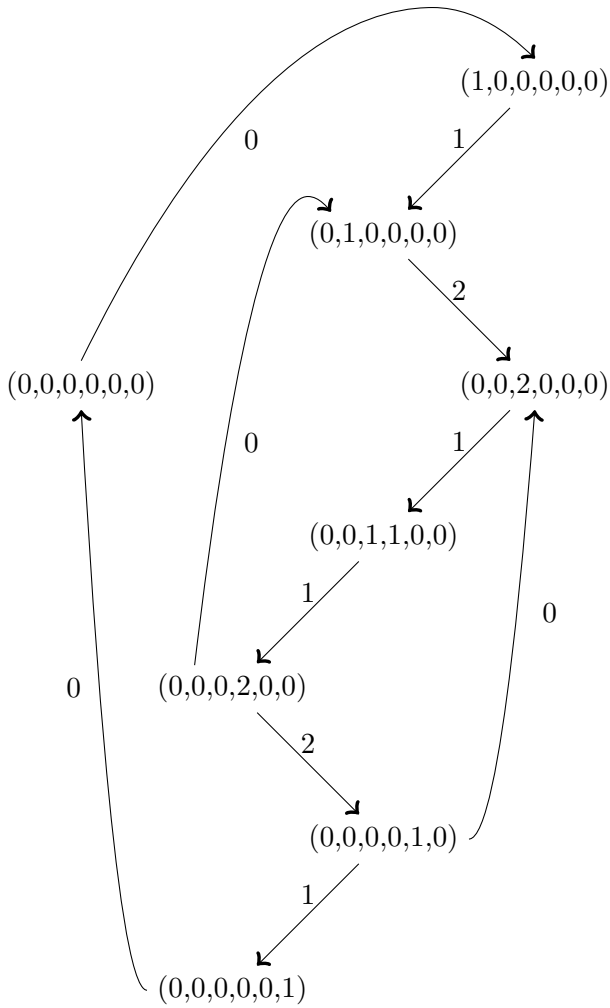

Figure 4.2: Perfect Crystal Graph for $\mathcal{B}_1$

Given a $U_q(D_4^{(3)})$-perfect crystal, $\mathcal{B}_\ell$, we can constuct a path realization of $\mathcal{B}_\ell$.

Let $V(\lambda)$ be the irreducible highest weight $U_q'(D_4^{(3)})$-module with highest weight λ and highest weight vector u_λ. Let $\mathcal{B}(\lambda)$ be the crystal graph of $V(\lambda)$ and $\mathcal{B}_\ell$ be the associated perfect crystal of level ℓ.

Recall, the ground state path of weight λ is

$$\mathbf{p}_\lambda = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0,$$

where

$$\lambda_0 = \lambda, \qquad \lambda_{k+1} = \varepsilon(b_{\lambda_k});$$

$$b_0 = b_\lambda, \qquad b_{k+1} = b_{\lambda_{k+1}}.$$

Example 4.2.2. For the $U_q'(D_4^{(3)}$ perfect crystal of level one given in Example 4.2.1, the ground state path is,

$$\mathbf{p}_{\Lambda_0} = \cdots \otimes b_0 \otimes b_0 \otimes b_0,$$

where $b_0 = (0,0,0,0,0,0) \in \mathcal{B}_1$.

By considering all λ-paths, $\mathcal{P}(\lambda)$, we use Theorem 3.3.3 to define a crystal structure on $\mathcal{P}(\lambda)$. Then, by Theorem 3.3.4, $\mathcal{B}(\lambda) \cong \mathcal{P}(\lambda)$ as $U_q'(D_4^{(3)})$-crystals.

Example 4.2.3. The irreducible highest weight $U_q'(D_4^{(3)})$-module of highest weight λ such that the perfect crystal has level one is $V(\Lambda_0)$. The (partial) path realization is given below. See Example 4.2.1 for notation.

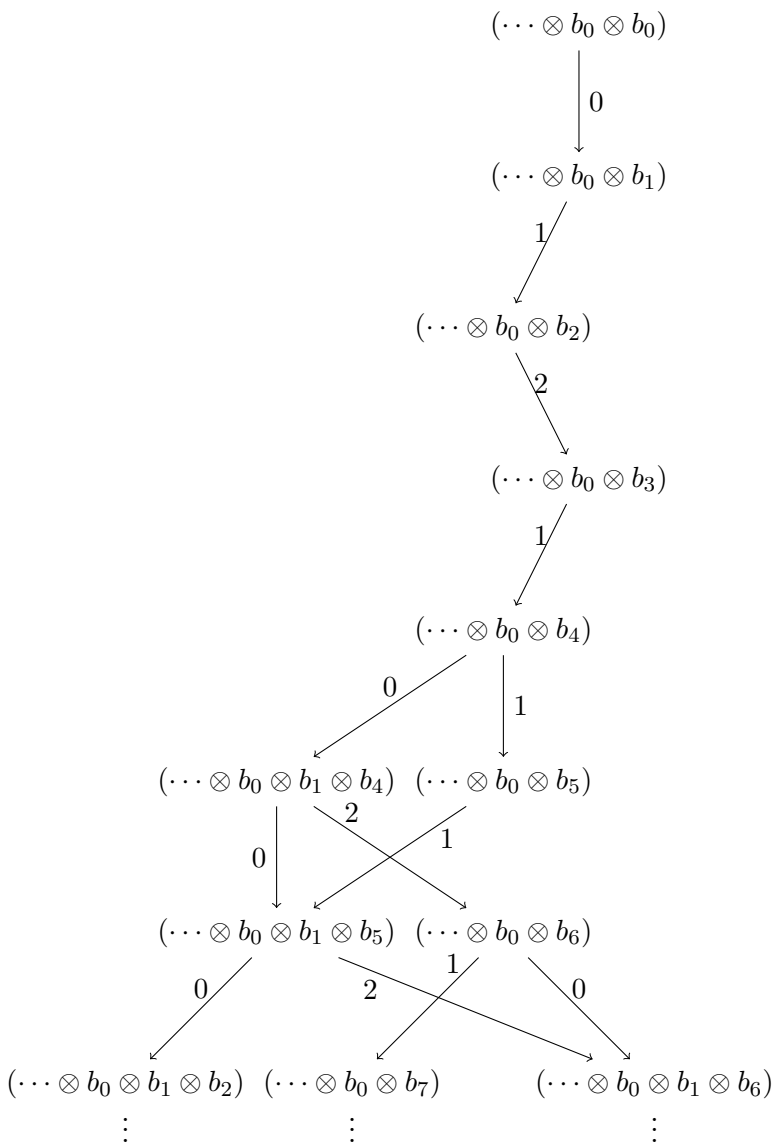

Figure 4.3: (Partial) Path Realization of $V(\Lambda_0)$, a $U_q'(D_4^{(3)})$-module.

Chapter 5

$U_q(D_4^{(3)})$-Demazure Crystals

In this chapter, we consider $U_q(D_4^{(3)})$-Demazure crystals for $\lambda = \ell\Lambda_2$, $\lambda = \ell\Lambda_1$, and $\lambda = \ell(\Lambda_0 + \Lambda_1)$. We will use the perfect crystal graphs and path realizations to prove Theorem 3.4.3 when $\lambda = \ell\Lambda_2$, and give a conjecture with some evidence that Theorem 3.4.3 holds when $\lambda = \ell\Lambda_1$ and $\lambda = \ell(\Lambda_0 + \Lambda_1)$.

5.1 Case 1: $\lambda = \ell\Lambda_2$

We begin by considering $\lambda = \ell\Lambda_2$ and the irreducible highest weight $U_q'(D_4^{(3)})$-module with highest weight $\ell\Lambda_2$. Note that $(\ell\Lambda_2)(c) = 3\ell$, so we will use the associated perfect crystal $\mathcal{B} = \mathcal{B}_{3\ell}$. The $\ell\Lambda_2$-minimal element in $\mathcal{B}$ is $\bar{b} = (0, \ell, \ell, \ell, \ell, 0)$. Also, $\lambda_j = \lambda = \ell\Lambda_2$ for $j \geq 1$, and hence $b_j = \bar{b}$. Thus, the ground-state path is $\mathbf{p}_\lambda = \cdots \otimes \bar{b} \otimes \bar{b} \otimes \bar{b}$.

Set $d = 6$ and define the sequence $\{i_a^{(j)} \mid j \geq 1, 1 \leq a \leq 6\} \subset \{0, 1, 2\}$ as follows,

$$i_1^{(j)} = i_3^{(j)} = 2,\ i_2^{(j)} = i_4^{(j)} = i_6^{(j)} = 1,\ i_5^{(j)} = 0.$$

By the action of $\tilde{f}_i$ on $\mathcal{B}$, we have

$$\begin{aligned}
b_0^{(j)} &= (0, \ell, \ell, \ell, \ell, 0), \\
b_1^{(j)} &= \tilde{f}_2^{\max}(b_0^{(j)}) = (0, 0, 3\ell, \ell, \ell, 0), \\
b_2^{(j)} &= \tilde{f}_1^{\max}(b_1^{(j)}) = (0, 0, 0, 4\ell, \ell, 0), \\
b_3^{(j)} &= \tilde{f}_2^{\max}(b_2^{(j)}) = (0, 0, 0, 0, 3\ell, 0), \\
b_4^{(j)} &= \tilde{f}_1^{\max}(b_3^{(j)}) = (0, 0, 0, 0, 0, 3\ell), \\
b_5^{(j)} &= \tilde{f}_0^{\max}(b_4^{(j)}) = (3\ell, 0, 0, 0, 0, 0), \\
b_6^{(j)} &= \tilde{f}_1^{\max}(b_5^{(j)}) = (0, 3\ell, 0, 0, 0, 0).
\end{aligned}$$

Using the notation from Theorem 3.4.3, yields the following Lemma.

Lemma 5.1.1. *Define conditions P and Q_n, $1 \leq n \leq 6$ for $b \in \mathcal{B}$ as follows,*

$$\begin{cases}
(P) : z_3 \geq 0, z_3 + 3z_4 \geq (-2z_2)_+, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) < 3\ell; \\
(Q_1) : z_3 < 0, z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 \geq 0, t(b) \leq 2\ell, s(b) \leq 3\ell; \\
(Q_2) : z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, z_1 + 2z_2 + z_3 \geq 0, s(b) \leq 3\ell; \\
(Q_3) : z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_2 < 0, z_2 + z_3 \geq 0, s(b) \leq 3\ell \\
(Q_4) : z_2 \geq 0, z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) \leq 3\ell; \\
(Q_5) : x_1 > 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 \geq 0, \\
\qquad t(b) < 2\ell, s(b) \leq 3\ell; \\
(Q_6) : x_1 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_2 + 3z_4 < 0, z_2 + 3z_4 > 0, z_2 + z_3 \geq 0, t(b) < 2\ell, s(b) \leq 3\ell.
\end{cases}$$

Then, the subsets $\{B_a^{(j)} \mid j \geq 1, 0 \leq a \leq 6\}$ *of* $\mathcal{B}$ *are given by,*

$$
\begin{aligned}
B_0^{(j)} &= \{(0,\ell,\ell,\ell,\ell,0)\} \\
B_1^{(j)} &= B_0^{(j)} \cup \{(0,x_2,x_3,\ell,\ell,0) \mid z_3 > 0, s(b) = 3\ell\} \\
B_2^{(j)} &= B_1^{(j)} \cup \{(0,x_2,x_3,\bar{x}_3,\ell,0) \mid z_2 < 0, z_3 \geq 0, s(b) = 3\ell\} \\
B_3^{(j)} &= B_2^{(j)} \cup \{(0,x_2,x_3,\bar{x}_3,\bar{x}_2,0) \mid z_3 \geq 0, z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) = 3\ell\} \\
B_4^{(j)} &= B_3^{(j)} \cup \{(0,x_2,x_3,\bar{x}_3,\bar{x}_2,\bar{x}_1) \mid \bar{x}_1 > 0, z_3 \geq 0, z_3 + 3z_4 \geq (-2z_2)_+, t(b) < 2\ell, s(b) = 3\ell\} \\
B_5^{(j)} &= B_4^{(j)} \cup C \cup D_1 \cup D_2 \cup \cdots \cup D_6 \\
B_6^{(j)} &= \mathcal{B},
\end{aligned}
$$

where $C = \{(0,x_2,x_3,\bar{x}_3,\bar{x}_2,\bar{x}_1) \mid (P) \text{ holds}\}$ *and for* $1 \leq n \leq 6$, $D_n = \{(x_1,x_2,x_3,\bar{x}_3,\bar{x}_2,\bar{x}_1) \mid (Q_n) \text{ holds}\}$.

Proof. By definition, $B_0^{(j)} = \{b_j\}$, where $b_j = (0,\ell,\ell,\ell,\ell,0)$ when $\lambda = \ell\Lambda_2$. Then, $B_a^{(j)} = \bigcup_{k\geq 0} \tilde{f}_{i_a^{(j)}}^k B_{a-1}^{(j)} \setminus \{0\}$. To obtain $B_1^{(j)}$, we apply $\tilde{f}_2$ repeatedly to $(0,\ell,\ell,\ell,\ell,0)$. Since $z_4 = 0$, applying $\tilde{f}_2$ repeatedly yields new elements of the form $\{(0,x_2,x_3,\ell,\ell,0) \mid z_3 > 0, s(b) = 3\ell\}$. Thus, $B_1^{(j)} = B_0^{(j)} \cup \{(0,x_2,x_3,\ell,\ell,0) \mid z_3 > 0, s(b) = 3\ell\}$.

To obtain $B_2^{(j)}$, we apply $\tilde{f}_1$ repeatedly to elements in $B_1^{(j)}$. Since $z_2 = 0$ and $z_3 \geq 0$ for all elements in $B_1^{(j)}$, applying $\tilde{f}_1$ repeatedly yields new elements of the form $\{(0,x_2,x_3,\bar{x}_3,\ell,0) \mid z_2 < 0, z_3 \geq 0, s(b) = 3\ell\}$. Thus, $B_2^{(j)} = B_1^{(j)} \cup \{(0,x_2,x_3,\bar{x}_3,\ell,0) \mid z_2 < 0, z_3 \geq 0, s(b) = 3\ell\}$.

Next, to obtain $B_3^{(j)}$, we apply $\tilde{f}_2$ repeatedly to elements in $B_2^{(j)}$. Since $B_1^{(j)}$ contain elements obtained by applying $\tilde{f}_2$ repeatedly, we only need to examine the action of $\tilde{f}_2$ to elements in $\{(0,x_2,x_3,\bar{x}_3,\ell,0) \mid z_2 < 0, z_3 \geq 0, s(b) = 3\ell\}$. If $z_4 \leq 0$, then applying $\tilde{f}_2$ repeatedly yields elements of the form $\{(0,x_2,x_3,\bar{x}_3,\ell,0) \mid z_2 < 0, z_3 > 0, z_4 < 0, s(b) = 3\ell\}$. Each of these elements is already contained in $B_2^{(j)}$, so no new elements are obtained. If $z_4 > 0$, then applying $\tilde{f}_2$ repeatedly yields new elements of the form $\{(0,x_2,x_3,\bar{x}_3,\bar{x}_2,0) \mid z_3 \geq 0, z_4 \geq 0, z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) = 3\ell\}$. These elements are not contained in $B_2^{(j)}$ since $t(b) < 2\ell$ here, whereas $t(b) = 2\ell$ for all elements in $B_2^{(j)}$.

Then, if $z_4 = 0$ in the set above, we again repeatedly apply $\tilde{f}_2$ to yield new elements of the form $\{(0,x_2,x_3,\bar{x}_3,\bar{x}_2,0) \mid z_3 \geq 0, z_4 < 0, z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) = 3\ell\}$. Again, these elements are not contained in $B_2^{(j)}$ by the same reasoning above, and they are also not contained in the set $\{(0,x_2,x_3,\bar{x}_3,\bar{x}_2,0) \mid z_3 \geq 0, z_4 \geq 0, z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) = 3\ell\}$, since $z_4 < 0$

now. We cannot apply $\tilde{f}_2$ anymore and obtain new elements. Thus, combining the two sets above gives $B_3^{(j)} = B_2^{(j)} \cup \{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_3 \geq 0, z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) = 3\ell\}$.

To obtain $B_4^{(j)}$, we apply $\tilde{f}_1$ repeatedly to elements in $B_3^{(j)}$. Since $B_2^{(j)}$ contain elements obtained by applying $\tilde{f}_1$ repeatedly to $B_1^{(j)}$, we only need to examine the action of $\tilde{f}_1$ to elements in $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_3 \geq 0, z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) = 3\ell\}$. If $z_2 \leq 0$ and $z_3 = 0$, then we obtain no new elements since $x_1 = 0$ and we cannot subtract from x_1. If $z_2 \leq 0$ and $z_3 > 0$, then applying $\tilde{f}_1$ repeatedly yields elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) = 3\ell\}$. Each of these elements is alreay contained in $B_3^{(j)}$, so no new elements are obtained. If $z_2 > 0$, applying $\tilde{f}_1$ repeatedly yields new elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 \geq 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) = 3\ell\}$. These elements are not contained in $B_3^{(j)}$ since $\bar{x}_1 > 0$.

Then, if $z_2 = 0$ and $z_3 > 0$ in the set above, we again repeatedly apply $\tilde{f}_1$ to yield new elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 < 0, z_3 \geq 0, z_3 + 3z_4 > 0, 2z_2 + z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) = 3\ell\}$. These elements are not contained in $B_3^{(j)}$ since $\bar{x}_1 > 0$; also, they are not contained in the set above since $z_2 < 0$. We cannot apply $\tilde{f}_1$ anymore since $x_1 = 0$. Thus, combining the two sets above yields $B_4^{(j)} = B_3^{(j)} \cup \{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_3 \geq 0, z_3 + 3z_4 \geq (-2z_2)_+, t(b) < 2\ell, s(b) = 3\ell\}$.

To obtain $B_5^{(j)}$, we apply $\tilde{f}_0$ repeatedly to elements in $B_4^{(j)}$.

(C) Consider elements from $B_4^{(j)}$ of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_3 \geq 0, z_3 + 3z_4 \geq (-2z_2)_+, z_1 + z_2 + z_3 + 3z_4 > 0, t(b) < 2\ell, s(b) = 3\ell\}$. We have $z_1 > 0$ and $z_1 + z_3 > 0$. Then, if $z_2 \geq 0$, $z_3 + 3z_4 \geq 0$, implying $z_1 + z_3 + 3z_4 > 0$. On the other hand, if $z_2 < 0$, then $z_1 + z_2 + z_3 + 3z_4 > 0$ implies $z_1 + z_3 + 3z_4 > 0$. So, these elements satisfy condition ($\mathcal{F}_6$); by repeatedly applying $\tilde{f}_0$ to elements of this form, we obtain new elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_3 \geq 0, z_3 + 3z_4 \geq (-2z_2)_+, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) < 3\ell\}$. These elements are not contained in $B_4^{(j)}$ since $s(b) \neq 3\ell$. This set is precisely C.

(D_1) Consider elements from $B_4^{(j)}$ of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_3 > 0, z_4 > 0, z_1 + z_2 + 3z_4 > 0, 2z_2 + z_3 + 3z_4 \geq 0, t(b) \leq 2\ell, s(b) = 3\ell\}$. This set is contained in $B_4^{(j)}$ since if $z_2 \geq 0$, then $z_3 > 0$ and $z_4 > 0$ implies $z_3 + 3z_4 \geq (-2z_2)_+$. Conversely, if $z_2 < 0$, then $2z_2 + z_3 + 3z_4 \geq 0$ implies $z_3 + 3z_4 \geq (-2z_2) = (-2z_2)_+$. Also, if $t(b) = 2\ell$, then $s(b) = 3\ell$ implies $\bar{x}_2 + \bar{x}_1 = \ell$, so $\bar{x}_2 < \ell$. Then $2z_2 + z_3 + 3z_4 \geq 0$ imples $\bar{x}_2 \geq \ell$. This is a contradiction, so $t(b) < 2\ell$. Then, since $z_1 > 0$, the conditions for ($\mathcal{F}_6$) are satisfied; by applying $\tilde{f}_0$ repeatedly, we obtain new elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_3 > 0, z_4 > 0, z_1 + z_2 + 3z_4 \geq 0, 2z_2 + z_3 + 3z_4 \geq 0, t(b) \leq 2\ell, s(b) < 3\ell\}$. Then, if $z_1 + z_2 + 3z_4 > 0$, the conditions for ($\mathcal{F}_5$) are satisfied. Applying $\tilde{f}_0$ repeatedly yields elements of the form

$\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_3 \geq 0, z_4 > 0, z_1 + z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 \geq 0, t(b) \leq 2\ell, s(b) < 3\ell\}$. Finally, if $z_3 = 0$, then the conditions for $(\mathcal{F}_4)$ are satisfied. Applying $\tilde{f}_0$ repeatedly yields elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_3 < 0, z_4 \geq 0, z_1+z_2+3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 \geq 0, t(b) \leq 2\ell, s(b) < 3\ell\}$. We also consider elements from $B_4^{(j)}$ of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_3 > 0, z_4 > 0, z_1 + z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 \geq 0, t(b) \leq 2\ell, s(b) = 3\ell\}$. By the same arguments as above, these elements are part of $B_4^{(j)}$. The conditions for $(\mathcal{F}_5)$ are satisfied, so applying $\tilde{f}_0$ repeatedly yields elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_3 \geq 0, z_4 > 0, z_1 + z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 \geq 0, t(b) \leq 2\ell, s(b) = 3\ell\}$. Then, if $z_3 = 0$, the conditions for $(\mathcal{F}_4)$ are satisfied, yielding new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_3 < 0, z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 \geq 0, t(b) \leq 2\ell, s(b) = 3\ell\}$. Combining this set with the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_3 < 0, z_4 \geq 0, z_1 + z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 \geq 0, t(b) \leq 2\ell, s(b) < 3\ell\}$, we obtain the set D_1. These elements are not contained in $B_4^{(j)}$ since $z_3 < 0$.

(D_2) Consider elements from $B_4^{(j)}$ of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 \geq 0, z_4 \leq 0, z_3 + 3z_4 > 0, z_1 + z_2 > 0, z_1 + 2z_2 + z_3 > 0, s(b) = 3\ell\}$. This set is contained in $B_4^{(j)}$ since $z_4 \leq 0$ and $z_3 + 3z_4 > 0$ implies $z_3 > 0$. Also, $z_3 + 3z_4 > 0$ and $z_1 + z_2 > 0$ imply that $2t(b) = 2x_2 + x_3 + \bar{x}_3 < 4\bar{x}_1 + 4\bar{x}_2$. Then, $s(b) = 3\ell$ implies that $2t(b) < 4(\bar{x}_1 + \bar{x}_2) = 12\ell - 4t(b)$. So, $6t(b) < 12\ell$ implying $t(b) < 2\ell$. Then since $z_1 > 0$, the conditions for $(\mathcal{F}_6)$ are satisfied. By repeatedly applying $\tilde{f}_0$ to elements of this form, we obtain the set $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 \geq 0, z_4 \leq 0, z_3 + 3z_4 > 0, z_1 + z_2 \geq 0, z_1 + 2z_2 + z_3 \geq 0, s(b) < 3\ell\}$. Since $z_1 = 0$, the conditions for $(\mathcal{F}_5)$ are now satisfied. Applying $\tilde{f}_0$ repeatedly yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 \geq 0, z_4 \leq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 \geq 0, z_1 + 2z_2 + z_3 \geq 0, s(b) < 3\ell\}$. If $z_3 + 3z_4 = 0$, $z_2 > 0$, and $z_1 + z_2 > 0$, then the conditions for $(\mathcal{F}_3)$ are satisfied. So, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, z_1 + 2z_2 + z_3 \geq 0, s(b) < 3\ell\}$. We also consider elements from $B_4^{(j)}$ of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 \geq 0, z_4 \leq 0, z_3 + 3z_4 > 0, z_1 + z_2 \geq 0, z_1 + 2z_2 + z_3 \geq 0, s(b) = 3\ell\}$. These elements are elements of $B_4^{(j)}$ by the same argument above. Then the conditions for $(\mathcal{F}_5)$ are satisfied, so applying $\tilde{f}_0$ repeatedly yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 \geq 0, z_4 \leq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 \geq 0, z_1 + 2z_2 + z_3 \geq 0, s(b) = 3\ell\}$. Finally, if $z_3 + 3z_4 = 0$, $z_2 > 0$, and $z_1 + z_2 > 0$, the conditions for $(\mathcal{F}_3)$ are satisfied. Repeatedly applying $\tilde{f}_0$ yields elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, z_1 + 2z_2 + z_3 \geq 0, s(b) = 3\ell\}$. Combining this set with the set, $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, z_1 + 2z_2 + z_3 \geq 0, s(b) < 3\ell\}$, we obtain the set D_2. These elements are not contained in $B_4^{(j)}$ since $z_3 + 3z_4 < 0$.

(D_3) Consider elements from $B_4^{(j)}$ of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 > 0, z_3 = 0, z_4 > 0, z_2 + z_3 \geq 0, s(b) = 3\ell\}$. We see that $z_2 > 0$, $z_3 = 0$, and $z_4 > 0$ imply that

$t(b) < 2\bar{x}_2$. Then, $s(b) = 3\ell$ implies that $t(b) < 2\ell$; so these elements are from $B_4^{(j)}$. Then, since $z_1 > 0$, the conditions of $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ yields $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 > 0, z_3 = 0, z_4 > 0, z_2 + z_3 \geq 0, s(b) < 3\ell\}$. Then, $z_1 = 0$, so the conditions of $(\mathcal{F}_4)$ are now satisfied. Applying $\tilde{f}_0$ yields $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 > 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 \geq 0, s(b) < 3\ell\}$. If $z_4 = 0$, then the conditions for $(\mathcal{F}_3)$ are satisfied, so we obtain elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_2 + z_3 \geq 0, s(b) < 3\ell\}$. Finally, if $z_2 = 0$, the conditions for $(\mathcal{F}_1)$ are satisfied. Applying $\tilde{f}_0$ to these such elements and adding the new elements to this set gives $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_2 + z_3 \geq 0, z_1 + z_2 < 0, s(b) \leq 3\ell\}$. This set is precisely D_3, and these elements are not contained in $B_4^{(j)}$ since $z_3 + 3z_4 < 0$.

(D_4) Consider elements from $B_4^{(j)}$ of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 \geq 0, z_3 > 0, z_1 + z_2 + z_3 + 3z_4 > 0, z_3 + 3z_4 > 0, t(b) < 2\ell, s(b) = 3\ell\}$. This set is clearly a subset of $B_4^{(j)}$ since $(-2z_2)_+ = 0$. Since $z_1 > 0$, the conditions for $(\mathcal{F}_6)$ are satisfied. So, applying $\tilde{f}_0$ repeatedly gives elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 \geq 0, z_3 > 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_3 + 3z_4 > 0, t(b) < 2\ell, s(b) < 3\ell\}$. Then, if $z_1 + z_2 + z_3 + 3z_4 > 0$, the conditions for $(\mathcal{F}_5)$ are satisfied. We obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 \geq 0, z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) < 3\ell\}$. Finally, if $z_1 + z_2 + z_3 + 3z_4 = 0$ and $z_1 + z_2 \leq 0$, then the conditions for $(\mathcal{F}_1)$ hold. Repeatedly applying $\tilde{f}_0$ yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 \geq 0, z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) \leq 3\ell\}$. This set is precisely D_4, and these elements are not contained in $B_4^{(j)}$ since $x_1 > 0$.

(D_5) Consider elements from $B_4^{(j)}$ of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_3 > 0, z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 > 0, 2z_2 + z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) = 3\ell\}$. This set is clearly a subset of $B_4^{(j)}$ since if $z_2 \geq 0$, then $z_3 + 3z_4 \geq 0 = (-2z_2)_+$; or if $z_2 < 0$, then $2z_2 + z_3 + 3z_4 \geq 0$ implies $z_3 + 3z_4 \geq (-2z_2)_+$. Since $z_1 > 0$, the conditions for $(\mathcal{F}_6)$ are satisfied. Applying $\tilde{f}_0$ repeatedly yields elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_3 > 0, z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, 2z_2 + z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) < 3\ell\}$. Then, if $z_1 + z_2 + z_3 + 3z_4 > 0$, the conditions for $(\mathcal{F}_5)$ hold. We obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) < 3\ell\}$. We also consider elements from $B_4^{(j)}$ of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_3 > 0, z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 > 0, 2z_2 + z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) = 3\ell\}$. These elements are part of $B_4^{(j)}$ by the same arguments as above. Then, since $z_1 = 0$, the conditions for $(\mathcal{F}_5)$ are satisfied. Applying $\tilde{f}_0$ repeatedly yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) = 3\ell\}$. Combining this set with $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 \geq 0, t(b) < 2\ell, s(b) < 3\ell\}$, we obtain the set D_5. These elements are not contained in $B_4^{(j)}$ since $x_1 > 0$.

(D_6) Finally, consider elements from $B_4^{(j)}$ of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 < 0, z_3 > 0, z_4 > 0, z_1 + z_2 + 3z_4 > 0, z_2 + z_3 \geq 0, 2z_2 + z_3 + 3z_4 \geq 0, t(b) \leq 2\ell, s(b) = 3\ell\}$. Since $z_2 < 0$ and $2z_2 + z_3 + 3z_4 \geq 0$, we have $z_3 + 3z_4 \geq (-2z_2)_+$. Also, if $t(b) = 2\ell$, then $s(b) = 3\ell$ implies $\bar{x}_2 + \bar{x}_1 = \ell$, so $\bar{x}_2 < \ell$. Then, $2z_2 + z_3 + 3z_4 \geq 0$ implies $\bar{x}_2 \geq \ell$. This is a contradiction, so $t(b) < 2\ell$. Thus, these elements are contained in $B_4^{(j)}$. Since $z_1 > 0$, the conditions for ($\mathcal{F}_6$) are satisfied. Applying $\tilde{f}_0$ repeatedly yields elements of the form $\{(0, x_2, x_2, \bar{x}_3, \bar{x}_2, 0) \mid z_2 < 0, z_3 > 0, z_4 > 0, z_1 + z_2 + 3z_4 \geq 0, z_2 + 3z_4 \geq 0, z_2 + z_3 \geq 0, t(b) \leq 2\ell, s(b) < 3\ell\}$. Then since $z_1 = 0$, the conditions for ($\mathcal{F}_5$) are satisfied. So, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_3 \geq 0, z_4 > 0, z_1 + z_2 + 3z_4 \geq 0, z_2 + 3z_4 > 0, z_2 + z_3 \geq 0, z_1 + z_2 < 0, t(b) \leq 2\ell, s(b) < 3\ell\}$. If $z_3 = 0$ and $z_1 + z_2 + 3z_4 > 0$, then the conditions for ($\mathcal{F}_4$) hold. Repeatedly applying $\tilde{f}_0$ yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_3 < 0, z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, z_2 + 3z_4 > 0, z_2 + z_3 \geq 0, t(b) \leq 2\ell, s(b) < 3\ell\}$. Finally, if $z_1 + z_2 + 3z_4 = 0$, then $z_1 + z_2 \leq 0$ and the conditions for ($\mathcal{F}_1$) hold. We obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_2 + 3z_4 < 0, z_2 + 3z_4 > 0, z_2 + z_3 \geq 0, t(b) \leq 2\ell, s(b) \leq 3\ell\}$. This set is precisely D_6, and these elements are not contained in $B_4^{(j)}$ since $x_1 > 0$.

Therefore, by repeatedly applying $\tilde{f}_0$ to elements in $B_4^{(j)}$, we obtain $B_5^{(j)} = B_4^{(j)} \cup C \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6$.

To obtain $B_6^{(j)}$, we apply $\tilde{f}_1$ repeatedly to elements in $B_5^{(j)}$. We want to show that $B_6^{(j)} = \mathcal{B}$, so we list the elements in $\mathcal{B}$ that are not contained in $B_5^{(j)}$:

(L_1) $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_3 + 3z_4 < (-2z_2)_+, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$

(L_2) $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_3 < 0, z_3 + 3z_4 \geq (-2z_2)_+, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$

(L_3) $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 \geq 0, z_3 \geq 0, z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$

(L_4) $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 < 0, z_3 < 0, 2z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$

(L_5) $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_2 < 0, z_3 \geq 0, 2z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$

(L_6) $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_2 \geq 0, z_3 < 0, z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$

(M_1) $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_2 + z_3 < 0, z_1 + 2z_2 + z_3 < 0, s(b) \leq 3\ell\}$

(M_2) $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 \geq 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 < 0, z_1 + 2z_2 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$

(M_3) $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 < 0, z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$

(M_4) $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 < 0, z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$

(M_5) $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 < 0, z_3 \geq 0, z_3+3z_4 \geq 0, z_1+z_2+z_3+3z_4 \geq 0, z_1+2z_2+z_3+3z_4 < 0, s(b) \leq 3\ell\}$

(N) $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1 > 0, \bar{x}_1 > 0, s(b) \leq 3\ell\}$

We will show that these elements are obtained by repeatedly applying $\tilde{f}_1$ to elements in $B_5^{(j)}$.

(L_1) Consider elements from $B_5^{(j)}$ of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 > 0, z_3 \geq 0, z_4 < 0, z_3+3z_4 < 0, z_1+z_2+z_3+3z_4 \geq 0, s(b) \leq 3\ell\}$. Since $z_1+z_2+z_3+3z_4 \geq 0$ and $z_3+3z_4 < 0$, we have $z_1 + z_2 \geq 0$. Thus, these elements are contained in D_2 in $B_5^{(j)}$. Since $z_3 \geq 0$ and $z_2 > 0$, we have $z_2 > (-z_3)_+$. Applying the third condition for $\tilde{f}_1$ repeatedly, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 \geq 0, z_3 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Then, if $z_2 = 0$ and $z_3 > 0$, we apply the second condition for $\tilde{f}_1$ repeatedly. This yields elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 < 0, z_3 \geq 0, 2z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. We stop if $x_1 = 0$. For $x_1 \neq 0$ and $z_3 = 0$, we apply the first condition for $\tilde{f}_1$ repeatedly. We obtain elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 < 0, z_3 < 0, 2z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Combining this set with the previous one where $x_1 = 0$ yields $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 < 0, 2z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. We also consider elements in $B_5^{(j)}$ of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 > 0, z_2 + z_3 > 0, z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. If $z_4 < 0$, then this set is contained in D_2 in $B_5^{(j)}$ by the same argument above. If $z_4 \geq 0$, then $z_3 + 3z_4 < 0$ implies $z_3 < 0$. Then, we have $z_1 + z_2 + z_3 + 3z_4 \geq 0$, implying $z_1 + z_2 + 3z_4 \geq 0$ and $z_1 + 2z_2 + z_3 + 3z_4 \geq 0$. All we need to show is $t(b) \leq 2\ell$ in order to show this set is a subset of $B_5^{(j)}$. So, assume $t(b) > 2\ell$. Since $s(b) \leq 3\ell$, we have $x_1 + t(b) + \bar{x}_2 \leq 3\ell$. If $t(b) > 2\ell$, then $x_1 + \bar{x}_2 \leq \ell$, implying $\bar{x}_2 \leq \ell$. Now, $z_3 + 3z_4 < 0$ implies $3\bar{x}_3 < x_3 + 2x_2$. Then $2t(b) = 2x_2 + x_3 + \bar{x}_3 < 4\bar{x}_3$. Also, $z_2 > 0$ means $\bar{x}_2 > \bar{x}_3$. So, $2t(b) < 4\bar{x}_3 < 4\bar{x}_2 \leq 4\ell$. This means $t(b) < 2\ell$, which is a contradiction. Therefore, $t(b) \leq 2\ell$ and so, these elements are contained in D_1 in $B_5^{(j)}$. Since $z_2 > 0$ and $z_2 + z_3 > 0$, we have $z_2 > (-z_3)_+$. Applying the third condition for $\tilde{f}_1$ repeatedly, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 \geq 0, z_2 + z_3 \geq 0, z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. If $x_1 = 0$, we stop. For $x_1 \neq 0$ and $z_2 + z_3 = 0$, we have $(z_2)_+ \geq -z_3$. So repeatedly applying the first condition for $\tilde{f}_1$ yields $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 \geq 0, z_2 + z_3 < 0, z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Combining this set with the previous one yields

$\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 \geq 0, z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Then, we combine these two final sets to obtain L_1.

(L_2) Consider elements from $B_5^{(j)}$ of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_3 < 0, z_2+z_3 > 0, z_3+3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Since $z_3 < 0$ and $z_3 + 3z_4 \geq 0$, we have $z_4 > 0$. Also, since $z_1 + z_2 + z_3 + 3z_4 \geq 0$, subtracting z_3 yields $z_1 + z_2 + 3z_4 \geq 0$. Then, since $z_2 + z_3 > 0$, we have $z_2 > 0$ and so, $z_1 + 2z_2 + z_3 + 3z_4 \geq 0$. Thus, we only need to show that $t(b) \leq 2\ell$ in order for this set to be contained in $B_5^{(j)}$. Assume $t(b) > 2\ell$. Since $s(b) \leq 3\ell$, we have $t(b) + \bar{x}_2 \leq 3\ell$. This implies $\bar{x}_2 < \ell$. Then $z_3 + 3z_4 \geq 0$ implies $3\bar{x}_3 \geq x_3 + 2x_2$. So, $2t(b) = 2x_2 + x_3 + \bar{x}_3 \leq 4\bar{x}_3$. But, $z_2 > 0$ implies $\bar{x}_2 > \bar{x}_3$. This means $2t(b) \leq 4\bar{x}_3 < 4\bar{x}_2 < 4\ell$. Dividing by 2 yields $t(b) < 2\ell$, which contradicts our assumption. Thus, $t(b) \leq 2\ell$ and the set above is contained in $B_5^{(j)}$. Since $z_3 < 0$, $z_2 < 0$, and $z_2+z_3 > 0$, we have $z_2 > (-z_3)_+$. Applying the third condition for $\tilde{f}_1$ repeatedly, we obtain elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_3 < 0, z_2+z_3 \geq 0, z_3+3z_4 \geq 0, z_1+z_2+z_3+3z_4 \geq 0, s(b) \leq 3\ell\}$. We also consider elements in $B_5^{(j)}$ of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 > 0, z_3 < 0, z_2 + z_3 > 0, z_3 + 3z_4 \geq x_1, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Since $z_3 < 0$ and $z_3 + 3z_4 \geq x_1 > 0$, we have $z_4 > 0$. By the same argument as above, we see that these elements are contained in D_1 in $B_5^{(j)}$. Then $z_3 < 0$, $z_2 > 0$, and $z_2 + z_3 > 0$. This means $z_2 > (-z_3)_+$, so we apply the third condition for $\tilde{f}_1$ repeatedly. This yields elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 \geq 0, z_3 < 0, z_2 + z_3 \geq 0, z_3 + 3z_4 \geq x_1, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Then if $z_2 + z_3 = 0$, we repeatedly apply the first condition for $\tilde{f}_1$ to yield $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 \geq 0, z_3 < 0, z_2 + z_3 < 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Finally, consider elements in $B_5^{(j)}$ of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 > 0, z_3 > 0, z_3 + 3z_4 > x_1, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Since $z_3 + 3z_4 > x_1 > 0$, we have $z_3 + 3z_4 > 0$. Also, since $z_3 + 3z_4 > 0$ and $z_2 > 0$, we have $t(b) < 2\ell$ by a similar argument as above. Thus, these elements are contained in D_5 in $B_5^{(j)}$. Since $z_2 > 0$ and $z_3 > 0$, we have $z_2 > (-z_3)_+$. Applying the third condition for $\tilde{f}_1$ repeatedly, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 \geq 0, z_3 > 0, z_3 + 3z_4 > x_1, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Then, if $z_2 = 0$, we repeatedly apply the second condition for $\tilde{f}_1$. This yields elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 < 0, z_3 \geq 0, z_3 + 3z_4 > x_1, 2z_2 + z_3 + 3z_4 > x_1, z_1 + z_2 + z_3 + 3z_4 > 0, s(b) \leq 3\ell\}$. If $z_3 = 0$, we have $(z_2)_+ \leq -z_3$, so we repeatedly apply the first condition of $\tilde{f}_1$. This gives elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 < 0, z_3 < 0, z_3 + 3z_4 > 0, 2z_2 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 > 0, s(b) \leq 3\ell\}$. After combining these three sets, we obtain L_2.

(L_3) Consider elements from $B_5^{(j)}$ of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 > 0, z_3 \geq 0, z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Since $z_3 \geq 0$ and $z_3 + 3z_4 < 0$, we have $z_4 < 0$. Also,

$z_1 > 0$, so $z_1 + z_2 \geq 0$ and $z_1 + 2z_2 + z_3 \geq 0$. Thus, these elements are contained in D_2 in $B_5^{(j)}$. Since $z_2 > 0$ and $z_3 \geq 0$, we have $z_2 > (-z_3)_+$. Applying the third condition for $\tilde{f}_1$ repeatedly, we obtain elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 \geq 0, z_3 \geq 0, z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. This set is precisely L_3.

(L_4) Consider elements from $B_5^{(j)}$ of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 > 0, z_3 > 0, z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Since $z_3 > 0$ and $z_3 + 3z_4 < 0$, $z_4 < 0$. Thus, this set is a subset of D_2 in $B_5^{(j)}$. Since $z_2 > 0$ and $z_3 > 0$, we have $z_2 > (-z_3)_+$. Applying the third condition for $\tilde{f}_1$ repeatedly, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 \geq 0, z_3 > 0, z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Then, if $z_2 = 0$, we repeatedly apply the second condition for $\tilde{f}_1$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_1 \geq 0, z_2 < 0, z_3 \geq 0, 2z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Finally, if $z_3 = 0$, then we repeatedly apply the first condition for $\tilde{f}_1$. We obtain elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 < 0, z_3 < 0, 2z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. This set is precisely L_4.

(L_5) Consider elements from $B_5^{(j)}$ of the form $\{(0, x_2, x_2, \bar{x}_3, \bar{x}_2, 0) \mid z_2 > 0, z_3 > 0, z_3 + 3z_4 < 0, z_1 + z_2 + 2z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Since $z_3 > 0$ and $z_3 + 3z_4 < 0$, we have $z_4 < 0$. Therefore, this set is contained in D_2 in $B_5^{(j)}$. Since $z_2 > 0$ and $z_3 > 0$, we have $z_2 > (-z_3)_+$. Applying the third condition for $\tilde{f}_1$ repeatedly, we obtain elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 \geq 0, z_3 > 0, z_3 + 3z_4 < 0, z_1 + z_2 + 2z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Then, if $z_2 = 0$, we repeatedly apply the second condition for $\tilde{f}_1$. This yields the set $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 < 0, z_3 \geq 0, 2z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + 2z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Then, since $z_3 \geq 0$ and $z_1 + z_2 + 2z_3 + 3z_4 < 0$, we have $z_1 + z_2 + z_3 + 3z_4 < -z_3 \leq 0$. Thus, the set becomes $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_1 > 0, z_2 < 0, z_3 \geq 0, 2z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Now we consider elements from $B_5^{(j)}$ of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 = 0, z_3 > 0, 2z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + 2z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. By a similar argument as above, we see that this set is contained in D_2 in $B_5^{(j)}$. Then, $z_2 = 0$ and $z_3 > 0$, so we repeatedly apply the second condition for $\tilde{f}_1$. This yields elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 < 0, z_3 \geq 0, 2z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + 2z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Again, these conditions imply that $z_1 + z_2 + z_3 + 3z_4 < 0$, so the set becomes $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 < 0, z_3 \geq 0, 2z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Then, combining the two final sets above, we obtain L_5.

(L_6) Consider elements from $B_5^{(j)}$ of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 > 0, z_3 < 0, z_2 + z_3 > 0, z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Since $z_2 + z_3 > 0$ and $z_1 = 0$, $z_1 + z_2 + z_3 + 3z_4 < 0$ implies that $z_4 < 0$. Thus, these elements are contained in D_2 in $B_5^{(j)}$.

Since $z_2 > 0$, $z_3 < 0$, and $z_2 + z_3 > 0$, we have $z_2 > (-z_3)_+$. Applying the third condition for $\tilde{f}_1$ repeatedly, we obtain elements of the form $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_2 \geq 0, z_3 < 0, z_2 + z_3 \geq 0, z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. We also consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 > 0, z_3 \leq 0, z_4 < 0, z_2 + z_3 > 0, z_3 + 3z_4 < 0, z_1 + z_2 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. This set is clearly contained in D_3 in $B_5^{(j)}$. Then, $z_2 > 0$, $z_3 \leq 0$, and $z_2 + z_3 > 0$ imply $z_2 > (-z_3)_+$. Applying the third condition for $\tilde{f}_1$ repeatedly, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1 > 0, z_2 \geq 0, z_3 \leq 0, z_4 < 0, z_2 + z_3 \geq 0, z_3 + 3z_4 < 0, z_1 + z_2 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Then, if $z_2 + z_3 = 0$, we repeatedly apply the first condition for $\tilde{f}_1$. This yields the set $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_2 \geq 0, z_3 < 0, z_4 < 0, z_2 + z_3 < 0, z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Finally, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 \geq 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 > 0, z_1 + z_2 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. In order to show that this set is contained in D_6 in $B_5^{(j)}$, we only need to show $t(b) \leq 2\ell$. Assume $t(b) > 2\ell$. Since $s(b) \leq 3\ell$, we have $x_1 + t(b) + \bar{x}_2 \leq 3\ell$. This implies $x_1 + \bar{x}_2 < \ell$ and so, $\bar{x}_2 < \ell$. Then, $z_2 > 0$ implies $l > \bar{x}_2 > \bar{x}_3$. Also, $z_4 \geq 0$ implies $l > \bar{x}_3 \geq x_3$. Finally, $z_2 + z_3 > 0$ implies $\bar{x}_2 + x_3 > \bar{x}_3 + x_2$. Then, $2t(b) = 2x_2 + x_3 + \bar{x}_3 \leq 2x_2 + 2\bar{x}_3 < 2\bar{x}_2 + 2x_3 < 2\ell + 2\ell = 4\ell$. Thus, $t(b) < 2\ell$, which contradicts our assumption. So, $t(b) \leq 2\ell$, making the set above a subset of D_6 in $B_5^{(j)}$. Since $z_2 > 0$, $z_3 < 0$, and $z_2 + z_3 > 0$, we have $z_2 > (-z_3)_+$. Applying the third condition for $\tilde{f}_1$ repeatedly, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1 > 0, z_2 \geq 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 \geq 0, z_1 + z_2 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Then, if $z_2 + z_3 = 0$, we repeatedly apply the first condition for $\tilde{f}_1$. This yields the set $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_2 \geq 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Since $z_1 \geq 0$,, $z_2 \geq 0$, and $z_1 + z_2 + z_3 + 3z_4 < 0$, we have $z_3 + 3z_4 < 0$. So, we add this condition to the set above to yield $\{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_2 \geq 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 < 0, z_3 + 3z_4 < 0 z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Then, combining the three final sets above, we obtain L_6.

(M_1) Consider elements from $B_5^{(j)}$ of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 1, z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_2 + z_3 = 0, z_1 + 2z_2 + z_3 < 0, s(b) \leq 3\ell\}$. This set is clearly contained in D_3 in $B_5^{(j)}$. Since $z_2 \geq 0$ and $z_2 + z_3 = 0$, we have $z_3 \leq 0$. This imples $(z_2)_+ \leq -z_3$ and we repeatedly apply the first condition for $\tilde{f}_1$. This yields elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 \geq 0, z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_2 + z_3 < 0, z_1 + 2z_2 + z_3 < 0, s(b) \leq 3\ell\}$. This set is precisely M_1.

(M_2) Consider elements from $B_5^{(j)}$ of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 1, z_2 \geq 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 = 0, z_1 + 2z_2 + z_3 < 0, z_1 + z_2 + 3z_4 < 0, s(b) \leq 3\ell\}$. We see that this set is contained in D_6 in $B_5^{(j)}$ if $t(b) \leq 2\ell$. Assume $t(b) > 2\ell$. Then $s(b) \leq 3\ell$ implies $x_1 + t(b) + \bar{x}_2 \leq 3\ell$. So, $x_1 + \bar{x}_2 < \ell$, implying $\bar{x}_2 < \ell$. Since $z_2 > 0$, we have $l > \bar{x}_2 > \bar{x}_3$. Also, since $z_4 \geq 0$, we

have $\ell > \bar{x}_3 \geq x_3$. Finally, since $z_2 + z_3 = 0$, we have $\bar{x}_2 = \bar{x}_3 + x_2 - x_3 \geq x_3 + x_2 - x_3 = x_2$. So, $l > \bar{x}_2 \geq x_2$. Therefore, $2t(b) = 2x_2 + x_3 + \bar{x}_3 < 2\ell + \ell + \ell = 4\ell$. So, $t(b) < 2\ell$, which contradicts our assumption. Thus, $t(b) \leq 2\ell$ and the set above is contained in D_6 in $B_5^{(j)}$. Since $z_2 \geq 0$ and $z_2 + z_3 = 0$, we have $(z_2)_+ \leq -z_3$. Applying the first condition for $\tilde{f}_1$ repeatedly, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 \geq 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 < 0, z_1 + 2z_2 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. This set is precisely M_2.

(M_3) Consider elements from $B_5^{(j)}$ of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 = 0, z_3 > 0, z_3 + 3z_4 < 0, z_1 + z_2 + 2z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Since $z_3 > 0$ and $z_3 + 3z_4 < 0$, we have $z_4 < 0$. Also, $z_1 < 0$, so $z_1 + z_2 = z_1 < 0$. Finally, $z_2 + z_3 = z_3 > 0$. Thus, this set is contained in D_3 in $B_5^{(j)}$. Since $z_2 = 0$ and $z_3 > 0$, we repeatedly apply the second condition for $\tilde{f}_1$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 < 0, z_3 \geq 0, z_1 + z_2 + 2z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Since $z_3 \geq 0$ and $z_1 + z_2 + 2z_3 + 3z_4 < 0$, we have $z_1 + z_2 + z_3 + 3z_4 < -z_3 \leq 0$. So, our set becomes $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 < 0, z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$, which is precisely M_3.

(M_4) Conisder elements from $B_5^{(j)}$ of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 = 0, z_3 > 0, z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. This set is contained in D_3 in $B_5^{(j)}$ by the same reasoning as the set considered in M_3. Since $z_2 = 0$ and $z_3 > 0$, we repeatedly apply the second condition for $\tilde{f}_1$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 < 0, z_3 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Then, if $z_3 = 0$, we repeatedly apply the first condition for $\tilde{f}_1$. This yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 < 0, z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$, which is precisely M_4.

(M_5) Consider elements from $B_5^{(j)}$ of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 = 0, z_3 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 = -1, s(b) \leq 3\ell\}$. This set is contained in D_3 in $B_5^{(j)}$ by the same reasoning as the set considered in M_3. Since $z_2 = 0$ and $z_3 > 0$, we repeatedly apply the second condtion for $\tilde{f}_1$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 < 0, z_3 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Since $z_1 \geq 0$, $z_2 < 0$, and $z_1 + z_2 + z_3 + 3z_4 \geq 0$, we have $z_3 + 3z_4 \geq 0$. Thus, the set becomes $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_2 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$, which is precisely M_5.

(N) Conisder elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 > 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. This set is contained in D_5 in $B_5^{(j)}$ if $t(b) < 2\ell$. So, assume $t(b) \geq 2\ell$. Since $s(b) \leq 3\ell$, we have $x_1 + t(b) + \bar{x}_2 \leq 3\ell$. This implies $x_1 + \bar{x}_2 \leq \ell$, so $\bar{x}_2 \leq \ell$. Also, $z_3 + 3z_4 \geq 0$ implies $3\bar{x}_3 \geq x_3 + 2x_2$. Also, $z_2 > 0$ implies $\bar{x}_2 > \bar{x}_3$. Then, $2t(b) = 2x_2 + x_3 + \bar{x}_3 \leq 4\bar{x}_3 < 4\bar{x}_2 \leq 4\ell$. Thus, $t(b) < 2\ell$, which contradicts our

assumption. Therefore, $t(b) < 2\ell$ and the set above is contained in D_5 in $B_5^{(j)}$. Since $z_2 > 0$ and $z_3 \geq 0$, we have $z_2 > (-z_3)_+$. By repeatedly applying the third condition for $\tilde{f}_1$, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 \geq 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Next, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 > 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + 2z_3 + 3z_4 \leq 0, s(b) \leq 3\ell\}$. Since $z_3 > 0$ and $z_1 + z_2 + 2z_3 + 3z_4 \leq 0$,w e have $z_1 + z_2 + z_3 + 3z_4 \leq -z_3 < 0$. This set is contained in D_4 in $B_5^{(j)}$ provided $t(b) < 2\ell$. Since $z_2 > 0$, $z_3 + 3z_4 \geq 0$, and $s(b) \leq 3\ell$, we have $t(b) < 2\ell$ by the same argument as above. Therefore, this set is contained in D_4 in $B_5^{(j)}$. Since $z_2 > 0$ and $z_3 \geq 0$, we have $z_2 > (-z_3)_+$. By repeatedly applying the third condition for $\tilde{f}_1$, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 \geq 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. We now consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 > 0, z_3 \geq 0, z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, s(b) \leq 3\ell\}$. Since $z_3 \geq 0$ and $z_3 + 3z_4 < 0$, we have $z_4 < 0$. This implies the set is contained in D_2 in $B_5^{(j)}$. Since $z_2 > 0$ and $z_3 \geq 0$, we have $z_2 > (-z_3)_+$. Repeatedly applying the third condition for $\tilde{f}_1$ yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 \geq 0, z_3 \geq 0, z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, s(b) \leq 3\ell\}$. We finally consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 > 0, z_3 \geq 0, z_3 + 3z_4 < 0, z_1 + z_2 < 0, s(b) \leq 3\ell\}$. Since $z_3 \geq 0$ and $z_3 + 3z_4 < 0$, we have $z_4 < 0$. This implies the set is contained in D_3 in $B_5^{(j)}$. Since $z_2 > 0$ and $z_3 \geq 0$, the condition $z_2 > (-z_3)_+$ is satisfied. So, by repeatedly applying the third condition for $\tilde{f}_1$, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 \geq 0, z_3 \geq 0, z_3 + 3z_4 < 0, z_1 + z_2 < 0, s(b) \leq 3\ell\}$. Then, combining the four final sets above yields the set $S_1 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 \geq 0, z_3 \geq 0, s(b) \leq 3\ell\}$.

Now, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 > 0, z_3 > 0, z_3 + 3z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Since $z_2 > 0$, $z_3 + 3z_4 \geq 0$, and $s(b) \leq 3\ell$, we have $t(b) < 2\ell$ by a similar argument as above. This implies this set is contained in D_5 in $B_5^{(j)}$. Since $z_2 > 0$ and $z_3 > 0$, we repeatedly apply the third condition for $\tilde{f}_1$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 \geq 0, z_3 > 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Then, if $z_2 = 0$, we repeatedly apply the second condition for $\tilde{f}_1$. This yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Since $z_2 < 0$ and $z_1 + 2z_2 + z_3 + 3z_4 \geq 0$, we have $z_1 + z_2 + z_3 + 3z_4 \geq -z_2 > 0$. We also consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 > 0, z_3 > 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + 2z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Since $z_3 > 0$ and $z_1 + z_2 + 2z_3 + 3z_4 \leq 0$, we have $z_1 + z_2 + z_3 + 3z_4 \leq -z_3 < 0$. Then, this set is contained in D_4 in $B_5^{(j)}$ provided $t(b) < 2\ell$. Since $z_2 > 0$, $z_3 + 3z_4 \geq 0$, and $s(b) \leq 3\ell$, we have $t(b) < 2\ell$ by the same argument as above. Therefore, this set is contained in D_4 in $B_5^{(j)}$. Since $z_2 > 0$ and $z_3 > 0$, we repeatedly apply the third condition for $\tilde{f}_1$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid$

$x_1, \bar{x}_1 > 0, z_2 \geq 0, z_3 > 0, z_3+3z_4 \geq 0, z_1+z_2+2z_3+3z_4 \leq 0, s(b) \leq 3\ell\}$. Then, if $z_2 = 0$, we repeatedly apply the second condition for $\tilde{f}_1$. This yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + 2z_3 + 3z_4 \leq 0, s(b) \leq 3\ell\}$. Since $z_3 \geq 0$ and $z_1 + z_2 + 2z_3 + 3z_4 \leq 0$, we have $z_1 + z_2 + z_3 + 3z_4 \leq -z_3 \leq 0$. Finally, we consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_2 > 0, z_3 > 0, 3z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, s(b) \leq 3\ell\}$. Since $z_3 > 0$ and $3z_3 + 3z_4 < 0$, we have $z_4 < 0$. So, this set is contained in D_2 in $B_5^{(j)}$. Since $z_2 > 0$ and $z_3 > 0$, we repeatedly apply the third condition for $\tilde{f}_1$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 \geq 0, z_3 > 0, 3z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, s(b) \leq 3\ell\}$. Then, if $z_2 = 0$, we repeatedly apply the second condition for $\tilde{f}_1$. This yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 < 0, z_3 \geq 0, 3z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Since $z_3 \geq 0$ and $3z_3 + 3z_4 < 0$, we have $z_3 + 3z_4 < -2z_3 \leq 0$. So, the set becomes $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 < 0, z_3 \geq 0, z_3 + 3z_4 < 0, s(b) \leq 3\ell\}$. Then, combining the three final sets above yields the set $S_2 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 < 0, z_3 \geq 0, s(b) \leq 3\ell\}$.

We now consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_3 < 0, z_4 < 0, z_2 + z_3 > 0, z_1 + z_2 \geq 0, s(b) \leq 3\ell\}$. Since $z_3 < 0$ and $z_2 + z_3 > 0$, we have $z_2 > 0$. This implies that the set is contained in D_2 in $B_5^{(j)}$. Since $z_2 > 0$, $z_3 < 0$, and $z_2 + z_3 > 0$, we have $z_2 > (-z_3)_+$. Applying the third condition for $\tilde{f}_1$, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_3 < 0, z_4 < 0, z_2 + z_3 \geq 0, z_1 + z_2 \geq 0, s(b) \leq 3\ell\}$. Next, we consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_3 < 0, z_4 < 0, z_2 + z_3 > 0, z_1 + z_2 < 0, s(b) \leq 3\ell\}$. Again, since $z_3 < 0$ and $z_2 + z_3 > 0$, we have $z_2 > 0$. This implies that the set is contained in D_3 in $B_5^{(j)}$. Since $z_2 > 0$, $z_3 < 0$, and $z_2 + z_3 > 0$, we have $z_2 > (-z_3)_+$. Repeatedly applying the third condition for $\tilde{f}_1$, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_3 < 0, z_4 < 0, z_2 + z_3 \geq 0, z_1 + z_2 < 0, s(b) \leq 3\ell\}$. We now consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 > 0, z_1 + z_2 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Since $z_2 + z_3 > 0$ and $z_1 + z_2 + 3z_4 \geq 0$, adding the two inequalities yields $z_1 + 2z_2 + z_3 + 3z_4 \geq 0$. So, this set is contained in D_1 provided $t(b) \leq 2\ell$. Assume $t(b) > 2\ell$. Then $s(b) \leq 3\ell$ implies $x_1 + t(b) + \bar{x}_2 \leq 3\ell$. So, $x_1 + \bar{x}_2 < \ell$, implying $\bar{x}_2 < \ell$. Also, since $z_3 < 0$ and $z_2 + z_3 > 0$, we have $z_2 > 0$. This means $\ell > \bar{x}_2 > \bar{x}_3$. Then, $z_4 \geq 0$ implies $\bar{x}_3 \geq x_3$. Since $z_2 + z_3 > 0$, $\bar{x}_2 > \bar{x}_3 + x_2 - x_3 \geq x_3 + x_2 - x_3 = x_2$. So, $\ell > \bar{x}_2 > x_2$. Finally, $z_3 < 0$ implies $x_3 < x_2 < \ell$. Putting this all together gives $2t(b) = 2x_2 + x_3 + \bar{x}_3 < 2\ell + \ell + \ell = 4\ell$. So, $t(b) < 2\ell$, which contradicts our assumption. Therefore, $t(b) \leq 2\ell$ and the set is contained in D_1 in $B_5^{(j)}$. Since $z_2 > 0$, $z_3 < 0$ and $z_2 + z_3 > 0$, we have $z_2 > (-z_3)_+$. Repeatedly applying the third condition for $\tilde{f}_1$ yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 \geq 0, z_1 + z_2 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Finally, we consider elements of the form

$\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 > 0, z_1 + z_2 + 3z_4 < 0, s(b) \leq 3\ell\}$. Since $z_3 < 0$ and $z_2 + z_3 > 0$, we have $z_2 > 0$. So, $z_2 + 3z_4 > 0$. Also, since $z_2 > 0$, $z_3 < 0$, $z_4 \geq 0$, $z_2 + z_3 > 0$, and $s(b) \leq 3\ell$, we have $t(b) \leq 2\ell$ by the same argument as above. Thus, this set is contained in D_6 in $B_5^{(j)}$. Since $z_2 > 0$, $z_3 < 0$, and $z_2 + z_3 > 0$, we have $z_2 > (-z_3)_+$. Repeatedly applying the third condition for $\tilde{f}_1$ yields elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 \geq 0, z_1 + z_2 + 3z_4 < 0, s(b) \leq 3\ell\}$. Then, combining the four final sets above yields the set $S_3 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_3 < 0, z_2 + z_3 \geq 0, s(b) \leq 3\ell\}$.

Lastly, we consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 1, z_3 \leq 0, z_4 < 0, z_2 + z_3 > 0, z_1 + z_2 \geq 0, s(b) \leq 3\ell\}$. Since $z_3 \leq 0$ and $z_2 + z_3 > 0$, we have $z_2 > 0$. This implies the set above is contained in D_5 in $B_5^{(j)}$. Since $z_2 > 0$, $z_3 \leq 0$, and $z_2 + z_3 > 0$, we have $z_2 > (-z_3)_+$. By repeatedly applying the third condition for $\tilde{f}_1$, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1 > 1, \bar{x}_1 > 0, z_3 \leq 0, z_4 < 0, z_2 + z_3 \geq 0, z_1 + z_2 \geq 0, s(b) \leq 3\ell\}$. Then, if $z_2 + z_3 = 0$, we repeatedly apply the first condition for $\tilde{f}_1$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_3 < 0, z_4 < 0, z_2 + z_3 < 0, z_1 + z_2 > 0, s(b) \leq 3\ell\}$. Next, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 1, z_3 \leq 0, z_4 < 0, z_2 + z_3 > 0, z_1 + z_2 < 0, s(b) \leq 3\ell\}$. Since $z_3 \leq 0$ and $z_2 + z_3 > 0$, we have $z_2 > 0$. This implies the set is contained in D_3 in $B_5^{(j)}$. Since $z_2 > 0$, $z_3 \leq 0$, and $z_2 + z_3 > 0$, we have $z_2 > (-z_3)_+$. Repeatedly applying the third condition for $\tilde{f}_1$ yields elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1 > 1, \bar{x}_1 > 0, z_3 \leq 0, z_4 < 0, z_2 + z_3 \geq 0, z_1 + z_2 < 0, s(b) \leq 3\ell\}$. Then, if $z_2 + z_3 = 0$, we repeatedly apply the first condition for $\tilde{f}_1$. We obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_3 < 0, z_4 < 0, z_2 + z_3 < 0, z_1 + z_2 \leq 0, s(b) \leq 3\ell\}$. We now consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 1, z_3 < 0, z_4 \geq 0, z_2 + z_3 > 0, z_1 + z_2 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Since $z_2 + z_3 > 0$ and $z_1 + z_2 + 3z_4 \geq 0$, we add the two inequalities to obtain the condition $z_1 + 2z_2 + z_3 + 3z_4 > 0$. Then, since $z_3 < 0$ and $z_2 + z_3 > 0$, we have $z_2 > 0$. With $z_2 > 0$, $z_3 < 0$, $z_4 \geq 0$, $z_2 + z_3 > 0$, and $s(b) \leq 3\ell$, we can show that $t(b) \leq 2\ell$ by a similar argument as above. Thus, this set is contained in D_1 in $B_5^{(j)}$. Since $z_2 > 0$, $z_3 < 0$, and $z_2 + z_3 > 0$, we have $z_2 > (-z_3)_+$. By repeatedly applying the third condition for $\tilde{f}_1$, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1 > 1, \bar{x}_1 > 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 \geq 0, z_1 + z_2 + 3z_4 \geq 0, s(b) \leq 3\ell\}$. Then, if $z_2 + z_3 = 0$, we repeatedly apply the first condition for $\tilde{f}_1$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 < 0, z_1 + z_2 + 3z_4 > 0, s(b) \leq 3\ell\}$. Finally, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid x_1 > 1, z_3 < 0, z_4 \geq 0, z_2 + z_3 > 0, z_1 + z_2 + 3z_4 < 0, s(b) \leq 3\ell\}$. Since $z_3 < 0$ and $z_2 + z_3 > 0$, we have $z_2 > 0$. This implies $z_2 + 3z_4 > 0$. Also, since $z_2 > 0$, $z_3 < 0$, $z_4 \geq 0$, $z_2 + z_3 > 0$, and $s(b) \leq 3\ell$, we have $t(b) \leq 2\ell$ by a similar

argument as above. Thus, the set is contained in D_6 in $B_5^{(j)}$. Since $z_2 > 0$, $z_3 < 0$, and $z_2 + z_3 > 0$, we have $z_2 > (-z_3)_+$. By repeatedly applying the third condition for $\tilde{f}_1$, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1 > 1, \bar{x}_1 > 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 \geq 0, z_1 + z_2 + 3z_4 < 0, s(b) \leq 3\ell\}$. Then, if $z_2 + z_3 = 0$, we repeatedly apply the first condition for $\tilde{f}_1$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_3 < 0, z_4 \geq 0, z_2 + z_3 < 0, z_1 + z_2 + 3z_4 \leq 0, s(b) \leq 3\ell\}$ Then, combining the four final sets above yields the set $S_4 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_3 < 0, z_2 + z_3 < 0, s(b) \leq 3\ell\}$.

We see that by combining $S_1 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 \geq 0, z_3 \geq 0, s(b) \leq 3\ell\}$, $S_2 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_2 < 0, z_3 \geq 0, s(b) \leq 3\ell\}$,
$S_3 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_3 < 0, z_2 + z_3 \geq 0, s(b) \leq 3\ell\}$, and $S_4 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid x_1, \bar{x}_1 > 0, z_3 < 0, z_2 + z_3 < 0, s(b) \leq 3\ell\}$, we obtain the set N_1.

Therefore, the missing elements in $\mathcal{B}$ are obtained by repeatedly applying $\tilde{f}_1$ to elements in $B_5^{(j)}$. So, $B_6^{(j)} = \mathcal{B}$ and we have now constructed the sets $B_a^{(j)}$ for $0 \leq a \leq 6$. □

By direct calculations of the simple reflections r_i $(i = 0, 1, 2)$ on Λ_2, we obtain the following Lemma, which we will use along with Proposition 3.4.4 to prove Theorem 3.4.3 for $\lambda = \ell\Lambda_2$.

Lemma 5.1.2. *Let $k \in \mathbb{Z}_{>0}$ and $k = 6(j-1) + a$ for $1 \leq a \leq 6$. Then $w^{(k)}\Lambda_2 = \Lambda_2 - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2$, where*

$$m_0 = \begin{cases} 3j^2 + 3j & \text{if } a = 1, 2, 3, 4 \\ 3j^2 + 9j + 6 & \text{if } a = 5, 6 \end{cases}$$

$$m_1 = \begin{cases} 6j^2 + 3j & \text{if } a = 1 \\ 6j^2 + 9j + 3 & \text{if } a = 2, 3 \\ 6j^2 + 12j + 6 & \text{if } a = 4, 5 \\ 3j^2 + 15j + 9 & \text{if } a = 6 \end{cases}$$

$$m_2 = \begin{cases} 3j^2 + 3j + 1 & \text{if } a = 1, 2 \\ 3j^2 + 6j + 3 & \text{if } a = 3, 4, 5, 6 \end{cases}$$

The main result for $\lambda = \ell\Lambda_2$ is given below.

Theorem 5.1.3. *For $\lambda = \ell\Lambda_2$, $\ell \geq 1$ and the given perfect crystal $\mathcal{B} = \mathcal{B}_{3\ell}$ for the quantum affine algebra, $U_q(D_4^{(3)})$, with $d = 6$ and the sequence $\{i_a^{(j)}\}$ given above, conditions (1), (2), and (3) in Theorem 3.4.3 hold with $\kappa = 1$. Hence, we have path realizations of the corresponding Demazure crystals $B_{w^{(k)}}(\ell\Lambda_2)$ for $U_q(D_4^{(3)})$ with tensor product-like structures.*

Proof. It is sufficient to show that conditions (1), (2), and (3) hold in Theorem 3.4.3. We have already shown, by explicit construction of the subsets $B_a^{(j)}$ in Lemma 5.1.1 that $B_6^{(j)} = \mathcal{B}$, and hence, condition (1) holds for $\kappa = 1$.

In order to show condition (2), we consider $\langle \ell\Lambda_2, h_{i_a^{(j)}} \rangle$. Observe that $\langle \ell\Lambda_2, h_{i_a^{(j)}} \rangle = 0 \leq \varepsilon_{i_a^{(j)}}(b)$ for all $b \in B_{a-1}^{(j)}$, $a = 2, 4, 5, 6$. Also, $\langle \ell\Lambda_2, h_{i_a^{(j)}} \rangle = \ell$ for $a = 1, 3$. We must consider $\varepsilon_2(b)$ for all $b \in B_0^{(j)}$ and $B_2^{(j)}$. Note that for all $b \in B_0^{(j)}$ or $B_2^{(j)}$, we have $\bar{x}_2 = \ell$. Hence, $\varepsilon_2(b) = \bar{x}_2 + \frac{1}{2}(x_3 - \bar{x}_3)_+ = l + \frac{1}{2}(x_3 - \bar{x}_3)_+ \geq \ell$ for all $b \in B_0^{(j)}$ and $B_2^{(j)}$. Thus, $\langle \ell\Lambda_2, h_{i_a^{(j)}} \rangle \leq \varepsilon_{i_a^{(j)}}(b)$ for all $b \in B_{a-1}^{(j)}$, and condition (2) holds.

To prove condition (3), we use Lemma 5.1.2. For $k = 6(j-1) + a$, $j \geq 1$, $1 \leq a \leq 6$, we have

$$\langle w^{(k)}\Lambda_2, h_{i_{a+1}^{(j)}} \rangle = \begin{cases} 3j+2 & \text{if } a = 2 \\ 3j+3 & \text{if } a = 3, 5 \\ 3j+4 & \text{if } a = 6 \\ 6j+3 & \text{if } a = 1 \\ 6j+6 & \text{if } a = 4. \end{cases}$$

Hence, for positive j, $\langle w^{(k)}\Lambda_2, h_{i_{a+1}^{(j)}} \rangle$ is greater than zero. By Proposition 3.4.4, this implies $w^{(k+1)} = r_{i_{a+1}^{(j)}} w^{(k)} \succ w^{(k)}$. Thus, the sequence of Weyl group elements, $\{w^{(k)}\}_{k \geq 0}$, is increasing with respect to the Bruhat order, satisfying condition (3).

Since conditions (1), (2), and (3) hold in Theorem 3.4.3, we have $B_{w^{(k)}}(\ell\Lambda_2) \cong \mathcal{P}^{(k)}(\ell\Lambda_2, \mathcal{B})$. □

5.2 Case 2: $\lambda = \ell\Lambda_1$

Now, consider $\lambda = \ell\Lambda_1$ and the irreducible highest weight $U_q'(D_4^{(3)})$-module with highest weight $\ell\Lambda_1$. Note that $(\ell\Lambda_1)(c) = 2\ell$, so we will use the associated perfect crystal $\mathcal{B} = \mathcal{B}_{2\ell}$. The $\ell\Lambda_1$-minimal element in $\mathcal{B}$ is $\bar{b} = (\ell, 0, 0, 0, 0, \ell)$. Also, $\lambda_j = \lambda = \ell\Lambda_1$ for $j \geq 1$, and hence $b_j = \bar{b}$. Thus, the ground-state path is $\mathbf{p}_\lambda = \cdots \otimes \bar{b} \otimes \bar{b} \otimes \bar{b}$.

Set $d = 6$ and define the sequence $\{i_a^{(j)} \mid j \geq 1, 1 \leq a \leq 6\} \subset \{0, 1, 2\}$ as follows,

$$i_1^{(j)} = i_3^{(j)} = i_5^{(j)} = 1,\ i_2^{(j)} = i_6^{(j)} = 2,\ i_4^{(j)} = 0.$$

By the action of $\tilde{f}_i$ on $\mathcal{B}$, we have

$$\begin{aligned}
b_0^{(j)} &= (\ell, 0, 0, 0, 0, \ell), \\
b_1^{(j)} &= \tilde{f}_1^{\max}(b_0^{(j)}) = (0, \ell, 0, 0, 0, \ell), \\
b_2^{(j)} &= \tilde{f}_2^{\max}(b_1^{(j)}) = (0, 0, 2\ell, 0, 0, \ell), \\
b_3^{(j)} &= \tilde{f}_1^{\max}(b_2^{(j)}) = (0, 0, 0, 2\ell, 0, \ell), \\
b_4^{(j)} &= \tilde{f}_0^{\max}(b_3^{(j)}) = (\ell, \ell, 0, 0, 0, 0), \\
b_5^{(j)} &= \tilde{f}_1^{\max}(b_4^{(j)}) = (0, 2\ell, 0, 0, 0, 0), \\
b_6^{(j)} &= \tilde{f}_2^{\max}(b_5^{(j)}) = (0, 0, 4\ell, 0, 0, 0).
\end{aligned}$$

Using the notation from Theorem 3.4.3, yields the following Lemma.

Lemma 5.2.1. *Define conditions P_m, Q_n, and R_s for $b \in \mathcal{B}$ as follows,*

$$\begin{cases}
(P_1): z_1 \geq 0, z_2 < 0, z_4 \leq 0, z_1 + z_3 + 3z_4 = 0, s(b) = 2\ell; \\
(P_2): z_1 \geq 0, z_2 < 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \\
\qquad t(b) + x_1 = \ell, s(b) < 2\ell; \\
(P_3): z_1 \geq 0, z_2 < -1, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 = \ell, \\
\qquad s(b) < 2\ell; \\
(P_4): z_1 < 0, z_2 < 0, z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 \geq \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell; \\
(P_5): z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \\
\qquad t(b) + x_1 = \ell, s(b) < 2\ell; \\
(P_6): z_1 \geq 0, z_2 < 0, z_4 = 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, x_1 + x_2 = \ell, \\
\qquad s(b) < 2\ell; \\
(P_7): z_1 < 0, z_2 < 0, z_3 < 0, z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 > \ell, \bar{x}_1 + x_2 = \ell, \\
\qquad s(b) \leq 2\ell; \\
(P_8): z_1 < 0, z_2 = 0, z_3 < 0, z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 = \ell, s(b) \leq 2\ell; \\
(P_9): z_1 \geq 0, z_2 < 0, z_4 > 0, z_1 + z_3 < 0, z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \\
\qquad t(b) + \bar{x}_1 > \ell, \bar{x}_1 + \bar{x}_3 < \ell, x_1 + x_2 - x_3 + \bar{x}_3 = \ell, s(b) < 2\ell; \\
(P_{10}): z_1 < 0, z_3 < 0, z_4 > 0, z_1 + z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \\
\qquad x_2 - x_3 + \bar{x}_3 + \bar{x}_1 = \ell, s(b) \leq 2\ell; \\
(P_{11}): z_1 < 0, z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell; \\
(P_{12}): z_1 < -1, z_2 < 0, z_3 \geq 0, z_2 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \\
\qquad \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell;
\end{cases}$$

$$
\begin{cases}
(Q_1): z_1 \geq 0, z_2 < 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, s(b) = 2\ell; \\
(Q_2): z_1 \geq 0, z_2 < -1, z_1 + z_3 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 > \ell, \\
\qquad s(b) < 2\ell; \\
(Q_3): z_2 < 0, z_3 < 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell; \\
(Q_4): z_1 < 0, z_2 < 0, z_3 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 \geq \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) \leq 2\ell; \\
(Q_5): z_2 < 0, z_3 < 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 > \ell, \\
\qquad \bar{x}_1 + \bar{x}_3 > \ell, s(b) \leq 2\ell; \\
(Q_6): z_2 \leq 0, z_3 < 0, z_4 = 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 > \ell, \\
\qquad x_1 + x_2 > \ell, \bar{x}_1 + x_2 > \ell, s(b) \leq 2\ell; \\
(Q_7): z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \\
\qquad t(b) + x_1 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 > \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 > \ell, s(b) \leq 2\ell; \\
(Q_8): z_1 < -1, z_2 < 0, z_3 \geq 0, z_2 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \\
\qquad t(b) + \bar{x}_1 < \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) \leq 2\ell; \\
(Q_9): z_2 < 0, z_3 < 0, z_2 + 3z_4 > 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \\
\qquad \bar{x}_1 + \bar{x}_3 > \ell, s(b) \leq 2\ell;
\end{cases}
$$

$$
\begin{cases}
(R_1): z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, s(b) - z_1 \geq \ell + x_1, s(b) + z_2 + z_3 + 3z_4 \geq \ell + x_1, s(b) \leq 2\ell; \\
(R_2): z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, s(b) - z_1 \geq \ell + x_1, s(b) - z_1 + z_2 + 3z_4 \geq \ell + x_1, \\
\qquad s(b) + z_2 + 3z_4 \geq \ell + x_1, s(b) \leq 2\ell; \\
(R_3): z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, s(b) - z_1 \geq \ell + x_1, s(b) + z_2 + z_3 + 3z_4 \geq \ell + x_1, \\
\qquad s(b) \leq 2\ell; \\
(R_4): z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, s(b) - z_1 \geq \ell + x_1, s(b) - z_1 + z_2 \geq \ell + x_1, \\
\qquad s(b) + z_2 \geq \ell + x_1, s(b) \leq 2\ell.
\end{cases}
$$

Then, the subsets $\{B_a^{(j)} \mid j \geq 1, 0 \leq a \leq 6\}$ *of* $\mathcal{B}$ *are given by,*

$$
\begin{aligned}
B_0^{(j)} &= \{(\ell, 0, 0, 0, 0, \ell)\} \\
B_1^{(j)} &= B_0^{(j)} \cup \{(x_1, x_2, 0, 0, 0, \ell) \mid z_1 > 0, z_1 + z_3 = 0, s(b) = 2\ell\} \\
B_2^{(j)} &= B_1^{(j)} \cup \{(x_1, x_2, x_3, 0, 0, \ell) \mid z_1 > 0, z_4 < 0, z_1 + z_3 + 3z_4 = 0, s(b) = 2\ell\} \\
B_3^{(j)} &= B_2^{(j)} \cup \{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, \\
&\qquad\qquad s(b) = 2\ell\} \\
B_4^{(j)} &= B_3^{(j)} \cup C_1 \cup C_2 \cup \cdots \cup C_{12} \\
B_5^{(j)} &= B_4^{(j)} \cup D_1 \cup D_2 \cup \cdots \cup D_9 \\
B_6^{(j)} &= F_1 \cup F_2 \cup F_3 \cup F_4,
\end{aligned}
$$

where $C_n = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid (P_n)$ *holds*$\}$ *for* $1 \leq n \leq 12$*,* $D_n = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid (Q_n)$ *holds*$\}$ *for* $1 \leq n \leq 9$*, and* $F_n = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid (R_n)$ *holds*$\}$ *for* $1 \leq n \leq 4$*.*

Proof. By definition, $B_0^{(j)} = \{b_j\}$, where $b_j = (\ell, 0, 0, 0, 0, \ell)$ when $\lambda = \ell(\Lambda_0 + \Lambda_1)$. Then, $B_a^{(j)} = \bigcup_{k \geq 0} \tilde{f}_{i_a^{(j)}}^k B_{a-1}^{(j)} \setminus \{0\}$. To obtain $B_1^{(j)}$, we apply $\tilde{f}_1$ repeatedly to $(\ell, 0, 0, 0, 0, \ell)$. Since $z_2 = z_3 = 0$, we apply the first condition for $\tilde{f}_1$ repeatedly to yield new elements of the form $\{(x_1, x_2, 0, 0, 0, \ell) \mid z_1 > 0, z_1 + z_3 = 0, s(b) = 2\ell\}$. Thus, $B_1^{(j)} = B_0^{(j)} \cup \{(x_1, x_2, 0, 0, 0, \ell) \mid z_1 > 0, z_1 + z_3 = 0, s(b) = 2\ell\}$.

To obtain $B_2^{(j)}$, we apply $\tilde{f}_2$ repeatedly to elements in $B_1^{(j)}$. Note that $x_2 = \bar{x}_3 = 0$ for $(\ell, 0, 0, 0, 0, \ell) \in B_0^{(j)}$, so we cannot apply $\tilde{f}_2$ to $B_0^{(j)}$. Then, since $z_4 = 0$ for all elements in $B_1^{(j)}$, applying $\tilde{f}_2$ repeatedly yields new elements of the form $\{(x_1, x_2, x_3, 0, 0, \ell) \mid z_1 > 0, z_4 < 0, z_1 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. These elements are not contained in $B_1^{(j)}$ since $x_3 > 0$. Thus, $B_2^{(j)} = B_1^{(j)} \cup \{(x_1, x_2, x_3, 0, 0, \ell) \mid z_1 > 0, z_4 < 0, z_1 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$.

Next, to obtain $B_3^{(j)}$, we apply $\tilde{f}_1$ repeatedly to elements in $B_2^{(j)}$. Since $B_1^{(j)}$ contains elements obtained by applying $\tilde{f}_1$ repeatedly, we only need to examine the action of $\tilde{f}_1$ to elements in $\{(x_1, x_2, x_3, 0, 0, \ell) \mid z_1 > 0, z_4 < 0, z_1 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. If $z_3 \leq 0$, then $(z_2)_+ \leq -z_3$ and applying $\tilde{f}_1$ repeatedly yields elements of the form $\{(x_1, x_2, x_3, 0, 0, \ell) \mid z_1 > 0, z_3 < 0, z_4 < 0, z_1 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. Each of these elements is already contained in $B_2^{(j)}$, so no new elements are obtained. If $z_3 > 0$, then we have $z_2 = 0 < z_3$, so applying $\tilde{f}_1$ repeatedly yields new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_3 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. Then, if $z_3 = 0$, the first condition for $\tilde{f}_1$ holds and

we repeatedly apply $\tilde{f}_1$ to obtain new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_3 < 0, z_1+z_3 > 0, z_1+2z_2+z_3+3z_4 = 0, s(b) = 2\ell\}$. We combine the two sets above to obtain $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1+z_3 > 0, z_1+2z_2+z_3+3z_4 = 0, s(b) = 2\ell\}$. These elements are not contained in $B_2^{(j)}$ since $\bar{x}_3 > 0$ here, whereas $\bar{x}_3 = 0$ for all elements in $B_2^{(j)}$.Therefore, we have $B_3^{(j)} = B_2^{(j)} \cup \{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$.

To obtain $B_4^{(j)}$, we apply $\tilde{f}_0$ repeatedly to elements in $B_3^{(j)}$.

(C_1) Consider the subset of $B_1^{(j)}$, $\{(x_1, x_2, 0, 0, 0, \ell) \mid z_1 > 0, z_1 + z_3 = 0, s(b) = 2\ell\}$. The conditions for $(\mathcal{F}_2)$ are satisfied, so we repeatedly apply $\tilde{f}_0$ to obtain new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 = 0, z_1 + z_3 = 0, s(b) = 2\ell\}$. Also, consider the subset of $B_2^{(j)}$, $\{(x_1, x_2, x_3, 0, 0, \ell) \mid z_1 > 0, z_4 < 0, z_1 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. Again, the conditions for $(\mathcal{F}_2)$ are satisfied, so we repeatedly apply $\tilde{f}_0$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 < 0, z_1+z_3+3z_4 = 0, s(b) = 2\ell\}$. Combining these two sets yields the set C_1. We note that C_1 is not contained in B_3 since $z_1 + z_3 + 3z_4 = 0$ and $s(b) = 2\ell$ imply $\bar{x}_1 + \bar{x}_3 = \ell$, and $z_2 < 0$ implies that $\bar{x}_1 < \ell$.

(C_2) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is clearly a subset of $B_3^{(j)}$. Then, since $z_2 < 0$ and $z_1 + 2z_2 + z_3 + 3z_4 = 0$, we have $z_1 + z_3 + 3z_4 > 0$ and $z_1 + z_2 + z_3 + 3z_4 > 0$. So, conditions for $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. This set is precisely C_2. Note that C_2 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$.

(C_3) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is clearly a subset of $B_3^{(j)}$. Then, since $z_2 < 0$ and $z_1 + 2z_2 + z_3 + 3z_4 = 0$, we have $z_1 + z_3 + 3z_4 > 0$ and $z_1 + z_2 + z_3 + 3z_4 > 0$. So, conditions for $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1+2z_2+z_3+3z_4 < 0, t(b)+x_1 = \ell, s(b) < 2\ell\}$. Then, if $z_1 > 0$ and $z_1+z_2+z_3+3z_4 = 0$, the conditions for $(\mathcal{F}_2)$ are satisfied since $z_2 + 3z_4 = -z_1 - z_3 < 0$. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < -1, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. We note that the condition $\bar{x}_1 + \bar{x}_3 = \ell$ is obtained from $z_1 + z_2 + z_3 + 3z_4 = 0$ and $t(b) + x_1 = \ell$. Then, this set is precisely C_3 and is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$.

(C_4) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_1 + 2z_2 +$

$z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is clearly a subset of $B_3^{(j)}$. Then, since $z_2 < 0$ and $z_1 + 2z_2 + z_3 + 3z_4 = 0$, we have $z_1 + z_3 + 3z_4 > 0$ and $z_1 + z_2 + z_3 + 3z_4 > 0$. So, conditions for $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Then, if $z_1 > 0$ and $z_1 + z_2 + z_3 + 3z_4 = 0$, the conditions for $(\mathcal{F}_2)$ are satisfied since $z_2 + 3z_4 = -z_1 - z_3 < 0$. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_2 < -1, z_3 \geq 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. Since $z_1 = 0$, the conditions for $(\mathcal{F}_1)$ are satisfied. Repeatedly applying $\tilde{f}_0$ yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < -1, z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 > \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell\}$. Now, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_2 + z_3 + 3z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is clearly a subset of $B_3^{(j)}$. Then, since $z_2 < 0$ and $z_1 + 2z_2 + z_3 + 3z_4 = 0$, we have $z_1 + z_3 + 3z_4 > 0$ and $z_1 + z_2 + z_3 + 3z_4 > 0$. So, conditions for $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_2 < 0, z_3 \geq 0, z_1 + z_3 \geq 0, z_2 + z_3 + 3z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Then since $z_1 = 0$ and $z_2 + z_3 + 3z_4 = 0$, conditions for $(\mathcal{F}_1)$ are satisfied. We repeatedly apply $\tilde{f}_0$ to obtain new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 = \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell\}$. Combining the two final sets above yields the set C_4. We note that this set is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$.

(C_5) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is clearly a subset of $B_3^{(j)}$. Then, since $z_2 < 0$ and $z_1 + 2z_2 + z_3 + 3z_4 = 0$, we have $z_1 + z_3 + 3z_4 > 0$ and $z_1 + z_2 + z_3 + 3z_4 > 0$. So, conditions for $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_1 + z_3 = 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Since $z_1 + z_3 = 0$ and $z_2 + 3z_4 > 0$, we have $z_1 + z_2 + z_3 + 3z_4 > 0$, satisfying the conditions for $(\mathcal{F}_4)$. Repeatedly applying $\tilde{f}_0$ to elements of this form yields $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Next, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_3 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is clearly a subset of $B_3^{(j)}$. Then, since $z_2 < 0$ and $z_1 + 2z_2 + z_3 + 3z_4 = 0$, we have $z_1 + z_3 + 3z_4 > 0$ and $z_1 + z_2 + z_3 + 3z_4 > 0$. So, conditions for $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_2 < 0, z_3 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Since $z_1 = 0$, $z_3 > 0$, and $z_2 + 3z_4 > 0$, we have $z_1 + z_2 + z_3 + 3z_4 > 0$ and conditions for $(\mathcal{F}_5)$ are satisfied. We repeatedly apply $\tilde{f}_0$ to these elements to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 = 0, z_1 + z_2 + 3z_4 >$

$0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Then, since $z_3 = 0$ and $z_1 + z_2 + z_3 + 3z_4 > 0$, the conditions for $(\mathcal{F}_4)$ are satisfied and we repeatedly apply $\tilde{f}_0$ to these elements. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 < 0, z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Combining the two final sets above yields the set C_5. We note that C_5 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$.

(C_6) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is clearly a subset of $B_3^{(j)}$. Then, since $z_2 < 0$ and $z_1 + 2z_2 + z_3 + 3z_4 = 0$, we have $z_1 + z_3 + 3z_4 > 0$ and $z_1 + z_2 + z_3 + 3z_4 > 0$. So, conditions for $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_1 + z_3 = 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Since $z_1 + z_3 = 0$ and $z_2 + 3z_4 > 0$, we have $z_1 + z_2 + z_3 + 3z_4 > 0$, satisfying the conditions for $(\mathcal{F}_4)$. Repeatedly applying $\tilde{f}_0$ to elements of this form yields $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_4 = 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Then, since $z_4 = 0$, $z_2 + 3z_4 \geq 0$, and $z_2 \leq 0$, we must have $z_2 = 0$. If $z_1 > 0$, then the conditions for $(\mathcal{F}_2)$ are satisfied and we repeatedly apply $\tilde{f}_0$ to yield the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 = 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, x_1 + x_2 = \ell, s(b) < 2\ell\}$. We obtain the condition $x_1 + x_2 = \ell$ from $z_2 = z_4 = 0$ and $t(b) + x_1 = \ell$. Then, this set is precisely C_6. We note that C_6 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$.

(C_7) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is clearly a subset of $B_3^{(j)}$. Then, since $z_2 < 0$ and $z_1 + 2z_2 + z_3 + 3z_4 = 0$, we have $z_1 + z_3 + 3z_4 > 0$ and $z_1 + z_2 + z_3 + 3z_4 > 0$. So, conditions for $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_1 + z_3 = 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Since $z_1 + z_3 = 0$ and $z_2 + 3z_4 > 0$, we have $z_1 + z_2 + z_3 + 3z_4 > 0$, satisfying the conditions for $(\mathcal{F}_4)$. Repeatedly applying $\tilde{f}_0$ to elements of this form yields $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_4 = 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Then, since $z_4 = 0$, $z_2 + 3z_4 \geq 0$, and $z_2 \leq 0$, we must have $z_2 = 0$. If $z_1 > 0$, then the conditions for $(\mathcal{F}_2)$ are satisfied and we repeatedly apply $\tilde{f}_0$ to yield the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_2 < 0, z_3 < 0, z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, x_1 + x_2 = \ell, s(b) < 2\ell\}$. Since $z_1 = 0$, the conditions for $(\mathcal{F}_1)$ are satisfied and we repeatedly apply $\tilde{f}_0$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_3 < 0, z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 > \ell, \bar{x}_1 + x_2 = \ell, s(b) \leq 2\ell\}$, which is precisely C_7. Again, note that C_7 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$.

(C_8) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is clearly a subset of $B_3^{(j)}$. Then, since $z_2 < 0$ and $z_1 + 2z_2 + z_3 + 3z_4 = 0$, we have $z_1 + z_3 + 3z_4 > 0$ and $z_1 + z_2 + z_3 + 3z_4 > 0$. So, conditions for $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_1 + z_3 = 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Since $z_1 + z_3 = 0$ and $z_2 + 3z_4 > 0$, we have $z_1 + z_2 + z_3 + 3z_4 > 0$, satisfying the conditions for $(\mathcal{F}_4)$. Repeatedly applying $\tilde{f}_0$ to elements of this form yields $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_3 < 0, z_4 = 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Since $z_4 = 0$, $z_2 + 3z_4 \geq 0$, and $z_2 \leq 0$, we must have $z_2 = 0$. Then, $z_1 = 0$ implies the conditions for $(\mathcal{F}_1)$ are satisfied. We repeatedly apply $\tilde{f}_0$ to these elements to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 = 0, z_3 < 0, z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 = \ell, s(b) \leq 2\ell\}$. This set is precisely C_8 and is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$.

(C_9) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is clearly a subset of $B_3^{(j)}$. Then, since $z_2 < 0$ and $z_1 + 2z_2 + z_3 + 3z_4 = 0$, we have $z_1 + z_3 + 3z_4 > 0$ and $z_1 + z_2 + z_3 + 3z_4 > 0$. So, conditions for $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_1 + z_3 = 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Since $z_1 + z_3 = 0$ and $z_2 + 3z_4 > 0$, we have $z_1 + z_2 + z_3 + 3z_4 > 0$, satisfying the conditions for $(\mathcal{F}_4)$. Repeatedly applying $\tilde{f}_0$ to elements of this form yields $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Then, if $z_1 > 0$ and $z_4 > 0$, the conditions for $(\mathcal{F}_2)$ are satisfied and we repeatedly apply $\tilde{f}_0$ to these elements. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 > 0, z_1 + z_3 < 0, z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 > \ell, \bar{x}_1 + \bar{x}_3 < \ell, x_1 + x_2 - x_3 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. The condition $\bar{x}_1 + \bar{x}_3 < \ell$ is obtained from $t(b) + x_1 = \ell$, $z_1 + z_3 < 0$, and $z_2 + 3z_4 = 0$. Also, the condition $x_1 + x_2 - x_3 + \bar{x}_3 = \ell$ is obtained from $t(b) + x_1 = \ell$ and $z_2 + 3z_4 = 0$. Then, this set is precisely C_9. We note that C_9 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$.

(C_{10}) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is clearly a subset of $B_3^{(j)}$. Then, since $z_2 < 0$ and $z_1 + 2z_2 + z_3 + 3z_4 = 0$, we have $z_1 + z_3 + 3z_4 > 0$ and $z_1 + z_2 + z_3 + 3z_4 > 0$. So, conditions for $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_1 + z_3 = 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Since $z_1 + z_3 = 0$ and $z_2 + 3z_4 > 0$, we have $z_1 + z_2 + z_3 + 3z_4 > 0$, satisfying the conditions for $(\mathcal{F}_4)$. Repeatedly applying $\tilde{f}_0$ to elements of this form yields $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 <$

$0, t(b)+x_1 = \ell, s(b) < 2\ell\}$. Then, if $z_1 > 0$ and $z_4 > 0$, the conditions for $(\mathcal{F}_2)$ are satisfied and we repeatedly apply $\tilde{f}_0$ to these elements. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_2 < 0, z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 > \ell, \bar{x}_1 + \bar{x}_3 < \ell, x_1 + x_2 - x_3 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. Since $z_1 = 0$, conditions for $(\mathcal{F}_1)$ are satisfied and we apply $\tilde{f}_0$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_3 < 0, z_4 > 0, z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 > \ell, \bar{x}_1 + \bar{x}_3 < \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 = \ell, s(b) \leq 2\ell\}$. Next, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is clearly a subset of $B_3^{(j)}$. Then, since $z_2 < 0$ and $z_1 + 2z_2 + z_3 + 3z_4 = 0$, we have $z_1 + z_3 + 3z_4 > 0$ and $z_1 + z_2 + z_3 + 3z_4 > 0$. So, conditions for $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_1 + z_3 = 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Since $z_1 + z_3 = 0$ and $z_2 + 3z_4 > 0$, we have $z_1 + z_2 + z_3 + 3z_4 > 0$, satisfying the conditions for $(\mathcal{F}_4)$. Repeatedly applying $\tilde{f}_0$ to elements of this form yields $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. If $z_4 > 0$, the conditions for $(\mathcal{F}_1)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to these elements yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_3 < 0, z_4 > 0, z_2 + 3z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 = \ell, s(b) \leq 2\ell\}$. Finally, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_3 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is clearly a subset of $B_3^{(j)}$. Then, since $z_2 < 0$ and $z_1 + 2z_2 + z_3 + 3z_4 = 0$, we have $z_1 + z_3 + 3z_4 > 0$ and $z_1 + z_2 + z_3 + 3z_4 > 0$. So, conditions for $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_2 < 0, z_3 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Since $z_1 = 0$, $z_3 > 0$, and $z_2 + 3z_4 > 0$, we have $z_1 + z_2 + z_3 + 3z_4 > 0$ and conditions for $(\mathcal{F}_5)$ are satisfied. We repeatedly apply $\tilde{f}_0$ to these elements to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 = 0, z_1 + z_2 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Then, since $z_3 = 0$ and $z_1 + z_2 + z_3 + 3z_4 > 0$, the conditions for $(\mathcal{F}_4)$ are satisfied and we repeatedly apply $\tilde{f}_0$ to these elements. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 < 0, z_4 \geq 0, z_1 + z_2 + 3z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Since $z_4 \geq 0$ and $z_1 + z_2 + 3z_4 = 0$, we have $z_1 + z_2 \leq 0$ and conditions for $(\mathcal{F}_1)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < -1, z_3 < 0, z_4 \geq 0, z_2 + 3z_4 > 0, z_1 + z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 = \ell, s(b) \leq 2\ell\}$. The condition $x_2 - x_3 + \bar{x}_3 + \bar{x}_1 = \ell$ is obtained from $t(b) + x_1 = \ell$ and $z_1 + z_2 + 3z_4 = 0$. Then, combining the three final sets above yields the set C_{10}. We note that C_{10} is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$.

(C_{11}) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_3 > 0, z_2 + 3z_4 >$

$0, z_1+2z_2+z_3+3z_4 = 0, s(b) = 2\ell\}$. This set is clearly a subset of $B_3^{(j)}$. Then, since $z_2 < 0$ and $z_1+2z_2+z_3+3z_4 = 0$, we have $z_1+z_3+3z_4 > 0$ and $z_1+z_2+z_3+3z_4 > 0$. So, conditions for $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_2 < 0, z_3 > 0, z_2+3z_4 > 0, z_1+2z_2+z_3+3z_4 < 0, t(b)+x_1 = \ell, s(b) < 2\ell\}$. Since $z_1 = 0$, $z_3 > 0$, and $z_2 + 3z_4 > 0$, we have $z_1 + z_2 + z_3 + 3z_4 > 0$ and conditions for $(\mathcal{F}_5)$ are satisfied. We repeatedly apply $\tilde{f}_0$ to these elements to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. This set is precisely C_{11}. Note that C_{11} is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$.

(C_{12}) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_3 > 0, z_2 + 3z_4 > 0, z_1+2z_2+z_3+3z_4 = 0, s(b) = 2\ell\}$. This set is clearly a subset of $B_3^{(j)}$. Then, since $z_2 < 0$ and $z_1+2z_2+z_3+3z_4 = 0$, we have $z_1+z_3+3z_4 > 0$ and $z_1+z_2+z_3+3z_4 > 0$. So, conditions for $(\mathcal{F}_6)$ are satisfied. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_2 < 0, z_3 > 0, z_2+3z_4 > 0, z_1+2z_2+z_3+3z_4 < 0, t(b)+x_1 = \ell, s(b) < 2\ell\}$. Since $z_1 = 0$, $z_3 > 0$, and $z_2 + 3z_4 > 0$, we have $z_1 + z_2 + z_3 + 3z_4 > 0$ and conditions for $(\mathcal{F}_5)$ are satisfied. We repeatedly apply $\tilde{f}_0$ to these elements to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_1+z_2+z_3+3z_4 = 0, z_1+2z_2+z_3+3z_4 < 0, t(b)+x_1 = \ell, s(b) < 2\ell\}$. Then, since $z_1+z_2+z_3+3z_4 = 0$, we have $z_1+z_2+3z_4 \leq 0$ and the conditions for $(\mathcal{F}_1)$ are satisfied. We repeatedly apply $\tilde{f}_0$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < -1, z_2 < 0, z_3 \geq 0, z_2 + z_3 + z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. The condition $\bar{x}_1 + \bar{x}_3 = \ell$ is obtained from $z_1 + z_2 + z_3 + 3z_4 = 0$ and $t(b) + x_1 = \ell$. Then, this set is precisely C_{12} and is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$.

Therefore, by repeatedly applying $\tilde{f}_0$ to elements in $B_3^{(j)}$, we obtain $B_4^{(j)} = B_3^{(j)} \cup C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7 \cup C_8 \cup C_9 \cup C_{10} \cup C_{11} \cup C_{12}$.

To obtain $B_5^{(j)}$, we apply $\tilde{f}_1$ repeatedly to elements in $B_4^{(j)}$. Since $B_3^{(j)}$ contains elements obtained by applying $\tilde{f}_1$ repeatedly to $B_2^{(j)}$, we only need to examine the action of $\tilde{f}_1$ on elements in $B_4^{(j)} \setminus B_3^{(j)} = C_1 \cup \cdots \cup C_{12}$.

(D_1) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_3 > 0, z_4 \leq 0, z_1 + z_3+3z_4 = 0, s(b) = 2\ell\}$. This is clearly a subset of C_1 in $B_4^{(j)}$. Since $z_2 < 0 < z_3$, the second condition for $\tilde{f}_1$ is satisfied. Repeatedly applying $\tilde{f}_1$ to this set yields $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_3 \geq 0, z_1 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, s(b) = 2\ell\}$. Then if $z_3 = 0$, the first condition for $\tilde{f}_1$ holds and we repeatedly apply $\tilde{f}_1$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 > 0, z_2 < 0, z_3 < 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, s(b) = 2\ell\}$. Combining these two sets yields the set D_1. We see that D_1 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$. Also, D_1 is not contained in C_1

in $B_4^{(j)}$ since $z_1 + z_3 + 3z_4 > 0$. D_1 is not contained in C_2, C_3, C_5, C_6, C_9, or C_{11} since $s(b) = 2\ell$. Finally, D_1 is not contained in C_4, C_7, C_8, C_{10}, or C_{12} since $z_1 \geq 0$. Thus, D_1 is not contained in $B_4^{(j)}$.

(D_2) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < -1, z_3 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. This set is a subset of C_3 in $B_4^{(j)}$. Since $z_2 < 0 < z_3$, we repeatedly apply the second condition for $\tilde{f}_1$ to obtain new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < -1, z_3 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) < 2\ell\}$. Then, if $z_3 = 0$, the first condition for $\tilde{f}_1$ is satisfied. Repeatedly applying $\tilde{f}_1$ yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 > 0, z_2 < -1, z_3 < 0, z_1 + z_3 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) < 2\ell\}$. Combining these two sets yields the set D_2. We note that D_2 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$. Also, D_2 is not contained in C_1 in $B_4^{(j)}$ since $s(b) < 2\ell$. Then D_2 is not contained in C_2, C_5, or C_{11} since $t(b) + x_1 > \ell$. D_2 is not contained in C_3, C_4, C_6, C_7, C_9, or C_{12} since $\bar{x}_1 + \bar{x}_3 > \ell$. Finally, D_2 is not contained in C_8 or C_{10} since $z_1 \geq 0$. Thus, D_2 is not contained in $B_4^{(j)}$.

(D_3) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_3 = 0, z_2 + 3z_4 \leq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 \geq \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell\}$. This is clearly a subset of C_4 in $B_4^{(j)}$. Since $z_3 = 0$ and $z_2 < 0$, we have $(z_2)_+ \leq -z_3$. Repeatedly applying the first condition for $\tilde{f}_1$ yields elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 < 0, z_3 < 0, z_1 + z_3 < 0, z_2 + 3z_4 \leq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 > \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell\}$. Next consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < -1, z_2 < 0, z_3 = 0, z_2 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell\}$. This set is contained in C_{12} in $B_4^{(j)}$. We have $z_2 < 0 = z_3$, so $(z_2)_+ \leq -z_3$ and we repeatedly apply the first condition for $\tilde{f}_1$. This yields $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 < 0, z_3 < 0, z_1 + z_2 < 0, z_2 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell\}$. Combining the two sets above yields the set D_3. We note that D_3 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$. Also, D_3 is not contained in C_1, C_6, C_7, or C_8 in $B_4^{(j)}$ since $\bar{x}_1 + \bar{x}_3 = \ell$ and $s(b) \leq 2\ell$ implies that $z_1 + z_3 + 3z_4 \geq 0$ which then implies that $z_4 > 0$ since $z_1 + z_3 < 0$. Then, D_3 is not contained in C_2 or C_3 since $z_1 + z_3 < 0$. D_3 is not contained in C_4, C_{11}, or C_{12} since $z_3 < 0$. Next, D_3 is not contained in C_5 since $t(b) + x_1 > \ell$. D_3 is not contained in C_9 since $\bar{x}_1 + \bar{x}_3 = \ell$. Finally, D_3 is not contained in C_{10} since $x_2 - x_3 + \bar{x}_3 + \bar{x}_1 \neq \ell$. Thus, D_3 is not contained in $B_4^{(j)}$.

(D_4) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_3 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 \geq \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell\}$. This is clearly a subset of C_4 in $B_4^{(j)}$. Since $z_2 < 0 < z_3$, the second condition for $\tilde{f}_1$ is satisfied. Repeatedly applying $\tilde{f}_1$ to elements

in this set yields $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_3 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 \geq \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) \leq 2\ell\}$. This set is precisely D_4. Note that D_4 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$. Also, D_4 is not contained in C_1, C_2, C_3, C_6, or C_9 since $z_1 < 0$. Then D_4 is not contained in C_4 or C_{12} since $\bar{x}_1 + \bar{x}_3 > \ell$. D_4 is not contained in C_5, C_7, C_8, or C_{10} since $z_3 \geq 0$. Finally, D_4 is not contained in C_{11} since $t(b) + x_1 > \ell$. Thus, D_4 is not contained in $B_4^{(j)}$.

(D_5) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_3 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 \geq \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell\}$. This is a subset of C_4 in $B_4^{(j)}$. Since $z_2 < 0 < z_3$, the second condition for $\tilde{f}_1$ is satisfied. Repeatedly applying $\tilde{f}_1$ to elements in this set yields $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_3 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 \geq \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) \leq 2\ell\}$. Then $z_3 = 0$, so we have $(z_2)_+ \leq -z_3$. Repeatedly applying the first condition for $\tilde{f}_1$ yields the set $\{x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 < 0, z_3 < 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 > \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) \leq 2\ell\}$. The condition $t(b) + x_1 > \ell$ is obtained from $t(b) + \bar{x}_1 > \ell$ and $z_1 < 0$. This set is precisely D_5. We note that D_5 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$. Also, D_5 is not contained in C_1, C_6, C_7, C_9, or C_{12} since $\bar{x}_1 + \bar{x}_3 > \ell$ and $s(b) \leq 2\ell$ imply that $z_1 + z_3 + 3z_4 > 0$. D_5 is not contained in C_2 or C_3 since $z_1 + z_3 < 0$, and D_5 is not contained in C_4 or C_{11} since $z_3 < 0$. Then D_5 is not contained in C_5 or C_8 since $t(b) + x_1 > \ell$ and $t(b) + \bar{x}_1 > \ell$. D_5 is not contained in C_{10} since $x_2 - x_3 + \bar{x}_3 + \bar{x}_1 \neq \ell$. Thus, D_5 is not contained in $B_4^{(j)}$.

(D_6) Consider $C_7 = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_3 < 0, z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 > \ell, \bar{x}_1 + x_2 = \ell, s(b) \leq 2\ell\}$ in $B_4^{(j)}$. Since $z_2 < 0$ and $z_3 < 0$, we have $(z_2)_+ \leq -z_3$ so we repeatedly apply the first condition for $\tilde{f}_1$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 < 0, z_3 < 0, z_4 = 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 > \ell, x_1 + x_2 > \ell, \bar{x}_1 + x_2 = \ell, s(b) \leq 2\ell\}$. The conditions $t(b) + x_1 > \ell$ and $x_1 + x_2 > \ell$ are obtained from the conditions $t(b) + \bar{x}_1 > \ell$, $\bar{x}_1 + x_2 = \ell$, and $z_1 < 0$. Next, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 = 0, z_3 < 0, z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 = \ell, \bar{x}_1 + x_2 = \ell, s(b) \leq 2\ell\}$ in $B_4^{(j)}$. This set is clearly contained in C_8 in $B_4^{(j)}$. Since $z_3 < 0 = z_2$, we apply the first condition for $\tilde{f}_1$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 = 0, z_3 < 0, z_4 = 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 > \ell, x_1 + x_2 > \ell, \bar{x}_1 + x_2 > \ell, s(b) \leq 2\ell\}$. Again, the conditions $t(b) + x_1 > \ell$ and $x_1 + x_2 > \ell$ are obtained from $t(b) + \bar{x}_1 = \ell$, $\bar{x}_1 + x_2 > \ell$, and $z_1 < 0$. Combining these two sets yields the set D_6. We note that D_6 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$. Also, D_6 is not contained in C_1 since $z_1 + z_3 + 3z_4 < 0$. D_6 is not contained in C_2 or C_3 since $z_1 + z_3 < 0$, and D_6 is not contained in C_4, C_{11}, or C_{12} since $z_3 < 0$. Then D_6 is not contained in C_5 or C_8 since

$t(b) + x_1 > \ell$ and $t(b) + \bar{x}_1 > \ell$. Finally, D_6 is not contained in C_6, C_7, C_9, or C_{10} since $x_1 + x_2 > \ell$, $\bar{x}_1 + x_2 > \ell$, and $z_4 = 0$. Thus, D_6 is not contained in $B_4^{(j)}$.

(D_7) Consider $C_{10} = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 < 0, z_4 > 0, z_1 + z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 = \ell, s(b) \leq 2\ell\}$ in $B_4^{(j)}$. Since $z_2 \leq 0$ and $z_3 < 0$, we have $(z_2)_+ \leq -z_3$. So, we repeatedly apply the first condition for $\tilde{f}_1$ to obtain new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 < \ell, x_1 + x_2 - x_3 + \bar{x}_3 > \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 > \ell, s(b) \leq 2\ell\}$. The conditions $\bar{x}_1 + \bar{x}_3 < \ell$ and $x_1 + x_2 - x_3 + \bar{x}_3 > \ell$ are obtained from the conditions $z_1 < 0$, $z_3 < 0$, and $x_2 - x_3 + \bar{x}_3 + \bar{x}_1 = \ell$. We see that this set is precisely D_7. Note that D_7 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$. Also, D_7 is not contained in C_1, C_6, C_7, or C_8 since $z_4 > 0$. Then D_7 is not contained in C_2, C_5, or C_{11} since $t(b) + x_1 > \ell$ and D_7 is not contained in C_3 since $z_1 + z_3 < 0$. D_7 is not contained in C_4 or C_{12} since $z_3 < 0$. Finally, D_7 is not contained in C_9 or C_{10} since $x_1 + x_2 - x_3 + \bar{x}_3 > \ell$ and $x_2 - x_3 + \bar{x}_3 + \bar{x}_1 > \ell$. Thus, D_7 is not contained in $B_4^{(j)}$.

(D_8) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < -1, z_2 < 0, z_3 > 0, z_2 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 < \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell\}$. This set is a subset of C_{12} in $B_4^{(j)}$. Since $z_2 < 0 < z_3$, we repeatedly apply the second condition for $\tilde{f}_1$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < -1, z_2 < 0, z_3 \geq 0, z_2 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 < \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) \leq 2\ell\}$. This set is precisely D_8. We note that D_8 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$. Also, D_8 is not contained in C_1, C_2, C_3, C_6 or C_9 since $z_1 < -1$. Then D_8 is not contained in C_4 or C_{12} since $\bar{x}_1 + \bar{x}_3 > \ell$, and D_8 is not contained in C_5, C_7, C_8 or C_{10} since $z_3 \geq 0$. Finally, D_8 is not contained in C_{11} since $t(b) + x_1 > \ell$. Thus, D_8 is not contained in $B_4^{(j)}$.

(D_9) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < -1, z_2 < 0, z_3 > 0, z_2 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 < \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell\}$. This set is a subset of C_{12} in $B_4^{(j)}$. Since $z_2 < 0 < z_3$, we repeatedly apply the second condition for $\tilde{f}_1$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < -1, z_2 < 0, z_3 = 0, z_2 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 < \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) \leq 2\ell\}$. Then since $z_3 = 0$, we have $(z_2)_+ \leq -z_3$ and we repeatedly apply the first condition of $\tilde{f}_1$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 < 0, z_3 < 0, z_1 + z_3 < 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 \leq \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) \leq 2\ell\}$. We need the condition $t(b) + \bar{x}_1 \leq \ell$ in order to be disjoint from D_5. Then, this set is precisely D_9 and we see that D_9 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$. Also, D_9 is not contained in C_1, C_6, C_7, or C_8 since $z_4 > 0$ and D_9 is not contained in C_9 since $z_2 + 3z_4 > 0$. Then D_9 is not contained in C_2 or C_3 since $z_1 + z_3 < 0$, and D_9 is not contained in C_4 or C_{11} since $z_3 < 0$.

D_9 is not contained in C_5 since $t(b) + x_1 > \ell$. Finally, D_9 is not contained in C_{10} and C_{12} since $z_3 < 0$ and $\bar{x}_1 + \bar{x}_3 > \ell$ imply that $x_2 - x_3 + \bar{x}_3 + \bar{x}_1 > \ell$. Thus, D_9 is not contained in $B_4^{(j)}$.

Therefore, by repeatedly applying $\tilde{f}_1$ to elements in $B_4^{(j)}$, we obtain $B_5^{(j)} = B_4^{(j)} \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6 \cup D_7 \cup D_8 \cup D_9$.

To obtain $B_6^{(j)}$, we apply $\tilde{f}_2$ repeatedly to elements in $B_5^{(j)}$. Since $B_2^{(j)}$ contains elements obtained by applying $\tilde{f}_2$ repeatedly to B_1, we only need to examine the action of $\tilde{f}_1$ on elements in $B_5^{(j)} \setminus B_2^{(j)}$. We first show that $B_5^{(j)} = B_4^{(j)} \cup L \cup M_1 \cup M_2 \cup N_1 \cup N_2 \cup N_3 \cup \cdots \cup N_{18}$, where $L = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \ell) \mid (X) \text{ holds}\}$, $M_m = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid (Y_m) \text{ holds}\}$, and $N_n = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid (Z_n) \text{ holds}\}$. The conditions for X, Y_m, and Z_n ($1 \le m \le 2$, $1 \le n \le 18$) are,

$$(X)\ \bar{x}_2 > 0, z_1 > 0, z_1 + z_3 > 0, z_1 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 > 0, s(b) = 2\ell;$$

$$\begin{cases} (Y_1)\ z_2 = 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 \ge \ell, \\ \qquad t(b) + \bar{x}_1 \ge \ell, s(b) \le 2\ell\}; \\ (Y_2)\ z_2 < 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 > \ell, \\ \qquad s(b) - z_1 + z_2 \ge \ell + x_1, s(b) + z_2 \ge \ell + x_1, s(b) \le 2\ell; \end{cases}$$

$$
\begin{cases}
(Z_1)\ \bar{x}_2 > 0, z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, \\
\qquad s(b) < 2\ell; \\
(Z_2)\ \bar{x}_2 > 0, z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, \\
\qquad \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) < 2\ell; \\
(Z_3)\ \bar{x}_2 > 0, z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 + \bar{x}_2 \geq \ell, \\
\qquad \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) \leq 2\ell; \\
(Z_4)\ \bar{x}_2 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, \\
\qquad s(b) < 2\ell; \\
(Z_5)\ \bar{x}_2 > 0, z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, \\
\qquad s(b) < 2\ell; \\
(Z_6)\ \bar{x}_2 > 0, z_1 \leq 0, z_3 < 0, z_4 \geq 0, z_1 + z_2 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, s(b) + z_2 + 3z_4 = \ell + x_1, \\
\qquad s(b) \leq 2\ell; \\
(Z_7)\ \bar{x}_2 > 0, z_1 = 0, z_2 < 0, z_4 < 0, z_1 + z_+ 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) + \bar{x}_1 + \bar{x}_2 > \ell, \\
\qquad \bar{x}_1 + \bar{x}_3 < \ell, s(b) - z_1 + z_2 + 3z_4 < \ell + x_1, s(b) + z_2 \leq \ell + x_1, s(b) < 2\ell; \\
(Z_8)\ \bar{x}_2 > 0, z_1 < 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + z_2 + 3z_4 < 0, \\
\qquad z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, s(b) + z_2 = \ell + x_1, s(b) \leq 2\ell; \\
(Z_9)\ \bar{x}_2 > 0, z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell; \\
(Z_{10})\ \bar{x}_2 > 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_2 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, \\
\qquad \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) \leq 2\ell; \\
(Z_{11})\ \bar{x}_2 > 0, z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, t(b) - z_1 + \bar{x}_2 > 0, s(b) = 2\ell; \\
(Z_{12})\ \bar{x}_2 > 0, z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) - z_1 + \bar{x}_2 > 0, \\
\qquad \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) < 2\ell; \\
(Z_{13})\ \bar{x}_2 > 0, z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, t(b) + \bar{x}_1 + \bar{x}_2 \geq \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, \\
\qquad s(b) \leq 2\ell; \\
(Z_{14})\ \bar{x}_2 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) - z_1 + \bar{x}_2 > 0, \\
\qquad \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) \leq 2\ell; \\
(Z_{15})\ \bar{x}_2 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, \\
\qquad s(b) - z_1 + z_2 + 3z_4 > \ell + x_1, s(b) + z_2 + 3z_4 > \ell + x_1, s(b) \leq 2\ell; \\
(Z_{16})\ \bar{x}_2 > 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, \\
\qquad s(b) - z_1 + z_2 > \ell + x_1, s(b) + z_2 > \ell + x_1, s(b) \leq 2\ell; \\
(Z_{17})\ \bar{x}_2 > 0, z_1 < -1, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_2 + z_3 + 3z_4 > 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) + \bar{x}_1 + \bar{x}_2 < \ell, \\
\qquad t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) \leq 2\ell.
\end{cases}
$$

(L) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_4 > 0, z_1 + z_3 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is clearly contained in $B_3^{(j)}$. Then since $z_4 > 0$, we apply the second condition for $\tilde{f}_2$ to obtain new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \ell) \mid \bar{x}_2 > 0, z_1 >$

$0, z_4 \geq 0, z_1 + z_3 > 0, z_1 + z_2 + z_3 + 3z_4 > 0, s(b) = 2\ell\}$. Then if $z_4 = 0$, the first condition for $\tilde{f}_2$ is satisfied. Repeatedly applying $\tilde{f}_2$ yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \ell) \mid \bar{x}_2 > 0, z_1 > 0, z_4 < 0, z_1 + z_3 > 0, z_1 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 > 0, s(b) = 2\ell\}$. Combining these two sets yields the set L.

(M_1) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 = 0, z_3 < 0, z_4 = 0, z_1 + z_3 < 0, z_1 + z_2 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. This is clearly a subset of C_5 in $B_4^{(j)}$. Since $z_4 = 0$, we repeatedly apply the first condition for $\tilde{f}_2$ to obtain new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 = 0, z_4 < 0, z_1 + z_2 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Next consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 = 0, z_3 < 0, z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 = \ell, s(b) < 2\ell\}$. This is a subset of C_8 in $B_4^{(j)}$, and since $z_4 = 0$, we repeatedly apply the first condition for $\tilde{f}_2$. This yields elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 = 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 = \ell, s(b) < 2\ell\}$. Lastly, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 = 0, z_3 < 0, z_4 = 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 > \ell, s(b) \leq 2\ell\}$. This set is contained in D_6 in $B_5^{(j)}$. Then since $z_4 = 0$, we repeatedly apply the first condition for $\tilde{f}_2$ to yield the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 = 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 > \ell, s(b) \leq 2\ell\}$. Combining these three sets yields the set M_1.

(M_2) Consider $C_6 = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 = 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, x_1 + x_2 = \ell, s(b) < 2\ell\}$ in $B_4^{(j)}$. Since $z_4 = 0$, we repeatedly apply the first condition for $\tilde{f}_2$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, x_1 + x_2 - z_4 = \ell, s(b) < 2\ell\}$. Next consider $C_7 = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_3 < 0, z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 > \ell, \bar{x}_1 + x_2 = \ell, s(b) < 2\ell\}$ in $B_4^{(j)}$. Again, $z_4 = 0$, so we repeatedly apply the first condition for $\tilde{f}_2$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 > \ell, \bar{x}_1 + x_2 - z_4 = \ell, s(b) < 2\ell\}$. Finally, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 < 0, z_3 < 0, z_4 = 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 > \ell, x_1 + x_2 > \ell, \bar{x}_1 + x_2 > \ell, s(b) \leq 2\ell\}$. This is a subset of D_6 in $B_5^{(j)}$. Then $z_4 = 0$, so repeatedly applying the first condition for $\tilde{f}_2$ yields $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 < 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 > \ell, x_1 + x_2 - z_4 > \ell, \bar{x}_1 + x_2 - z_4 > \ell, s(b) \leq 2\ell\}$. Combining the three final sets above yields the set M_2.

(N_1) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 > 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. This is clearly a subset of C_2 in $B_4^{(j)}$. Then since $z_4 > 0$, we repeatedly apply the second $\tilde{f}_2$ condition to obtain new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_4 \geq 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. Then if $z_4 = 0$, we repeatedly apply the first condition for $\tilde{f}_2$. This yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_4 < 0, z_1 + z_3 > 0, z_1 + z_2 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. Combining these two sets yields the set N_1.

(N_2) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < -1, z_4 > 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. This set is contained in C_3 in $B_4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_4 \geq 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 >$

$\ell, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) < 2\ell\}$. Then if $z_4 = 0$, we repeatedly apply the first condition for $\tilde{f}_2$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_4 < 0, z_1 + z_3 > 0, z_1 + z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) < 2\ell\}$. Combining these two sets yields the set N_2.

(N_3) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_3 \geq 0, z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 \geq \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell\}$. This set is contained in C_4 in $B_4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 \geq 0, z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 + \bar{x}_2 \geq \ell, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) \leq 2\ell\}$. Then if $z_4 = 0$, we repeatedly apply the first condition for $\tilde{f}_2$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 > 0, z_4 < 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 + \bar{x}_2 \geq \ell, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) \leq 2\ell\}$. Combining these two sets yields the set N_3.

(N_4) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. This set is contained in C_5 in $B_4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. This set is precisely N_4.

(N_5) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. This set is a subset of C_5 in $B_4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_2 \geq 0, z_3 < 0, z_4 = 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. Then $z_4 = 0$ so repeatedly applying the first condition for $\tilde{f}_2$ yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. This set is precisely N_5.

(N_6) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_2 < 0, z_3 < 0, z_4 > 0, z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 > \ell, \bar{x}_1 + \bar{x}_3 < \ell, x_1 + x_2 - x_3 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. This is a subset of C_9 in $B - 4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 = 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) + \bar{x}_1 + \bar{x}_2 > \ell, \bar{x}_1 + \bar{x}_3 < \ell, x_1 + x_2 - x_3 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) < 2\ell\}$. Next consider $C_{10} = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 < 0, z_4 > 0, z_1 + z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 = \ell, s(b) \leq 2\ell\}$ in $B_4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 < 0, z_4 \geq 0, z_1 + z_2 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, x_2 - x_3 + \bar{x}_3 + 2\bar{x}_2 + \bar{x}_1 = \ell, s(b) \leq 2\ell\}$. Combining the two final sets above yields the set N_6.

(N_7) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_2 < 0, z_3 < 0, z_4 > 0, z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 > \ell, \bar{x}_1 + \bar{x}_3 < \ell, x_1 + x_2 - x_3 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. This is a subset of C_9 in $B - 4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 = 0, z_2 < 0, z_4 = 0, z_1 + z_3 < 0, z_2 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) + \bar{x}_1 + \bar{x}_2 > \ell, \bar{x}_1 + \bar{x}_3 < \ell, x_1 + x_2 - x_3 + \bar{x}_3 + 2\bar{x}_2 = \ell, x_2 + 2\bar{x}_2 + \bar{x}_1 - z_4 = \ell, s(b) < 2\ell\}$. Then since $z_4 = 0$, we repeatedly apply the first condition for $\tilde{f}_2$ which yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 = 0, z_2 < 0, z_4 < 0, z_1 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 >$

$\ell, t(b) + \bar{x}_1 + \bar{x}_2 > \ell, \bar{x}_1 + \bar{x}_3 < \ell, x_1 + x_2 - x_3 + \bar{x}_3 + 2\bar{x}_2 < \ell, x_2 + 2\bar{x}_2 + \bar{x}_1 - z_4 \leq \ell, s(b) < 2\ell\}$. This set is precisely N_7.

(N_8) Consider $C_{10} = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 < 0, z_4 > 0, z_1 + z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 = \ell, s(b) \leq 2\ell\}$ in $B_4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 < 0, z_4 = 0, z_1 + z_2 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, x_2 - x_3 + \bar{x}_3 + 2\bar{x}_2 + \bar{x}_1 = \ell, s(b) \leq 2\ell\}$. Then $z_4 = 0$, so repeatedly applying the first condition for $\tilde{f}_2$ yields elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + z_2 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, x_2 + 2\bar{x}_2 + \bar{x}_1 - z_4 = \ell, s(b) \leq 2\ell\}$. This set is precisely N_8.

(N_9) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_4 > 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. This is set is clearly contained in C_{11} in $B_4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 \geq 0, z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. Then if $z_4 = 0$, the first condition of $\tilde{f}_2$ is satisfied. Repeatedly applying $\tilde{f}_2$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 > 0, z_4 < 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. Combining these two sets yields the set N_9.

(N_{10}) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < -1, z_2 < 0, z_3 \geq 0, z_4 > 0, z_2 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell\}$. This is clearly a subset of C_{12} in $B_4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < -1, z_3 \geq 0, z_4 \geq 0, z_2 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) \leq 2\ell\}$. Then if $z_4 = 0$, the first condition of $\tilde{f}_2$ is satisfied. Repeatedly applying $\tilde{f}_2$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < -1, z_3 > 0, z_4 < 0, z_3 + 3z_4 \geq 0, z_2 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) \leq 2\ell\}$. Combining these two sets yields the set N_{10}.

(N_{11}) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_3 \geq 0, z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, s(b) = 2\ell\}$. This set is clearly contained in D_1 in $B_5^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_3 \geq 0, z_4 \geq 0, t(b) - z_1 + \bar{x}_2 > 0, s(b) = 2\ell\}$. The condition $t(b) - z_1 + \bar{x}_2 > 0$ is obtained from the conditions $z_1 + 2z_2 + z_3 + 3z_4 < 0$ and $s(b) = 2\ell$. Then if $z_4 = 0$, the first condition for $\tilde{f}_2$ is satisfied. Repeatedly applying $\tilde{f}_2$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_3 > 0, z_4 < 0, z_3 + 3z_4 \geq 0, t(b) - z_1 + \bar{x}_2 > 0, s(b) = 2\ell\}$. Combining the two sets above yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, t(b) - z_1 + \bar{x}_2 > 0, s(b) = 2\ell\}$. Next, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 > 0, z_2 < 0, z_3 < 0, z_4 > 0, z_1 + z_3 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, s(b) = 2\ell\}$. This set is clearly contained in D_1 in $B_5^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 \geq 0, t(b) - z_1 + \bar{x}_2 > 0, s(b) = 2\ell\}$. Then if $z_4 = 0$, the first condition for $\tilde{f}_2$ is satisfied. Repeatedly applying $\tilde{f}_2$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 > 0, z_4 < 0, z_1 + z_3 > 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 \geq 0, t(b) - z_1 + \bar{x}_2 > 0, s(b) = 2\ell\}$. Combining the two sets above yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 > 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, t(b) - z_1 + \bar{x}_2 > 0, s(b) = 2\ell\}$. Finally, we combine the two final sets above to obtain the

set N_{11}.

(N_{12}) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < -1, z_3 \geq 0, z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) < 2\ell\}$. This set is clearly contained in D_2 in $B_5^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_3 \geq 0, z_4 \geq 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) < 2\ell\}$. The condition $t(b) - z_1 + \bar{x}_2 > 0$ is obtained from the conditions $z_1 + 2z_2 + z_3 + 3z_4 < 0$ and $s(b) < 2\ell$. Then if $z_4 = 0$, the first condition for $\tilde{f}_2$ is satisfied. Repeatedly applying $\tilde{f}_2$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_3 > 0, z_4 < 0, z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) < 2\ell\}$. Combining the two sets above yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) < 2\ell\}$. Next, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 > 0, z_2 < -1, z_3 < 0, z_4 > 0, z_1 + z_3 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) < 2\ell\}$. This set is again contained in D_2 in $B_5^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 \geq 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_3 > \ell, s(b) < 2\ell\}$. Then if $z_4 = 0$, the first condition for $\tilde{f}_2$ is satisfied. Repeatedly applying $\tilde{f}_2$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 > 0, z_4 < 0, z_1 + z_3 > 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 \geq 0, t(b) + z_1 + \bar{x}_2 > \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) < 2\ell\}$. Combining the two sets above yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 > 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) < 2\ell\}$. Finally, we combine the two final sets above to obtain the set N_{12}.

(N_{13}) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_3 \geq 0, z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 \geq \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) \leq 2\ell\}$. This is a subset of D_4 in $B_5^{(j)}$. Since $z_4 > 0$, we can repeatedly apply the second condition for $\tilde{f}_2$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 \geq 0, z_4 \geq 0, t(b) + \bar{x}_1 + \bar{x}_2 \geq \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) \leq 2\ell\}$. The condition $t(b) - z_1 + \bar{x}_2 > 0$ is obtained from the conditions $z_1 + 2z_2 + z_3 + 3z_4 < 0$ and $s(b) \leq 2\ell$. Then if $z_4 = 0$, we repeatedly apply the first condition for $\tilde{f}_2$ to obtain new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 > 0, z_4 < 0, z_3 + 3z_4 \geq 0, t(b) + \bar{x}_1 + \bar{x}_2 \geq \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) \leq 2\ell\}$. Combining these two sets yields the set N_{13}.

(N_{14}) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 < 0, z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 > \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) \leq 2\ell\}$. This is a subset of D_5 in $B_5^{(j)}$. Since $z_4 > 0$, we can repeatedly apply the second condition for $\tilde{f}_2$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) + \bar{x}_1 + \bar{x}_2 > \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) \leq 2\ell\}$. The condition $t(b) - z_1 + \bar{x}_2 > 0$ is obtained from the conditions $z_1 + 2z_2 + z_3 + 3z_4 < 0$ and $s(b) \leq 2\ell$. Next, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 < 0, z_3 < 0, z_2 + 3z_4 > 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 \leq \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) \leq 2\ell\}$. This is a subset of D_9 in $B_5^{(j)}$. Since $z_4 > 0$, we can repeatedly apply the second condition for $\tilde{f}_2$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_3 < 0, z_4 \geq 0, z_2 + 3z_4 > 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) + \bar{x}_1 + \bar{x}_2 \leq \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) \leq 2\ell\}$. Combining the two final sets above yields the set N_{14}.

(N_{15}) Consider $D_7 = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 > \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 > \ell, s(b) \leq 2\ell\}$ in $B_5^{(j)}$. Since $z_4 > 0$, we can repeatedly apply the second condition for $\tilde{f}_2$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 + 2\bar{x}_2 > \ell, x_2 - x_3 + \bar{x}_3 + 2\bar{x}_2 + \bar{x}_1 > \ell, s(b) \leq 2\ell\}$. This set is precisely N_{16}.

(N_{16}) Consider $D_7 = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 > \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 > \ell, s(b) \leq 2\ell\}$ in $B_5^{(j)}$. Since $z_4 > 0$, we can repeatedly apply the second condition for $\tilde{f}_2$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_3 < 0, z_4 = 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 + 2\bar{x}_2 > \ell, x_2 - x_3 + \bar{x}_3 + 2\bar{x}_2 + \bar{x}_1 > \ell, s(b) \leq 2\ell\}$. Then $z_4 = 0$, so we repeatedly apply the first condition for $\tilde{f}_2$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, x_1 + x_2 + 2\bar{x}_2 - z_4 > \ell, x_2 + 2\bar{x}_2 + \bar{x}_1 - z_4 > \ell, s(b) \leq 2\ell\}$. This set is precisely N_{17}.

(N_{17}) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < -1, z_2 < 0, z_3 \geq 0, z_4 > 0, z_2 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 < \ell, \bar{x}_1 + \bar{x}_3 > \ell, s(b) \leq 2\ell\}$. This set is contained in D_8 in $B_5^{(j)}$. Since $z_4 > 0$, we can repeatedly apply the second condition for $\tilde{f}_2$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < -1, z_3 \geq 0, z_4 \geq 0, z_2 + z_3 + 3z_4 > 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) + \bar{x}_1 + \bar{x}_2 < \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) \leq 2\ell\}$. The condition $t(b) - z_1 + \bar{x}_2 > 0$ is obtained from the conditions $z_1 + 2z_2 + z_3 + 3z_4 < 0$ and $s(b) \leq 2\ell$. Then if $z_4 = 0$, the first condition for $\tilde{f}_2$ is satisfied. Repeatedly applying $\tilde{f}_2$ yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < -1, z_3 > 0, z_4 < 0, z_3 + 3z_4 \geq 0, z_2 + z_3 + 3z_4 > 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) + \bar{x}_1 + \bar{x}_2 < \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) \leq 2\ell\}$. Combining the two sets above yields the set N_{18}.

Therefore, by repeatedly applying $\tilde{f}_2$ to elements in $B_5^{(j)}$, we obtain $B_6^{(j)} = B_5^{(j)} \cup L \cup M_1 \cup M_2 \cup N_1 \cup N_2 \cup \cdots \cup N_{17}$. Now, we work to show that $B_6^{(j)} = F_1 \cup F_2 \cup F_3 \cup F_4$.

(F_1) We first combine C_4 in $B_4^{(j)}$ with N_3 in $B_6^{(j)}$ to obtain the set $F_{1,1} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 + \bar{x}_2 \geq \ell, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) \leq 2\ell\}$. Then, combine C_{11} in $B_4^{(j)}$ with N_9 in $B_6^{(j)}$ yielding the set $F_{1,2} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. Next, combine C_{12} in $B_4^{(j)}$ with N_{10} in $B_6^{(j)}$ to obtain the set $F_{1,3} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_3 \geq 0, z_3 + 3z_4 \geq 0, z_2 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) \leq 2\ell\}$. Finally, combine D_8 in $B_5^{(j)}$ with N_{17} in $B_6^{(j)}$ yielding the set $F_{1,4} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_2 + z_3 + 3z_4 > 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) + \bar{x}_1 + \bar{x}_2 < \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) \leq 2\ell\}$. Combining the sets $N_{13} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, t(b) + \bar{x}_1 + \bar{x}_2 \geq \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) \leq 2\ell\}$, $F_{1,1} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + \bar{x}_1 + \bar{x}_2 \geq \ell, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) \leq 2\ell\}$, $F_{1,2} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$, $F_{1,3} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_3 \geq 0, z_3 + 3z_4 \geq 0, z_2 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) \leq 2\ell\}$, and $F_{1,4} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_2 + z_3 + 3z_4 > 0, t(b) + x_1 + \bar{x}_2 >$

$\ell, t(b) + \bar{x}_1 + \bar{x}_2 < \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) \leq 2\ell\}$, we obtain the set F_1 as desired.

(F_2) First combine C_6, C_7 and C_8 in $B_4^{(j)}$ with D_6 in $B_5^{(j)}$ to obtain the set $F_{2,1} = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 \leq 0, z_3 < 0, z_4 = 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, t(b) + \bar{x}_1 \geq \ell, x_1 + x_2 \geq \ell, \bar{x}_1 + x_2 \geq \ell, s(b) \leq 2\ell\}$. Then, we combine C_9 and C_{10} in $B_4^{(j)}$ with D_5, D_7, and D_9 in $B_5^{(j)}$ yielding the set $F_{2,2} = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 \geq \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 \geq \ell, s(b) \leq 2\ell\}$. Next, combine C_5 in $B_4^{(j)}$ with N_4 in $B_6^{(j)}$ to obtain the set $F_{2,3} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. Combining the sets $N_6 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \leq 0, z_3 < 0, z_4 \geq 0, z_1 + z_2 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, s(b) + z_2 + 3z_4 = \ell + x_1, s(b) \leq 2\ell\}$, $N_{14} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) \leq 2\ell\}$, $N_{15} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) \leq 2\ell\}$ in $B_6^{(j)}$ with $F_{2,1}$, $F_{2,2}$, and $F_{2,3}$ given above, we obtain the set F_2 as desired.

(F_3) We first can combine the disjoint sets in $B_3^{(j)}$ to obtain $B_3^{(j)} = \{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. Next, combine C_3 in $B_4^{(j)}$ with N_2 in $B_6^{(j)}$ to obtain the set $F_{3,1} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) < 2\ell\}$. Then, combine D_2 in $B_5^{(j)}$ with N_{12} in $B_6^{(j)}$ yielding $F_{3,2} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) < 2\ell\}$. We combine C_1 in $B_4^{(j)}$ with D_1 in $B_5^{(j)}$ and N_{11} in $B_6^{(j)}$ to obtain the set $F_{3,3} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, t(b) - z_1 + \bar{x}_2 > 0, s(b) = 2\ell\}$. Finally, combine $B_3^{(j)}$, C_2 in $B_4^{(j)}$, and L and N_1 in $B_6^{(j)}$ yielding the set $F_{3,4} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) \leq 2\ell\}$. Combining the sets $F_{3,1} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) < 2\ell\}$, $F_{3,2} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 > \ell, t(b) - z_1 + \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, s(b) < 2\ell\}$, $F_{3,3} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, t(b) - z_1 + \bar{x}_2 > 0, s(b) = 2\ell\}$, and $F_{3,4} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) \leq 2\ell\}$, we obtain the set F_3 as desired.

(F_4) We first combine M_1, M_2, N_8, and N_{16} in $B_6^{(j)}$ to obtain the set $F_{4,1} = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 \geq \ell, s(b) - z_1 + z_2 \geq \ell + x_1, s(b) + z_2 \geq \ell + x_1, s(b) \leq 2\ell\}$. Then, combining $F_{4,1}$ above with $N_5 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$ from $B_6^{(j)}$ yields the set F_4 as desired.

Thus, $B_6^{(j)} = F_1 \cup F_2 \cup F_3 \cup F_4$ and we have now constructed the set $B_a^{(j)}$ for $0 \leq a \leq 6$. □

Remark 5.2.2. By the explicit construction of $B_a^{(j)}$ in Lemma 5.2.1, we can see that $B_6^{(j)} \neq \mathcal{B}$. For example, the element $(0, 0, 0, 0, 0, 0)$ is not contained in $B_6^{(j)}$, but is contained in $\mathcal{B}$.

Therefore, in Theorem 3.4.3, $\kappa \neq 1$.

We now show conditions (2) and (3) in Theorem 3.4.3 are satisfied with the sequence $\{i_a^{(j)} \mid j \geq 1, 1 \leq a \leq 6\}$ given above for $\lambda = \ell\Lambda_1$.

By direct calculations of the simple reflections r_i $(i = 0, 1, 2)$ on Λ_1, we obtain the following Lemma, which we will use along with Proposition 3.4.4 to provide justification for condition (3) in Theorem 3.4.3.

Lemma 5.2.3. *Let $k \in \mathbb{Z}_{>0}$ and $k = 6(j-1) + a$ for $1 \leq a \leq 6$. Then $w^{(k)}\Lambda_1 = \Lambda_1 - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2$, where*

$$m_0 = \begin{cases} 2j^2 - 3j + 1 & \text{if } a = 1, 2, 3 \\ 2j^2 + j & \text{if } a = 4, 5, 6 \end{cases}$$

$$m_1 = \begin{cases} 4j^2 - 4j + 1 & \text{if } a = 1, 2 \\ 4j^2 - 2j + 1 & \text{if } a = 3, 4 \\ 4j^2 & \text{if } a = 5, 6 \end{cases}$$

$$m_2 = \begin{cases} 2j^2 - 3j + 1 & \text{if } a = 1 \\ 2j^2 - j & \text{if } a = 2, 3, 4, 5 \\ 2j^2 + j & \text{if } a = 6 \end{cases}$$

Proposition 5.2.4. Conditions (2) and (3) in Theorem 3.4.3 are satisfied with the sequence $\{i_a^{(j)} \mid j \geq 1, 1 \leq a \leq 6\}$ and the Weyl group sequence $\{w^{(k)}\}_{k \geq 0}$ given above for $\lambda = \ell\Lambda_1$.

Proof. In order to show condition (2), we consider $\langle \ell\Lambda_1, h_{i_a^{(j)}} \rangle$. Observe that $\langle \ell\Lambda_1, h_{i_a^{(j)}} \rangle = 0 \leq \varepsilon_{i_a^{(j)}}(b)$ for all $b \in B_{a-1}^{(j)}$, $a = 2, 4, 6$. Also, $\langle \ell\Lambda_1, h_{i_a^{(j)}} \rangle = \ell$ for $a = 1, 3, 5$. We must consider $\varepsilon_1(b)$ for all $b \in B_0^{(j)}$, $B_2^{(j)}$, and $B_4^{(j)}$. First, note that for all $b \in B_0^{(j)}$ or $B_2^{(j)}$, we have $\bar{x}_1 = \ell$. Hence, $\varepsilon_1(b) = \bar{x}_1 + (\bar{x}_3 - \bar{x}_2 + (x_2 - x_3)_+)_+ = l + (\bar{x}_3 - \bar{x}_2 + (x_2 - x_3)_+)_+ \geq \ell$ for all $b \in B_0^{(j)}$ and $B_2^{(j)}$. Now, consider $b \in B_4^{(j)}$. Note that for all $b \in B_4^{(j)}$, $\bar{x}_2 = 0$, so $\varepsilon_1(b) = \bar{x}_1 + (\bar{x}_3 + (x_2 - x_3)_+)_+ = \bar{x}_1 + \bar{x}_3 + (x_2 - x_3)_+$. We consider each subset in $B_4^{(j)}$.

- If $b \in B_3^{(j)}$, then $\bar{x}_1 = \ell$ and $\varepsilon_1(b) \geq \ell$ by the same reasoning as above.
- If $b \in C_1$, then $z_1 + z_3 + 3z_4 = 0$ and $s(b) = 2\ell$. The condition $z_1 + z_3 + 3z_4 = 0$ implies $\bar{x}_1 + \frac{3}{2}\bar{x}_3 = x_1 + x_2 + \frac{1}{2}x_3 = s(b) - \bar{x}_1 - \frac{1}{2}\bar{x}_3$. So, $2\bar{x}_1 + 2\bar{x}_3 = s(b) = 2\ell$, implying $\bar{x}_1 + \bar{x}_3 = \ell$. Thus, $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + (x_2 - x_3)_+ \geq \ell$.
- If $b \in C_2$, then $z_1 + z_2 + z_3 + 3z_4 \geq 0$ and $t(b) + x_1 = \ell$. Since $z_1 + z_2 + z_3 + 3z_4 \geq 0$, we have $\bar{x}_1 + \frac{1}{2}\bar{x}_3 \geq x_1 + x_2 + \frac{1}{2}x_3$. This means $\bar{x}_1 + \bar{x}_3 \geq x_1 + x_2 + \frac{1}{2}x_3 + \frac{1}{2}\bar{x}_3 = t(b) + x_1 = \ell$. Thus, $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + (x_2 - x_3)_+ \geq \ell$.

- If $b \in C_3, C_4$, then $\bar{x}_1 + \bar{x}_3 = \ell$. This implies $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + (x_2 - x_3)_+ \geq \ell$.

- If $b \in C_5$, then $z_3 < 0$, $z_1 + z_2 + 3z_4 \geq 0$, and $t(b) + x_1 = \ell$. Since $z_3 < 0$, $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + x_2 - x_3$. Then $z_1 + z_2 + 3z_4 \geq 0$ implies $\bar{x}_1 + \frac{1}{2}\bar{x}_3 > \frac{3}{2}x_3 + x_1$. Thus, $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + x_2 - x_3 > \frac{1}{2}\bar{x}_3 + \frac{1}{2}x_3 + x_2 + x_1 = t(b) + x_1 = \ell$.

- If $b \in C_6$, then $z_1 \geq 0$, $z_4 = 0$, $z_1 + z_3 < 0$, and $x_1 + x_2 = \ell$. Since $z_1 \geq 0$ and $z_1 + z_3 < 0$, we have $z_3 < 0$. So, $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + x_2 - x_3$. Then, $z_4 = 0$, implying $x_3 = \bar{x}_3$. This makes $\varepsilon_1(b) = \bar{x}_1 + x_2 \geq x_1 + x_2 = \ell$ since $z_1 \geq 0$.

- If $b \in C_7$, then $z_3 < 0$, $z_4 = 0$, and $\bar{x}_1 + x_2 = \ell$. Since $z_3 < 0$, $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + x_2 - x_3$. Then $z_4 = 0$ implies $\bar{x}_3 = x_3$ and hence, $\varepsilon_1(b) = \bar{x}_1 + x_2 = \ell$.

- If $b \in C_8$, then $z_2 = 0$, $z_3 < 0$, $z_4 = 0$, and $t(b) + \bar{x}_1 = \ell$. Since $z_3 < 0$, $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + x_2 - x_3$. Then $z_4 = 0$ implies $\bar{x}_3 = x_3$, so $\varepsilon_1(b) = \bar{x}_1 + x_2$. Also, $z_2 = 0$, so $\bar{x}_3 = x_3 = 0$. This implies $\ell = t(b) + \bar{x}_1 = x_2 + \bar{x}_1$. Thus, $\varepsilon_1(b) = \ell$.

- If $b \in C_9$, then $z_1 \geq 0$, $z_1 + z_3 < 0$, and $x_1 + x_2 - x_3 + \bar{x}_3 = \ell$. The conditions $z_1 \geq 0$ and $z_1 + z_3 < 0$ imply that $z_3 < 0$. So, $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + x_2 - x_3 \geq x_1 + x_2 - x_3 + \bar{x}_3 = \ell$.

- If $b \in C_{10}$, then $z_3 < 0$ and $x_2 - x_3 + \bar{x}_3 + \bar{x}_1 = \ell$. Since $z_3 < 0$, we have $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + x_2 - x_3 = \ell$.

- If $b \in C_{11}$, then $z_1 + z_2 + z_3 + 3z_4 \geq 0$ and $t(b) + x_1 = \ell$. Since $z_1 + z_2 + z_3 + 3z_4 \geq 0$, we have $\bar{x}_1 + \frac{1}{2}\bar{x}_3 \geq x_1 + x_2 + \frac{1}{2}x_3$. This means $\bar{x}_1 + \bar{x}_3 \geq x_1 + x_2 + \frac{1}{2}x_3 + \frac{1}{2}\bar{x}_3 = t(b) + x_1 = \ell$. Thus, $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + (x_2 - x_3)_+ \geq \ell$.

- If $b \in C_{12}$, then $z_3 \geq 0$ and $\bar{x}_1 + \bar{x}_3 = \ell$. Since $z_3 \geq 0$, we have $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 = \ell$.

Therefore, $\varepsilon_1(b) \geq \ell$ for all $b \in B_0^{(j)}$, $B_2^{(j)}$, and $B_4^{(j)}$. This implies $\langle \ell\Lambda_1, h_{i_a^{(j)}} \rangle \leq \varepsilon_{i_a^{(j)}}(b)$ for all $b \in B_{a-1}^{(j)}$, and condition (2) holds.

To prove condition (3), we use Lemma 5.2.3. For $k = 6(j-1) + a$, $j \geq 1$, $1 \leq a \leq 6$, we have

$$\langle w^{(k)}\Lambda_1, h_{i_{a+1}^{(j)}} \rangle = \begin{cases} 2j - 1 & \text{if } a = 1, 4 \\ 2j & \text{if } a = 2, 5 \\ 4j - 1 & \text{if } a = 3 \\ 4j + 1 & \text{if } a = 6. \end{cases}$$

Hence, for positive j, $\langle w^{(k)}\Lambda_1, h_{i_{a+1}^{(j)}} \rangle$ is greater than zero. By Proposition 3.4.4, this implies $w^{(k+1)} = r_{i_{a+1}}^{(j)} w^{(k)} \succ w^{(k)}$. Thus, the sequence of Weyl group elements, $\{w^{(k)}\}_{k \geq 0}$, is increasing

with respect to the Bruhat order, satisfying condition (3).

$\square$

We have shown above that $B_6^{(j)} \neq \mathcal{B}$ and hence, $\kappa \neq 1$ in Theorem 3.4.3. Due to the previous conjecture that $\kappa \leq 2$, we have conjectured that $\kappa = 2$ for $\lambda = \ell\Lambda_1$. We provide the conjecture and evidence toward the conjecture below.

Conjecture 5.2.5. *For $\lambda = \ell\Lambda_1$, $\ell \geq 1$ and the given perfect crystal $\mathcal{B} = \mathcal{B}_{2\ell}$ for the quantum affine algebra, $U_q(D_4^{(3)})$, with $d = 6$ and the sequence $\{i_a^{(j)}\}$ given above, we conjecture that conditions (1), (2), and (3) in Theorem 3.4.3 hold with $\kappa = 2$. This would imply that path realizations of the corresponding Demazure crystals $B_{w^{(k)}}(\ell\Lambda_1)$ for $U_q(D_4^{(3)})$ have tensor product-like structures.*

We provide justification for this conjecture by giving the sets $B_a^{(j+1,j)}$, $0 \leq a \leq 6$ for $\ell = 1, 2$ and show that $B_6^{(j+1,j)} = B_6^{(j)} \otimes \mathcal{B}$ in these two cases. This implies condition (1) in Theorem 3.4.3 holds with $\kappa = 2$.

5.2.1 $\ell = 1$ Case

First, by Lemma 5.2.1, we have the following sets $B_a^{(j)}$ for $0 \leq a \leq 6$.

$$
\begin{aligned}
B_0^{(j)} &= \{(1,0,0,0,0,1)\} \\
B_1^{(j)} &= B_0^{(j)} \cup \{(0,1,0,0,0,1)\} \\
B_2^{(j)} &= B_1^{(j)} \cup \{(0,0,2,0,0,1)\} \\
B_3^{(j)} &= B_2^{(j)} \cup \{(0,0,0,2,0,1),(0,0,1,1,0,1)\} \\
B_4^{(j)} &= B_3^{(j)} \cup \{(0,0,0,2,0,0),(0,0,1,1,0,0),(0,0,3,1,0,0),(0,1,0,0,0,0),(0,1,1,1,0,0), \\
&\qquad (1,0,1,1,0,0),(1,1,0,0,0,0)\} \\
B_5^{(j)} &= B_4^{(j)} \cup \{(0,0,0,4,0,0),(0,0,1,3,0,0),(0,0,2,2,0,0),(0,1,0,2,0,0),(0,2,0,0,0,0), \\
&\qquad (1,0,0,2,0,0)\} \\
B_6^{(j)} &= B_5^{(j)} \cup \{(0,0,0,0,1,0),(0,0,0,0,1,1),(0,0,0,0,2,0),(0,0,0,2,1,0),(0,0,1,1,1,0), \\
&\qquad (0,0,2,0,0,0),(0,0,2,0,1,0),(0,0,4,0,0,0),(0,1,0,0,1,0),(0,1,2,0,0,0), \\
&\qquad (1,0,0,0,1,0),(1,0,2,0,0,0)\}.
\end{aligned}
$$

Note the elements in $\mathcal{B}$ that are missing in $B_6^{(j)}$ are $(0,0,0,0,0,0)$, $(0,0,0,0,0,1)$, $(0,0,0,0,0,2)$, $(1,0,0,0,0,0)$, and $(2,0,0,0,0,0)$. Thus, $B_6^{(j)} \neq \mathcal{B}$, and hence $\kappa \neq 1$.

We now construct the sets $B_a^{(j+1,j)}$, $0 \leq a \leq 6$ explicitly.

By definition, $B_0^{(j+1,j)} = B_0^{(j+1)} \otimes B_6^{(j)}$. Since $B_0^{(j)} = \{b_j\}$ and $b_j = (1,0,0,0,0,1)$ for all $j \geq 1$, we have $B_0^{(j+1)} = \{(1,0,0,0,0,1)\}$. So,

$$B_0^{(j+1,j)} = \{(1,0,0,0,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \in B_6^{(j)}\}.$$

Then we use the recursive formula,

$$B_a^{(j+1,j)} = \bigcup_{k \geq 0} \tilde{f}^k_{i_a^{(j+1)}} B_{a-1}^{(j+1,j)} \setminus \{0\},$$

to determine $B_a^{(j+1,j)}$ for $1 \leq a \leq 6$. We use the tensor product rule defined in Theorem 3.1.7 along with the operators, $\varepsilon_i,\ \varphi_i$, $i = 0,1,2$ defined in Chapter 4 to do this.

We define the following conditions for $(x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \in \mathcal{B}$,

$$\begin{cases} (P_1)\ \bar{x}_1' + \bar{x}_2' + \frac{1}{2}(\bar{x}_3' + x_3') > 0, \\ (P_2)\ \bar{x}_1' + \bar{x}_2' + \frac{1}{2}\bar{x}_3' > 0, \\ (P_3)\ \bar{x}_1' + \bar{x}_2' > 0, \\ (P_4)\ z_3' \geq 0, \\ (P_5)\ z_3' \geq -1. \end{cases}$$

Thus, the sets $B_a^{(j+1,j)}$ for $\ell = 1$ are constructed as follows,

$$
\begin{aligned}
B_1^{(j+1,j)} &= B_0^{(j+1,j)} \cup \{(1,0,0,0,0,1) \otimes (0,0,0,0,0,2)\} \\
&\qquad \cup \{(0,1,0,0,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (P_1) \text{ holds}\}, \\
B_2^{(j+1,j)} &= B_1^{(j+1,j)} \cup \{(0,0,2,0,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (P_2) \text{ holds}\}, \\
B_3^{(j+1,j)} &= B_2^{(j+1,j)} \cup \{(0,0,1,1,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (P_2) \text{ holds}\} \\
&\qquad \cup \{(0,0,0,2,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (P_3) \text{ holds}\}, \\
B_4^{(j+1,j)} &= \{(1,0,0,0,0,1)\} \otimes \mathcal{B} \cup \{(0,1,0,0,0,1)\} \otimes \mathcal{B} \cup \{(0,1,1,1,0,0)\} \otimes \mathcal{B} \\
&\quad \cup \{(0,0,2,0,0,1)\} \otimes \mathcal{B} \cup \{(0,0,3,1,0,0)\} \otimes \mathcal{B} \cup \{(0,0,1,1,0,0)\} \otimes \mathcal{B} \\
&\quad \cup \{(1,0,1,1,0,0)\} \otimes \mathcal{B} \cup \{(0,0,0,2,0,0)\} \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (P_3) \text{ holds}\} \\
&\quad \cup \{(x_1, x_2, 0,0,0,0) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid x_2 = 1,\ s(b) \le 2,\ (P_3) \text{ holds}\} \\
&\quad \cup \{(0,1,0,0,0,0) \otimes (x_1', x_2', x_3', 0,0,0) \mid (P_4) \text{ holds}\} \\
&\quad \cup \{(1,1,0,0,0,0) \otimes (x_1', x_2', x_3', 0,0,0) \mid (P_5) \text{ holds}\} \\
&\quad \cup \{(1,1,0,0,0,0) \otimes (x_1', x_2', x_3', 1,0,0) \mid s(b)' \le 2\} \\
&\quad \cup \{(1,1,0,0,0,0) \otimes (0,0,0,2,0,0)\}, \\
B_5^{(j+1,j)} &= \{(x_1, x_2, x_3, \bar{x}_3, 0, 1) \mid s(b) = 2\} \otimes \mathcal{B} \\
&\quad \cup \{(x_1, x_2, 0,0,0,0) \mid x_2 > 0,\ s(b) \le 2\} \otimes \mathcal{B} \\
&\quad \cup \{(x_1, x_2, x_3, \bar{x}_3, 0, 0) \mid z_2 < 0,\ s(b) \le 2\} \otimes \mathcal{B}, \\
B_6^{(j+1,j)} &= B_5^{(j+1,j)} \cup \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0,\ s(b) \le 2\} \otimes \mathcal{B} \\
&\qquad \cup \{(x_1, x_2, x_3, 0,0,0) \mid z_4 < 0,\ s(b) \le 2\} \otimes \mathcal{B}.
\end{aligned}
$$

We see that by combining the sets in $B_6^{(j+1,j)}$, we obtain $B_6^{(j+1,j)} = B_6^{(j)} \otimes \mathcal{B}$. Thus, condition (1) in Theorem 3.4.3 holds for $\lambda = \Lambda_1$ with $\kappa = 2$. This implies Conjecture 5.2.5 holds when $\ell = 1$, and path realizations of the corresponding Demazure crystals $B_{w^{(k)}}(\Lambda_1)$ for $U_q(D_4^{(3)})$ have tensor product-like structures.

5.2.2 $\ell = 2$ Case

First, by Lemma 5.2.1, we have the following sets $B_a^{(j)}$ for $0 \le a \le 6$.

$$
\begin{aligned}
B_0^{(j)} &= \{(2,0,0,0,0,2)\} \\
B_1^{(j)} &= B_0^{(j)} \cup \{(x_1,x_2,0,0,0,2) \mid z_1 > 0, z_1 + z_3 = 0, s(b) = 4\} \\
B_2^{(j)} &= B_1^{(j)} \cup \{(x_1,x_2,x_3,0,0,2) \mid z_1 > 0, z_4 < 0, z_1 + z_3 + 3z_4 = 0, s(b) = 4\} \\
B_3^{(j)} &= B_2^{(j)} \cup \{(x_1,x_2,x_3,\bar{x}_3,0,2) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, \\
&\qquad\qquad s(b) = 4\} \\
B_4^{(j)} &= B_3^{(j)} \cup C_1 \cup C_2 \cup \cdots \cup C_{12} \\
B_5^{(j)} &= B_4^{(j)} \cup D_1 \cup D_2 \cup \cdots \cup D_9 \\
B_6^{(j)} &= F_1 \cup F_2 \cup F_3 \cup F_4,
\end{aligned}
$$

where $C_n = \{(x_1,x_2,x_3,\bar{x}_3,0,\bar{x}_1) \mid (P_n) \text{ holds}\}$ for $1 \le n \le 12$, $D_n = \{(x_1,x_2,x_3,\bar{x}_3,0,\bar{x}_1) \mid (Q_n) \text{ holds}\}$ for $1 \le n \le 9$, and $F_n = \{(x_1,x_2,x_3,\bar{x}_3,\bar{x}_2,\bar{x}_1) \mid (R_n) \text{ holds}\}$ for $1 \le n \le 4$ with $\ell = 2$.

We note that $B_6^{(j)} \neq \mathcal{B}$ since $(0,0,0,0,0,0) \notin B_6^{(j)}$, and hence $\kappa \neq 1$. We now construct the sets $B_a^{(j+1,j)}$, $0 \le a \le 6$ explicitly.

By definition, $B_0^{(j+1,j)} = B_0^{(j+1)} \otimes B_6^{(j)}$. Since $B_0^{(j)} = \{b_j\}$ and $b_j = (2,0,0,0,0,2)$ for all $j \ge 1$, we have $B_0^{(j+1)} = \{(2,0,0,0,0,2)\}$. So,

$$B_0^{(j+1,j)} = \{(2,0,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \in B_6^{(j)}\}.$$

Then we use the recursive formula,

$$B_a^{(j+1,j)} = \bigcup_{k \ge 0} \tilde{f}^k_{i_a^{(j+1)}} B_{a-1}^{(j+1,j)} \setminus \{0\},$$

to determine $B_a^{(j+1,j)}$ for $1 \le a \le 6$. We use the tensor product rule defined in Theorem 3.1.7 along with the operators, ε_i, φ_i, $i = 0,1,2$ defined in Chapter 4 to do this.

We define the following conditions for $(x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \in \mathcal{B}$,

$$\begin{cases} (P_1)\ \bar{x}_1' + \bar{x}_2' > 0, \\ (P_2)\ \frac{1}{2}(x_3' + \bar{x}_3') \geq 2, \\ (P_3)\ \bar{x}_1' + \bar{x}_2' + \frac{1}{2}\bar{x}_3' > 2, \\ (P_4)\ x_2' > 0, x_3' > 0, \frac{1}{2}(x_3' + \bar{x}_3') \leq 2, \\ (P_5)\ \frac{1}{2}(x_3' + \bar{x}_3') \leq 1, s(b)' \geq 2, \\ (P_6)\ \bar{x}_1' \leq 1, \bar{x}_1' + \bar{x}_2' \leq 2, x_1' + x_2' > 0, \\ (P_7)\ \frac{1}{2}(x_3' + \bar{x}_3') = 1, x_1' + x_2' > 0, \end{cases}$$

$$\begin{cases} (Q_1)\ \bar{x}_1' + 2\bar{x}_2' \geq 2, \\ (Q_2)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, \bar{x}_1' + \bar{x}_2' > 0, \\ (Q_3)\ \frac{1}{2}(x_3' + \bar{x}_3') \geq 2, \\ (Q_4)\ \frac{1}{2}(x_3' + \bar{x}_3') = 1, x_2' > 0, x_3' > 0, \\ (Q_5)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, \bar{x}_1' \geq 2, \\ (Q_6)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, \bar{x}_2' > 0, \\ (Q_7)\ \bar{x}_3' \geq 2, \frac{1}{2}(x_3' + \bar{x}_3') \geq 3, \\ (Q_8)\ \bar{x}_3' \geq 2, x_2' \leq 1, x_3' \geq 1, \frac{1}{2}(x_3' + \bar{x}_3') = 2, \\ (Q_9)\ 1 \leq \bar{x}_2' \leq 2, \bar{x}_3' \geq 1, \bar{x}_2' + \frac{1}{2}(x_3' + \bar{x}_3') \geq 3. \end{cases}$$

$$\begin{cases} (R_1)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, 2\bar{x}_1' + \bar{x}_2' > 1, \\ (R_2)\ \bar{x}_2' \leq 1, \bar{x}_3' \geq 1, \bar{x}_2 + \frac{1}{2}(x_3' + \bar{x}_3') \geq 2, \\ (R_3)\ \bar{x}_3' \geq 1, \frac{1}{2}(x_3' + \bar{x}_3') \geq 2, \\ (R_4)\ x_2' > 0, x_3' \geq 1, \bar{x}_3' \leq 1, \frac{1}{2}(x_3' + \bar{x}_3') = 1, \\ (R_5)\ 1 \leq x_2', x_3' \leq 2, 2 \leq \bar{x}_3' \leq 3, \frac{1}{2}(x_3' + \bar{x}_3') = 2, \\ (R_6)\ \bar{x}_1' + \bar{x}_3' \geq 2, \frac{1}{2}(x_3' + \bar{x}_3') + \bar{x}_1' \geq 3, \\ (R_7)\ x_3' \leq 1, \bar{x}_3' \geq 1, \frac{1}{2}(x_3' + \bar{x}_3') = 1. \end{cases}$$

Thus, the sets $B_a^{(j+1,j)}$ for $\ell = 1$ are constructed as follows,

$$
\begin{aligned}
B_1^{(j+1,j)} = B_0^{(j+1,j)} &\cup \{(2,0,0,0,0,2) \otimes \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid \bar{x}_1' \geq 3,\ s(b)' = 4\} \\
&\cup \{(0,2,0,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (P_1) \text{ holds},\ 1 \leq s(b)' \leq 4\} \\
&\cup \{(0,2,0,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (P_2) \text{ holds}\} \\
&\cup \{(1,1,0,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (P_3) \text{ holds}\} \\
&\cup \{(1,1,0,0,0,2) \otimes (x_1', x_2', 0,0,0,2) \mid (P_1) \text{ holds}\} \\
&\cup \{(1,1,0,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0,0) \mid (P_4) \text{ holds}\} \\
&\cup \{(1,1,0,0,0,2) \otimes (x_1', 0,4,0,0,0) \mid x_1' \leq 1\} \\
&\cup \{(1,1,0,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 1,0) \mid (P_5) \text{ holds}\} \\
&\cup \{(1,1,0,0,0,2) \otimes (x_1', x_2', 0,0, \bar{x}_2', \bar{x}_1') \mid (P_6) \text{ holds}\} \\
&\cup \{(1,1,0,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0,1) \mid (P_7) \text{ holds},
\end{aligned}
$$

$$
\begin{aligned}
B_2^{(j+1,j)} = B_1^{(j+1,j)} &\cup \{(0,0,4,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (Q_1) \text{ holds}\} \\
&\cup \{(0,0,4,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0,1) \mid 2 \leq s(b)' \leq 4\} \\
&\cup \{(0,0,4,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0,0) \mid \bar{x}_3' \geq 2\} \\
&\cup \{(0,0,4,0,0,2) \otimes (x_1', x_2', x_3', 1,0,0) \mid t(b)' \geq 2,\ s(b)' \leq 4\} \\
&\cup \{(0,1,2,0,0,2) \otimes (0,0,0,0, \bar{x}_2', \bar{x}_1') \mid 2 \leq s(b)' \leq 4\} \\
&\cup \{(0,1,2,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (Q_2) \text{ holds}\} \\
&\cup \{(0,1,2,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0,0) \mid (Q_3) \text{ holds}\} \\
&\cup \{(0,1,2,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0,0) \mid (Q_4) \text{ holds}\} \\
&\cup \{(1,0,2,0,0,2) \otimes (0,0,0,0, \bar{x}_2', \bar{x}_1') \mid 3 \leq s(b)' \leq 4\} \\
&\cup \{(1,0,2,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (Q_5) \text{ holds}\} \\
&\cup \{(1,0,2,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', 1) \mid (Q_6) \text{ holds}\} \\
&\cup \{(1,0,2,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0,1) \mid \bar{x}_3' > 0,\ 3 \leq s(b)' \leq 4\} \\
&\cup \{(1,0,2,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0,0) \mid (Q_7) \text{ holds}\} \\
&\cup \{(1,0,2,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0,0) \mid (Q_8) \text{ holds}\} \\
&\cup \{(1,0,2,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', 0) \mid (Q_9) \text{ holds}\} \\
&\cup \{(1,0,2,0,0,2) \otimes (x_1', x_2', 1,1,1,0) \mid x_2' > 0,\ s(b)' \leq 4\} \\
&\cup \{(1,0,2,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 3,0) \mid s(b)' = 4\}
\end{aligned}
$$

$$
\begin{aligned}
B_3^{(j+1,j)} = B_2^{(j+1,j)} &\cup \{(0,0,0,4,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid \bar{x}_1' + \bar{x}_2' \geq 2,\ s(b)' \leq 4\} \\
&\cup \{(0,0,0,4,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 1, 0) \mid \bar{x}_3' > 0,\ s(b)' \leq 4\} \\
&\cup \{(0,0,x_3,\bar{x}_3,0,2) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid z_3 > 0,\ z_2 < 0,\ s(b) = 4, \\
&\qquad 2 \leq s(b)' \leq 4\} \\
&\cup \{(0,0,x_3,\bar{x}_3,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid z_3 > 0,\ z_2 < 0,\ s(b) = 4, \\
&\qquad (R_1) \text{ holds}\} \\
&\cup \{(0,0,x_3,\bar{x}_3,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', 0) \mid z_3 > 0,\ z_2 < 0,\ s(b) = 4, \\
&\qquad (R_2) \text{ holds}\} \\
&\cup \{(0,1,0,2,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid \bar{x}_1' + \bar{x}_2' \geq 3,\ s(b)' \leq 4\} \\
&\cup \{(0,1,0,2,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid \bar{x}_1' + \bar{x}_2' = 2,\ \bar{x}_3' \geq 1,\ s(b)' \leq 4\} \\
&\cup \{(x_1,x_2,x_3,\bar{x}_3,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 1, 0) \mid x_1 + x_2 = 1, z_2 + z_3 + 3z_4 = 0, \\
&\qquad s(b) = 4,\ (R_3) \text{ holds}\} \\
&\cup \{(x_1,x_2,0,2,0,2) \otimes (x_1', x_2', 1, 1, 1, 0) \mid x_1 + x_2 = 1,\ \bar{x}_2' \geq 1, 3 \leq s(b)' \leq 4\} \\
&\cup \{(0,1,1,1,0,2) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid 2 \leq s(b)' \leq 4\} \\
&\cup \{(0,1,1,1,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (Q_2) \text{ holds}\} \\
&\cup \{(0,1,1,1,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, 0) \mid (Q_3) \text{ holds}\} \\
&\cup \{(0,1,1,1,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, 0) \mid (R_4) \text{ holds}\} \\
&\cup \{(1,0,x_3,\bar{x}_3,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid z_2 < 0, s(b) = 4, 3 \leq s(b)' \leq 4\} \\
&\cup \{(1,0,x_3,\bar{x}_3,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid z_2 < 0,\ s(b) = 4,\ (Q_9) \text{ holds}\} \\
&\cup \{(1,0,1,1,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, 0) \mid (R_5) \text{ holds}\} \\
&\cup \{(1,0,1,1,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, \bar{x}_1') \mid (R_6) \text{ holds} \\
&\cup \{(1,0,1,1,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, 1) \mid (R_7) \text{ holds}\},
\end{aligned}
$$

$$
\begin{aligned}
B_4^{(j+1,j)} &= \{(0,0,x_3,\bar{x}_3,0,\bar{x}_1) \mid \bar{x}_3 \geq 3, z_4 > 0, s(b) = 2\} \otimes \mathcal{B} \\
&\cup \{(x_1,x_2,x_3,2,0,0) \mid x_2 \leq 2, z_3 \geq 0, 2 \leq s(b) \leq 4\} \otimes \mathcal{B} \\
&\cup \{(x_1,2,x_3,\bar{x}_3,0,0) \mid x_3 = \bar{x}_3, x_1 \leq 1, s(b) \leq 4\} \otimes \mathcal{B} \\
&\cup \{(x_1,x_2,x_3,2,0,1) \mid s(b) = 3\} \otimes \mathcal{B} \\
&\cup \{(x_1,x_2,x_3,1,0,1) \mid x_1 \leq 1, 3 \leq s(b) \leq 4\} \otimes \mathcal{B} \\
&\cup \{(x_1,x_2,0,0,0,1) \mid 1 \leq x_2 \leq 2, 3 \leq s(b) \leq 4\} \otimes \mathcal{B} \\
&\cup \{(x_1,x_2,x_3,\bar{x}_3,0,2) \mid \bar{x}_3 \leq 2, s(b) = 4\} \otimes \mathcal{B}, \\
B_5^{(j+1,j)} &= \{(x_1,x_2,x_3,\bar{x}_3,0,0) \mid \bar{x}_3 \geq 2, 2 \leq s(b) \leq 4\} \otimes \mathcal{B} \\
&\cup \{(x_1,x_2,x_3,\bar{x}_3,0,0) \mid x_3 = \bar{x}_3 \leq 1, x_2 \geq 2, s(b) \leq 4\} \otimes \mathcal{B} \\
&\cup \{(x_1,x_2,x_3,\bar{x}_3,0,1) \mid \bar{x}_3 > 0, 3 \leq s(b) \leq 4\} \otimes \mathcal{B} \\
&\cup \{(x_1,x_2,0,0,0,1) \mid x_2 > 0, 3 \leq s(b) \leq 4\} \otimes \mathcal{B} \\
&\cup \{(x_1,x_2,x_3,\bar{x}_3,0,2) \mid \bar{x}_3 \leq 2, s(b) = 4\} \otimes \mathcal{B}, \\
B_6^{(j+1,j)} &= B_6^{(j)} \otimes \mathcal{B}.
\end{aligned}
$$

From above, we see that condition (1) in Theorem 3.4.3 holds for $\lambda = 2\Lambda_1$ with $\kappa = 2$. This implies Conjecture 5.2.5 holds when $\ell = 2$, and path realizations of the corresponding Demazure crystals $B_{w^{(k)}}(2\Lambda_1)$ for $U_q(D_4^{(3)})$ have tensor product-like structures.

5.3 Case 3: $\lambda = \ell(\Lambda_0 + \Lambda_1)$

We begin by considering $\lambda = \ell(\Lambda_0 + \Lambda_1)$ and the irreducible highest weight $U_q'(D_4^{(3)})$-module with highest weight $\ell(\Lambda_0 + \Lambda_1)$. Note that $\ell(\Lambda_0 + \Lambda_1)(c) = 3\ell$, so we will use the associated perfect crystal $\mathcal{B} = \mathcal{B}_{3\ell}$. The $\ell(\Lambda_0 + \Lambda_1)$-minimal element in $\mathcal{B}$ is $\bar{b} = (\ell, 0, 0, 0, 0, \ell)$. Also, $\lambda_j = \lambda = \ell(\Lambda_0 + \Lambda_1)$ for $j \geq 1$, and hence $b_j = \bar{b}$. Thus, the ground-state path is $\mathbf{p}_\lambda = \cdots \otimes \bar{b} \otimes \bar{b} \otimes \bar{b}$.

Set $d = 6$ and define the sequence $\{i_a^{(j)} \mid j \geq 1, 1 \leq a \leq 6\} \subset \{0, 1, 2\}$ as follows,

$$
i_1^{(j)} = i_3^{(j)} = i_5^{(j)} = 1, \ i_2^{(j)} = i_6^{(j)} = 2, \ i_4^{(j)} = 0.
$$

By the action of $\tilde{f}_i$ on $\mathcal{B}$, we have

$$\begin{aligned}
b_0^{(j)} &= (\ell, 0, 0, 0, 0, \ell), \\
b_1^{(j)} &= \tilde{f}_1^{\max}(b_0^{(j)}) = (0, \ell, 0, 0, 0, \ell), \\
b_2^{(j)} &= \tilde{f}_2^{\max}(b_1^{(j)}) = (0, 0, 2\ell, 0, 0, \ell), \\
b_3^{(j)} &= \tilde{f}_1^{\max}(b_2^{(j)}) = (0, 0, 0, 2\ell, 0, \ell), \\
b_4^{(j)} &= \tilde{f}_0^{\max}(b_3^{(j)}) = (\ell, \ell, 0, 0, 0, 0), \\
b_5^{(j)} &= \tilde{f}_1^{\max}(b_4^{(j)}) = (0, 2\ell, 0, 0, 0, 0), \\
b_6^{(j)} &= \tilde{f}_2^{\max}(b_5^{(j)}) = (0, 0, 4\ell, 0, 0, 0).
\end{aligned}$$

Using the notation from Theorem 3.4.3, yields the following Lemma.

Lemma 5.3.1. *Define conditions P_m, Q_n, and R_s for $b \in \mathcal{B}$ as follows,*

$$\begin{cases}
(P_1): z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \\
\qquad t(b) + x_1 = \ell, s(b) < 2\ell; \\
(P_2): z_1 < 0, z_3 < 0, z_4 \geq 0, z_1 + z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 < \ell, \\
\qquad t(b) + x_1 > \ell, t(b) + \bar{x}_1 < 2\ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 = \ell, s(b) \leq 3\ell; \\
(P_3): z_1 \geq 0, z_2 < 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, \\
\qquad z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell; \\
(P_4): z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \\
\qquad \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 3\ell; \\
(P_5): z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, \\
\qquad s(b) < 2\ell; \\
(P_6): z_1 \geq 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 < \ell, t(b) + x_1 > \ell, \\
\qquad x_1 + x_2 - x_3 + \bar{x}_3 = \ell, s(b) < 2\ell; \\
(P_7): z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \\
\qquad t(b) + x_1 = \ell, s(b) < 2\ell;
\end{cases}$$

$$\begin{cases} (Q_1): z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 \leq \ell, \\ \qquad t(b) + x_1 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 > \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 > \ell, s(b) \leq 3\ell; \\ (Q_2): z_1 \geq 0, z_2 < 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 > \ell, \\ \qquad t(b) + x_1 > \ell, s(b) \leq 2\ell; \\ (Q_3): z_1 < 0, z_3 \geq 0, z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 > \ell, t(b) + x_1 > \ell, \\ \qquad s(b) + z_1 \leq 2\ell, s(b) \leq 3\ell; \\ (Q_4): z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 > \ell, t(b) + x_1 > \ell, \\ \qquad s(b) + z_1 + z_3 \leq 2\ell, s(b) \leq 3\ell; \end{cases}$$

$$\begin{cases} (R_1): z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 \geq \ell, s(b) - z_1 \geq \ell + x_1, s(b) \leq 2\ell; \\ (R_2): z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 \geq \ell, s(b) - z_1 \geq \ell + x_1, s(b) + z_1 \leq 2\ell, \\ \qquad s(b) \leq 3\ell; \\ (R_3): z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, s(b) - z_1 \geq \ell + x_1, s(b) - z_1 + z_2 + 3z_4 \geq \ell + x_1, \\ \qquad s(b) + z_2 + 3z_4 \geq \ell + x_1, s(b) + z_1 + z_3 \leq 2\ell, s(b) \leq 3\ell; \\ (R_4): z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, s(b) - z_1 \geq \ell + x_1, s(b) - z_1 + z_2 \geq \ell + x_1, \\ \qquad s(b) + z_2 \geq \ell + x_1, s(b) + z_1 + z_3 + 3z_4 \leq 2\ell, s(b) \leq 3\ell. \end{cases}$$

Then, the subsets $\{B_a^{(j)} \mid j \geq 1, 0 \leq a \leq 6\}$ *of* $\mathcal{B}$ *are given by,*

$$\begin{aligned} B_0^{(j)} &= \{(\ell, 0, 0, 0, 0, \ell)\} \\ B_1^{(j)} &= B_0^{(j)} \cup \{(x_1, x_2, 0, 0, 0, \ell) \mid z_1 > 0, z_1 + z_3 = 0, s(b) = 2\ell\} \\ B_2^{(j)} &= B_1^{(j)} \cup \{(x_1, x_2, x_3, 0, 0, \ell) \mid z_1 > 0, z_4 < 0, z_1 + z_3 + 3z_4 = 0, s(b) = 2\ell\} \\ B_3^{(j)} &= B_2^{(j)} \cup \{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, \\ &\qquad\qquad s(b) = 2\ell\} \\ B_4^{(j)} &= B_3^{(j)} \cup C_1 \cup C_2 \cup \cdots \cup C_7 \\ B_5^{(j)} &= B_4^{(j)} \cup D_1 \cup D_2 \cup D_3 \cup D_4 \\ B_6^{(j)} &= F_1 \cup F_2 \cup F_3 \cup F_4, \end{aligned}$$

where $C_n = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid (P_n)$ *holds*$\}$ *for* $1 \leq n \leq 7$, $D_n = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid (Q_n)$ *holds*$\}$ *for* $1 \leq n \leq 4$, *and* $F_n = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid (R_n)$ *holds*$\}$ *for* $1 \leq n \leq 4$.

Proof. By definition, $B_0^{(j)} = \{b_j\}$, where $b_j = (\ell, 0, 0, 0, 0, \ell)$ when $\lambda = \ell(\Lambda_0 + \Lambda_1)$. Then,

$B_a^{(j)} = \bigcup_{k\geq 0} \tilde{f}_{i_a^{(j)}}^k B_{a-1}^{(j)} \setminus \{0\}$. We first note that the sequence $\{i_a^{(j)} \mid j \geq 1, 1 \leq a \leq 6\}$ for $\lambda = \ell(\Lambda_0 + \Lambda_1)$ is the same as that for $\lambda = \ell\Lambda_1$. Also, we have $b_j = (\ell, 0, 0, 0, 0, \ell)$ when $\lambda = \ell\Lambda_1$. Thus, the sets $B_0^{(j)}$, $B_1^{(j)}$, $B_2^{(j)}$, and $B_3^{(j)}$ are the same as those for $\lambda = \ell\Lambda_1$ since $\tilde{f}_1$ and $\tilde{f}_2$ do not change $s(b) = 2\ell$.

To obtain $B_4^{(j)}$, we apply $\tilde{f}_0$ repeatedly to elements in $B_3^{(j)}$.

(C_1) Consider elements from $B_3^{(j)}$ of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_3 \leq 0, z_4 > 0, z_1 + z_3 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, t(b) + x_1 = \ell, s(b) = 2\ell\}$. This is a subset of $B_3^{(j)}$ since $z_1 + z_3 > 0$, $z_4 > 0$, and $z_1 + 2z_2 + z_3 + 3z_4 = 0$ imply that $z_2 < 0$. This implies $z_1 + z_2 + z_3 + 3z_4 = -z_2 > 0$; hence, these elements satisfy condition ($\mathcal{F}_6$). Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_3 \leq 0, z_4 > 0, z_1 + z_3 = 0, z_2 + 3z_4 > 0, z_1 + z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0 t(b) + x_1 = \ell, s(b) < 2\ell\}$. Then, condition ($\mathcal{F}_4$) is satisfied and we repeatedly apply $\tilde{f}_0$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Now, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_3 > 0, z_4 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, t(b) + x_1 = \ell, s(b) = 2\ell\}$. This is a subset of $B_3^{(j)}$ by the same reasoning as above. We also have that $z_1 + z_2 + z_3 + 3z_4 = -z_2 > 0$, so these elements satisfy condition ($\mathcal{F}_6$). Repeatedly applying $\tilde{f}_0$ to these elements yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_3 > 0, z_4 > 0, z_1 + z_3 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Then the conditions for ($\mathcal{F}_5$) are satisfied. By repeatedly applying $\tilde{f}_0$, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 = 0, z_4 > 0, z_2 + 3z_4 > 0, z_1 + z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Finally, the conditions for ($\mathcal{F}_4$) are satisfied, so we repeatedly apply $\tilde{f}_0$ to this set to yield $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Combining the two final sets above yields the set C_1. We note that this set is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 \neq 0$.

(C_2) Consider elements from $B_3^{(j)}$ of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_3 < 0, z_4 > 0, z_1 + z_3 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) = 2\ell\}$. This is a subset of $B_3^{(j)}$ since $z_1 + z_3 > 0$, $z_4 > 0$, and $z_1 + 2z_2 + z_3 + 3z_4 = 0$ imply that $z_2 < 0$. This implies $z_1 + z_2 + z_3 + 3z_4 = -z_2 > 0$; hence, these elements satisfy condition ($\mathcal{F}_6$). Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 > 0, z_1 + z_3 = 0, z_1 + z_2 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Then, the conditions for ($\mathcal{F}_4$) are satisfied and we repeatedly apply $\tilde{f}_0$. This yields elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 =$

$0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Since $z_4 \geq 0$ and $z_2 + 3z_4 = 0$, we have $z_2 \leq 0$ and conditions for $(\mathcal{F}_2)$ are satisfied. Repeatedly applying $\tilde{f}_0$ yields $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 < \ell, t(b) + x_1 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. Finally, the conditions for $(\mathcal{F}_1)$ are satisfied, so we repeatedly apply $\tilde{f}_0$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 < 0, z_4 \geq 0, z_2 + 3z_4 < 0, z_1 + z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 < \ell, t(b) + x_1 > \ell, t(b) + \bar{x}_1 < 2\ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 = \ell, s(b) \leq 3\ell\}$. Now, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_3 > 0, z_4 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, t(b) + x_1 = \ell, s(b) = 2\ell\}$. This is a subset of $B_3^{(j)}$ by the same reasoning as above. We also have that $z_1 + z_2 + z_3 + 3z_4 = -z_2 > 0$, so these elements satisfy condition $(\mathcal{F}_6)$. Repeatedly applying $\tilde{f}_0$ to these elements yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_3 > 0, z_4 > 0, z_1 + z_3 > 0, z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Then the conditions for $(\mathcal{F}_5)$ are satisfied. By repeatedly applying $\tilde{f}_0$, we obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 = 0, z_4 > 0, z_2 + 3z_4 > 0, z_1 + z_2 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. The conditions for $(\mathcal{F}_4)$ are satisfied, so we repeatedly apply $\tilde{f}_0$ to this set to yield $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 < 0, z_4 \geq 0, z_2 + 3z_4 \geq 0, z_1 + z_2 + 3z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Finally, the conditions for $(\mathcal{F}_1)$ are satisfied. Repeatedly applying $\tilde{f}_0$ yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 < 0, z_4 \geq 0, z_2 + 3z_4 > 0, z_1 + z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 < \ell, t(b) + x_1 > \ell, t(b) + \bar{x}_1 < \ell < 2\ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 = \ell, s(b) \leq 3\ell\}$. Combining the two final sets above yields the set C_2. We note that this set is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 \neq 0$.

(C_3) Consider elements from $B_3^{(j)}$ of the form $\{(x_1, x_2, 0, 0, 0, \ell) \mid z_1 > 0, z_1 + z_3 = 0, s(b) = 2\ell$. This set is a subset of $B_3^{(j)}$ since it is a subset of $B_1^{(j)}$. The conditions for $(\mathcal{F}_2)$ are satisfied, so we repeatedly apply $\tilde{f}_0$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 = 0, z_1 + z_3 = 0, \bar{x}_1 + \bar{x}_3 = \ell, s(b) = 2\ell\}$. Next, consider elements from $B_3^{(j)}$ of the form $\{(x_1, x_2, x_3, 0, 0, \ell) \mid z_1 > 0, z_4 < 0, z_1 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is a subset of $B_2^{(j)}$, so it is contained in $B_3^{(j)}$. Again, conditions for $(\mathcal{F}_2)$ are met, so repeatedly applying $\tilde{f}_0$ yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 < 0, z_1 + z_3 + 3z_4 = 0, \bar{x}_1 + \bar{x}_3 = \ell, s(b) = 2\ell\}$. Finally, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_3 < 0, z_1 + z_3 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This is clearly a subset of $B_3^{(j)}$. Also, $z_1 + z_2 + z_3 + 3z_4 = -z_2 > 0$; hence, these elements satisfy condition $(\mathcal{F}_6)$. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 < 0, z_3 < 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Then conditions for $(\mathcal{F}_2)$ are satisfied, so we repeatedly apply $\tilde{f}_0$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 >$

$0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. Combining the three final sets above yields the set C_3. We note that this set is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 \neq 0$.

(C_4) First, consider $B_0^{(j)} = \{(\ell, 0, 0, 0, 0, \ell)\}$. The conditions for $(\mathcal{F}_1)$ are satisfied, so we repeatedly applying $\tilde{f}_0$ to obtain the set $\{(x_1, 0, 0, 0, 0, \ell) \mid z_1 < 0, 2\ell < s(b) \leq 3\ell\}$. Now, consider elements from $B_3^{(j)}$ of the form $\{(x_1, x_2, 0, 0, 0, \ell) \mid z_1 > 0, z_1 + z_3 = 0, s(b) = 2\ell$. This set is a subset of $B_3^{(j)}$ since it is a subset of $B_1^{(j)}$. The conditions for $(\mathcal{F}_2)$ are satisfied, so we repeatedly apply $\tilde{f}_0$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_2 < 0, z_4 = 0, z_1 + z_3 = 0, \bar{x}_1 + \bar{x}_3 = \ell, s(b) = 2\ell\}$. Then, the conditions for $(\mathcal{F}_1)$ are satisfied. Repeatedly applying $\tilde{f}_0$ yields $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_3 = 0, z_4 = 0, \bar{x}_1 + \bar{x}_3 = \ell, 2\ell < s(b) \leq 3\ell\}$. Next, consider elements from $B_3^{(j)}$ of the form $\{(x_1, x_2, x_3, 0, 0, \ell) \mid z_1 > 0, z_4 < 0, z_1 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This set is a subset of $B_2^{(j)}$, so it is contained in $B_3^{(j)}$. Again, conditions for $(\mathcal{F}_2)$ are met, so repeatedly applying $\tilde{f}_0$ yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_2 < 0, z_4 < 0, z_1 + z_3 + 3z_4 = 0, \bar{x}_1 + \bar{x}_3 = \ell, s(b) = 2\ell\}$. Then conditions for $(\mathcal{F}_1)$ are satisfied. We repeatedly apply $\tilde{f}_0$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_4 < 0, z_3 + 3z_4 = 0, \bar{x}_1 + \bar{x}_3 = \ell, 2\ell < s(b) \leq 3\ell\}$. Now, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_3 < 0, z_1 + z_3 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This is clearly a subset of $B_3^{(j)}$. Also, $z_1 + z_2 + z_3 + 3z_4 = -z_2 > 0$; hence, these elements satisfy condition $(\mathcal{F}_6)$. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 < 0, z_3 < 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Then conditions for $(\mathcal{F}_2)$ are satisfied, so we repeatedly apply $\tilde{f}_0$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_2 < 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. We see that conditions for $(\mathcal{F}_1)$ are satisfied, and we repeatedly apply $\tilde{f}_0$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_2 < 0, z_3 \geq 0, z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 3\ell\}$. Finally, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_3 < 0, z_1 + z_3 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This is a subset of $B_3^{(j)}$ since by the same reasoning as above. We have $z_2 > 0$, so these elements satisfy condition $(\mathcal{F}_6)$. Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_2 < 0, z_3 > 0, z_1 + z_2 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. The conditions for $(\mathcal{F}_5)$ are met, so repeatedly applying $\tilde{f}_0$ yields elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Then, conditions for $(\mathcal{F}_1)$ hold and we repeatedly apply $\tilde{f}_0$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_2 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 3\ell\}$.

Combining the five final sets above yields the set C_4. We note that this set is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 \neq 0$.

(C_5) Consider elements from $B_3^{(j)}$ of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_1+2z_2+z_3+3z_4 = 0, s(b) = 2\ell\}$. This is clearly a subset of $B_3^{(j)}$. Also, $z_1+z_2+z_3+3z_4 = -z_2 > 0$; hence, these elements satisfy condition ($\mathcal{F}_6$). Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. This set is precisely C_5 and is not contained in $B_3^{(j)}$ because $z_1 + 2z_2 + z_3 + 3z_4 \neq 0$.

(C_6) Consider elements from $B_3^{(j)}$ of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_3 < 0, z_4 > 0, z_1 + z_3 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This is clearly a subset of $B_3^{(j)}$. Also, $z_1 + z_2 + z_3 + 3z_4 = -z_2 > 0$; hence, these elements satisfy condition ($\mathcal{F}_6$). Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_3 < 0, z_4 > 0, z_1 + z_3 = 0, z_2 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0t(b) + x_1 = \ell, s(b) < 2\ell\}$. Then, condition ($\mathcal{F}_4$) is satisfied and we repeatedly apply $\tilde{f}_0$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 = 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Finally, conditions for ($\mathcal{F}_2$) are met, so we repeatedly apply $\tilde{f}_0$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 < \ell, t(b) + x_1 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. This set is precisely C_6 and is not contained in $B_3^{(j)}$ because $z_1 + 2z_2 + z_3 + 3z_4 \neq 0$.

(C_7) Consider elements from $B_3^{(j)}$ of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_3 > 0, z_2 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This is clearly a subset of $B_3^{(j)}$. Also, $z_1 + z_2 + z_3 + 3z_4 = -z_2 > 0$; hence, these elements satisfy condition ($\mathcal{F}_6$). Repeatedly applying $\tilde{f}_0$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 = 0, z_2 < 0, z_3 > 0, z_2+z_3+3z_4 > 0, z_1+z_2+z_3+3z_4 \geq 0, z_1+2z_2+z_3+3z_4 < 0, t(b)+x_1 = \ell, s(b) < 2\ell\}$. Then conditions for ($\mathcal{F}_5$) are satisfied and we repeatedly apply $\tilde{f}_0$ to yield the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_3+3z_4 \geq 0, z_1+z_2+z_3+3z_4 \geq 0, z_1+2z_2+z_3+3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. This set is precisely C_7 and is not contained in $B_3^{(j)}$ because $z_1 + 2z_2 + z_3 + 3z_4 \neq 0$.

Therefore, by repeatedly applying $\tilde{f}_0$ to elements in $B_3^{(j)}$, we obtain $B_4^{(j)} = B_3^{(j)} \cup C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7$.

To obtain $B_5^{(j)}$, we apply $\tilde{f}_1$ repeatedly to elements in $B_4^{(j)}$. Since $B_3^{(j)}$ contains elements obtained by applying $\tilde{f}_1$ repeatedly to $B_2^{(j)}$, we only need to examine the action of $\tilde{f}_1$ on elements in $B_4^{(j)} \setminus B_3^{(j)} = C_1 \cup \cdots \cup C_7$.

(D_1) Consider the set $C_2 = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 < 0, z_4 \geq 0, z_1 + z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 < \ell, t(b) + x_1 > \ell, t(b) + \bar{x}_1 < 2\ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 = \ell, s(b) \leq 3\ell\}$. Since $z_2 \leq 0$ and $z_3 < 0$, we have $(z_2)_+ \leq -z_3$. So, applying the first condition of $\tilde{f}_1$ yields elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 < \ell, t(b) + x_1 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 > \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 > \ell, s(b) \leq 3\ell\}$. Also, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 = 0, z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 3\ell\}$. This set is clearly contained in C_4 in $B_4^{(j)}$. Then $z_2 \leq 0$ and $z_3 = 0$, so $(z_2)_+ \leq -z_3$. Repeatedly applying the first condition of $\tilde{f}_1$ yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 3\ell\}$. Combining the two final sets above yields the set D_1. We note that D_1 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$. Also, D_1 is not contained in C_1, C_5, or C_7 because $t(b) + x_1 > \ell$. Then, D_1 is not contained in C_2 or C_6 since $x_1 + x_2 - x_3 + \bar{x}_3 > \ell$ and $x_2 - x_3 + \bar{x}_3 + \bar{x}_1 > \ell$. D_1 is not contained in C_3 since $z_1 + z_3 < 0$, and finally, D_1 is not contained in C_4 since $z_3 < 0$. Therefore, D_1 is not contained in $B_4^{(j)}$.

(D_2) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_3 > 0, z_1 + z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 = \ell, t(b) + x_1 > \ell, s(b) \leq 2\ell\}$. This set is clearly contained in C_3 in $B_4^{(j)}$. We have $z_2 < 0 < z_3$, so we repeatedly apply the second condition for $\tilde{f}_1$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_3 \geq 0, z_1 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 > \ell, t(b) + x_1 > \ell, s(b) \leq 2\ell\}$. Then if $z_3 = 0$, we have $(z_2)_+ \leq -z_3$ and we repeatedly apply the first condtion for $\tilde{f}_1$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 < 0, z_3 < 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 > \ell, t(b) + x_1 > \ell, s(b) \leq 2\ell\}$. Combining these two sets yields the set D_2. We note that D_2 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$. Also, D_2 is not contained in C_1, C_5, or C_7 because $t(b) + x_1 > \ell$. Then, D_2 is not contained in C_2, C_3, C_4 or C_6 since $\bar{x}_1 + \bar{x}_3 > \ell$. Therefore, D_2 is not contained in $B_4^{(j)}$.

(D_3) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 > 0, z_3 + z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 = \ell, t(b) + x_1 > \ell s(b) \leq 3\ell\}$. This set is clearly contained in C_4 in $B_4^{(j)}$. Then $z_2 \leq 0 < z_3$, so we repeatedly apply the second condition of $\tilde{f}_1$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 > \ell, t(b) + x_1 > \ell, s(b) + z_1 \leq 2\ell, s(b) \leq 3\ell\}$. Note that $s(b) + z_1 \leq 2\ell$ since $z_3 + 3z_4 \geq 0$ and $\bar{x}_1 + \bar{x}_3 = \ell$ initially. This set is precisely D_3. We note that D_3 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$. Also, D_2 is not contained in C_1, C_5, or C_7 because $t(b) + x_1 > \ell$. Then, D_3 is not contained in C_2, C_3, C_4 or C_6 since $\bar{x}_1 + \bar{x}_3 > \ell$. Therefore, D_3 is not contained in $B_4^{(j)}$.

(D_4) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 > 0, z_3 + z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 = \ell, t(b) + x_1 > \ell, s(b) \leq 3\ell\}$. This set is clearly contained in C_4 in $B_4^{(j)}$. Then $z_2 \leq 0 < z_3$, so we repeatedly apply the second condition of $\tilde{f}_1$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 > \ell, t(b) + x_1 > \ell, s(b) + z_1 \leq 2\ell, s(b) \leq 3\ell\}$. Note that $s(b) + z_1 \leq 2\ell$ since $z_3 + 3z_4 \geq 0$ and $\bar{x}_1 + \bar{x}_3 = \ell$ initially. Then if $z_3 = 0$, we have $(z_2)_+ \leq -z_3$ and we repeatedly apply the first condition of $\tilde{f}_1$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 > \ell, t(b) + x_1 > \ell, s(b) + z_1 + z_3 \leq 2\ell, s(b) \leq 3\ell\}$. We obtain the condition $s(b) + z_1 + z_3 \leq 2\ell$ from $s(b) + z_1 \leq 2\ell$ and $z_3 = 0$. This set is precisely D_4. We note that D_4 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$. Also, D_4 is not contained in C_1, C_5, or C_7 because $t(b) + x_1 > \ell$. Finally, D_4 is not contained in C_2, C_3, C_4, or C_6 since $\bar{x}_1 + \bar{x}_3 > \ell$. Thus, D_4 is not contained in $B_4^{(j)}$.

Therefore, by repeatedly applying $\tilde{f}_1$ to elements in $B_4^{(j)}$, we obtain $B_5^{(j)} = B_4^{(j)} \cup D_1 \cup D_2 \cup D_3 \cup D_4$.

To obtain $B_6^{(j)}$, we apply $\tilde{f}_2$ repeatedly to elements in $B_5^{(j)}$. Since $B_2^{(j)}$ contains elements obtained by applying $\tilde{f}_2$ repeatedly to B_1, we only need to examine the action of $\tilde{f}_1$ on elements in $B_5^{(j)} \setminus B_2^{(j)}$. We first show that $B_5^{(j)} = B_4^{(j)} \cup G_1 \cup G_2 \cup G_3 \cup G_4$, where $G_n = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid (S_n) \text{ holds}\}$. The conditions for S_n, $1 \leq n \leq 4$ are,

$$\begin{cases} (S_1)\ z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, s(b) - z_1 \geq \ell + x_1, s(b) - z_1 + z_2 \geq \ell + x_1, \\ \qquad s(b) + z_2 \geq \ell + x_1, s(b) + z_1 + z_3 + 3z_4 \leq 2\ell, s(b) \leq 3\ell; \\ (S_2)\ \bar{x}_2 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, s(b) - z_1 \geq \ell + x_1, s(b) - z_1 + z_2 + 3z_4 \geq \ell + x_1, \\ \qquad s(b) + z_2 + 3z_4 \geq \ell + x_1, s(b) + z_1 + z_3 \leq 2\ell, s(b) \leq 3\ell; \\ (S_3)\ \bar{x}_2 > 0, z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 \geq \ell, s(b) - z_1 \geq \ell + x_1, \\ \qquad s(b) \leq 2\ell; \\ (S_4)\ \bar{x}_2 > 0, z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 \geq \ell, s(b) - z_1 \geq \ell + x_1, s(b) + z_1 \leq 2\ell, \\ \qquad s(b) \leq 3\ell. \end{cases}$$

(G_1) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 \geq 0, z_3 < 0, z_4 = 0, z_1 + z_3 < 0, z_1 + z_2 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. This set is clearly a subset of C_1 in $B_4^{(j)}$. Then, since $z_4 = 0$, we repeatedly apply the first condition of $\tilde{f}_2$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_2 = 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Next consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + z_2 + 3z_4 \geq$

$0, z_1+2z_2+z_3+3z_4 < 0, t(b)+x_1 = \ell, s(b) < 2\ell\}$. Again, this is a subset of C_1 in $B_4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_2 < 0, z_2 + 3z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. Then if $z_4 = 0$, the first condition for $\tilde{f}_2$ is satisfied. Repeatedly applying $\tilde{f}_2$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. Combining these two sets yields the set $T_1 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$.

Now consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 < 0, z_4 = 0, z_1 + z_2 < 0, z_1+2z_2+z_3+3z_4 < 0, \bar{x}_1+\bar{x}_3 < \ell, t(b)+x_1 > \ell, t(b)+\bar{x}_1 < 2\ell, x_2-x_3+\bar{x}_3+\bar{x}_1 = \ell, s(b) \leq 3\ell\}$. This is clearly a subset of C_2 in $B_4^{(j)}$. Since $z_4 = 0$, we repeatedly apply the first condition of $\tilde{f}_2$ to yield the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_4 < 0, z_3+3z_4 < 0, z_1+z_2 < 0, z_1+2z_2+z_3+3z_4 < 0, \bar{x}_1+\bar{x}_3 < \ell, t(b)+x_1 > \ell, t(b)+\bar{x}_1 < 2\ell, x_2+\bar{x}_1-z_4 = \ell, s(b) \leq 3\ell\}$. Next consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 < 0, z_4 > 0, z_1 + z_2 + 3z_4 < 0, z_1+2z_2+z_3+3z_4 < 0, \bar{x}_1+\bar{x}_3 < \ell, t(b)+x_1 > \ell, t(b)+\bar{x}_1 < 2\ell, x_2-x_3+\bar{x}_3+\bar{x}_1 = \ell, s(b) \leq 3\ell\}$. Again, this is a subset of C_2 in $B_4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ and obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 < 0, z_4 \geq 0, z_1+z_2+3z_4 < 0, \bar{x}_1+\bar{x}_3+2\bar{x}_2 < \ell, t(b)+x_1+\bar{x}_2 > \ell, t(b)+\bar{x}_1+\bar{x}_2 < 2\ell, x_2 - x_3 + \bar{x}_3 + 2\bar{x}_2 + \bar{x}_1 = \ell, s(b) \leq 3\ell\}$. Then, if $z_4 = 0$, we repeatedly apply the first condition for $\tilde{f}_2$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_2 < 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 < \ell, t(b) + x_1 + \bar{x}_2 > \ell, t(b) + \bar{x}_1 + \bar{x}_2 < 2\ell, x_2 + 2\bar{x}_2 + \bar{x}_1 - z_4 = \ell, s(b) \leq 3\ell\}$. Now consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 = 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 \leq \ell, t(b) + x_1 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 > \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 > \ell, s(b) \leq 3\ell\}$. This is clearly a subset of D_1 in $B_5^{(j)}$. Then, $z_4 = 0$, so we repeatedly apply the first condition for $\tilde{f}_2$ to yield the set $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1+2z_2+z_3+3z_4 < 0, \bar{x}_1+\bar{x}_3 \leq \ell, t(b)+x_1 > \ell, x_1+x_2-z_4 > \ell, x_2+\bar{x}_1-z_4 > \ell, s(b) \leq 3\ell\}$. Then, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_1+z_2+z_3+3z_4 < 0, z_1+2z_2+z_3+3z_4 < 0, \bar{x}_1+\bar{x}_3 \leq \ell, t(b)+x_1 > \ell, x_1+x_2-x_3+\bar{x}_3 > \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 > \ell, s(b) \leq 3\ell\}$. This is clearly a subset of D_1 in $B_5^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 \leq \ell, t(b) + x_1 + \bar{x}_2 > \ell, x_1+x_2-x_3+\bar{x}_3+2\bar{x}_2 > \ell, x_2-x_3+\bar{x}_3+2\bar{x}_2+\bar{x}_1 > \ell, s(b) \leq 3\ell\}$. Then, if $z_4 = 0$, the first condition for $\tilde{f}_2$ is satisfied. Repeatedly applying $\tilde{f}_2$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, \bar{x}_1+\bar{x}_3+2\bar{x}_2 \leq \ell, t(b)+x_1+\bar{x}_2 > \ell, x_1+x_2+2\bar{x}_2-z_4 > \ell, x_2+2\bar{x}_2+\bar{x}_1-z_4 > \ell, s(b) \leq 3\ell\}$.

Lastly, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 > \ell, t(b) + x_1 > \ell, s(b) + z_1 + z_3 \leq 2\ell, s(b) \leq 3\ell\}$. This set is clearly contained in D_4 in $B_5^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to obtain new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, t(b) + x_1 + \bar{x}_2 > \ell, s(b) + z_1 + z_3 \leq 2\ell, s(b) \leq 3\ell\}$. Then, if $z_4 = 0$, the first condition for $\tilde{f}_2$ is satisfied. Repeatedly applying $\tilde{f}_2$ yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, t(b) + x_1 + \bar{x}_2 > \ell, s(b) + z_1 + z_3 + 3z_4 \leq 2\ell, s(b) \leq 3\ell\}$. Combining the five final sets above yields the set $T_2 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, x_1 + x_2 + 2\bar{x}_2 - z_4 > \ell, x_2 + 2\bar{x}_2 + \bar{x}_1 - z_4 \geq \ell, s(b) + z_1 + z_3 + 3z_4 \leq 2\ell, s(b) \leq 3\ell\}$.

Lastly, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 = 0, z_1 + z_3 < 0, \bar{x}_1 + \bar{x}_3 < \ell, t(b) + x_1 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. This is clearly a subset of C_6 in $B_4^{(j)}$. Since $z_4 = 0$, we repeatedly apply the first condition for $\tilde{f}_2$ to obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 < 0, z_1 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 < \ell, t(b) + x_1 > \ell, x_1 + x_2 - z_4 = \ell, s(b) < 2\ell\}$. Next, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_4 > 0, z_1 + z_3 < 0, z_2 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 < \ell, t(b) + x_1 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. This is also a subset of C_6 in $B_4^{(j)}$. We have $z_4 > 0$, so we repeatedly apply the second condition for $\tilde{f}_2$ to yield the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 < \ell, t(b) + x_1 + \bar{x}_2 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) < 2\ell\}$. Then if $z_4 = 0$, we repeatedly apply the first condition for $\tilde{f}_2$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_2 < 0, z_4 < 0, z_1 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 < \ell, t(b) + x_1 + \bar{x}_2 > \ell, x_1 + x_2 + 2\bar{x}_2 - z_4 = \ell, s(b) < 2\ell\}$. Combining the two final sets above yields the set $T_3 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 < 0, z_1 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 < \ell, t(b) + x_1 + \bar{x}_2 > \ell, x_1 + x_2 + 2\bar{x}_2 - z_4 = \ell, s(b) < 2\ell\}$.

We see that by combining $T_1 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_2 \geq 0, z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, z_1 + z_2 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$, $T_2 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_4 < 0, z_3 + 3z_4 < 0, z_1 + z_3 + 3z_4 < 0, t(b) + x_1 + \bar{x}_2 > \ell, x_1 + x_2 + 2\bar{x}_2 - z_4 > \ell, x_2 + 2\bar{x}_2 + \bar{x}_1 - z_4 \geq \ell, s(b) + z_1 + z_3 + 3z_4 \leq 2\ell, s(b) \leq 3\ell\}$, and $T_3 = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 < 0, z_1 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 < \ell, t(b) + x_1 + \bar{x}_2 > \ell, x_1 + x_2 + 2\bar{x}_2 - z_4 = \ell, s(b) < 2\ell\}$, we obtain the set G_1. We note that G_1 is not contained in $B_5^{(j)}$ if $\bar{x}_2 > 0$. Then, if $\bar{x}_2 = 0$ (and hence, $z_2 \leq 0$), G_1 is not contained in $B_3^{(j)}$ since $z_1 + 2z_2 + z_3 + 3z_4 < 0$. Also, G_1 is not contained in C_1, C_2, C_6, D_1, or D_4 since $z_4 < 0$. G_1 is not contained in C_3, C_5, C_7, or D_2 since $z_1 + z_3 + 3z_4 < 0$. Finally, G_1 is not contained in C_4 or D_3 since $z_3 + 3z_4 < 0$.

(G_2) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_2 + 3z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. Again, this is a

subset of C_1 in $B_4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_2 < 0, z_2 + 3z_4 \geq 0, z_1 + z_2 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. Next, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 < 0, z_4 > 0, z_1 + z_2 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 < \ell, t(b) + x_1 > \ell, t(b) + \bar{x}_1 < 2\ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 = \ell, s(b) \leq 3\ell\}$. This is a subset of C_2 in $B_4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ and obtain elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 < 0, z_4 \geq 0, z_1 + z_2 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 < \ell, t(b) + x_1 + \bar{x}_2 > \ell, t(b) + \bar{x}_1 + \bar{x}_2 < 2\ell, x_2 - x_3 + \bar{x}_3 + 2\bar{x}_2 + \bar{x}_1 = \ell, s(b) \leq 3\ell\}$. Next, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_4 > 0, z_1 + z_3 < 0, z_2 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 < \ell, t(b) + x_1 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 = \ell, s(b) < 2\ell\}$. This is clearly a subset of C_6 in $B_4^{(j)}$. We have $z_4 > 0$, so we repeatedly apply the second condition for $\tilde{f}_2$ to yield the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_4 \geq 0, z_1 + z_3 < 0, z_2 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 < \ell, t(b) + x_1 + \bar{x}_2 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) < 2\ell\}$. Then, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 \leq \ell, t(b) + x_1 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 > \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 > \ell, s(b) \leq 3\ell\}$. This is clearly a subset of D_1 in $B_5^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$. This yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_1 + z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 \leq \ell, t(b) + x_1 + \bar{x}_2 > \ell, x_1 + x_2 - x_3 + \bar{x}_3 + 2\bar{x}_2 > \ell, x_2 - x_3 + \bar{x}_3 + 2\bar{x}_2 + \bar{x}_1 > \ell, s(b) \leq 3\ell\}$. Lastly, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 > 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 > \ell, t(b) + x_1 > \ell, s(b) + z_1 + z_3 \leq 2\ell, s(b) \leq 3\ell\}$. This set is clearly contained in D_4 in $B_5^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to obtain new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, t(b) + x_1 + \bar{x}_2 > \ell, s(b) + z_1 + z_3 \leq 2\ell, s(b) \leq 3\ell\}$. Combining the five final sets above yields the set G_2. We note that G_2 is not contained in $B_5^{(j)}$ since $\bar{x}_2 > 0$.

(G_3) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 > 0, z_2 < 0, z_4 > 0, z_1 + z_3 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. This is clearly a subset of $B_3^{(j)}$. Then since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to yields new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \ell) \mid \bar{x}_2 > 0, z_1 > 0, z_4 \geq 0, z_1 + z_3 > 0, z_1 + z_2 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 > 0, s(b) = 2\ell\}$. Then if $z_4 = 0$, we repeatedly apply the first condition for $\tilde{f}_2$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \ell) \mid \bar{x}_2 > 0, z_1 > 0, z_4 < 0, z_1 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 > 0, s(b) = 2\ell\}$. Combining these two sets yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \ell) \mid \bar{x}_2 > 0, z_1 > 0, z_1 + z_3 > 0, z_1 + z_3 + 3z_4 > 0, z_1 + z_2 + z_3 + 3z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 > 0, s(b) = 2\ell\}$. Next, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 > 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 2\ell\}$. This is clearly a subset of C_3 in $B_4^{(j)}$. Since

$z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_4 \geq 0, z_1+z_3 \geq 0, z_1+z_2+z_3+3z_4 < 0, \bar{x}_1+\bar{x}_3+2\bar{x}_2 = \ell, s(b) \leq 2\ell\}$. Then, if $z_4 = 0$, the first condition for $\tilde{f}_2$ is satisfied. Repeatedly applying $\tilde{f}_2$ yields new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_4 < 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, z_1+z_2+z_3+3z_4 < 0, \bar{x}_1+\bar{x}_3+2\bar{x}_2 = \ell, s(b) \leq 2\ell\}$. Combining these two sets yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_1+z_3 \geq 0, z_1+z_3+3z_4 \geq 0, z_1+z_2+z_3+3z_4 < 0, \bar{x}_1+\bar{x}_3+2\bar{x}_2 = \ell, s(b) \leq 2\ell\}$. Then, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_4 > 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 = \ell, s(b) < 2\ell\}$. This is a subset of C_5 in $B_4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_4 \geq 0, z_1 + z_3 \geq 0, z_1+z_2+z_3+3z_4 \geq 0, t(b)+x_1+\bar{x}_2 = \ell, s(b) < 2\ell\}$. Then if $z_4 = 0$, the first condition for $\tilde{f}_2$ is satisfied. Repeatedly applying $\tilde{f}_2$ yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_4 < 0, z_1+z_3 \geq 0, z_1+z_3+3z_4 \geq 0, z_1+z_2+z_3+3z_4 \geq 0, t(b)+x_1+\bar{x}_2 = \ell, s(b) < 2\ell\}$. Combining these two sets yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. Lastly, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_4 > 0, z_1 + z_3 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 > \ell, t(b) + x_1 > \ell, s(b) \leq 2\ell\}$. This is clearly a subset of D_2 in $B_5^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to obtain new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_4 \geq 0, z_1 + z_3 \geq 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, t(b) + x_1 + \bar{x}_2 > \ell, s(b) \leq 2\ell\}$. Then if $z_4 = 0$, the first condition for $\tilde{f}_2$ is satisfied. Applying $\tilde{f}_2$ repeatedly yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_4 < 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, t(b) + x_1 + \bar{x}_2 > \ell, s(b) \leq 2\ell\}$. Combining these two sets yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, t(b) + x_1 + \bar{x}_2 > \ell, s(b) \leq 2\ell\}$. Then, combining the four final sets above yields the set G_3. We note that G_3 is not contained in $B_5^{(j)}$ because $\bar{x}_2 > 0$.

(G_4) Consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_4 > 0, z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 = \ell, s(b) \leq 3\ell\}$. This set is clearly a subset of C_4 in $B_4^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to yield $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 \geq 0, z_4 \geq 0, z_1+z_2+z_3+3z_4 < 0, \bar{x}_1+\bar{x}_3+2\bar{x}_2 = \ell, s(b) \leq 3\ell\}$. Then if $z_4 = 0$, the first condition for $\tilde{f}_2$ is satisfied. Repeatedly applying $\tilde{f}_2$ to elements of this form yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 > 0, z_4 < 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 = \ell, s(b) \leq 3\ell\}$. Combining these two sets yields $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 \geq 0, z_3+3z_4 \geq 0, z_1+z_2+z_3+3z_4 < 0, \bar{x}_1+\bar{x}_3+2\bar{x}_2 = \ell, s(b) \leq 3\ell\}$. Next, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_4 > 0, z_1+z_2+z_3+3z_4 \geq 0, z_1+2z_2+z_3+3z_4 < 0, t(b)+x_1 = \ell, s(b) < 2\ell\}$.

This is clearly a subset of C_7 in $B_4^{(j)}$. Then, $z_4 > 0$, so we repeatedly apply the second condition for $\tilde{f}_2$. This yields new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 \geq 0, z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. If $z_4 = 0$, we repeatedly apply the first condition for $\tilde{f}_2$ to obtain the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 \geq 0, z_4 < 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. Combining these two sets yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, t(b) + x_1 + \bar{x}_2 = \ell, s(b) < 2\ell\}$. Lastly, consider elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_4 > 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 > \ell, t(b) + x_1 > \ell, s(b) + z_1 \leq 2\ell, s(b) \leq 3\ell\}$. This set is contained in D_3 in $B_5^{(j)}$. Since $z_4 > 0$, we repeatedly apply the second condition for $\tilde{f}_2$ to obtain new elements of the form $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 \geq 0, z_4 \geq 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, t(b) + x_1 + \bar{x}_2 > \ell, s(b) + z_1 \leq 2\ell, s(b) \leq 3\ell\}$. Then if $z_4 = 0$, the first condition for $\tilde{f}_2$ is satisfied. Repeatedly applying $\tilde{f}_2$ yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 > 0, z_4 < 0, z_3 + 3z_4 \geq 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, t(b) + x_1 + \bar{x}_2 > \ell, s(b) + z_1 \leq 2\ell, s(b) \leq 3\ell\}$. Combining these two sets yields the set $\{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \bar{x}_2 > 0, z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, \bar{x}_1 + \bar{x}_3 + 2\bar{x}_2 > \ell, t(b) + x_1 + \bar{x}_2 > \ell, s(b) + z_1 \leq 2\ell, s(b) \leq 3\ell\}$. Finally, combining the three final sets above yields the set G_4. We note that G_4 is not contained in $B_5^{(j)}$ because $\bar{x}_2 > 0$.

Therefore, by repeatedly applying $\tilde{f}_2$ to elements in $B_5^{(j)}$, we obtain $B_6^{(j)} = B_5^{(j)} \cup G_1 \cup G_2 \cup G_3 \cup G_4$. Now, we work to show that $B_6^{(j)} = F_1 \cup F_2 \cup F_3 \cup F_4$.

(F_1) First, we can combine the disjoint sets in $B_3^{(j)}$ to obtain $B_3^{(j)} = \{(x_1, x_2, x_3, \bar{x}_3, 0, \ell) \mid z_1 \geq 0, z_2 \leq 0, z_1 + z_3 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, s(b) = 2\ell\}$. Then, we can combine $B_3^{(j)}$ with C_5 in $B_4^{(j)}$ to obtain the set $F_{1,1} = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_1 + z_3 \geq 0, z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 \leq 0, t(b) + x_1 = \ell, s(b) \leq 2\ell\}$. Next, we can combine C_3 in $B_4^{(j)}$ and D_2 in $B_5^{(j)}$ to yield the set $F_{1,2} = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 \geq 0, z_2 < 0, z_1 + z_3 \geq 0, z_1 + z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 \geq \ell, t(b) + x_1 > \ell, s(b) \leq 2\ell\}$. Finally, if we combine $F_{1,1}$, $F_{1,2}$, and G_3 in $B_6^{(j)}$, we obtain the set F_1 as desired.

(F_2) We can first combine C_4 and C_7 in $B_4^{(j)}$ with D_3 in $B_5^{(j)}$ to obtain the set $F_{2,1} = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_1 < 0, z_3 \geq 0, z_3 + 3z_4 \geq 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, \bar{x}_1 + \bar{x}_3 \geq \ell, t(b) + x_1 \geq \ell, s(b) + z_1 \leq 2\ell, s(b) \leq 3\ell\}$. Then, if we combine $F_{2,1}$ with G_4 in $B_6^{(j)}$, we obtain the set F_2 as desired.

(F_3) We can combine C_1, C_2, and C_6 in $B_4^{(j)}$ with D_1 and D_4 in $B_5^{(j)}$ to obtain the set $F_{3,1} = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid z_3 < 0, z_4 \geq 0, z_1 + z_3 < 0, z_1 + 2z_2 + z_3 + 3z_4 < 0, t(b) + x_1 \geq \ell, x_1 + x_2 - x_3 + \bar{x}_3 \geq \ell, x_2 - x_3 + \bar{x}_3 + \bar{x}_1 \geq \ell, s(b) + z_1 + z_3 \leq 2\ell, s(b) \leq 3\ell\}$. Then, if we combine $F_{3,1}$ with G_2 in $B_6^{(j)}$, we obtain the set F_3 as desired.

(F_4) The set F_4 is precisely G_1 in $B_6^{(j)}$.

Thus, $B_6^{(j)} = F_1 \cup F_2 \cup F_3 \cup F_4$ and we have now constructed the set $B_a^{(j)}$ for $0 \leq a \leq 6$. $\square$

Remark 5.3.2. By the explicit construction of $B_a^{(j)}$ in Lemma 5.3.1, we can see that $B_6^{(j)} \neq \mathcal{B}$. For example, the element $(0,0,0,0,0,0)$ is not contained in $B_6^{(j)}$, but is contained in $\mathcal{B}$. Therefore, in Theorem 3.4.3, $\kappa \neq 1$.

We now show conditions (2) and (3) in Theorem 3.4.3 are satisfied with the sequence $\{i_a^{(j)} \mid j \geq 1, 1 \leq a \leq 6\}$ and the Weyl group sequence $\{w^{(k)}\}_{k \geq 0}$ for $\lambda = \ell(\Lambda_0 + \Lambda_1)$.

By direct calculations of the simple reflections r_i $(i = 0, 1, 2)$ on $\Lambda_0 + \Lambda_1$, we obtain the following Lemma, which we will use along with Proposition 3.4.4 to provide justification for condition (3) in Theorem 3.4.3.

Lemma 5.3.3. *Let $k \in \mathbb{Z}_{>0}$ and $k = 6(j-1) + a$ for $1 \leq a \leq 6$. Then $w^{(k)}(\Lambda_0 + \Lambda_1) = \Lambda_0 + \Lambda_1 - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2$, where*

$$m_0 = \begin{cases} 3j^2 - 5j + 2 & \text{if } a = 1,2,3 \\ 3j^2 + j & \text{if } a = 4,5,6 \end{cases}$$

$$m_1 = \begin{cases} 6j^2 - 7j + 2 & \text{if } a = 1,2 \\ 6j^2 - 4j + 1 & \text{if } a = 3,4 \\ 6j^2 - j & \text{if } a = 5,6 \end{cases}$$

$$m_2 = \begin{cases} 3j^2 - 5j + 2 & \text{if } a = 1 \\ 3j^2 - 2j & \text{if } a = 2,3,4,5 \\ 3j^2 + j & \text{if } a = 6 \end{cases}$$

Proposition 5.3.4. Conditions (2) and (3) in Theorem 3.4.3 are satisfied with the sequence $\{i_a^{(j)} \mid j \geq 1, 1 \leq a \leq 6\}$ and the Weyl group sequence $\{w^{(k)}\}_{k \geq 0}$ given above for $\lambda = \ell(\Lambda_0 + \Lambda_1)$.

Proof. In order to show condition (2), we consider $\langle \ell(\Lambda_0 + \Lambda_1), h_{i_a^{(j)}} \rangle$. Observe that $\langle \ell(\Lambda_0 + \Lambda_1), h_{i_a^{(j)}} \rangle = 0 \leq \varepsilon_{i_a^{(j)}}(b)$ for all $b \in B_{a-1}^{(j)}$, $a = 2, 6$. Also, $\langle \ell(\Lambda_0 + \Lambda_1), h_{i_a^{(j)}} \rangle = \ell$ for $a = 1, 3, 4, 5$. For $a = 1, 3, 5$, we must consider $\varepsilon_1(b)$ for all $b \in B_0^{(j)}$, $B_2^{(j)}$, and $B_4^{(j)}$. First, note that for all $b \in B_0^{(j)}$ or $B_2^{(j)}$, we have $\bar{x}_1 = \ell$. Hence, $\varepsilon_1(b) = \bar{x}_1 + (\bar{x}_3 - \bar{x}_2 + (x_2 - x_3)_+)_+ = l + (\bar{x}_3 - \bar{x}_2 + (x_2 - x_3)_+)_+ \geq \ell$ for all $b \in B_0^{(j)}$ and $B_2^{(j)}$. Now, consider $b \in B_4^{(j)}$. Note that for all $b \in B_4^{(j)}$, $\bar{x}_2 = 0$, so $\varepsilon_1(b) = \bar{x}_1 + (\bar{x}_3 + (x_2 - x_3)_+)_+ = \bar{x}_1 + \bar{x}_3 + (x_2 - x_3)_+$. We consider each subset in $B_4^{(j)}$.

- If $b \in B_3^{(j)}$, then $\bar{x}_1 = \ell$ and $\varepsilon_1(b) \geq \ell$ by the same reasoning as above.

- If $b \in C_1$, then $z_3 < 0$, $z_1 + z_2 + 3z_4 \geq 0$, and $t(b) + x_1 = \ell$. Since $z_3 < 0$, $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + x_2 - x_3$. Then $z_1 + z_2 + 3z_4 \geq 0$ implies $\bar{x}_1 + \frac{1}{2}\bar{x}_3 > \frac{3}{2}x_3 + x_1$. Thus, $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + x_2 - x_3 > \frac{1}{2}\bar{x}_3 + \frac{1}{2}x_3 + x_2 + x_1 = t(b) + x_1 = \ell$.

- If $b \in C_2$, then $z_3 < 0$ and $x_2 - x_3 + \bar{x}_3 + \bar{x}_1 = \ell$. Since $z_3 < 0$, we have $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + x_2 - x_3 = \ell$.

- If $b \in C_3, C_4$, then $\bar{x}_1 + \bar{x}_3 = \ell$. This implies $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + (x_2 - x_3)_+ \geq \ell$.

- If $b \in C_5$, then $z_1 + z_2 + z_3 + 3z_4 \geq 0$ and $t(b) + x_1 = \ell$. Since $z_1 + z_2 + z_3 + 3z_4 \geq 0$, we have $\bar{x}_1 + \frac{1}{2}\bar{x}_3 \geq x_1 + x_2 + \frac{1}{2}x_3$. This means $\bar{x}_1 + \bar{x}_3 \geq x_1 + x_2 + \frac{1}{2}x_3 + \frac{1}{2}\bar{x}_3 = t(b) + x_1 = \ell$. Thus, $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + (x_2 - x_3)_+ \geq \ell$.

- If $b \in C_6$, then $z_1 \geq 0$, $z_1 + z_3 < 0$, and $x_1 + x_2 - x_3 + \bar{x}_3 = \ell$. The conditions $z_1 \geq 0$ and $z_1 + z_3 < 0$ imply that $z_3 < 0$. So, $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + x_2 - x_3 \geq x_1 + x_2 - x_3 + \bar{x}_3 = \ell$.

- If $b \in C_7$, then $z_1 + z_2 + z_3 + 3z_4 \geq 0$ and $t(b) + x_1 = \ell$. Since $z_1 + z_2 + z_3 + 3z_4 \geq 0$, we have $\bar{x}_1 + \frac{1}{2}\bar{x}_3 \geq x_1 + x_2 + \frac{1}{2}x_3$. This means $\bar{x}_1 + \bar{x}_3 \geq x_1 + x_2 + \frac{1}{2}x_3 + \frac{1}{2}\bar{x}_3 = t(b) + x_1 = \ell$. Thus, $\varepsilon_1(b) = \bar{x}_1 + \bar{x}_3 + (x_2 - x_3)_+ \geq \ell$.

Therefore, $\varepsilon_1(b) \geq \ell$ for all $b \in B_0^{(j)}$, $B_2^{(j)}$, and $B_4^{(j)}$.

For $a = 4$, we consider $\varepsilon_0(b)$ for all $b \in B_3^{(j)}$. By definition,

$$\varepsilon_0(b) = 3\ell - s(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4),$$

where $A = (0, z_1, z_1 + z_2, z_1 + z_2 + 3z_4, z_1 + z_2 + z_3 + 3z_4, 2z_1 + z_2 + z_3 + 3z_4)$. If $b \in B_0^{(j)}$, then $\varepsilon_0(b) = 3\ell - 2\ell + 0 - 0 = \ell$. If $b \in B_1^{(j)} \setminus B_0^{(j)}$, then $\varepsilon_0(b) = 3\ell - 2\ell + (2z_1 + z_2 + z_3 + 3z_4) - (2z_1 + z_2 + z_3 + 3z_4) = \ell$, since $z_1 > 0$, $z_1 + z_3 = 0$, and $z_2 = z_4 = 0$. If $b \in B_2^{(j)} \setminus B_1^{(j)}$, then $\varepsilon_0(b) = 3\ell - 2\ell + z_1 - z_1 = \ell$, since $z_1 > 0$, $z_4 < 0$, $z_1 + z_3 + 3z_4 = 0$, and $z_2 = 0$. Finally, if $b \in B_3^{(j)} \setminus B_2^{(j)}$, then $\varepsilon_0(b) = 3\ell - 2\ell + \max\{z_1 + z_2 + 3z_4, z_1 - z_2\} - (z_1 - z_2) = \ell$, since $z_1 > 0$ and $z_2 < 0$. Then $\varepsilon_0(b) \geq \ell + (z_1 - z_2) - (z_1 - z_2) = \ell$. Therefore, $\varepsilon_0(b) \geq \ell$ for all $b \in B_3^{(j)}$.

Thus, $\langle \ell(\Lambda_0 + \Lambda_1), h_{i_a^{(j)}} \rangle \leq \varepsilon_{i_a^{(j)}}(b)$ for all $b \in B_{a-1}^{(j)}$, and condition (2) holds.

To prove condition (3), we use Lemma 5.1.2. For $k = 6(j-1)+a$, $j \geq 1$, $1 \leq a \leq 6$, we have

$$\langle w^{(k)}(\Lambda_0 + \Lambda_1), h_{i_{a+1}^{(j)}} \rangle = \begin{cases} 3j-2 & \text{if } a = 1 \\ 3j-1 & \text{if } a = 2, 4 \\ 3j & \text{if } a = 5 \\ 6j-2 & \text{if } a = 3 \\ 6j+1 & \text{if } a = 6. \end{cases}$$

Hence, for positive j, $\langle w^{(k)}(\Lambda_0 + \Lambda_1), h_{i_{a+1}^{(j)}} \rangle$ is greater than zero. By Proposition 3.4.4, this implies $w^{(k+1)} = r_{i_{a+1}}^{(j)} w^{(k)} \succ w^{(k)}$. Thus, the sequence of Weyl group elements, $\{w^{(k)}\}_{k \geq 0}$, is increasing with respect to the Bruhat order, satisfying condition (3). $\square$

We have shown above that $B_6^{(j)} \neq \mathcal{B}$ and hence, $\kappa \neq 1$ in Theorem 3.4.3. Due to the previous conjecture that $\kappa \leq 2$, we have conjectured that $\kappa = 2$ for $\lambda = \ell(\Lambda_0 + \Lambda_1)$. We provide the conjecture and evidence toward the conjecture below.

Conjecture 5.3.5. *For $\lambda = \ell(\Lambda_0 + \Lambda_1)$, $\ell \geq 1$ and the given perfect crystal $\mathcal{B} = \mathcal{B}_{3\ell}$ for the quantum affine algebra, $U_q(D_4^{(3)})$, with $d = 6$ and the sequence $\{i_a^{(j)}\}$ given above, we conjecture that conditions (1), (2), and (3) in Theorem 3.4.3 hold with $\kappa = 2$. This would imply that path realizations of the corresponding Demazure crystals $B_{w^{(k)}}(\ell(\Lambda_0 + \Lambda_1))$ for $U_q(D_4^{(3)})$ have tensor product-like structures.*

5.3.1 $\ell = 1$ Case

First, by Lemma 5.3.1, we have the following sets $B_a^{(j)}$ for $0 \leq a \leq 6$.

$$\begin{aligned}
B_0^{(j)} &= \{(1,0,0,0,0,1)\} \\
B_1^{(j)} &= B_0^{(j)} \cup \{(0,1,0,0,0,1)\} \\
B_2^{(j)} &= B_1^{(j)} \cup \{(0,0,2,0,0,1)\} \\
B_3^{(j)} &= B_2^{(j)} \cup \{(0,0,0,2,0,1), (0,0,1,1,0,1)\} \\
B_4^{(j)} &= B_3^{(j)} \cup C_1 \cup C_2 \cup \cdots \cup C_7 \\
B_5^{(j)} &= B_4^{(j)} \cup D_1 \cup D_2 \cup D_3 \cup D_4 \\
B_6^{(j)} &= F_1 \cup F_2 \cup F_3 \cup F_4,
\end{aligned}$$

where $C_n = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid (P_n) \text{ holds}\}$ for $1 \leq n \leq 7$, $D_n = \{(x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1) \mid$

(Q_n) holds$\}$ for $1 \leq n \leq 4$, and $F_n = \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid (R_n) \text{ holds}\}$ for $1 \leq n \leq 4$ as defined in Lemma 5.3.1 with $\ell = 1$.

Note the elements in $\mathcal{B}$ that are missing in $B_6^{(j)}$ are

$$\begin{aligned}
&\{(0,0,0,0,0,\bar{x}_1) \mid s(b) \leq 3\},\\
&\{(0,0,0,0,\bar{x}_2,\bar{x}_1) \mid \bar{x}_2 > 0,\ \bar{x}_1 + \bar{x}_2 = 3,\ s(b) \leq 3\},\\
&\{(0,0,0,\bar{x}_3,\bar{x}_2,\bar{x}_1) \mid \bar{x}_3 > 0,\ s(b) = 3\},\\
&\{(0,0,x_3,\bar{x}_3,\bar{x}_2,\bar{x}_1) \mid x_3 > 0,\ z_3 \leq 4,\ s(b) = 3\},\\
&\{(0,1,x_3,\bar{x}_3,\bar{x}_2,\bar{x}_1) \mid \bar{x}_2 \leq 1,\ z_2 > -4,\ s(b) = 3\},\\
&\{(1,0,x_3,\bar{x}_3,\bar{x}_2,\bar{x}_1) \mid \bar{x}_1 > 0,\ z_3 \leq 1,\ s(b) = 3\},\\
&\{(x_1,0,0,0,0,0) \mid x_1 > 0,\ s(b) \leq 3\}.
\end{aligned}$$

Thus, $B_6^{(j)} \neq \mathcal{B}$, and hence $\kappa \neq 1$. We now construct the sets $B_a^{(j+1,j)}$, $0 \leq a \leq 6$ explicitly.

By definition, $B_0^{(j+1,j)} = B_0^{(j+1)} \otimes B_6^{(j)}$. Since $B_0^{(j)} = \{b_j\}$ and $b_j = (1,0,0,0,0,1)$ for all $j \geq 1$, we have $B_0^{(j+1)} = \{(1,0,0,0,0,1)\}$. So,

$$B_0^{(j+1,j)} = \{(1,0,0,0,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \in B_6^{(j)}\}.$$

Then we use the recursive formula,

$$B_a^{(j+1,j)} = \bigcup_{k \geq 0} \tilde{f}^k_{i_a^{(j+1)}} B_{a-1}^{(j+1,j)} \setminus \{0\},$$

to determine $B_a^{(j+1,j)}$ for $1 \leq a \leq 6$. We use the tensor product rule defined in Theorem 3.1.7 along with the operators, $\varepsilon_i,\ \varphi_i$, $i = 0,1,2$ defined in Chapter 4 to do this.

We define the following conditions for $(x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \in \mathcal{B}$,

$$\begin{cases}
(P_1)\ z_2' \leq -2, z_3' > 0, s(b)' = 3,\\
(P_2)\ x_3' > 0, z_4' = 0, \bar{x}_1' + 2\bar{x}_2' \leq 2, t(b)' + 2\bar{x}_2' \leq 3,\\
(P_3)\ \bar{x}_2' > 0, z_3' + 3z_4' \geq 0, s(b)' = 3,\\
(P_4)\ \bar{x}_1' + \bar{x}_2' + \bar{x}_3' > 0, \bar{x}_1' < 3,\\
(P_5)\ \bar{x}_1' + \bar{x}_2' + \bar{x}_3' > 0,\\
(P_6)\ x_3 > 0, \bar{x}_3 > 0, z_4 \neq 0, t(b) = 2.
\end{cases}$$

Thus, the sets $B_a^{(j+1,j)}$ for $\ell = 1$ are constructed as follows,

$$
\begin{aligned}
B_1^{(j+1,j)} = B_0^{(j+1,j)} &\cup \{(1,0,0,0,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, 1) \mid z_2' < 0,\ s(b)' = 3\} \\
&\cup \{(1,0,0,0,0,1) \otimes (0, x_2', x_3', \bar{x}_3', 0, 0) \mid (P_1) \text{ holds}\} \\
&\cup \{(0,1,0,0,0,1) \otimes (x_1', x_2', 0, 0, 0, \bar{x}_1') \mid \bar{x}_1' > 0,\ s(b)' \leq 3\} \\
&\cup \{(0,1,0,0,0,1) \otimes (x_1', x_2', 0, 0, \bar{x}_2', 0) \mid 0 < \bar{x}_2' < 3,\ s(b)' \leq 3\} \\
&\cup \{(0,1,0,0,0,1) \otimes (x_1', x_2', 0, 0, 1, 1) \mid s(b)' \leq 3\} \\
&\cup \{(0,1,0,0,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (P_2) \text{ holds}\} \\
&\cup \{(0,1,0,0,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, \bar{x}_1') \mid z_4' \neq 0,\ s(b)' \leq 3\} \\
&\cup \{(0,1,0,0,0,1) \otimes (x_1', x_2', 0, \bar{x}_3', 1, \bar{x}_1') \mid z_4' > 0,\ s(b)' \leq 3\} \\
&\cup \{(0,1,0,0,0,1) \otimes (x_1', x_2', 2, 0, \bar{x}_2', 0) \mid \bar{x}_2' > 0,\ s(b)' \leq 3\} \\
B_2^{(j+1,j)} = B_1^{(j+1,j)} &\cup \{(1,0,0,0,0,1) \otimes (0, 0, x_3', \bar{x}_3', \bar{x}_2', 0) \mid (P_3) \text{ holds}\} \\
&\cup \{(1,0,0,0,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', 1) \mid \bar{x}_2' > 0,\ s(b)' = 3\} \\
&\cup \{(1,0,0,0,0,1) \otimes (0,1,1,1,1,0)\} \\
&\cup \{(0,0,2,0,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (P_4) \text{ holds}\} \\
B_3^{(j+1,j)} = B_2^{(j+1,j)} &\cup \{(1,0,0,0,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid \bar{x}_1' \geq 2,\ s(b)' = 3\} \\
&\cup \{(0,1,0,0,0,1) \otimes (0, 0, x_3', \bar{x}_3', \bar{x}_2', 0) \mid (P_3) \text{ holds}\} \\
&\cup \{(0,1,0,0,0,1) \otimes (0,1,1,1,1,0)\} \\
&\cup \{(0,1,0,0,0,1) \otimes (0, 0, 0, 0, \bar{x}_2', \bar{x}_1') \mid \bar{x}_1' > 0,\ s(b)' = 3\} \\
&\cup \{(0,1,0,0,0,1) \otimes (0, 0, x_3', \bar{x}_3', 1, 1) \mid \bar{x}_3' > 0,\ s(b)' = 3\} \\
&\cup \{(0,0,2,0,0,1) \otimes (0,0,0,0,0,3)\} \\
&\cup \{(0,0,2,0,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (P_5) \text{ holds}\} \\
&\cup \{(0,0,2,0,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid \bar{x}_1' + \bar{x}_2' > 0,\ s(b)' \leq 3\}
\end{aligned}
$$

$$
\begin{aligned}
B_4^{(j+1,j)} = {} & \{(x_1,0,0,0,0,1) \mid x_1 > 0,\ s(b) \le 3\} \otimes \mathcal{B} \cup \{(0,1,0,0,0,1)\} \otimes \mathcal{B} \\
& \cup \{(x_1,x_2,1,1,0,0) \mid x_2 \le 1,\ s(b) \le 3\} \otimes \mathcal{B} \\
& \cup \{(x_1,0,3,1,0,0) \mid s(b) \le 3\} \otimes \mathcal{B} \\
& \cup \{(0,0,x_3,\bar{x}_3,0,1) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid z_3 > 0,\ s(b) = 2,\ (P_5) \text{ holds}\} \\
& \cup \{(0,0,0,2,0,\bar{x}_1) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid \bar{x}_1 \le 1, s(b) \le 3, \bar{x}_1 + \bar{x}_2 > 0, s(b)' \le 3\} \\
& \cup \{(x_1,1,0,0,0,0) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid s(b) \le 3,\ \bar{x}_1 + \bar{x}_2 > 0,\ s(b)' \le 3\} \\
& \cup \{(0,1,0,0,0,0) \otimes (x_1',x_2',x_3',0,0,0) \mid z_3' \ge 0,\ s(b)' \le 3\} \\
& \cup \{(1,1,0,0,0,0) \otimes (x_1',x_2',x_3',0,0,0) \mid z_3' \ge -1,\ s(b)' \le 3\} \\
& \cup \{(1,1,0,0,0,0) \otimes (x_1',x_2',x_3',1,0,0) \mid s(b)' \le 3\} \\
& \cup \{(1,1,0,0,0,0) \otimes (0,x_2',0,2,0,0) \mid s(b)' < 3\} \\
& \cup \{(2,1,0,0,0,0) \otimes (x_1',x_2',x_3',0,0,0) \mid s(b)' \le 3\} \\
& \cup \{(2,1,0,0,0,0) \otimes (x_1',x_2',x_3',\bar{x}_3',0,0) \mid z_2' < 0,\ s(b)' \le 2\} \\
& \cup \{(2,1,0,0,0,0) \otimes (x_1',x_2',x_3',1,0,0) \mid s(b)' = 3\} \\
& \cup \{(2,1,0,0,0,0) \otimes (x_1',x_2',2,2,0,0) \mid s(b)' = 3\} \\
& \cup \{(2,1,0,0,0,0) \otimes (2,0,0,2,0,0)\} \\
B_5^{(j+1,j)} = {} & \{(x_1,x_2,0,\bar{x}_3',0,0) \mid \bar{x}_3' > 0,\ s(b) \le 3\} \otimes \mathcal{B} \\
& \cup \{(0,0,x_3,\bar{x}_3,0,1) \mid z_4 \ne 0,\ s(b) = 2\} \otimes \mathcal{B} \\
& \cup \{(x_1,0,x_3,\bar{x}_3,0,0) \mid (P_6) \text{ holds}\} \otimes \mathcal{B} \\
& \cup \{(x_1,x_2,0,0,0,0) \mid x_2 > 0,\ s(b) \le 3\} \otimes \mathcal{B} \\
& \cup \{(x_1,x_2,0,0,0,1) \mid 1 \le s(b) \le 3\} \otimes \mathcal{B} \\
& \cup \{(x_1,x_2,1,1,0,0) \mid x_2 \le 1,\ s(b) \le 3\} \otimes \mathcal{B} \\
& \cup \{(0,0,1,1,0,1),(0,0,2,2,0,0)\} \otimes \mathcal{B} \\
B_6^{(j+1,j)} = {} & B_6^{(j)} \otimes \mathcal{B}
\end{aligned}
$$

From above, we see that condition (1) in Theorem 3.4.3 holds for $\lambda = \Lambda_0 + \Lambda_1$ with $\kappa = 2$. This implies Conjecture 5.3.5 holds when $\ell = 1$, and path realizations of the corresponding Demazure crystals $B_{w^{(k)}}(\Lambda_0 + \Lambda_1)$ for $U_q(D_4^{(3)})$ have tensor product-like structures.

5.3.2 $\ell = 2$ Case

First, by Lemma 5.3.1, we have the following sets $B_a^{(j)}$ for $0 \le a \le 6$.

$$
\begin{aligned}
B_0^{(j)} &= \{(2,0,0,0,0,2)\} \\
B_1^{(j)} &= B_0^{(j)} \cup \{(x_1,x_2,0,0,0,2) \mid z_1 > 0, z_1 + z_3 = 0, s(b) = 4\} \\
B_2^{(j)} &= B_1^{(j)} \cup \{(x_1,x_2,x_3,0,0,2) \mid z_1 > 0, z_4 < 0, z_1 + z_3 + 3z_4 = 0, s(b) = 4\} \\
B_3^{(j)} &= B_2^{(j)} \cup \{(x_1,x_2,x_3,\bar{x}_3,0,2) \mid z_1 > 0, z_2 < 0, z_1 + z_3 > 0, z_1 + 2z_2 + z_3 + 3z_4 = 0, \\
&\qquad\qquad s(b) = 4\} \\
B_4^{(j)} &= B_3^{(j)} \cup C_1 \cup C_2 \cup \cdots \cup C_{12} \\
B_5^{(j)} &= B_4^{(j)} \cup D_1 \cup D_2 \cup \cdots \cup D_9 \\
B_6^{(j)} &= F_1 \cup F_2 \cup F_3 \cup F_4,
\end{aligned}
$$

where $C_n = \{(x_1,x_2,x_3,\bar{x}_3,0,\bar{x}_1) \mid (P_n) \text{ holds}\}$ for $1 \le n \le 7$, $D_n = \{(x_1,x_2,x_3,\bar{x}_3,0,\bar{x}_1) \mid (Q_n) \text{ holds}\}$ for $1 \le n \le 4$, and $F_n = \{(x_1,x_2,x_3,\bar{x}_3,\bar{x}_2,\bar{x}_1) \mid (R_n) \text{ holds}\}$ for $1 \le n \le 4$ as defined in Lemma 5.3.1 with $\ell = 1$.

We note that $B_6^{(j)} \neq \mathcal{B}$ since $(0,0,0,0,0,0) \notin B_6^{(j)}$, and hence $\kappa \neq 1$. We now construct the sets $B_a^{(j+1,j)}$, $0 \le a \le 6$ explicitly.

By definition, $B_0^{(j+1,j)} = B_0^{(j+1)} \otimes B_6^{(j)}$. Since $B_0^{(j)} = \{b_j\}$ and $b_j = (2,0,0,0,0,2)$ for all $j \ge 1$, we have $B_0^{(j+1)} = \{(2,0,0,0,0,2)\}$. So,

$$B_0^{(j+1,j)} = \{(2,0,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \in B_6^{(j)}\}.$$

Then we use the recursive formula,

$$B_a^{(j+1,j)} = \bigcup_{k \ge 0} \tilde{f}^k_{i_a^{(j+1)}} B_{a-1}^{(j+1,j)} \setminus \{0\},$$

to determine $B_a^{(j+1,j)}$ for $1 \le a \le 6$. We use the tensor product rule defined in Theorem 3.1.7 along with the operators, ε_i, φ_i, $i = 0,1,2$ defined in Chapter 4 to do this.

We define the following conditions for $(x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \in \mathcal{B}$,

$$\begin{cases} (P_1)\ \bar{x}_3' > 0, \bar{x}_1' + \bar{x}_3' \geq 3, \\ (P_2)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, 2 \leq \bar{x}_1' \leq 3, \\ (P_3)\ \bar{x}_2', \bar{x}_3' > 0, \frac{1}{2}(x_3' + \bar{x}_3') \leq 3, 2 \leq s(b)' \leq 4, \\ (P_4)\ \bar{x}_3' > 0, \frac{1}{2}(x_3' + \bar{x}_3') \leq 2, \\ (P_5)\ \bar{x}_3', \bar{x}_2', \bar{x}_1' > 0, \bar{x}_1' + \bar{x}_2' = 3, \frac{1}{2}(x_3' + \bar{x}_3') = 1, \\ (P_6)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, 0 < \bar{x}_1' \leq 4, \\ (P_7)\ \bar{x}_3' > 0, \bar{x}_2' \leq 1, \frac{1}{2}(x_3' + \bar{x}_3') + \bar{x}_2' \geq 2, \\ (P_8)\ \bar{x}_2', \bar{x}_3' > 0, \bar{x}_1' + \bar{x}_2' = 3, s(b)' = 4, \end{cases}$$

$$\begin{cases} (Q_1)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, \bar{x}_1' \leq 2, \bar{x}_1' + \bar{x}_2' \geq 4, \\ (Q_2)\ \bar{x}_3' > 0, 0 < \bar{x}_2' \leq 3, \frac{1}{2}(x_3' + \bar{x}_3') + \bar{x}_2' \geq 5, \\ (Q_3)\ \bar{x}_3' > 0, 1 \leq \bar{x}_2', \bar{x}_1' \leq 2, \frac{1}{2}(x_3' + \bar{x}_3') \geq 2, \bar{x}_1' + \bar{x}_2' \leq 3, \\ (Q_4)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, \bar{x}_1' \leq 3, 2\bar{x}_1' + \bar{x}_2' > 2, \\ (Q_5)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, \bar{x}_2' > 0, \\ (Q_6)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, \bar{x}_1' \leq 4, \\ (Q_7)\ 1 \leq \bar{x}_2' \leq 2, \bar{x}_1' + \bar{x}_2' = 4, 5 \leq s(b)' \leq 6, \\ (Q_8)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, 2\bar{x}_1' + \bar{x}_2' \geq 2, \\ (Q_9)\ \bar{x}_3' > 0, 1 \leq \bar{x}_2' \leq 2, \frac{1}{2}(x_3' + \bar{x}_3') + \bar{x}_2' \geq 2, \end{cases}$$

$$\begin{cases} (R_1)\ \bar{x}_3', \bar{x}_2' > 0, 2 \leq \bar{x}_1' \leq 3, 3 \leq \bar{x}_1' + \bar{x}_2' \leq 4, 5 \leq s(b)' \leq 6, \\ (R_2)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, 2\bar{x}_1' + \bar{x}_2' \geq 3, \\ (R_3)\ \bar{x}_3' > 0, 1 \leq \bar{x}_2' \leq 2, \frac{1}{2}(x_3' + \bar{x}_3') + \bar{x}_2' \geq 3, \\ (R_4)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, \bar{x}_1' + \bar{x}_2' \geq 3, \\ (R_5)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, \bar{x}_1' + \bar{x}_3' > 0, \\ (R_6)\ \bar{x}_3', \bar{x}_2' > 0, \bar{x}_1' + \bar{x}_2' \leq 2, \frac{1}{2}(x_3' + \bar{x}_3') + \bar{x}_2' + \bar{x}_1' \geq 3, \\ (R_7)\ \bar{x}_3' > 0, \bar{x}_2' \leq 1, \frac{1}{2}(x_3' + \bar{x}_3') \geq 2, \\ (R_8)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, \bar{x}_1' + \bar{x}_2' \geq 2, \end{cases}$$

$$\begin{cases} (S_1)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, \bar{x}_1' + 2\bar{x}_2' \geq 2, \\ (S_2)\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > 0, 1 \leq \bar{x}_2' \leq 2, \\ (S_3)\ x_3' > 0, \bar{x}_1' \leq 1, x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') \geq 2, \\ (S_4)\ \bar{x}_3', \bar{x}_1' \leq 1, x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') \geq 3, \\ (S_5)\ \bar{x}_1' \leq 2, \bar{x}_1' + \bar{x}_3' \leq 3, x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') \geq 4. \end{cases}$$

Thus, the sets $B_a^{(j+1,j)}$ for $\ell = 1$ are constructed as follows,

$$\begin{aligned} B_1^{(j+1,j)} = B_0^{(j+1,j)} &\cup \{(2,0,0,0,0,2) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid \bar{x}_1' \geq 3,\ s(b)' = 4\} \\ &\cup \{(2,0,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',0,3) \mid s(b)' = 4\} \\ &\cup \{(2,0,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',0,\bar{x}_1') \mid (P_1) \text{ holds}\} \\ &\cup \{(1,1,0,0,0,2) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid \bar{x}_1' \leq 4,\ 3 \leq s(b)' \leq 4\} \\ &\cup \{(1,1,0,0,0,2) \otimes (0,0,0,0,2,0)\} \\ &\cup \{(1,1,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',0,\bar{x}_1') \mid (P_2) \text{ holds}\} \\ &\cup \{(1,1,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',0) \mid (P_3) \text{ holds}\} \\ &\cup \{(1,1,0,0,0,2) \otimes (x_1',x_2',x_3',1,\bar{x}_2',0) \mid 1 \leq \bar{x}_2' \leq 2,\ s(b)' \leq 3\} \\ &\cup \{(1,1,0,0,0,2) \otimes (x_1',x_2',1,1,3,0) \mid s(b)' \leq 3\} \\ &\cup \{(1,1,0,0,0,2) \otimes (x_1',x_2',x_3',0,\bar{x}_2',\bar{x}_1') \mid x_3' > 0,\ 1 \leq \bar{x}_2',\bar{x}_1' \leq 2,\ s(b)' \leq 6\} \\ &\cup \{(1,1,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',1,1) \mid (P_4) \text{ holds}\} \\ &\cup \{(1,1,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid (P_5) \text{ holds}\} \\ &\cup \{(1,1,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',0,\bar{x}_1') \mid \bar{x}_3' > 0,\ \bar{x}_1' \leq 1,\ 3 \leq s(b)' \leq 6\} \\ &\cup \{(1,1,0,0,0,2) \otimes (x_1',x_2',x_3',0,0,\bar{x}_1') \mid x_3' > 0,\ \bar{x}_1' \leq 1,\ 3 \leq s(b)' \leq 6\} \\ &\cup \{(0,2,0,0,0,2) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid \bar{x}_1' \leq 4,\ 2 \leq s(b)' \leq 4\} \\ &\cup \{(0,2,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',0,\bar{x}_1') \mid (P_6) \text{ holds}\} \\ &\cup \{(0,2,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',0) \mid (P_7) \text{ holds}\} \\ &\cup \{(0,2,0,0,0,2) \otimes (x_1',x_2',x_3',0,0,0) \mid x_3' > 0,\ 2 \leq s(b)' \leq 6\} \\ &\cup \{(0,2,0,0,0,2) \otimes (x_1',x_2',x_3',0,\bar{x}_2',\bar{x}_1') \mid x_3',\bar{x}_2' > 0,\ 2 \leq s(b)' \leq 6\} \\ &\cup \{(0,2,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',1,1) \mid \bar{x}_3' > 0,\ s(b)' \leq 6\} \\ &\cup \{(0,2,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',2,0) \mid \bar{x}_3' \leq 2,\ s(b)' \leq 4\} \\ &\cup \{(0,2,0,0,0,2) \otimes (x_1',x_2',x_3',1,2,0) \mid s(b)' \leq 6\} \\ &\cup \{(0,2,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid (P_8) \text{ holds}\}, \end{aligned}$$

$$
\begin{aligned}
B_2^{(j+1,j)} = B_1^{(j+1,j)} &\cup \{(2,0,0,0,0,2) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid \bar{x}_1' \leq 2,\ s(b)' \geq 5\} \\
&\cup \{(2,0,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid (Q_1) \text{ holds}\} \\
&\cup \{(2,0,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',0) \mid (Q_2) \text{ holds}\} \\
&\cup \{(2,0,0,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid (Q_3) \text{ holds}\} \\
&\cup \{(1,0,2,0,0,2) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid \bar{x}_1' \leq 3,\ 3 \leq s(b)' \leq 6\} \\
&\cup \{(1,0,2,0,0,2) \otimes (0,0,0,0,0,4)\} \\
&\cup \{(1,0,2,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid (Q_4) \text{ holds}\} \\
&\cup \{(1,0,2,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',0) \mid \bar{x}_2' \leq 2,\ 3 \leq s(b)' \leq 6\} \\
&\cup \{(1,0,2,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',0,1) \mid \bar{x}_3' > 0,\ 3 \leq s(b)' \leq 6\} \\
&\cup \{(0,1,2,0,0,2) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid \bar{x}_1' \leq 4,\ 2 \leq s(b)' \leq 4\} \\
&\cup \{(0,1,2,0,0,2) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid \bar{x}_1' \leq 1,\ 5 \leq s(b)' \leq 6\} \\
&\cup \{(0,1,2,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',0) \mid (Q_5) \text{ holds}\} \\
&\cup \{(0,1,2,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',0,\bar{x}_1') \mid (Q_6) \text{ holds}\} \\
&\cup \{(0,1,2,0,0,2) \otimes (x_1',x_2',x_3',0,\bar{x}_2',\bar{x}_1') \mid (Q_7) \text{ holds}\} \\
&\cup \{(0,1,2,0,0,2) \otimes (x_1',x_2',x_3',0,1,2) \mid 4 \leq s(b)' \leq 6\} \\
&\cup \{(0,1,2,0,0,2) \otimes (x_1',x_2',0,2,1,2) \mid 4 \leq s(b)' \leq 6\} \\
&\cup \{(0,1,2,0,0,2) \otimes (x_1',0,1,1,1,2) \mid 4 \leq s(b)' \leq 6\} \\
&\cup \{(0,0,4,0,0,2) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid \bar{x}_1' \leq 4,\ 2 \leq s(b)' \leq 6\} \\
&\cup \{(0,0,4,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid (Q_7) \text{ holds}\} \\
&\cup \{(0,0,4,0,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',0) \mid (Q_8) \text{ holds}\},
\end{aligned}
$$

$$
\begin{aligned}
B_3^{(j+1,j)} = B_2^{(j+1,j)} &\cup \{(2,0,0,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid \bar{x}_1' \geq 3, \bar{x}_1' + \bar{x}_2' \geq 4, \\
&\qquad 5 \leq s(b)' \leq 6\} \\
&\cup \{(1,0,2,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid \bar{x}_1' \geq 4,\ 5 \leq s(b)' \leq 6\} \\
&\cup \{(0,1,2,0,0,2) \otimes (0,0,0,0,\bar{x}_2', \bar{x}_1') \mid \bar{x}_1' \geq 2,\ 5 \leq s(b)' \leq 6\} \\
&\cup \{(0,x_2,x_3,0,0,) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, 5) \mid 1 \leq x_2 \leq 2, t(b) = 2, s(b) = 4, s(b)' = 6\} \\
&\cup \{(0,1,2,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid 2 \leq \bar{x}_1' \leq 4,\ \bar{x}_1' + \bar{x}_2' = 5,\ s(b)' = 6\} \\
&\cup \{(0,1,2,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (R_1) \text{ holds}\} \\
&\cup \{(1,1,0,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid \bar{x}_1' + 2\bar{x}_2' \geq 2,\ 5 \leq s(b)' \leq 6\} \\
&\cup \{(x_1,x_2,x_3,\bar{x}_3,0,2) \otimes (0,0,0,0,\bar{x}_2', \bar{x}_1') \mid x_1 + x_2 = 1,\ 0 < x_3 \leq 2,\ x_2 + x_3 \leq 1, \\
&\qquad s(b) = 4,\ 3 \leq s(b)' \leq 6\} \\
&\cup \{(1,0,x_3,\bar{x}_3,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid \bar{x}_3 > 0, t(b) = 1, s(b) = 4, (R_2) \text{ holds}\} \\
&\cup \{(1,0,1,1,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid \bar{x}_3' > 0,\ 2\bar{x}_1' + \bar{x}_2' \leq 2,\ 3 \leq s(b)' \leq 6\} \\
&\cup \{(1,0,0,2,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', 0) \mid (R_3) \text{ holds}\} \\
&\cup \{(0,x_2,x_3,\bar{x}_3,0,2) \otimes (0,0,0,0,\bar{x}_2', \bar{x}_1') \mid 1 \leq x_2 \leq 2,\ z_4 = 0,\ t(b) = 2, \\
&\qquad s(b) \leq 6,\ 5 \leq s(b)' \leq 6\} \\
&\cup \{(0,2,0,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (R_4) \text{ holds}\} \\
&\cup \{(0,2,0,0,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 2, 0) \mid \bar{x}_3' > 0,\ \frac{1}{2}(x_3' + \bar{x}_3') \geq 3,\ s(b)' \leq 6\} \\
&\cup \{(0,1,1,1,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (R_5) \text{ holds}\} \\
&\cup \{(0,1,1,1,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, 0) \mid \bar{x}_3' > 0,\ \frac{1}{2}(x_3' + \bar{x}_3') \geq 2,\ s(b)' \leq 6\} \\
&\cup \{(0,1,1,1,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, 0) \mid x_2' > 0, \bar{x}_3' \leq 1, \frac{1}{2}(x_3' + \bar{x}_3') = 1, s(b)' \leq 6\} \\
&\cup \{(0,1,0,2,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (R_4) \text{ holds}\} \\
&\cup \{(0,1,0,2,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (R_6) \text{ holds}\} \\
&\cup \{(0,0,x_3,\bar{x}_3,0,2) \otimes (0,0,0,0,\bar{x}_2', \bar{x}_1') \mid \bar{x}_3 > 0,\ s(b) = 4,\ 2 \leq s(b)' \leq 6\} \\
&\cup \{(0,0,x_3,\bar{x}_3,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid x_3, \bar{x}_3 > 0,\ s(b) = 4,\ (Q_8) \text{ holds}\} \\
&\cup \{(0,0,x_3,\bar{x}_3,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', 0) \mid x_3, \bar{x}_3 > 0,\ s(b) = 4,\ (R_7) \text{ holds}\} \\
&\cup \{(0,0,x_3,\bar{x}_3,0,2) \otimes (x_1', x_2', 1, 1, 0, 0) \mid x_3, \bar{x}_3 > 0,\ s(b) = 4,\ x_2' > 0,\ s(b)' \leq 6\} \\
&\cup \{(0,0,0,4,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (R_8) \text{ holds}\} \\
&\cup \{(0,0,0,4,0,2) \otimes (x_1', x_2', x_3', \bar{x}_3', 1, 0) \mid \bar{x}_3' > 0,\ s(b)' \leq 6\},
\end{aligned}
$$

$$
\begin{aligned}
B_4^{(j+1,j)} = \{&(x_1,0,0,0,0,2) \mid x_1 \geq 2,\ s(b) \leq 6\} \otimes \mathcal{B} \\
&\cup \{(x_1,x_2,0,0,0,2) \mid x_2 > 0,\ s(b) = 4\} \otimes \mathcal{B} \\
&\cup \{(x_1,x_2,x_3,0,0,2) \mid x_3 > 0,\ s(b) = 4\} \otimes \mathcal{B} \\
&\cup \{(x_1,x_2,x_3,\bar{x}_3,0,\bar{x}_1) \mid x_3 > 0, 1 \leq \bar{x}_3, \bar{x}_1 \leq 1, t(b) = 2,3, s(b) \leq 6\} \otimes \mathcal{B} \\
&\cup \{(x_1,0,x_3,\bar{x}_3,0,\bar{x}_1) \mid x_1 > 0,\ x_3, \bar{x}_1 \leq 1,\ t(b) = 1,\ s(b) \leq 6\} \otimes \mathcal{B} \\
&\cup \{(0,0,x_3,\bar{x}_3,0,\bar{x}_1) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid \bar{x}_3 > 0,\ \bar{x}_1 \leq 2,\ s(b) = 4,\ 2 \leq s(b)' \leq 6\} \\
&\cup \{(0,0,x_3,\bar{x}_3,0,\bar{x}_1) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid x_3, \bar{x}_3 > 0,\ \bar{x}_1 \leq 2, \\
&\qquad s(b) = 4,\ (Q_8) \text{ holds}\} \\
&\cup \{(0,0,x_3,\bar{x}_3,0,\bar{x}_1) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',0) \mid x_3, \bar{x}_3 > 0,\ \bar{x}_1 \leq 2, \\
&\qquad s(b) = 4,\ (R_7) \text{ holds}\} \\
&\cup \{(0,0,x_3,\bar{x}_3,0,\bar{x}_1) \otimes (x_1',x_2',1,1,0,0) \mid x_3, \bar{x}_3 > 0,\ \bar{x}_1 \leq 2,\ s(b) = 4, \\
&\qquad x_2' > 0,\ s(b)' \leq 6\} \\
&\cup \{(0,0,0,4,0,\bar{x}_1) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid \bar{x}_1 \leq 2,\ s(b) \leq 4,\ (R_8) \text{ holds}\} \\
&\cup \{(0,0,0,4,0,\bar{x}_1) \otimes (x_1',x_2',x_3',\bar{x}_3',1,0) \mid \bar{x}_1 \leq 2,\ s(b) \leq 4,\ \bar{x}_3' > 0,\ s(b)' \leq 6\} \\
&\cup \{(0,1,0,2,0,\bar{x}_1) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid 1 \leq \bar{x}_1 \leq 2,\ 3 \leq s(b)' \leq 6\} \\
&\cup \{(0,1,0,2,0,\bar{x}_1) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid 1 \leq \bar{x}_1 \leq 2,\ (R_4) \text{ holds}\} \\
&\cup \{(0,1,0,2,0,\bar{x}_1) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid 1 \leq \bar{x}_1 \leq 2,\ (R_6) \text{ holds}\} \\
&\cup \{(0,1,1,1,0,2) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid 5 \leq s(b)' \leq 6\} \\
&\cup \{(0,1,1,1,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid (R_5) \text{ holds}\} \\
&\cup \{(0,1,1,1,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',0,0) \mid \bar{x}_3' > 0,\ \frac{1}{2}(x_3' + \bar{x}_3') \geq 2,\ s(b)' \leq 6\} \\
&\cup \{(0,1,1,1,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',0,0) \mid x_2' > 0, \bar{x}_3' \leq 1, \frac{1}{2}(x_3' + \bar{x}_3') = 1, s(b)' \leq 6\} \\
&\cup \{(1,0,0,2,0,\bar{x}_1) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid 1 \leq \bar{x}_1 \leq 2,\ 3 \leq s(b)' \leq 6\} \\
&\cup \{(1,0,0,2,0,\bar{x}_1) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid 1 \leq \bar{x}_1 \leq 2,\ (R_2) \text{ holds}\} \\
&\cup \{(1,0,0,2,0,\bar{x}_1) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',0) \mid 1 \leq \bar{x}_1 \leq 2,\ (R_3) \text{ holds}\} \\
&\cup \{(1,0,1,1,0,2) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid 3 \leq s(b)' \leq 6\} \\
&\cup \{(1,0,1,1,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid (R_2) \text{ holds}\} \\
&\cup \{(1,0,1,1,0,2) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid \bar{x}_3' > 0,\ 2\bar{x}_1' + \bar{x}_2' \leq 2,\ 3 \leq s(b)' \leq 6\} \\
&\cup \{(0,1,0,2,0,0) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid 2 \leq s(b)' \leq 6\} \\
&\cup \{(0,1,0,2,0,0) \otimes (x_1',x_2',x_3',\bar{x}_3',\bar{x}_2',\bar{x}_1') \mid (R_8) \text{ holds}\} \\
&\cup \{(0,1,0,2,0,0) \otimes (x_1',x_2',x_3',\bar{x}_3',1,0) \mid \bar{x}_3' > 0,\ s(b)' \leq 6\} \\
&\cup \{(0,2,0,0,0,0) \otimes (0,0,0,0,\bar{x}_2',\bar{x}_1') \mid s(b)' \leq 6\}
\end{aligned}
$$

$\cup \{(0,2,0,0,0,0) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid (S_1) \text{ holds}\}$

$\cup \{(0,2,x_3,\bar{x}_3,0,0) \otimes (x_1', x_2', x_3', 0,0,0) \mid x_3 \le 1, z_4 = 0, s(b) \le 6, x_3' > 0, s(b)' \le 6\}$

$\cup \{(0,2,0,0,0,1) \otimes (0,0,0,0,\bar{x}_2', \bar{x}_1') \mid 3 \le s(b)' \le 6\}$

$\cup \{(0,2,x_3,\bar{x}_3,0,\bar{x}_1) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid x_3 \le 1,\ z_4 = 0,\ \bar{x}_1 + \bar{x}_3 = 1,$
$s(b) \le 6,\ (R_4) \text{ holds}\}$

$\cup \{(0,2,0,0,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid \bar{x}_3' > 0,\ \bar{x}_1' \le 1,\ \bar{x}_1' + \bar{x}_2' = 2,\ s(b)' \le 6\}$

$\cup \{(0,2,0,0,0,1) \otimes (x_1', x_2', x_3', \bar{x}_3', 1,0) \mid \bar{x}_3' > 0,\ \frac{1}{2}(x_3' + \bar{x}_3') \ge 2,\ s(b)' \le 6\}$

$\cup \{(x_1,2,1,1,0,0) \otimes (0,0,0,0,\bar{x}_2', \bar{x}_1') \mid s(b) \le 6,\ s(b)' \le 6\}$

$\cup \{(0,2,1,1,0,0) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', 0) \mid (S_2) \text{ holds}\}$

$\cup \{(0,2,1,1,0,0) \otimes (x_1', x_2', x_3', \bar{x}_3', 1,1) \mid \bar{x}_3' > 0,\ s(b)' \le 6\}$

$\cup \{(x_1,x_2,0,0,0,\bar{x}_1) \otimes (0,0,0,0,\bar{x}_2', \bar{x}_1') \mid x_1 > 0,\ 0 < x_2 \le 2,\ \bar{x}_1 \le 1,$
$s(b) \le 6,\ s(b)' \le 6\}$

$\cup \{(x_1,x_2,0,0,0,\bar{x}_1) \otimes (x_1', x_2', x_3', \bar{x}_3', \bar{x}_2', \bar{x}_1') \mid x_1 > 0,\ 0 < x_2 \le 2,\ \bar{x}_1 \le 1,$
$s(b) \le 6,\ (S_1) \text{ holds}\}$

$\cup \{(x_1,x_2,0,0,0,\bar{x}_1) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, \bar{x}_1') \mid x_1 > 0,\ 0 < x_2 \le 2,\ \bar{x}_1 \le 1,\ s(b) \le 6,$
$0 < x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') \le x_1,\ \bar{x}_1' \le 1,\ s(b)' \le 6\}$

$\cup \{(x_1,x_2,0,0,0,\bar{x}_1) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, \bar{x}_1') \mid x_1 > 0,\ 0 < x_2 \le 2,\ \bar{x}_1 \le 1,\ s(b) \le 6,$
$\bar{x}_3', \bar{x}_1' \le 1,\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > x_1,\ s(b)' \le 6\}$

$\cup \{(x_1,x_2,0,0,0,\bar{x}_1) \otimes (x_1', x_2', x_3', \bar{x}_3', 0,0) \mid x_1 > 0,\ 0 < x_2 \le 2,\ \bar{x}_1 \le 1,\ s(b) \le 6,$
$2 \le \bar{x}_3' \le x_1,\ x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') > x_1,\ s(b)' \le 6\}$

$\cup \{(x_1,x_2,0,0,0,\bar{x}_1) \otimes (x_1', x_2', x_3', \bar{x}_3', 0,1) \mid x_1 > 0,\ 0 < x_2 \le 2,\ \bar{x}_1 \le 1,\ s(b) \le 6,$
$x_3' \le x_1 + 1,\ 2 \le \bar{x}_3' \le x_1,\ s(b)' \le 6\}$

$\cup \{(x_1,2,1,1,0,0) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, \bar{x}_1') \mid x_1 > 0,\ s(b) \le 6,\ \bar{x}_1' \ge 3,\ s(b)' \le 6\}$

$\cup \{(x_1,2,1,1,0,0) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, \bar{x}_1') \mid x_1 > 0,\ s(b) \le 6,\ \bar{x}_1' \le 2,$
$0 < x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') \le x_1,\ s(b) \le 6\}$

$\cup \{(1,2,1,1,0,0) \otimes (x_1', x_2', x_3', 0,0, \bar{x}_1') \mid (S_3) \text{ holds}\}$

$\cup \{(1,2,1,1,0,0) \otimes (x_1', x_2', x_3', 1,0,0) \mid x_1' + x_2' + \frac{1}{2}(x_3' + \bar{x}_3') \ge 2,\ s(b)' \le 6\}$

$\cup \{(2,2,1,1,0,0) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, \bar{x}_1') \mid (S_4) \text{ holds}\}$

$\cup \{(3,2,1,1,0,0) \otimes (x_1', x_2', x_3', \bar{x}_3', 0, \bar{x}_1') \mid (S_5) \text{ holds}\},$

$$
\begin{aligned}
B_5^{(j+1,j)} = {} & \{(x_1, x_2, x_3, \bar{x}_3', 0, 2) \mid s(b) = 4\} \otimes \mathcal{B} \\
& \cup \{(x_1, x_2, 0, 0, 0, 2) \mid 5 \leq s(b) \leq 6\} \otimes \mathcal{B} \\
& \cup \{(x_1, x_2, 0, 0, 0, 1) \mid x_2 > 0,\ s(b) \leq 6\} \otimes \mathcal{B} \\
& \cup \{(0, 0, x_3, \bar{x}_3, 0, 1) \mid \bar{x}_3 > 0,\ 3 \leq s(b) \leq 4\} \otimes \mathcal{B} \\
& \cup \{(x_1, 0, x_3, \bar{x}_3, 0, 1) \mid x_1, \bar{x}_3 > 0,\ \frac{1}{2}(x_3 + \bar{x}_3) \leq 2,\ s(b) \leq 6\} \otimes \mathcal{B} \\
& \cup \{(0, x_2, x_3, \bar{x}_3, 0, 1) \mid x_2, \bar{x}_3 > 0, ,\ \frac{1}{2}(x_3 + \bar{x}_3) = 1,\ s(b) \leq 3\} \otimes \mathcal{B} \\
& \cup \{(0, 1, x_3, \bar{x}_3, 0, 1) \mid \bar{x}_3 > 0,\ \frac{1}{2}(x_3 + \bar{x}_3) = 2,\ s(b) \leq 6\} \otimes \mathcal{B} \\
& \cup \{(0, x_2, 0, 4, 0, 1) \mid x_2 > 0,\ s(b) \leq 6\} \otimes \mathcal{B} \\
& \cup \{(x_1, 0, x_3, \bar{x}_3, 0, 0) \mid \bar{x}_3 \geq 2,\ \frac{1}{2}(x_3 + \bar{x}_3) \leq 4,\ 2 \leq s(b)' \leq 6\} \otimes \mathcal{B} \\
& \cup \{(x_1, x_2, x_3, \bar{x}_3, 0, 0) \mid x_2 > 0, \bar{x}_3 \geq 2, \frac{1}{2}(x_3 + \bar{x}_3) \leq 3,\ 2 \leq s(b)' \leq 6\} \otimes \mathcal{B} \\
& \cup \{(x_1, x_2, 0, 8, 0, 0) \mid x_2 > 0,\ s(b) \leq 6\} \otimes \mathcal{B} \\
& \cup \{(x_1, x_2, 1, 1, 0, 0) \mid x_2 > 1,\ s(b) \leq 6\} \otimes \mathcal{B}, \\
B_6^{(j+1,j)} = {} & B_6^{(j)} \otimes \mathcal{B}
\end{aligned}
$$

From above, we see that condition (1) in Theorem 3.4.3 holds for $\lambda = 2(\Lambda_0 + \Lambda_1)$ with $\kappa = 2$. This implies Conjecture 5.3.5 holds when $\ell = 2$, and path realizations of the corresponding Demazure crystals $B_{w^{(k)}}(2(\Lambda_0 + \Lambda_1))$ for $U_q(D_4^{(3)})$ have tensor product-like structures.

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