ABSTRACT

WHEELESS, WILLIAM BENNETT. Additional Symmetries of the Extended Toda Hierarchy. (Under the direction of Bojko Bakalov.)

Some integrable hierarchies, such as the Korteweg-de Vries (KdV) hierarchy and the Toda hierarchy, arise by modeling a physical process with a base differential equation then adding other evolutionary equations that commute with it. This allows one to create a flow of solutions of the first equation. In addition to this, Gromov-Witten invariants of certain projective manifolds (or more generally, orbifolds) can be organized into a generating function that is a solution to an integrable hierarchy. The extended Toda hierarchy (ETH) was introduced to find such a hierarchy that governs the Gromov-Witten invariants of $\mathbb{C}P^1$. Similarly, the extended bigraded Toda hierarchy (EBTH) was found to govern the Gromov-Witten theory of certain orbifolds of $\mathbb{C}P^1$ after its introduction.

The evolutionary differential equations that form a given integrable hierarchy produce actions that commute with one another and can be solved simultaneously; in other words, they are symmetries of one another. These equations form flows of solutions. For some hierarchies, such as KdV and Kadomtsev-Petviashvili (KP), these are not the only flows that commute with the flows of the hierarchy. These additional commuting flows are fittingly called additional symmetries. After reviewing the tools necessary to the discussion of additional symmetries in the context of the KdV and KP hierarchies, we discuss the Toda hierarchy, its symmetries, and the ETH. We define additional flows for the extended Toda in Lax form, proving that these flows are indeed symmetries of the ETH and that they commute with one another as the Virasoro algebra. We then show how these symmetries act on the tau function of the hierarchy. Finally, we move to the EBTH, defining additional symmetries for it and investigating these additional flows' actions on the hierarchy's Lax operator, wave functions, and tau function.

Additional Symmetries of the Extended Toda Hierarchy

by
William Bennett Wheeless

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APPROVED BY:

Naihuan Jing

Kailash Misra

Ernest Stitzinger

Stephen Reynolds

Bojko Bakalov
Chair of Advisory Committee

DEDICATION

To my parents, who instilled in me a desire to learn and a love for mathematics.

BIOGRAPHY

The author was born in in Raleigh, NC on November 10, 1987 to Mark and Sara Wheeless. He has an older sister, Amy, and a younger brother, Daniel. For undergraduate studies, William also attended North Carolina State University where, in 2010, he earned Bachelor of Science degrees in Mathematics and Physics with a Minor in Computer Programming.

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CHAPTER

1 INTRODUCTION

Integrable hierarchies, such as those looked at in this thesis, are formulated in several ways. The goal of this chapter is to introduce two such formulations, the Lax pair form and the tau function, as well as what it means for two differential equations to be symmetries of one another. This is done within the context of the Korteweg-de Vries and Kadomtsev-Petviashvili hierarchies so that these concepts will be well-established when we introduce the corresponding notions for the Toda lattice and extended Toda hierarchies in Chapter 2. The majority of this chapter's material is adapted from [7] and [15].

1.1 Symmetries

A central concept of this thesis is what it means for a set of evolutionary partial differential equations to be symmetries of one another. We will first look at what this means for two such equations:

Definition 1.1. *Suppose we have an evolutionary partial differential equation on u*

$$\partial_t u = K(u), \tag{1.1}$$

with K only depending directly on x and its derivatives (or other like operators). Let $u(x,t)$ be a *solution of (1.1) and use it as an initial condition (i.e. $u(x,t,0) = u(x,t)$) of a second evolutionary partial differential equation on u*

$$\partial_s u = H(u), \tag{1.2}$$

*again with H only depending directly on x and its derivatives. We say that (1.2) is a **symmetry** of (1.1) if there is some neighborhood of $s = 0$ for which $u(x,t,s)$ is a solution to (1.1) for all s in that neighborhood.*

Note that this definition is symmetric in the sense that (1.2) is a symmetry of (1.1) if and only if (1.1) is a symmetry of (1.2).

Another way of saying (1.2) is a symmetry of (1.1) is that these two equations are ***compatible***: they can be simultaneously solved, which is equivalent to saying

$$\partial_t\big(H(u)\big) = \partial_t \partial_s u = \partial_s \partial_t u = \partial_s\big(K(u)\big). \tag{1.3}$$

Taking the middle equivalence and taking the difference, we get the following definition of two differential operators, ∂_t and ∂_s, being symmetries of one another:

$$[\partial_t, \partial_s]u = 0, \tag{1.4}$$

where $[\cdot,\cdot]$ is the commutator bracket used in Lie algebras; for two operators, A and B,

$$[A,B] = A \circ B - B \circ A,$$

where $\circ$ denotes composition (this will be suppressed for much of the remainder of the thesis). Equation (1.4) is the definition of symmetry that we will use for the rest of the thesis, replacing u with various operators and functions.

If we can find such a symmetry for our base equation (1.1), then our common solution $u(x,t,s)$ gives us a flow of solutions for this equation; $u(x,t,0)$ is a solution, but so is $u(x,t,\varepsilon)$, where ε is in the neighborhood of $s = 0$ referenced in Definition 1.1 above. This allows us to find new solutions from known solutions by traveling along these so-called differential flows.

A set of such differential equations, $\partial_{t_n} u = K_n(u)$, are all symmetries of one another if (1.4) is satisfied for $t = t_n$ and $s = t_m$, for any n, m in the set that indexes these equations. In this way, these t_n's can be thought as variables in our common solution.

Example: The KdV Equation

To illustrate how finding these symmetries can be accomplished, consider the Korteweg-de Vries (abbreviated KdV) equation:

$$\partial_t u = K(u) = \frac{3}{2} u u_x + \frac{1}{4} u_{xxx}. \tag{1.5}$$

This equation provides a model for the movement of waves through narrow, shallow channels, such as those found in blood vessels. To find a symmetry for this equation, we first need to introduce a grading that makes the right-hand side homogeneous. Letting $\deg(u) = 2$ and $\deg(\partial_x) = 1$ makes the right side homogeneous of degree 5. Assuming the $H(u)$ is also homogeneous yields the first non-trivial symmetry to be of degree 7:

$$\partial_s = H(u) = c_1 u^2 u_x + c_2 u u_{xxx} + c_3 u_x u_{xx} + c_4 u_{xxxxx}.$$

For the sake of brevity, only part of the calculation is shown:

$$\begin{aligned}\partial_s \partial_t u = \partial_s\big(K(u)\big) = \partial_s\big(\frac{3}{2} u u_x + \frac{1}{4} u_{xxx}\big)& \\ = \frac{3}{2} u_s u_x + \frac{3}{2} u(u_s)_x + \frac{1}{4}(u_s)_{xxx}& \\ = \frac{3}{2} H(u) u_x + \frac{3}{2} u(H(u))_x + \frac{1}{4}(H(u))_{xxx}.&\end{aligned}$$

Writing out $H(u)$ and taking the appropriate number of derivatives, doing the same process for the left side (1.3), then matching like terms allows us to come to the following relationship between the constants c_k:

$$2c_1 = 6c_2 = 3c_3 = 60c_4,$$

giving us the first non-trivial symmetry of (1.5) as

$$\partial_s = H(u) = 30u^2 u_x + 10u u_{xxx} + 20u_x u_{xx} + u_{xxxxx} \tag{1.6}$$

up to some arbitrary non-zero multiple of the right-hand side.

1.2 Lax Form and Formation of a Hierarchy

The study of such hierarchies of differential equations as the KdV hierarchy, KP hierarchy, and Toda hierarchies rely heavily on Lax pairs. For this reason, the vast majority of original results found in this thesis are given in Lax pair form. We will realize the concept with the KdV hierarchy, then generalize it to the larger KP (Kadomtsev-Petviashvili) and other such hierarchies.

In general, we consider an operator L of some form that incorporates the solution of the partial differential equation in which we are interested; this is what will be called the ***Lax operator***. In the case of KdV, this operator is the 1-D Schrödinger operator, $L = \partial_x^2 + u$. Consider the following:

$$L\psi = z^2\psi, \tag{1.7}$$

where $\psi = \psi(x,z)$ is an eigenfunction of L of eigenvalue z^2, also called a ***wave function***. In general, the power of z depends on the degree of the operator L; since L for KdV is of degree 2 according to the previous grading scheme, the eigenvalue is z^2.

For KdV, we find a formal solution for the family of eigenfunctions

$$\psi = w(x,z)e^{xz} = e^{xz}\left(1 + w_1 z^{-1} + w_2 z^{-2} + \cdots\right),$$

whose coefficients can be found in terms of u and its derivatives. In addition to imposing (1.7), we also want to find a symmetry of the form

$$\partial_t\psi = B\psi, \tag{1.8}$$

where B is another operator independent of z and its derivatives of finite order. Combining (1.7) and (1.8), we get

$$\begin{aligned}
\partial_t\left(L\psi\right) &= z^2\partial_t\psi \\
\frac{\partial L}{\partial t}\psi + L\left(\partial_t\psi\right) &= z^2\partial_t\psi \\
\frac{\partial L}{\partial t}\psi + LB\psi &= z^2 B\psi = B\left(z^2\psi\right) = BL\psi \\
\left(\frac{\partial L}{\partial t} - BL + LB\right)\psi &= 0 \\
\left(\frac{\partial L}{\partial t} - [B,L]\right)\psi &= 0.
\end{aligned}$$

Since the operator acting on ψ in the last equality is of finite order and independent of z, there must be a finite number of independent ψ that satisfy this equation. However, since there is an independent ψ for each z, there are an infinite number of independent ψ. Thus, this operator must be 0. This gives us

$$\frac{\partial L}{\partial t} = [B,L]. \tag{1.9}$$

This is called the ***Lax equation*** and (L, B) is a ***Lax pair***.

In particular for KdV, we assume that B takes the form

$$B = \partial_x^3 + b_1 \partial_x + b_0 \tag{1.10}$$

and impose (1.9) for $\dfrac{\partial L}{\partial t} = u_t$. Then, we find that $b_1 = \dfrac{3}{2}u$ and $b_0 = \dfrac{1}{4}u_x$ up to the addition of arbitrary constants (assumed to be 0); the KdV equation (1.5) comes from the constant coefficient (the ∂_x^0 term) of the right-hand side. Note that this solution makes B homogeneous of degree 3.

Pseudo-differential Operators

To arrive at a more general method of finding these symmetries in Lax form, we need to introduce the concept of pseudo-differential operators.

Definition 1.2. *An operator A is a **pseudo-differential operator of order** n ($n \in \mathbb{Z}$) if it is of the form*

$$A = \sum_{i=-\infty}^{n} a_i \partial_x^i, \quad a_i = a_i(x),$$

where $a_n(x) \neq 0$.

Note that for any function $f(x)$ and $n \in \mathbb{Z}$, we have the *Leibniz rule* for pseudo-differential operators:

$$\partial_x^n \circ f = \sum_{i=0}^{\infty} \binom{n}{i} (\partial_x^i f) \partial_x^{n-i}.$$

We also have for any $n \in \mathbb{Z}$ that $\partial_x^n e^{xz} = z^n e^{xz}$. Thus, for any pseudo-differential operator A, as given above, we have

$$Ae^{ez} = \left(\sum_{i=-\infty}^{n} a_i z^i \right) e^{xz}.$$

Define the decomposition of the space of pseudo-differential operators into "plus" and "minus" parts as follows: for such an operator A as defined before,

$$A_+ = \sum_{i=0}^{n} a_i(x)\partial_x^i, \quad A_- = \sum_{i=-\infty}^{-1} a_i(x)\partial_x^i.$$

Finally, suppose we have a pseudo-differential operator of the form

$$A = \sum_{i=0}^{\infty} a_i(x)\partial_x^{-i},$$

where $a_0(x) \neq 0$. Then A has a unique inverse, $A^{-1} = \sum_{i=0}^{\infty} a_i^{-1} \partial_x^{-i}$, such that

$$A \circ A^{-1} = A^{-1} \circ A = 1$$

and whose coefficients can be solved for recursively.

Wave Operator

Define the pseudo-differential operator W, called the ***wave operator*** (or sometimes ***dressing operator***), as

$$W = \sum_{i=0}^{\infty} w_i(x) \partial_x^{-i}, \tag{1.11}$$

where $w_0(x) = 1$ and the $w_i(x)$ are the same as those of $w(x,z)$ in the wave function ψ. Then we have that

$$\psi = w(x,z) \exp\left(xz + \sum_{i=1}^{\infty} t_i z^i \right) = W \exp\left(xz + \sum_{i=1}^{\infty} t_i z^i \right).$$

Combining this observation with (1.7) gives us

$$\begin{aligned} LWe^{xz+\cdots} &= z^2 We^{xz+\cdots} \\ &= W\left(z^2 e^{xz+\cdots}\right) \\ &= W\partial_x^2 e^{xz+\cdots}, \end{aligned}$$

which means that

$$L = W \partial_x^2 W^{-1}. \tag{1.12}$$

The Hierarchy

In order to present a Lax pair representation of a family of symmetries to (1.5), define the pseudo-differential operator

$$P = \partial_x + p_1(x) \partial_x^{-1} + p_2(x) \partial_x^{-2} + \cdots \tag{1.13}$$

such that $P = W \partial_x W^{-1}$, making $P^2 = L$. Due to this definition of P, we have that ψ is also an eigenfunction of P with eigenvalue z. We also have a well defined concept of the inverse of P: $P^{-1} = W \partial_x^{-1} W^{-1}$ with $P^{-1}\psi = z^{-1}\psi$.

Solving for the first few coefficients of W and then P, one can show that $B = (P^3)_+$, for B defined as before in (1.10).

We define a hierarchy of evolutionary partial differential equations based upon the KdV equation as follows:

Definition 1.3. *The **KdV hierarchy** (in Lax form) is given by*

$$\frac{\partial L}{\partial t_n} = [(P^n)_+, L], \quad n = 1, 3, 5, \ldots, \tag{1.14}$$

where the t_n are now new variables for L.

Each of these equations are symmetries of one another and of the KdV equation, which arises from the equation for $n = 3$. We also have that $\partial_{t_1} = \partial_x$, allowing us to interchange x and t_1 freely. Note that since $P^2 = L$, $(P^{2k})_+ = L^k$, making $\dfrac{\partial L}{\partial t_{2k}} = 0$.

The KP Hierarchy

The flows of (1.14) induce similar actions on the operator P:

$$\frac{\partial P}{\partial t_n} = [(P^n)_+, P], \quad n = 1, 3, 5, \ldots, \tag{1.15}$$

where the restriction to positive odd integers stems from the condition that P^2 is a purely differential operator. The broader KP (Kadomtsev-Petviashvili) hierarchy comes about by removing this restriction on P and using it as the Lax operator:

Definition 1.4. *On a pseudo-differential operator $P = \partial_x + p_1\partial_x^{-1} + p_2\partial_x^{-2} + \cdots$, the flows of the **KP hierarchy** are defined to be*

$$\frac{\partial P}{\partial t_n} = [(P^n)_+, P], \quad n \in \mathbb{Z}_{\geq 1}. \tag{1.16}$$

Now, all previous functions and operator, including w, W, and ψ, depend on $\mathbf{t} = (t_1, t_2, t_3, \ldots)$.

Restricting different powers of P to be purely differential operators gives us different reductions of this hierarchy. As already seen, doing so for P^2 gives us the KdV hierarchy; doing so for P^3 gives us the *Boussinesq hierarchy*; doing so for any $N \in \mathbb{Z}_{\geq 2}$ gives us the N^{th} *Gelfand-Dickey hierarchy*.

Finally, the KP hierarchy has a mirrored concept of a wave function and wave operator, and the flows induce actions on each:

$$\frac{\partial \psi}{\partial t_n} = (P^n)_+\psi, \tag{1.17}$$

$$\frac{\partial W}{\partial t_n} = -(P^n)_- W. \tag{1.18}$$

For the remainder of this chapter, we will focus on the KP hierarchy; statements about it can also be made for the KdV hierarchy by restricting $(P^2)_- = 0$.

1.3 Tau Function

The last major object discussed in this dissertation is the tau function. For a hierarchy such as the ones studied in this thesis, the tau function is a single function with which the entire hierarchy can be described.

Existence

For a function f of $\mathbf{t} = (t_1, t_2, t_3, \ldots)$, define the operator $G(z)$ acting on f as

$$G(z) f(\mathbf{t}) = f(\mathbf{t} - [z^{-1}]), \tag{1.19}$$

where

$$[z] = \left(z, \frac{1}{2} z^2, \frac{1}{3} z^3, \ldots \right). \tag{1.20}$$

More explicitly,

$$G(z) = \exp\left(- \sum_{i=1}^{\infty} \frac{1}{i} z^{-i} \frac{\partial}{\partial t_i} \right).$$

Using this operator, we have the following theorem by Mikio Sato:

Theorem 1.1. *For the KP hierarchy, there exists a function, $\tau(\mathbf{t})$, called the **tau function**, such that*

$$w(\mathbf{t}, z) = \frac{G(z)\tau(\mathbf{t})}{\tau(\mathbf{t})}. \tag{1.21}$$

Due to the nature of the definition of the coefficients of W and w and the result of Theorem 1.1, τ is unique only up to multiplication by $c \exp\left(\sum_{i=1}^{\infty} c_i t_i \right)$, where c, c_1, c_2, … are arbitrary constants with respect to the t_i variables.

1.4 Additional Symmetries of KP

The flows of the KP hierarchy (and its reductions, such as KdV) are symmetries of one another. However, these are not the only flows whose actions commute with the hierarchy. These flows are called ***additional symmetries***, and there are an infinite number of them. These are not included in the hierarchy itself because these extra flows do not commute amongst themselves.

Lax Form

Define a new operator, M, such that $\partial_z \psi = M\psi$. This gives us

$$M := W x W^{-1} + \sum_{n=2}^{\infty} n t_n L^{n-1} \tag{1.22}$$

and, for any $\ell \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$,

$$P^{\ell} M^m \psi = \partial_z^m z^{\ell} \psi, \quad M^m P^{\ell} \psi = z^{\ell} \partial_z^m \psi.$$

It is important to note that this operator is only well defined when only finitely many of the t_n variables are nonzero.

Then, for such ℓ and m, define a new set of differential equations

$$\partial_{t^*_{\ell m}} W = -(M^m P^{\ell})_- W, \tag{1.23}$$

where $t^*_{\ell m}$ are some new set of variables. The flows on the wave operator in (1.23) lead directly to the actions on P:

$$\partial_{t^*_{\ell m}} P = [-(M^m P^{\ell})_-, P]. \tag{1.24}$$

We then have the following theorem by Orlov and Schulman from [16]:

Theorem 1.2. *The flows defined in (1.23) and (1.24) give the Lax representation of additional symmetries for the KP hierarchy; i.e.*

$$[\partial_{t_n}, \partial_{t^*_{\ell m}}]\Phi = 0,$$

where Φ *can be any one of* W, P, *or* ψ.

Action on τ

The actions of the additional symmetries of the KP hierarchy on the wave function induce corresponding actions on the tau function by means of the Adler-Shiota-van Moerbeke formula (as seen in [1, 2]:

$$\frac{\partial_{t^*_{\ell m}} \psi}{\psi} = G(z)\left(\frac{\partial_{t^*_{\ell m}} \tau}{\tau}\right) - \frac{\partial_{t^*_{\ell m}} \tau}{\tau}. \tag{1.25}$$

Note that the previous formula can be found by treating $\partial_{t^*_{\ell m}}$ as a derivation and using the quotient rule from calculus. The previous formula holds regardless of the hierarchy or the symmetry, as long as there is a representation of the coefficients of the wave operator in terms of τ similar to that found in (1.21).

The corresponding actions of the additional symmetries on τ are more easily represented within a generating function: Define the operator

$$X(\mathbf{t},\lambda,\mu) = \exp\left(\sum_{n=1}^{\infty}(\mu^n - \lambda^n)t_n\right)\exp\left(\sum_{n=1}^{\infty}(\lambda^{-n} - \mu^{-n})\frac{1}{n}\frac{\partial}{\partial t_n}\right). \tag{1.26}$$

The actions of $\partial_{t^*_{\ell m}}$ on τ are then given by

$$\partial_{t^*_{\ell+m,m}}\tau = \frac{W_\ell^{(m+1)}\tau}{m+1}, \tag{1.27}$$

where, expanding $X(\mathbf{t},\lambda,\mu)$ in all powers of λ and nonnegative powers of $\mu-\lambda$, we have

$$X(\mathbf{t},\lambda,\mu) = \sum_{k=0}^{\infty}\frac{(\mu-\lambda)^k}{k!}\sum_{j\in\mathbb{Z}}\lambda^{-j-k}W_j^{(k)}.$$

These $W_j^{(k)}$ operators satisfy the Adler-Shiota-van Moerbeke formula for the actions of the additional symmetries.

1.5 Outline of this Thesis

The remainder of this thesis introduces and discusses its main topic, the extended Toda hierarchy. Chapter 2 introduces the Toda lattice hierarchy, its Lax form and relevant operators, and its symmetries, similar to how this chapter does the same for the KP hierarchy. It then goes on the extend this hierarchy to the extended Toda hierarchy. Chapter 3 contains the main original results of this thesis, involving the definitions of the additional symmetries of the extended Toda hierarchy as well as their actions on the Lax operator, wave functions, and tau function. We also show that these flows commute with each other as the Virasoro algebra. Finally, Chapter 4 briefly discusses a larger hierarchy originally defined in [4], the extended bigraded Toda hierarchy. We then define its additional symmetries in Lax form and investigate how these symmetries act on its tau function.

CHAPTER

2

EXTENDED TODA HIERARCHY

This thesis' main result involves additional symmetries of the extended Toda hierarchy. As the name implies, this is an extension of a smaller hierarchy, the Toda lattice hierarchy. In this chapter, we discuss the Toda lattice hierarchy, the functions and operators associated with it, its additional symmetries, and ultimately, its extension. Material found in Sections 2.1 and 2.2 primarily comes from [3], [5], and Section 6 of [20]. Section 2.3 contains material adapted from [5] and [17].

2.1 The Toda Lattice Hierarchy

In 1967, Morikazu Toda introduced a model describing the movement of particles in an infinite one-dimensional crystal lattice in [18]. A simplified version of this model, rescaling variables so that the masses of the particles do not appear, is

$$\frac{\partial^2 q_n}{\partial t^2} = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}}, \quad n \in \mathbb{Z} \tag{2.1}$$

where $q_n = q_n(t)$ represents the position of the n^{th} particle at time t. This is the ***Toda lattice equation***.

In order to use the tools and theory developed for KP on Toda, we must change this second order equation into a first order one by way of a common method in solving partial differential equations. Define the dependent variables v_n and u_n by

$$v_n = -\frac{\partial q_n}{\partial t}, \quad u_n = q_{n-1} - q_n.$$

Using these variables, the Toda lattice equation can be recast into a system of first order equations:

$$\begin{cases} \dfrac{\partial v_n}{\partial t} & = e^{u_{n+1}} - e^{u_n} \\ \dfrac{\partial u_n}{\partial t} & = v_n - v_{n-1} \end{cases}, \quad n \in \mathbb{Z}. \tag{2.2}$$

Lax Form

The KP hierarchy considers pseudo-differential operators. In analogy with these types of operators, the Toda lattice makes use of the shift operator Λ in place of ∂_x:

Definition 2.1. *Let $f : \mathbb{Z} \to \mathbb{C}$ be a function on the infinite lattice. Then the* ***shift operator*** *Λ is defined such that*

$$\Lambda f_n = f_{n+1}.$$

Note that Λ^{-1} is well defined:

$$\Lambda^{-1} f_n = f_{n-1}.$$

In the context of this thesis, a ***difference operator*** *is any formal power series in Λ and Λ^{-1} with coefficients functions on the lattice.*

Also in analogy with KP, for any such difference operator $A = \sum_{i \in \mathbb{Z}} a_i \Lambda^i$, define the "plus" and "minus" parts of A, denoted A_+ and A_- respectively, as

$$A_+ = \sum_{i \geq 0} a_i \Lambda^i, \quad A_- = \sum_{i<0} a_i \Lambda^i.$$

With these concepts in mind, define the Lax operator L for the Toda lattice as

$$L = L_n = \Lambda + v_n + e^{u_n} \Lambda^{-1}. \tag{2.3}$$

Then the Toda lattice equation can be rewritten in Lax form as

$$\frac{\partial L}{\partial t} = [L_+, L]. \tag{2.4}$$

Indeed, a direct calculation shows that the first equation of (2.2) follows from the Λ^0 terms of each side of (2.4) while the second equation follows from the Λ^{-1} terms.

In order to discuss the formulation of the Toda lattice hierarchy and, ultimately, its additional symmetries, we temporarily reformulate the Lax operator as an infinite matrix. We will return to the form in (2.3) in Section 2.3.

Infinite Matrices

Define $\tilde{g\ell}_\infty$ to be the algebra of all $\mathbb{Z}\times\mathbb{Z}$ matrices with only finitely many nonzero diagonals. It is clear that multiplication of any finite number of such matrices is well defined. Define $\Lambda \in \tilde{g\ell}_\infty$ (the ***shift matrix***) such that $\Lambda_{k,k+1} = 1$ for any $k \in \mathbb{Z}$ and all other entries are 0:

$$\Lambda = \begin{pmatrix} \ddots & & & & & & \\ & 0 & 1 & 0 & 0 & 0 & \\ & 0 & 0 & 1 & 0 & 0 & \\ & 0 & 0 & 0 & 1 & 0 & \\ & 0 & 0 & 0 & 0 & 1 & \\ & 0 & 0 & 0 & 0 & 0 & \\ & & & & & & \ddots \end{pmatrix}.$$

This matrix has the property that for any infinite column vector $f = (f_n)_{n\in\mathbb{Z}}$, $(\Lambda f)_n = f_{n+1}$, similar to the shift operator in Definition 2.1. Also, define the infinite column vector $\zeta(z) := (z^n)_{n\in\mathbb{Z}}$. Note that $\Lambda^i\zeta = z^i\zeta$ for $i \in \mathbb{Z}$. Similar to how KP considers operators that have finitely many positive powers of ∂_x, the Toda lattice will only consider such difference operators with only finitely many positive powers of Λ.

Similar to how P is defined in the KP case, define the Lax matrix operator $L = \Lambda + a_0 + a_1\Lambda^{-1}$ (just as in (2.3)) , where a_0 and a_1 are matrices with only main diagonal entries:

$$L = \begin{pmatrix} \ddots & & & & & & \\ & L_{-2,-2} & 1 & 0 & 0 & 0 & \\ & L_{-1,-2} & L_{-1,-1} & 1 & 0 & 0 & \\ & 0 & L_{0,-1} & L_{0,0} & 1 & 0 & \\ & 0 & 0 & L_{1,0} & L_{1,1} & 1 & \\ & 0 & 0 & 0 & L_{2,1} & L_{2,2} & \\ & & & & & & \ddots \end{pmatrix}.$$

Also as in the case with KP and the definition in the previous section, we consider the splitting of such infinite matrices into their upper triangular (with diagonal) part, denoted as $()_+$, and their lower triangular part (without diagonal), denoted as $()_-$. It is clear that $A_+ + A_- = A$, for any $A \in \tilde{g\ell}_\infty$.

Definition 2.2. *The* ***Toda lattice hierarchy*** *(in Lax matrix form) is the set of differential flows of $t_1, t_2, \dots$ on L given by (similar to KP)*

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L]. \tag{2.5}$$

Note that for $n = 1$, the right-hand sides of (2.5) and (2.3) are the same. Because of this observation, we say that $t = t_1$. There is an equivalent definition of the hierarchy using a modified Lax matrix that will be discuss when talking about additional symmetries in Section 2.2.

Wave Matrix, Wave Vector, and Tau Vector

As in the KP case, the Toda hierarchy has a wave matrix operator $W = I + w_1\Lambda^{-1} + w_2\Lambda^{-2} + \cdots$ such that

$$L = W\Lambda W^{-1}, \quad \frac{\partial W}{\partial t_n} = -(L^n)_- W$$

We also have the wave vector $\psi = (\psi_n)_{n\in\mathbb{Z}}$ defined as

$$\psi = We^{\xi(\mathbf{t},z)}\zeta,$$

where $\xi(\mathbf{t}, z) = \sum_{i=1}^{\infty} t_i z^i$. Note that these definitions give us that $L\psi = z\psi$ and $\frac{\partial \psi}{\partial t_n} = (L^n)_+\psi$.

Finally, just as in KP, we have the concept of a τ (vector) function for the Toda lattice hierarchy. There exists $\tau = (\tau_n)_{n\in\mathbb{Z}}$ such that we have the following relations on L and ψ for all $n \in \mathbb{Z}$:

$$L_{n,n} = -\frac{\partial}{\partial t}\left(\log \frac{\tau_{n+1}}{\tau_n}\right), \quad L_{n,n-1} = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \quad \psi_n = e^{\xi(\mathbf{t},z)} z^n \frac{G(z)\tau_n}{\tau_n},$$

where $G(z)$ is the same shifting operator used in KP and KdV.

2.2 Additional Symmetries of Toda

In order to define the additional symmetries of the Toda lattice hierarchy, we must first introduce an equivalent definition of the hierarchy. Define the diagonal matrix $\gamma = \text{diag}(\dots, \gamma_{-1}, \gamma_0, \gamma_1, \dots)$ with

$\gamma_k = \sqrt{\frac{\tau_{k+1}}{\tau_k}}$. Using this, define the following modified operator matrices and vectors:

$$\tilde{L} = \gamma^{-1} L \gamma,$$
$$\tilde{W} = \gamma^{-1} W,$$
$$\tilde{\psi} = \gamma^{-1} \psi.$$

By using the τ representation of the entries of L, it is clear that $\tilde{L}$ is a symmetric matrix and that the diagonal entries of L and $\tilde{L}$ are equal. We also have that $\tilde{L} = \tilde{W} \Lambda \tilde{W}^{-1}$, $\tilde{\psi} = \tilde{W} e^{\xi(\mathbf{t},z)} \zeta$, and $\tilde{L}\tilde{\psi} = z\tilde{\psi}$.

Introduce a different splitting of $\tilde{g\ell}_\infty$ into anti-symmetric part, denoted $()_s$, and lower Borel (lower triangular with diagonal) part, denoted $()_b$. To demonstrate this splitting, consider the 2×2 matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Then

$$A_s = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \quad A_b = \begin{pmatrix} 1 & 0 \\ 5 & 4 \end{pmatrix}.$$

If A is symmetric, then we have that $A_b = 2A_- + A_0$, where A_0 denotes the diagonal part of A.

Using this splitting and the modified Lax matrix, the following is an equivalent definition of the Toda lattice hierarchy:

$$\frac{\partial \tilde{L}}{\partial t_n} = \frac{1}{2}[(L^n)_s, \tilde{L}]. \tag{2.6}$$

The actions of the Toda lattice flows on the modified wave function and operator matrix are similar to those on the original wave function and operator matrix:

$$\frac{\partial \tilde{W}}{\partial t_n} = -\frac{1}{2}(L^n)_b \tilde{W},$$
$$\frac{\partial \tilde{\psi}}{\partial t_n} = \frac{1}{2}(L^n)_s \tilde{\psi}.$$

Lax Form of Additional Symmetries

Just as in the KP case, we define an operator, $\tilde{M}$, such that $\tilde{M}\tilde{\psi} = \partial_z \tilde{\psi}$:

$$\tilde{M} := \tilde{W} \Delta \tilde{W}^{-1} + \sum_{n=1}^{\infty} n\, t_n \tilde{L}^{n-1},$$

where $\Delta \in \tilde{g\ell}_\infty$ is defined

$$\Delta := \begin{pmatrix} \ddots & & & & & & \\ & 0 & 0 & 0 & 0 & 0 & \\ & -1 & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 1 & 0 & 0 & \\ & 0 & 0 & 0 & 2 & 0 & \\ & & & & & & \ddots \end{pmatrix}$$

such that $\Delta\zeta = \partial_z \zeta$. Just as with M for the KP hierarchy in (1.22), $\tilde{M}$ is only well defined when finitely many of the t_n variables are nonzero.

Define flows $\frac{\partial}{\partial t^*_{\ell,m}}$ on the modified Lax operator $\tilde{L}$ as follows:

$$\frac{\partial \tilde{L}}{\partial t^*_{\ell,m}} = [-(\tilde{M}^m \tilde{L}^\ell)_b, \tilde{L}]. \tag{2.7}$$

These flows have the corresponding actions on the modified wave vector and wave operator matrix:

$$\frac{\partial \tilde{W}}{\partial t^*_{\ell,m}} = -(\tilde{M}^m \tilde{L}^\ell)_b \tilde{W},$$
$$\frac{\partial \tilde{\psi}}{\partial t^*_{\ell,m}} = -(\tilde{M}^m \tilde{L}^\ell)_b \tilde{\psi}.$$

However, recall that L and $\tilde{L}$ are tridiagonal matrices and that $\tilde{L}$ is symmetric. So, actions that are to act as derivatives on these matrices must preserve these properties. The only such actions that do this are when $m = 0$, $\ell \in \mathbb{Z}$ and when $m = 1$, $\ell \in \mathbb{Z}_{\geq 0}$.

With these flows, we have the following theorem by Adler and van Moerbeke given in [3]:

Theorem 2.1. *The flows defined in (2.7) are additional symmetries of the Toda lattice hierarchy; i.e. when $m = 0$, $\ell \in \mathbb{Z}$ or when $m = 1$, $\ell \in \mathbb{Z}_{\geq 0}$, we have*

$$[\partial_{t_n}, \partial_{t^*_{\ell,m}}]\Phi = 0,$$

where Φ *can be any one of* $\tilde{W}$, $\tilde{L}$, *or* $\tilde{\psi}$. *They commute with each other as follows:*

$$[\partial_{t^*_{\ell,0}}, \partial_{t^*_{k,0}}] = -\frac{\ell}{2}\delta_{\ell+k,0},$$
$$[\partial_{t^*_{\ell,0}}, \partial_{t^*_{k,1}}] = -\ell\,\partial_{t^*_{\ell+k,0}},$$
$$[\partial_{t^*_{\ell,1}}, \partial_{t^*_{k,1}}] = -(\ell-k)\partial_{t^*_{\ell+k,1}}.$$

Note that the nonzero commutation relations of these symmetries mean that solutions of the Toda lattice hierarchy are not functions of the $t^*_{\ell,m}$ variables.

Action on τ

Similar to KP and KdV, the flows defined in (2.7) induce actions on the τ vector function. They are constructed from operators $W_j^{(k)}$ generated by expanding the following operator in powers of λ and $\mu-\lambda$:

$$X(\mathbf{t},\lambda,\mu) = \exp\left(\frac{1}{2}\sum_{n=1}^{\infty}(\mu^n-\lambda^n)t_n\right)\exp\left(\sum_{n=1}^{\infty}(\lambda^{-n}-\mu^{-n})\frac{1}{n}\frac{\partial}{\partial t_n}\right)$$
$$= \sum_{k=0}^{\infty}\frac{(\mu-\lambda)^k}{k!}\sum_{j\in\mathbb{Z}}\lambda^{-j-k}W_j^{(k)}.$$

Note the factor of $\frac{1}{2}$ in front of the t_n term, differing this operator from (1.27). Expanding this operator gives the first three terms:

$$W_j^{(0)} = J_n^{(0)} = \delta_{n,0},$$
$$W_j^{(1)} = J_n^{(1)},$$
$$W_j^{(2)} = J_n^{(2)} - (n+1)J_n^{(1)},$$

where

$$J_n^{(1)} = \begin{cases} \partial_{t_n} & , \quad n>0 \\ 0 & , \quad n=0 \\ \frac{1}{2}(-n)t_{-n} & , \quad n<0 \end{cases}$$

and

$$J_n^{(2)} = \sum_{i+j=n}\partial_{t_i}\partial_{t_j} + \sum_{-i+j=n} i\,t_i\partial_{t_j} + \frac{1}{4}\sum_{-i-j=n}(i\,t_i)(j\,t_j).$$

Finally, to more easily formulate the actions of these flows on τ, Adler and van Moerbeke define the following operators:

$$\mathcal{L}^{j}_{\ell+1,1} := W^{(2)}_{\ell} + 2jW^{(1)}_{\ell} + (j^2 - j)W^{(0)}_{\ell},$$
$$\mathcal{L}^{j}_{\ell,0} := 2W^{(1)}_{\ell} + 2jW^{(0)}_{\ell}.$$

Using these operators, we have the following theorem from [3]:

Theorem 2.2. *The actions of the additional symmetries $\partial_{t^*_{\ell,m}}$ ($m = 0$, $\ell \in \mathbb{Z}$ or $m = 1$, $\ell \in \mathbb{Z}_{\geq 0}$) on $\tau = (\tau_j)_{j\in\mathbb{Z}}$ for the Toda lattice hierarchy are induced by the actions on $\psi = (\psi_j)_{j\in\mathbb{Z}}$ by the following:*

$$\frac{\partial_{t^*_{\ell,m}} \psi_j}{\psi_j} = (G(z)-1)\frac{\mathcal{L}^{j}_{\ell,m}\tau_j}{\tau_j} + \frac{1}{2}\left(\frac{\mathcal{L}^{j}_{\ell,m}\tau_j}{\tau_j} - \frac{\mathcal{L}^{j+1}_{\ell,m}\tau_{j+1}}{\tau_{j+1}}\right). \tag{2.8}$$

2.3 The Extended Toda Hierarchy

To extend an integrable hierarchy such as Toda, one needs to introduce a new set of variables with corresponding evolutionary flows. The actions of these flows must commute with the actions of the original hierarchy, just like additional symmetries. However, they must also commute with each other, allowing them to be added to the original flows to make a larger hierarchy. As is briefly discussed in [5], the flows of the Toda lattice hierarchy form a complete family of such commuting flows when all functions are functions of discrete points on the lattice. New symmetries arise when we view them as functions of a continuous variable, $x \in \mathbb{R}$.

The Interpolated Toda Hierarchy

For every function on the lattice $f : \mathbb{Z} \to \mathbb{C}$, interpolate a corresponding function of a continuous variable $x \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{C}$, so that $f(n\epsilon) = f_n$, $n \in \mathbb{Z}$. Here, ϵ is called the *string coupling constant*, which is a formal parameter that can be thought of as the lattice mesh. This interpolation also changes our shift operator Λ so that $\Lambda = e^{\epsilon\partial_x}$:

$$\Lambda f(x) = f(x+\epsilon).$$

Typically, ϵ is kept as a formal parameter, as in [5]. However, in [17] and the material presented in this thesis, $\epsilon = 1$ to simplify the notation. A change of variables allows one to recover the explicit dependence on ϵ. To differentiate the notations and remain consistent with the notation in [17],

$s = \dfrac{x}{\epsilon}$ will be used as the continuous variable instead of x. Thus, we have

$$\Lambda f(s) = f(s+1).$$

The Lax operator for Toda then becomes

$$L = L(s) = \Lambda + v(s) + e^{u(s)}\Lambda^{-1}, \tag{2.9}$$

and the splitting of such difference operators into "plus" and "minus" parts remains unchanged.

We use the following definition from [17]:

Definition 2.3. *The* ***(interpolated) Toda hierarchy*** *is defined to be*

$$\frac{\partial L}{\partial t_n} = [A_n, L], \quad n \in \mathbb{Z}_{n\geq 1}, \tag{2.10}$$

where

$$\begin{aligned} A_n &= \frac{1}{2}(L^n)_+ - \frac{1}{2}(L^n)_- \\ &= (L^n)_+ - \frac{1}{2}L^n \\ &= \frac{1}{2}L^n - (L^n)_-. \end{aligned}$$

Note that (2.10) amounts to $\partial_{t_n} L = [(L^n)_+, L] = [-(L^n)_-, L]$ since integer powers of L commute with L.

This interpolation is what allows new symmetries to be added to the hierarchy. The simplest symmetry of the Toda hierarchy that is not found within the hierarchy itself is translation along the continuous variable s, $s \to s + c$ for some constant c, found by acting with ∂_s.

Wave Operators

Just as in the KP and Toda lattice hierarchies, we have the concept of a wave or dressing operator. However, unlike in the Toda lattice, we consider all difference operators rather than just those that have finitely many positive powers of Λ. Because we expand the operators that we are considering, L no long has a well defined inverse as it did in the Toda lattice; only those operators of the form $A = \sum_{i\geq 0} a_i \Lambda^i$ or $A = \sum_{i\leq 0} a_i \Lambda^i$ with $a_0 \neq 0$ have well defined inverses.

Thus, we have two wave operators for the Toda hierarchy,

$$W = 1 + \sum_{i=1}^{\infty} w_i(s)\Lambda^{-i}, \quad \bar{W} = \sum_{i=0}^{\infty} \bar{w}_i(s)\Lambda^{i}, \quad \bar{w}_0(s) \neq 0,$$

such that

$$L = W\Lambda W^{-1} = \bar{W}\Lambda^{-1}\bar{W}^{-1}. \tag{2.11}$$

The flows of the Toda hierarchy act on these dressing operator as

$$\begin{aligned} \frac{\partial W}{\partial t_n} &= A_n W - \frac{1}{2} W\Lambda^n = -(L^n)_- W, \\ \frac{\partial \bar{W}}{\partial t_n} &= A_n \bar{W} + \frac{1}{2} \bar{W}\Lambda^{-n} = (L^n)_+ \bar{W}. \end{aligned} \tag{2.12}$$

Logarithm of L and the Extension

In order to extend the Toda hierarchy, a new Lax operator must be introduced. The authors of [5] define the so-called ***logarithm of*** L as follows

$$\log L = \frac{1}{2} W\partial_s W^{-1} - \frac{1}{2}\bar{W}\partial_s \bar{W}^{-1} = -\frac{1}{2}\frac{\partial W}{\partial s} W^{-1} + \frac{1}{2}\frac{\partial \bar{W}}{\partial s}\bar{W}^{-1}. \tag{2.13}$$

The ∂_s in each term of $\log L$ cancels, showing that this operator is indeed a difference operator. Also, using each representation of L in (2.11), it clear that L commutes with each term of $\log L$. Thus, L and $\log L$ commute.

Using this new difference operator, we have the following:

Definition 2.4. *Introduce a new set of variables, $x_0, x_1, x_2, \ldots$, with corresponding evolutionary flows on L given by*

$$\frac{\partial L}{\partial x_n} = [L^n \partial_s + P_n, L], \quad n \in \mathbb{Z}_{\geq 0}, \tag{2.14}$$

where

$$\begin{aligned} P_n &= -\left(L^n \frac{\partial W}{\partial s} W^{-1}\right)_+ - \left(L^n \frac{\partial \bar{W}}{\partial s}\bar{W}^{-1}\right)_- \\ &= L^n W\partial_s W^{-1} - (2L^n \log L)_- - L^n\partial_s \\ &= L^n \bar{W}\partial_s \bar{W}^{-1} + (2L^n \log L)_+ - L^n \partial_s. \end{aligned}$$

These new flows along with the flows of the Toda hierarchy in (2.10) represent the ***extended Toda***

***hierarchy** (abbreviated ETH) in Lax pair form.*

Again, note that (2.14) can be rewritten as $\partial_{x_n} L = [(2L^n \log L)_+, L] = [-(2L^n \log L)_-, L]$.

The actions of these flows on the wave operators W and $\bar{W}$ are similar to those of the original Toda flows:

$$\begin{aligned} \frac{\partial W}{\partial x_n} &= (L^n \partial_s + P_n)W - W\Lambda^n \partial_s = -(2L^n \log L)_- W, \\ \frac{\partial \bar{W}}{\partial x_n} &= (L^n \partial_s + P_n)\bar{W} - \bar{W}\Lambda^{-n} \partial_s = (2L^n \log L)_+ \bar{W}. \end{aligned} \tag{2.15}$$

Note that when comparing this to (2.12), $L^n \partial_s + P_n$ plays the role of A_n and $\Lambda^{\pm n}\partial_s$ plays the role of $\pm\frac{1}{2}\Lambda^{\pm n}$.

Consider the action on W and let $n = 0$. $(2\log L)_- = -\frac{\partial W}{\partial s} W^{-1}$, making

$$\frac{\partial W}{\partial x_0} = \frac{\partial W}{\partial s}.$$

Similar formulas hold for $\bar{W}$ and L. For this reason, we can think of $x_0 = s$, and thus, translation along s is a symmetry present in the new hierarchy.

The Wave Functions

The final pieces of information for the extended Toda that we need to continue are the wave functions and tau function. There are two different wave operators, W and $\bar{W}$. So, there are two different wave functions ψ and $\bar{\psi}$ for the extended Toda. Recall from the Toda lattice that we had an infinite column vector $\zeta = (z^n)_{n\in\mathbb{Z}}$ such that $\Lambda\zeta = z\zeta$. Interpolating along the lattice clearly leads to the function z^s, with $\Lambda z^s = z^{s+1} = z \cdot z^s$. To expand upon this observation, for any difference operator $A = \sum_{i\in\mathbb{Z}} a_i \Lambda^i$,

$$Az^s = \left(\sum_{i\in\mathbb{Z}} a_i \Lambda^i\right) z^s = \left(\sum_{i\in\mathbb{Z}} a_i z^i\right) z^s.$$

In [17], Takasaki generates the exponential terms of these wave functions by acting on z^s with the following exponentials:

$$\exp\left(\pm\frac{1}{2}\sum_{n=1}^{\infty} t_n \Lambda^{\pm n}\right) z^s = z^s e^{\pm\frac{1}{2}\xi(\mathbf{t}, z^{\pm 1})},$$
$$\exp\left(\sum_{n=1}^{\infty} x_n \Lambda^{\pm n}\partial_s\right) z^s = z^{s+\xi(\mathbf{x}, z^{\pm 1})},$$

recalling that $\xi(\mathbf{t},z)=\sum_{i=1}^{\infty} t_i z^i$. Note that these generators, $\Lambda^{\pm n}$ and $\Lambda^{\pm n}\partial_s$, are pulled from the second terms in (2.12) and (2.15), respectively.

We then have the following definitions for the wave functions for the extended Toda hierarchy:

$$\begin{aligned} \psi(s,\mathbf{t},\mathbf{x},z) &= W\chi = w(s,z)\chi, \\ \bar{\psi}(s,\mathbf{t},\mathbf{x},z) &= \bar{W}\bar{\chi} = \bar{w}(s,z)\bar{\chi}, \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} \chi &= z^{s+\xi(\mathbf{x},z)}e^{\frac{1}{2}\xi(\mathbf{t},z)}, \\ \bar{\chi} &= z^{s+\xi(\mathbf{x},z^{-1})}e^{-\frac{1}{2}\xi(\mathbf{t},z^{-1})}, \end{aligned} \tag{2.17}$$

and

$$w(s,z)=w(s,\mathbf{t},\mathbf{x};z)=1+\sum_{i=1}^{\infty} w_i(s)z^{-i},$$
$$\bar{w}(s,z)=\bar{w}(s,\mathbf{t},\mathbf{x};z)=\sum_{i=0}^{\infty} \bar{w}_i(s)z^{i}.$$

These wave functions are formulated in such a way that

$$L\psi = z\psi, \quad L\bar{\psi} = z^{-1}\bar{\psi}, \tag{2.18}$$

and that the actions of the flows of the extended Toda hierarchy on them are

$$\begin{aligned} \frac{\partial\psi}{\partial t_n} &= A_n\psi, & \frac{\partial\bar{\psi}}{\partial t_n} &= A_n\bar{\psi}, \\ \frac{\partial\psi}{\partial x_n} &= (L^n\partial_s + P_n)\psi, & \frac{\partial\bar{\psi}}{\partial x_n} &= (L^n\partial_s + P_n)\bar{\psi}. \end{aligned} \tag{2.19}$$

The Tau Function

Now that we have explicit formulas for the wave functions, we can define the tau function for the extended Toda hierarchy in terms of them. These definitions follow from the work in [17] but were not explicitly written. The tau function for the extended Toda hierarchy is represented in terms of the wave functions as:

$$\begin{aligned} \psi &= \frac{\tau(s,\mathbf{t}-[z^{-1}],\mathbf{x})}{\tau(s,\mathbf{t},\mathbf{x})}\chi &&= \frac{G(z)\tau(s,\mathbf{t},\mathbf{x})}{\tau(s,\mathbf{t},\mathbf{x})}\chi, \\ \bar{\psi} &= \frac{\tau(s+1,\mathbf{t}+[z],\mathbf{x})}{\tau(s,\mathbf{t},\mathbf{x})}\bar{\chi} &&= \frac{\bar{G}(z)\tau(s,\mathbf{t},\mathbf{x})}{\tau(s,\mathbf{t},\mathbf{x})}\bar{\chi}, \end{aligned} \tag{2.20}$$

where $[z]$ and $G(z)$ are defined in Chapter 1 in (1.20) and (1.19), respectively, and $\bar{G}(z)$ is defined on functions of s and $\mathbf{t}$ as

$$\bar{G}(z)f(s,\mathbf{t}) = f(s+1,\mathbf{t}+[z]), \tag{2.21}$$

or more explicitly,

$$\bar{G}(z) = \exp\left(\partial_s + \sum_{n=1}^{\infty} \frac{z^n}{n}\partial_{t_n}\right).$$

Dividing by χ and $\bar{\chi}$ respectively, we get the following relationships between the tau function and the coefficients of the dressing operators:

$$\begin{aligned} w(s,z) &= \frac{G(z)\tau}{\tau}, \\ \bar{w}(s,z) &= \frac{\bar{G}(z)\tau}{\tau}. \end{aligned} \tag{2.22}$$

Change of Variables

A final note in this section is the matter of differences in notation. The standard formulation of the extended Toda hierarchy in Lax form (as introduced in [5]) is as follows:

$$\epsilon \frac{\partial L}{\partial t^{\alpha,k}} = [A_{\alpha,k}, L], \quad \alpha = 1,2, \; k \in \mathbb{Z}_{\geq 0}, \tag{2.23}$$

where

$$\begin{aligned} A_{1,k} &= \frac{2}{k!}\big(L^k(\log L - c_k)\big)_+, \\ A_{2,k} &= \frac{1}{(k+1)!}(L^{k+1})_+, \end{aligned} \tag{2.24}$$

and

$$c_0 = 0, \quad c_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \tag{2.25}$$

are the *harmonic numbers*.

As Takasaki points out in [17], this formulation is useful in 2D topological field theory, but the differences between this formulation and that used in this thesis are superficial. It is for this reason and the simplified notation that we use the definition of the hierarchy found in [17]. One can shift from the notation of (2.23) to that used in this thesis by way of the following change of variables:

$$\begin{aligned} x &= \epsilon s, \\ t^{1,k} &= \epsilon k! x_k, \qquad\qquad k \in \mathbb{Z}_{\geq 0}, \\ t^{2,k} &= \epsilon (k+1)! \big(t_{k+1} + 2 c_{k+1} x_{k+1} \big). \end{aligned} \tag{2.26}$$

This change of variables creates the following relationship between the derivatives with respect to the standard variables and the derivatives of the variables used in this thesis:

$$\begin{aligned} \epsilon \frac{\partial}{\partial x} &= \frac{\partial}{\partial s}, \\ \epsilon \frac{\partial}{\partial t^{1,k}} &= \frac{1}{k!} \left(\frac{\partial}{\partial x_k} - 2 c_k \frac{\partial}{\partial t_k} \right), \\ \epsilon \frac{\partial}{\partial t^{2,k}} &= \frac{1}{(k+1)!} \frac{\partial}{\partial t_{k+1}}. \end{aligned}$$

It is clear that this change of variables can be used to recover the standard definition of the extended Toda hierarchy.

CHAPTER

3

SYMMETRIES OF THE ETH

This chapter contains the primary results of this thesis. We begin by going through the process of constructing the additional symmetries of the extended Toda in Lax form in Section 3.1. Section 3.2 involves proving that these flows are indeed symmetries of the ETH as well as how these symmetries commute with one another. Finally, in Section 3.3, we discuss the actions these symmetries have on the tau function of the hierarchy.

3.1 Constructing the Symmetries

M and $\bar{M}$

Recalling the actions of L on the wave functions in (2.18), we begin by constructing two corresponding operators, M and $\bar{M}$, such that

$$M\psi = \partial_z \psi, \quad \bar{M}\bar{\psi} = \partial_{z^{-1}}\bar{\psi}.$$

Note that since the wave operators W and $\bar{W}$ do not depend on z, we will focus on the functions χ and $\bar{\chi}$ first. For these functions

$$\begin{aligned} \Lambda^n \chi &= z^n \chi, \qquad \Lambda^n \bar{\chi} = z^n \bar{\chi}, \quad (n \in \mathbb{Z}) \\ \partial_s \chi &= \log(z)\chi, \qquad \partial_s \bar{\chi} = \log(z)\bar{\chi}. \end{aligned}$$

First, we derive an operator Γ such that $\Gamma\chi = \partial_z \chi$:

$$\begin{aligned} \partial_z \chi &= \partial_z \left(z^{s+\xi(\mathbf{x},z)} e^{\frac{1}{2}\xi(\mathbf{t},z)} \right) \\ &= \partial_z \left(e^{\log(z)\left(s+\xi(\mathbf{x},z)\right)+\frac{1}{2}\xi(\mathbf{t},z)} \right) \\ &= \partial_z \left(\log(z)\big(s+\xi(\mathbf{x},z)\big) + \frac{1}{2}\xi(\mathbf{t},z) \right) \cdot e^{\log(z)\left(s+\xi(\mathbf{x},z)\right)+\frac{1}{2}\xi(\mathbf{t},z)} \\ &= \left(s z^{-1} + \sum_{n\geq 1} \left(x_n z^{n-1} + n x_n z^{n-1} \log(z) + \frac{1}{2} n t_n z^{n-1} \right) \right) \chi \\ &= \left(s\Lambda^{-1} + \sum_{n\geq 1} \left(x_n + n x_n \partial_s + \frac{1}{2} k t_n \right) \Lambda^{n-1} \right) \chi. \end{aligned}$$

We define Γ as the operator acting on χ in the last line above:

$$\Gamma := s\Lambda^{-1} + \sum_{n\geq 1} \left(x_n + n x_n \partial_s + \frac{1}{2} n t_n \right) \Lambda^{n-1}. \tag{3.1}$$

Then, we define the operator M by "dressing" (conjugating) Γ with the wave operator W:

$$M := W\Gamma W^{-1} = W s\Lambda^{-1} W^{-1} + \sum_{n\geq 1} \left(x_n + n x_n W \partial_s W^{-1} + \frac{1}{2} n t_n \right) L^{n-1}. \tag{3.2}$$

Similarly, to define $\bar{M}$, we first find an operator $\bar{\Gamma}$ such that $\bar{\Gamma}\bar{\chi} = \partial_{z^{-1}} \bar{\chi}$:

$$\begin{aligned} \partial_{z^{-1}} \bar{\chi} &= \partial_{z^{-1}} \left(z^{s+\xi(\mathbf{x},z^{-1})} e^{-\frac{1}{2}\xi(\mathbf{t},z^{-1})} \right) \\ &= -z^2 \partial_z \left(e^{\log(z)\left(s+\xi(\mathbf{x},z^{-1})\right)-\frac{1}{2}\xi(\mathbf{t},z^{-1})} \right) \\ &= -z^2 \partial_z \left(\log(z)\big(s+\xi(\mathbf{x},z^{-1})\big) - \frac{1}{2}\xi(\mathbf{t},z^{-1}) \right) \cdot e^{\log(z)\left(s+\xi(\mathbf{x},z^{-1})\right)-\frac{1}{2}\xi(\mathbf{t},z^{-1})} \\ &= -z^2 \left(s z^{-1} + \sum_{n\geq 1} \left(x_n z^{-n-1} - n x_n z^{-n-1} \log(z) + \frac{1}{2} n t_n z^{-n-1} \right) \right) \bar{\chi} \end{aligned}$$

$$= -\left(s\Lambda + \sum_{n\geq 1}\left(x_n - n x_n \partial_s + \frac{1}{2} n t_n \right)\Lambda^{-(n-1)} \right)\bar{\chi}.$$

Just as before, $\bar{\Gamma}$ is the operator acting on $\bar{\chi}$ in the last line:

$$\bar{\Gamma} := -\left(s\Lambda + \sum_{n\geq 1}\left(x_n - n x_n \partial_s + \frac{1}{2} n t_n \right)\Lambda^{-(n-1)} \right), \tag{3.3}$$

and $\bar{M}$ is defined by dressing $\bar{\Gamma}$ with $\bar{W}$:

$$\bar{M} := \bar{W}\bar{\Gamma}\bar{W}^{-1} = -\left(\bar{W} s\Lambda \bar{W}^{-1} + \sum_{n\geq 1}\left(x_n - n x_n \bar{W}\partial_s \bar{W}^{-1} + \frac{1}{2} n t_n \right) L^{n-1} \right). \tag{3.4}$$

Once again, note that M and $\bar{M}$ are only well defined when only finitely many of the t_n and x_n are nonzero, just as with KP and the Toda lattice.

Based on the definitions of M and $\bar{M}$, we have the following lemma:

Lemma 3.1. *We have* $[L, M] = [L, \bar{M}] = 1$.

Proof. Note that since L commutes with itself and does not affect t_n or x_n, $[L, M] = [L, W s\Lambda^{-1} W^{-1}]$ and $[L, \bar{M}] = [L, -\bar{W} s\Lambda \bar{W}^{-1}]$. Then

$$\begin{aligned}
[L, M] &= [L, W s\Lambda^{-1} W^{-1}] \\
&= W[\Lambda, s\Lambda^{-1}]W^{-1} \\
&= W\left(\Lambda s\Lambda^{-1} - s\right)W^{-1} \\
&= W(s + 1 - s)W^{-1} \\
&= W W^{-1} = 1.
\end{aligned}$$

$$\begin{aligned}
[L, \bar{M}] &= [L, -\bar{W} s\Lambda \bar{W}^{-1}] \\
&= \bar{W}[\Lambda^{-1}, -s\Lambda]\bar{W}^{-1} \\
&= \bar{W}\left(-\Lambda^{-1} s\Lambda + s\right)\bar{W}^{-1} \\
&= \bar{W}(-(s-1) + s)\bar{W}^{-1} \\
&= \bar{W}\bar{W}^{-1} = 1.
\end{aligned}$$

□

This lemma and the Leibniz rule for commutator brackets lead to the following corollary, given without proof:

Corollary 3.1. $[L^n, M] = [L^n, \bar{M}] = nL^{n-1}$.

Next, we determine the action of the flows of the extended Toda hierarchy on M and $\bar{M}$.

Lemma 3.2. *The actions of the original flows of the Toda hierarchy on the operators M and $\bar{M}$ are*

$$\begin{aligned} \partial_{t_n} M &= [A_n, M], \\ \partial_{t_n} \bar{M} &= [A_n, \bar{M}]. \end{aligned} \tag{3.5}$$

Proof.

$$\begin{aligned} \partial_{t_n} M &= \partial_{t_n} \left(W \Gamma W^{-1} \right) \\ &= (\partial_{t_n} W) \Gamma W^{-1} + W (\partial_{t_n} \Gamma) W^{-1} + W \Gamma (\partial_{t_n} W^{-1}) \\ &= -(L^n)_- W \Gamma W^{-1} + W \left(\frac{1}{2} n \Lambda^{n-1} \right) W^{-1} + W \Gamma W^{-1} (L^n)_- W W^{-1} \\ &= -(L^n)_- M + \frac{1}{2} n L^{n-1} + M (L^n)_- \\ &= [-(L^n)_-, M] + \frac{1}{2} [L^n, M] = [A_n, M]. \end{aligned}$$

$$\begin{aligned} \partial_{t_n} \bar{M} &= \partial_{t_n} \left(\bar{W} \bar{\Gamma} \bar{W}^{-1} \right) \\ &= (\partial_{t_n} \bar{W}) \bar{\Gamma} \bar{W}^{-1} + \bar{W} (\partial_{t_n} \bar{\Gamma}) \bar{W}^{-1} + \bar{W} \bar{\Gamma} (\partial_{t_n} \bar{W}^{-1}) \\ &= (L^n)_+ \bar{W} \bar{\Gamma} \bar{W}^{-1} - \bar{W} \left(\frac{1}{2} n \Lambda^{-n+1} \right) \bar{W}^{-1} - \bar{W} \bar{\Gamma} \bar{W}^{-1} (L^n)_+ \bar{W} \bar{W}^{-1} \\ &= (L^n)_+ \bar{M} - \frac{1}{2} n L^{n-1} - \bar{M} (L^n)_+ \\ &= [(L^n)_+, \bar{M}] - \frac{1}{2} [L^n, \bar{M}] = [A_n, \bar{M}]. \end{aligned}$$

□

Lemma 3.3. *The actions of the logarithmic flows of the extended Toda hierarchy on the operators M and $\bar{M}$ are*

$$\begin{aligned} \partial_{x_n} M &= [L^n \partial_s + P_n, M], \\ \partial_{x_n} \bar{M} &= [L^n \partial_s + P_n, \bar{M}]. \end{aligned} \tag{3.6}$$

Proof.

$$\begin{aligned}
\partial_{x_n} M &= \partial_{x_n}\left(W\Gamma W^{-1}\right)\\
&=(\partial_{x_n} W)\Gamma W^{-1} + W(\partial_{x_n}\Gamma)W^{-1} + W\Gamma(\partial_{x_n} W^{-1})\\
&=(L^n\partial_s + P_n)M - L^n W\partial_s W^{-1}M + L^{n-1} + nL^{n-1}W\partial_s W^{-1} - M(L^n\partial_s + P_n) + ML^n W\partial_s W^{-1}\\
&=[L^n\partial_s + P_n, M] - [L^n W\partial_s W^{-1}, M] + L^{n-1} + nL^{n-1}W\partial_s W^{-1}\\
&=[L^n\partial_s + P_n, M] - L^n[W\partial_s W^{-1}, M] - [L^n, M]W\partial_s W^{-1} + L^{n-1} + nL^{n-1}W\partial_s W^{-1}\\
&=[L^n\partial_s + P_n, M] - L^n W[\partial_s, \Gamma]W^{-1} - [L^n, M]W\partial_s W^{-1} + L^{n-1} + nL^{n-1}W\partial_s W^{-1}\\
&=[L^n\partial_s + P_n, M] - L^{n-1} - nL^{n-1}W\partial_s W^{-1} + L^{n-1} + nL^{n-1}W\partial_s W^{-1}\\
&=[L^n\partial_s + P_n, M].
\end{aligned}$$

$$\begin{aligned}
\partial_{x_n}\bar{M} &= \partial_{x_n}\left(\bar{W}\bar{\Gamma}\bar{W}^{-1}\right)\\
&=(\partial_{x_n}\bar{W})\bar{\Gamma}\bar{W}^{-1} + \bar{W}(\partial_{x_n}\bar{\Gamma})\bar{W}^{-1} + \bar{W}\bar{\Gamma}(\partial_{x_n}\bar{W}^{-1})\\
&=(L^n\partial_s + P_n)\bar{M} - L^n\bar{W}\partial_s\bar{W}^{-1}\bar{M} - L^{n-1} + nL^{n-1}\bar{W}\partial_s\bar{W}^{-1} - \bar{M}(L^n\partial_s + P_n) + \bar{M}L^n\bar{W}\partial_s\bar{W}^{-1}\\
&=[L^n\partial_s + P_n, \bar{M}] - [L^n\bar{W}\partial_s\bar{W}^{-1}, \bar{M}] - L^{n-1} + nL^{n-1}\bar{W}\partial_s\bar{W}^{-1}\\
&=[L^n\partial_s + P_n, \bar{M}] - L^n[\bar{W}\partial_s\bar{W}^{-1}, \bar{M}] - [L^n, \bar{M}]\bar{W}\partial_s\bar{W}^{-1} - L^{n-1} + nL^{n-1}\bar{W}\partial_s\bar{W}^{-1}\\
&=[L^n\partial_s + P_n, \bar{M}] - L^n\bar{W}[\partial_s, \bar{\Gamma}]\bar{W}^{-1} - [L^n, \bar{M}]\bar{W}\partial_s\bar{W}^{-1} - L^{n-1} + nL^{n-1}\bar{W}\partial_s\bar{W}^{-1}\\
&=[L^n\partial_s + P_n, \bar{M}] + L^{n-1} - nL^{n-1}\bar{W}\partial_s\bar{W}^{-1} - L^{n-1} + nL^{n-1}\bar{W}\partial_s\bar{W}^{-1}\\
&=[L^n\partial_s + P_n, \bar{M}].
\end{aligned}$$

□

Finally, it is useful to consider the difference of the operators M and $\bar{M}$:

$$M - \bar{M} = W s\Lambda^{-1}W^{-1} + \bar{W}s\Lambda\bar{W}^{-1} + \sum_{n\geq 1}\left(2x_n + 2nx_n\log L + nt_n\right)L^{n-1}. \tag{3.7}$$

Lemma 3.1 clearly leads to the following about this operator:

Corollary 3.2. *We have* $[L, M - \bar{M}] = 0$.

Note that, while the presence of the ∂_s terms in M and $\bar{M}$ makes these operators not difference operators, the terms cancel in $M - \bar{M}$. So, $M - \bar{M}$ is in fact a difference operator.

Definition of Additional Symmetries

Define additional flows, denoted $\partial_{t_\ell^*}$ ($\ell \in \mathbb{Z}_{\geq 0}$), on the wave operators as follows:

$$\begin{aligned} \partial_{t_\ell^*} W &= -(B_\ell)_- W, \\ \partial_{t_\ell^*} \bar{W} &= (B_\ell)_+ \bar{W} \end{aligned} \tag{3.8}$$

where

$$B_\ell := (M - \bar{M}) L^\ell + \bar{W} \Lambda^{-(\ell-1)} \bar{W}^{-1}. \tag{3.9}$$

Note that $[B_\ell, L] = 0$ due to Corollary 3.2. Also, because of Lemmas 3.2 and 3.3, Equations (2.10) and (2.14), and the Leibniz rule, we have the following:

Corollary 3.3. *We have*

$$\partial_{t_n} B_\ell = [A_n, B_\ell], \tag{3.10}$$

$$\partial_{x_n} B_\ell = [P_n + L^n \partial_s, B_\ell]. \tag{3.11}$$

The flows defined in (3.8) have corresponding actions on the wave functions:

$$\begin{aligned} \partial_{t_\ell^*} \psi &= -(B_\ell)_- \psi, \\ \partial_{t_\ell^*} \bar{\psi} &= (B_\ell)_+ \bar{\psi}; \end{aligned} \tag{3.12}$$

as well as actions on the Lax operator L and the operators M and $\bar{M}$:

$$\partial_{t_\ell^*} L = [-(B_\ell)_-, L] = [(B_\ell)_+, L], \tag{3.13}$$

$$\begin{aligned} \partial_{t_\ell^*} M &= [-(B_\ell)_-, M], \\ \partial_{t_\ell^*} \bar{M} &= [(B_\ell)_+, \bar{M}]. \end{aligned} \tag{3.14}$$

3.2 First Main Theorem

Next, we will show that the actions of these additional flows commute with the actions of the hierarchy. To aid us in proving this, we will use Lemmas 3.4 and 3.5. The proof Lemma 3.4 follows that of Lemma 2.1 in [2]:

Lemma 3.4. *Let W and $\bar{W}$ be operators of the forms $W = \sum_{k\geq 0} w_k \Lambda^{-k}$ and $\bar{W} = \sum_{k \geq 0} \bar{w}_k \Lambda^k$. Also, let a* *and b be difference operators of the form*

$$a = A - \bar{A}, \quad b = B - \bar{B}$$

such that A and B contain only finitely many positive powers of Λ and $\bar{A}$ and $\bar{B}$ contain only finitely many negative powers of Λ. Suppose the vector fields ∂_a^* and ∂_b^* act on A, B, $\bar{A}$, $\bar{B}$, W, and $\bar{W}$ in the following ways:

$$\begin{aligned} \partial_a^* W &= -a_- W, & \partial_b^* W &= -b_- W, \\ \partial_a^* \bar{W} &= a_+ \bar{W}, & \partial_b^* \bar{W} &= b_+ \bar{W}, \\ \partial_a^* B &= [-a_- + a_\bullet, B], & \partial_b^* A &= [-b_-, A], \\ \partial_a^* \bar{B} &= [a_+ + \bar{a}_\bullet, \bar{B}], & \partial_b^* \bar{A} &= [b_+, \bar{A}], \end{aligned}$$

where $a_\bullet$ and $\bar{a}_\bullet$ are operators with finitely many positive and negative powers of Λ, respectively. Then

$$\begin{aligned} [\partial_a^*, \partial_b^*] W &= \partial_c^* W = -c_- W, \\ [\partial_a^*, \partial_b^*] \bar{W} &= \partial_c^* \bar{W} = c_+ \bar{W}, \end{aligned} \tag{3.15}$$

where $c = [A, B] - [\bar{A}, \bar{B}] - [a_\bullet, B] + [\bar{a}_\bullet, \bar{B}]$.

Proof. Note that for any operators α and β,

$$[\alpha_+, \beta_+]_- = [\alpha_-, \beta_-]_+ = 0.$$

Using the methods found in the proof of Lemma 2.1 in [2], we find that

$$[\partial_a^*, \partial_b^*] W = -C_- W$$

where

$$\begin{aligned} C &= -\partial_a^* b + \partial_b^* a - [a_-, b_-] \\ &= -[-a_- + a_\bullet, B] + [a_+ + \bar{a}_\bullet, \bar{B}] + [-b_-, A] - [b_+, \bar{A}] - [a_-, b_-] \\ &= [A_-, B] - [\bar{A}_-, B] - [a_\bullet, B] + [A_+, \bar{B}] - [\bar{A}_+, \bar{B}] + [\bar{a}_\bullet, \bar{B}] \\ &\quad - [B_-, A] + [\bar{B}_-, A] - [B_+, \bar{A}] + [\bar{B}_+, \bar{A}] \\ &\quad - [A_-, B_-] + [A_-, \bar{B}_-] + [\bar{A}_-, B_-] - [\bar{A}_-, \bar{B}_-] \end{aligned}$$

$$
\begin{aligned}
&= +[A_-, B] + [A, B_-] - [A_-, B_-] \\
&\quad - [\bar{A}_-, B] + [\bar{A}, B_+] + [\bar{A}_-, B_-] \\
&\quad + [A_+, \bar{B}] - [A, \bar{B}_-] + [A_-, \bar{B}_-] \\
&\quad - [\bar{A}_+, \bar{B}] - [\bar{A}, \bar{B}_+] - [\bar{A}_-, \bar{B}_-] \\
&\quad - [a_\bullet, B] + [\bar{a}_\bullet, \bar{B}] \\
&= [A_-, B_+] + [A, B_-] - [\bar{A}_-, B_+] + [\bar{A}, B_+] \\
&\quad + [A_+, \bar{B}] - [A_+, \bar{B}_-] - [\bar{A}_+, \bar{B}] - [\bar{A}_+, \bar{B}_+] - [\bar{A}_-, \bar{B}] \\
&\quad - [a_\bullet, B] + [\bar{a}_\bullet, \bar{B}] \\
&= [A, B] - [A_+, B_+] + [\bar{A}_+, B_+] + [A_+, \bar{B}_+] - [\bar{A}, \bar{B}] - [\bar{A}_+, \bar{B}_+] \\
&\quad - [a_\bullet, B] + [\bar{a}_\bullet, \bar{B}].
\end{aligned}
$$

Since $[\alpha_+, \beta_+]_- = 0$ for any operators α and β, we can safely drop any term of that form from C, giving us

$$[\partial_a^*, \partial_b^*]W = -c_- W,$$

where $c = [A, B] - [\bar{A}, \bar{B}] - [a_\bullet, B] + [\bar{a}_\bullet, \bar{B}]$. Thus ends the proof for W.

The proof for $\bar{W}$ is similar: we have that

$$[\partial_a^*, \partial_b^*]\bar{W} = D_+ \bar{W}$$

where

$$
\begin{aligned}
D &= -\partial_a^* b + \partial_b^* a + [a_+, b_+] \\
&= -[-a_- + a_\bullet, B] + [a_+ + \bar{a}_\bullet, \bar{B}] + [-b_-, A] - [b_+, \bar{A}] + [a_+, b_+] \\
&= [A_-, B] - [\bar{A}_-, B] - [a_\bullet, B] + [A_+, \bar{B}] - [\bar{A}_+, \bar{B}] + [\bar{a}_\bullet, \bar{B}] \\
&\quad - [B_-, A] + [\bar{B}_-, A] - [B_+, \bar{A}] + [\bar{B}_+, \bar{A}] \\
&\quad + [A_+, B_+] - [A_+, \bar{B}_+] - [\bar{A}_+, B_+] + [\bar{A}_+, \bar{B}_+]
\end{aligned}
$$

$$\begin{aligned}
&= [A_-, B] + [A, B_-] + [A_+, B_+] \\
&\quad - [\bar{A}_-, B] + [\bar{A}, B_+] - [\bar{A}_+, B_+] \\
&\quad + [A_+, \bar{B}] - [A, \bar{B}_-] - [A_+, \bar{B}_+] \\
&\quad - [\bar{A}_+, \bar{B}] - [\bar{A}, \bar{B}_+] + [\bar{A}_+, \bar{B}_+] \\
&\quad - [a_\bullet, B] + [\bar{a}_\bullet, \bar{B}] \\
&= [A_-, B] + [A_+, B] + [A_-, B_-] - [\bar{A}_-, B] + [\bar{A}_-, B_+] \\
&\quad + [A_+, \bar{B}_-] - [A, \bar{B}_-] + [\bar{A}_+, \bar{B}] - [\bar{A}_-, \bar{B}_+] \\
&= [A, B] + [A_-, B_-] - [\bar{A}_-, B_-] - [A_-, \bar{B}_-] - [\bar{A}, \bar{B}] + [\bar{A}_-, \bar{B}_-] \\
&\quad - [a_\bullet, B] + [\bar{a}_\bullet, \bar{B}].
\end{aligned}$$

Since $[\alpha_-, \beta_-]_+ = 0$ for any operators α and β, we can safely drop any term of that form from D, giving us

$$[\partial_a^*, \partial_b^*]\bar{W} = c_+ \bar{W},$$

where $c = [A, B] - [\bar{A}, \bar{B}] - [a_\bullet, B] + [\bar{a}_\bullet, \bar{B}]$. Thus ends the proof for $\bar{W}$. □

Given that we have the previous lemma, we also need to determine how these flows act on L and similar operators:

Lemma 3.5. *Let ∂_a^* be a vector field, S be an invertible difference operator, and $K = S\Omega S^{-1}$ such that $\partial_a^*\Omega = 0$. Suppose that $\partial_a^* S = AS$ for some operator A. Then $\partial_a^* K = [A, K]$.*

Proof. Note that for any derivation ∂_a^* and invertible operator S, $\partial_a^* S^{-1} = -S^{-1}(\partial_a^* S)S^{-1}$. Then

$$\begin{aligned}
\partial_a^* K &= \partial_a^*(S\Omega S^{-1}) \\
&= (\partial_a^* S)\Omega S^{-1} - S\Omega S^{-1}(\partial_a^* S)S^{-1} \\
&= AS\Omega S^{-1} - S\Omega S^{-1} A \\
&= AK - KA = [A, K].
\end{aligned}$$

□

First Main Theorem

We split the main theorem into two propositions proving that the $\partial_{t_\ell^*}$ flows commute with the ∂_{t_k} and ∂_{x_k} flows, respectively.

Proposition 3.1. *The actions of the flows $\partial_{t_\ell^*}$ commute with the actions of the original flows of the Toda hierarchy, i.e.*

$$[\partial_{t_n}, \partial_{t_\ell^*}]\Phi = 0,$$

where $n \in \mathbb{Z}_{\geq 1}$ and Φ can be W, $\bar{W}$, or L.

Proof. Recall that

$$A_n = -(L^n)_- + \frac{1}{2}L^n = (L^n)_+ - \frac{1}{2}L^n.$$

So, using Lemma 3.4 and setting

$$\begin{aligned} A &= L^n, & \bar{A} &= 0, \\ a_\bullet &= \frac{1}{2}L^n, & \bar{a}_\bullet &= -\frac{1}{2}L^n, \\ B &= ML^\ell, & \bar{B} &= \bar{M}L^\ell - \bar{W}\Lambda^{-\ell+1}\bar{W}^{-1}, \end{aligned}$$

we have that

$$\begin{aligned} [\partial_{t_n}, \partial_{t_\ell^*}]W &= \partial_c^* W = -c_- W, \\ [\partial_{t_n}, \partial_{t_\ell^*}]\bar{W} &= \partial_c^* \bar{W} = c_+ \bar{W}, \end{aligned}$$

and, using Lemma 3.5,

$$[\partial_{t_n}, \partial_{t_\ell^*}]L = [-c_-, L],$$

where

$$\begin{aligned} c &= [L^n, ML^\ell] - \frac{1}{2}[L^n, ML^\ell] - \frac{1}{2}[L^n, \bar{M}L^\ell - \bar{W}\Lambda^{-\ell+1}\bar{W}^{-1}] \\ &= +[L^n, M]L^\ell + \frac{1}{2}[L^n, M]L^\ell - \frac{1}{2}[L^n, \bar{M}]L^\ell \\ &= +nL^{\ell+n-1} - \frac{1}{2}nL^{\ell+n-1} - \frac{1}{2}nL^{\ell+n-1} = 0. \end{aligned}$$

□

Proposition 3.2. *The actions of the flows $\partial_{t_\ell^*}$ commute with the actions of the logarithmic flows of the extended Toda hierarchy, i.e.*

$$[\partial_{x_n}, \partial_{t_\ell^*}]\Phi = 0,$$

where $n \in \mathbb{Z}_{\geq 0}$ and Φ can be W, $\bar{W}$, or L.

Proof. Recall that

$$P_n + L^n \partial_s = -(2L^n \log L)_- + L^n W \partial_s W^{-1} = (2L^n \log L)_+ + L^n \bar{W} \partial_s \bar{W}^{-1}.$$

So, using Lemma 3.4 and setting

$$\begin{aligned} A &= a_\bullet = L^n W \partial_s W^{-1}, \\ \bar{A} &= \bar{a}_\bullet = L^n \bar{W} \partial_s \bar{W}^{-1}, \\ B &= M L^\ell, \\ \bar{B} &= \bar{M} L^\ell - \bar{W} \Lambda^{-\ell+1} \bar{W}^{-1}, \end{aligned}$$

we have that

$$\begin{aligned} [\partial_{x_n}, \partial_{t^*_\ell}] W &= \partial^*_c W = -c_- W, \\ [\partial_{x_n}, \partial_{t^*_\ell}] \bar{W} &= \partial^*_c \bar{W} = c_+ \bar{W}, \end{aligned}$$

and, using Lemma 3.5,

$$[\partial_{x_n}, \partial_{t^*_\ell}] L = [-c_-, L],$$

where

$$\begin{aligned} c = &+[L^n W \partial_s W^{-1}, M L^\ell] - [L^n \bar{W} \partial_s \bar{W}^{-1}, \bar{M} L^\ell - \bar{W} \Lambda^{-\ell+1} \bar{W}^{-1}] \\ &-[L^n W \partial_s W^{-1}, M L^\ell] + [L^n \bar{W} \partial_s \bar{W}^{-1}, \bar{M} L^\ell - \bar{W} \Lambda^{-\ell+1} \bar{W}^{-1}] = 0. \end{aligned}$$

□

Combining Propositions 3.1 and 3.2 proves the first main theorem of this thesis:

Theorem 3.1. *The flows $\partial_{t^*_\ell}$, whose actions on the wave operators are given in (3.8) and whose actions on the Lax operator L are given in (3.13), define additional symmetries of the extended Toda hierarchy. In other words, the actions of the flows commute with the actions of the flows of the hierarchy when acting on the wave operators and on L, i.e.*

$$[\partial_y, \partial_{t^*_\ell}] \Phi = 0,$$

where y can be any of the t_n or x_n variables and Φ can be W, $\bar{W}$, or L.

How Additional Symmetries Commute

Now that we have proven that the flows defined in (3.8) and (3.13) are indeed additional symmetries of the extended Toda hierarchy, it is natural to consider how their actions commute with each other when acting on W, $\bar{W}$, and L.

Theorem 3.2. *The map $\partial_{t^*_\ell} \longmapsto \mathcal{L}_{\ell-1}$, $\ell \in \mathbb{Z}_{\geq 0}$ defines a homomorphism between the flows of (3.8) and (3.13) acting on W, $\bar{W}$, and L and the Virasoro algebra; i.e.,*

$$[\partial_{t^*_\ell}, \partial_{t^*_p}]\Phi = (\ell - p)\partial_{t^*_{\ell+p-1}}\Phi, \tag{3.16}$$

where Φ can be W, $\bar{W}$, or L.

Proof. In Lemma 3.4, let $A = ML^\ell$, $\bar{A} = \bar{M}L^\ell - \bar{W}\Lambda^{-\ell+1}\bar{W}^{-1}$, $B = ML^p$, $\bar{B} = \bar{M}L^p - \bar{W}\Lambda^{-p+1}\bar{W}^{-1}$, and $a_\bullet = \bar{a}_\bullet = 0$. Then,

$$\begin{aligned}
[\partial_{t^*_\ell}, \partial_{t^*_p}]W &= \partial^*_c W = -c_- W \\
[\partial_{t^*_\ell}, \partial_{t^*_p}]\bar{W} &= \partial^*_c \bar{W} = c_+ \bar{W}
\end{aligned}$$

and Lemma 3.5 gives us that

$$[\partial_{t^*_\ell}, \partial_{t^*_p}]L = [-c_-, L] = [c_+, L],$$

where

$$\begin{aligned}
c &= [ML^\ell, ML^p] - [\bar{M}L^\ell - \bar{W}\Lambda^{-\ell+1}\bar{W}^{-1}, \bar{M}L^p - \bar{W}\Lambda^{-p+1}\bar{W}^{-1}] \\
&= M[L^\ell, M]L^p + M[M, L^p]L^\ell - \bar{M}[L^\ell, \bar{M}]L^p - \bar{M}[\bar{M}, L^p]L^\ell \\
&\quad + [\bar{M}, \bar{W}\Lambda^{-p+1}\bar{W}^{-1}]L^\ell + [\bar{W}\Lambda^{-\ell+1}\bar{W}^{-1}, \bar{M}]L^p \\
&= \ell ML^{\ell+p-1} - pML^{\ell+p-1} - \ell\bar{M}L^{\ell+p-1} + p\bar{M}L^{\ell+p-1} \\
&\quad - (p-1)\bar{W}\Lambda^{-\ell-p}\bar{W}^{-1} + (\ell-1)\bar{W}\Lambda^{-\ell-p}\bar{W}^{-1} \\
&= (\ell - p)\big((M - \bar{M})L^{\ell+p-1} + \bar{W}\Lambda^{-\ell-p}\bar{W}^{-1}\big) = (\ell - p)B_{\ell+p-1}.
\end{aligned}$$

Since $\psi = W\chi$, $\bar{\psi} = \bar{W}\bar{\chi}$, and $\partial_{t^*_\ell}\chi = \partial_{t^*_\ell}\bar{\chi} = 0$, the statement for ψ and $\bar{\psi}$ also holds true. □

3.3 Actions on the Tau Function

The actions of the $\partial_{t^*_\ell}$ flows on the wave functions induce actions on the tau function by means of the Adler-Shiota-van Moerbeke formula (1.25). More explicitly in the case of the extended Toda, we want to find operators, $\mathcal{L}_{\ell-1}$, of the t_k and x_k variables and their derivatives such that $\partial_{t_\ell^*}\tau = \mathcal{L}_{\ell-1}\tau$ and

$$\begin{aligned}\partial_{t_\ell^*}\psi &= \left(G(z)\left(\frac{\mathcal{L}_{\ell-1}\tau}{\tau}\right) - \frac{\mathcal{L}_{\ell-1}\tau}{\tau}\right)\frac{G(z)\tau}{\tau}\chi,\\ \partial_{t_\ell^*}\bar{\psi} &= \left(\bar{G}(z)\left(\frac{\mathcal{L}_{\ell-1}\tau}{\tau}\right) - \frac{\mathcal{L}_{\ell-1}\tau}{\tau}\right)\frac{\bar{G}(z)\tau}{\tau}\bar{\chi},\end{aligned} \tag{3.17}$$

recalling the definitions of $G(z)$ and $\bar{G}(z)$ from (1.19) and (2.21), respectively.

Note that due to the manner in which we shift the variables t_k and s in these previous equations, these $\mathcal{L}_p$ operators are only unique up to the addition of a function of $\mathbf{x}$ periodic in $s = x_0$ of period 1. This is because

$$\begin{aligned}G(z)\left(\frac{(\mathcal{L}_p + f(s,\mathbf{x}))\tau}{\tau}\right) - \frac{(\mathcal{L}_p + f(s,\mathbf{x}))\tau}{\tau} &= G(z)\left(\frac{\mathcal{L}_p\tau}{\tau}\right) + \frac{f(s,\mathbf{x})\tau}{\tau} - \frac{\mathcal{L}_p\tau}{\tau} - \frac{f(s,\mathbf{x})\tau}{\tau}\\ &= G(z)\left(\frac{\mathcal{L}_p\tau}{\tau}\right) - \frac{\mathcal{L}_p\tau}{\tau},\end{aligned}$$

as well as a similar statement involving $\bar{G}(z)$. We will derive the minimal such $\mathcal{L}_p$ operators that satisfy (3.17), beginning with the first four. However, we will first present a few helpful lemmas.

Lemma 3.6. *For $n \in \mathbb{Z}_{\geq 0}$,*

$$\frac{-(L^n)_-\psi}{\chi} = \partial_{t_n} w, \quad \frac{(L^n)_+\bar{\psi}}{\bar{\chi}} = \partial_{t_n}\bar{w} + \delta_{n,0}\bar{w}. \tag{3.18}$$

Recall that if either ∂_{t_n} or t_n for $n \leq 0$ are present in a term, that term is 0.

Proof. We have

$$\begin{aligned}\frac{-(L^n)_-\psi}{\chi} &= \frac{-(L^n)_- W\chi}{\chi}\\ &= \frac{(-(L^n)_- W)\chi}{\chi}\\ &= \frac{(\partial_{t_n} W)\chi}{\chi}\\ &= \frac{(\partial_{t_n} w)\chi}{\chi}\\ &= \partial_{t_n} w.\end{aligned}$$

Note that $(L^0)_+ = 1$. Then we have

$$\begin{aligned}\frac{(L^n)_+\bar{\psi}}{\bar{\chi}} &= \frac{(L^n)_+\bar{W}\bar{\chi}}{\bar{\chi}}\\ &= \frac{((L^n)_+\bar{W})\bar{\chi}}{\bar{\chi}}\\ &= \frac{(\partial_{t_n}\bar{W})\bar{\chi}}{\bar{\chi}} + \delta_{n,0}\frac{\bar{W}\bar{\chi}}{\bar{\chi}}\\ &= \frac{(\partial_{t_n}\bar{w})\bar{\chi}}{\bar{\chi}} + \delta_{n,0}\frac{\bar{w}\bar{\chi}}{\bar{\chi}}\\ &= \partial_{t_n}\bar{w} + \delta_{n,0}\bar{w}.\end{aligned}$$

□

Lemma 3.7. *For $n \in \mathbb{Z}_{\geq 0}$,*

$$\frac{-(2L^n \log L)_-\psi}{\chi} = \partial_{x_n} w, \quad \frac{(2L^n \log L)_+\bar{\psi}}{\bar{\chi}} = \partial_{x_n}\bar{w}. \tag{3.19}$$

Proof.

$$\begin{aligned}\frac{-(2L^n \log L)_-\psi}{\chi} &= \frac{-(2L^n \log L)_- W\chi}{\chi}\\ &= \frac{(-(2L^n \log L)_- W)\chi}{\chi}\\ &= \frac{(\partial_{x_n} W)\chi}{\chi}\\ &= \frac{(\partial_{x_n} w)\chi}{\chi}\\ &= \partial_{x_n} w.\end{aligned}$$

An identical proof is used for the second statement. □

Lemma 3.8. *For $y = t_n$ or $y = x_n$,*

$$\begin{aligned}\partial_y w &= \left(G(z)\left(\frac{\partial_y \tau}{\tau}\right) - \frac{\partial_y \tau}{\tau}\right)\frac{G(z)\tau}{\tau},\\ \partial_y \bar{w} &= \left(\bar{G}(z)\left(\frac{\partial_y \tau}{\tau}\right) - \frac{\partial_y \tau}{\tau}\right)\frac{\bar{G}(z)\tau}{\tau}.\end{aligned} \tag{3.20}$$

Proof. First, note that $\partial_y G(z) = \partial_y \bar{G}(z) = 0$. The proof of this lemma is a simple application of the quotient rule from calculus:

$$\begin{aligned}
\partial_y w &= \partial_y \left(\frac{G(z)\tau}{\tau} \right) \\
&= \frac{\tau \partial_y (G(z)\tau) - (G(z)\tau)(\partial_y \tau)}{\tau^2} \\
&= \frac{G(z)(\partial_y \tau)}{G(z)\tau} \frac{G(z)\tau}{\tau} - \frac{\partial_y \tau}{\tau} \frac{G(z)\tau}{\tau} \\
&= \left(G(z) \left(\frac{\partial_y \tau}{\tau} \right) - \frac{\partial_y \tau}{\tau} \right) \frac{G(z)\tau}{\tau}.
\end{aligned}$$

An identical process proves the statement for $\bar{w} = \dfrac{\bar{G}(z)\tau}{\tau}$. □

Corollary 3.4. *For $n \in \mathbb{Z}_{\geq 1}$ and $p \in \mathbb{Z}_{\geq -1}$,*

$$\begin{aligned}
n t_n \partial_{t_{n+p}} w &= \left(G(z) \left(\frac{n t_n \partial_{t_{n+p}} \tau}{\tau} \right) - \frac{n t_n \partial_{t_{n+p}} \tau}{\tau} + z^{-n} G(z) \left(\frac{\partial_{t_{n+p}} \tau}{\tau} \right) \right) \frac{G(z)\tau}{\tau}, \\
n t_n \partial_{t_{n+p}} \bar{w} &= \left(\bar{G}(z) \left(\frac{n t_n \partial_{t_{n+p}} \tau}{\tau} \right) - \frac{n t_n \partial_{t_{n+p}} \tau}{\tau} - z^{n} \bar{G}(z) \left(\frac{\partial_{t_{n+p}} \tau}{\tau} \right) \right) \frac{\bar{G}(z)\tau}{\tau}.
\end{aligned} \tag{3.21}$$

Proof. Using Lemma 3.8, we have

$$\begin{aligned}
n t_n \partial_{t_{n+p}} w &= \left(n t_n G(z) \left(\frac{\partial_{t_{n+p}} \tau}{\tau} \right) - n t_n \frac{\partial_{t_{n+p}} \tau}{\tau} \right) \frac{G(z)\tau}{\tau} \\
&= \left(n \left(t_n - \frac{z^{-n}}{n} \right) G(z) \left(\frac{\partial_{t_{n+p}} \tau}{\tau} \right) - n t_n \frac{\partial_{t_{n+p}} \tau}{\tau} + n \frac{z^{-n}}{n} G(z) \left(\frac{\partial_{t_{n+p}} \tau}{\tau} \right) \right) \frac{G(z)\tau}{\tau} \\
&= \left(G(z) \left(\frac{n t_n \partial_{t_{n+p}} \tau}{\tau} \right) - \frac{n t_n \partial_{t_{n+p}} \tau}{\tau} + z^{-n} G(z) \left(\frac{\partial_{t_{n+p}} \tau}{\tau} \right) \right) \frac{G(z)\tau}{\tau}.
\end{aligned}$$

A similar proof is used for the second statement. □

We use the previous lemmas and corollary in the proof of the following:

Corollary 3.5. *For $p \in \mathbb{Z}_{\geq -1}$, let S_p be the summation portion of $(M - \bar{M})L^{p+1}$; i.e.*

$$S_p = \sum_{n=1}^{\infty} \left(2x_n + 2n x_n \log L + n t_n \right) L^{n+p}, \quad p \in \mathbb{Z}_{\geq -1}.$$

Then

$$\begin{aligned} \frac{-(S_p)_-\psi}{\chi} &= \left(G(z)\left(\frac{D_p\tau}{\tau}\right) - \frac{D_p\tau}{\tau} + \sum_{n=1}^{\infty} z^{-n} G(z)\left(\frac{\partial_{t_{n+p}}\tau}{\tau}\right)\right)\frac{G(z)\tau}{\tau}, \\ \frac{(S_p)_+\bar{\psi}}{\bar{\chi}} &= \left(\bar{G}(z)\left(\frac{\bar{D}_p\tau}{\tau}\right) - \frac{\bar{D}_p\tau}{\tau} - \sum_{n=1}^{\infty} z^{n}\bar{G}(z)\left(\frac{\partial_{t_{n+p}}\tau}{\tau}\right) - \delta_{p,-1} z(x_0+1)\right)\frac{\bar{G}(z)\tau}{\tau}, \end{aligned} \tag{3.22}$$

where

$$D_p = \sum_{n=1}^{\infty} 2x_n\partial_{t_{n+p}} + n x_n \partial_{x_{n+p}} + n t_n \partial_{t_{n+p}} \tag{3.23}$$

and $\bar{D}_p = D_p + \delta_{p,-1}(t_1 x_0 + 2x_0 x_1)$.

Proof. Combining Lemmas 3.6, 3.7, and 3.8, Corollary 3.4, and linearity, gives us the first equality. We also combine them for the second statement:

$$\begin{aligned} \frac{(S_p)_+\bar{\psi}}{\bar{\chi}} &= \left(\bar{G}(z)\left(\frac{D_p\tau}{\tau}\right) - \frac{D_p\tau}{\tau} - \sum_{n=1}^{\infty} z^{n}\bar{G}(z)\left(\frac{\partial_{t_{n+p}}\tau}{\tau}\right) + \delta_{p,-1}(t_1+2x_1)\right)\frac{\bar{G}(z)\tau}{\tau} \\ &= \left(\bar{G}(z)\left(\frac{D_p\tau}{\tau}\right) - \frac{D_p\tau}{\tau} - \sum_{n=1}^{\infty} z^{n}\bar{G}(z)\left(\frac{\partial_{t_{n+p}}\tau}{\tau}\right)\right. \\ &\quad \left. + \delta_{p,-1}\big((t_1+z)(x_0+1) + 2(x_0+1)x_1 - t_1 x_0 - 2x_0 x_1 - z(x_0+1)\big)\right)\frac{\bar{G}(z)\tau}{\tau} \\ &= \left(\bar{G}(z)\left(\frac{\bar{D}_p\tau}{\tau}\right) - \frac{\bar{D}_p\tau}{\tau} - \sum_{n=1}^{\infty} z^{n}\bar{G}(z)\left(\frac{\partial_{t_{n+p}}\tau}{\tau}\right) - \delta_{p,-1} z(x_0+1)\right)\frac{\bar{G}(z)\tau}{\tau}. \end{aligned}$$

□

Lemma 3.9. *For* $p \in \mathbb{Z}_{\geq -1}$,

$$\begin{aligned} \frac{-(W s\Lambda^p W^{-1})\psi}{\chi} &= \left(-s z^p - \sum_{n=1}^{\infty} z^{p-n} G(z)\left(\frac{\partial_{t_n}\tau}{\tau}\right)\right)\frac{G(z)\tau}{\tau}, \\ \frac{(\bar{W} s\Lambda^{-p}\bar{W}^{-1})\bar{\psi}}{\bar{\chi}} &= \left(s z^{-p} + \sum_{n=1}^{\infty} z^{-p+n}\bar{G}(z)\left(\frac{\partial_{t_n}\tau}{\tau}\right)\right)\frac{\bar{G}(z)\tau}{\tau}. \end{aligned} \tag{3.24}$$

Proof.

$$
\begin{aligned}
\frac{-(W s\Lambda^p W^{\text{-}1})\psi}{\chi} &= \frac{-(W s\Lambda^p W^{\text{-}1})W\chi}{\chi} = \frac{-W s\Lambda^p \chi}{\chi} = \frac{-W s z^p \chi}{\chi} \\
&= \frac{-s z^p W\chi}{\chi} + z^p \sum_{i=0}^{\infty} \frac{i w_i \Lambda^{\text{-}i}\chi}{\chi} \\
&= \frac{-s z^p w\chi}{\chi} + z^p \sum_{i=0}^{\infty} \frac{i w_i z^{\text{-}i}\chi}{\chi} \\
&= -s z^p w + z^p \sum_{i=0}^{\infty} i w_i z^{\text{-}i} \\
&= -s z^p w - z^{p+1} \sum_{i=0}^{\infty} (\text{-}i) w_i z^{\text{-}i-1} \\
&= -s z^p w - z^{p+1}\partial_z w \\
&= -s z^p \frac{G(z)\tau}{\tau} - z^{p+1}\partial_z\left(\frac{G(z)\tau}{\tau}\right) \\
&= -s z^p \frac{G(z)\tau}{\tau} - z^{p+1}\sum_{n=1}^{\infty} \frac{G(z)\partial_{t_n}\tau}{\tau} z^{\text{-}n-1} \\
&= \left(-s z^p - \sum_{n=1}^{\infty} z^{p-n} G(z)\left(\frac{\partial_{t_n}\tau}{\tau}\right)\right)\frac{G(z)\tau}{\tau}.
\end{aligned}
$$

A similar process is performed for $\bar{\psi}$:

$$
\begin{aligned}
\frac{(\bar{W} s\Lambda^{\text{-}p} \bar{W}^{\text{-}1})\bar{\psi}}{\bar{\chi}} &= \frac{(\bar{W} s\Lambda^{\text{-}p} \bar{W}^{\text{-}1})\bar{W}\bar{\chi}}{\bar{\chi}} = \frac{\bar{W} s\Lambda^{\text{-}p}\bar{\chi}}{\bar{\chi}} = \frac{\bar{W} s z^{\text{-}p}\bar{\chi}}{\bar{\chi}} \\
&= \frac{s z^{\text{-}p}\bar{W}\bar{\chi}}{\bar{\chi}} + z^{\text{-}p}\sum_{i=0}^{\infty}\frac{i\bar{w}_i\Lambda^i\bar{\chi}}{\bar{\chi}} \\
&= \frac{s z^{\text{-}p}\bar{w}\bar{\chi}}{\bar{\chi}} + z^{\text{-}p}\sum_{i=0}^{\infty}\frac{i\bar{w}_i z^i\bar{\chi}}{\bar{\chi}} \\
&= s z^{\text{-}p}\bar{w} + z^{\text{-}p}\sum_{i=0}^{\infty} i\bar{w}_i z^i \\
&= s z^{\text{-}p}\bar{w} + z^{\text{-}p+1}\sum_{i=0}^{\infty} i\bar{w}_i z^{i-1} \\
&= s z^{\text{-}p}\bar{w} + z^{\text{-}p+1}\partial_z\bar{w}
\end{aligned}
$$

$$
\begin{aligned}
&= s z^{-p} \frac{\bar{G}(z)\tau}{\tau} + z^{-p+1}\partial_z\left(\frac{\bar{G}(z)\tau}{\tau}\right) \\
&= s z^{-p} \frac{\bar{G}(z)\tau}{\tau} + z^{-p+1}\sum_{n=1}^{\infty}\frac{\bar{G}(z)\partial_{t_n}\tau}{\tau} z^{n-1} \\
&= \left(s z^{-p} + \sum_{n=1}^{\infty} z^{-p+n}\bar{G}(z)\left(\frac{\partial_{t_n}\tau}{\tau}\right)\right)\frac{\bar{G}(z)\tau}{\tau}.
\end{aligned}
$$

□

We now can show how the additional symmetries act on the tau function. The next four propositions give the actions of the first four symmetries on the tau function and are proven by construction.

Proposition 3.3. *The action of $\partial_{t_0^*}$ on the tau function for the extended Toda hierarchy is given by $\partial_{t_0^*}\tau = \mathcal{L}_{-1}\tau$, where*

$$
\mathcal{L}_{-1} = \sum_{n=1}^{\infty}\left(2x_n\partial_{t_{n-1}} + n x_n\partial_{x_{n-1}} + n t_n\partial_{t_{n-1}}\right) + t_1 x_0 + 2x_0 x_1. \tag{3.25}
$$

Proof. Note that $(W s\Lambda^{-1}W^{-1})_- = W s\Lambda^{-1}W^{-1}$ and $(\bar{W} s\Lambda\bar{W}^{-1})_- = (\bar{W}\Lambda\bar{W}^{-1})_- = 0$. Then

$$
\begin{aligned}
\frac{\partial_{t_0^*}\psi}{\chi} &= \frac{-(M-\bar{M}+\bar{W}\Lambda\bar{W}^{-1})_-\psi}{\chi} \\
&= \frac{-(W s\Lambda^{-1}W^{-1} + \bar{W}s\Lambda\bar{W}^{-1} + S_{-1} + \bar{W}\Lambda\bar{W}^{-1})_-\psi}{\chi} \\
&= \frac{-W s\Lambda^{-1}W^{-1}\psi}{\chi} + \frac{-(S_{-1})_-\psi}{\chi} \\
&= \left(-s z^{-1} - \sum_{n=1}^{\infty} z^{-n-1}G(z)\left(\frac{\partial_{t_n}\tau}{\tau}\right) + G(z)\left(\frac{D_{-1}\tau}{\tau}\right) - \frac{D_{-1}\tau}{\tau} + \sum_{n=1}^{\infty} z^{-n}G(z)\left(\frac{\partial_{t_{n-1}}\tau}{\tau}\right)\right)\frac{G(z)\tau}{\tau} \\
&= \left(x_0(t_1 - z^{-1} + 2x_1 - t_1 - 2x_1) + G(z)\left(\frac{D_{-1}\tau}{\tau}\right) - \frac{D_{-1}\tau}{\tau}\right)\frac{G(z)\tau}{\tau} \\
&= \left(G(z)\left(\frac{(D_{-1}+t_1x_0+2x_0x_1)\tau}{\tau}\right) - \frac{(D_{-1}+t_1x_0+2x_0x_1)\tau}{\tau}\right)\frac{G(z)\tau}{\tau} \\
&= \left(G(z)\left(\frac{\mathcal{L}_{-1}\tau}{\tau}\right) - \frac{\mathcal{L}_{-1}\tau}{\tau}\right)\frac{G(z)\tau}{\tau}.
\end{aligned}
$$

In the steps above, we can add any arbitrary function of the x_k variables, allowing us to obtain the $2x_0x_1$ term we find when we perform similar steps on $\bar{\psi}$:

$$
\begin{aligned}
\frac{\partial_{t_0^*}\bar{\psi}}{\bar{\chi}} &= \frac{(M-\bar{M}+\bar{W}\Lambda\bar{W}^{-1})_+\psi}{\bar{\chi}} \\
&= \frac{(W s\Lambda^{-1}W^{-1}+\bar{W}s\Lambda\bar{W}^{-1}+S_{-1}+\bar{W}\Lambda\bar{W}^{-1})_+\bar{\psi}}{\bar{\chi}} \\
&= \frac{\bar{W}s\Lambda\bar{W}^{-1}\bar{\psi}}{\bar{\chi}}+\frac{\bar{W}\Lambda\bar{W}^{-1}\bar{\psi}}{\bar{\chi}}+\frac{(S_{-1})_+\bar{\psi}}{\bar{\chi}} \\
&= \frac{\bar{W}s\Lambda\bar{W}^{-1}\bar{\psi}}{\bar{\chi}}+\frac{z\bar{W}\bar{\chi}}{\bar{\chi}}+\frac{(S_{-1})_+\bar{\psi}}{\bar{\chi}} \\
&= \frac{\bar{W}s\Lambda\bar{W}^{-1}\bar{\psi}}{\bar{\chi}}+z\bar{w}+\frac{(S_{-1})_+\bar{\psi}}{\bar{\chi}} \\
&= \Bigg((s+1)z+\sum_{n=1}^{\infty}z^{n+1}\bar{G}(z)\left(\frac{\partial_{t_n}\tau}{\tau}\right)+\bar{G}(z)\left(\frac{\bar{D}_{-1}\tau}{\tau}\right)-\frac{\bar{D}_{-1}\tau}{\tau} \\
&\quad -\sum_{n=1}^{\infty}z^n\bar{G}(z)\left(\frac{\partial_{t_{n-1}}\tau}{\tau}\right)-z(x_0+1)\Bigg)\frac{\bar{G}(z)\tau}{\tau} \\
&= \left(\bar{G}(z)\left(\frac{\bar{D}_{-1}\tau}{\tau}\right)-\frac{\bar{D}_{-1}\tau}{\tau}\right)\frac{\bar{G}(z)\tau}{\tau} \\
&= \left(\bar{G}(z)\left(\frac{\mathcal{L}_{-1}\tau}{\tau}\right)-\frac{\mathcal{L}_{-1}\tau}{\tau}\right)\frac{\bar{G}(z)\tau}{\tau}.
\end{aligned}
$$

In either case, we get that $\partial_{t_0^*}\tau = \mathcal{L}_{-1}\tau$. □

Proposition 3.4. *The action of $\partial_{t_1^*}$ on the tau function for the extended Toda hierarchy is given by $\partial_{t_1^*}\tau = \mathcal{L}_0\tau$, where*

$$
\mathcal{L}_0 = \sum_{n=1}^{\infty}\left(2x_n\partial_{t_n}+n x_n\partial_{x_n}+n t_n\partial_{t_n}\right)+x_0^2. \tag{3.26}
$$

Proof. We have

$$
\begin{aligned}
\frac{\partial_{t_1^*}\psi}{\chi} &= \frac{-((M-\bar{M})L+1)_-\psi}{\chi} \\
&= \frac{-(W s W^{-1}+\bar{W}s\bar{W}^{-1}+S_0+1)_-\psi}{\chi} \\
&= \frac{-W s W^{-1}\psi}{\chi}+\frac{(W s^{-1}W^{-1})_+\psi}{\chi}+\frac{-(S_0)_-\psi}{\chi} \\
&= \frac{-W s W^{-1}\psi}{\chi}+\frac{s w\chi}{\chi}+\frac{-(S_0)_-\psi}{\chi}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(-s - \sum_{n=1}^{\infty} z^{-n} G(z) \left(\frac{\partial_{t_n} \tau}{\tau} \right) + s + G(z) \left(\frac{D_0 \tau}{\tau} \right) - \frac{D_0 \tau}{\tau} + \sum_{n=1}^{\infty} z^{-n} G(z) \left(\frac{\partial_{t_n} \tau}{\tau} \right) \right) \frac{G(z)\tau}{\tau} \\
&= \left(G(z) \left(\frac{(D_0 + x_0^2)\tau}{\tau} \right) - \frac{(D_0 + x_0^2)\tau}{\tau} \right) \frac{G(z)\tau}{\tau} \\
&= \left(G(z) \left(\frac{\mathcal{L}_0 \tau}{\tau} \right) - \frac{\mathcal{L}_0 \tau}{\tau} \right) \frac{G(z)\tau}{\tau}.
\end{aligned}
$$

Just as with $\mathcal{L}_{-1}$, we can add the x_0^2 to each term without affecting the action on ψ to give us the operator obtained when we perform similar actions on $\bar{\psi}$:

$$
\begin{aligned}
\frac{\partial_{t_1^*} \bar{\psi}}{\bar{\chi}} &= \frac{((M - \bar{M})L + 1)_+ \bar{\psi}}{\bar{\chi}} \\
&= \frac{(W s W^{-1} + \bar{W} s \bar{W}^{-1} + S_0 + 1)_+ \bar{\psi}}{\bar{\chi}} \\
&= \frac{(W s W^{-1})_+ \bar{\psi}}{\bar{\chi}} + \frac{\bar{W} s \bar{W}^{-1} \bar{\psi}}{\bar{\chi}} + \frac{\bar{\psi}}{\bar{\chi}} + \frac{(S_0)_+ \bar{\psi}}{\bar{\chi}} \\
&= \frac{s \bar{w} \bar{\chi}}{\bar{\chi}} + \frac{\bar{W} s \bar{W}^{-1} \bar{\psi}}{\bar{\chi}} + \frac{\bar{w} \bar{\chi}}{\bar{\chi}} + \frac{(S_0)_+ \bar{\psi}}{\bar{\chi}} \\
&= \left(2s + 1 + \sum_{n=1}^{\infty} z^{n} \bar{G}(z) \left(\frac{\partial_{t_n} \tau}{\tau} \right) + \bar{G}(z) \left(\frac{D_0 \tau}{\tau} \right) - \frac{D_0 \tau}{\tau} - \sum_{n=1}^{\infty} z^{n} \bar{G}(z) \left(\frac{\partial_{t_n} \tau}{\tau} \right) \right) \frac{\bar{G}(z)\tau}{\tau} \\
&= \left(\bar{G}(z) \left(\frac{(D_0 + x_0^2)\tau}{\tau} \right) - \frac{(D_0 + x_0^2)\tau}{\tau} \right) \frac{\bar{G}(z)\tau}{\tau} \\
&= \left(\bar{G}(z) \left(\frac{\mathcal{L}_0 \tau}{\tau} \right) - \frac{\mathcal{L}_0 \tau}{\tau} \right) \frac{\bar{G}(z)\tau}{\tau}.
\end{aligned}
$$

In either case, we get that $\partial_{t_1^*} \tau = \mathcal{L}_0 \tau$. □

Before we present the $\mathcal{L}_1$ and $\mathcal{L}_2$ operators, we need to explicitly expand the τ definitions of $w(s,z)$ and $\bar{w}(s,z)$ from (2.22) in powers of z^{-1} and z, respectively. Doing so yields the following formulas for the first couple of coefficients of W and $\bar{W}$:

$$
w_1(s) = -\frac{\partial_{t_1} \tau}{\tau},
$$

$$
\begin{aligned}
w_2(s) &= \frac{1}{2}\left(\frac{\partial_{t_1}^2\tau}{\tau} - \frac{\partial_{t_2}\tau}{\tau}\right) \\
&= \frac{1}{2}\left(\partial_{t_1}\left(\frac{\partial_{t_1}\tau}{\tau}\right) + \left(\frac{\partial_{t_1}\tau}{\tau}\right)^2 - \frac{\partial_{t_2}\tau}{\tau}\right) \\
&= \frac{1}{2}\left(-\partial_{t_1} w_1(s) + w_1(s)^2 - \frac{\partial_{t_2}\tau}{\tau}\right) \\
\bar{w}_0(s) &= \frac{\tau(s+1)}{\tau}, \\
\bar{w}_1(s) &= \frac{\partial_{t_1}\tau(s+1)}{\tau}.
\end{aligned}
$$

We also need the first coefficients of W^{-1} and $\bar{W}^{-1}$ in terms of the coefficients of W and $\bar{W}$, respectively:

$$
\begin{aligned}
w_1^{-1}(s) &= -w_1(s), \\
\bar{w}_0^{-1}(s) &= \frac{1}{\bar{w}_0(s)}.
\end{aligned}
$$

Finally, the derivations for $\mathcal{L}_{-1}$ and $\mathcal{L}_0$ were both shown from acting on both ψ and $\bar{\psi}$ to show that the extra terms added ($2x_0x_1$ for $\mathcal{L}_{-1}$ and x_0^2 for $\mathcal{L}_0$) were not arbitrary. For $\mathcal{L}_1$ and $\mathcal{L}_2$, we will only show the derivation from the action on ψ; no such additions are necessary in these cases.

Proposition 3.5. *The action of $\partial_{t_2^*}$ on the tau function for the extended Toda hierarchy is given by $\partial_{t_2^*}\tau = \mathcal{L}_1\tau$, where*

$$
\mathcal{L}_1 = \sum_{n=1}^{\infty}\left(2x_n\partial_{t_{n+1}} + n x_n \partial_{x_{n+1}} + n t_n \partial_{t_{n+1}}\right) + 2x_0\partial_{t_1}. \tag{3.27}
$$

Proof. We'll begin by looking at a portion of the proof to come and rewriting it as an operator applied to $w = \dfrac{G(z)\tau}{\tau}$:

$$
\begin{aligned}
\frac{(W s\Lambda W^{-1})_+\psi}{\chi} &= \frac{\left(sL - \sum_{i=1}^{\infty} i\, w_i(s)\Lambda^{-i+1}W^{-1}\right)_+\psi}{\chi} \\
&= \frac{sL_+\psi}{\chi} - \frac{w_1(s)\psi}{\chi} \\
&= s\frac{L\psi}{\chi} + s\frac{-L_-\psi}{\chi} - \frac{w_1(s)\psi}{\chi}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(sz + s\partial_{t_1} - w_1(s)\right)w \\
&= \left(sz + G(z)\left(\frac{s\partial_{t_1}\tau}{\tau}\right) - \frac{s\partial_{t_1}\tau}{\tau} + \frac{\partial_{t_1}\tau}{\tau}\right)\frac{G(z)\tau}{\tau};
\end{aligned}
$$

$$
\begin{aligned}
\frac{(\bar{W}s\Lambda^{-1}\bar{W}^{-1})_-\psi}{\chi} &= \frac{\left(sL + \sum_{i=1}^{\infty} i\bar{w}_i(s)\Lambda^{i-1}\bar{W}^{-1}\right)_-\psi}{\chi} \\
&= s\frac{L_-\psi}{\chi} \\
&= -\left(\bar{G}(z)\left(\frac{s\partial_{t_1}\tau}{\tau}\right) - \frac{s\partial_{t_1}\tau}{\tau}\right)\frac{\bar{G}(z)\tau}{\tau}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\frac{\partial_{t_2^*}\psi}{\chi} &= \frac{-((M-\bar{M})L^2 + L)_-\psi}{\chi} \\
&= \frac{-(Ws\Lambda W^{-1} + \bar{W}s\Lambda^{-1}\bar{W}^{-1} + S_1 + L)_-\psi}{\chi} \\
&= \frac{-Ws\Lambda W^{-1}\psi}{\chi} + \frac{(Ws\Lambda W^{-1})_+\psi}{\chi} - \frac{(\bar{W}s\Lambda^{-1}\bar{W}^{-1})_-\psi}{\chi} + \frac{-(S_1)_-\psi}{\chi} + \frac{-L_-\psi}{\chi} \\
&= \Bigg(-sz - \sum_{n=1}^{\infty} z^{-n+1}G(z)\left(\frac{\partial_{t_n}\tau}{\tau}\right) + sz + G(z)\left(\frac{2s\partial_{t_1}\tau}{\tau}\right) - \frac{2s\partial_{t_1}\tau}{\tau} + \frac{\partial_{t_1}\tau}{\tau} \\
&\quad + G(z)\left(\frac{D_1\tau}{\tau}\right) - \frac{D_1\tau}{\tau} + \sum_{n=1}^{\infty} z^{-n}G(z)\left(\frac{\partial_{t_{n+1}}\tau}{\tau}\right) + G(z)\left(\frac{\partial_{t_1}\tau}{\tau}\right) - \frac{\partial_{t_1}\tau}{\tau}\Bigg)\frac{G(z)\tau}{\tau} \\
&= \left(G(z)\left(\frac{(D_1 + 2x_0\partial_{t_1})\tau}{\tau}\right) - \frac{(D_1 + 2x_0\partial_{t_1})\tau}{\tau}\right)\frac{G(z)\tau}{\tau} \\
&= \left(G(z)\left(\frac{\mathcal{L}_1\tau}{\tau}\right) - \frac{\mathcal{L}_1\tau}{\tau}\right)\frac{G(z)\tau}{\tau}
\end{aligned}
$$

Thus, we get that $\partial_{t_2^*}\tau = \mathcal{L}_1\tau$. □

Proposition 3.6. *The action of $\partial_{t_3^*}$ on the tau function for the extended Toda hierarchy is given by $\partial_{t_3^*}\tau = \mathcal{L}_2\tau$, where*

$$
\mathcal{L}_2 = \sum_{n=1}^{\infty}\left(2x_n\partial_{t_{n+2}} + nx_n\partial_{x_{n+2}} + nt_n\partial_{t_{n+2}}\right) + 2x_0\partial_{t_2} + \partial_{t_1}^2. \tag{3.28}
$$

Proof. We will begin by determining the form of the term that would produce the $\partial_{t_1}^2$ term when acting on w:

$$\begin{aligned}
\partial_{t_1}^2 w &= \partial_{t_1}^2\left(\frac{G(z)\tau}{\tau}\right) \\
&= \partial_{t_1}\left(\left(G(z)\left(\frac{\partial_{t_1}\tau}{\tau}\right) - \frac{\partial_{t_1}\tau}{\tau}\right)\frac{G(z)\tau}{\tau}\right) \\
&= \left(G(z)\left(\frac{\partial_{t_1}^2\tau}{\tau}\right) - \frac{\partial_{t_1}^2\tau}{\tau} + 2\left(\frac{\partial_{t_1}\tau}{\tau}\right)^2 - 2\frac{\partial_{t_1}\tau}{\tau}G(z)\left(\frac{\partial_{t_1}\tau}{\tau}\right)\right)\frac{G(z)\tau}{\tau} \\
&= \left(G(z)\left(\frac{\partial_{t_1}^2\tau}{\tau}\right) - \frac{\partial_{t_1}^2\tau}{\tau} - 2\frac{\partial_{t_1}\tau}{\tau}\left(G(z)\left(\frac{\partial_{t_1}\tau}{\tau}\right) - \frac{\partial_{t_1}\tau}{\tau}\right)\right)\frac{G(z)\tau}{\tau} \\
&= \left(G(z)\left(\frac{\partial_{t_1}^2\tau}{\tau}\right) - \frac{\partial_{t_1}^2\tau}{\tau} + 2w_1\partial_{t_1}\right)\frac{G(z)\tau}{\tau}.
\end{aligned}$$

So, we have that

$$\partial_{t_1}^2 w - 2w_1\partial_{t_1} w = \left(G(z)\left(\frac{\partial_{t_1}^2\tau}{\tau}\right) - \frac{\partial_{t_1}^2\tau}{\tau}\right)\frac{G(z)\tau}{\tau}.$$

The $\partial_{t_1}^2 w$ portion of this would derive from

$$\begin{aligned}
\partial_{t_1}^2 W &= \partial_{t_1}\left(-L_- W\right) \\
&= \left([L_-, L]_- + (L_-)^2\right)W \\
&= \left(L_- v(s) - v(s)L_- + (L_-)^2\right)W.
\end{aligned}$$

As in the case of $\mathcal{L}_1$, we begin by looking at portions of the proof to come and rewriting them separately. First, based on the definition of L in terms of W, recall that

$$v(s) = w_1(s) + w_1^{-1}(s+1) = w_1(s) - w_1(s+1).$$

Also, using the definitions of the action of ∂_{t_1} on W, we see that

$$\partial_{t_1} w_1(s) = -e^{u(s)} = -L_-\Lambda = -L_-(L - v(s) - L_-) = -L_-L + L_-v(s) + (L_-)^2.$$

Finally, recall that

$$2w_2(s) - w_1(s)^2 = -\partial_{t_1} w_1(s) - \frac{\partial_{t_2}\tau}{\tau}.$$

Then

$$
\begin{aligned}
(W s\Lambda^2 W^{-1})_+ &= sL^2 - \sum_{i=1}^{\infty} i\, w_i(s)\Lambda^{-i+2} W^{-1} \\
&= s(L^2)_+ - (2w_2(s) + w_1(s)w_1^{-1}(s+1) + w_1(s)\Lambda) \\
&= sL^2 - s(L^2)_- - (2w_2(s) - w_1(s)^2) - w_1(s)(v(s)+\Lambda) \\
&= sL^2 - s(L^2)_- + \partial_{t_1} w_1(s) + \frac{\partial_{t_2}\tau}{\tau} - w_1(s)L_+ \\
&= sL^2 - s(L^2)_- - L_-L + L_-v(s) + (L_-)^2 + \frac{\partial_{t_2}\tau}{\tau} - w_1(s)L + w_1(s)L_- \\
&= sL^2 - s(L^2)_- - L_-L + L_-v(s) - v(s)L_- + (L_-)^2 + \frac{\partial_{t_2}\tau}{\tau} \\
&\quad - w_1(s)L + 2w_1(s)L_- - w_1(s+1)L_- \\
&= sL^2 - s(L^2)_- - L_-L + \left([L_-, L]_- + (L_-)^2\right) + \frac{\partial_{t_2}\tau}{\tau} - w_1(s)L + 2w_1(s)L_- - w_1(s+1)L_-.
\end{aligned}
$$

This gives us

$$
\begin{aligned}
\frac{(W s\Lambda^2 W^{-1})_+\psi}{\chi} &= \left(sz^2 + s\partial_{t_2} + z\partial_{t_1} + \partial_{t_1}^2 - 2w_1(s)\partial_{t_1} - z\,w_1(s) + w_1(s+1)\partial_{t_1} + \frac{\partial_{t_2}\tau}{\tau}\right) w \\
&= \left(G(z)\left(\frac{(s\partial_{t_2} + \partial_{t_1}^2)\tau}{\tau}\right) - \frac{(s\partial_{t_2} + \partial_{t_1}^2)\tau}{\tau} + sz^2 + zG(z)\left(\frac{\partial_{t_1}\tau}{\tau}\right)\right. \\
&\quad \left. -z\frac{\partial_{t_1}\tau}{\tau} - z\,w_1(s) - \frac{\partial_{t_1}\tau(s+1)}{\tau(s+1)}\partial_{t_1} + \frac{\partial_{t_2}\tau}{\tau}\right)\frac{G(z)\tau}{\tau}
\end{aligned}
$$

Next, based on the definition of L in terms of $\bar{W}$, recall that

$$
L_- = e^{u(s)}\Lambda^{-1} = \bar{w}_0(s)\bar{w}_0^{-1}(s-1)\Lambda^{-1}
$$

and that $\bar{w}_0^{-1}(s) = (\bar{w}_0(s))^{-1}$. Then

$$
\begin{aligned}
\frac{(\bar{W} s\Lambda^{-2}\bar{W}^{-1})_-\psi}{\chi} &= \frac{\left(sL^2 + \sum_{i=1}^{\infty} i\,\bar{w}_i(s)\Lambda^{i-2}\bar{W}^{-1}\right)_-\psi}{\chi} \\
&= s\frac{(L^2)_-\psi}{\chi} + \frac{\bar{w}_1(s)\bar{w}_0^{-1}(s)(\bar{w}_0(s)\bar{w}_0^{-1}(s-1)\Lambda^{-1})\psi}{\chi}
\end{aligned}
$$

$$
\begin{aligned}
&= s\frac{(L^2)_-\psi}{\chi} + \frac{\bar{w}_1(s)\bar{w}_0^{-1}(s)L_-\psi}{\chi} \\
&= \left(-s\partial_{t_2} - \bar{w}_1(s)\bar{w}_0^{-1}(s)\partial_{t_1}\right) w \\
&= -\left(G(z)\left(\frac{s\partial_{t_2}\tau}{\tau}\right) - \frac{s\partial_{t_2}\tau}{\tau} + \frac{\partial_{t_1}\tau(s+1)}{\tau(s+1)}\partial_{t_1}\right)\frac{G(z)\tau}{\tau}.
\end{aligned}
$$

Using Lemma 3.9, Corollary 3.5, and the previous results, we find that

$$
\begin{aligned}
\frac{\partial_{t_3^*}\psi}{\chi} &= \frac{-((M-\bar{M})L^3 + L^2)_-\psi}{\chi} \\
&= \frac{-(W s\Lambda^2 W^{-1} + \bar{W} s\Lambda^{-2}\bar{W}^{-1} + S_2 + L^2)_-\psi}{\chi} \\
&= \frac{-W s\Lambda^2 W^{-1}\psi}{\chi} + \frac{(W s\Lambda^2 W^{-1})_+\psi}{\chi} - \frac{(\bar{W} s\Lambda^{-2}\bar{W}^{-1})_-\psi}{\chi} + \frac{-(S_2)_-\psi}{\chi} + \frac{-(L^2)_-\psi}{\chi} \\
&= \left(-\sum_{n=1}^{\infty} z^{-n+2}G(z)\left(\frac{\partial_{t_n}\tau}{\tau}\right) + G(z)\left(\frac{(D_2 + 2s\partial_{t_2} + \partial_{t_1}^2)\tau}{\tau}\right) - \frac{(D_2 + 2s\partial_{t_2} + \partial_{t_1}^2)\tau}{\tau} + \frac{\partial_{t_2}\tau}{\tau}\right. \\
&\quad \left. + \sum_{n=1}^{\infty} z^{-n}G(z)\left(\frac{\partial_{t_{n+2}}\tau}{\tau}\right) + zG(z)\left(\frac{\partial_{t_1}\tau}{\tau}\right) - z\frac{\partial_{t_1}\tau}{\tau} - z\,w_1(s) + G(z)\left(\frac{\partial_{t_2}\tau}{\tau}\right) - \frac{\partial_{t_2}\tau}{\tau}\right)\frac{G(z)\tau}{\tau} \\
&= \left(G(z)\left(\frac{\mathcal{L}_2\tau}{\tau}\right) - \frac{\mathcal{L}_2\tau}{\tau} + \frac{\partial_{t_2}\tau}{\tau} - zG(z)\left(\frac{\partial_{t_1}\tau}{\tau}\right) - G(z)\left(\frac{\partial_{t_2}\tau}{\tau}\right)\right. \\
&\quad \left. + zG(z)\left(\frac{\partial_{t_1}\tau}{\tau}\right) - z\frac{\partial_{t_1}\tau}{\tau} + z\frac{\partial_{t_1}\tau}{\tau} + G(z)\left(\frac{\partial_{t_2}\tau}{\tau}\right) - \frac{\partial_{t_2}\tau}{\tau}\right)\frac{G(z)\tau}{\tau} \\
&= \left(G(z)\left(\frac{\mathcal{L}_2\tau}{\tau}\right) - \frac{\mathcal{L}_2\tau}{\tau}\right)\frac{G(z)\tau}{\tau}.
\end{aligned}
$$

Thus, we see that $\partial_{t_3^*}\tau = \mathcal{L}_2\tau$. □

While we can derive these operators for specific symmetries, it is clear that the process will become more complex as we progress to higher symmetries. Since the first four symmetries have been derived, the next two propositions allow us to find the actions of the remaining flows on the tau function without the need for such computations.

Proposition 3.7. *Suppose we have a family of operators $\mathcal{L}_k$, $k \in \mathbb{Z}_{\geq -1}$ that commute as the Virasoro algebra and so that $\partial_{t_\ell^*}\mathcal{L}_p = 0$ for $\ell \in \mathbb{Z}_{\geq 0}$. Also, suppose that two of these operators, $\mathcal{L}_p$ and $\mathcal{L}_q$, $p \neq q$, can be derived from the Adler-Shiota-van Moerbeke formula (3.17) so that $\partial_{t_{p+1}^*}\tau = \mathcal{L}_p\tau$ and*

$\partial_{t^*_{q+1}}\tau = \mathcal{L}_q\tau$. Then $\mathcal{L}_{p+q}$ can be derived from the Adler-Shiota-van Moerbeke formula as well; i.e.

$$\frac{\partial_{t^*_{p+q+1}}\psi}{\chi} = \left(G(z)\left(\frac{\mathcal{L}_{p+q}\tau}{\tau}\right) - \frac{\mathcal{L}_{p+q}\tau}{\tau}\right)\frac{G(z)\tau}{\tau}$$

and $\partial_{t^*_{p+q-1}}\tau = \mathcal{L}_{p+q}\tau$.

Proof. Using similar methods to those used to prove Lemma 3.1 in [3], we have that

$$[\partial_{t^*_{p+1}}, \partial_{t^*_{q+1}}]\psi = \left(G(z)\left(\frac{[\mathcal{L}_p,\mathcal{L}_q]\tau}{\tau}\right) - \frac{[\mathcal{L}_p,\mathcal{L}_q]\tau}{\tau}\right)\frac{G(z)\tau}{\tau}\chi.$$

Since the $\mathcal{L}_p$ commute as Virasoro by assumption and by the result of Theorem 3.16, the previous equation becomes

$$(p-q)\partial_{t^*_{p+q+1}}\psi = (p-q)\left(G(z)\left(\frac{\mathcal{L}_{p+q}\tau}{\tau}\right) - \frac{\mathcal{L}_{p+q}\tau}{\tau}\right)\frac{G(z)\tau}{\tau}\chi,$$

or, with common terms canceled,

$$\frac{\partial_{t^*_{p+q+1}}\psi}{\chi} = \left(G(z)\left(\frac{\mathcal{L}_{p+q}\tau}{\tau}\right) - \frac{\mathcal{L}_{p+q}\tau}{\tau}\right)\frac{G(z)\tau}{\tau}.$$

Thus, $\mathcal{L}_{p+q}$ can be derived from the Adler-Shiota-van Moerbeke formula and $\partial_{t^*_{p+q+1}}\tau = \mathcal{L}_{p+q}\tau$. □

Proposition 3.8. *Define operators $\mathcal{L}_p$ for $p \in \mathbb{Z}_{\geq\text{-}1}$ as*

$$\begin{aligned}
\mathcal{L}_{\text{-}1} &= \sum_{n=1}^{\infty}\left(2x_n\partial_{t_{n-1}} + n t_n\partial_{t_{n-1}} + n x_n\partial_{x_{n-1}}\right) + t_1x_0 + 2x_0x_1 \\
\mathcal{L}_0 &= \sum_{n=1}^{\infty}\left(2x_n\partial_{t_n} + n t_n\partial_{t_n} + n x_n\partial_{x_n}\right) + x_0^2 \\
\mathcal{L}_p &= \sum_{n=1}^{\infty}\left(2x_n\partial_{t_{n+p}} + n t_n\partial_{t_{n+p}} + n x_n\partial_{x_{n+p}}\right) + 2x_0\partial_{t_p} + \sum_{n=1}^{p-1}\partial_{t_n}\partial_{t_{p-n}}, \quad p \in \mathbb{Z}_{\geq 1}.
\end{aligned} \tag{3.29}$$

These operators commute as the Virasoro algebra; i.e.

$$[\mathcal{L}_p, \mathcal{L}_q] = (p-q)\mathcal{L}_{p+q}.$$

Note that the operators derived in Propositions 3.3, 3.4, 3.5, and 3.6 are part of these $\mathcal{L}_p$ operators.

Proof. Clearly, the formula holds for $p = q$. So, let $p \neq q$. We will first compute $[D_p, D_q]$, where these are defined in (3.23) as the summation portion of $\mathcal{L}_p$ and $\mathcal{L}_q$, respectively. Take note of the following non-zero commutators of the summands in D_p and D_q:

$$\begin{aligned}
[2x_n\partial_{t_{n+p}}, m t_m\partial_{t_{m+q}}] &= \delta_{m,n+p}2(n+p)x_n\partial_{t_{n+p+q}},\\
[2x_n\partial_{t_{n+p}}, m x_m\partial_{x_{m+q}}] &= -\delta_{n,m+q}2m x_m\partial_{t_{m+p+q}} + \delta_{q,\text{-}1}\delta_{n,1}\delta_{m,1}2x_1\partial_{t_p},\\
[n t_n\partial_{t_{n+p}}, m t_m\partial_{t_{m+q}}] &= \delta_{m,n+p}n(n+p)t_n\partial_{t_{n+p+q}} - \delta_{n,m+q}m(m+q)t_m\partial_{t_{m+p+q}},\\
[n x_n\partial_{x_{n+p}}, m x_m\partial_{x_{m+q}}] &= \delta_{m,n+p}n(n+p)x_n\partial_{x_{n+p+q}} - \delta_{n,m+q}m(m+q)x_m\partial_{x_{m+p+q}}.
\end{aligned}$$

We then have that

$$\begin{aligned}
[D_p, D_q] = &\sum_{n=1}^{\infty} 2(n+p)x_n\partial_{t_{n+p+q}} + 2n x_n\partial_{t_{n+p+q}} + n(n+p)t_n\partial_{t_{n+p+q}} + n(n+p)x_n\partial_{x_{n+p+q}}\\
&-\sum_{n=1}^{\infty} 2(n+q)x_n\partial_{t_{n+p+q}} + 2n x_n\partial_{t_{n+p+q}} + n(n+q)t_n\partial_{t_{n+p+q}} + n(n+q)x_n\partial_{x_{n+p+q}}\\
&-\delta_{p,\text{-}1}x_1\partial_{t_q} + \delta_{q,\text{-}1}x_1\partial_{t_p}\\
=&\,(p-q)\sum_{n=1}^{\infty} 2x_n\partial_{t_{n+p+q}} + n x_n\partial_{x_{n+p+q}} + n t_n\partial_{t_{n+p+q}} - \delta_{p,\text{-}1}x_1\partial_{t_q} + \delta_{q,\text{-}1}x_1\partial_{t_p}\\
=&\,(p-q)D_{p+q} - \delta_{p,\text{-}1}x_1\partial_{t_q} + \delta_{q,\text{-}1}x_1\partial_{t_p}.
\end{aligned}$$

Similarly, we compute that $[D_p, 2x_0\partial_{t_q}] = -2q x_0\partial_{t_{p+q}} + \delta_{p,\text{-}1}2x_1\partial_{t_q}$.

Since D_p does not involve any ∂_{t_n} or ∂_{x_n} for $p \geq n$,

$$\begin{aligned}
[D_p, t_1x_0 + 2x_0x_1] &= \delta_{p,0}(t_1x_0 + 4x_0x_1), \quad p \geq 0,\\
[D_p, x_0^2] &= \delta_{p,\text{-}1}2x_0x_1.
\end{aligned}$$

Next, we want to compute $\sum_{n=1}^{p-1}[\partial_{t_n}\partial_{t_{p-n}}, D_q]$ for $p \geq 2$: for $q \neq \text{-}1$,

$$\begin{aligned}
\sum_{n=1}^{p-1}[\partial_{t_n}\partial_{t_{p-n}}, D_q] &= \sum_{n=1}^{p-1} n\partial_{t_{p-n}}\partial_{t_{n+q}} + (p-n)\partial_{t_n}\partial_{t_{p+q-n}}\\
&= \sum_{n=1}^{p-1}(p-n)\partial_{t_n}\partial_{t_{p+q-n}} + \sum_{n=q}^{p+q-1}(n-q)\partial_{t_n}\partial_{t_{p+q-n}},
\end{aligned}$$

and for $q = \text{-}1$,

$$\begin{aligned}\sum_{n=1}^{p-1}[\partial_{t_n}\partial_{t_{p-n}}, D_{\text{-}1}] &= \sum_{n=2}^{p-1} n\partial_{t_{p-n}}\partial_{t_{n-1}} + \sum_{n=1}^{p-2}(p-n)\partial_{t_n}\partial_{t_{p-n-1}} \\ &= \sum_{n=1}^{p-2}(n+1)\partial_{t_{p-n-1}}\partial_{t_n} + (p-n)\partial_{t_n}\partial_{t_{p-n-1}} \\ &= (p+1)\sum_{n=1}^{p-2}\partial_{t_n}\partial_{t_{p-1-n}}.\end{aligned}$$

We also have that

$$\sum_{n=1}^{p-1}[\partial_{t_n}\partial_{t_{p-n}}, t_1 x_0] = 2x_0\partial_{t_{p-1}}$$

and that

$$[2x_0\partial_{t_p}, t_1 x_0] = \delta_{p,1}2x_0^2.$$

So, let us compute $[\mathcal{L}_0, \mathcal{L}_{\text{-}1}]$, $[\mathcal{L}_1, \mathcal{L}_{\text{-}1}]$, and $[\mathcal{L}_p, \mathcal{L}_{\text{-}1}]$ for $p \geq 2$:

$$\begin{aligned}[\mathcal{L}_0, \mathcal{L}_{\text{-}1}] &= [D_0 + x_0^2, D_{\text{-}1} + t_1 x_0 + 2x_0 x_1] \\ &= [D_0, D_{\text{-}1}] + [D_0, t_1 x_0 + 2x_0 x_1] + [x_0^2, D_{\text{-}1}] \\ &= D_{\text{-}1} + t_1 x_0 + 4x_0 x_1 - 2x_0 x_1 \\ &= D_{\text{-}1} + t_1 x_0 + 2x_0 x_1 = \mathcal{L}_{\text{-}1};\end{aligned}$$

$$\begin{aligned}[\mathcal{L}_1, \mathcal{L}_{\text{-}1}] &= [D_1 + 2x_0\partial_{t_1}, D_{\text{-}1} + t_1 x_0 + 2x_0 x_1] \\ &= [D_1, D_{\text{-}1}] + [2x_0\partial_{t_1}, D_{\text{-}1}] + [2x_0\partial_{t_1}, t_1 x_0] \\ &= 2D_0 + x_1\partial_{t_1} - 2x_1\partial_{t_1}2x_0^2 \\ &= 2\left(D_0 + x_0^2\right) = 2\mathcal{L}_0;\end{aligned}$$

$$\begin{aligned}
[\mathcal{L}_p, \mathcal{L}_{-1}] &= [D_p + 2x_0\partial_{t_p} + \sum_{n=1}^{p-1}\partial_{t_n}\partial_{t_{p-n}}, D_{-1} + t_1 x_0 + 2x_0 x_1] \\
&= [D_p, D_{-1}] + [2x_0\partial_{t_p}, D_{-1}] + \sum_{n=1}^{p-1}[\partial_{t_n}\partial_{t_{p-n}}, D_{-1}] + \sum_{n=1}^{p-1}[\partial_{t_n}\partial_{t_{p-n}}, t_1 x_0] \\
&= (p+1)D_{p-1} + x_1\partial_{t_p} - x_1\partial_{t_p} + 2p x_0\partial_{t_{p-1}} + (p+1)\sum_{n=1}^{p-2}\partial_{t_n}\partial_{t_{p-1-n}} + 2x_0\partial_{t_{p-1}} \\
&= (p+1)\left(D_{p-1} + 2x_0\partial_{t_{p-1}} + \sum_{n=1}^{p-2}\partial_{t_n}\partial_{t_{p-1-n}}\right) = (p+1)\mathcal{L}_{p-1}.
\end{aligned}$$

So, the relations hold when one of the operators is $\mathcal{L}_{-1}$.

Next we compute $[\mathcal{L}_p, \mathcal{L}_0]$ for $p \in \mathbb{Z}_{\geq 1}$ to show it holds for either operator being $\mathcal{L}_0$:

$$\begin{aligned}
[\mathcal{L}_p, \mathcal{L}_0] &= [D_p + 2x_0\partial_{t_p} + \sum_{n=1}^{p-1}\partial_{t_n}\partial_{t_{p-n}}, D_0 + x_0^2] \\
&= [D_p, D_0] + [2x_0\partial_{t_p}, D_0] + \sum_{n=1}^{p-1}[\partial_{t_n}\partial_{t_{p-n}}, D_0] \\
&= pD_p + 2p x_0\partial_{t_p} + p\sum_{n=1}^{p-1}\partial_{t_n}\partial_{t_{p-n}} \\
&= p\left(D_p + 2x_0\partial_{t_p} + \sum_{n=1}^{p-1}\partial_{t_n}\partial_{t_{p-n}}\right) = p\mathcal{L}_p.
\end{aligned}$$

Finally, we compute $[\mathcal{L}_p, \mathcal{L}_q]$ for $p, q \in \mathbb{Z}_{\geq 1}$:

$$\begin{aligned}
[\mathcal{L}_p, \mathcal{L}_q] &= [D_p + 2x_0\partial_{t_p} + \sum_{n=1}^{p-1}\partial_{t_n}\partial_{t_{p-n}}, D_q + 2x_0\partial_{t_q} + \sum_{n=1}^{q-1}\partial_{t_n}\partial_{t_{q-n}}] \\
&= (p-q)D_{p+q} + 2p x_0\partial_{t_{p+q}} + \sum_{n=1}^{p-1}(p-n)\partial_{t_n}\partial_{t_{p+q-n}} + \sum_{n=q}^{p+q-1}(n-q)\partial_{t_n}\partial_{t_{p+q-n}} \\
&\quad - 2q x_0\partial_{t_{p+q}} - \sum_{n=1}^{q-1}(q-n)\partial_{t_n}\partial_{t_{p+q-n}} - \sum_{n=p}^{p+q-1}(n-p)\partial_{t_n}\partial_{t_{p+q-n}} \\
&= (p-q)\left(D_{p+q} + 2x_0\partial_{t_{p+q}} + \sum_{n=1}^{p+q-1}\partial_{t_n}\partial_{t_{p+q-n}}\right) = (p-q)\mathcal{L}_{p+q}.
\end{aligned}$$

Thus, the commutation relations hold for any such $\mathcal{L}_p$, $\mathcal{L}_q$. □

The combination of Propositions 3.7 and 3.8 prove for us the second main theorem of this thesis:

Theorem 3.3. *The actions of the additional flows on the wave functions in (3.12) induce actions on the tau function:*

$$\partial_{t^*_\ell}\tau = \mathcal{L}_{\ell-1}\tau, \tag{3.30}$$

where $\mathcal{L}_{\ell-1}$ are defined in (3.29).

Finally, recall from (2.26) that we performed a change of variables to simplify the notation for the extended Toda hierarchy. If we perform the reverse change of variables,

$$\begin{aligned} s &= \frac{x}{\epsilon}, \\ t_{n+1} &= \frac{1}{\epsilon(n+1)!}\left(t^{2,n} - 2c_{n+1}t^{1,n+1}\right), \\ x_n &= \frac{1}{\epsilon n!}t^{1,n}, \end{aligned} \tag{3.31}$$

on the Virasoro operators of (3.29), we obtain the following formulas for $\mathcal{L}_p$:

$$\begin{aligned} \mathcal{L}_{\text{-}1} &= \sum_{n=1}^{\infty} t^{\alpha,n}\partial_{t^{\alpha,n-1}} + \frac{1}{\epsilon^2}t^{1,0}t^{2,0}, \\ \mathcal{L}_0 &= \sum_{n=1}^{\infty} n(t^{1,n}\partial_{t^{1,n}} + t^{2,n-1}\partial_{t^{2,n-1}}) + 2t^{1,n}\partial_{t^{2,n-1}} + \frac{1}{\epsilon}\left(t^{1,0}\right)^2, \\ \mathcal{L}_p &= \epsilon^2\sum_{n=1}^{p-1} k!(p-n)!\partial_{t^{2,n-1}}\partial_{t^{2,p-n-1}} + \sum_{n=1}^{\infty}\frac{(p+n)!}{(n-1)!}\left(t^{1,n}\partial_{t^{1,p+n}} + t^{2,n-1}\partial_{t^{2,p+n-1}}\right) \\ &\quad +2\sum_{n=0}^{\infty}\alpha_p(n)t^{1,n}\partial_{t^{2,p+n-1}}, \end{aligned} \tag{3.32}$$

where

$$\alpha_p(0) = p!, \quad \alpha_p(n) = \frac{(p+n)!}{(n-1)!}\sum_{j=n}^{p+n}\frac{1}{j}, \quad n > 0,$$

and sums that include $t^{\alpha,*}$ are implied to be sums over $\alpha = 1, 2$. These $\mathcal{L}_p$ of (3.32) precisely match those given by Dubrovin and Zhang in [9].

CHAPTER

4

EXTENDED BIGRADED TODA

Attention is now turned toward a hierarchy of which the extended Toda is a special case. Instead of considering a Lax operator whose powers of Λ stop at 1 and -1, we consider difference operators whose highest power of Λ is k and whose lowest power is -m, where $k, m \in \mathbb{Z}_{\geq 1}$. This hierarchy was first introduced by Carlet in [4] as a generalization of the extended Toda and was further investigated by Carlet and van de Leur in [6]. Using these two papers, Section 4.1 will consist of the definitions of the bigraded Toda and extended bigraded Toda. We define the additional symmetries of the hierarchy in Lax form in Section 4.2, very similar to what was done in Chapter 3. In Section 4.3, we determine the action of these flows on the tau function.

4.1 Definition

Just as with the ETH, the extended bigraded Toda hierarchy is an extension of a smaller hierarchy, the bigraded Toda hierarchy. Thus, this section will first define this hierarchy and then move on to define the extended bigraded Toda.

Bigraded Toda Hierarchy

Consider a Lax operator of the form

$$L = \Lambda^k + u_{k-1}\Lambda^{k-1} + \cdots + u_{-m+1}\Lambda^{-m+1} + u_{-m}\Lambda^{-m}, \tag{4.1}$$

where $k, m \in \mathbb{Z}_{\geq 1}$ and $u_{-m} \neq 0$. In the special case where $k = m = 1$, it is clear that this form reduces to the Lax operator of the Toda hierarchy.

Just as with the Toda and extended Toda hierarchies, the Lax operator has representations using dressing operators,

$$W = 1 + \sum_{i=1}^{\infty} w_i(s)\Lambda^{-i}$$

and

$$\bar{W} = \sum_{i=0}^{\infty} \bar{w}_i(s)\Lambda^{i}, \quad \bar{w}_0(s) \neq 0:$$

$$L = W\Lambda^k W^{-1} = \bar{W}\Lambda^{-m}\bar{W}^{-1}. \tag{4.2}$$

Similar to how the KP hierarchy and its Lax operator were introduced in Chapter 1, (4.2) allows us to define fractional power of L:

$$L^{\frac{1}{k}} = W\Lambda W^{-1} = \Lambda + \sum_{i\geq 0} a_i(s)\Lambda^{-i}, \quad L^{\frac{1}{m}} = \bar{W}\Lambda^{-1}\bar{W}^{-1} = \sum_{i\geq -1} b_i(s)\Lambda^{i}. \tag{4.3}$$

Note that L commutes with both of these operators and that we have

$$\left(L^{\frac{1}{k}}\right)^k = \left(L^{\frac{1}{m}}\right)^m = L.$$

It is heavily stressed that, due to the powers of Λ involved in each operator, $L^{\frac{1}{k}} \neq L^{\frac{1}{m}}$, unless we are in the notable case $k = m = 1$. We will use the notation

$$L^{\frac{p}{k}} = \left(L^{\frac{1}{k}}\right)^p = W\Lambda^p W^{-1}, \quad L^{\frac{p}{m}} = \left(L^{\frac{1}{m}}\right)^p = \bar{W}\Lambda^{-p}\bar{W}^{-1}, \quad p \in \mathbb{Z}.$$

With these new operators in mind, we have the following:

Definition 4.1. *Define variables t_n and $\bar{t}_n$, $n \in \mathbb{Z}_{\geq 1}$, with a system of flows on the Lax operator L*

given by

$$\begin{aligned}\frac{\partial L}{\partial t_n} &= [(L^{\frac{n}{k}})_+, L] = [-(L^{\frac{n}{k}})_-, L], \\ \frac{\partial L}{\partial \bar{t}_n} &= [(L^{\frac{n}{m}})_+, L] = [-(L^{\frac{n}{m}})_-, L],\end{aligned} \tag{4.4}$$

where $(\ )_+$ *and* $(\ )_-$ *are defined as in Toda and extended Toda. These flows define the* ***bigraded Toda hierarchy***.

Using this notation, it is important to note that we consider $t_{nk} = \bar{t}_{nm}$ since their flows on L are identical. The flows of (4.4) also induce flows on the dressing operators:

$$\begin{aligned}\frac{\partial W}{\partial t_n} &= -(L^{\frac{n}{k}})_- W, \qquad \frac{\partial \bar{W}}{\partial t_n} = (L^{\frac{n}{k}})_+ \bar{W}, \\ \frac{\partial W}{\partial \bar{t}_n} &= -(L^{\frac{n}{m}})_- W, \qquad \frac{\partial \bar{W}}{\partial \bar{t}_n} = (L^{\frac{n}{m}})_+ \bar{W}.\end{aligned} \tag{4.5}$$

Logarithm of L and Definition of the Hierarchy

Extending the bigraded Toda involves the same steps as with the Toda hierarchy. Define the ***logarithm of*** L as before:

$$\log L = \frac{1}{2}\left(W \partial_s W^{-1} - \bar{W} \partial_s \bar{W}^{-1}\right).$$

Just as before, this operator commutes with L and all integral powers of L. However, the composition of a fractional power of L and $\log L$ is not well defined.

Using $\log L$, we have the following:

Definition 4.2. *The* ***extended bigraded Toda hierarchy*** *(abbreviated EBTH) consists of a system of flows given in Lax pair form by (4.4) and by*

$$\frac{\partial L}{\partial x_n} = [(2L^n \log L)_+, L] = [-(2L^n \log L)_-, L], \quad n \in \mathbb{Z}_{\geq 0}. \tag{4.6}$$

Just as with extended Toda, these new flows induce corresponding actions on the dressing operators:

$$\frac{\partial W}{\partial x_n} = -(2L^n \log L)_- W, \quad \frac{\partial \bar{W}}{\partial x_n} = (2L^n \log L)_+ \bar{W}. \tag{4.7}$$

Change of Variables

The version of the extended bigraded Toda hierarchy presented in this thesis is not the version originally introduced by Carlet in [4] nor the one used in [6] by Carlet and van de Leur. We will compare our version with [6] since that is the more recent publication. In that paper, the authors consider a Lax operator of the form

$$L = \Lambda^{-k} + u_1\Lambda^{-k+1} + \cdots + u_{k+m}\Lambda^{m}, \quad u_{k+m} \neq 0,$$

the reverse of what this thesis considers. Recalling the use of the spectral parameter ϵ from Section 2.3, the form of the Lax operator used in this thesis can be obtained by making the change $\epsilon \to -\epsilon$. Doing so also changes $\Lambda \to \Lambda^{-1}$ and $\zeta \to \zeta^{-1}$ (z in this thesis) in [6].

After making this change, the authors define the extended bigraded Toda hierarchy as the following flows on L:

$$\frac{\partial L}{\partial q_n^{\beta}} = [(B_n^{\beta})_+, L] = -[(B_n^{\beta})_-, L], \quad n \in \mathbb{Z}_{\geq 0} \tag{4.8}$$

where

$$\begin{aligned} B_n^{\beta} &= \frac{1}{\epsilon k}\binom{-\frac{\beta}{k}}{-n-1} L^{n-\frac{\beta}{k}+1}, && \beta = 1, 2, \ldots, k-1, \\ B_n^{\beta} &= \frac{1}{\epsilon m}\binom{\frac{\beta-k}{m}-1}{-n-1} L^{\frac{\beta-k}{m}+n}, && \beta = k+1, k+2, \ldots, k+m, \\ B_n^{k} &= \frac{2}{\epsilon n!}\left(L^n\left(\log L - \frac{c_n}{2}\left(\frac{1}{k}+\frac{1}{m}\right)\right)\right), \end{aligned} \tag{4.9}$$

the harmonic numbers c_n are defined in (2.25), and for $\ell \in \mathbb{Z}_{\geq 1}$,

$$\begin{aligned} (q)_0 &= 1, \\ (q)_\ell &= \prod_{i=1}^{\ell}(q-i+1), \\ (q)_{-\ell} &= \prod_{i=-\ell+1}^{0}(q-i+1)^{-1} = \frac{1}{(q+\ell)_\ell}. \end{aligned}$$

Using Takasaki's logic again, the use of the harmonic numbers and the use of $(q)_n$ are necessary for the study of the underlying manifold. Otherwise, they are superficial and are dropped for this thesis. As with the ETH, we have the following change of variables allowing us to convert from the definition of [6] to the one used here:

$$\begin{aligned} x &= \epsilon s, \\ q_n^{k-\alpha} &= \epsilon k \left(n + \frac{\alpha}{k}\right)_{n+1} t_{nk+\alpha}, \quad \alpha = 1, 2, \ldots, k-1,\, n \in \mathbb{Z}_{\geq 0}, \\ q_n^{k+\beta} &= \epsilon m \left(n + \frac{\beta}{m}\right)_{n+1} \bar{t}_{nm+\beta}, \quad \beta = 1, 2, \ldots, m-1, \\ q_n^{k+m} &= \epsilon m (n+1)! \left(t_{(n+1)k} + c_{n+1} \left(\frac{1}{k} + \frac{1}{m}\right) x_{k+1} \right), \\ q_n^k &= \epsilon n! x_n, \end{aligned} \tag{4.10}$$

keeping in mind that $t_{nk} = \bar{t}_{nm}$. The inverse of this change can be used to translate the results of the next section back to the version of the hierarchy used by Carlet and van de Leur.

4.2 Additional Symmetries

In this section, we use the methods found in Chapter 3 to formulate additional symmetries for the EBTH in Lax form. As a result, many of the propositions and theorems found here have proofs very similar to those found in the previous chapter. Those proofs will be omitted, and any significant differences will be highlighted instead.

Wave Functions

To derive additional symmetries of the EBTH as was done for the ETH in Chapter 3, we need to introduce an appropriate pair of wave functions, ψ and $\bar{\psi}$. Similar to how Takasaki does this for the ETH in [17], the exponential terms of the wave functions are generated by acting on z^s with the undressed generators of the flows of the hierarchy:

$$\exp\left(\sum_{n=1}^{\infty} t_n\Lambda^n - \frac{1}{2}t_{nk}\Lambda^{nk}\right)z^s = z^s e^{\sum t_n z^n - \frac{1}{2}\sum t_{nk}z^{nk}},$$
$$\exp\left(\sum_{n=1}^{\infty} x_n\Lambda^{nk}\partial_s\right)z^s = z^{s+\sum x_n z^{nk}},$$

$$\exp\left(\sum_{n=1}^{\infty} \bar{t}_n\Lambda^{-n} - \frac{1}{2}\bar{t}_{nm}\Lambda^{-nm}\right)z^s = z^s e^{-\sum \bar{t}_n z^{-n} + \frac{1}{2}\sum \bar{t}_{nm}z^{-nm}},$$
$$\exp\left(\sum_{n=1}^{\infty} x_n\Lambda^{-nm}\partial_s\right)z^s = z^{s+\sum x_n z^{-nm}}.$$

This gives us the following definitions for wave functions of the EBTH:

$$\begin{aligned} \psi &:= W\chi = w\chi, \\ \bar{\psi} &:= \bar{W}\bar{\chi} = \bar{w}\bar{\chi}, \end{aligned} \tag{4.11}$$

where $w = w(s,z)$ and $\bar{w} = \bar{w}(s,z)$ are defined as in the extended Toda and

$$\chi = z^{s+\sum x_n z^{kn}} e^{\sum t_n z^n - \frac{1}{2}\sum t_{nk}z^{nk}},$$
$$\bar{\chi} = z^{s+\sum x_n z^{-mn}} e^{-\sum \bar{t}_n z^{-n} + \frac{1}{2}\sum t_{nk}z^{-nm}}.$$

We thus have that

$$L\psi = z^k\psi,$$
$$L\bar{\psi} = z^{-m}\bar{\psi}$$

and that the flows of the hierarchy act on these wave functions as follows:

$$\begin{aligned}
\frac{\partial \psi}{\partial t_n} &= (L^{\frac{n}{k}})_+ \psi, \quad n \notin k\mathbb{Z}_{\geq 1}, \\
\frac{\partial \psi}{\partial t_{nk}} &= \left(-(L^n)_- + \frac{1}{2} L^n\right) \psi, \\
\frac{\partial \psi}{\partial \bar{t}_n} &= -(L^{\frac{n}{m}})_- \psi, \quad n \notin m\mathbb{Z}_{\geq 1}, \\
\frac{\partial \psi}{\partial x_n} &= (L^n \partial_s + P_n) \psi,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \bar{\psi}}{\partial t_n} &= (L^{\frac{n}{k}})_+ \bar{\psi}, \quad n \notin k\mathbb{Z}_{\geq 1}, \\
\frac{\partial \bar{\psi}}{\partial \bar{t}_{nm}} &= \left((L^n)_+ - \frac{1}{2} L^n\right) \bar{\psi}, \\
\frac{\partial \bar{\psi}}{\partial \bar{t}_n} &= -(L^{\frac{n}{m}})_- \bar{\psi}, \quad n \notin m\mathbb{Z}_{\geq 1}, \\
\frac{\partial \bar{\psi}}{\partial x_n} &= (L^n \partial_s + P_n) \bar{\psi}.
\end{aligned}$$

Here, P_n is defined as in the ETH in Definition 2.4.

M and $\bar{M}$

With the introduction of wave functions for the hierarchy, we can now derive the operators M and $\bar{M}$. Just like before, we want these operators defined in such a way that

$$[L, M] = [L, \bar{M}] = 1.$$

To this end, M and $\bar{M}$ are to act on their respective wave functions as follows:

$$M\psi = \partial_{z^k} \psi, \quad \bar{M}\bar{\psi} = \partial_{z^{-m}} \bar{\psi}.$$

Mirroring the derivations from Section 3.1, we have the following definitions for M and $\bar{M}$:

$$M := \frac{1}{k} W s \Lambda^{-k} W^{-1} + \sum_{n=1}^{\infty} \left(\frac{1}{k} x_n L^{n-1} + n x_n W \partial_s W^{-1} L^{n-1} + \frac{n}{k} t_n L^{\frac{n-k}{k}} - \frac{1}{2} n t_{nk} L^{n-1} \right), \tag{4.12}$$

$$\bar{M} := -\left(\frac{1}{m} \bar{W} s \Lambda^{m} \bar{W}^{-1} + \sum_{n=1}^{\infty} \left(\frac{1}{m} x_n L^{n-1} - n x_n \bar{W} \partial_s \bar{W}^{-1} L^{n-1} + \frac{n}{m} \bar{t}_n L^{\frac{n-m}{m}} - \frac{1}{2} n \bar{t}_{nm} L^{n-1} \right) \right). \tag{4.13}$$

The flows of the EBTH act on M and $\bar{M}$ as follows:

$$\begin{aligned}
\partial_{t_n} M &= [-(L^{\frac{n}{k}})_- + L^{\frac{n}{k}}, M], \quad n \notin k\mathbb{Z}_{\geq 1}, \\
\partial_{t_{nk}} M &= [-(L^{n})_- + \frac{1}{2} L^{n}, M], \\
\partial_{\bar{t}_n} M &= [-(L^{\frac{n}{m}})_-, M], \quad n \notin m\mathbb{Z}_{\geq 1}, \\
\partial_{x_n} M &= [L^n \partial_s + P_n, M]
\end{aligned}$$

$$\begin{aligned}
\partial_{t_n} \bar{M} &= [(L^{\frac{n}{k}})_+, M], \quad n \notin k\mathbb{Z}_{\geq 1}, \\
\partial_{t_{nk}} \bar{M} &= [(L^{n})_+ - \frac{1}{2} L^{n}, \bar{M}], \\
\partial_{\bar{t}_n} \bar{M} &= [(L^{\frac{n}{m}})_+ - L^{\frac{n}{m}}, \bar{M}], \quad n \notin m\mathbb{Z}_{\geq 1}, \\
\partial_{x_n} \bar{M} &= [L^n \partial_s + P_n, \bar{M}].
\end{aligned}$$

Finally, we have the following difference operator that commutes with L:

$$\begin{aligned}
M - \bar{M} \quad &= \frac{1}{k} W s \Lambda^{-k} W^{-1} + \frac{1}{m} \bar{W} s \Lambda^{m} \bar{W}^{-1} + \sum_{n=1}^{\infty} \left(\left(\frac{1}{k} + \frac{1}{m} \right) x_n L^{n-1} + n x_n (2 L^{n-1} \log L) \right. \\
&\quad \left. + \frac{n}{k} t_n L^{\frac{n-k}{k}} + \frac{n}{m} \bar{t}_n L^{\frac{n-m}{m}} - n t_{nk} L^{n-1} \right).
\end{aligned} \tag{4.14}$$

Additional Symmetries

With the previous two sections in mind, the additional symmetries for the extended bigraded Toda hierarchy are defined identically to those of the ETH: define additional flows denoted by $\partial_{t^*_\ell}$ that act on the Lax operator L by

$$\partial_{t^*_\ell} L = [(B_\ell)_+, L] = [-(B_\ell)_-, L], \tag{4.15}$$

where

$$B_\ell = (M - \bar{M})L^\ell - \frac{k-1}{2k} W\Lambda^{k(\ell-1)}W^{-1} + \frac{m+1}{2m}\bar{W}\Lambda^{-m(\ell-1)}\bar{W}^{-1}.$$

These flows induce actions on the wave operators, wave functions, and M and $\bar{M}$:

$$\partial_{t^*_\ell} W = -(B_\ell)_- W, \quad \partial_{t^*_\ell} \bar{W} = (B_\ell)_+ \bar{W}; \tag{4.16}$$

$$\partial_{t^*_\ell} \psi = -(B_\ell)_- \psi, \quad \partial_{t^*_\ell} \bar{\psi} = (B_\ell)_+ \bar{\psi}. \tag{4.17}$$

$$\partial_{t^*_\ell} M = [-(B_\ell)_-, M], \quad \partial_{t^*_\ell} \bar{M} = [(B_\ell)_+ \bar{M}]. \tag{4.18}$$

The main result of this chapter is the following theorem:

Theorem 4.1. *The flows $\partial_{t^*_\ell}$ defined in (4.15) on L and in (4.16) on W and $\bar{W}$ define additional symmetries in Lax pair form for the extended bigraded Toda hierarchy; i.e. for Φ being L, W, or $\bar{W}$ and y being any of the t_n, $\bar{t}_n$, or x_n variables, we have that*

$$[\partial_y, \partial_{t^*_\ell}]\Phi = 0.$$

The proof of this theorem is fundamentally the same as the proof for Theorem 3.1 and is therefore omitted. These additional symmetries commute with one another as the symmetries of the ETH do:

Theorem 4.2. *The map $\partial_{t^*_\ell} \longmapsto \mathcal{L}_{\ell-1}$, $\ell \in \mathbb{Z}_{\geq 0}$ defines a homomorphism between the flows of (4.16) and (4.15) acting on W, $\bar{W}$, and L of the extended bigraded Toda hierarchy and the Virasoro algebra; i.e.,*

$$[\partial_{t^*_\ell}, \partial_{t^*_p}]\Phi = (\ell - p)\partial_{t^*_{\ell+p-1}}\Phi, \tag{4.19}$$

where Φ can be any of W, $\bar{W}$, and L.

Again the proof of this theorem is fundamentally the same as the proof for Theorem 3.2 and is omitted.

4.3 Symmetries on Tau Function

Continuing with the process set forth in Chapter 3, the next step is to use the Adler-Shiota-van Moerbeke formula in (1.25) to determine these symmetries' actions on the tau function of the EBTH. To do this, we require the representation of the coefficients of W and $\bar{W}$ in terms of the tau function, such as (2.22) for the ETH. Carlet and van de Leur provide such formulas in [6], and after the appropriate change of variables, we have the following:

$$w(s,z) = \frac{G_1(z)\tau}{\tau}, \quad \bar{w}(s,z) = \frac{G_2(z)\tau}{\tau}. \tag{4.20}$$

Here, $G_1(z)$ and $G_2(z)$ are defined on functions of s, $\mathbf{t} = (t_1, t_2, \ldots)$, $\bar{\mathbf{t}} = (\bar{t}_1, \bar{t}_2, \ldots)$, and $\mathbf{x} = (x_1, x_2, \ldots)$ by

$$G_1(z) f(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}) = f(s, \mathbf{t} - [z^{-1}], \bar{\mathbf{t}}, \mathbf{x})$$

and

$$G_2(z) f(s, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}) = f(s+1, \mathbf{t}, \bar{\mathbf{t}} + [z], \mathbf{x})$$

where $[z]$ is defined in (1.20).

We again want to find operators $\mathcal{L}_p$, $p \in \mathbb{Z}_{\geq -1}$ such that $\partial_{t^*_{p+1}}\tau = \mathcal{L}_p \tau$ and

$$\partial_{t^*_{p+1}}\psi = \left(G_1(z)\left(\frac{\mathcal{L}_p \tau}{\tau}\right) - \frac{\mathcal{L}_p \tau}{\tau}\right)\psi$$

and

$$\partial_{t^*_{p+1}}\bar{\psi} = \left(G_2(z)\left(\frac{\mathcal{L}_p \tau}{\tau}\right) - \frac{\mathcal{L}_p \tau}{\tau}\right)\bar{\psi}.$$

Actions of $\mathcal{L}_{-1}$ and $\mathcal{L}_0$

Proposition 4.1. *The first additional symmetry of the EBTH acts on the tau function as* $\partial_{t^*_0}\tau = \mathcal{L}_{-1}\tau$, *where*

$$\begin{aligned}\mathcal{L}_{-1} &= \sum_{n=1}^{\infty}\left(\left(\frac{1}{k}+\frac{1}{m}\right)x_n\partial_{t_{k(n-1)}} + n x_n \partial_{x_{n-1}} + \frac{n}{k} t_n \partial_{t_{n-k}} + \frac{n}{m}\bar{t}_n \partial_{\bar{t}_{n-m}} - n t_{nk}\partial_{t_{k(n-1)}}\right) \\ &\quad + t_k x_0 + \left(\frac{1}{k}+\frac{1}{m}\right)x_0 x_1 + \frac{1}{2k}\sum_{\alpha=1}^{k-1}(\alpha t_\alpha)\big((k-\alpha)t_{k-\alpha}\big) + \frac{1}{2m}\sum_{\beta=1}^{m-1}(\beta \bar{t}_\beta)\big((m-\beta)\bar{t}_{m-\beta}\big).\end{aligned} \tag{4.21}$$

Since this proof and the proof for $\mathcal{L}_0$ use many of the same methods that were used in Chapter 3, many details have been omitted. Also, note that, even though it shifts the $\bar{t}_{nm}$ variables since $\bar{t}_{nm} = t_{nk}$, the presence of the $n t_{nk}\partial_{t_{k(n-1)}}$ term in $\mathcal{L}_{-1}$ and similar terms in the remaining Virasoro operators effectively makes the shifting operator $G_1(z)$ only affect the $\frac{n}{k} t_n \partial_{t_{n-k}}$ terms. Similarly, $G_2(z)$ effectively only shifts the $\frac{n}{m}\bar{t}_n \partial_{\bar{t}_{n-m}}$ terms of $\mathcal{L}_{-1}$.

Proof. We have

$$\begin{aligned}
\frac{\partial_{t_0^*}\psi}{\chi} &= \frac{1}{\chi}\left(-(M-\bar{M})_- + \frac{k-1}{2k} W\Lambda^{-k}W^{-1}\right)W\chi \\
&= \frac{1}{\chi}\left(-\frac{1}{k}Ws\Lambda^{-k}W^{-1} - \sum_{n=1}^{\infty}\left(\frac{n}{k}t_n(L^{\frac{n-k}{k}})_- + \cdots\right) + \frac{k-1}{2k}W\Lambda^{-k}W^{-1}\right)W\chi \\
&= \left(-s\frac{z^{-k}}{k} - \frac{z^{-k+1}}{k}\partial_z - \frac{1}{k}\sum_{\alpha=1}^{k-1}\alpha t_\alpha z^{\alpha-k} + \frac{k-1}{2k}z^{-k} + \sum_{n=1}^{\infty}\left(\frac{n}{k}t_n\partial_{t_{n-1}} + \cdots\right)\right)w \\
&= \Bigg(s\left(t_k - \frac{z^{-k}}{k}\right) - s\,t_k - \frac{1}{k}\sum_{n=1}^{\infty}z^{-k-n}G_1(z)\left(\frac{\partial_{t_n}\tau}{\tau}\right) \\
&\quad + \frac{1}{2k}\sum_{\alpha=1}^{k-1}\left(\alpha\left(t_\alpha - \frac{z^{-\alpha}}{\alpha}\right)(k-\alpha)\left(t_{k-\alpha} - \frac{z^{\alpha-k}}{k-\alpha}\right) - \alpha t_\alpha(k-\alpha)t_{k-\alpha}\right) \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{n}{k}t_n\partial_{t_{n-1}}\left(G_1(z)\left(\frac{\partial_{t_{n-k}}\tau}{\tau}\right) - \frac{\partial_{t_{n-k}}\tau}{\tau}\right) + \cdots\right)\Bigg)\frac{G_1(z)\tau}{\tau} \\
&= \left(G_1(z)\left(\frac{\mathcal{L}_{-1}\tau}{\tau}\right) - \frac{\mathcal{L}_{-1}\tau}{\tau}\right)\frac{G_1(z)\tau}{\tau}
\end{aligned}$$

since the $G_1(z)$ shift operator does not affect any of the x_n or the strictly $\bar{t}_n$ variables. A similar calculation involving the action on $\bar{\psi}$ gives rise to the $\sum_{\beta=1}^{m-1}\beta\,\bar{t}_\beta(m-\beta)\bar{t}_{m-\beta}$ and $\left(\frac{1}{k}+\frac{1}{m}\right)x_0x_1$ terms, thus proving the proposition.

□

Proposition 4.2. *The second additional symmetry of the EBTH acts on the tau function as $\partial_{t_1^*}\tau = \mathcal{L}_0\tau$, where*

$$\begin{aligned}
\mathcal{L}_0 &= \sum_{n=1}^{\infty}\left(\left(\frac{1}{k}+\frac{1}{m}\right)x_n\partial_{t_{kn}} + n x_n\partial_{x_n} + \frac{n}{k}t_n\partial_{t_n} + \frac{n}{m}\bar{t}_n\partial_{\bar{t}_n} - n t_{nk}\partial_{t_{kn}}\right) \\
&\quad + \frac{1}{2}\left(\frac{1}{k}+\frac{1}{m}\right)x_0^2 + \frac{k^2-1}{24k} + \frac{m^2-1}{24m}.
\end{aligned} \tag{4.22}$$

Proof. We have

$$\begin{aligned}
\frac{\partial_{t_1^*}\psi}{\chi} &= \frac{1}{\chi}\left(-((M-\bar{M})L)_- W\chi\right) \\
&= \frac{1}{\chi}\left(-\frac{1}{k}WsW^{-1} + \frac{s}{k} - \sum_{n=1}^{\infty}\left(\frac{n}{k}t_n(L^{\frac{n}{k}})_- + \cdots\right)\right)W\chi
\end{aligned}$$

$$
\begin{aligned}
&= \left(-\frac{1}{k} z \partial_z + \sum_{n=1}^{\infty} \left(\frac{n}{k} t_n \partial_{t_n} + \cdots \right) \right) w \\
&= \left(-\frac{1}{k} \sum_{n=1}^{\infty} z^{-n} G_1(z) \left(\frac{\partial_{t_n} \tau}{\tau} \right) + \sum_{n=1}^{\infty} \left(\frac{n}{k} t_n \left(G_1(z) \left(\frac{\partial_{t_n} \tau}{\tau} \right) - \frac{\partial_{t_n} \tau}{\tau} \right) + \cdots \right) \right) \frac{G_1(z)\tau}{\tau} \\
&= \left(G_1(z) \left(\frac{\mathcal{L}_0 \tau}{\tau} \right) - \frac{\mathcal{L}_0 \tau}{\tau} \right) \frac{G_1(z)\tau}{\tau}.
\end{aligned}
$$

The same calculation using the action on $\bar{\psi}$ and the shift operator $G_2(z)$ gives rise to the $\frac{1}{2}\left(\frac{1}{k} + \frac{1}{m}\right) x_0^2$ term. Note that the last two terms of $\mathcal{L}_0$ vanish when converting from the action on τ back to the action on ψ or $\bar{\psi}$. Thus, their presences are not necessary when viewing the action of $\mathcal{L}_0$ alone. However, to maintain the Virasoro commutation relations so that $[\mathcal{L}_1, \mathcal{L}_{-1}] = 2\mathcal{L}_0$, these two terms are added to $\mathcal{L}_0$. □

The Remaining Actions

Finally we present the following theorem for the actions of the remaining symmetries on the tau function, the proof of which to be presented in a future publication:

Theorem 4.3. *For $p \in \mathbb{Z}_{\geq 1}$, the actions of the additional symmetries $\partial_{t^*_{p+1}}$ on the tau function are given by*

$$
\begin{aligned}
\mathcal{L}_p \; &= \sum_{n=0}^{\infty} \left(\left(\frac{1}{k} + \frac{1}{m} \right) x_n \partial_{t_{k(n+p)}} + n x_n \partial_{x_{n+p}} + \frac{n}{k} t_n \partial_{t_{n+pk}} + \frac{n}{m} \bar{t}_n \partial_{\bar{t}_{n+pm}} - n t_{nk} \partial_{t_{k(n+p)}} \right) \\
&\quad + \frac{1}{2k} \sum_{\alpha=1}^{pk-1} \partial_{t_\alpha} \partial_{t_{pk-\alpha}} + \frac{1}{2m} \sum_{\beta=1}^{pm-1} \partial_{\bar{t}_\beta} \partial_{\bar{t}_{pm-\beta}}.
\end{aligned} \tag{4.23}
$$

Combining a constructive proof for $\mathcal{L}_2$, verification that these $\mathcal{L}_p$ operators commute as Virasoro, and Proposition 3.7 would prove this theorem. However, as of the writing of this thesis, no such proof of $\mathcal{L}_2$ has been produced. The proof of this theorem that will be given in the future involves using the commutators of the operator $\mathcal{L}_{-1}$ and the Heisenberg algebra to show that the remaining $\mathcal{L}_p$ operators must be of the form given in (4.23). The last statement given in this thesis is these operators do in fact commute as the Virasoro algebra:

Proposition 4.3. *The $\mathcal{L}_p$ operators of (4.21), (4.22), and (4.23) commute with one another as the Virasoro algebra; i.e.*

$$
[\mathcal{L}_p, \mathcal{L}_q] = (p-q)\mathcal{L}_{p+q}, \quad p, q \in \mathbb{Z}_{\geq -1}.
$$

Since many of the calculations for this proposition are virtually identical to those given in the proof for Proposition 3.8, we will not include the rigorous proof of this proposition. Instead, only the particular calculations that are wholly different from those previously shown will be given below.

Note that the presence of the $n\,t_{nk}\partial_{t_{k(n+p)}}$ term in $\mathcal{L}_p$ $(p \in \mathbb{Z}_{\geq -1})$ ensures that there is only one such term in the operator. This effectively allows partial derivatives involving t_n to only affect the $\frac{n}{k}t_n\partial_{t_{n+pk}}$ term and those involving $\bar{t}_n$ to only affect the $\frac{n}{m}\bar{t}_n\partial_{\bar{t}_{n+pm}}$ term. Because of this observation and that the $\mathcal{L}_p$ operators do not mix the t_n and $\bar{t}_n$ variables, we will omit the $\bar{t}_n$ variables with these calculations.

We will determine the commutator of a double derivative term, such as those found in $\mathcal{L}_p$ for $p \in \mathbb{Z}_{\geq 1}$, and the new term found in $\mathcal{L}_{-1}$. For the terms from $\mathcal{L}_1$ and $\mathcal{L}_{-1}$, we have

$$\begin{aligned}\frac{1}{4k^2}\sum_{\alpha,\beta=1}^{k-1}[\partial_{t_\alpha}\partial_{t_{k-\alpha}}, \beta\, t_\beta(k-\beta)t_{k-\beta}] &= \frac{1}{2k^2}\sum_{\alpha=1}^{k-1}\alpha(k-\alpha)t_\alpha\partial_{t_\alpha} + \frac{1}{2k^2}\sum_{\alpha=1}^{k-1}\alpha(k-\alpha)\\ &= \frac{1}{2k^2}\sum_{\alpha=1}^{k-1}\alpha(k-\alpha)t_\alpha\partial_{t_\alpha} + \frac{k^2-1}{12k}.\end{aligned}$$

Recalling that $[\mathcal{L}_1, \mathcal{L}_{-1}] = 2\mathcal{L}_0$, the last term provides the constant terms added to $\mathcal{L}_0$ in (4.22). For the terms from $\mathcal{L}_p$ $(p \geq 2)$ and $\mathcal{L}_{-1}$, we have

$$\begin{aligned}\frac{1}{4k^2}\sum_{\alpha=1}^{pk-1}\sum_{\beta=1}^{k-1}[\partial_{t_\alpha}\partial_{t_{pk-\alpha}}, \beta\, t_\beta(k-\beta)t_{k-\beta}] &= \frac{1}{k^2}\sum_{\alpha=1}^{k-1}\alpha(k-\alpha)t_{k-\alpha}\partial_{t_{pk-\alpha}}\\ &= \frac{1}{k^2}\sum_{\alpha=1}^{k-1}\alpha(k-\alpha)t_\alpha\partial_{t_{(p-1)k+\alpha}}.\end{aligned}$$

The summations in each of these results combine with other terms in their respective commutators to give the $\frac{n}{k}t_n\partial_{t_{n+pk}}$ terms for $1 \leq n \leq k-1$ after taking into account that $[\mathcal{L}_p, \mathcal{L}_{-1}] = (p+1)\mathcal{L}_{p-1}$.

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