Abstract

IVY, SAMUEL JAMAL. Classifying the Fine Structures of Involutions Acting on Root Systems. (Under the direction of Aloysius Helminck.)

Symbolic Computation is a growing and exciting intersection of Mathematics and Computer Science that provides a vehicle for illustrations and algorithms for otherwise difficult to describe mathematical objects. In particular, symbolic computation lends an invaluable hand to the area of symmetric spaces. Symmetric spaces, as the name suggests, offers the study of symmetries. Indeed, it can be realized as spaces acted upon by a group of symmetries or motions (a Lie Group). Though the presence of symmetric spaces reaches to several other areas of Mathematics and Physics, the point of interest reside in the realm of Lie Theory. More specifically, much can be determine and described about the Lie algebra/group from the root system. This dissertation focuses on the algebraic and combinatorial structures of symmetric spaces including the action of involutions on the underline root systems.

The characterization of the orbits of parabolic subgroups acting on these symmetric spaces involves the action of both the symmetric space involution θ on the maximal k-split tori and their root system and its opposite $-\theta$. While the action of θ is often known, the action of $-\theta$ is not well understood. This thesis focuses on building results and algorithms that enable one to derive the root system structure related to the action of $-\theta$ from the root system structure related to θ. This work involves algebraic group theory, combinatorics, and symbolic computation.

Classifying the Fine Structures of Involutions Acting on Root Systems

by
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Biography

The author was born and raised in Memphis, TN while spending some of his developmental years within his parent's home town of Holly Springs, MS. He graduate among the top of his class from Central High School in May of 2005 before matriculating in Morehouse College in Atlanta, GA. There, he developed a passion for Mathematics and great mentorships that helped to set his trajectory to graduate school and his doctoral efforts. While graduating Magma Cum Laude, he received his Bachelors of Science in Mathematics with a minor in Economics. The author was accepted into the Ph.D program at North Carolina State University and sought to begin a new chapter in his life. While at NC State, he received his Master's degree and continued in passions in Mathematics and the recruitment of underrepresented groups in STEM. Further, he gain much experience in teaching and realized his desire to be in the classroom and continue to introduce students to the vastness of Mathematics. Samuel received his Ph.D at NC State under the supervision of Dr. Aloysius Helminck.

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On a more personal note, I would first like to thank my wife, Jasmine Ivy for her continual love and support throughout this journey. Her faith and confidence in me help to fuel my ambition and drive to complete this program. Many thanks to my family and friends for their continued support, love and prayers, and a special thanks to my parents and siblings for their encouragement.

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INTRODUCTION

1.1 Motivation

Symmetric spaces describe a variety of symmetries in nature and are significant in many fields of science, with a larger focus and influence in mathematics and physics. In general, one can think of symmetric spaces as nice spaces acted on by a group, where the group is a set of symmetries or motions (a Lie group). In Mathematics, these symmetric spaces arise in areas such as differential geometry, representation theory, number theory, algebraic geometry, invariant theory, and singularity theory. In physics, symmetric spaces play an intricate role in integrable quantum field theory, elementary particle physics, and many other areas.

Due to their usefulness in these areas, many scientists utilize a symbolic computation program to help in tedious calculations of examples and conjecture verification for specific cases. Because much of the structure of symmetric spaces is very combinatorial in nature, it is well suited for symbolic computation.

A large portion of the fundamental structure of generalized symmetric spaces is encoded on the root system, which is a finite set of vectors that characterizes a Lie algebra/group (an infinite set). By lifting from this finite set to the infinite set, we can perform many of the calculations on the infinite set (in this case a Lie algebra, Lie group or symmetric space), and turn many otherwise stringent problems into combinatorially manageable ones.

For Lie algebras, the combinatorial structure on the root system level has been implemented in several symbolic manipulation packages, including the package LiE written by CAN as well as the Maple packages Coxeter and Weyl by J. Stembridge. There are also algorithms for Lie theory in GAP4 and Magma. Due to my interests in Maple, the algorithms introduced in my work build upon Stembridges Coxeter and Weyl packages.

Symmetric spaces are defined by an involution σ of the group G. In the study of these symmetric spaces, it is important how this involution as well as $-\theta$ acts on the maximal $\mathbb{R}$-split tori and their root systems. While much is known about the action of θ on the root systems, not much is known about the related action of $-\theta$.

1.2 The Problem

Let G be a connected reductive linear algebraic group over 4 field $\Bbbk$ of characteristic not 2. Let $\sigma \in \mathrm{Aut}(G)$ be an invlutional $\Bbbk$-automorphism of G and $H = G^\sigma = \{g \in G \mid \sigma(g) = g\}$ the set of fixed points of σ. Denote the set of $\Bbbk$raitional points of G by $G_\Bbbk$. We define $G_\Bbbk / H_\Bbbk$ to be the generalized symmetric space ($\Bbbk$-varieties).

To study the representations of symmetric $\Bbbk$-varieties, one must consider the orbits of these varieties. In particular, we study the orbits of a minimal parabolic $\Bbbk$-subgroup P acting on the symmetric $\Bbbk$-variety $G_\Bbbk / H_\Bbbk$. In [22], a characterization of this orbital action is described.

Theorem 1.2.1 ([22])**.** *Let $\{T_i \mid i \in I\}$ be the representatives of $H_\Bbbk$-conjugacy classes of σ-stable maximal $\Bbbk$-split tori in G. Then*

$$P_\Bbbk \backslash G_\Bbbk / H_\Bbbk \cong \bigcup_{i \in I} W_{G_\Bbbk}(T_i) \backslash W_{H_\Bbbk}(T_i).$$

As the theorem suggests, it suffices to determine the conjugacy classes of σ-stable maximal $\Bbbk$-split tori and understand their associated Weyl groups. However, this task is difficult calculate in general. Hence the need for algorithms and computer algebra packages that will perform calculations within symmetric $\Bbbk$-varieties.

Nonetheless, there exists a series of correspondences that causes this characterization to be a bit more manageable. In fact, the determination of these conjugacy classes reduces to the determination of parabolic subsets $\Delta(-w)$ where $w \in W(\Phi)$ is an involution. The indexing set I, posed in Theorem 2.3.3 can be realized a subposet of diagrams that represent an ordering of Weyl group conjugacy classes of involutions in $W(\Phi)$. This paper provides a foundation in creating an algorithm to determine a "top" and "bottom" of a sub-poset and the possible conjugacy classes that lie in between. To characterize the top and bottom elements of poset, it suffices to determine maximal involutions in $\Phi(\mathfrak{a}) \cap \Phi(\mathfrak{a}_\sigma^-)$ and the same for the opposite involution $-\sigma$.This will involve the study of quasi $\mathbb{R}$-split tori and their conjugacy classes. The information of quasi $\mathbb{R}$-split tori and their conjugacy classes is embedding within (θ, σ)-diagrams. In total, there are 171 isomorphy classes of (θ, σ). In [21], these 171 cases (in which Theorem 3.1.1 applies), have been listed. In particular, these cases address the conjugacy classes of quasi $\mathbb{R}$-split tori in $\mathfrak{h}$, while providing the types and maximal involutions of $\overline{\Phi}_{(\theta,\sigma)} \cap \overline{\Phi}_\theta$.

In [21], a characterization of these conjugacy classes have been provided. To add to that characterization, this exposition provides a characterization of the action of opposing involutions on root systems, and in particular, address the cases that allow for diagram automorphism to exists within the $+1$ and -1 eigenspaces relative to the involution.

1.3 Summary of Results

The following result addresses the question of when a diagram automorphism will exists based on the involution and parabolic subset. The root systems of interest are those of type A_ℓ, $D_{2\ell+1}$ and E_6.

Theorem 1.3.1. *Let Φ be of type A_ℓ, $\theta \in Aut(\Phi)$ an involution and Δ be a $(-\theta)$-basis. Then the following are equivalent.*

(i) $\Delta_0(\theta)$ *is of type* $r \cdot A_1$.

(ii) $\Delta_0(-\theta)$ *is of type* $A_{\ell-2r}$.

(iii) $\theta^* = id$

(iv) $(-\theta)^* \neq id$

Proof. The result follows from Table **??** and Propositions 5.1.1 and 5.1.2. □

Proposition 1.3.2. *Let Φ be of type $D_{2\ell+1}$ and $\theta \in \mathrm{Aut}(\Phi)$ be an involution. Suppose $\Delta_0(\theta)$ is of type $r \cdot A_1$ and $\Delta_0(\theta) \subset D_{2\ell+1} - D_2$. Then $\theta^* = id$ if and only if $(-\theta)^* \neq id$.*

Proof. If $\theta^* = id$, then $\theta = -w_0(\theta) = -s_1 s_3 \cdots s_{2\ell-1}$. Since $\Delta_0(-\theta)$ is also of type $r \cdot A_1$, $w_0(-\theta)$ is also the product strongly orthogonal reflections. Thus, $(-\theta)^* = \theta w_0(-\theta) = -w_0(\theta) w_0(-\theta) \neq id$. The argument for the converse is the same. This proves the statement. □

Proposition 1.3.3. *Let Φ be of type $D_{2\ell+1}$ and $\theta \in \mathrm{Aut}(\Phi)$ be an involution. Suppose $\Delta_0(\theta)$ is of type D_n $(n < 2\ell+1)$. Then $\theta^* = id$ if and only if $(-\theta)^* \neq id$.*

Proof. From Theorem 4.3.5, $(-\theta)^* = id$ if and only if $\Delta_0(-\theta)$ is of type D with even rank if and only if $\Delta_0(\theta)$ is of type D with odd rank if and only if $w_0(\theta)|\Delta_0(\theta) \neq -id$ if and only if $\theta^8 \neq id$. $\square$

Theorem 1.3.4. *Let Δ_1 and Δ_2 be θ- and $(-\theta)$-bases, respective. Let θ^* and $(-\theta)^*$ be defined as above. If Φ is of type A_ℓ, $D_{2\ell+1}$ and E_6, then $\theta^* = id$ if and only if $(-\theta)^* \neq id$.*

CHAPTER

2

PRELIMINARIES AND RECOLLECTIONS

2.1 Symmetric Spaces

Let G be a connected reductive algebraic group over a field $\mathbb{k}$ with characteristic not 2.

Definition 2.1.1. *Let $\sigma \in \mathrm{Aut}(G)$. Then σ is a $\mathbb{k}$-involution if $\sigma^2 = id$, $\sigma \neq id$ and is defined over $\mathbb{k}$.*

Definition 2.1.2. *Let G be a group and σ a $\mathbb{k}$-involution. Then we define the fixed point group of G as*

$$H = G^\sigma = \{g \in G \,|\, \sigma(g) = g\}$$

From this, we can define a generalized reductive symmetric space as G/H. Notice that if we let $\tau : G \longrightarrow G$ be a map defined as $\tau(g) = g\sigma(g)^{-1}$ where $Q = \tau(G) = \{g\sigma(g)^{-1} \,|\, g \in G\}$, then it follows that $Q \cong G/H$. Here, Q is called the generalized symmetric variety. If G is semisimple, then Q is called a semisimple symmetric space.

For a general field $\mathbb{k}$, we denote the set of $\mathbb{k}$-rational points of G (respectively H) as $G_\mathbb{k}$ (respectively $H_\mathbb{k}$). Thus, the symmetric $\mathbb{k}$-variety $G_\mathbb{k}/H_\mathbb{k}$ is isomorphic to $Q_\mathbb{k} = \{g\sigma(g)^{-1} \,|\, g \in$

$G_{\mathbb{k}}\}$. When $k = \mathbb{R}$, $G_{\mathbb{R}}/H_{\mathbb{R}}$ is called a real reductive symmetric space. Though we shall introduce results for a general field $\mathbb{k}$ with characteristic not 2, the focus is on the case when $\mathbb{k} = \mathbb{R}$.

Nonetheless, the study of symmetric spaces is that of both breath and depth. Associated with G over an algebraically closed field is a natural fine structure of a root system and its corresponding Weyl group in which both are dependent on a maximal torus of G. When we move to a field that is not algebraically closed, there arises a second root system and Weyl group that characterizes the $\mathbb{k}$-structure of group G. This additional root system stems from a maximal $\mathbb{k}$-split torus $A \subset G$ with an associated Weyl group along with the multiplicities of the roots. These two components help in playing a large role in the representation of symmetric spaces.

2.2 Tori Definitions

As we are considering real symmetric spaces, the understanding of tori is paramount. Hence, we provide the following definitions.

Definition 2.2.1. *A subgroup $T \subset G$ is called a torus if T is connected, abelian, and consists of semisimple elements. Moreover, T is a $\mathbb{k}$-torus if it is defined over $\mathbb{k}$.*

Definition 2.2.2. *Let $T \subset G$ be a torus and $\sigma \in \operatorname{Aut}(G)$ an involution. Then T is called σ-stable if $\sigma(T) = T$. Further, $T = T_\sigma^+ T_\sigma^-$ where T_σ^+ is the connected component of $(T \cap H)$ and T_σ^- is the connected component of $\{x \in T \mid \sigma(x) = x^{-1}\}$.*

Definition 2.2.3. *Let $T \subset G$ be a torus and $\sigma \in \operatorname{Aut}(G)$ an involution. Then*

1. *T is $\mathbb{k}$-split if all elements of T can be diagonalized over $\mathbb{k}$. On the other hand, T is called $\mathbb{k}$-anisotropic if T has no subtori that are $\mathbb{k}$-split.*
2. *T is called σ-split if $\sigma(t) = t^{-1}$ for all $t \in T$.*
3. *T is a G-quasi (or H-quasi) $\mathbb{k}$-split torus if T is G-conjugate (or H-conjugate) to a $\mathbb{k}$-split torus.*
4. *T is called σ-fixed if $\sigma(t) = t$ for all $t \in T$.*

Note that a torus is called $(\sigma, \mathbb{k})$-split if the torus is both σ-split and $\mathbb{k}$-split. Also for a torus T, T_σ^+ is called the σ-stable component and T_σ^- is the σ-split component. From this definition, we can consider the following decomposition of a torus.

Theorem 2.2.4. *Let $T \subset G$ be a $\mathbb{k}$-torus. Then $T = T_a T_s$ where $T_a \cap T_s$ is finite and T_a is the maximal anisotropic subtorus and T_s is the maximal $\mathbb{k}$-split subtorus.*

2.2.1 Tori in Lie Algebras

Unfortunately, calculations within linear reductive groups (or Lie groups) can be quite complicated. Luckily, we can consider the the Lie algebra that underlies that of a goup G. Here, studying the structure of our symmetric space is analogous to the structure within the Lie algebra, called the local symmetric space. Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G. By abuse of notation, we will use σ as an involution of $\mathfrak{g}$; i.e. $\sigma \in \mathrm{Aut}(\mathfrak{g})$ and $\sigma^2 = id$. This σ is induced by the σ that acts on G and defines the symmetric space. Hence the definitions of a torus T of G can be translated to the Lie algebra level.

Definition 2.2.5. *Let $\mathfrak{t} \subset \mathfrak{g}$ be a torus. Then $\mathfrak{t}$ is called σ-stable if $\sigma(\mathfrak{t}) = \mathfrak{t}$. Further, $\mathfrak{t} = \mathfrak{t}_\sigma^+ \oplus \mathfrak{t}_\sigma^-$ where $\mathfrak{t}_\sigma^+ = (\mathfrak{t} \cap \mathfrak{h})$ and $\mathfrak{t}_\sigma^- = \{x \in \mathfrak{t} \,|\, \sigma(x) = -x\}$.*

For consistency with our notation on the group level, we denote this toral decomposition in the following manner.

Definition 2.2.6. *Let $\mathfrak{g}$ be a Lie algebra. Then*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$$

where $\mathfrak{h} = \{x \in \mathfrak{g} \,|\, \sigma(x) = x\}$ and $\mathfrak{q} = \{x \in \mathfrak{g} \,|\, \sigma(x) = -x\}$. Here, $\mathfrak{q}$ is called the local symmetric space of $\mathfrak{g}$ relative to σ.

In other words, $\mathfrak{h} = \mathrm{Lie}(H)$ (the $+1$ eigenspace of σ) and $\mathfrak{q} = \mathrm{Lie}(Q)$ (the -1 eigenspace of σ). Relative to the Killing form, $\mathfrak{h}$ and $\mathfrak{q}$ are orthogonal spaces [7]. Note that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ while $\mathfrak{q}$ is not.

2.3 Orbits of Minimal Parabolic Subgroups Acting on Symmetric Spaces

To study the representations of symmetric $\Bbbk$-varieties, one must consider the orbits of these varieties. In particular, we study the orbits of a minimal parabolic $\Bbbk$-subgroup P acting on the symmetric $\Bbbk$-variety $G_\Bbbk / H_\Bbbk$.

Definition 2.3.1. *A subgroup $B \subset G$ is called a Borel subgroup if B is connected and a maximal solvable subgroup.*

Definition 2.3.2. *A parabolic $\Bbbk$-subgroup $P_\Bbbk$ is a connected subgroup that contains a Borel subgroup and is defined over $\Bbbk$.*

The orbit of a minimal parabolic $\Bbbk$-subgroup $P_\Bbbk$ acting on the symmetric $\Bbbk$-variety $G_\Bbbk/H_\Bbbk$ is represented by the double coset $P_\Bbbk\backslash G_\Bbbk/H_\Bbbk$. Consequently, there are several ways of characterizing this double coset.

1. It can be characterized using the θ-twisted action: i.e. for $x, g \in G$, define $g * x := gx\theta(g)^{-1}$.
2. It can be viewed as the $H_\Bbbk$-orbtis acting on the flag variety $G_\Bbbk/P_\Bbbk$ by conjugation.
3. It can be viewed as the $P_\Bbbk \times H_\Bbbk$-orbits on $G_\Bbbk$.
4. It can be viewed as the set $P_\Bbbk\backslash G_\Bbbk/H_\Bbbk$ of $(P_\Bbbk, H_\Bbbk)$-double cosets in $G_\Bbbk$.

All these characterizations are quite similar. Springer characterized these orbits when $\Bbbk$ is algebraically closed and the Borel and parabolic subgroups are the same, [6]. Brion and Helminck, characterized these orbits when $\Bbbk$ is algebraically closed and P is a general parabolic subgroup, [[5],[4]]. Matsuki [3] and Rossmann [2] characterized these orbits when $\Bbbk = \mathbb{R}$ and P is a minimal parabolic $\mathbb{R}$-subgroup. Helminck and Wang characterized these orbits over general local fields, [22]. The focus of this thesis is on another characterization of these orbit introduced by Helminck and Wang.

Theorem 2.3.3 ([22])**.** *Let $\{T_i \mid i \in I\}$ be the representatives of $H_\Bbbk$-conjugacy classes of σ-stable maximal $\Bbbk$-split tori in G. Then*

$$P_\Bbbk\backslash G_\Bbbk/H_\Bbbk \cong \bigcup_{i\in I} W_{G_\Bbbk}(T_i)\backslash W_{H_\Bbbk}(T_i).$$

As the theorem suggests, it suffices to determine the conjugacy classes of σ-stable maximal $\Bbbk$-split tori and understand their associated Weyl groups. To begin, we introduce the definition of a Weyl group for a torus T of G.

Definition 2.3.4. *Let $T \subset G$ be a torus and H the fixed point group of G. Then the Weyl group of T with respect to H is*

$$W_H(T) = N_H(T)/Z_H(T)$$

where $N_H(T) = \{x \in H \mid xTx^{-1} \subset T\}$ and $Z_H(T) = \{x \in H \mid xt = tx \text{ for all } t \in T\}$ are the normalizer and centralizer of T, respectively.

Remark 2.3.5. *If $T \subset G$ is a maximal torus, then $T = Z(T) = N(T)$.*

2.3.1 Weyl Group for Toral Lie Algebras

Let $\mathfrak{g} \subset \mathfrak{gl}(\ell, \mathbb{C})$ be a complex semismple Lie subalgebra and $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$. Note that $\mathfrak{u} = \mathrm{Lie}(K)$ where K is compact subgroup of $GL(\ell, \mathbb{C})$. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{u}$. We can associated the Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{t}$. So define the Weyl group of $\mathfrak{g}$ as follows:

$$W(\mathfrak{t}) = N(\mathfrak{t})/Z(\mathfrak{t})$$

where $Z(\mathfrak{t}) = \{A \in K \,|\, \mathrm{Ad}_A(H) = H \text{ for all } H \in \mathfrak{t}\}$ and $N(\mathfrak{t}) = \{A \in K \,|\, \mathrm{Ad}_A(H) \subset \mathfrak{t} \text{ for all } H \in \mathfrak{t}\}$.

For $w \in W(\mathfrak{t})$, let A be the corresponding equivalence class in $N(\mathfrak{t})$. Then the action of $W(\mathfrak{t})$ on $\mathfrak{t}$ is defined as follows. Let $H \in \mathfrak{t}$. Then $w \cdot H = \mathrm{Ad}_A(H)$. We can extend this action to the Cartan subalgebra $\mathfrak{h}$: i.e. $w \cdot H$ for $H \in \mathfrak{h}$.

It is typical to view the Weyl group $W(\mathfrak{t})$ as the group generated by simple reflections s_α for all simple roots α. Recall that the reflection is defined as follows.

$$s_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}.$$

Proposition 2.3.6. *For each root, there exists an element $w_\alpha \in W$ such that $w_\alpha \cdot \alpha = -\alpha$ and $w_\alpha \cdot H = H$ for all $H \in \mathfrak{h}$ with $(\alpha, H) = 0$.*

Proposition 2.3.7. *The Weyl group W is generated by elements w_α as α rangers over all roots.*

2.4 Commuting Pairs of Involutions and Semisimple Symmetric Spaces

Since $\mathbb{k} = \mathbb{R}$, we seek to investigate the $H_{\mathbb{R}}$-conjugacy classes of σ-stable maximal $\mathbb{R}$-split tori. However, the challenge arises when considering the $\mathbb{R}$-split condition. To remedy this, we must transition to commuting paris of involution over $\mathbb{C}$ guaranteeing that the splitting condition is satisfied.

Suppose G_0 is a real semisimple Lie group and $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$ is the Lie algebra associated to G_0. For an involution $\sigma \in \mathrm{Aut}(G_0)$, we have the fixed point group of σ denoted $(G_0)^\sigma$.

Definition 2.4.1. *Let $H \subset G_0$ be a closed subgroup with associated Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$ such that $(G_0^\sigma)^\circ \subset H \subset (G_0)^\sigma$. Then the pair (G_0, H) is called a semisimple symmetric pair and $(\mathfrak{g}_0, \mathfrak{h})$ is a semisimple locally symmetric pair.*

Hence, we obtain the affine (or semisimple) symmetric space G_0/H. Likewise if G_0 is reductive, then we have the reductive symmetric pair (G_0, H) and reductive locally symmetric pair $(\mathfrak{g}_0, \mathfrak{h})$.

Definition 2.4.2. *Let $(\mathfrak{g}_0, \mathfrak{h}_1)$ and $(\mathfrak{g}_0, \mathfrak{h}_2)$ be semisimple locally symmetric pairs. Then they are isomorphic if there exists $\varphi \in \mathrm{Aut}(\mathfrak{g}_0)$ such that $\varphi(\mathfrak{g}_0) = \mathfrak{g}_0$ and $\varphi(\mathfrak{h}_1) = \mathfrak{h}_2$.*

Definition 2.4.3. *Let $\theta \in \mathrm{Aut}(\mathfrak{g}_0)$ be an involution and $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}$ where $=\{x \in \mathfrak{g}_0 \,|, \theta(x) = x\}$ and $\mathfrak{p} = \{x \in \mathfrak{g}_0 \,|\, \theta(x) = -x\}$. If $\mathfrak{t}_0$ is a maximal compact subalgebra of $\mathfrak{g}_0$, then θ is called a Cartan involution.*

Note that by compact we mean that the Killing form is negative definite. Also, we will refer to $\mathfrak{k}$ and $\mathfrak{p}$ as the $+1$ and -1 eigenspaces of $\mathfrak{g}_0$ with respect to θ. With the following result, we get that Cartan involutions for real semisimple Lie algebra are unique up to inner automorphism. On the group G_0, a Cartan involution has a maximal $\mathbb{k}$-anisotropic (compact) fixed point group. Hence, θ-split implies $\mathbb{R}$-split.

Proposition 2.4.4. *Let θ_1, $\theta_2 \in \mathrm{Aut}(\mathfrak{g}_0)$ be Cartan involutions. Then there exists $\varphi \in$ Int$(\mathfrak{g}_0)$ such that $\varphi\theta_1\varphi^{-1} = \theta_2$.*

With this, we can determine the involution that commutes with our fixed involution σ.

Proposition 2.4.5. *[1] Let $\mathfrak{g}_0$ be a real semisimple Lie algebra with a given involution σ. Let θ be a Cartan involution. Then there exists $\varphi \in$ Int$(\mathfrak{g}_0)$ such that $\varphi\theta\varphi^{-1}$ commutes with σ.*

From Proposition 2.4.4, it follows that $\varphi\theta\varphi^{-1}$ is a Cartan involution. By abuse of notation, we denote this Cartan involution as θ. Hence, we can find a Cartan involution θ such that $\theta\sigma = \sigma\theta$.

Corollary 2.4.5.1. *Let $(\mathfrak{g}_0, \mathfrak{h})$ be a semisimple locally symmetric pari with the associated involution σ. Then there exists a Cartan involution θ of $\mathfrak{g}_0$ such that $\theta\sigma = \sigma\theta$.*

We also see that a semisimple locally symmetric pair determines the pair of commuting involutions. Now we will show that this pair of commuting involutions are involutions of the complexification of $\mathfrak{g}_0$.

Definition 2.4.6. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and $\mathfrak{g}^{\mathbb{R}}$ be the real Lie algebra. A real form of $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}^{\mathbb{R}}$ such that each $z \in \mathfrak{g}$ can be uniquely written as $z = x + iy$ with $x, y \in \mathfrak{g}_0$. This $\mathfrak{g}$ would be the complexification to $\mathfrak{g}_0$. A compact real form $\mathfrak{u}$ is a real form of $\mathfrak{g}$ having a negative definite Killing form.*

Definition 2.4.7. *Let $\mathfrak{g}_0$ be the real form of $\mathfrak{g}$. Define the mapping $\tau : \mathfrak{g} \longrightarrow \mathfrak{g}$ as $\tau(x+iy) = x - iy$ for $x, y \in \mathfrak{g}_0$. Then τ is called the conjugation map with respect to $\mathfrak{g}_0$ having the following properties.*

(i) $\tau(\tau(x)) = x$

(ii) $\tau(x+y) = x+y$

(iii) $\tau(cx) = \bar{c}x$ *for some scalar c.*

(iv) $\tau([x,y]) = [\tau(x), \tau(y)]$

Proposition 2.4.8 ([22])**.** *Let θ be a Cartan $\mathbb{k}$-involution of $\mathfrak{g}$ and $\mathfrak{m} \subset \mathfrak{g}$ a θ-stable Lie subalgebra. Then $\theta|\mathfrak{m}$ is a Cartan involution of $\mathfrak{m}$ over k.*

Theorem 2.4.9 ([24])**.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\theta_i \in \mathrm{Aut}(\mathfrak{g})$ be commuting pairs of involutions for $i = 1, \ldots, n$. Then there exists a compact real form $\mathfrak{u}$ of $\mathfrak{g}$, with the conjugation map τ, such that $\theta_i\tau = \tau\theta_i$ for $i = 1, \ldots, n$.*

Corollary 2.4.9.1. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\theta \in \mathrm{Aut}(\mathfrak{g})$ be an involution. Then there exists a unique θ-stable compact real form of $\mathfrak{g}$.*

Proof. Let τ be the conjugation mapping of the compact real form $\mathfrak{u}$. It follows from Theorem 2.4.9, that $\theta\tau = \tau\theta$. As these are linear maps, they share the same eigenspace. Thus, we get $\theta(\mathfrak{u}) = \mathfrak{u}$. Set $\rho = \theta\tau$. Then ρ is also a conjugation mapping of $\mathfrak{u}$. Let $\mathfrak{g}_\rho = \{x \in \mathfrak{g} \mid \rho(x) = x\}$. From Proposition 2.4.8, $\theta|\mathfrak{g}_\rho$ is a Cartan involution. Therefore, $\mathfrak{g}_\rho = \mathfrak{u} \cap \mathfrak{g}_\rho^+ \oplus i\mathfrak{u} \cap \mathfrak{g}_\rho^-$. □

Note that the linear algebra result used in the previous proof is actually a statement of equivalence; i.e. $\theta(\mathfrak{u}) = \mathfrak{u}$ if and only if $\theta\tau = \tau\theta$.

For commuting pairs of involutions, order matters. In particular, the order helps to signify the involution defining the symmetric space with the other being the Cartan involution for the real form of $\mathfrak{g}$. Hence we denote the ordered pair as (θ, σ) where the first involution determines the real form (the Cartan involution).

Let $\mathfrak{u}$ be a (θ, σ)-stable compact real form of $\mathfrak{g}$ with conjugation map τ. Set $\overline{\theta} = \theta\tau$ and $\overline{\sigma} = \sigma\tau$. Then it follows that $(\mathfrak{g}_{\overline{\theta}}, \sigma|\mathfrak{g}_{\overline{\sigma}})$ is a semisimple locally symmetric space corresponding to (θ, σ). Moreover, [14] yields that the isomorphy class of $(\mathfrak{g}_{\overline{\theta}}, \sigma|\mathfrak{g}_{\overline{\sigma}})$ does not depend on a choice of $\mathfrak{u}$.

Remark 2.4.10. *Pairs of commuting involutions of $\mathfrak{g}$ correspond bijectively to pairs of commuting involutions in $\mathrm{Aut}(\mathfrak{g})$.*

The isomorphy of these pairs correspond directly to affine symmetric spaces that are associated with the semisimple locally symmetric pair.

Theorem 2.4.11 ([24]). *The inner (resp. outer) isomorphism classes of the semisimple locally symmetric pairs* $(\mathfrak{g}_0, \mathfrak{h})$ *correspond bijectively to the inner (resp. outer) isomorphsm classes of ordered pairs of commuting involutions* (θ, σ) *of* $\mathfrak{g}$ *or* $\mathrm{Aut}(\mathfrak{g})^\circ$.

Many have studied the structures associated with these semisimple locally symmetric pairs such as Berget, Cartan, Helminck, Oshima and Sekiguchi. See [21] for further discussion and notation.

2.5 More on Cartan Involutions

We also get the following results for Cartan involution θ on the group level. Once again, we abuse notation for θ.

Lemma 2.5.1. *If* T *is a maximal* θ*-split* $\mathbb{k}$*-torus of* G*, then* T *is a maximal* θ*-split torus of* G*.*

Lemma 2.5.2. *Let* $\varphi : G \longrightarrow G$ *be the map defined as* $\varphi(g) = g^{-1}\theta(x)$ *for all* $g \in G$*. Assume there is a given involution* θ *and that* $\tau(G_{\mathbb{k}})$ *consists of* $\mathbb{k}$*-split semisimple elements. Then any maximal* θ*-split* $\mathbb{k}$*-torus of* G *is maximal* $(\theta, \mathbb{k})$*-split.*

It follows that σ-stable maximal $\mathbb{R}$-split tori of $\mathfrak{g}$ can be viewed as a (σ, θ)-stable maximal $\mathbb{R}$-split tori (or θ-split tori) of $\mathfrak{g}$ [22].

When discussing the action of the Cartan involution θ on a reductive group G (or equivalently $\mathrm{Lie}(G)$ and the corresponding root system), it is important to address notation. We denote the general action of θ as $X_a^b(Type, \epsilon_i)$ where θ acts on a root system of type $X = A, B, \ldots, G$ with $rank(X) = a$. Moreover, b is the dimension of the (-1)-eigenspace of θ while ϵ_i represents the quadratic element associated when classifying pairs on involutions. We shall elaborate on the topic of quadratic elements later in the chapter. The following table shows the type of involutions, their diagram representations, restricted roots and the type of restricted root systems.

Table 2.1 Involution Diagrams

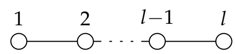
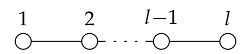
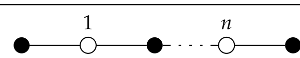
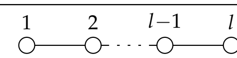
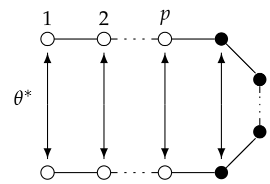
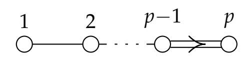
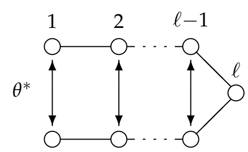
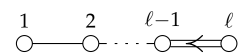
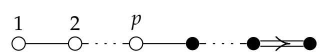
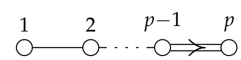
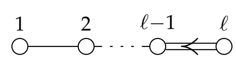
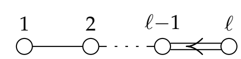
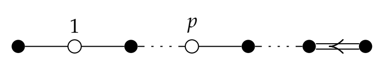
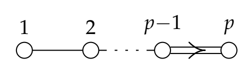
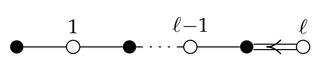
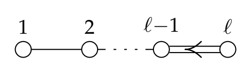
θ Type	θ-Diagram	Δ_θ	$\overline{\Phi}_\theta$ Type
AI			A_l
AII			A_l
$AIII_a$ $(AIV\ (p=1))$ $(1 \le 2p \le l)$			BC_p
$AIII_b$ $(l \ge 2)$			C_l
BI $(BII\ (p=1))$ $(l \ge 2,\ 1 \le p \le l)$			BC_p
CI			C_l
CII_a $(l \ge 3)$ $(1 \le p \le \frac{1}{2}(l-1))$			BC_p
CII_b $(l \ge 2)$			C_l

Table 2.1 – Continued

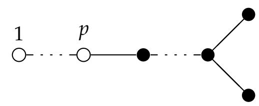
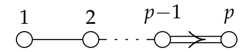
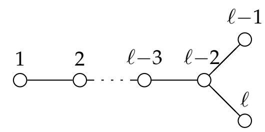
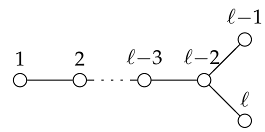
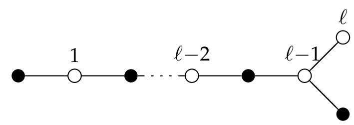
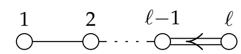
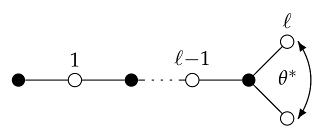
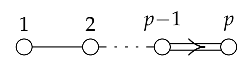
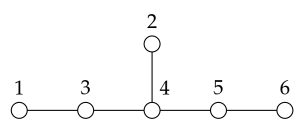
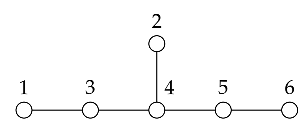
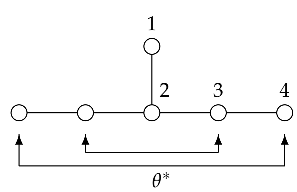
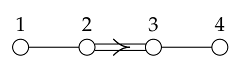
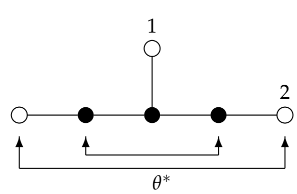
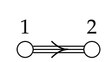
DI_a $(DII\,(p=1))$ $(l \geq 4,\ 1 \leq p \leq l-1)$			B_p
DI_b $(l \geq 4)$			D_l
$DIII_a$ $(l \geq 2)$			C_l
$DIII_b$ $(l \geq 2)$			B_l
EI			E_6
EII			F_4
$EIII$			B_2

Table 2.1 – Continued

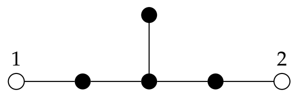
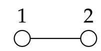
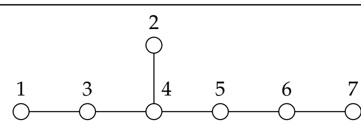
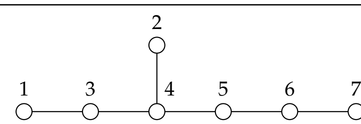
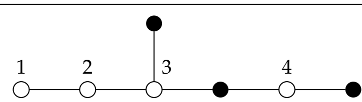
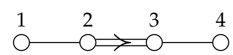
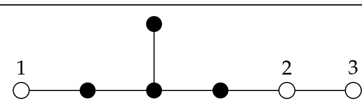
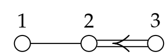
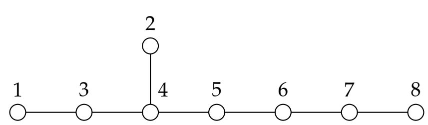
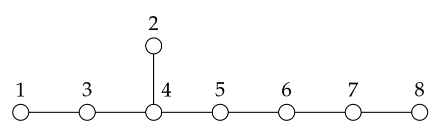
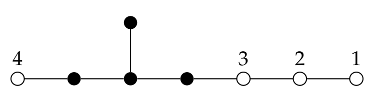
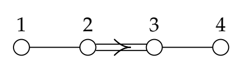
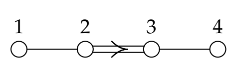
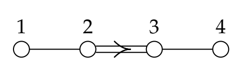
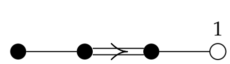
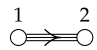
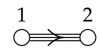
EIV			A_2
EV			E_7
EVI			F_4
$EVII$			C_3
$EVIII$			E_8
$EVIX$			F_4
FI			F_4
FII			A_1

Table 2.1 – Continued

G			G_2

This notation has been adopted from Helminck and corresponds with the notation of Oshima-Sekiguchi in the following manner. Note that within Table 2.3, we have the following relations.

Helminck	Oshima-Sekiguchi	Note
$A_\ell^p(III_a, \epsilon_i),\ i \neq \ell$	$BC_{p,i}^{2m,2,1},\ i \neq \ell$	$m = \ell + 1 - 2p$
$B_\ell^p(I_a, \epsilon_i)$	$B_{p,i}^{m,1}$	$m = 2\ell + 1 - 2p$
$C_\ell^p(II_a, \epsilon_i)$	$BC_{p,i}^{4m,4,3}$	$m = \ell - 2p$
$D_\ell^p(I_a, \epsilon_i)$	$B_{p,i}^{m,1}$	$m = 2\ell - 2p$

Table 2.3 Notation for Cartan Involutions

Cartan	Helminck	Oshima-Sekiguchi	$\mathfrak{g}$	$\mathfrak{h}$
AI	$A_\ell^\ell(I, \epsilon_i)$	$A_{\ell_i}^\ell$	$sl(\ell, \mathbb{R})$	$so(\ell+1-i, i)$
AII	$A_{2\ell+1}^\ell(II, \epsilon_i)$	$A_{\ell,i}^4$	$su^*(2\ell+2)$	$sp(\ell+1-i, i)$
$AIII$	$A_{2\ell-1}^\ell(III_b, \epsilon_i)\ i \neq \ell$	$C_{\ell,i}^{2,1},\ i =\neq \ell$	$su(\ell, \ell)$	$su(\ell+1-i, i) + su(\ell-i, i) + so(2)$
	$A_{2\ell-1}^\ell(III_b, \epsilon_i)$	$C_{\ell,A}^{2,1}$	$su(\ell, \ell)$	$sl(\ell, \mathbb{C}) + \mathbb{R}$
	$A_\ell^p(III_a, \epsilon_i)$	$BC_{p,i}^{2m,2,1}$	$su(\ell-p+1, p)$	$su(\ell-p+1-i) + su(p-i, i) + so(2)$
BI	$B_\ell^p(I_a, \epsilon_i)$	$B_{p,i}^{m,1}$	$so(2\ell+1-p, p)$	$so(2\ell+1-p-i) + so(p-i, i)$
CI	$C_\ell^\ell(I, \epsilon_i)\ i \neq \ell$	$C_{\ell,i}^{1,1},\ i \neq \ell$	$sp(\ell, \mathbb{R})$	$su(\ell-i, i) + so(2)$
	$C_\ell^\ell(I, \epsilon_\ell)$	$C_{\ell,A}^{1,1}\ i \neq \ell$	$sp(\ell, \mathbb{R})$	$sl(\ell, \mathbb{R}) + \mathbb{R}$
CII	$C_\ell^p(II_a, \epsilon_i)$	$BC_{p,i}^{4m,4,3}$	$sp(\ell-p, p)$	$sp(\ell-p-i, i) + sp(p-i, i)$
	$C_{2\ell}^\ell(II_b, \epsilon_i)\ i \neq \ell$	$C_{\ell,i}^{4,3}$	$sp(\ell, \ell)$	$sp(\ell-i, i) + sp(\ell-i, i)$
	$C_{2\ell}^\ell(II_b, \epsilon_\ell)$	$C_{\ell,A}^{4,3}$	$sp(\ell, \ell)$	$sp(\ell, \mathbb{C})$
DI	$D_\ell^p(I_a, \epsilon_i)$	$B_{p,i}^{m,1}$	$so(2\ell-p, p)$	$so(2\ell-p-i, i) + so(p-i, i)$
	$D_\ell^\ell(I_b, \epsilon_i)\ i \neq \ell$	$D_{\ell,i}^1\ i \neq \ell$	$so(\ell, \ell)$	$so(\ell-i, i) + so(\ell-i, i)$
	$D_\ell^\ell(I_b, \epsilon_\ell)$	$D_{\ell,i}^1$	$so(\ell, \ell)$	$su(\ell, \mathbb{C})$
$DIII$	$D_{2\ell}^\ell(III_a, \epsilon_i)\ i \neq \ell$	$C_{\ell,i}^{4,1}\ i \neq \ell$	$so^*(4\ell)$	$su(2\ell-2i, 2i) + so(2)$
	$D_{2\ell}^\ell(III_a, \epsilon_\ell)$	$C_{\ell,A}^{4,1}$	$so^*(4\ell)$	$su^*(2\ell) + \mathbb{R}$

Table 2.3 – Continued

	$D^{\ell}_{2\ell+1}(III_b,\epsilon_i)$	$BC^{4,4,1}_{\ell,i}$	$so^*(4\ell+2)$	$su^*(2\ell+1-2i,2i)+so(2)$
EI	$E^6_6(I,\epsilon_1)$	$E^1_{6,D}$	$e_{6(6)}$	$sp(2,2)$
	$E^6_6(I,\epsilon_2)$	$E^1_{6,A}$	$e_{6(6)}$	$sp(4,\mathbb{R})$
EII	$E^4_6(II,\epsilon_1)$	$F^{2,1}_{4,C}$	$e_{6(6)}$	$su(3,3)+sl(2,\mathbb{R})$
	$E^4_6(II,\epsilon_4)$	$F^{2,1}_{4,B}$	$e_{6(2)}$	$su(4,2)+su(2)$
$EIII$	$E^2_6(III,\epsilon_1)$	$BC^{8,6,1}_{2,A}$	$e_{6(-14)}$	$so^*(10)+so(2)$
	$E^2_6(III,\epsilon_2)$	$BC^{8,6,1}_{2,B}$	$e_{6(-14)}$	$so(8,2)+so(2)$
EIV	$E^2_6(IV,\epsilon_i)$	$A^8_{2,A}$	$e_{6(-26)}$	$f_{4(-20)}$
EV	$E^7_7(V,\epsilon_1)$	$E^1_{7,D}$	$e_{7(7)}$	$su(4,4)$
	$E^7_7(V,\epsilon_2)$	$E^1_{7,A}$	$e_{7(7)}$	$sl(8,\mathbb{R})$
	$E^7_7(V,\epsilon_7)$	$E^1_{7,E}$	$e_{7(7)}$	$su^*(8)$
EVI	$E^4_7(VI,\epsilon_1)$	$F^{4,1}_{4,C}$	$e_{7(-5)}$	$so^*(12)+sl(2,\mathbb{R})$
	$E^4_7(VI,\epsilon_4)$	$F^{4,1}_{4,B}$	$e_{7(-5)}$	$so(8,4)+su(2)$
$EVII$	$E^3_7(VII,\epsilon_1)$	$C^{8,1}_{3,A}$	$e_{7(-25)}$	$e_{6(-26)}+sl(2,\mathbb{R})$
	$E^3_7(VII,\epsilon_2)$	$C^{8,1}_{3,B}$	$e_{7(-25)}$	$e_{6(-14)}+so(2)$
$EVIII$	$E^8_8(VIII,\epsilon_1)$	$E^1_{8,D}$	$e_{8(8)}$	$so(8,8)$

Table 2.3 – Continued

	$E_8^8(VIII, \epsilon_8)$	$E_{8,E}^1$	$e_{8(8)}$	$so^*(16)$
$EVIX$	$E_8^4(IX, \epsilon_1)$	$F_{4,C}^{8,1}$	$e_{8(-24)}$	$e_{7(-25)} + sl(2,\mathbb{R})$
	$E_8^4(IX, \epsilon_2)$	$F_{4,B}^{8,1}$	$e_{8(-24)}$	$e_{7(-5)} + su(2)$
FI	$F_4^4(I, \epsilon_1)$	$F_{4,C}^1$	$f_{4(4)}$	$sp(3,\mathbb{R}) + sl(2,\mathbb{R})$
	$F_4^4(I, \epsilon_4)$	$F_{4,B}^1$	$f_{4(4)}$	$sp(2,1) + su(2)$
FII	$F_4^1(I, \epsilon_1)$	$BC_{1,A}^{8,7}$	$f_{4(-20)}$	$so(8,1)$
G	$G_2^2(\epsilon_i)$	G_2^1	$g_{2(2)}$	$sl(2,\mathbb{R}) + sl(2,\mathbb{R})$

CHAPTER

3

I-POSET CONSTRUCTION

To apply Theorem 2.3.3, we must classify the $H_{\mathbb{k}}$ conjugacy classes of σ-stable maximal $\mathbb{k}$-split tori. With an associated order, the indexing set I is realized as a poset representing these conjugacy classes of σ-stable maximal $\mathbb{k}$-split tori. See Figure 3.1.

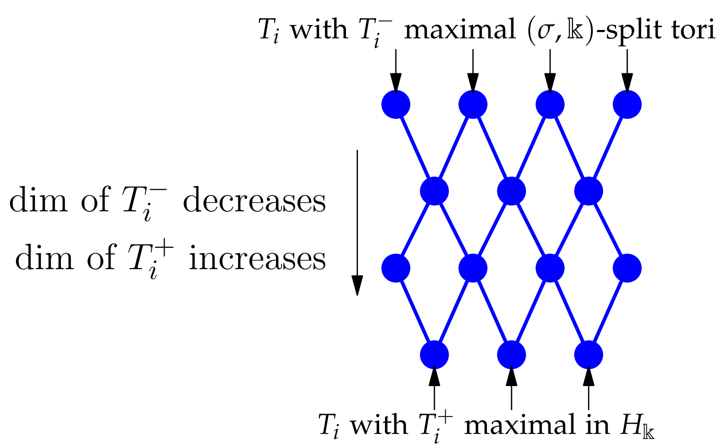


Figure 3.1 The poset of $H_{\mathbb{k}}$-conjugacy classes of θ-stable maximal $\mathbb{k}$-split tori

In Figure 3.1, each node represents the $H_{\Bbbk}$ conjugacy classes of σ-stable maximal $\Bbbk$-split tori. As each tori T_i can be written as $T_i = T_i^- T_i^+$, each row of the poset corresponds to the dimensions of T_i^+ and T_i^-. At each level, the tori representatives for that row share the same dimensions for their T_i^+ and T_i^- parts. We orient the poset so that the representative tori with a maximal T_i^- part is on top, while those with a maximal T_i^+ part are situated in the bottom level. Thus as we transition from top to bottom, the dimension of T_i^- decreases and that of T_i^+ increases.

This paper establishes a foundation in creating an algorithm to determine a sub-poset of this I-poset. In particular, we seek to determine a "top" and "bottom" of this sub-poset and the possible conjugacy classes that lie in between. But before we delve into that intent and algorithm, more foundation must be established.

3.1 $P_{\Bbbk}\backslash G_{\Bbbk}/H_{\Bbbk}$ Modfication

We continue the usual set up for σ-stable maximal $\Bbbk$-split tori. Recall $\overline{\theta} = \theta\sigma$ where θ is the Cartan involution. Then $\mathfrak{g}_{\overline{\theta}} = \mathfrak{k} \oplus \mathfrak{p}$ would be the Cartan decomposition into its $+1$ and -1 eigenspaces, respectively. Similarly, $\mathfrak{g}_{\overline{\theta}} = \mathfrak{h} \oplus \mathfrak{q}$ is the decomposition into $+1$ and -1 eigenspaces of $\sigma|\mathfrak{g}_{\overline{\theta}}$, respectively. Here, we obtain the correspondence between θ-split (resp. σ-split and (θ,σ)-split) tori of G and Cartan subspaces $\mathfrak{t}$ of $\mathfrak{p}$ (resp. $\mathfrak{q}$ and $\mathfrak{p} \cap \mathfrak{q}$). This provides a realization of semisimple locally symmetric pairs $(\mathfrak{g}_0, \sigma)$ as (θ,σ)-stable Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}_0$ such that $\mathfrak{t} \cap \mathfrak{p}$ (resp. $\mathfrak{t} \cap \mathfrak{p}$ and $\mathfrak{t} \cap \mathfrak{p} \cap \mathfrak{q}$) is maximal and abelian in $\mathfrak{p}$ (resp. $\mathfrak{q}$ and $\mathfrak{p} \cap \mathfrak{q}$).

It follows that pairs of commuting involutions of $\mathfrak{g}$ correspond bijectively with those of G. By abuse of notation, we continue to denote the ordered commuting pair as (θ,σ) for involutions and lifted involutions of the Lie algebra level.

Let $\mathfrak{a} \subset \mathfrak{g}$ be a σ-stable $\mathbb{R}$-split tori. We know there exists an involution θ such that $\theta\sigma = \sigma\theta$ thereby making $\mathfrak{a}$ θ-stable. From Corollary 2.4.9.1, there exists (θ,σ)-stable compact real form of $\mathfrak{g}$. Then the θ decomposition of $\mathfrak{a}$ would be $\mathfrak{a} = \mathfrak{a}_\theta^+ \oplus \mathfrak{a}_\theta^-$. Note that $\mathfrak{a}_\theta^+$ is contained in the compact real form of $\mathfrak{g}$. Yet $\mathfrak{a}$ being $\mathbb{R}$-split forces $\mathfrak{a} = \mathfrak{a}_\theta^-$. Hence σ-stable $\mathbb{R}$-split tori can be realized as (θ,σ)-stable maximal θ-split tori for commuting involutions (θ,σ).

This observation allows us to modify Theorem 2.3.3. Let $H = G^\sigma$ and $K = G^\theta$. Set $H^+ = (H \cap K)^\circ$. Then $H_{\Bbbk}^+$ is the set $\Bbbk$-rational points of H^+.

Theorem 3.1.1 ([22]). *Let K be the fixed point group of θ and H the $\Bbbk$-open subgroup of the fixed point group of σ. Let $H^+ = (H \cap K)^\circ$. Then*

$$P_{\Bbbk}\backslash G_{\Bbbk}/H_{\Bbbk}^+ \cong \bigcup_{i \in I} W_{G_{\Bbbk}}(A_i)\backslash W_{H_{\Bbbk}^+}(A_i)$$

where $\{A_i \mid i \in I\}$ are the representatives of $H_{\Bbbk}^+$-conjugacy classes of (θ,σ)-stable maximal $\Bbbk$-split tori of G.

Now it suffices to consider the semisimple locally symmetric space $(\mathfrak{g}_{\overline{\theta}}, \sigma|\mathfrak{g}_{\overline{\sigma}})$ corresponding to (θ, σ), and determine the the corresponding maximal (θ, σ)-stable θ-split tori.

Proposition 3.1.2 ([22]). *Let θ, $\sigma \in \mathrm{Aut}(G)$ be $\Bbbk$-involutions where θ is a Cartan involution and $\theta\sigma = \sigma\theta$. Let $H = G^{\theta}$. If two (θ, σ)-stable maximal $\Bbbk$-split tori are $H_{\Bbbk}$-conjugate, then they are also $H_{\Bbbk}^{+}$-conjugate.*

Since, we now want to consider (θ, σ)-stable maximal $\mathbb{R}$-split tori (or θ-split tori) of $\mathfrak{g}$, we can revisit the classification theorem introduced by Helminck and Wang.

3.2 Standard Tori & Involutions

Recall we have modified Theorem 2.3.3.

Theorem 3.2.1. *Let K be the fixed point group of θ and H the $\Bbbk$-open subgroup of the fixed point group of σ. Let $H^{+} = (H \cap K)^{\circ}$. Then*

$$P_{\Bbbk} \backslash G_{\Bbbk} / H_{\Bbbk} \cong P_{\Bbbk} \backslash G_{\Bbbk} / H_{\Bbbk}^{+} \cong \bigcup_{i \in I} W_{G_{\Bbbk}}(A_i) \backslash W_{H_{\Bbbk}^{+}}(A_i)$$

where $\{A_i \mid i \in I\}$ are the representatives of $H_{\Bbbk}^{+}$-conjugacy classes of (θ, σ)-stable maximal $\Bbbk$-split tori of G.

Our focus for this section will center on this classification within the Lie algebra. Let $\mathfrak{g} = Lie(G)$. We denote $\mathfrak{A}_{\Bbbk}^{(\theta,\sigma)}$ to be the set of all (θ, σ)-stable maximal $\Bbbk$-split tori. From Theorem 3.1.1, we now want the $\mathfrak{h}_{\Bbbk}^{+}$-conjugacy classes and will denote this set as $\mathfrak{A}_{\Bbbk}^{(\theta,\sigma)}/\mathfrak{h}_{\Bbbk}^{+}$. Before finding this , we determine the $\mathfrak{h}$-conjugacy classes of (θ, σ)-stable maximal quasi $\Bbbk$-split tori. We denote this set as $\mathfrak{A}^{(\theta,\sigma)}/\mathfrak{h}$.

Definition 3.2.2. *Let $\mathfrak{t} \subset \mathfrak{g}$ be a torus. Then $\mathfrak{t}$ is called a quasi $\Bbbk$-split torus if $\mathfrak{t}$ is $\mathfrak{g}$-conjugate with a $\Bbbk$-split torus of $\mathfrak{g}$.*

This leads to another set of interest for this characterization. Let $\mathfrak{A}_0^{\theta,\sigma}$ be the set of quasi $\Bbbk$-split tori that are $\mathfrak{h}$-conjugate with a $\Bbbk$-split torus. Let $\mathfrak{A}_0^{(\theta,\sigma)}/\mathfrak{h}$ denote the set of $\mathfrak{h}$-conjugacy classes of $\mathfrak{A}_0^{(\theta,\sigma)}$.

To determine the sets $\mathfrak{A}_{\Bbbk}^{(\theta,\sigma)}/\mathfrak{h}$, $\mathfrak{A}^{(\theta,\sigma)}/\mathfrak{h}$ and $\mathfrak{A}_{\Bbbk}^{(\theta,\sigma)}/\mathfrak{h}_{\Bbbk}^{+}$, we use the notion of standard tori and standard involutions. Consider the natural map $\eta : \mathfrak{A}_{\Bbbk}^{(\theta,\sigma)}/\mathfrak{h}_{\Bbbk}^{+} \to \mathfrak{A}_{\Bbbk}^{(\theta,\sigma)}/\mathfrak{h}$ that sends each $\mathfrak{h}_{\Bbbk}^{+}$-conjugacy classes of a σ-stable maximal $\Bbbk$-split torus to its $\mathfrak{h}$-conjugacy class. Note that the image of η is in fact $\mathfrak{A}_0^{(\theta,\sigma)}$. Hence the $\mathfrak{h}$-conjugacy classes of σ-stable maximal quasi $\Bbbk$-split tori are all $\mathfrak{h}$-conjugate to a σ-stable maximal k-split torus. Luckily, this map η is injective when $\Bbbk = \mathbb{R}$. So we are able to classify $\mathfrak{A}_{\Bbbk}^{(\theta,\sigma)}/\mathfrak{h}_{\Bbbk}^{+}$ by classifying the image and fibers of η. To help, we start the discussion on standard tori.

Definition 3.2.3. *Let $\mathfrak{a}_1$, $\mathfrak{a}_1 \in \mathfrak{A}_{\Bbbk}^{(\theta,\sigma)}$. The pair $(\mathfrak{a}_1, \mathfrak{a}_2)$ is called standard if $\mathfrak{a}_1^{-} \subset \mathfrak{a}_2^{-}$ and $\mathfrak{a}_1^{+} \supset \mathfrak{a}_2^{+}$. For this, we say that $\mathfrak{a}_1$ is standard with respect to $\mathfrak{a}_2$.*

Given a single involution, all elements of $\mathfrak{A}_{\mathbb{R}}^{(\theta,\sigma)}$ and $\mathfrak{A}^{(\theta,\sigma)}$ can be placed in standard position. So by fixing $\mathfrak{a} \in \mathfrak{A}_{\mathbb{k}}^{(\theta,\sigma)}$ with $\mathfrak{a}_\sigma^-$ maximal, we can put $\mathfrak{a}_1$ and $\mathfrak{a}_2$ is standard position with $\mathfrak{a}$. When there is no confusion, we will write $\mathfrak{a}_1^-$ instead of $(\mathfrak{a}_1)_\sigma^-$.

Proposition 3.2.4 ([21], Theorem 4.1.4). *Let $(\mathfrak{a}_1, \mathfrak{a}_2)$ be a standard pair of (θ,σ)-stable $\mathbb{R}$-split (or quasi $\mathbb{R}$-split) tori of $\mathfrak{g}$. Then we have the following conditions:*

1. *There exists $g \in Z_{\mathfrak{g}}(\mathfrak{a}_1^- \oplus \mathfrak{a}_2^+)$ such that $ad_g(\mathfrak{a}_1) = \mathfrak{a}_2$.*
2. *If $n_1 = ad_{g^{-1}\sigma(g)}$ and $n_2 = ad_{\sigma(g)g^{-1}}$, then $n_1 \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{a}_1)$ and $n_2 \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{a}_2)$.*
3. *Let w_1 and w_2 be the images of n_1 and n_2 in $W(\mathfrak{a}_1)$ and $W(\mathfrak{a}_2)$, respectively. Then $w_1^2 = w_2^2 = e$ and $(\mathfrak{a}_1)_{w_1}^+ = (\mathfrak{a}_2)_{w_2}^+ = \mathfrak{a}_1^- \oplus \mathfrak{a}_2^+$ which characterizes w_1 and w_2.*

Remark 3.2.5. *From part 3 of Proposition 3.2.4, the involutions w_1 and $_2$ are independent of the choice of the element $g \in Z_{\mathfrak{g}}(\mathfrak{a}_1^- \oplus \mathfrak{a}_2^+)$ with $ad_g(\mathfrak{a}_1) = \mathfrak{a}_2$.*

This leads to the following definition of standard involutions.

Definition 3.2.6. *Let w_1 and w_2 be defined as above. Then w_1 and w_2 are called the $\mathfrak{a}_i$-standard involutions of $W(\mathfrak{a}_i)$ for $i = 1, 2$.*

Let (θ, σ) be the ordered commuting pair of involutions of $\mathfrak{g}$ and $\mathfrak{h}$. We mention the results from various propositions in [22] and [15].

1. (Lemma 11.5) Any maximal θ-split $\mathbb{k}$-torus of $\mathfrak{g}$ is maximal $(\theta, \mathbb{k})$-split.
2. (Proposition 2.14) All maximal $(\sigma, \mathbb{k})$-split tori are $\mathfrak{h}_{\mathbb{k}}$-conjugate.
3. (Proposition 11.3 & 11.4) Any θ-stable maximal $\mathbb{k}$-split torus is θ-split.

Theorem 3.2.7. *There is only one $\mathfrak{h}_{\mathbb{R}}$-conjugacy class of (θ,σ)-stable maximal $(\sigma, \mathbb{R})$-split tori and one class of (θ,σ)-stable maximal $\mathbb{R}$-split, σ-fixed tori.*

From the previous theorem, it follows that top and bottom of the I-poset will only consist of one vertex.

Consequently by fixing $\mathfrak{a} \in \mathfrak{A}_{\mathbb{k}}^{(\theta,\sigma)}$ so that $\mathfrak{a}_\sigma^-$ is maximal, we can put any torus in $\mathfrak{A}_{\mathbb{R}}^{(\theta,\sigma)}$ in standard position with $\mathfrak{a}$. Since $\mathfrak{A}_{\mathbb{R}}^{(\theta,\sigma)} \subset \mathfrak{A}^{(\theta,\sigma)}$, this observation also holds for $\mathfrak{A}^{(\theta,\sigma)}$. That is, any torus in $\mathfrak{A}^{(\theta,\sigma)}$ can be placed in standard position.

Also for some fixed $\mathfrak{a} \in \mathfrak{A}_{\mathbb{k}}^{(\theta,\sigma)}$, we can find a torus that is $\mathfrak{h}_{\mathbb{k}}$-conjugate to $\mathfrak{a}$. Hence, we can modify Proposition 3.2.4

Corollary 3.2.7.1 ([21], Corollary 4.1.7). *Let $\mathfrak{a}_1$ be put in standard position with $\mathfrak{a}$ where $\mathfrak{a}^-$ is a maximal $(\sigma, \mathbb{R})$-split torus of $\mathfrak{g}$. Then we have the following conditions:*

1. *There exists $g \in Z_{\mathfrak{g}}(\mathfrak{a}_1^- \oplus \mathfrak{a}+)$ such that $ad_g(\mathfrak{a}_1) = \mathfrak{a}$.*

2. *If $n = ad_{g^{-1}\sigma(g)}$, then $n \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})$.*

3. *Let w be the image of n in $W(\mathfrak{a})$. Then $w^2 = e$ and $(\mathfrak{a})^+_w = \mathfrak{a}_1^- \oplus \mathfrak{a}^+$ which characterizes w.*

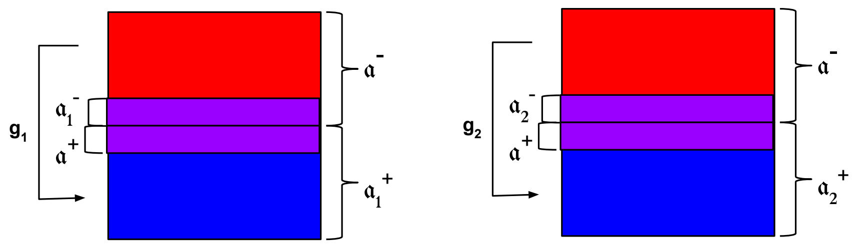


Figure 3.2 Action of g_1 and g_2 on the standard pair in Corollary

3.2.1 Weyl Group Connection for Standard Tori

From Corollary 3.2.7.1, we can associate a Weyl group element w with the torus $\mathfrak{a} \in \mathfrak{A}^{\theta,\sigma}_{\mathbb{k}}$, We call w an $\mathfrak{a}$-standard involution. Now the issue of characterizing $H^+_{\mathbb{k}}$-conjugacy classes translates to determining conjugacy of Weyl group elements which are $\mathfrak{a}$-standard involutions.

Proposition 3.2.8 ([21]). *Suppose $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{A}^{\theta,\sigma}_{\mathbb{R}}$ are both standard with respect to $\mathfrak{a}$. Let w_1 and w_2 be the $\mathfrak{a}_1$-standard and $\mathfrak{a}_2$-standard involutions, respectively, in $W(\mathfrak{a})$. Then $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are $\mathfrak{h}^+_{\mathbb{R}}$-conjugate if and only if w_1 and w_2 are conjugate under $W(\mathfrak{a}, \mathfrak{h}^+_{\mathbb{R}})$.*

Corollary 3.2.8.1 ([21]). *Suppose $\mathfrak{a}_1', \mathfrak{a}_2' \in \mathfrak{A}^{\theta,\sigma}$ are both standard with respect to $\mathfrak{a}$. Let w_1' and w_2' be the $\mathfrak{a}_1'$-standard and $\mathfrak{a}_2'$-standard involutions, respectively, in $W(\mathfrak{a})$. Then $\mathfrak{a}_1'$ and $\mathfrak{a}_2'$ are $\mathfrak{h}_{\mathbb{k}}$-conjugate if and only if w_1 and w_2 are conjugate under $W(\mathfrak{a}, \mathfrak{h})$.*

Essentially, these standard involutions define a one-dimensional piece of the (-1)-eigenspace of σ that can be flipped thereby adding a dimension to the $(+1)$-eigenspace of σ acting on the torus. In fact, we can determine the involution within the Weyl group that performs this flipping action of eigenspaces.

Theorem 3.2.9 ([26]). *Let $(\mathfrak{a}_1, \mathfrak{a}_2)$ be a standard pair of (θ, σ)-stable $\mathbb{R}$-split (or quasi $\mathbb{R}$-split) tori of $\mathfrak{g}$. Then there exists $g \in Z(\mathfrak{a}_1^- \oplus \mathfrak{a}_2^+)$ such that $w = \sigma(g)g^{-1} \in W(\mathfrak{a})$.*

Using the finite set of involutions in $W(\mathfrak{a}, \mathfrak{h}^+_{\mathbb{R}})$, we can determine which (θ, σ)-stable maximal quasi k-split tori are $\mathfrak{h}_{\mathbb{R}}$-conjugate. A natural next step would be to determine the involutions $w \in W(\mathfrak{a})$ that are $\mathfrak{a}_i$-standard involutions for some $\mathfrak{a}_i \in \mathfrak{A}^{(\theta,\sigma)}_{\mathbb{R}}$. Moreover, what would $\Phi(\mathfrak{a}, \mathfrak{h}^+_{\mathbb{R}})$ and the conjugacy classes of the involutions in $W(\mathfrak{a}, \mathfrak{h}^+_{\mathbb{R}})$ look like?

3.3 $\mathbb{R}$-Involutions Acting on Root Systems

Fortunately within the study of real symmetric spaces, much of its fundamental structure is encoded on root systems. With this switch in focus, we take the time to recall background information regarding root systems. This foundation is adopted from [23]. Let $\mathbb{E}$ be a finite dimensional vector space over $\mathbb{R}$. With this space, we associate a positive definite symmetric bilinear form $(\cdot, \cdot)$.

To determine maximal (θ, σ)-stable tori, we consider the root systems of the reductive groups and, consequently, their associated Lie algebra. We first introduce the notion of a root datum.

Definition 3.3.1. *A root datum is a quadruple $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ where X, $X^\vee$ are free abelian groups of finite ranks, in duality by the pairing $X \times X^\vee \to \mathbb{Z}$ denoted by the bilinear form $\langle \cdot, \cdot \rangle$, Φ and $\Phi^\vee$ are finite subsets of X and $X^\vee$ with a bijection $\alpha \to \alpha^\vee$ of Φ onto $\Phi^\vee$. If $\alpha \in \Phi$, we define endomorphisms s_α and $s_{\alpha^\vee}$ of X and $X^\vee$, respectively, by $s_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha$ and $s_{\alpha^\vee}(\gamma) = \chi - \langle \alpha, \gamma \rangle \alpha^\vee$ for all $\chi \in X$ and $\gamma \in X^\vee$. The following two axioms are imposed:*

(1) If $\alpha \in \Phi$, then $\langle \alpha, \alpha^\vee \rangle = 2$.

(2) $f\, \alpha \in \Phi$, then $s_\alpha(\Phi) \subset \Phi$, $s_{\alpha^\vee}(\Phi^\vee) \subset \Phi^\vee$.

Note that $X = X^*(T)$ is considered to be the additively written group of rational characters of the torus T and $X^\vee = X_*(T)$ is the group of rational one-parameter multiplicative subgroups of T; i.e. $X^\vee = \mathrm{Hom}(GL_1(\mathbb{k}), T)$.

When $T \subset G$ (resp. $\mathfrak{t} \subset \mathrm{Lie}(G)$) is a maximal torus or maximal θ-split torus, the root datum is guaranteed. We denote $\Phi(T)$ (rest. $\Phi(\mathfrak{t})$) to be the root system of T (resp. $\mathfrak{t}$) with the associated Weyl group $W(T)$ (resp. $W(\mathfrak{t})$). We continue the abuse of notation and call the involutions acting on the root system θ and σ.

Notation 3.3.2. *Let $\Phi = \Phi(T)$ where T is a maximal torus and $\Phi \subset X$. Let $\theta, \sigma \in Aut(\Phi)$ be involutions. Then we have the following definition.*

$$\begin{aligned}
X_0(\theta) &= \{\chi \in X \mid \chi - \theta(\chi) = 0\} \\
X_0(\theta, \sigma) &= \{\alpha \in X | \alpha - \sigma(\alpha) - \theta(\alpha) - \sigma\theta(\alpha) = 0\} \\
\Phi_0(\theta) &= \{\alpha \in \Phi | \alpha - \theta(\alpha) = 0\} \\
\Phi_0(\theta, \sigma) &= \{\alpha \in \Phi | \alpha - \sigma(\alpha) - \theta(\alpha) - \sigma\theta(\alpha) = 0\} \\
\overline{\Phi}_\theta &= \Phi / \Phi_0(\theta) = \Phi(\mathfrak{a}) \\
\overline{\Phi}_{\theta,\sigma} &= \Phi / \Phi_0(\theta, \sigma)
\end{aligned}$$

The restricted root system $\overline{\Phi}_\theta$ helps in determining the maximal θ-split tori, or equivalently, maximal $\mathbb{R}$-split tori as we discussed earlier. Moreover for $\overline{\Phi}$, there exists a natural projection map

$\pi_\theta : \Phi \to \overline{\Phi}_\theta$ defined as $\pi_\theta(\alpha) = \frac{1}{2}(\alpha - \theta(\alpha))$. Likewise, we define the natural projection map $\pi_{\theta,\sigma} : \Phi \to \overline{\Phi}_{\theta,\sigma}$ as $\pi_{\theta,\sigma}(\alpha) = \frac{1}{4}(\alpha - \sigma(\alpha) - \theta(\alpha) - \sigma\theta(\alpha))$. For a basis Δ, the basis for $\overline{\Phi}_{\theta,\sigma}$ is defined as $\overline{\Delta}_{\theta,\sigma} = \pi_{\theta,\sigma}(\Phi - \Phi_0(\theta,\sigma))$ and $\Delta_0(\theta,\sigma) = \Delta \cap \Phi_0(\theta,\sigma)$ for $\Phi_0(\theta,\sigma)$.

Lemma 3.3.3. *Let $\Phi_0(\theta)$, $\Phi_0(\theta,\sigma)$, $X_0(\theta)$ and $X_0(\theta,\sigma)$ be defined as above. Then we have the following.*

(i) *$X_0(\theta)$ and $\Phi_0(\theta)$ are θ-stable.*

(ii) *$\Phi_0(\theta,\sigma)$ and $X_0(\theta,\sigma)$ are (θ,σ)-stable.*

(iii) *$\Phi_0(\theta)$ and $\Phi_0(\theta,\sigma)$ are closed subsystems of Φ.*

3.3.1 Opposing Involutions

When discussing θ-stable maximal $\mathbb{k}$-split tori, we must consider both action of θ and opposite involution $-id \cdot \theta = -\theta$. We have defined the action of θ in Notation 3.3.2. Similary, we can define the action of $-\theta$. Hence for a θ-stable maximal quasi k-split torus $\mathfrak{a} \subset \mathfrak{g}$, let $X = X(\mathfrak{a})$ and $\Phi = \Phi(\mathfrak{a})$. Then we have the following.

$$\begin{aligned}
X_0(\theta) &= \{\chi \in X \mid \theta(\chi) = \chi\} \\
X_0(-\theta) &= \{\chi \in X \mid \theta(\chi) = -\chi\} \\
\Phi_0(\theta) &= \Phi \cap X_0(\theta) \\
\Phi_0(-\theta) &= \Phi \cap X_0(-\theta) \\
\overline{\Phi}_\theta &= \Phi/\Phi_0(\theta) = \pi_\theta(\Phi - \Phi_0(\theta)) \\
\overline{\Phi}_{-\theta} &= \Phi/\Phi_0(-\theta) = \pi_{-\theta}(\Phi - \Phi_0(-\theta))
\end{aligned}$$

We now denote the natural projection from Φ to $\overline{\Phi}_\theta$ as π_θ and $\pi_{-\theta}$ as the natural projection from Φ to $\overline{\Phi}_{-\theta}$. Further, we define an order dependent on both involutions.

Definition 3.3.4. *Let $\succ$ be the linear order on Φ. The order $\succ$ is called a θ^+-order if it has the property*

$$\text{if } \chi \in X,\ \chi \succ 0\ \&\ \chi \notin X_0(\theta),\ \text{then } \theta(\chi) \prec 0.$$

The order $\succ$ is called a θ^--order if it has the property

$$\text{if } \chi \in X,\ \chi \succ 0\ \&\ \chi \notin X_0(\theta),\ \text{then } \theta(\chi) \succ 0.$$

A θ^+-order on X will be called a θ-order on X. Likewise, a θ^--order on X is called a $(-\theta)$-order on X. Note that a θ^- order on X is a θ-order on X for $-\theta$ of (X,Φ).

Definition 3.3.5. *A basis $\Delta \subset \Phi$ with respect to a $(\pm\theta)$-order is called a $(\pm\theta)$-basis of Φ.*

Notation 3.3.6. *Let X, $X_0(\pm\theta)$, Φ, $\Phi_0(\pm\theta)$ and $\overline{\Phi}_{\pm\theta}$ and be defined as above. For a θ-basis Δ,*

- $\Delta_0(\theta) = \Delta \cap \Phi_0(\theta)$
- $\overline{\Delta}_\theta = \pi_\theta(\Delta - \Delta_0(\theta))$
- $\Delta_0(-\theta) = \Delta \cap \Phi_0(-\theta)$
- $\overline{\Delta}_{-\theta} = \pi_{-\theta}(\Delta - \Delta_0(-\theta))$

Hence the ordered bases of $\Phi_0(\pm\theta)$ and $\overline{\Phi}_{\pm\theta}$, and are $\Delta_0(\pm\theta)$ and $\overline{\Delta}_{\pm\theta}$, respectively. We extend this notation on θ to coincide with that of the commuting involution pair (θ, σ).

Definition 3.3.7. *A linear order $\succ$ on Φ is called a (θ, σ)-order if*

$$\alpha \in \Phi,\ \alpha \succ 0,\ \textit{and}\ \alpha \notin \Phi_0(\theta, \sigma),\ \textit{then}\ \sigma(\alpha) \succ 0\ \textit{and}\ \theta(\alpha) \succ 0.$$

Hence a (θ, σ)-order on X includes orders on $\Phi_0(\theta, \sigma)$ and $\overline{\Phi}_{\theta,\sigma}$.

Definition 3.3.8. *A basis $\Delta \subset \Phi$ with respect to a (θ, σ)-order is called a (θ, σ)-basis of Φ.*

3.4 Weyl Group

Recall for a basis $\Delta \subset \Phi$, the Weyl group of Φ is defined as $W(\Phi) = \langle s_\alpha\ \alpha \in \Delta \rangle$ where

$$s_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha = \beta - \langle \alpha, \beta \rangle \alpha.$$

We denote the Weyl gorup of $\Phi_0(\theta)$ as $W_0(\theta) = W(\Phi_0(\theta)) \subset W(\Phi)$. With the natural projection map $\pi_\theta : X \to X_0(\theta)$, it follows that $\overline{X} = X/X_0(\theta)$. Let $W_1(\theta) = \{w \in W(\Phi) \mid w(X_0(\theta)) = X_0(\theta)\}$. Every $w \in W_1(\theta)$ induces an automorphism, denoted $\pi_\theta(w)$, of $\overline{X}_\theta$ where $\pi_\theta(w\chi) = \pi_\theta(w)\pi_\theta(\chi)$ for some $\chi \in X$. Define $\overline{W}_\theta = \{\pi_\theta(w) \mid w \in W_1(\theta)\}$. Then $\overline{W} \cong W_1(\theta)/W_0(\theta)$ (see [19], 2.1.3). This is called the restricted Weyl group with respect to the action of θ on X. It is important to note that $\overline{W}_\theta$ may not be a Weyl group in the sense of Bourbaki [9].

Analogously, we can consider the approach in defining the restricted Weyl group for the ordered commuting pair (θ, σ).

Notation 3.4.1. *Let $X_0(\theta, \sigma)$, $\Phi_0(\theta, \sigma)$ and $\overline{\Phi}_{\theta,\sigma}$ be defined as in Notation 3.3.2. Then we have the following.*

$$\begin{aligned}
W_0(\theta, \sigma) &= W(\Phi_0(\theta, \sigma)) = \{w \in W(\Phi) \mid w(\Phi_0(\theta, \sigma)) = \Phi_0(\theta, \sigma)\} \\
W_1(\theta, \sigma) &= \{w \in W(\Phi) \mid w(X_0(\theta, \sigma)) = X_0(\theta, \sigma)\} \\
\overline{X}_{\theta,\sigma} &= X/X_0(\theta, \sigma) = \{\chi \in X \mid \sigma(\chi) = \theta(\chi) = -\chi\} \quad (\textit{ with the natural map } \pi_{\theta,\sigma}) \\
\overline{W}_{\theta,\sigma} &= \{\pi_{\theta,\sigma}(w) \mid w \in W_1(\theta, \sigma)\}
\end{aligned}$$

Though $\overline{W}_{\theta,\sigma}$ may not be a Weyl group in the sense of Bourbak, we do have the following result.

Proposition 3.4.2. *Let G be a reductive group and $T \subset G$ a maximal $\Bbbk$-split torus of G. Let X, $X_0(\theta,\sigma)$, Φ, $\Phi_0(\theta,\sigma)$, $\overline{\Phi}_{\theta,\sigma}$, Δ, $\overline{\Delta}_{\theta,\sigma}$, $W_0(\theta,\sigma)$, $W_1(\theta,\sigma)$ and $\overline{W}_{\theta,\sigma}$ be defined as above and $A = \{t \in T \mid \chi(t) = e$ for all $\chi \in X_0(\theta,\sigma)\}$. Then*

1. *If $w \in W(\theta,\sigma)$, then $w(\Delta)$ is a (θ,σ)-basis.*
2. *Given $w \in W_1(\theta,\sigma)$, $w \in W_0(\theta,\sigma)$ if and only if $\pi_{\theta,\sigma}(w) = 1$ if and only if $\pi_{\theta,\sigma}(w)\overline{\Delta}_{\theta,\sigma} = \overline{\Delta}_{\theta,\sigma}$*
3. *$\overline{W}_{\theta,\sigma} \cong W_1(\theta,\sigma)/W_0(\theta,\sigma)$*
4. *$W_1(\theta,\sigma)/W_0(\theta,\sigma) \cong N_G(A)/Z_G(A)$ where $N_G(A)$ and $Z_G(A)$ are the normalizer and centralizer of A in G, respectively.*

Remark 3.4.3. *When A is a maximal $\Bbbk$-split, θ-split or $(\theta,\Bbbk)$-split torus, then $\overline{\Phi}_{\theta,\sigma}$ is a root system with Weyl group $\overline{W}_{\theta,\sigma}$.*

3.4.1 Weyl Group Conjugate

Let $\mathbb{E}$ be the Euclidean space. For $w \in W(\mathfrak{a})$, let $E(w,\zeta) \subset \mathbb{E}$ be the eigenspace of w with eigenvalue ζ. In particular, we will focus on cases when $\zeta = \pm 1$; i.e., $E(w,\pm 1) = \{\alpha \in \mathbb{E} \mid w(\alpha) = \pm\alpha\}$.

Lemma 3.4.4. *Let $\theta \in \mathrm{Aut}(\Phi)$ such that $\theta^2 = id$. Then $E(\theta,+1)$ and $E(\theta,-1)$ are orthogonal spaces.*

Proof. Let $\alpha \in \mathbb{E}(\theta,+1)$ and $\beta \in \mathbb{E}(\theta,-1)$. Then $(\alpha,\beta) = (\theta(\alpha),\theta(\beta)) = (\alpha,-\beta) = -(\alpha,\beta)$ where $(\cdot,\cdot)$ is the $\mathrm{Aut}(\Phi)$-invariant symmetric bilinear form associated with Φ . Hence $(\alpha,\beta) = 0$. ☐

We can determine the conjugacy classes of involutions $w \in W(\mathfrak{a})$ with $E(w,-1) \subset E(\sigma-1)$ and $w\sigma = \sigma w$. This follow from the fact that $\mathfrak{a}_\sigma^+$ is a maximal $(\sigma,\mathbb{R})$-split torus and $\mathfrak{a}_w^- \subset \mathfrak{a}_\sigma^-$. Each such $w \in W(\mathfrak{a})$ yields a subset of a basis Δ_1 of Φ. Let $\Phi = \Phi(\mathfrak{a})$ and $W = W(\mathfrak{a}) = W(\Phi(\mathfrak{a}))$.

The conjugacy of Δ-standard involutions can be restricted to the conjugacy of their corresponding parabolic subsets.

Definition 3.4.5. *Let Δ be a basis of Φ.*

(a) Two subsets Δ_1, $\Delta_2 \subset \Delta$ are called Aut(Φ)-conjugate if there exists $\varphi \in \mathrm{Aut}(\Phi)$ such that $\varphi(\Delta_1) = \Delta_2$. If $\varphi \in W(\Phi)$, then Δ_1 and Δ_2 are $W(\Phi)$-conjugate.

(b) An involution $w \in W$ is called Δ-standard if Δ is a $(-w)$-basis of Φ (i.e. $E(w,-1) \cap \Delta = \Delta_0(-w)$).

(c) Two involutions θ_1, $\theta_2 \in \mathrm{Aut}(\Phi)$ are called Aut(Φ)-conjugate (resp. $W(\Phi)$-conjugate) if there exists $\varphi \in \mathrm{Aut}(\Phi)$ (resp. $w \in W(\Phi)$-conjugate) such that $\varphi\theta_1\varphi^{-1} = \theta_2$ (resp. $w\theta_1 w^{-1} = \theta_2$).

Proposition 3.4.6. *[26] Let $\Delta \subset \Phi$ be a basis and w_1, w_2 be Δ-standard involutions in W. Then w_1, w_2 are W-conjugate (resp. Aut(Φ)-conjugate) if and only if $\Delta(w_1)$, $\Delta(w_2)$ are W-conjugate (resp. Aut(Φ)-conjugate)*

Definition 3.4.7. *Let $w \in W$ be an involution. Then we have the following.*

(a) w is a maximal involution if $E(w,-1)$ is maximal; i.e. w is the longest involution of W with respect to the basis Δ.

(b) w is a σ-maximal involution of W if $E(w,-1) \subset E(\sigma,-1)$ and w is a maximal involution of $W(\sigma)$.

Proposition 3.4.8 (Proposition 2.11, [26])**.** *Let $w \in W$ be a maximal involution. Then we have the following.*

1. *There is a basis $\Delta \subset \Phi$ such that w is the opposition involution of W with respect to Δ.*
2. *All involutions $w \in W$ with $E(w,-1)$ maximal are conjugate.*

Definition 3.4.9. *Let Δ be a $(-\theta)$-basis of Φ and w_0 a fixed θ-maximal involution of $W(\theta)$ which is Δ-standard. An involution $w \in W$ is called (Δ, w_0)-standard if w is Δ-standard and $\Delta(w) \subset \Delta(w_0)$.*

Corollary 3.4.9.1 ([26])**.** *Let $\theta \in \mathrm{Aut}(\Phi)$ be an involution, Δ a $(-\theta)$-basis of Φ, $w_0 \in W(\theta)$ a θ-maximal involution, which is Δ-standard. If $w_1, w_2 \in W$ are (Δ, w_0)-standard involutions, then w_1, w_2 are W-conjugate (resp. Aut(Φ)-conjugate) if and only if $\Delta(w_1)$ and $\Delta(w_2)$ are W-conjugate (resp. Aut(Φ)-conjugate).*

From this corollary, we find that the classification of the conjugacy classes of $w \in W(\mathfrak{a})$ with $E(w,-1) \subset E(\sigma,-1)$ actually reduces to studying parabolic subsets of $\Phi_0(-w_0)$ where w_0 is a σ-maximal involution in $W(\mathfrak{a})$. Helminck provides a characterization of parabolic subsets in the following proposition.

Lemma 3.4.10 ([26])**.** *Let Φ be irreducible and $w \in W$ an involution. Then $\Phi_0(-w)$ is of type $r \cdot A_1 + X_\ell$ where either $X_\ell = \emptyset$ or one of $B_\ell(\ell \geq 1)$, $C_\ell(\ell \geq 1)$, $D_{2\ell}(\ell \geq 1)$, E_7, E_8, F_4, or G_2. Here, $r \cdot A_1 = A_1 + A_1 + \cdots + A_1$ (r times).*

Let Φ be irreducible and $w \in W$ an involution. The conjugacy of involutions (and consequently that of the parabolic subsets) is based on looking at orthogonal components $\Delta_0(w)$ and $\Delta_0(-w)$. The interpretation of these classes can be described within a diagram.

Denote $\mathcal{W}$ as the set of W-conjugacy classes of involutions in W. Define an order $>$ on $\mathcal{W}$ so that for for $[w_1], [w_2] \in \mathcal{W}$ we have $[w_1] > [w_2]$ if and only if $\Delta(w_1) \subset \Delta(w_2)$ for some representatives w_i of $[w_i](i = 1, 2)$. This order allows us to construct this diagram, I-poset. These diagrams are denoted as $\mathcal{L}(\Phi_0(-w))$. Some of the diagrams $\mathcal{L}(\Phi)$ are provided within the following table. For more context, see [26].

Table 3.1 Diagrams of $W(\Phi)$ Conjugacy Classes

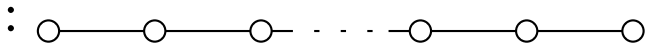
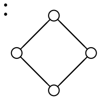
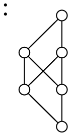
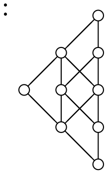
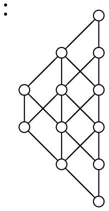
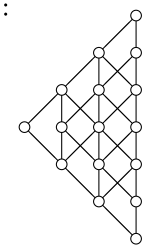
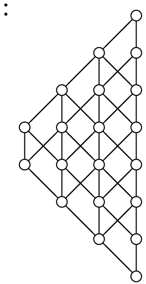
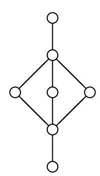
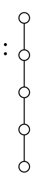
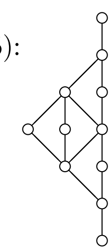
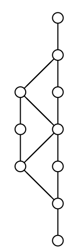
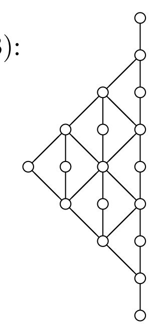
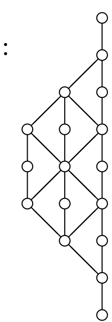
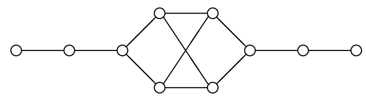
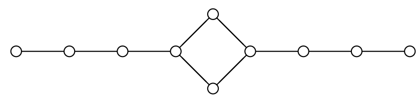
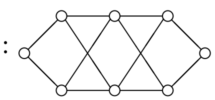
A_l $(l \geq 1)$	$\mathcal{L}(A_l) = \mathcal{L}^*(A_l) = A(n)$ with $n = [\frac{l+1}{2}]$ $A(n)$: 
B_l $(l \geq 2)$	$\mathcal{L}(B_l) = \mathcal{L}^*(B_l) = B(l)$ $B(2)$:  $B(3)$:  $B(4)$:  $B(5)$:  $B(6)$:  $B(7)$:  etc.
C_l $(l \geq 3)$	$\mathcal{L}(C_l) = \mathcal{L}^*(C_l) = B(l)$
D_{2l} $(l \geq 2)$	$\mathcal{L}(D_{2l}) = D(2l)$ $\quad \mathcal{L}^*(D_{2l}) = D^*(2l)$ $D(4)$:  $D^*(4)$:  $D(6)$:  $D^*(6)$:  $D(8)$:  $D^*(8)$:  etc.

Table 3.1 – Continued

D_{2l+1} $(l \geq 2)$	$\mathcal{L}(D_{2l+1}) = \mathcal{L}^*(D_{2l+1}) = D^*(2l)$
E_6	$\mathcal{L}(E_6) = \mathcal{L}^*(E_6) = A(4)$
E_7	$\mathcal{L}(E_7) = \mathcal{L}^*(E_7) = E(7)$ $E(7)$: 
E_8	$\mathcal{L}(E_8) = \mathcal{L}^*(E_8) = E(8)$ $E(8)$: 
F_4	$\mathcal{L}(F_4) = \mathcal{L}^*(F_4) = F(4)$ $F(4)$: 
G_2	$\mathcal{L}(G_2) = \mathcal{L}^*(G_2) = B(2)$

Note that the I poset, mentioned earlier in this chapter is contained in the poset of all involutions $\mathcal{L}(\Phi)$. By identifying the $\mathfrak{a}_i$-standard involutions in $W(\mathfrak{a})$, we are able to employ the diagram and describe the conjugacy classes of the tori $\mathfrak{a}_1$ and $\mathfrak{a}_2$. So for $\mathfrak{a}_1$ and $\mathfrak{a}_2$-standard involutions $w_1, w_2 \in W(\mathfrak{a})$, there is the correspondence

$$\mathfrak{a}_1^- \subset \mathfrak{a}_2^- \iff \mathfrak{a}_{w_1}^- \supset \mathfrak{a}_{w_2}^-$$

Hence given representatives of these conjugacy classes,

$$[\mathfrak{a}_1] < [\mathfrak{a}_2] \iff [w_1] < [w_2] \iff E(w_1, -1) \supset E(w_2, -1)$$

CHAPTER

4

CHARACTERIZATION OF PARABOLIC SUBSETS OF THE ROOT SYSTEM

4.1 Quasi $\mathbb{R}$-split Tori and Singular Involutions

We have described the connection between standard tori and Weyl group elements. In particular, for each torus in standard position, we can determine a Weyl group involution. Hence the conjugacy of these σ-stable maximal $\mathbb{k}$-split tori translates to the conjugacy of the associated standard involutions.

We can determine which Weyl group elements are related to standard tori, H-conjugacy classes (resp. $\mathfrak{h}$-conjugacy classes), and $H_{\mathbb{k}}$-conjugacy classes (resp. $\mathfrak{h}_{\mathbb{k}}$-conjugacy classes) for $\mathbb{k}$-split and quasi $\mathbb{k}$-split tori.

Definition 4.1.1. *Let $\mathfrak{a} \in \mathfrak{A}^{(\theta,\sigma)}$, $w \in W(\mathfrak{a})$ and $\mathfrak{g}_w = Z(\mathfrak{a}_w^+)$. Then w is called σ-singular if the following hold.*

1. $w^2 = e$
2. $\sigma w = w\sigma$
3. $\sigma|[\mathfrak{g}_w, \mathfrak{g}_w]$ *is $\mathbb{k}$-split*

4. $[\mathfrak{g}_w, \mathfrak{g}_w] \cap \mathfrak{h}$ *contains a maximal quasi* $\Bbbk$*-split torus of* $[\mathfrak{g}_w, \mathfrak{g}_w]$.

A root $\alpha \in \Phi(\mathfrak{a})$ *is called* σ*-singular if the corresponding reflection* $s_\alpha \in W(\mathfrak{a})$ *is* σ*-singular. Or equivalently, if* $[\mathfrak{g}_{s_\alpha}, \mathfrak{g}_{s_\alpha}] \not\subset \mathfrak{h}$ *then* α *is* σ*-singular with* $\sigma(\alpha) = \pm\alpha$.

Proposition 4.1.2. *Let* $\mathfrak{a} \in \mathfrak{A}_k^{(\theta,\sigma)}$ *with* a_σ^- *maximal. Then there is a one to one correspondence between the* $W(\mathfrak{a}, \mathfrak{h})$*-conjugacy classes of* a_i*-standard involutions in* $W(\mathfrak{a})$ *and the* $W(\mathfrak{a}, \mathfrak{h})$*-conjugacy classes of* σ*-singular involutions in* $W(\mathfrak{a})$.

This proposition provides an avenue of addressing the original problem of classification. We now need to consider involutions in $W(\mathfrak{a})$ and determine if they are σ-singular involutions. We can obtain much information from the θ-diagram where θ is the Cartan involution; i.e., determining $\Phi(\mathfrak{a})$ when projected to the (-1)-eigenspace. Note that $\Phi(\mathfrak{a}) = \overline{\Phi}_\theta$ and $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-) = \Phi(\mathfrak{a}) \cap \Phi(\mathfrak{a}_\sigma^-)$.

Lemma 4.1.3. *Let* $\mathfrak{a} \in \mathfrak{A}_{\mathbb{R}}^{(\theta,\sigma)}$ *with* $\mathfrak{a}_\sigma^-$ *a maximal* $(\sigma, \mathbb{R})$*-split torus of* $\mathfrak{g}$ *and* $w \in W(\mathfrak{a})$ *where* $w^2 = e$*. Then the following are equivalent.*

1. w *is* σ*-singular*
2. $\mathfrak{a}_w^- \subset \mathfrak{a}_\sigma^-$

Corollary 4.1.3.1. *Let* $\alpha \in \Phi(\mathfrak{a})$ *such that* $\sigma(\alpha) = \pm\alpha$*. Then* α *is a* σ*-singular root if and only if* $\alpha \in \Phi(\mathfrak{a}) \cap \Phi(\mathfrak{a}_\sigma^-)$.

Theorem 4.1.4. *Let* $\mathfrak{a} \in \mathfrak{A}_{\mathbb{R}}^{(\theta,\sigma)}$ *with* $\mathfrak{a}_\sigma^-$ *a maximal* $(\sigma, \mathbb{R})$*-split torus of* $\mathfrak{g}$*. Then there is a one to one correspondence between the* $W(\mathfrak{a})$*-conjugacy classes of* σ*-singular involutions in* $W(\mathfrak{a})$ *and* $W(\mathfrak{a})$*-conjugacy classes of elements in* $W(\mathfrak{a}, \mathfrak{a}_\sigma^-) = W(\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-))$.

Given an ordered commuting pair (θ, σ), $\Phi(\mathfrak{a})$ and $W(\mathfrak{a})$ can be determined from the θ-diagram. By restricting σ to $\Phi(\mathfrak{a})$, we can find the roots of the maximal σ-split torus in $\Phi(\mathfrak{a})$. Using Table 3.1, we can determine the conjugacy classes of elements in $W(\mathfrak{a}, \mathfrak{a}_\sigma^-) \subset W(\mathfrak{a})$. These maximal involutions in $W(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ provides a location within the poset of $W(\mathfrak{a})$ to derive the structure of classes.

Definition 4.1.5. *Let* $\mathfrak{a}$ *be a* θ*-stable maximal quasi* $\Bbbk$*-split torus of* $\mathfrak{g}$ *with* $\Phi = \Phi(\mathfrak{a})$ *and* $\alpha \in \Phi$ *a* θ*-singular root. Then we have the following.*

- α *is called complex relative to* θ *if* $\theta(\alpha) = \pm\alpha$.
- α *is called real relative to* θ *if* $\theta(\alpha) = -\alpha$.
- α *is called imaginary relative to* θ *if* $\theta(\alpha) = \alpha$.

The complex roots provide a means of determine maximality for a θ-stable maximal quasi $\Bbbk$ torus of $\mathfrak{g}$.

Proposition 4.1.6. *Let $\mathfrak{a}$ be a θ-stable maximal quasi $\Bbbk$-split torus of $\mathfrak{g}$ with $\Phi = \Phi(\mathfrak{a})$ and $\alpha \in \Phi$ a θ-singular root. Then we have the following.*

1. *$\mathfrak{a}^+$ is maximal if and only if $\Phi(\mathfrak{a})$ has no real roots.*

2. *$\mathfrak{a}^-$ is maximal if and only if $\Phi(\mathfrak{a})$ has θ-singular roots.*

3. *If $\alpha \in \Phi(\mathfrak{a})$ where $\theta(\alpha) = -\alpha$, then α is real.*

These θ-singular roots provide the capability of moving within the I-poset. In particular, we are able to move step by step through the poset using, at each step, a real or θ-singular imaginary root of a maximal torus that corresponds to that node.

Corollary 4.1.6.1 ([25])**.** *Let $\mathfrak{a} \in \mathfrak{A}_{\mathbb{R}}^{(\theta,\sigma)}$ with $\mathfrak{a}_\sigma^-$ a maximal $(\sigma, \mathbb{R})$-split torus of $\mathfrak{g}$ and $w \in W(\mathfrak{a})$ where $w^2 = e$. Then we have the following.*

1. *Every involution in $W(\mathfrak{a}) \cap W(\mathfrak{a}_\sigma^-)$ is σ-singular.*

2. *Let $\alpha \in \Phi(\mathfrak{a})$. Then α is σ-singular if and only if α is a real root.*

4.2 Conjugacy of Maximal $\mathbb{R}$-split Tori

Now from the previous section, there is a way to determine the $\mathfrak{h}$-conjugacy classes of (θ, σ)-stable maximal quasi $\mathbb{R}$-split tori. Recall that this avenue was pursued to eventually determine $\mathfrak{h}_{\mathbb{R}}$-conjugacy classes of (θ, σ)-stable maximal $\mathbb{R}$-split tori. We shall discuss this connection and classification now.

Definition 4.2.1. *Let (θ, σ) be the usual ordered commuting pair of involutions. Then the ordered pair $(\theta, \sigma\theta)$ is called the associated pair.*

It is obvious that the associated pair commutes and $(\sigma\theta)^2 = id$. We borrow the diagram from [15] and [12] to elaborate on the relationship between the original pairs we have discussed and these new associated pairs.

$$\begin{array}{ccccc}
\begin{matrix}(\mathfrak{g}, \mathfrak{h}) \\ (\theta, \sigma)\end{matrix} & \leftarrow \text{associated} \rightarrow & \begin{matrix}(\mathfrak{g}, \mathfrak{h}^a) \\ (\theta, \sigma\theta)\end{matrix} & \leftarrow \text{dual} \rightarrow & \begin{matrix}(\mathfrak{g}^{ad}, \mathfrak{h}^d) \\ (\sigma\theta, \theta)\end{matrix} \\
\uparrow & & & & \uparrow \\
\text{dual} & & & & \text{associated} \\
\downarrow & & & & \downarrow \\
\begin{matrix}(\mathfrak{g}^d, \mathfrak{h}^d) \\ (\theta, \sigma)\end{matrix} & \leftarrow \text{associated} \rightarrow & \begin{matrix}(\mathfrak{g}^d, \mathfrak{h}^a) \\ (\sigma, \sigma\theta)\end{matrix} & \leftarrow \text{dual} \rightarrow & \begin{matrix}(\mathfrak{g}^{ad}, \mathfrak{h}) \\ (\sigma\theta, \sigma)\end{matrix}
\end{array}$$

Figure 4.1 Associated and Dual Pairs

From [21] and [24], a list of associated pairs for each original pair is described. These associated pairs aid in our effort to preserve $\mathbb{R}$-splitness while yielding much information on maximal θ-split ($\mathbb{R}$-split) tori within H^σ. As we maintain θ-splitness, we now consider the projections onto the (-1)-eigenspaces of θ, σ and $\sigma\theta$.

With the focus turning to $\sigma\theta$, note that a new torus (from this involution) is produced. Nonetheless, we mimic the actions described above for the quasi case. We now consider the action of $\sigma\theta$ on the restricted root system $\overline{\Phi}_\theta$ to find the $\sigma\theta$-split part inside the θ-split torus. Thus, we shall denote the maximal $\mathbb{R}$-split torus for $(\theta, \sigma\theta)$ as $\mathfrak{m}$. Then $\mathfrak{m}^-_{\sigma\theta}$ is maximal $\sigma\theta$-split. It also follows that while being θ-split $\mathfrak{m}^-_{\sigma\theta}$ is equivalent to the maximal torus $\mathfrak{m}^+_\sigma \subset \mathfrak{h}$.

This provides more information on the structure within the poset. In particular, we are able to determine a "top" (maximal σ-split torus inside a θ-split) and "bottom" (maximal σ-fixed torus inside a θ-split) to the sub-poset within the larger poset. We can calculate the number of levels in between once we know the ranks of our "top" and "bottom" conjugacy class representative. We will call the difference in rank of a (θ, σ)-stable maximal $(\sigma, \mathbb{R})$-split torus and a (θ, σ)-stable maximal $\mathbb{R}$-split, σ-fixed torus the singular rank.

Lemma 4.2.2 (10)**.** *Let $\mathfrak{m}$ and $\mathfrak{a}$ be defined as above. Then*

$$\textit{singular rank } = \dim(\mathfrak{a}^+_\sigma) + \dim(\mathfrak{m}^-_{\sigma\theta}) - \dim(\mathfrak{a})$$

Luckily, much of the process used in the quasi $\mathbb{R}$-split torus case can be translated and applied to that of $\mathbb{R}$-split tori.

Definition 4.2.3. *Let $\mathfrak{a} \in \mathfrak{A}^{(\theta,\sigma)}_{\mathbb{R}}$ and $w \in W(\mathfrak{a})$. Then w is called (θ, σ)-singular if the following hold.*

1. $w^2 = e$
2. $\sigma w = w\sigma$
3. *the involutions $\sigma|[\mathfrak{g}_w, \mathfrak{g}_w]$ and $\sigma\theta|[\mathfrak{g}_w, \mathfrak{g}_w]$ are $\mathbb{k}$-split*

A root $\alpha \in \Phi(\mathfrak{a})$ is called (θ, σ)-singular if the corresponding reflection $s_\alpha \in W(\mathfrak{a})$ is (θ, σ)-singular.

Proposition 4.2.4. *Let $w \in W(\mathfrak{a})$ be an involution. Then w is a (θ, σ)-singular involution if and only if w is an $\mathfrak{a}_i$-standard involution for some $\mathfrak{a}_i \in \mathfrak{A}^{(\sigma,\theta)}_k$*

Proposition 4.2.5. *Let $\mathfrak{a} \in \mathfrak{A}^{(\theta,\sigma)}_k$ with $\mathfrak{a}^-_\sigma$ a maximal $(\sigma, \mathbb{R})$-split torus of $\mathfrak{g}$. Then there is a one to one correspondence between the $\mathfrak{h}^+_k$-conjugacy classes of $\mathfrak{A}^{(\theta,\sigma)}_k$ and the $W(\mathfrak{a}, \mathfrak{h}^+_k)$-conjugacy classes of (θ, σ)-singular roots of $W(\mathfrak{a})$.*

Hence, we build on the understanding and calculation of σ-singular involutions and roots. Consequently, the only condition of interest becomes $\sigma\theta|[\mathfrak{g}_w, \mathfrak{g}_w]$ is $\mathbb{k}$-split.

Recall that all σ-singular roots are contained in $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-)$. So (θ, σ)-singular involutions can be found in $W(\mathfrak{a})$. It follows that we can realize $\overline{\Phi}_\theta$ as the root system of a maximal θ-split torus and $\overline{\Phi}_{\theta,\sigma}$ as that of the maximal $(\sigma, \mathbb{R})$-split torus. Now with the associated pair $(\theta, \sigma\theta)$ we can find (θ, σ)-singular (resp. $(\theta, \sigma\theta)$-singular) roots in $\overline{\Phi}_\theta \cap \overline{\Phi}_{\theta,\sigma}$ (resp. $\overline{\Phi}_\theta \cap \overline{\Phi}_{\theta,\sigma\theta}$). To determine the conjugacy classes within $W(\overline{\Phi}_\theta \cap \overline{\Phi}_{\theta,\sigma})$, we start with the following lemmas.

Lemma 4.2.6. *Let w_m be a maximal (θ, σ)-singular involution in $W(\overline{\Phi}_\theta \cap \overline{\Phi}_{\theta,\sigma})$. Every (θ, σ)- involution in $W(\overline{\Phi}_\theta \cap \overline{\Phi}_{\theta,\sigma})$ is conjugate under $W(\mathfrak{a}, \mathfrak{h}_{\mathbb{R}}^+)$ with a (θ, σ)-singular involution w satisfying $\mathfrak{a}_w^- \subset \mathfrak{a}_{w_m}^-$.*

Lemma 4.2.7. *Let w_m be a maximal (θ, σ)-singular involution in $W(\overline{\Phi}_\theta \cap \overline{\Phi}_{\theta,\sigma})$ and w a (θ, σ)-singular involution in $W(\overline{\Phi}_\theta \cap \overline{\Phi}_{\theta,\sigma})$ with $\mathfrak{a}_w^- \subset \mathfrak{a}_{w_m}^-$. Then $w \cdot w_m$ is a (θ, σ)-singular involution in $W(\overline{\Phi}_\theta \cap \overline{\Phi}_{\theta,\sigma})$.*

Recall $\mathfrak{m}$ is a (θ, σ)-stable maximal $\mathbb{R}$-split torus of $\mathfrak{g}$ and $\mathfrak{m}_\sigma^+$ is maximal. Let $\mathfrak{m}$ be in standard position with respect to $\mathfrak{a}$. Let $\Delta(\mathfrak{m})$ be a $(-\sigma)$-basis of $\Phi(\mathfrak{m})$. We know this induces an order on $\Phi_0(\sigma)$. Denote $w_0(\sigma)$ as the longest element in $W(\Phi_0(\sigma))$. Then $w_0(\sigma)(\Phi^+(\mathfrak{m})) = \sigma(\Phi^+(\mathfrak{m}))$. Let $\Upsilon \subset \Phi(\sigma)$ such that $\Upsilon \supset \Phi_0(\sigma)$. Then we can realize $w_0(\sigma)$ as an element in Υ, say $\tilde{w}_0(\sigma)$. Let $w(\sigma)$ be a maximal involution in $W(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ such that $\mathfrak{a}_{\tilde{w}_0(\sigma)} \subset \mathfrak{a}_{w(\sigma)}^-$. Then $w_m = w(\sigma) \cdot \tilde{w}_0(\sigma)$ is a maximal (θ, σ)-singular involution. So by the equation, it suffices to determine $\Phi_0(\sigma)$ and σ-maximal involutions.

Proposition 4.2.8. *Let w_1 and w_2 be (θ, σ)-singular involutions. Then the following are equivalent.*

1. *w_1 and w_2 are conjugate under $W(\mathfrak{a}, \mathfrak{h}_{\mathbb{k}}^+)$.*
2. *w_1 and w_2 are conjugate under $W(\mathfrak{a}_\sigma^-, \mathfrak{h}_{\mathbb{k}}^+)$.*
3. *w_1 and w_2 are conjugate under $W(\mathfrak{a}, \mathfrak{g}_{\sigma\theta})$.*
4. *w_1 and w_2 are conjugate under $W(\mathfrak{a}_\sigma^-, \mathfrak{g}_{\sigma\theta})$.*

Remark 4.2.9. *[21]*

Case 1 *There are no (θ, σ)- or $(\theta, \theta\sigma)$-singular roots. This arises when $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-) = \emptyset$ or $\Phi(\mathfrak{m}, \mathfrak{m}_{\sigma\theta}^-) = \emptyset$. An empty interestion means there are no candidates to flip a one-dimensional piece from σ-split to the portion fixed by σ. Further if there are no (θ, σ)-singular roots then the singular rank is 0, we can conclude there is only one $\mathfrak{h}_k$-conjugacy class of σ-stable maximal $\mathbb{R}$-split tori in $\mathfrak{g}$. This occurs in 18 of the 171 cases.*

Case 2 *[10] Suppose the singular rank is r. Let $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-) = r \cdot A_1$ or $\Phi(\mathfrak{m}, \mathfrak{m}_{\sigma\theta}^-) = r \cdot A_1$. Then all (θ, σ)-stable involutions w with $\mathfrak{a}_w^- \subset \mathfrak{a}_{w_m}^-$ are (θ, σ)-singular. Though w_m being of type $r \cdot A_1$ and (θ, σ)-singular must be verified, $\Phi(\mathfrak{t})$ will be of type B_r, C_r, D_r, E_r ($r = 6, 7$ or 8), F_4 or G_2 in most cases. These conjugacy classes will not split.*

Case 3 *Suppose the singular rank is r. Let $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ or $\Phi(\mathfrak{m}, \mathfrak{m}_{\sigma\theta}^-)$ be of type B_r, C_r, D_r, E_r ($r = 6, 7$ or 8), F_4 or G_2. Then all involutions in the poset are (θ, σ)-singular if w_m is of type B_r, C_r, D_r, E_r ($r = 6, 7$ or 8), F_4 or G_2. It is possible that these involutions will split in $W(\mathfrak{a}, \mathfrak{h}_{\mathbb{R}}^+)$. For cases when $\Phi(\mathfrak{t}) = X_i + X_j$ where $X = A, B, C, BC, D$ or E, the roots will split. When this happens, the singular rank could be less than the maximal rank in $\Phi(\mathfrak{a}, \mathfrak{a}_\sigma^-)$ or $\Phi(\mathfrak{m}, \mathfrak{m}_{\sigma\theta}^-)$.*

4.3 Characterization of θ on θ-basis of Φ

Definition 4.3.1. *Let $\Delta \subset \Phi$ be a base. Then there exists a unique Weyl group element, w_0, such that $w_0(\Delta) = -\Delta$ called the longest element of Δ.*

Lemma 4.3.2. *Let $\mathfrak{t} \subset \mathfrak{g}$ be a θ-stable maximal torus, Δ a θ-basis of $\Phi(\mathfrak{t})$. Then $\theta = -id \cdot \theta^* \cdot w_0(\theta)$ where $w_0(\theta)$ is the longest element of $\Phi_0(\theta)$ with respect to a θ-ordered basis $\Delta_0(\theta)$ and θ^* is a Dynkin diagram automorphism of $\Phi(\mathfrak{t})$.*

From this Lemma, we get the characterization of θ on a θ-basis of Φ. Note that $\theta^* \in \mathrm{Aut}(X, \Phi, \Delta, \Delta_0(\theta))$ where $\mathrm{Aut}(X, \Phi, \Delta, \Delta_0(\theta)) = \{\phi \in \mathrm{Aut}(X, \Phi) \,|\, \phi(\Delta) = \Delta \text{ and } \phi(\Delta_0(\theta)) = \Delta_0(\theta)\}$. So $\theta = -id \cdot \theta^* \cdot w_0(\theta)$ is called a characterization of θ on its $(+1)$-eigenspace. Similarly, $\theta = (-\theta)^* \cdot w_0(-\theta)$ is the characterization of θ on a $(-\theta)$-basis. This is called a characterization of θ on its (-1)-eigenspace.

In fact, this Dynkin diagram automorphism must be an involution.

Proposition 4.3.3.

$$\theta^* = \begin{cases} \textit{id} \\ \textit{Dynkin diagram automorphism of order } 2 \end{cases}$$

Definition 4.3.4. *Let $\alpha, \beta \in \Phi$. Then α and β are said to be strongly orthogonal if $(\alpha, \beta) = 0$ and $\alpha \pm \beta \notin \Phi$.*

Then we have the following characterization of involutions in W.

Theorem 4.3.5 ([26])**.** *Let $\theta \in \mathrm{Aut}(\Phi)$, $\theta^2 = id$ and let Δ be a $(-\theta)$-basis of Φ. If we write $\theta = (-\theta)^* w(\theta)$, then the following are equivalent.*

(i) $\theta \in W(\Phi)$

(ii) $\theta = s_{\alpha_1} \cdots s_{\alpha_\ell}$ *where* $\alpha_1, \ldots, \alpha_\ell \in \Phi(\theta)$ *are strongly orthongoal*

(iii) $(-\theta)^* = id$

(iv) $-id \in W(\Phi(\theta)) = W(\theta)$

(v) *There irreducible components of $\Phi_0(-\theta)$ are of type A_1, B_ℓ, C_ℓ, $D_{2\ell}$ $(\ell \geq 2)$, E_7, E_8, F_4, or G_2.*

This discussion on the characterization of θ on its $+1$ and -1-eigenspaces lends itself to the determination of the existence of θ^* and $(-\theta)^*$.

Theorem 4.3.6. *Let Δ_1 and Δ_2 be θ- and $(-\theta)$-bases, respective. Let θ^* and $(-\theta)^*$ be defined as above. If Φ is irreducible, then $\theta^* = id$ if and only if $(-\theta)^* \neq id$.*

We will prove this theorem in Chapter 5.

Definition 4.3.7. *Let $\theta = id$ and Δ be a basis of Φ. Then $\theta^*(\Delta) = -id \cdot w_0(id)$ is called the opposition involution of Δ. For this case, we will write $id^*(\Delta)$ for $\theta^*(\Delta)$ or simply id^*.*

4.4 The Index of θ

The action of θ on a root datum Ψ can be described by an index. These indices help to determine the fine structure of restricted root systems and essentially that of the symmetric variety G/H.

From [16], we get the following result.

Proposition 4.4.1. *Let X, $X_0(\theta)$, Φ, $\Phi_0(\theta)$, $\bar{\Phi}_\theta$, etc. be defined as above. Let Δ, Δ' be θ-bases of Φ. Then we have the following*

(i) $\Delta = \Delta'$ *if and only if* $\Delta_0(\theta) = \Delta'_0(\theta)$ *and* $\overline{\Delta}'_\theta = \overline{\Delta'}_\theta$.

(ii) If $\overline{\Delta}_\theta = \overline{\Delta}'_\theta$, *then there exists a unique* $w' \in W_0(\Phi)$ *such that* $\Delta' = w'\Delta$.

Definition 4.4.2. *Let Ψ be a semisimple root datum and $\theta \in \mathrm{Aut}(\Psi)$ be an involution. If Δ is a $(-\theta)$-basis, then the quadruple $(X, \Delta, \Delta_0(\theta), \theta^*)$ is called the index of θ where $\theta = -id \cdot \theta^* \cdot w_0(\theta)$.*

Definition 4.4.3. *Let Ψ be a root datum and θ_1, $\theta_2 \in \mathrm{Aut}(\Phi)$ be involutions. Let Δ_1 and Δ_2 be θ_1 and θ_2 bases. Then two indices $(X, \Delta_1, \Delta_0(\theta_1), \theta_1^*)$ and $(X, \Delta_2, \Delta_0(\theta_2), \theta_2^*)$ are said to be $W(\Phi)$-isomorphic (resp. Aut(Φ)-isomorphic) if there exists $w \in W(\Phi)$ (resp. $\varphi \in \mathrm{Aut}(\Phi)$) which maps $(\Delta_1, \Delta_0(\theta_1))$ to $(\Delta_2, \Delta_0(\theta_2))$ and satisfies $w\theta_1^* w^{-1} = \theta_2^*$ (resp. $\varphi\theta_1^*\varphi^{-1} = \theta_2^*$). When not confusing, we will only use the term isomorphic.*

For two θ-bases Δ and Δ', the corresponding indices $(X, \Delta, \Delta_0(\theta), \theta^*)$ and $(X, \Delta', \Delta'_0(\theta), \theta^*)$ may not necessarily be isomorphic. Nevertheless, this isomorphism occurs when $\overline{\Phi}_\theta$ is a root system with Weyl group $\overline{W}_\theta$.

Lemma 4.4.4 ([24], 2.13)**.** *Let Ψ be a semisimple root datum and $\theta \in \mathrm{Aut}(\Psi)$ be an involution such that $\overline{\Phi}_\theta$ is root system with Weyl group $\overline{W}_\theta$. Let Δ and Δ' be two θ-bases of Φ. Then $(X, \Delta, \Delta_0(\theta), \theta^*)$ and $(X, \Delta', \Delta'_0(\theta), \theta^*)$ are isomorphic.*

Proof. Consider $\overline{W}_\theta$. From Proposition 4.4.1, there is a unique $w \in W_1(\theta)$ such that $w(\Delta) = \Delta'$. It also follows that $w(\Delta_0(\theta)) = \Delta'_0(\theta)$. Let $\theta_1^* = \theta^*(\Delta)$ and $\theta_2^* = \theta^*(\Delta')$. We want to show that there exists $\bar{w} \in W(\Phi)$ such that $\bar{w}\theta_1^*\bar{w}^{-1} = \theta_2^*$.

Recall that $w_0(\theta)$ and $w'_0(\theta)$ are the longest Weyl group elements with respect to Δ_0 and Δ'_0, respectively. So $ww_0(\theta)w^{-1} = w'_0(\theta)$. By using the characterization of θ on Δ, we get

$$\begin{aligned} w\theta_1^* w^{-1} &= w(-id \cdot w_0(\theta) \cdot \theta)w^{-1} \\ &= -id \cdot w \cdot w_(\theta)w^{-1}\theta \\ &= -id \cdot w'_0(\theta)\theta \\ &= \theta_2^* \end{aligned}$$

□

Lemma 4.4.5. *Let θ_1, $\theta_2 \in \mathrm{Aut}(\Phi)$ be involutions and $\Delta_0(\theta_1)$, $\Delta_0(\theta_2) \subset \Delta$. If $(X, \Delta_1, \Delta_0(\theta_1), \theta_1^*)$ and $(X, \Delta_2, \Delta_0(\theta_2), \theta_2^*)$ are isomorphic, then θ_1 and θ_2 are $W(\Phi)$-conjugate.*

Proof. Let $w \in W(\Phi)$ be so that $w(\Delta_1) = \Delta_2$ and $w(\Delta_0(\theta_1)) = \Delta_0(\theta_2)$. Then $ww_0(\theta_1)w^{-1} = w_0(\theta_2)$ and $w\theta_1^* w^{-1} = \theta_2^*$. Since Δ_1 and Δ_2 are θ_1 and θ_2 bases, respectively, we have the characterizations of θ_1 and θ_2 on their respective bases. Therefore,

$$\begin{aligned} w\theta_1 w^{-1} &= w(-id \cdot \theta_1^* \cdot w_0(\theta_1))w^{-1} \\ &= (w \cdot -id \cdot w^{-1})(w \cdot \theta_1^* \cdot w^{-1})(w \cdot w_0(\theta_1) \cdot w^{-1}) \\ &= -id \cdot \theta_2^* \cdot w_0(\theta_2) \\ &= \theta_2 \end{aligned}$$

□

As in the works of Helminck, Tits and Satake, we adopt the diagrammatic representation of the index of θ. Here, nodes that represent roots in $\Delta_0(\theta)$ are colored black within the ordinary Dynkin diagram of θ, along with the indication the action of θ^* on $\Delta - \Delta_0(\theta)$ by arrows. We denote this representation as $\mathcal{D}(\theta)$.

Proposition 4.4.6. *Let $\theta_1, \theta_2 \in Aut(\Phi)$ be involutions. If $\Delta_0(\theta_1)$ and $\Delta_0(\theta_2)$ are $W(\Phi)$-conjugate such that $w\theta_1 w^{-1} = \theta_2$, then we have the following.*

(i) *$w\pi_1 w^{-1} = \pi_2$ for projections maps π_1 and π_2 corresponding to θ_1 and θ_2, respectively.*

(ii) *$\bar{\Delta}_{\theta_1}$ and $\bar{\Delta}_{\theta_2}$ are $W(\Phi)$-conjugate.*

Proof. Let $w \in Aut(\Phi)$ be so that $w(\Delta_0(\theta_1)) = \Delta_0(\theta_2)$ and $w\theta_1 w^{-1} = \theta_2$. For (i), let $\alpha \in \Phi$. Then

$$w\pi_1(\alpha) = w\left[\frac{1}{2}(\alpha - \theta_1(\alpha))\right] = \frac{1}{2}(w(\alpha) - w\theta_1(\alpha)) = \frac{1}{2}(w(\alpha) - \theta_2 w(\alpha)) = \pi_2 w(\alpha).$$

For (ii), it suffices to show that $\bar{\Phi}_{\theta_1}$ and $\bar{\Phi}_{\theta_2}$ are $W(\Phi)$-conjugate. So for $\alpha \in \Phi$,

$$w(\bar{\Phi}_{\theta_1}) = w\pi_1(\Phi - \Phi_0(\theta_1)) = \pi_2 w(\Phi - \Phi_0(\theta_1)) = \pi_2(\Phi - \Phi_0(\theta_2)) = \bar{\Phi}_{\theta_2}.$$

□

If the conditions hold from the previous proposition, we say that $\Delta_0(\theta_1)$ and $\Delta_0(\theta_2)$ are $W(\Phi)$-semi-isomorphic, denoted $\Delta_0(\theta_1) \simeq \Delta_0(\theta_2)$. Analogously if (i) and (ii) are satisfied, $\bar{\Delta}_{\theta_1}$ and $\bar{\Delta}_{\theta_2}$ are $W(\Phi)$-semi-isomorphic.

Proposition 4.4.7. *Let θ_1, $\theta_2 \in \mathrm{Aut}(\Phi)$ be involutions and $\Delta_0(\theta_1)$, $\Delta_0(\theta_2) \subset \Delta$. If $\Delta_0(\theta_1) \simeq \Delta_0(\theta_2)$ and $\bar{\Delta}_{\theta_1} \simeq \bar{\Delta}_{\theta_2}$, then the θ_1-basis Δ_1 and θ_2-basis Δ_2 are $W(\Phi)$-conjugate.*

Proof. Let $\succ_1$ and $\succ_2$ be the θ_1 and θ_2 orders, respectively. We define the positive roots as $\Phi_i^+ = \{\alpha \in \Phi \mid \alpha \succ_i 0\}$ for $i = 1, 2$. It suffices to show that Φ_1^+ and Φ_2^+ are $W(\Phi)$-conjugate. Let $w \in W(\Phi)$ be so that $w(\Delta_0(\theta_1)) = \Delta_0(\theta_2)$ and $w\theta_1 w^{-1} = \theta_2$. Take $\alpha \in \Delta_1$. If $\alpha \in \Delta_0(\theta_1)$, then $w(\alpha) \succ_2 0$. If $\alpha \notin \Delta_0(\theta_1)$, then $\pi_1(\alpha) \in \bar{\Delta}_{\theta_1}$ and $w\pi_1(\alpha) \succ_2 0$. Therefore, $w(\Phi_1^+) = \Phi_2^+$. $\square$

Theorem 4.4.8. *Let θ_1, $\theta_2 \in \mathrm{Aut}(\Phi)$ be involutions and $\Delta_0(\theta_1)$, $\Delta_0(\theta_2) \subset \Delta$. If $\Delta_0(\theta_1) \simeq \Delta_0(\theta_2)$, then the two indices $(X, \Delta_1, \Delta_0(\theta_1), \theta_1^*)$ and $(X, \Delta_2, \Delta_0(\theta_2), \theta_2^*)$ are isomorphic.*

Proof. It suffices to show that θ_1^* and θ_2^* are $W(\Phi)$-conjugate. Note that with linear orders $\succ_1$ and $\succ_2$, we can characterize the involutions θ_1 and θ_2 as $\theta_1 = -id \cdot \theta_1^* \cdot w_0(\theta_1)$ and $\theta_2 = -id \cdot \theta_2^* \cdot w_0(\theta_2)$. Hence

$$
\begin{aligned}
w\theta_1 w^{-1} &= \theta_2 \\
w\left(-id \cdot \theta_1^* w_0(\theta_1)\right) w^{-1} &= -id \cdot \theta_2^* \cdot w_0(\theta_2) \\
(w \cdot -id \cdot w^{-1})(w \cdot \theta_1^* \cdot w^{-1})(w \cdot w_0(\theta_1) w^{-1}) &= -id \cdot \theta_2^* \cdot w_0(\theta_2) \\
-id \cdot w \cdot \theta_1^* w^{-1} \cdot w_0(\theta_2) &= -id \cdot \theta_2^* \cdot w_0(\theta_2) \\
w \cdot \theta_1^* w^{-1} &= \theta_2^*.
\end{aligned}
$$

$\square$

From the previous two propositions, we gather that conjugacy of indices simply rely on that of there θ-fixed subsets.

4.5 Diagrams of (θ, σ)

When classifying maximal (θ, σ)-stable $\mathbb{R}$-split tori, we rely on the root system structure obtained by restricting to σ and θ to their common -1-eigenspaces. We identify $\overline{\Phi}_\theta$ (resp. $\overline{\Phi}_{\theta,\sigma}$) as the projection onto the -1-eigenspace of θ (resp. θ and σ). Note that different involutions will produce different systems.

Remark 4.5.1. *Let Δ be a (θ, σ)-basis of Φ, $w_0(\sigma)$, $w_0(\theta) \in W_0(\theta, \sigma)$ be involutions such that $w_0(\sigma)(\Delta_0(\sigma)) = -\Delta_0(\sigma)$ and $w_0(\theta)(\Delta_0(\theta)) = -\Delta_0(\theta)$. Then we have the following.*

1. *The characterization of θ and σ on a (θ, σ)-basis:*

$$
\begin{aligned}
\theta^* &= -id \cdot \theta \cdot w_0(\theta) \\
\sigma^* &= -id \cdot \sigma \cdot w_0(\sigma)
\end{aligned}
$$

2. *$w_0(\sigma)$, $w_0(\theta)$, θ^*, σ^* and $-id$ commute.*

3. $\theta^*, \sigma^* = \begin{cases} id \\ \text{Dynkin diagram automorphism of order } 2 \end{cases}$

This remark follows from results in Section 4.3, [24] and from the following result.

Lemma 4.5.2. *All Weyl group elements commute with $-id$.*

Proof. Let $w \in W(\Phi)$. We use the fact that w is a linear transformation. So for $\chi \in X$,

$$-id(w(\chi)) = w(-\chi) = w(-id(\chi)).$$

$\square$

Proposition 4.5.3 ([16], Proposition 5.26)**.** *Let Ψ be semisimple root datum and assume the pair (θ, σ) acts on (X, Φ). The following are equivalent.*

1. *(X, Φ) has a (θ, σ)-order.*
2. *$\Phi_0(\theta, \sigma) = \Phi_0(\sigma) \cap \Phi_0(\theta)$.*
3. *If $\Phi_1 \subset \Phi_0(\theta, \sigma)$ is an irreducible component, then $\Phi_1 \subset \Phi_0(\theta)$ or $\Phi_1 \subset \Phi_0(\theta)$.*

The (θ, σ)-order is completely determined by the θ-index and σ-index.

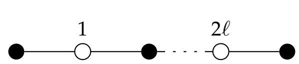
Definition 4.5.4. *Let Δ be a (θ, σ)-basis of Φ. Then the sextuple $(X, \Delta, \Delta_0(\sigma), \Delta_0(\theta), \sigma^*, \theta^*)$ is called the (θ, σ)-index where $\theta^* = -id \cdot \theta \cdot w_0(\theta)$ and $\sigma^* = -id \cdot \sigma \cdot w_0(\sigma)$.*

We will call the diagram representation associated with the (θ, σ)-index the (θ, σ)-diagram. Similar to the θ-diagram, we color black the vertices of the ordinary Dynkin diagram that represents roots in $\Delta_0(\theta, \sigma)$. For vertices that lie in $\Delta_0(\sigma) - (\Delta_0(\sigma) \cap \Delta_0(\theta))$, we label with θ. For vertices that lie in $\Delta_0(\theta) - (\Delta_0(\sigma) \cap \Delta_0(\theta))$, we label with σ. We indicate the actions of diagram automorphisms σ^* and θ^* with arrows if $\sigma^* \neq id$ and $\theta^* \neq id$.

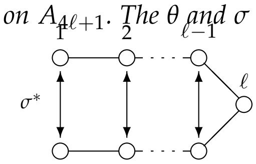
From the (θ, σ)-index, we can obtain information on both the θ- and σ-indices. The converse is also true. Consider the following example.

Example 4.5.5. *Let θ be of type $AIII_b$ and σ of type AII acton on $A_{4\ell+1}$. The θ and σ diagrams are as follows.*

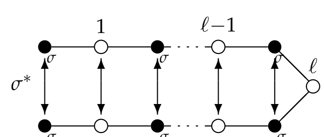
θ :



σ :



Thus, the (θ, σ)-diagram would be:



Definition 4.5.6. *Let $S = (X, \Delta, \Delta_0(\sigma), \Delta_0(\theta), \sigma^*, \theta^*)$ be the (θ, σ)-index. Then S is called irreducilbe if Δ is not the union of two mutually orthogonal θ^*- and σ^*-stable non-empty subsets Δ_1 and Δ_2. S is called absolutely irreducible if Δ is connected.*

Note that S is irreducible if and only if $\overline{\Delta}_{\theta,\sigma}$ is connected.

If S is absolutely irreducible, then denote the action of the pair (θ, σ) on Φ as $X_\ell^{q,p}(type\ \theta, type\ \sigma)$ where $\ell = rank(X)$, $p = rank(\overline{\Delta}_\sigma)$ and $q = rank(\overline{\Delta}_\theta)$. For example, $A_{4\ell+1}^{2\ell,2\ell+1}(II, III_b)$ represents Φ of type $A_{4\ell+1}$, θ is of type $AIII_b$ and σ is of type AII with $\overline{\Delta}_\theta = 2\ell$ and $\overline{\Delta}_\sigma = 2\ell + 1$. This is indeed the notation of the (θ, σ)-diagram mentioned in the above example.

4.5.1 Quadratic Elements & Multiplicities

As mentioned in Section 2.5, quadratic elements help in the classification of (θ, σ)-diagrams.

Definition 4.5.7. *Let $\mathfrak{t} \subset \mathfrak{g}$ be a (θ, σ)-split torus and $\epsilon \in \mathfrak{t}$. Then ϵ is called a quadratic element if $\epsilon^2 \in Z(\mathfrak{g})$, the centralizer of $\mathfrak{g}$.*

When using quadratic elements within the classification of commuting pairs of involutions, it follows that commuting involutions σ and $\sigma\mathrm{Int}(\epsilon)$ are two different involutions. A more detailed explanation of quadratic elements can be found in [24]. Nonetheless, it is a point of interest to consider the action of $\sigma\mathrm{Int}(\epsilon)$ on $\Phi(\mathfrak{t})$. This is largely due to the observation from Helminck that there exists a quadratic element ϵ such that the commuting pairs (θ_1, σ_1) and $(\sigma\mathrm{Int}(\epsilon), \sigma)$ are isomorphic; i.e. there exists $g \in \mathfrak{g}$ such that $\mathrm{Int}(g)\sigma_1\mathrm{Int}(g^{-1}) = \sigma$ and $\mathrm{Int}(g)\theta_1\mathrm{Int}(g^{-1}) = \sigma\mathrm{Int}(\epsilon)$ [[24], Chapter 5]. These elements can be described by using a basis of $\Phi(\mathfrak{a})$ for $\mathfrak{a} \subset \mathfrak{t}$.

Definition 4.5.8. *For all $\lambda \in \Phi(\mathfrak{a})$, define*

$$\Phi(\lambda) = \{\alpha \in \Phi(\mathfrak{t}) \mid \alpha|_{\mathfrak{a}} = \lambda\} = \{\alpha \in \Phi(\mathfrak{t}) \mid \pi_\theta(\alpha) = \lambda\}$$

.

Definition 4.5.9. *Let $\mathfrak{a} \subset \mathfrak{t}$ be a (θ, σ)-stable $\mathbb{R}$-split tori with $\mathfrak{a}_\sigma^-$ maximal. For $\lambda \in \Phi(\mathfrak{a})$, let $m(\lambda) = |\Phi(\lambda)|$. We call $m(\lambda)$ the multiplicity of λ.*

For all $\lambda \in \Phi(\mathfrak{a})$, let $\mathfrak{g}(\mathfrak{a}, \lambda) = \{x \in \mathfrak{g} \mid [h, x] = \lambda(h)x \text{ for all } h \in \mathfrak{a}\}$. Hence $\sigma\theta(\lambda) = \theta\sigma(\lambda)$ and $\sigma\theta(\mathfrak{g}(\mathfrak{a}, \lambda)) = \theta\sigma(\mathfrak{g}(\mathfrak{a}, \lambda))$. Let

$$\begin{aligned} \mathfrak{g}(\mathfrak{a}, \lambda)^\pm_{\theta,\sigma} &= \{x \in \mathfrak{g}(\mathfrak{a}, \lambda) \mid \theta\sigma(x) = \pm x\} \\ m^\pm(\lambda, \theta\sigma) &= \dim\left(\mathfrak{g}(\mathfrak{a}, \lambda)^\pm_{\theta,\sigma}\right) \end{aligned}$$

Now we can also realize the multiplicity of λ as

$$m(\lambda) = \dim\left(\mathfrak{g}(\mathfrak{a}, \lambda)\right) = m^+(\lambda, \theta\sigma) + m^-(\lambda, \theta\sigma).$$

The signature of λ is the pair $(m^+(\lambda, \theta\sigma), m^-(\lambda, \theta\sigma))$.

Remark 4.5.10. *If $\epsilon \in \mathfrak{a}$ is a quadratic element and $\lambda \in \Phi(\mathfrak{a})$ is such that $\lambda(\epsilon) = -1$, then $m^+(\lambda, \theta\sigma) = m^-(\lambda, \theta\sigma Int(\epsilon))$ and $m^-(\lambda, \theta\sigma) = m^+(\lambda, \theta\sigma Int(\epsilon))$*

From this remark, standard pairs can be defined using the signature of λ.

Definition 4.5.11. *The ordered commuting pair (θ, σ) is called a standard pair if $m^+(\lambda, \theta\sigma) \geq m^-(\lambda, \theta\sigma)$ for any maximal (θ, σ)-split torus $\mathfrak{a}$ of $\mathfrak{g}$ and any $\lambda \in \Phi(\mathfrak{a})$.*

So in addition to quadratic elements, the classification of symmetric spaces and their representation heavily relies on the fine structure of the restricted root system with multiplicities and Weyl groups. The multiplicities of λ for the different types of Cartan involutions can be found in [7] and [24].

4.5.2 Isomorphy Classes of (θ, σ)

Definition 4.5.12. *Two (θ, σ)-indices $(X, \Delta, \Delta_0(\sigma_1), \Delta_0(\theta_1), \sigma_1^*, \theta_1^*)$ and $(X, \Delta', \Delta_0'(\sigma_2), \Delta_0'(\theta_2), \sigma_2^*, \theta_2^*)$ are said to be isomorphic if there exists $w \in W(\Phi)$ which maps $(X, \Delta, \Delta_0(\sigma_1), \Delta_0(\theta_1))$ onto $(X, \Delta', \Delta_0'(\sigma_2), \Delta_0'(\theta_2)$ and satisfies*

$$w\theta_1^* w^{-1} = \theta_2^* \quad \textit{and} \quad w\sigma_1^* w^{-1} = \sigma_2^*$$

The isomorphy classes of (θ, σ) can be represented by (θ, σ)-diagrams. On the Lie algebra level, if $\mathfrak{g}$ is simple, then there are 88 types of local symmetric spaces and the (θ, σ)-diagrams are called absolutely irreducible. If $\mathfrak{g}$ is not simple, then there are 83 types of local symmetric spaces and the (θ, σ)-diagrams are irreducible, but not absolutely irreducible. In [21], these 171 cases (in which Theorem 3.1.1 applies), have been listed. In particular, these cases address the conjugacy classes of quasi $\mathbb{R}$-split tori in $\mathfrak{h}$, while providing the types and maximal involutions of $\overline{\Phi}_{(\theta,\sigma)} \cap \overline{\Phi}_\theta$. As this is relevant to the focus of this paper, that table is provided below.

Table 4.1 Classification of $A^{(\theta,\sigma)}/H^{+}$

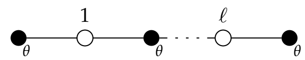
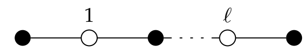
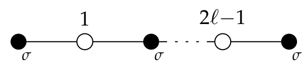
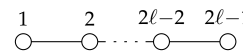
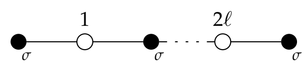
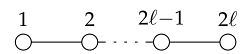
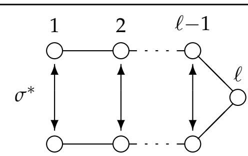
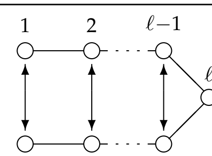
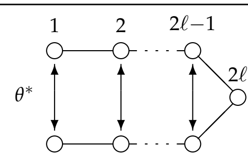
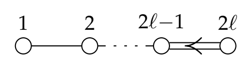
Type (θ,σ)	Diagram (θ,σ)	Type Φ_θ $\Phi(A)$	Diagram $\sigma\vert\Phi_\theta$	Type $\Phi_{\theta,\sigma}\cap\Phi_\theta$ $\Phi(A,A_\sigma^-)$	max. involution $\Phi_{\theta,\sigma}\cap\Phi_\theta$
$A_{2\ell+1}^{2\ell+1,\ell}(I,II)$		$A_{2\ell+1}$		$\emptyset$	id
$A_{4\ell-1}^{2\ell-1,4\ell-1}(II,I)$		$A_{2\ell-1}$		$A_{2\ell-1}$	$\ell\cdot A_1$
$A_{4\ell+1}^{2\ell,4\ell+1}(II,I)$		$A_{2\ell}$		$A_{2\ell}$	$\ell\cdot A_1$
$A_{2\ell-1}^{2\ell-1,\ell}(I,III_b,\epsilon_0)$		$A_{2\ell-1}$		$\ell\cdot A_1$	$\ell\cdot A_1$
$A_{4\ell-1}^{2\ell,4\ell-1}(III_b,I,\epsilon_0)$		$C_{2\ell}$		$C_{2\ell}$	$C_{2\ell}$

Table 4.1 – Continued

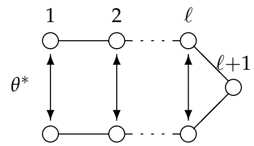
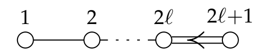
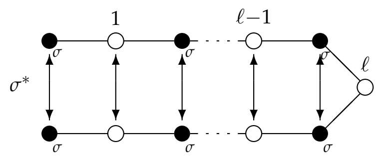
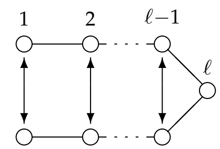
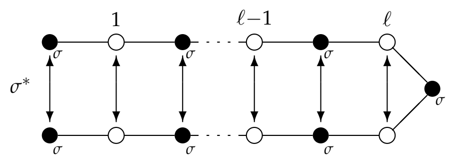
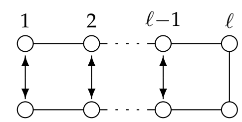
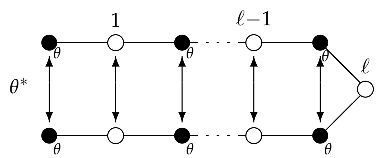
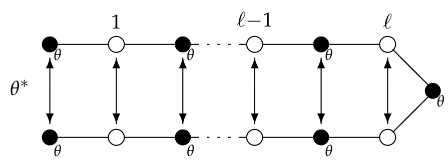
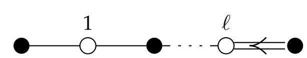
$A_{4\ell+1}^{2\ell+1,4\ell+1}(III_b, I, \epsilon_0)$		$C_{2\ell+1}$		$C_{2\ell+1}$	$C_{2\ell+1}$
$A_{4\ell-1}^{2\ell-1,2\ell}(II, III_b, \epsilon_\ell)$		$A_{2\ell-1}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$A_{4\ell+1}^{2\ell,2\ell+1}(II, III_b)$		$A_{2\ell}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$A_{4\ell-1}^{2\ell,2\ell-1}(III_b, II, \epsilon_\ell)$		$C_{2\ell}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$A_{4\ell+1}^{2\ell+1,2\ell}(III_b, II)$		$C_{2\ell+1}$		$\ell \cdot A_1$	$\ell \cdot A_1$

Table 4.1 – Continued

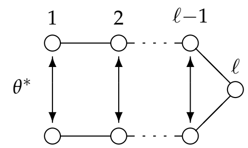
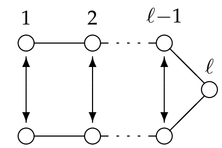
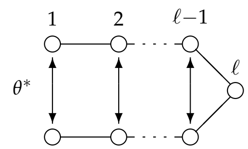
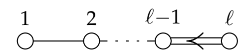
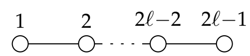
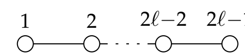
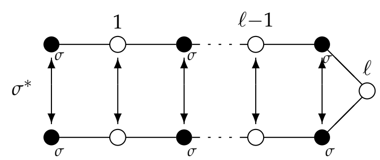
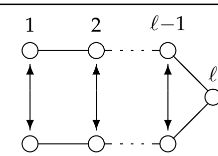
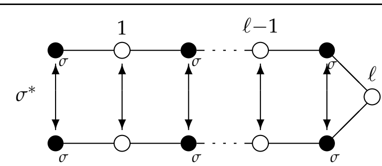
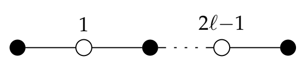
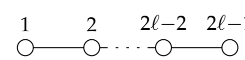
$A_{2\ell-1}^{2\ell-1,\ell}(I, III_b, \epsilon_\ell)$		$A_{2\ell-1}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$A_{2\ell-1}^{\ell,2\ell-1}(III_b, I, \epsilon_\ell)$		C_ℓ		C_ℓ	C_ℓ
$A_{2\ell-1}^{2\ell-1}(I, \epsilon_\ell)$		$A_{2\ell-1}$		$A_{2\ell-1}$	$\ell \cdot A_1$
$A_{4\ell-1}^{2\ell-1,2\ell}(II, III_b, \epsilon_0)$		$A_{2\ell-1}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$A_{4\ell-1}^{2\ell,2\ell-1}(III_b, II, \epsilon_0)$		$C_{2\ell}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$A_{4\ell-1}^{2\ell-1}(II, \epsilon_\ell)$		$A_{2\ell-1}$		$A_{2\ell-1}$	$\ell \cdot A_1$

Table 4.1 – Continued

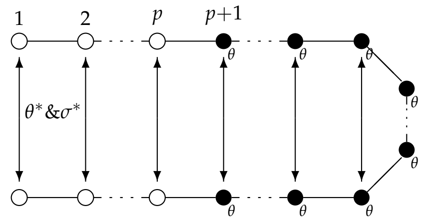
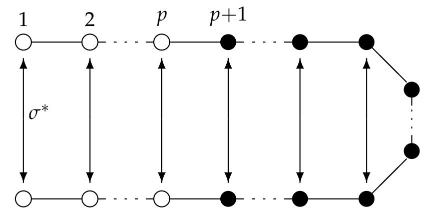
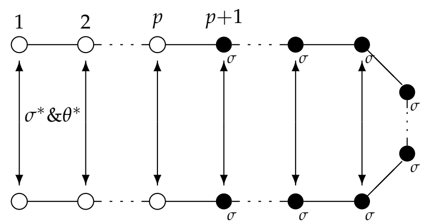
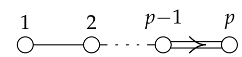
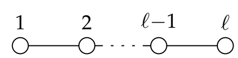
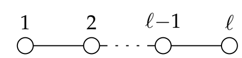
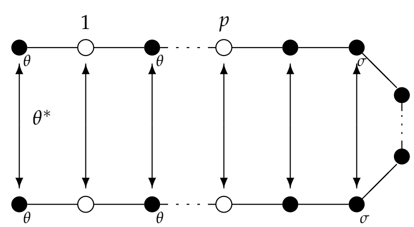
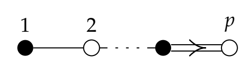
$A_\ell^{\ell,p}(I, III_a)$ $1 \leq 2p \leq \ell$		A_ℓ		$p \cdot A_1$	$p \cdot A_1$
$A_\ell^{p,\ell}(III_a, I)$ $1 \leq 2p \leq \ell$		BC_p		BC_p	BC_p
$A_\ell^{\ell}(I, \epsilon_p)$ $1 \leq 2p \leq \ell$		A_ℓ		A_ℓ	$\lfloor \frac{\ell+1}{2} \rfloor \cdot A_1$
$A_{4\ell-1}^{2p,2\ell-1}(III_a, II)$ $1 \leq 4p \leq 4\ell$		BC_{2p}		$\emptyset$	identity

Table 4.1 – Continued

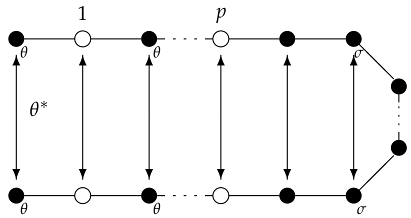
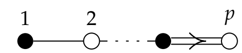
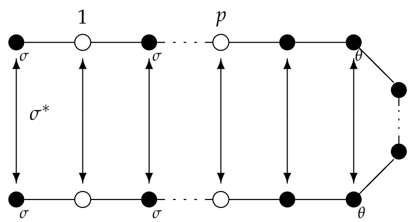
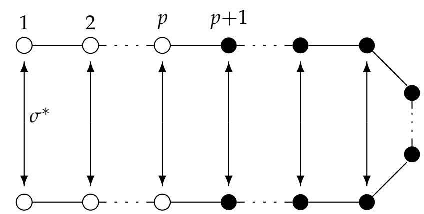
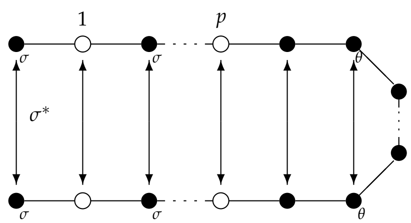
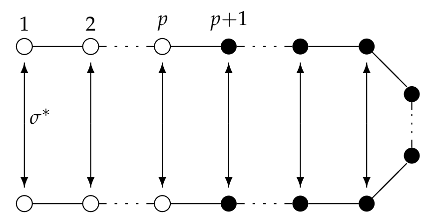
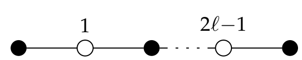
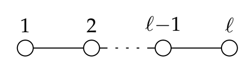
$A_{4\ell+1}^{2p,2\ell}(III_a, II)$ $1 \leq 4p \leq 4\ell$		BC_{2p}		$\emptyset$	identity
$A_{4\ell-1}^{2\ell-1,2p}(II, III_a)$ $1 \leq 4p \leq 4\ell+2$		$A_{2\ell-1}$		$p \cdot A_1$	$p \cdot A_1$
$A_{4\ell+1}^{2\ell,2p}(II, III_a)$ $1 \leq 4p \leq 4\ell+2$		$A_{2\ell}$		$p \cdot A_1$	$p \cdot A_1$
$A_{2\ell+1}^{\ell}(II, \epsilon_p)$ $1 \leq 2p \leq \ell+1$		A_ℓ		A_ℓ	$\lfloor \frac{\ell+1}{2} \rfloor \cdot A_1$

Table 4.1 – Continued

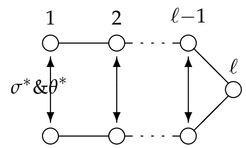
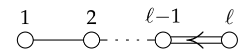
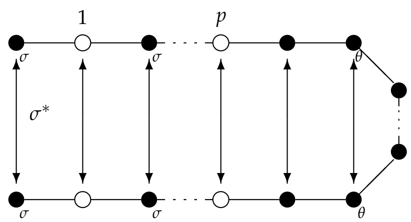
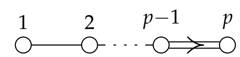
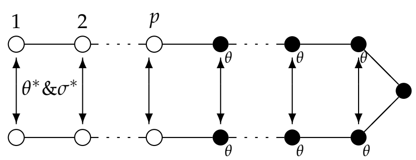
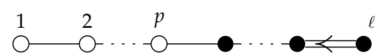
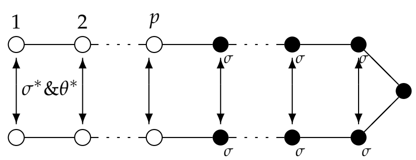
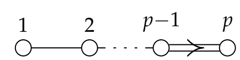
$A^{\ell,\ell}_{2\ell-1}(III_b, \epsilon_\ell)$		C_ℓ		C_ℓ	C_ℓ
$A^{p,p}_{\ell}(III_a, \epsilon_p)$ $1 \leq p < \ell$ $0 \leq i \leq p-1$		BC_p		BC_p	BC_p
$A^{\ell,p}_{2\ell-1}(III_b, III_a, \epsilon_i)$ $1 \leq p < \ell$ $0 \leq i \leq p-1$		C_ℓ		C_p	C_p
$A^{p,\ell}_{2\ell-1}(III_a, III_b, \epsilon_i)$ $1 \leq p < \ell$ $0 \leq i \leq p-1$		BC_p		BC_p	BC_p

Table 4.1 – Continued

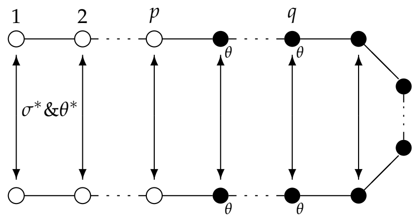
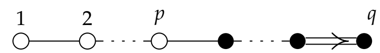
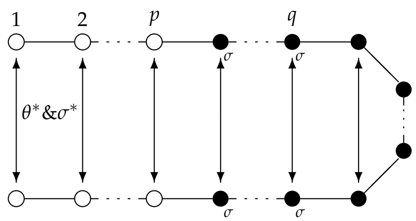
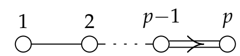
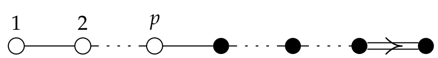
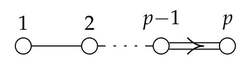
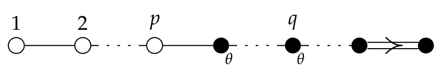
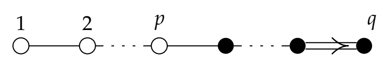
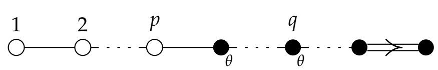
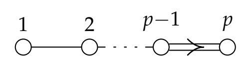
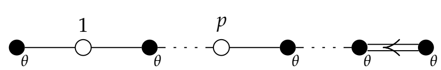
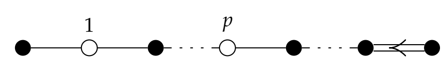
$A_\ell^{q,p}(III_a, III_a, \epsilon_i)$ $1 \leq p < q \leq \frac{1}{2}(\ell+1)$ $0 \leq i \leq p$		BC_q		BC_p	BC_p
$A_\ell^{p,q}(III_a, III_a, \epsilon_i)$ $1 \leq p < q \leq \frac{1}{2}(\ell+1)$ $0 \leq i \leq p$		BC_p		BC_p	BC_p
$B_\ell^{p,p}(I_a, \epsilon_i)$		B_p		B_p	B_p
$B_\ell^{q,p}(I_a, I_a, \epsilon_i)$ $1 \leq p < q \leq \ell$ $0 \leq i \leq p$		B_q		B_p	B_p
$B_\ell^{p,q}(I_a, I_a, \epsilon_i)$		B_p		B_p	B_p
$C_\ell^{\ell,p}(I, II_a)$ $(2p \leq \ell)$		C_ℓ		$p \cdot A_1$	$p \cdot A_p$

Table 4.1 – Continued

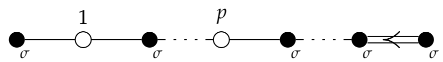
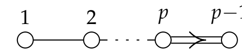
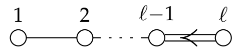
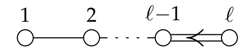
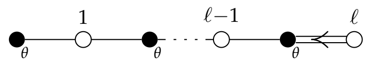
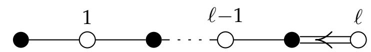
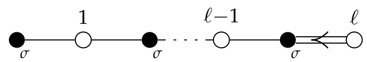
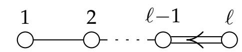
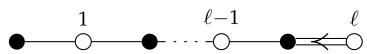
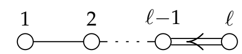
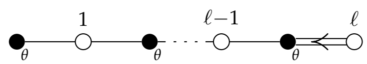
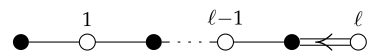
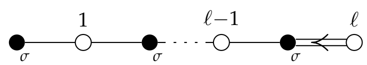
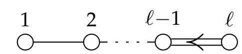
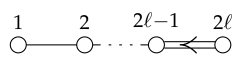
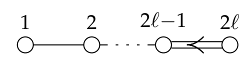
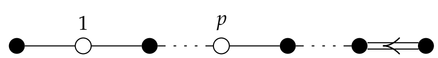
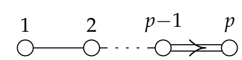
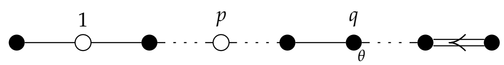
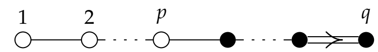
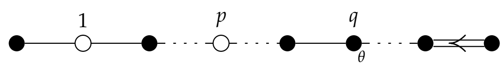
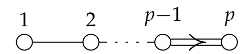
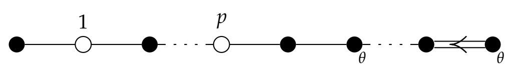
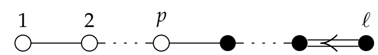
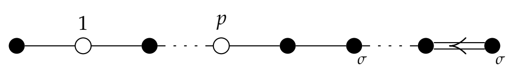
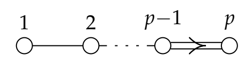
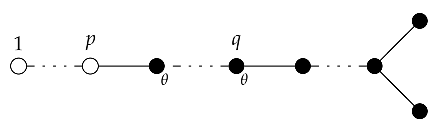
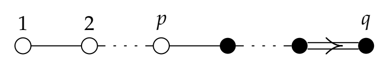
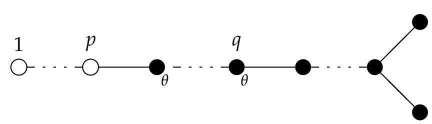
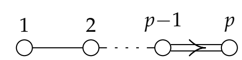
$C_{\ell}^{p,\ell}(II_a, I)$ $(2p \leq \ell)$		BC_p		BC_p	BC_p
$C_{\ell}^{\ell}(I, \epsilon_p)$ $(2p \leq \ell)$		C_{ℓ}		C_{ℓ}	C_{ℓ}
$C_{2\ell}^{2\ell,\ell}(I, II_b, \epsilon_0)$		$C_{2\ell}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$C_{2\ell}^{\ell,2\ell}(II_b, I, \epsilon_0)$		C_{ℓ}		C_{ℓ}	C_{ℓ}
$C_{2\ell}^{\ell,\ell}(II_b, \epsilon_{\ell})$		C_{ℓ}		C_{ℓ}	C_{ℓ}
$C_{2\ell}^{2\ell,\ell}(I, II_b, \epsilon_{\ell})$		$C_{2\ell}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$C_{2\ell}^{\ell,2\ell}(II_b, I, \epsilon_{\ell})$		C_{ℓ}		C_{ℓ}	C_{ℓ}
$C_{2\ell}^{2\ell,2\ell}(I, \epsilon_{\ell})$		$C_{2\ell}$		$C_{2\ell}$	$C_{2\ell}$
$C_{\ell}^{p,p}(II_a, \epsilon_p)$		BC_p		BC_p	BC_p

Table 4.1 – Continued

$C_{\ell}^{q,p}(II_a, II_a, \epsilon_i)$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$		BC_q		BC_p	BC_p
$C_{\ell}^{p,q}(II_a, II_a, \epsilon_i)$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$		BC_p		BC_p	BC_p
$C_{2\ell}^{\ell,p}(II_b, II_a, \epsilon_i)$ $0 \leq i \leq p-1$ $1 \leq p \leq l$		C_{ℓ}		C_p	C_p
$C_{2\ell}^{p,\ell}(II_a, II_b, \epsilon_i)$ $0 \leq i \leq p-1$ $1 \leq p \leq l$		BC_p		BC_p	BC_p
$D_{\ell}^{q,p}(I_a, I_a, \epsilon_i)$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p-1)$		B_q		B_p	B_p
$D_{\ell}^{p,q}(I_a, I_a, \epsilon_i)$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p-1)$		B_p		B_p	B_p

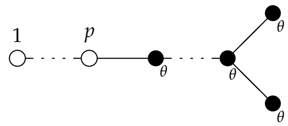
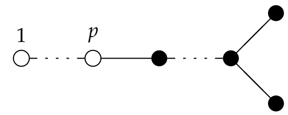
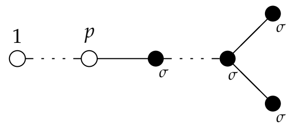
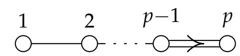
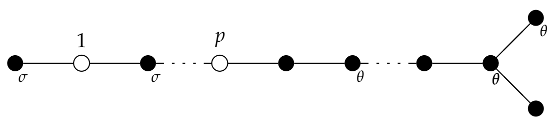
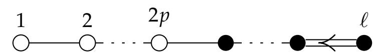
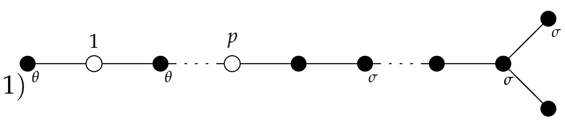
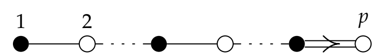
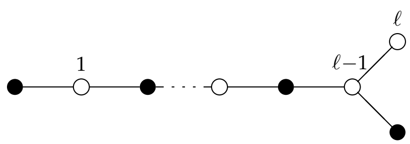
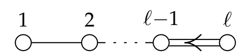
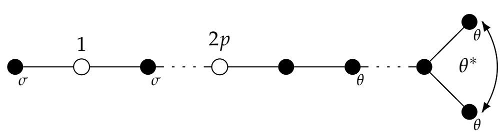
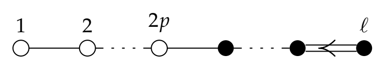
$D_\ell^{\ell,p}(I_b, I_a, \epsilon_i)$ $0 \le i \le p-1$ $1 \le p < \ell$		D_ℓ		D_p	D_p
$D_\ell^{p,\ell}(I_a, I_b, \epsilon_i)$ $0 \le i \le p-1$ $0 \le p < \ell$		B_p		B_p	B_p
$D_{2\ell}^{\ell,4p}(III_a, I_a)$ $(1 \le 2p \le \ell-1)$		C_ℓ		C_{2p}	C_{2p}
$D_{2\ell}^{2p,\ell}(I_a, III_a)$ $(1 \le 2p \le 2\ell-1)$		B_{2p}		$p \cdot A_1$	$p \cdot A_1$
$D_{2\ell}^{\ell,\ell}(III_a, \epsilon_p)$ $(1 \le 2p \le \ell-1)$		C_ℓ		C_ℓ	C_ℓ
$D_{2\ell+1}^{\ell,4p}(III_b, I_a)$ $(1 \le 2p \le \ell)$		BC_ℓ		BC_{2p}	BC_{2p}

Table 4.1 – Continued

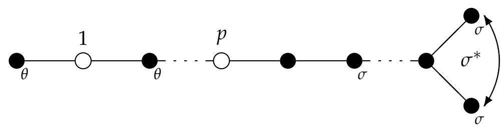
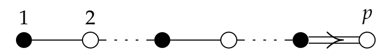
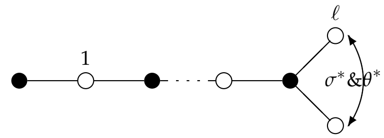
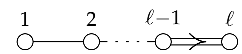
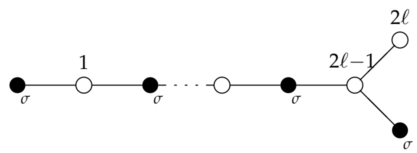
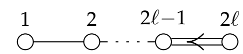
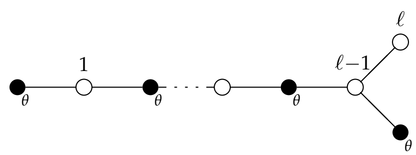
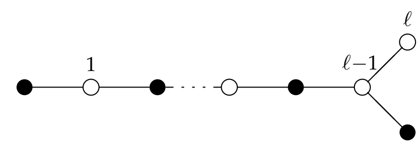
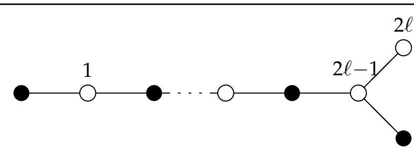
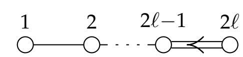
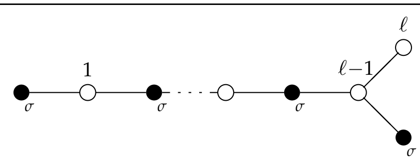
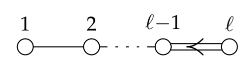
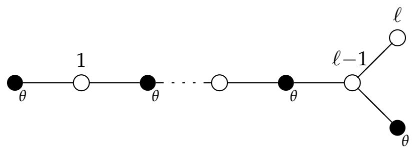
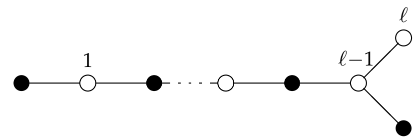
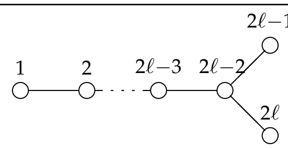
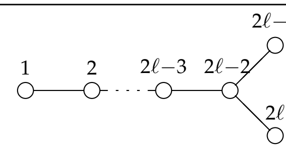
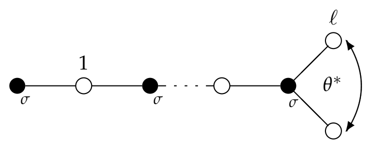
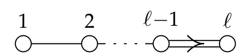
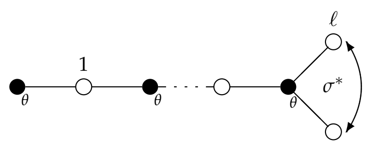
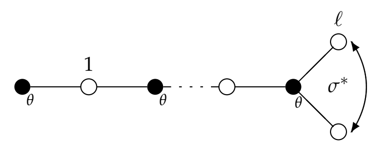
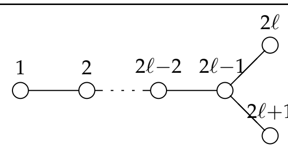
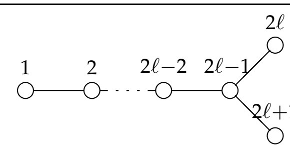
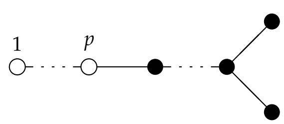
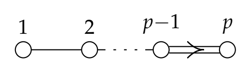
$D^{2p,\ell}_{2\ell+1}(I_a, III_b)$ $(1 \leq 2p \leq 2\ell)$		B_{2p}		$p \cdot A_1$	$p \cdot A_1$
$D^{\ell,\ell}_{2\ell+1}(III_b, \epsilon_p)$ $(1 \leq 2p \leq \ell)$		BC_ℓ		BC_ℓ	BC_ℓ
$D^{2\ell,4\ell}_{4\ell}(III_a, I_b, \epsilon_0)$		$C_{2\ell}$		$C_{2\ell}$	$C_{2\ell}$
$D^{2\ell,\ell}_{2\ell}(I_b, III_a, \epsilon_0)$		$D_{2\ell}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$D^{2\ell,2\ell}_{4\ell}(III_a, \epsilon_\ell)$		$C_{2\ell}$		$C_{2\ell}$	$C_{2\ell}$
$D^{\ell,2\ell}_{2\ell}(III_a, I_b, \epsilon_\ell)$		C_ℓ		C_ℓ	C_ℓ

Table 4.1 – Continued

$D_{2\ell}^{2\ell,\ell}(I_b, III_a, \epsilon_\ell)$		$D_{2\ell}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$D_{2\ell}^{2\ell,2\ell}(I_b, \epsilon_\ell)$		$D_{2\ell}$		$D_{2\ell}$	$D_{2\ell}$
$D_{2\ell+1}^{\ell,2\ell+1}(III_b, I_b)$		BC_ℓ		BC_ℓ	BC_ℓ
$D_{2\ell+1}^{2\ell+1,\ell}(I_b, III_b)$		$D_{2\ell+1}$		$\ell \cdot A_1$	$\ell \cdot A_1$
$D_{2\ell+1}^{2\ell+1,2\ell+1}(I_b, \epsilon_\ell)$		D_{2l+1}		$D_{2\ell+1}$	$D_{2\ell+1}$
$D_{\ell}^{p,p}(I_a, \epsilon_p)$		B_p		B_p	B_p

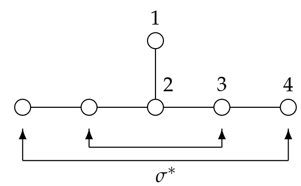
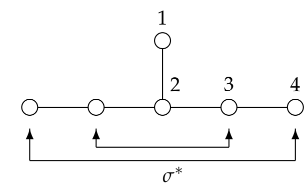
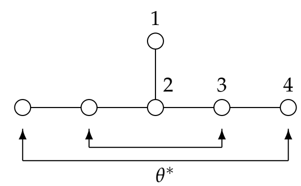
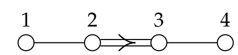
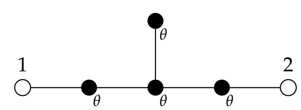
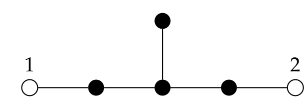
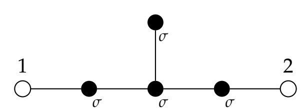
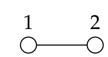
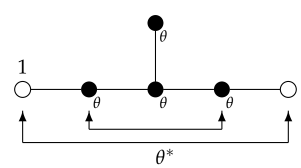
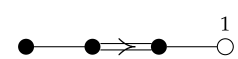
$E_6^{6,4}(I, II, \epsilon_0)$		E_6		D_4	D_4
$E_6^{4,6}(II, I, \epsilon_0)$		F_4		F_4	F_4
$E_6^{6,2}(I, IV)$		E_6		$\emptyset$	id
$E_6^{2,6}(IV, I)$		A_2		A_2	A_1
$E_6^{4,2}(II, IV)$		F_4		A_1	A_1

Table 4.1 – Continued

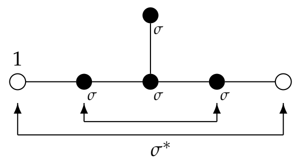
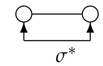
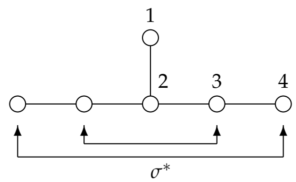
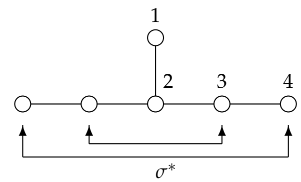
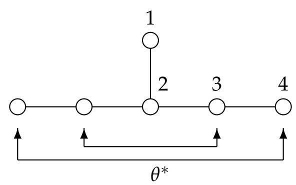
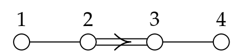
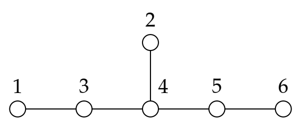
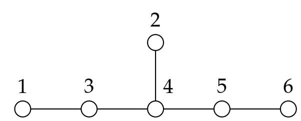
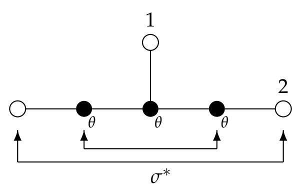
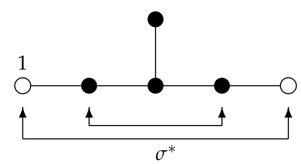
$E_6^{2,4}(IV, II)$		A_2		A_1	A_1
$E_6^{6,4}(I, II, \epsilon_1)$		E_6		D_4	D_4
$E_6^{4,6}(II, I, \epsilon_1)$		F_4		F_4	F_4
$E_6^{6,6}(I, \epsilon_i)$		E_6		E_6	D_4
$E_6^{6,2}(I, III)$		E_6		$2 \cdot A_1$	$2 \cdot A_1$

Table 4.1 – Continued

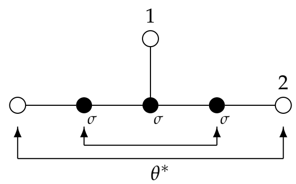
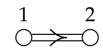
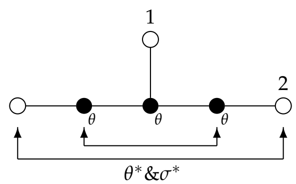
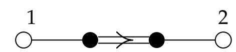
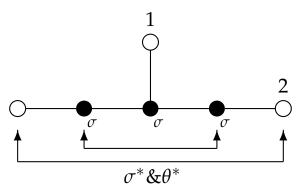
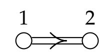
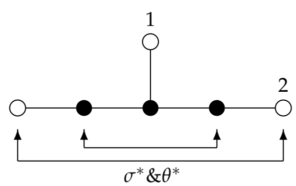
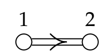
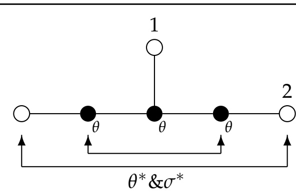
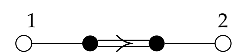
$E_6^{2,6}(III, I)$		BC_2		BC_2	BC_2
$E_6^{4,2}(II, III, \epsilon_0)$		F_4		$2 \cdot A_1$	$2 \cdot A_1$
$E_6^{2,4}(III, II, \epsilon_0)$		BC_2		BC_2	BC_2
$E_6^{2,2}(III, \epsilon_i)$		BC_2		BC_2	BC_2
$E_6^{4,2}(II, III, \epsilon_1)$		F_4		C_2	C_2

Table 4.1 – Continued

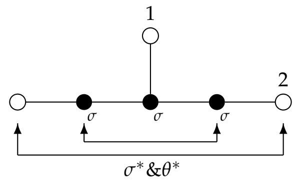
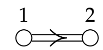
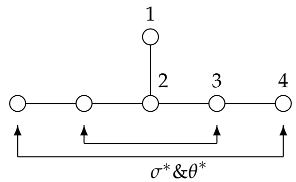
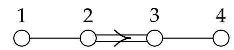
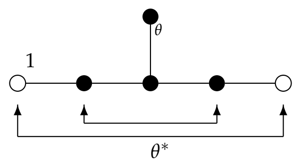
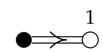
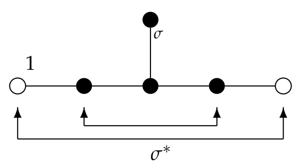
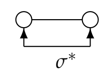
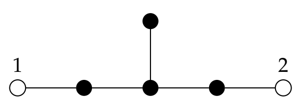
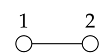
$E_6^{2,4}(III, II, \epsilon_i)$		BC_2		B_2	B_2
$E_6^{4,4}(II, \epsilon_i)$		F_4		F_4	F_4
$E_6^{2,2}(III, IV)$		BC_2		A_1	A_1
$E_6^{2,2}(IV, III)$		A_2		A_1	A_1
$E_6^{2,2}(IV, \epsilon_1)$		A_2		A_2	A_1

Table 4.1 – Continued

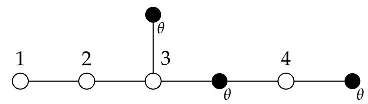
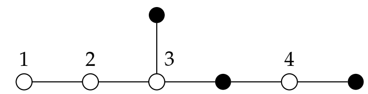
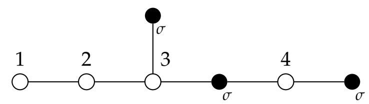
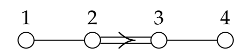
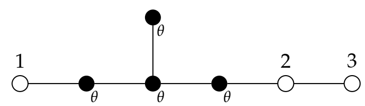
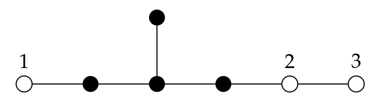
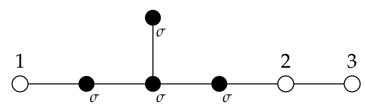
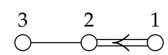
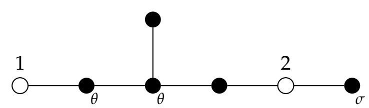
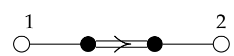
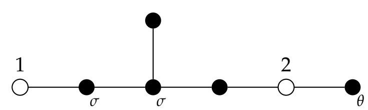
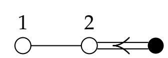
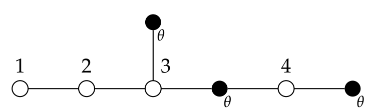
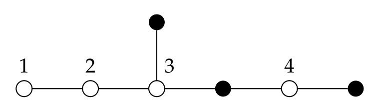
$E_7^{7,4}(V, VI, \epsilon_0)$		E_7		F_4	F_4
$E_7^{4,7}(VI, V, \epsilon_0)$		F_4		F_4	F_4
$E_7^{7,3}(V, VII)$		E_7		C_3	C_3
$E_7^{3,7}(VII, V)$		C_3		C_3	C_3
$E_7^{4,3}(VI, VII, \epsilon_1)$		F_4		C_2	C_2
$E_7^{3,4}(VII, VI, \epsilon_1)$		C_3		C_2	C_2
$E_7^{7,4}(V, VI, \epsilon_1)$		E_7		F_4	F_4

Table 4.1 – Continued

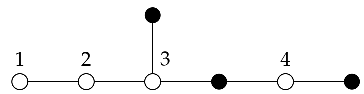
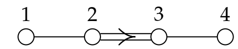
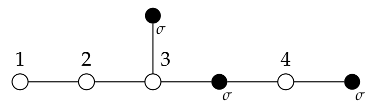
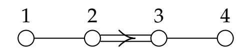
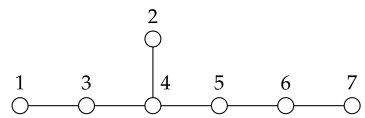
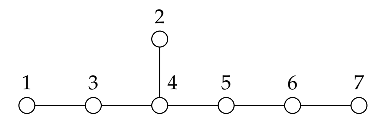
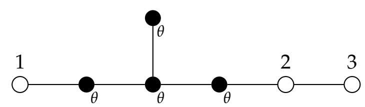
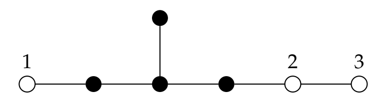
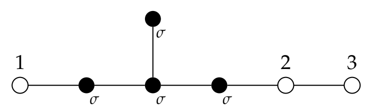
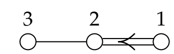
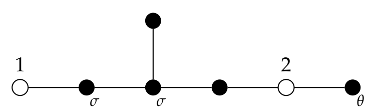
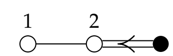
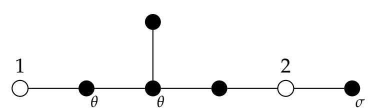
$E_7^{4,4}(VI, \epsilon_i) i = 1, 4$		F_4		F_4	F_4
$E_7^{4,7}(VI, V, \epsilon_1)$		F_4		F_4	F_4
$E_7^{7,7}(V, \epsilon_i), i = 1, 2, 7$		E_7		E_7	E_7
$E_7^{7,3}(V, VII, \epsilon_1)$		E_7		C_3	C_3
$E_7^{3,7}(VII, V, \epsilon_1)$		C_3		C_3	C_3
$E_7^{3,4}(VII, VI)$		C_3		C_2	C_2
$E_7^{4,3}(VI, VII)$		F_4	C_2	C_2	C_2

Table 4.1 – Continued

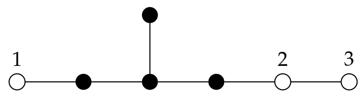
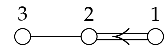
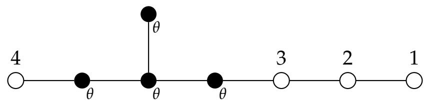
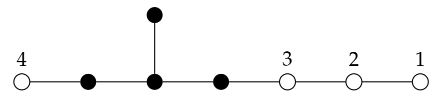
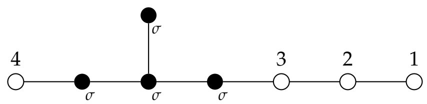
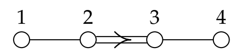
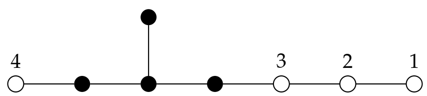
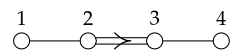
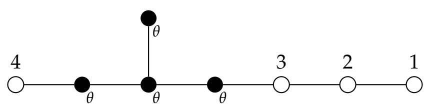
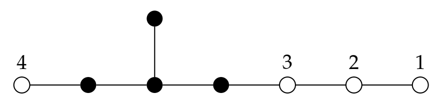
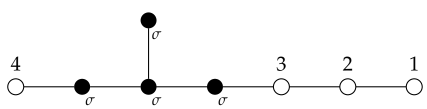
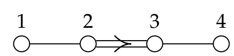
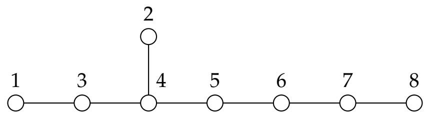
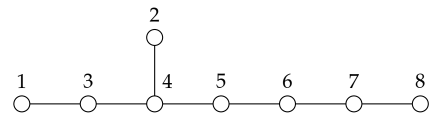
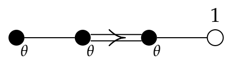
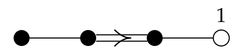
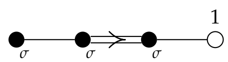
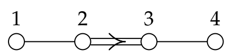
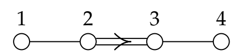
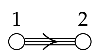
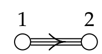
$E_7^{3,3}(VII, \epsilon_3)$		C_3		C_3	C_3
$E_8^{8,4}(VIII, IX, \epsilon_0)$		E_8		D_4	D_4
$E_8^{4,8}(IX, VIII, \epsilon_0)$		F_4		F_4	F_4
$E_8^{4,4}(IX, \epsilon_i), i = 1, 4$		F_4		F_4	F_4
$E_8^{8,4}(VIII, IX, \epsilon_1)$		E_8		D_4	D_4
$E_8^{4,8}(IX, VIII, \epsilon_1)$		F_4		F_4	F_4

Table 4.1 – Continued

$E_8^{8,8}(VIII, \epsilon_8)$		E_8		E_8	E_8
$F_4^{4,1}(I, II)$		F_4		A_1	A_1
$F_4^{1,4}(II, I)$		BC_1		BC_1	A_1
$F_4^{4,4}(I, \epsilon_i), i = 1, 4$		F_4		F_4	F_4
$G_2^{2,2}(I, \epsilon_1)$		G_2		G_2	G_2

4.6 Characterization of Parabolic Subsets

Recall from Subsection 3.4.1 that conjugacy classes of involutions $w \in W(\mathfrak{a})$ with $E(w, -1) \subset E(\theta, -1)$ and $w\theta = \theta w$ determine subsets of a basis $\Delta_1 \subset \Phi(w)$. These subsets are called parabolic subsets. Recall from Lemma 3.4.10, we have a foundation in considering these subsets of the basis $\Delta_0(-w)$ for some involution $w \in W$.

To continue in the classification of conjugacy classes in the Weyl group and the discussion of parabolic subsets, we must consider orthogonal complements of the basis $\Delta_0(-w)$. Let $\Upsilon \subset \Phi$ be a subset. Recall that $(\cdot, \cdot)$ is the symmetric bilinear form on $\mathbb{E}$ and is also Aut(Φ)-invariant. Define the orthogonal subset of Υ as $\Upsilon^{\perp} = \{\alpha \in \Phi \mid (\alpha, \beta) = 0 \text{ for all } \beta \in \Upsilon\}$

Let $R(\mathfrak{t}) = \text{span}_{\mathbb{Z}}\{\Phi(\mathfrak{t})\}$ for some torus $\mathfrak{t}$ of $\mathfrak{g}$. Then $R(\mathfrak{t})$ is considered to be the root lattice of $\Phi(\mathfrak{t})$. With this, we can realize the Euclidean space $\mathbb{E}$ as $\mathbb{E} = R(\mathfrak{t}) \otimes_{\mathbb{Z}} \mathbb{R}$. Likewise, $\Phi^{\perp} = \{\alpha \in \mathbb{E} \mid (\alpha, \beta) = 0 \text{ for all } \beta \in \mathbb{E}\}$.

Lemma 4.6.1. $\Upsilon^{\perp}$ *is closed subsystem of* Φ.

Proof. Let $\alpha, \beta \in \Upsilon^{\perp}$. Then $(\alpha, \gamma) = (\beta, \gamma) = 0$ for all $\gamma \in \Phi$ Since $(\cdot, \cdot)$ is bilinear, $(\alpha + \beta, \gamma) = 0$. This proves the statement. □

Lemma 4.6.2. *Let $w \in W$ be an involution and $\Delta_0(-w)$ be a basis of $\Phi_0(-w)$. Then*

1. $\Phi_0(-w)^{\perp} = \Delta_0(-w)^{\perp}$
2. $\Phi_0(-w)^{\perp} = \Phi_0(w)$

Proof. For statement (1), recall 4.4.1. Then $\Delta_0(-w) = \Delta \cap \Phi_0(-w)$. Hence $\Delta_0(-w)^{\perp} = (\Delta \cap \Phi_0(-w))^{\perp} = \Delta^{\perp} \cap \Phi_0(-w)^{\perp} = \Phi_0(-w)^{\perp}$.
For statement (2), note that $\Phi_0(-w) \subset E(w, -1)$ and $\Phi_0(w) \subset E(w, 1)$. From **??**, $\Phi_0(-w)^{\perp} \subset E(w, 1)$. So for $\alpha \in \Phi_0(-w)^{\perp}$, $w(\alpha) = \alpha$ and $\Phi_0(-w)^{\perp} \subset \Phi_0(w)$. Now let $\alpha \in \Phi_0(w)$ and $\beta \in \Phi_0(-w)$. Then $(\alpha, \beta) = (w(\alpha), w(\beta)) = (\alpha, -\beta) = -(\alpha, \beta)$. Thus, $(\alpha, \beta) = 0$. This proves the statment. □

Lemma 4.6.3 ([26]). *Let Φ be irreducible and Δ a basis of Φ. Also, let $w \in W$ be w Δ-standard involution and Δ_1 be an irreducible component of $\Delta_0(-w)$. Then $\Phi(\Delta_1^{\perp})$ is given in Table* **??**.

Note that in Table **??**, the subsystems A_{-1}, A_0 and D_1 are considered to be an empty subsystem of Φ.

Table 4.2 Table of Irreducible Components of $\Delta_0(-w)$

Type Φ	Type Δ_1	Type $\Delta_1^\perp$	Type Φ	Type Δ_1	Type $\Delta_1^\perp$
A_ℓ	A_1	$A_{\ell-2}$	E_8	A_1	E_7
$(\ell \geq 1)$				D_4	D_4
B_ℓ	A_1	$A_1 + B_{\ell-2}$		D_6	$2 \cdot A_1$
$(\ell \geq 2)$	B_k $(k < \ell)$	$B_{\ell-k}$		E_7	A_1
C_ℓ	A_1	$A_1 + C_{\ell-2}$		E_8	$\emptyset$
$(\ell \geq 2)$	C_k $(k < \ell)$	$C_{\ell-k}$	F_4	A_1 $(short)$	B_3
D_ℓ	A_1	$A_1 + D_{\ell-2}$		A_1 $(long)$	C_3
$(\ell \geq 4)$	D_{2k} $(2k < \ell)$	$D_{\ell-2k}$		B_2	B_2
E_6	A_1	A_5		B_3	A_1 $(short)$
	D_4	$\emptyset$		C_3	A_1 $(long)$
E_7	A_1	D_6		F_4	$\emptyset$
	D_4	$3 \cdot A_1$	G_2	A_1 $(short)$	A_1 $(long)$
	D_6	A_1		A_1 $(long)$	A_1 $(short)$
	E_7	$\emptyset$		G_2	$\emptyset$

Lemma 4.6.4 ([26],7.9)**.** *Let Φ be irreducible and $w \in W$ an involution. Then $\Phi_0(-w)^\perp$ is of type $r \cdot A_1 + X_\ell$ where $X_\ell = \emptyset$ or X_ℓ is irreducible.*

Table 4.3 Root System Types for $w \in W(\Phi)$

Type Φ	Type $\Delta(w)$		Type $\Delta(w)^{\perp}$
$A_n (n \geq 2)$	$r \cdot A_1$	$0 < 2r \leq n+1$	A_{n-2r}
$B_n (n \geq 2)$	$r \cdot A_1 + B_k$	$0 < k + 2r \leq n$	$r \cdot A_1 + B_{n-k-2r}$
$C_n (n \geq 2)$	$r \cdot A_1 + C_k$	$0 < k + 2r \leq n$	$r \cdot A_1 + C_{n-k-2r}$
$D_n (n \geq 4)$	$r \cdot A_1 + D_{2k}$	$0 < 2k + 2r \leq n$	$r \cdot A_1 + D_{n-2k-2r}$
E_6	$r \cdot A_1$	$0 < r \leq 3$	$(4-r) \cdot A_1$
	D_4		$\emptyset$
E_7	A_1		D_6
	$2 \cdot A_1$		$A_1 + D_4$
	$3 \cdot A_1$		D_4
			$4 \cdot A_1$
	$4 \cdot A_1$		$3 \cdot A_1$
	D_4		$3 \cdot A_1$
	$A_1 + D_4$		$2 \cdot A_1$
	D_6		A_1
	E_7		$\emptyset$
E_8	A_1		E_7
	$2 \cdot A_1$		D_6
	$3 \cdot A_1$		$A_1 + D_4$
	$4 \cdot A_1$		$4 \cdot A_1$
	D_4		D_4
	$A_1 + D_4$		$3 \cdot A_1$
	D_6		$2 \cdot A_1$
	E_7		A_1
	E_8		$\emptyset$
F_4	A_1 (short)		B_3
	A_1 (long)		C_3
	$2 \cdot A_1$		$2 \cdot A_1$
	B_2		B_2
	B_3		A_1 (short)
	C_3		A_1 (long)
	F_4		$\emptyset$
G_2	A_1 (short)		A_1 (long)
	A_1 (long)		A_1 (short)
	G_2		$\emptyset$

CHAPTER

5

ACTIONS OF OPPOSING INVOLUTIONS ON ROOT SYSTEMS

For this chapter we will consider $\theta \in \mathrm{Aut}(\Phi)$ where $\theta^2 = id$. We denote the opposite involution of θ as $-id \cdot \theta$. Helminck has provided characterizations of parabolic subgroups dependent on $w \in W(\Phi)$ and Δ which is a $(-w)$-basis. We build on this characterizations introduced in Chapters 3 and 4 now consider any involution θ.

We continue the notation from introduced in Section 3.3.1. Further, we provide more results involving parabolic subsets.

Lemma 5.0.1. *Let Φ, $\Phi_0(\pm\theta)$ and $\overline{\Phi}_{\pm\theta}$ be defined as above. Then we have the following.*

1. $\overline{\Phi}_\theta \cap \Phi = \Phi_0(-\theta)$.
2. $\Phi_0(\theta) = \Phi_0(-\theta)^\perp$.

Proof. (1) Let $\alpha \in \overline{\Phi}_\theta$. Then there exists $\beta \in \Phi$ such that $\pi(\beta) = \frac{1}{2}(\beta - \theta(\beta)) = \alpha$. So $-\theta(\alpha) = -\theta(\frac{1}{2}(\beta - \theta(\beta))) = -\frac{1}{2}\theta(\beta) + \frac{1}{2}\beta = \frac{1}{2}(\beta - \theta(\beta)) = \alpha$. Hence $\alpha \in \Phi_0(-\theta)$.
Let $\alpha \in \Phi_0(-\theta)$. Then $-\theta(\alpha) = \alpha$ which implies that $\alpha - \theta(\alpha) = 2\alpha$. Since $\frac{1}{2}(\alpha - \theta(\alpha)) = \alpha$, then $\pi(\alpha) = \alpha$. Thus $\alpha \in \bar{\Phi}_\theta$

(2)This follows from Lemma 4.6.2. $\square$

These relations provide a correspondence between the θ and $-\theta$-diagrams. It will also aid in producing an algorithm in determine one diagram from that of the opposing involution. Moreover, these relations coupled with the Helminck's tables of parabolic subsets help in determine these diagrams. What is left to be determined is whether these diagrams will have a diagram automorphism.

From Theorem 4.3.5, we are given that there does not exist a diagram automorphism when the irreducible components of $\Phi_0(-\theta)$ are of type A_1, B_ℓ, C_ℓ, $D_{2\ell}$ $\;(\ell \geq 2)$, E_7, E_8, F_4, or G_2 for a $(-\theta)$-basis of Φ and $\theta = (-\theta)^* w(\theta)$. Note this result also hold for a θ-basis with θ characterized as $\theta = -id \cdot \theta^* w_0(\theta)$ and the irreducible components of same type are within $\Phi_0(\theta)$. Thus, the focus shifts to consider root systems in which the irreducible component is of type A_ℓ, $D_{2\ell+1}$ and E_6. We shall address these cases individually.

Throughout the subsequent sections, we will use the following result from Humphreys.

Lemma 5.0.2 (Humpreys). *Let Φ be irreducible. Then all roots of the same length are conjugate under $W(\Phi)$. In other words, all roots of the same length create a single orbit under $W(\Phi)$.*

5.1 Parabolic Subsets of Type A_ℓ

Proposition 5.1.1. *Let Φ be irreducible, $\theta \in \mathrm{Aut}(\Phi)$ so that $\theta^2 = id$ and $\Delta_0(\theta) \subset \Delta$ where Δ is a $(-\theta)$-basis. If $\Delta_0(\theta)$ is of the type A_r for $r \geq 2$, then $\theta^* \neq id$.*

Proof. Let $\Delta_0(\theta) = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ where $(\alpha_i, \alpha_j) = 0$ if $|i - j| > 1$ and 0 otherwise. With Δ, we have the characterization $\theta^* = -id \cdot \theta \cdot w_0(\theta)$. From Bourbaki, $w_0(\theta)(\alpha_i) = -\alpha_{r+1-i}$ for all $i \leq r$,[9]. So $\theta^*(\alpha_i) = \alpha_{r+1-i}$ and hence $\theta^*|\Delta_0(\theta) \neq id$. Therefore, $\theta^* \neq id$. □

Proposition 5.1.2. *Let Φ be irreducible, $\theta \in \mathrm{Aut}(\Phi)$ so that $\theta^2 = id$ and $\Delta_0(\theta) \subset \Delta$ where Δ is a $(-\theta)$-basis. If $\Delta_0(\theta)$ is of the type $r \cdot A_1$ for $r \geq 2$, then $\theta^* = id$.*

Proof. With Δ, we have the characterization $\theta^* = -id \cdot \theta \cdot w_0(\theta)$. Let $\Delta_0(\theta) = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ where $(\alpha_i, \alpha_j) = 0$ for all $i \neq j$. It follows that $w_0(\theta) = s_1 s_2 \cdots s_r$ where $s_i = s_{\alpha_i}$. So $\theta^*|\Delta_0(\theta) = id$. Since $\overline{W}_\theta = W(\Phi)/W_0(\Phi)$, $w_0(\theta)|\overline{\Delta}_\theta = id$. Recall $\bar{\Phi}_\theta \subseteq \Phi_0(-\theta)$. Hence $\theta|\overline{\Delta}_\theta = -id$. So $\theta^*|\overline{\Delta}_\theta = id$. This proves the statement. □

Notice from the two previous propositions that we did not address the cases when $r = 0, 1$. Here, we find that both outcomes can occur; i.e. $\theta^* = id$ or $\theta^* \neq id$.

Proposition 5.1.3. *Suppose Φ be irreducible of type A_ℓ, E_6 or $D_{2\ell+1}$ and $\theta \in \mathrm{Aut}(\Phi)$ so that $\theta^2 = id$. Let $\Delta_0(\theta) \subset \Delta$ where Δ is a θ-basis. If $\Delta_0(\theta) = \emptyset$ or A_1, then $\theta^* = id$ if and only if $(-\theta)^* \neq id$.*

Proof. Suppose $\Delta_0(\theta) = \emptyset$. If $\theta^* = id$, then $\theta = -id$ and $w_0(\theta) = id$. So $\Delta_0(-\theta)$ is of type A_ℓ, $D_{2\ell+1}$ or E_6. In any of these cases, $w_0(-\theta) \neq id$. Hence $(-\theta)^* = \theta w_0(-\theta) \neq id$. Since these statements are equivalent, the converse holds.

Now suppose $\Delta_0(\theta) = A_1$. If $\theta^* = id$, then $\theta = -w_0(\theta) = -s_\alpha$ for $\Delta_0(\theta) = \{\alpha\}$. If Φ is of type A_ℓ, $\Delta_0(-\theta)$ is of type $A_{\ell-2}$. Hence $w_0(-\theta) \neq id$ and therefore, $(-\theta)^* = \theta w_(- \theta) = -s_\alpha w_0(-\theta) \neq id$. Since these statements are equivalent, the converse holds.

□

Lemma 5.1.4. *Suppose Φ is of type $A_{2\ell}$ and $\Delta \subset \Phi$ a θ-basis. Let $\theta \in \mathrm{Aut}(\Phi)$ so that $\theta^2 = id$ and $\Delta_0(\theta)$ is of type A_1. Then $\theta^* = id$.*

Proof. This statement follows from the fact that $\theta^* \in \mathrm{Aut}(X, \Phi, \Delta, \Delta_0(\theta))$. □

Theorem 5.1.5. *Let Φ be of type A_ℓ, $\theta \in Aut(\Phi)$ an involution and Δ be a $(-\theta)$-basis. Then the following are equivalent.*

(i) $\Delta_0(\theta)$ *is of type* $r \cdot A_1$.

(ii) $\Delta_0(-\theta)$ *is of type* $A_{\ell-2r}$.

(iii) $\theta^* = id$

(iv) $(-\theta)^* \neq id$

Proof. The result follows from Table **??** and Propositions 5.1.1 and 5.1.2. □

5.2 Parabolic Subsets of Type $D_{2\ell+1}$

Let Δ be the basis of the root system of type D_ℓ. Then $\Delta = \{\alpha_1, \ldots, \alpha_{\ell-1}, \alpha_\ell\}$. Denote the subset $\{\alpha_{\ell-1}, \alpha_\ell\}$ as D_2.

Lemma 5.2.1. *Let Φ be of type D_n $(n \geq 5)$ and $\theta \in \mathrm{Aut}(\Phi)$ be an involution. If $\Delta_0(\theta)$ is of type D_2, then $\Delta_0(\theta)$ is not $W(\Phi)$-conjugate to any subsystem of type $2 \cdot A_1$.*

Before the proof of this Lemma, we make the following observation.

Lemma 5.2.2. *Let $\theta_1, \theta_2 \in Aut(\Phi)$ be involutions. Then $\Delta_0(\theta_1)$ and $\Delta_0(\theta_2)$ are $W(\Phi)$-conjugate if and only if $\Delta_0(\theta_1)^\perp$ and $\Delta_0(\theta_2)^\perp$ are $W(\Phi)$-conjugate.*

Proof. This follows from Proposition 4.4.6 and Lemma 5.0.1. □

Lemma 5.2.1. Let $\Delta_0 = \{\alpha_1, \alpha_2\}$ be of type D_2 and $\Delta_0' = \{\beta_1, \beta_2\}$ of type $2 \cdot A_1$. Suppose that they are $W(\Phi)$-conjugate. By Lemma 5.2.2, $\Delta_0^\perp$ and $(\Delta_0')^\perp$ are $W(\Phi)$-conjugate. However from Table **??** , $\Delta_0^\perp$ is of type $D_{\ell-2}$ and $(\Delta_0')^\perp$ is of type $2A_1 + D_{\ell-4}$, which is impossible. □

The previous Lemma and proof are also used in [26] to prove a more general case.

Lemma 5.2.3 ([26],7.15). *Let Φ be irreducible of type D_ℓ ($\ell > 4$), Δ a basis of Φ and let w_1, w_2 be Δ-standard involutions of Φ. If $\Delta_0(-w_1)$ is of type $r \cdot A_1$ and $\Delta_0(-w_2)$ is of type $(r-2) \cdot A_1 + D_2$ ($r \geq 2$), then $\Delta_0(-w_1)$ and $\Delta_0(-w_2)$ are not $\mathrm{Aut}(\Phi)$-conjugate.*

See [26] for more results concerning $\mathrm{Aut}(\Phi)$-conjugacy within Φ of type D_ℓ.

Proposition 5.2.4. *Let Φ be of type D_ℓ and $\theta \in \mathrm{Aut}(\Phi)$ be an involution. Suppose $\Delta_0(\theta)$ is of type $r \cdot A_1$. If there exists $\alpha \in \Delta_0(\theta)$ such that $\alpha \in D_2$, then $\theta^* = id$.*

Proof. Let $\Delta = \{\alpha_1, \ldots, \alpha_{\ell-1}, \alpha_\ell\}$ be the base of D_ℓ. We use induction on r. Suppose $r = 1$. We can assume $\Delta_0(\theta) = \{\alpha_\ell\}$. Since $\theta^* \in \mathrm{Aut}(\Delta_0(\theta))$, $\theta^* = id$. Assume this holds for $\Delta_0(\theta) = \{\beta_1, \beta_2, \ldots, \beta_{r-1}\}$ of type $(r-1) \cdot A_1$ where $\beta_{r-1} = \alpha_\ell$. Let Δ_0' be of type $r \cdot A_1$ and containing $\Delta_0(\theta)$. Then $\theta^*|\Delta_0(\theta) = id$ which implies that $\theta^*|\Delta_0'$.

□

Corollary 5.2.4.1. *Let Φ be of type D_ℓ and $\theta \in \mathrm{Aut}(\Phi)$ be an involution. If $\Delta_0(\theta)$ is of type D_2, then $\theta^* = id$.*

Proposition 5.2.5. *Let Φ be of type $D_{2\ell+1}$ and $\theta \in \mathrm{Aut}(\Phi)$ be an involution. Suppose $\Delta_0(\theta)$ is of type $r \cdot A_1$ and $\Delta_0(\theta) \subset D_{2\ell+1} - D_2$. Then $\theta^* = id$ if and only if $(-\theta)^* \neq id$.*

Proof. If $\theta^* = id$, then $\theta = -w_0(\theta) = -s_1 s_3 \cdots s_{2\ell-1}$. Since $\Delta_0(-\theta)$ is also of type $r \cdot A_1$, $w_0(-\theta)$ is also the product strongly orthogonal reflections. Thus, $(-\theta)^* = \theta w_0(-\theta) = -w_0(\theta) w_0(-\theta) \neq id$. The argument for the converse is the same. This proves the statement. □

Proposition 5.2.6. *Let Φ be of type $D_{2\ell+1}$ and $\theta \in \mathrm{Aut}(\Phi)$ be an involution. Suppose $\Delta_0(\theta)$ is of type D_n ($n < 2\ell + 1$). Then $\theta^* = id$ if and only if $(-\theta)^* \neq id$.*

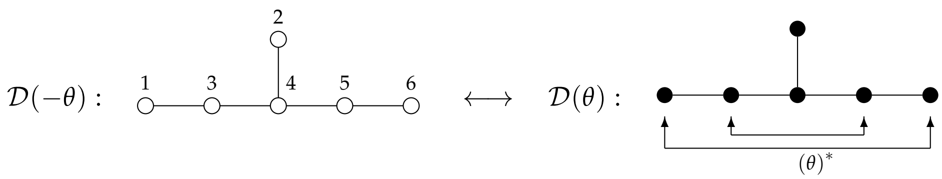
Proof. From Theorem 4.3.5, $(-\theta)^* = id$ if and only if $\Delta_0(-\theta)$ is of type D with even rank if and only if $\Delta_0(\theta)$ is of type D with odd rank if and only if $w_0(\theta)|\Delta_0(\theta) \neq -id$ if and only if $\theta^8 \neq id$. □

5.3 Parabolic Subsets of Type E_6

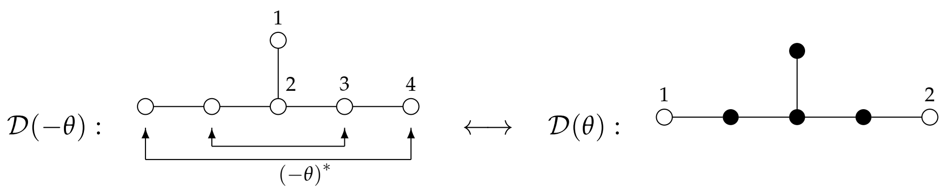
Let Φ be of type E_6. For this section, we take different approach to determine if the diagram automorphisms exist for opposing involutions. As Φ is or finite rank, it is not as tedious to consider each rank of the paraboilic subset individually; i.e. $|\Delta_0| = 0, 1, 2, 3, 4, 5$ and 6.

We start with the usual set up. Let Δ be a $(-\theta)$-basis of Φ and $\theta = (-\theta)^* w_0(-\theta)$. Let $\mathcal{D}(\pm\theta)$ denote the diagram representation for $\pm\theta$.

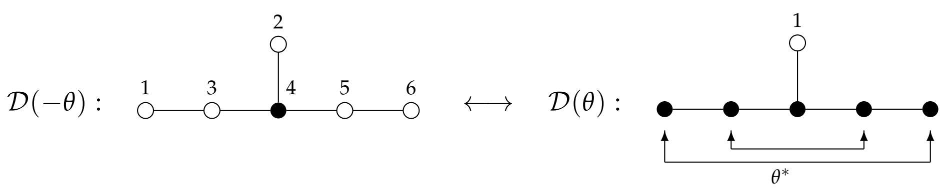
Rank 0,4,6 It follows that $w_0(-\theta) = id$ and $\theta = (-\theta)^*$. By Proposition 5.1.3, if $(-\theta)^* = id$, then $\theta^* \neq id$ and $\theta = id$. Hence $\Delta_0(\theta) = E_6$. The following is a representation of the opposing diagrams. Note this representation also addresses the rank 6 case.



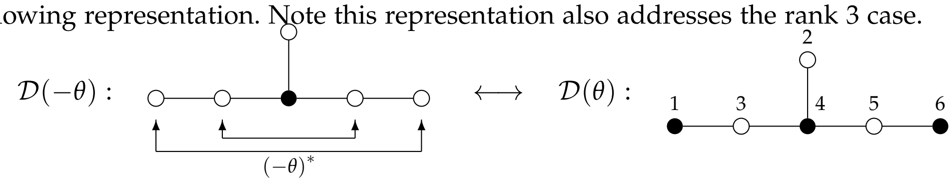
If $(-\theta)^* \neq id$, then $\theta \notin W(\Phi)$. Hence from Table **??**, $\Delta_0(\theta)$ is of type D_4. Then we have the following representation. Note this representation also addresses the rank 4 case.



Rank 1,5 It follows that $w_0(-\theta)$ is a simple reflection and $\theta = (-\theta)^* w_0(\theta)$. By Proposition 5.1.3, if $(-\theta)^* = id$, then $\theta^* \neq id$ and $\theta = s_\alpha \in W(\Phi)$ for $\alpha \in \Delta_0(-\theta)$. Since θ is a Δ-standard involution, $\Delta_0(\theta)$ is of type A_5. Thus we get the following representation. Note this representation also addresses the rank 5 case.



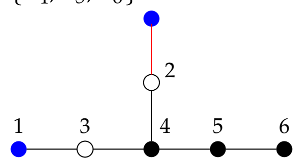
If $(-\theta)^* \neq id$, then $\theta \notin W(\Phi)$. Hence from Table **??**, $\Delta_0(\theta)$ is of type $3 \cdot A_1$. Thus we get the following representation. Note this representation also addresses the rank 3 case.



Rank 2 It follows that $w_0(-\theta)$ is a product of two simple orthogonal reflections and $\theta = (-\theta)^* w_0(-\theta)$. By Theorem 4.3.5, $(-\theta)^* = id$, then $\theta^* \neq id$ and $\theta = w_0(-\theta) \in W(\Phi)$. Due to the condition that $\theta^* \neq id$, Table **??** cannot be applied to determine $\Delta_0(\theta)$; this contradicts Theorem 5.1.5. Luckily, the parabolic subset can be derived from the extended Dynkin diagram of E_6. From Lemma 5.0.2, any two roots can be chosen to represent the two orthogonal reflections; say α_1 and $\tilde{\alpha}$ the highest root. Then the set of roots orthogonal to α_1 and $\tilde{\alpha}$ is $\{\alpha_4, \alpha_5, \alpha_6\}$. This set is of type A_3. See Figure

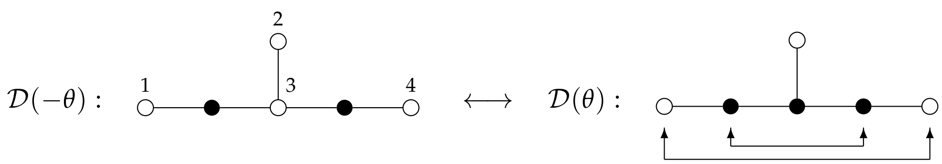
h!

Figure 5.1 E_6 Example: α_1 and $\tilde{\alpha}$ and $\{\alpha_4, \alpha_5, \alpha_6\}$ are fixed



??.

Thus we get the following representation. Note this representation also addresses the rank 3 case.



Theorem 5.3.1. *Let Φ be of type E_6, $\theta \in \operatorname{Aut}(\Phi)$ an involution and Δ be a $(\overset{\theta^*}{-}\theta)$-basis. Then we have the following.*

(i) If $\Delta_0(\theta)$ is of type $r \cdot A_1$, then $\theta^* = id$ if and only $(-\theta)^* \neq id$.

(ii) If $\Delta_0(-\theta)$ is of type A_ℓ, then $\theta^* = id$ if and only $(-\theta)^* \neq id$.

(iii) If $\Delta_0(-\theta)$ is of type D_4, then $\theta^* = id$ if and only $(-\theta)^* \neq id$.

With the conclusion of this section, we are now able to prove the following.

Theorem 5.3.2. *Let Δ_1 and Δ_2 be θ- and $(-\theta)$-bases, respective. Let θ^* and $(-\theta)^*$ be defined as above. If Φ is of type A_ℓ, $D_{2\ell+1}$ and E_6, then $\theta^* = id$ if and only if $(-\theta)^* \neq id$.*

Proof. The result is immediate from Theorem 5.1.5, Propositions 5.2.6, 5.2.5 and this section. □

Corollary 5.3.2.1. *Let Δ_1 and Δ_2 be θ- and $(-\theta)$-bases, respective. Let θ^* and $(-\theta)^*$ be defined as above. Then Φ is of type A_ℓ, $D_{2\ell+1}$ and E_6 if and only if the opposition involution id^* is nontrivial.*

Remark 5.3.3. *If Y_ℓ is of type B_ℓ, C_ℓ, E_7, E_8, F_2 and G_2, then $\theta^* = (-\theta)^* = id$.*

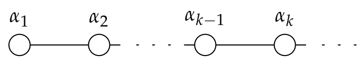
The I-posets have been determined for this case

5.4 Further Results: Parabolic Subsets

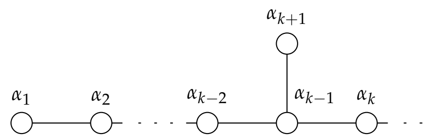
For this we section, we seek to provide further results for parabolic subsets of type $r \cdot A_1 + X_\ell$. By change of notation, we will denote the type of $\Delta_0(\theta_1)$ as $r \cdot A_1^{(1)} + Y_\ell^{(1)}$ and similarly for $\Delta_0(\theta_2)$ where Y_ℓ can be of type A_3 $(r = 0)$, A_5 $(r = 0)$, B_ℓ $(\ell \geq 2)$, C_ℓ $(\ell \geq 2)$, D_ℓ $(\ell \geq 4)$, E_6, E_7, E_8, F_4 and G_2.

But first, we consider the following propositions regarding parabolic subsets of type $r \cdot A_1$.

Lemma 5.4.1 ([26],7.10)**.** *Let Φ be irreducible, Δ a basis of Φ and $w \in W$ a Δ-standard involution. Let $\Delta_1 = \{\alpha\}$ be an irreducible component of $\Delta_0(w)$ of type A_1. Assume that the diagram of Φ contains a sub diagram of the form:*

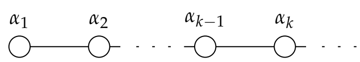


or

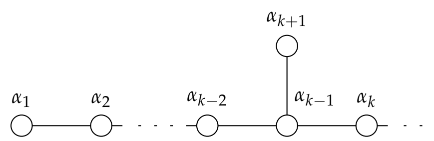


with α_1 an endpoint, $\alpha_k = \alpha$ and $\alpha_{k+1} \notin \Delta(w)$. Then there exists $r \in W$ such that $r(\Delta(w)) \subset \Delta$ and $r(\alpha) = \alpha_1$.

Proposition 5.4.2. *Let Φ be irreducible, Δ a basis of Φ and $w \in W$ a Δ-standard involution. Let $\Delta_1 = \{\beta_1, \beta_2, \ldots, \beta_r\}$ be an irreducible component of $\Delta_0(w)$ of type $r \cdot A_1$. Assume that the diagram of Φ contains a sub diagram of the form:*



or



with α_1 an endpoint, $\alpha_k = \beta_r$ and $\alpha_{k+1} \notin \Delta(w)$. Then there exists $\rho \in W$ such that $\rho(\Delta_0(w)) \subset \Delta$ and $\rho(\Delta_1) = \{\alpha_1, \alpha_3, \ldots, \alpha_{2s+1}\}$ for some s.

Proof. We shall use induction on r. The base step is simply Lemma 5.4.1. Suppose $\Delta_1 = \{\alpha, \beta\}$. Consider the subdiagram only containing α_1 and α where $\alpha = \alpha_k$ and $\alpha_{k+1} \notin \Delta_0(w)$. Then by Lemma 5.4.1, there exists $\rho \in W(\Phi)$ such that $\rho_1(\Delta_0(w)) \subset \Delta$ and $\rho_1(\alpha) = \alpha_1$. Now consider the subdiagram containing α_3 and β where α_3 is the endpoint within subdiagram. Let $\beta = \alpha_{k_1+1}$ with $k_1 > k$. Then $\alpha_{k_1+2} \notin \Delta_0(w)$. So by Lemma 5.4.1, there exists $\rho_2 \in W(\Phi)$ such that $\rho_2(\Delta_0(w)) \subset \Delta$ and $\rho_2(\beta) = \alpha_3$. Continue this process for each subsequent root in Δ_1 in which α_{2s+1} is the endpoint for some s, $\beta_i = \alpha_{k_s}$ and $\alpha_{k_s+1} \notin \Delta_0(w)$. For β_r, the size of the subdiagram should be minimal. If β_r is the other endpoint and $\alpha_{r-2} = \beta_{r-1}$, then we are done. If not, then apply Lemma 5.4.1 to the remaining subdiagram of type A_ℓ. Set $\rho = \rho_r \rho_{r-1} \cdots \rho_2 \rho_1$. This satisfies the two conditions. $\square$

As the proof suggest, Proposition 5.4.2 allows for the rearranging of black nodes. In other words, all parabolic subsets of type $r \cdot A_1$ with $r \neq 2$ are $W(\Phi)$-conjugate. So for $\Delta_1 = r_1 \cdot A_1$ and $\Delta_2 = r_2 \cdot A_2$, we write $w(r_1 \cdot A_1) = r_2 \cdot A_2$ to represent the rearranging of the r_1 black vertices of $r_1 \cdot A_1$ to $r_2 \cdot A_2$.

Proposition 5.4.3. *Let $\theta_1, \theta_2 \in \mathrm{Aut}(\Phi)$ be involutions and $\Delta_0(\theta_1), \Delta_0(\theta_2) \subset \Delta$ be of type $r \cdot A_1 + Y_\ell$ where Y_ℓ is of type B_ℓ, C_ℓ or D_ℓ. If $\Delta_0(\theta_1)$ and $\Delta_0(\theta_2)$ are $W(\Phi)$-conjugate for some $w \in W(\Phi)$, then $w \in W(Y_\ell^{(1)\perp})$.*

Proof. From the assumption $w\left(r \cdot A_1^{(1)}\right) = r \cdot A_1^{(2)}$ and $w\left(Y_\ell^{(1)}\right) = Y_\ell^{(2)}$. So there exists $w_1 \in W\left(Y_\ell^{(2)}\right)$ such that $w_1 w|Y_\ell^{(1)} = id$. Then $w_1\left(Y_\ell^{(2)\perp}\right) = Y_\ell^{(2)\perp} = Y_{n-\ell}^{(2)} \supset A_{n-\ell-1}^{(2)} \supset r \cdot A_1^{(2)}$. So $w_1\left(r \cdot A_1^{(2)}\right) =$

$r \cdot A_1^{(2)}$ and $w_1 w(\Delta_0(\theta_1)) = \Delta_0(\theta_2)$. Hence $w_1 | \left(r \cdot A_1^{(2)}\right) = id$. Note $w(Y_\ell^{(1)}) = w_1 w(Y_\ell^{(1)}) = w_1(Y_\ell^{(2)})$. So $w|Y_\ell^{(1)} = id$. Therefore, $w \in W(Y_\ell^{(1)\perp})$. □

Corollary 5.4.3.1. *Let $\Delta_0(\theta_1)$, $\Delta_0(\theta_2) \subset \Delta$ be of type $r \cdot A_1 + Y_\ell$ where Y_ℓ is of type B_ℓ, C_ℓ or D_ℓ. If $w(\Delta_0(\theta_1)) = \Delta_0(\theta_2)$ for some $w \in W(\Phi)$, then $Y_\ell^{(1)} = Y_\ell^{(2)}$.*

Theorem 5.4.4. *Suppose Δ_1, $\Delta_2 \subset \Phi$ be θ_1 and θ_2-bases, respectively. Let $\Delta_0(\theta_1)$ and $\Delta_0(\theta_2)$ be of the same type $r \cdot A_1 + Y_\ell$. Then the two indices $(X, \Delta_1, \Delta_0(\theta_1), \theta_1^*)$ and $(X, \Delta_2, \Delta_0(\theta_2), \theta_2^*)$ are isomorphic.*

Proof. Note from 4.4.8, it suffices to show that $\Delta_0(\theta_1)$ and $\Delta_0(\theta_2)$ are $W(\Phi)$-semi-isomorphic. Let rank($\Delta_0(\theta_1)$) =rank($\Delta_0(\theta_2)$) $= m$. We show this by considering different values of m. First, we consider the special cases where $m = 0$ and $m = 1$. From the bases, we obtain the characterizations of θ_1 and θ_2 as $\theta_1 = -id \cdot \theta_1^* \cdot w_0(\theta_1)$ and $\theta_2 = -id \cdot \theta_2^* \cdot w_0(\theta_2)$, respectively.

Set $m = 0$**.** Then $\Delta_0(\theta_1) = \Delta_0(\theta_2) = \emptyset$. It follows that $w_0(\theta_1) = w_0(\theta_2) = id$. Let $w \in W(\Phi)$ be such that $w(\Delta_0(\theta_1)) = \Delta_0(\theta_2)$. If there are no diagram automorphisms for Δ_1 and Δ_2, then $\theta_1 = \theta_2 = -id$. From 4.5.2, $w\theta_1 w^{-1} = \theta_2$. Thus, $\Delta_0(\theta_1) \simeq \Delta_0(\theta_2)$.

Now suppose θ_1^*, $\theta_2^* \neq id$. Notice that $\Delta_0(\theta_1)^\perp = \bar{\Delta}_{\theta_1} = \Delta_1$ and $\Delta_0(\theta_2)^\perp = \bar{\Delta}_{\theta_2} = \Delta_2$. Then there exists $\sigma \in W(\Phi)$ such that $\sigma(\Delta_1) = \Delta_2$. It follows that $\sigma\theta_1^*\sigma^{-1} = \theta_2^*$. Therefore, $(X, \Delta_1, \Delta_0(\theta_1), \theta_1^*)$ and $(X, \Delta_2, \Delta_0(\theta_2), \theta_2^*)$ are isomorphic.

Set $m = 1$**.** Let $\Delta_0(\theta_1) = \{\alpha\}$ and $\Delta_0(\theta_2) = \{\beta\}$. Although the parabolic subgroups can be of type A_1 or Y_1, it suffices to only consider the case when $\|\alpha\| = \|\beta\|$. If $\|\alpha\| \neq \|\beta\|$, then the resulting bases cannot share an isomorphy class. Since the roots have the same length, there exists $w \in W(\Phi)$ such that $w(\alpha) = \beta$ and hence $w(\Delta_0(\theta_1)) = \Delta_0(\theta_2)$. Note $w_0(\theta_1) = s_\alpha$ and $w_0(\theta_2) = s_\beta$. From [Humphreys' Lemma 9.2], $ws_\alpha w^{-1} = s_\beta$. If there are no diagram automorphisms, then $\theta_1 = -id \cdot s_\alpha$ and $\theta_2 = -id \cdot s_\beta$. By 4.5.2, $w\theta_1 w^{-1} = \theta_2$. Thus, the parabolic subgroups are $W(\Phi)$-semi-isomorphic.

Now suppose θ_1^*, $\theta_2^* \neq id$. Then we need only consider the cases where Φ is of type A_n, E_6 or D_{2n+1}.

Claim: $w\bar{\Phi}_{\theta_1} = \bar{\Phi}_{\theta_2}$.

Let $\gamma \in \bar{\Phi}_{\theta_1}$. Since $\Phi_0(\theta_1)^\perp = \bar{\Phi}_{\theta_1}$, $(\alpha, \gamma) = 0$. Recall that the inner product is Aut(Φ)-invariant. Hence $(\alpha, \gamma) = (w(\alpha), w(\gamma)) = (\beta, w(\gamma)) = 0$. So $w(\gamma) \in \Delta_0(\theta_2)^\perp = \bar{\Phi}_{\theta_2}$. Thus $w\bar{\Phi}_{\theta_1} \subset \bar{\Phi}_{\theta_2}$. Since w is injective and rank($\bar{\Phi}_{\theta_1}$)=rank($\bar{\Phi}_{\theta_2}$), this proves the claim.

It follows from the claim that $w\theta_1^* w^{-1} = \theta_2^*$. Therefore,

$$\begin{aligned} w\theta_1 w^{-1} &= (w \cdot -id \cdot w^{-1})(w \cdot \theta_1^* \cdot w^{-1})(w \cdot s_\alpha \cdot w^{-1}) \\ &= -id(w \cdot \theta_1^* \cdot w^{-1})s_\beta \\ &= -id \cdot \theta_2^* \cdot s_\beta \\ &= \theta_2. \end{aligned}$$

Set $m > 1$**.** We consider three subcases.

Case 1: Suppose $Y_\ell = \emptyset$. Then $m = r$. Let $\Delta_0(\theta_1) = \{\alpha_1, \ldots, \alpha_r\}$ and $\Delta_0(\theta_2) = \{\beta_1, \ldots, \beta_r\}$. Then $w_0(\theta_1) = s_{\alpha_1} \cdots s_{\alpha_r}$ and $w_0(\theta_2) = s_{\beta_1} \cdots s_{\beta_r}$. Since all roots have the same length, there exists $w \in w(\Phi)$ such that $w(\alpha_i) = \beta_i$ for all $i \leq r$. So $w(\Delta_0(\theta_1)) = \Delta_0(\theta_2)$ and $ww_0(\theta_1)w^{-1} = w_0(\theta_2)$. If Φ is not of type D_n, then $\theta_1^* = \theta_2^* = id$ from 5.1.2. Thus, $w\theta_1 w^{-1} = \theta_2$. Therefore, $\Delta_0(\theta_1) \simeq \Delta_0(\theta_2)$.

If Φ is of type D_n, then we know from 5.2.4 that the diagram automorphism is dependent on where the black dots are in D_n. If there exists a subset of $\Delta_0(\theta_1)$ of type D_2 but not in $\Delta_0(\theta_2)$, then the two parabolic subgroups are not of the same conjugacy class. If so, then $\theta_1^* = \theta_2^* = id$ and $w\theta_1^* w^{-1} = \theta_2^*$. Therefore, $\Delta_0(\theta_1) \simeq \Delta_0(\theta_2)$ if both parabolic subgroups contain a subset of type D_2. If both do not, then either $\theta_1^* = \theta_2^* = id$ or $\theta_1^*, \theta_2^* \neq id$. If the first holds, then we are done. So we assume the latter. Since $\Delta_0(\theta_1), \Delta_0(\theta_2) \subset A_{n-2}$, $\theta_i^* | \Delta_0(\theta_i) = id$ for $i = 1, 2$. Well $w(D_2) = D_2$, so it follows that $w\theta_1^* w^{-1} = \theta_2^*$. Thus, $\Delta_0(\theta_1) \simeq \Delta_0(\theta_2)$ if both parabolic subgroups do not contain a subset of type D_2.

Case 2: Suppose $r = 0$. Then $\Delta_0(\theta_1)$ and $\Delta_0(\theta_2)$ are both of type Y_m assuming $Y_m \neq D_m$. From Prop, there are no diagram automorphisms for both ± 1-eigenspaces. Since $W(Y_m) = \mathrm{Aut(Y_m)}$, $w\theta_1^* w^{-1} = id = \theta_2^*$ for all $w \in W(Y_m)$. Also, there exists $\sigma \in W(Y_m)$ such that $\sigma(\Delta_0(\theta_1)) = \Delta_0(\theta_2)$. Hence, $\sigma w_0(\theta_1)\sigma^{-1} = w_0(\theta_2)$. So, $\sigma\theta_1\sigma^{-1} = \theta_2$. Therefore, $\Delta_0(\theta_1) \simeq \Delta_0(\theta_2)$. Now assume $Y_m = D_m$. Let $\{\alpha_1, \alpha_2\} \subset \Delta_0(\theta_1)$ and $\{\beta_1, \beta_2\} \subset \Delta_0(\theta_2)$ be of type D_2. Since there exists $w \in W(D_m)$ such that $w(\Delta_0(\theta_1)) = \Delta_0(\theta_2)$, then $w(\alpha_1) = \beta_1$ and $w(\alpha_2) = \beta_2$ without loss of generality. Note $\theta_1^* | (\Delta_1 - \{\alpha_1, \alpha_2\}) = id$ and $\theta_2^* | (\Delta_2 - \{\beta_1, \beta_2\}) = id$. So $w\theta_1^* w^{-1}(\beta_i) = \theta_2^*(\beta_i)$ for $i = 1, 2$ which implies that $w\theta_1^* w^{-1}(\gamma) = \theta_2^*(\gamma)$ for all $\gamma \in \Delta_2$. Hence, $\Delta_0(\theta_1) \simeq \Delta_0(\theta_2)$.

Case 3: Suppose $r \neq 0$ and $Y_m \neq 0$. From 5.4.5, there exists $w \in W(\Phi)$ such that $w(\Delta_0(\theta_1)) = \Delta_0(\theta_2)$. So $ww_0(\theta_1)w^{-1} = w_0(\theta_2)$, $w(r \cdot A_1) = A_1$ and $w(Y_\ell) = Y_\ell$. From **??**, $\theta_i^* | (r \cdot A_1) = id$ for $i = 1, 2$. Then the result follows from Case 2.

□

Corollary 5.4.4.1. *Let Φ be irreducible and $\theta \in \mathrm{Aut}(\Phi)$ an involution. If $\Delta_0(\theta) \subset \Delta$ is of type $r \cdot A_1 + Y_\ell$ and*

$Y_\ell \neq \emptyset$, *then* Y_ℓ *determines the existence of a diagram automorphism.*

Proof. It follows from the proof of Theorem 5.4.4 and Proposition 4.3.5. $\square$

Corollary 5.4.4.2. *Let* Φ *be irreducible and* $\Phi_0(\theta_i)$ *is of the type* $r_i \cdot A_1 + Y_\ell$ $(i = 1, 2)$ *where either* $Y_\ell = \emptyset$ *or one of* $B_\ell(\ell \geq 1)$*,* $C_\ell(\ell \geq 1)$*,* $D_{2\ell}(\ell \geq 1)$*,* E_7*,* E_8*,* F_4*, or* G_2*. If there exists* $w \in W$ *such that* $\Phi_0(\theta_1) \cong \Phi_0(\theta_2)$*, then* w *leaves* Y_ℓ *invariant and* $w(r_1 \cdot A_1) = r_2 \cdot A_1$*.*

Proposition 5.4.5. *Let* Φ *be irreducible and* $\theta_1, \theta_2 \in \mathrm{Aut}(\Phi)$ *be involutions. Suppose* $\Delta_0(\theta_1)$*,* $\Delta_0(\theta_2) \subsetneq \Delta$ *are of type* $r \cdot A_1 + Y_\ell$*. Then there exists a* $w \in W(\Phi)$ *such that* $w(\Delta_0(\theta_1)) = \Delta_0(\theta_2)$*.*

Proof. If $Y_\ell = \emptyset$, then there exist k such that $r \cdot A_1 \subset A_k$. Hence, the result follows from **??**. If $r = 0$, then we first consider the rank(Y_ℓ) where Y_ℓ is of type B_ℓ, C_ℓ or D_ℓ. Let $\ell = 2$. Then $\Delta_0(\theta_1) = \{\alpha_1, \alpha_2\}$ and $\Delta_0(\theta_2) = \{\beta_1, \beta_2\}$. Since $\|\alpha_1\| = \beta_1\|$, there exists $w \in W(\Phi)$ such that $w(\alpha_1) = \beta_1$. Well,

$$\langle \alpha_1, \alpha_2 \rangle = \langle w(\alpha_1), w(\alpha_2) \rangle = \langle \beta_1, w(\alpha_2) \rangle = \langle \beta_1, \beta_2 \rangle$$

implies that $w(\alpha_2) = \beta_2$. Hence $w(\Delta_0(\theta_1)) = \Delta_0(\theta_2)$. For $\ell > 2$, note that $Y_\ell \supset A_{\ell-1}$. So from 5.0.2, there exists $w' \in W(\Phi)$ such that $w'\left(A^{(1)}_{\ell-1}\right) = A^{(2)}_{\ell-1}$ where $A^{(i)}_{\ell-1}$ represents the type of $\Delta_0(\theta_i)$ for $i = 1, 2$. From the base case, we see that if $w'(\alpha_{\ell-1}) = \beta_{\ell-1}$ then $w'(\alpha_\ell) = \beta_\ell$. Thus, $w'(\Delta_0(\theta_1)) = \Delta_0(\theta_2)$. Notice for the other types of Y_ℓ there exists k such that $A_k \subset Y_\ell$. Then there exists $w'' \in W(\Phi)$ such that $w''\left(A^{(1)}_k\right) = A^{(2)}_k$. It follows from a similar argument of Y_2 that $w''(\Delta_0(\theta_1)) = \Delta_0(\theta_2)$. Thus for the case where $Y_\ell \neq \emptyset$ and $r \neq 0$, we consider $A_q \subset r \cdot A_1 + Y_\ell$ where q is of maximal value. So for there exist $\sigma \in W(\Phi)$ such that $\sigma\left(A^{(1)}_q\right) = A^{(2)}_q$, and the result follows.

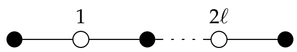
$\square$

5.5 Quasi $\mathbb{R}$-split Characterization for Opposing Involutions Example

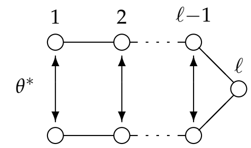
The results from this chapter can be applied to the classification for quasi $\mathbb{R}$-split tori. In [21], a complete table of this classification with the types of (θ, σ)-diagram, $\Phi(\mathfrak{a}) = \Phi_\theta$, $\Phi_{\sigma,\theta} \cap \Phi_\theta = \Phi(\mathfrak{a}, \mathfrak{a}^-_\sigma)$ and the maximal involution of $\Phi_{\sigma,\theta} \cap \Phi_\theta$. This paper contributes the diagrams of the maximal σ-split torus inside a θ-split $(\sigma|\Phi_\theta)$, maximal θ-split torus inside a σ-split $(\theta\Phi_\sigma)$ and the diagrams of their opposing involutions. A complete list is provided within Appendix A.

Recall the Example 4.5.5. Let θ be of type AII and σ of type $AIII_b$ acton on $A_{4\ell+1}$. The θ and σ diagrams are as follows.

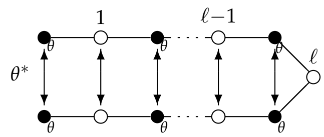
σ :



θ :

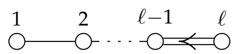


Thus, the (θ, σ)-diagram of type $A_{4\ell-1}^{2\ell,2\ell-1}(III_b, II, \epsilon_0)$ would be:

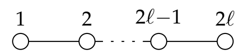


The restricted root systems relative to θ and σ are below.

$\overline{\Phi}_\theta$:

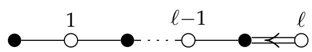


$\overline{\Phi}_\sigma$:



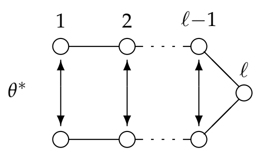
Now we can determine the action of $\pm\theta$ and $\pm\sigma$ on these restricted root systems.

$\sigma|\Phi_\theta$:

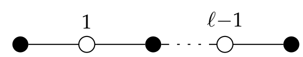


$-\sigma|\Phi_\theta$:

$\theta|\Phi_\sigma$:



$-\theta|\Phi_\sigma$:



A detailed list of the actions of $\pm\sigma$ and $\pm\theta$ on quasi $\mathbb{R}$-split tori have been provided in Appendix A. This list is an adaption of the classification tables that addresses the 171 isomorphy classes provided in [21].

References

[1] Marcel Berger. "Les espaces symetriques noncompacts". *Annales Scientifiques De L'E.N.S* **74**.85-177 (1957).

[2] W. Rossmann. "The Structure of Semisimple Symmetric Spaces". *Canadian Journal of Mathematics* **31**.1 (1979), pp. 157–180.

[3] Toshihiko Matsuki. "The Orbtis of Affine Symmetric Spaces under the action of Minimal Parabolic Subgroups". *Journal of Mathematical Society of Japan* **31**.2 (1979), pp. 331–357.

[4] A.G. Helminck. "Combinatorics Related to Orbit Closures of Symmetric Subgroups in Flag Varieties". *Invariant Theory in all Characteristics* (2004), pp. 71–90.

[5] Michel Brion & A.G. Helminck. "On Orbit Closures of Symmetric Subgroups in Flag Varieties". *Canadian Journal of Mathematics* **52**.2 (2000), pp. 265–292.

[6] T.A. Springer. *Some Results on Algebraic Groups with Involutions*. Mathematisch Institut der Rÿksuniversiteit, 1984.

[7] Jennifer Fowler. "Algorithms for Computations in Local Symmetric Spaces". PhD thesis. North Carolina State University, 2003.

[8] A.G. Helminck & Jennifer Daniel. "Computing the Fine Structure of Real Reductive Symmetric Spaces". *Journal of Symbolic Computation* **42**.5 (2007), pp. 497–510.

[9] Nicolas Bourbaki. *Lie Groups and Lie Algebras: Chapters 7-9 (Elements of Mathematics*. Springer, 2008.

[10] Kathryn Brenneman. "Lifting k-Involutions from the Root Systems to the Lie Algebra". PhD thesis. North Carolina State University, 2013.

[11] Robert Watson. "Lifting Automorphisms from Root Systems to Lie Algebras". PhD thesis. North Carolina State University, 2010.

[12] Toshio Oshima & Jiro Sekiguchi. "The Restricted Root System of a Semisimple Symmetric Pair". *Advanced Studies in Pure Mathematics* **4** (1984), pp. 433–497.

[13] Brian C. Hall. "Lie Groups, Lie Algebras, and Representations". *Springer* (2003).

[14] Sigurdur Helgason. "Differential Geometry, Lie Groups, and Symmetric Spaces". *American Mathematical Society* (2001).

[15] A.G. Helminck. "On Groups with a Cartan Involution". *Proceddings of the Hyderabad Conference on Algebraic Groups* (1992).

[16] A.G. Helminck. "On the Classification of k-Involutions". *Advances in Mathematics* **153**.1 (2000).

[17] Kennether Hoffman & Ray Kunze. *Linear Algebra*. 2nd. Prentice Hall, 2008.

[18] Andrew Baker. *Matrix Groups: An Introduction to Lie Group Theory*. Springer, 2003.

[19] Ichiro Satake. *Classification Theory of Semi-simple Algebraic Groups*. Marcel Dekker, 1971.

[20] Stacy Beun. "On the Classification of Orbits of Minimal Parabolic k-Subgroups Acting on Symmetric k-Varties of $SL(n,k)$". PhD thesis. North Carolina State University, 2008.

[21] Catherine Buell. "On Maximal $\mathbb{R}$-split Tori Invariant under an Involution". PhD thesis. North Carolina State University, 2011.

[22] A.G. Helminck & S. P Wang. "On Rationality Properties of Involutions of Reductive Groups". *Advances in Mathematics* (1993), pp. 26–97.

[23] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Ed. by P.R. Halmos. Springer-Verlag New York, 1972.

[24] Aloysius G. Helminck. "Algebraic Groups with a Commuting Pair of Involutions and Semisimple Symmetric Spaces". *Advances in Mathematics* **71**.1 (1988), pp. 21–91. ISSN: 0001-8708. DOI: 10.1016/0001-8708(88)90066-7. URL: http://dx.doi.org/10.1016/0001-8708(88)90066-7.

[25] A.G. Helminck. "Tori Invariant under an Involutorial Automorphism II". *Advances in Mathematics* **131**.1 (1997), pp. 1–92.

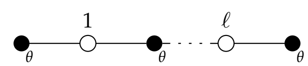
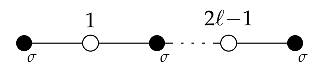
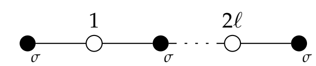
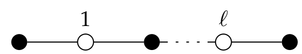
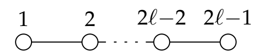
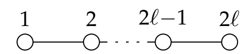
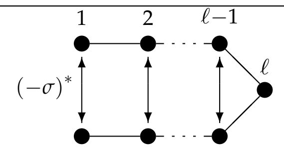
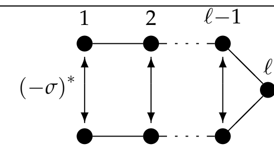
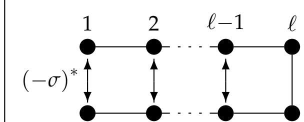
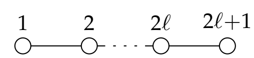
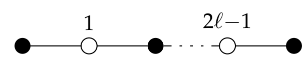
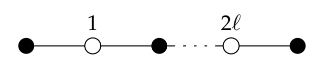
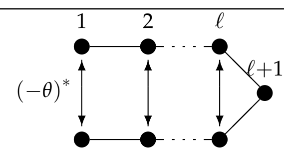
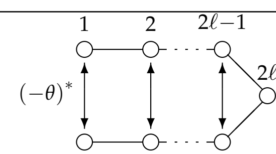
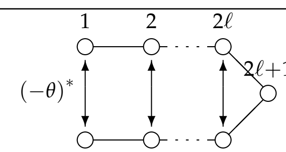
[26] A.G. Helminck. "Tori Invariant under an Involutorial Automorphism I". *Advances in Mathematics* 85 (1991), pp. 1–38.

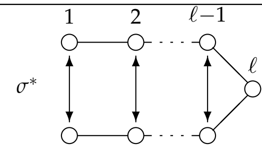
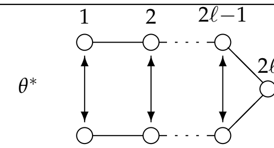
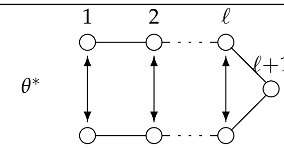
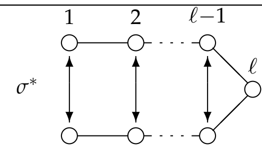
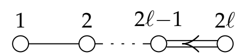
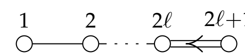
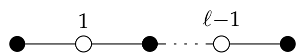
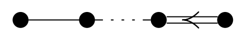
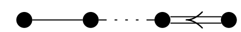
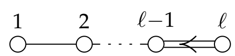
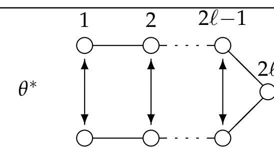
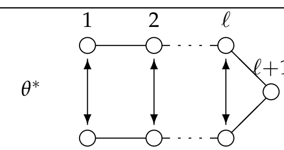
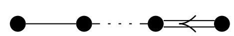
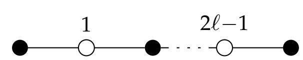
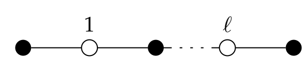
APPENDIX

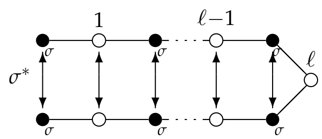
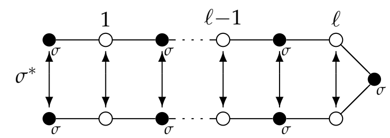
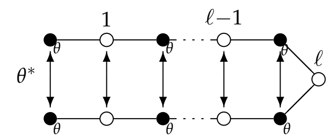
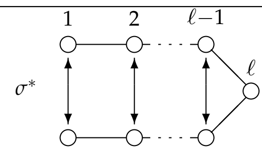
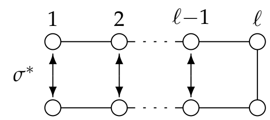
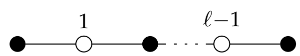
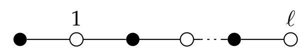
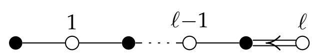
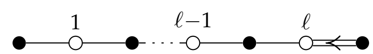
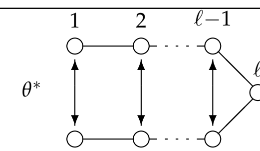
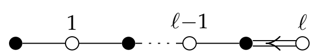
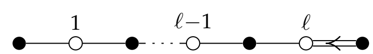
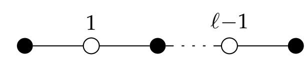
APPENDIX

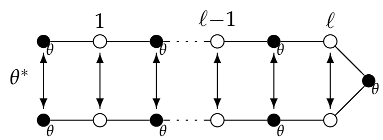
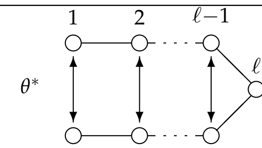
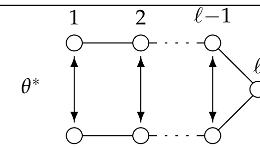
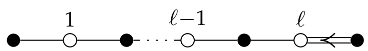
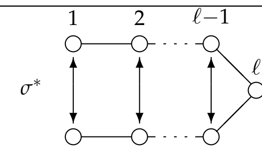
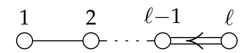
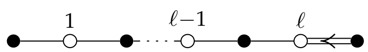
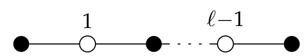
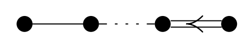
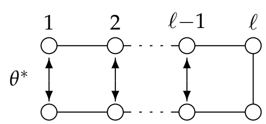
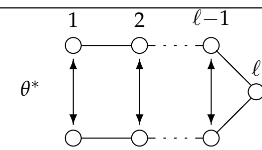
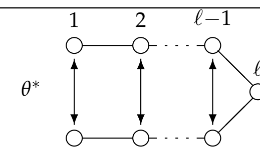
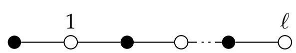
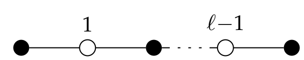
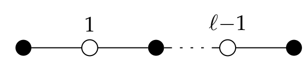
A

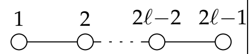
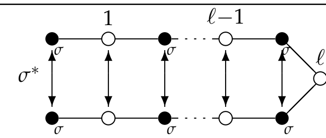
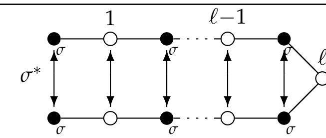
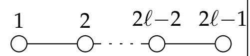
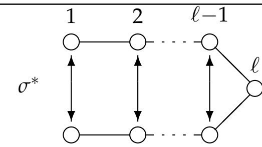
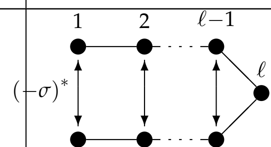
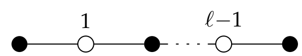
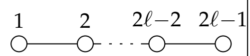
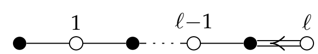
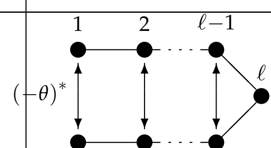
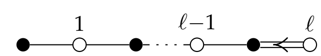
COMPLETE TABLES ON THE ACTION OF OPPOSING INVOLUTIONS ON QUASI $\mathbb{R}$-SPLIT TORI

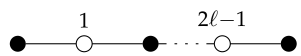
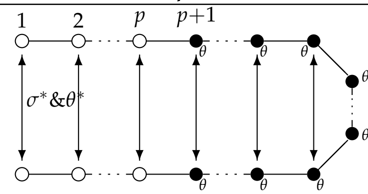
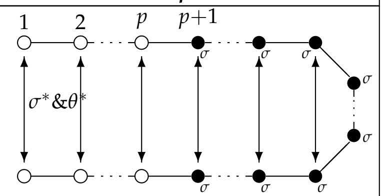
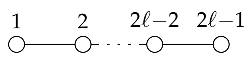
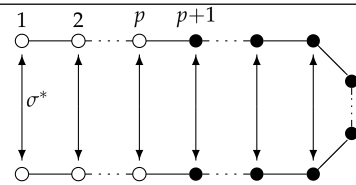
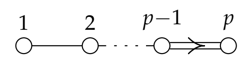
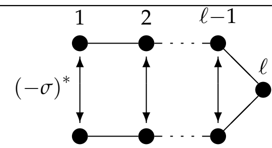
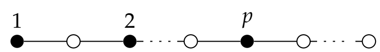
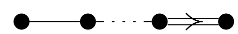
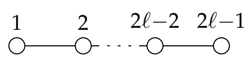
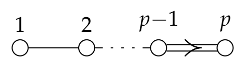
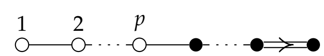
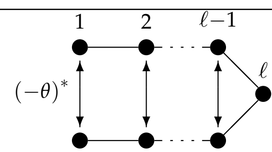
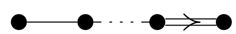
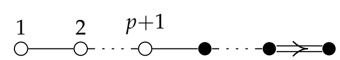
Type (θ, σ)	$A_{2\ell+1}^{2\ell+1,\ell}(I, II)$	$A_{4\ell-1}^{2\ell-1,4\ell-1}(II, I)$	$A_{4\ell+1}^{2\ell,4\ell+1}(II, I)$
(θ, σ)-Diagram			
$\sigma \vert \Phi_\theta$			
$-\sigma \vert \Phi_\theta$			
$\theta \vert \Phi_\sigma$			
$-\theta \vert \Phi_\sigma$			

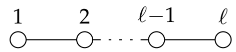
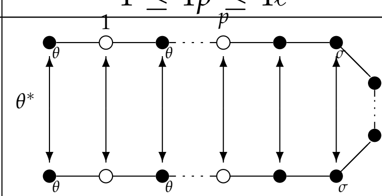
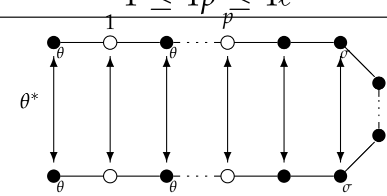
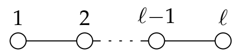
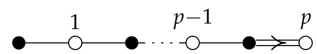
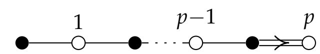
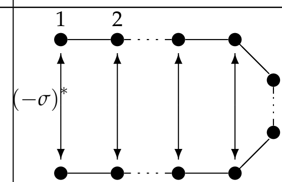
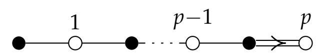
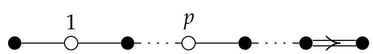
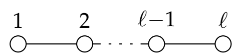
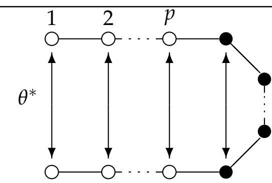
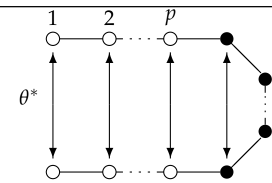
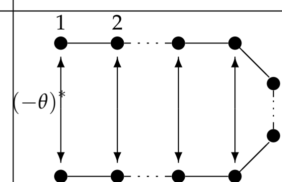
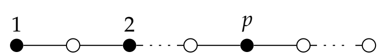
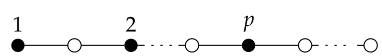
Type (θ,σ)	$A^{2\ell-1,\ell}_{2\ell-1}(I, III_b, \epsilon_0)$	$A^{2\ell,4\ell-1}_{4\ell-1}(III_b, I, \epsilon_0)$	$A^{2\ell+1,4\ell+1}_{4\ell+1}(III_b, I, \epsilon_0)$
(θ,σ)-Diagram			
$\sigma\vert\Phi_\theta$			
$-\sigma\vert\Phi_\theta$			
$\theta\vert\Phi_\sigma$			
$-\theta\vert\Phi_\sigma$			

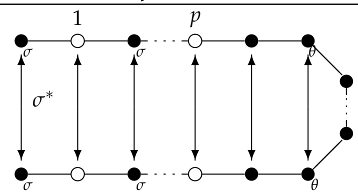
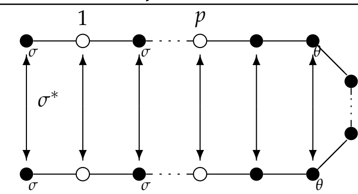
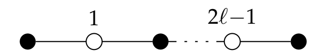
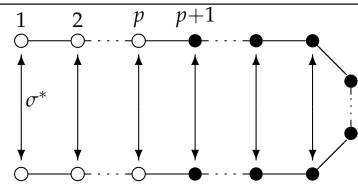
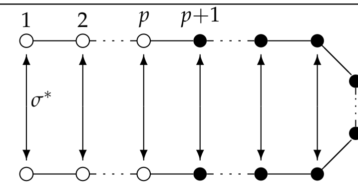
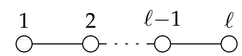
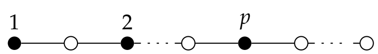
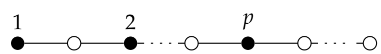
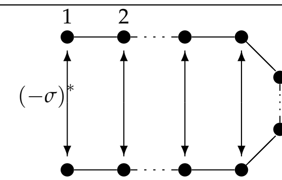
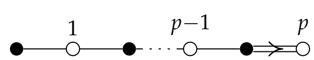
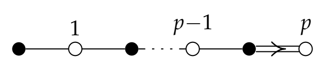
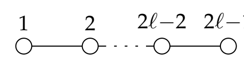
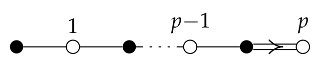
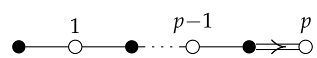
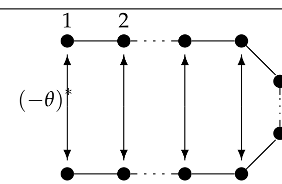
Type (θ, σ)	$A_{4\ell-1}^{2\ell-1,2\ell}(II, III_b, \epsilon_\ell)$	$A_{4\ell+1}^{2\ell,2\ell+1}(II, III_b)$	$A_{4\ell-1}^{2\ell,2\ell-1}(III_b, II, \epsilon_\ell)$
(θ, σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

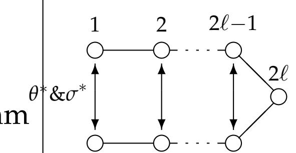
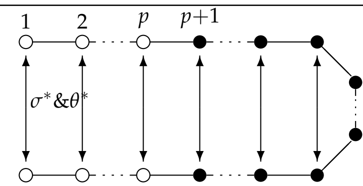
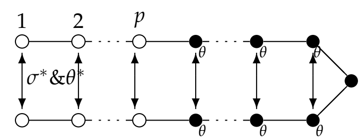
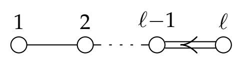
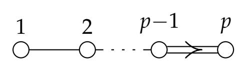
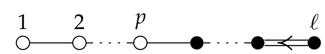
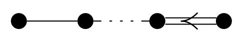
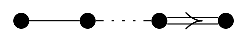
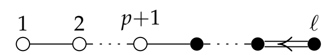
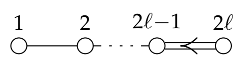
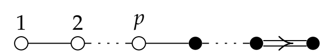
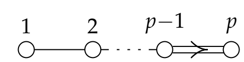
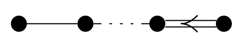
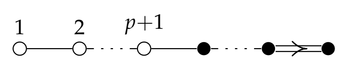
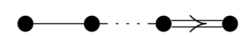
Type (θ,σ)	$A_{4\ell+1}^{2\ell+1,2\ell}(III_b, II)$	$A_{2\ell-1}^{2\ell-1,\ell}(I, III_b, \epsilon_\ell)$	$A_{2\ell-1}^{\ell,2\ell-1}(III_b, I, \epsilon_\ell)$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

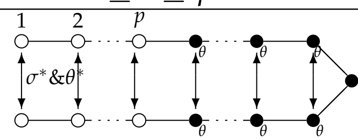
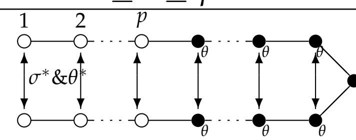
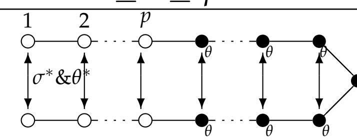
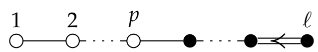
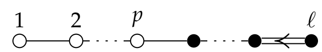
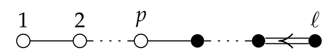
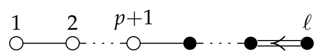
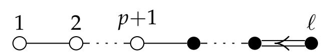
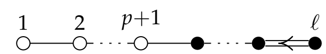
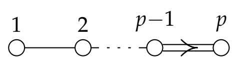
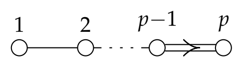
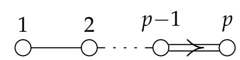
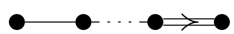
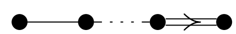
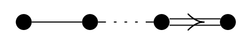
Type (θ,σ)	$A_{2\ell-1}^{2\ell-1,2\ell-1}(I,\epsilon)$	$A_{4\ell-1}^{2\ell-1,2\ell}(II,III_b,\epsilon_0)$	$A_{4\ell-1}^{2\ell,2\ell-1}(III_b,II,\epsilon_0)$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

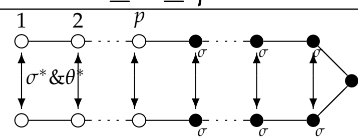
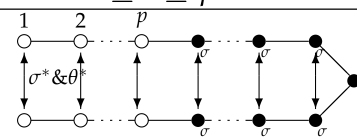
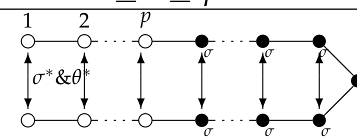
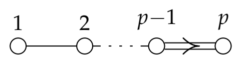
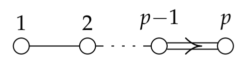
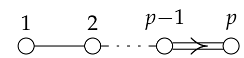
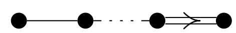
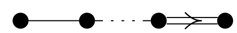
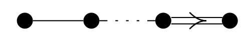
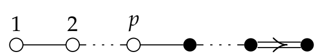
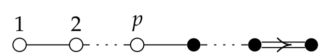
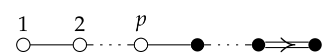
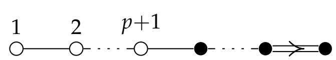
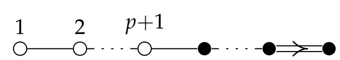
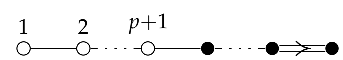
Type (θ,σ)	$A^{2\ell-1}_{4\ell-1}(II,\epsilon_\ell)$	$A^{\ell,p}_{\ell}(I,III_a)$ $1\le 2p\le \ell$	$A^{p,\ell}_{\ell}(III_a,I)$ $1\le 2p\le \ell$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

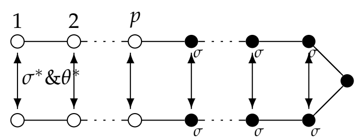
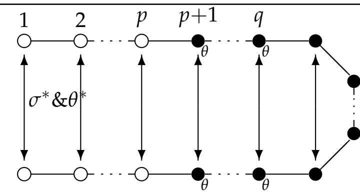
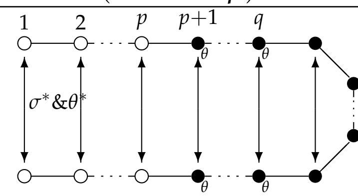
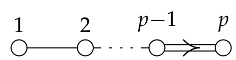
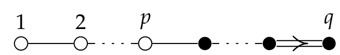
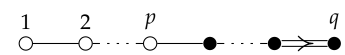
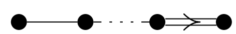
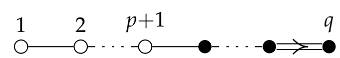
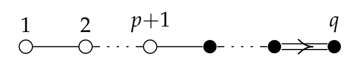
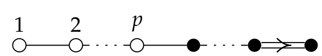
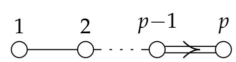
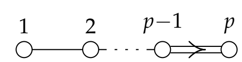
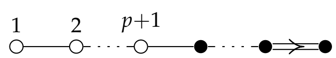
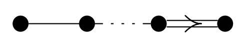
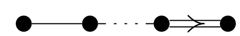
Type (θ,σ)	$A_\ell^\ell(I,\epsilon_p)$ $1 \le 2p \le \ell$	$A_{4\ell-1}^{2p,2\ell-1}(III_a,II)$ $1 \le 4p \le 4\ell$	$A_{4\ell+1}^{2p,2\ell}(III_a,II)$ $1 \le 4p \le 4\ell$
(θ,σ)-Diagram			
$\sigma\vert\Phi_\theta$			
$-\sigma\vert\Phi_\theta$			
$\theta\vert\Phi_\sigma$			
$-\theta\vert\Phi_\sigma$			

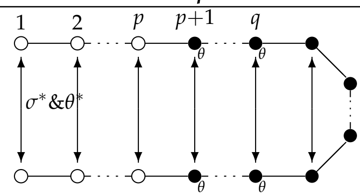
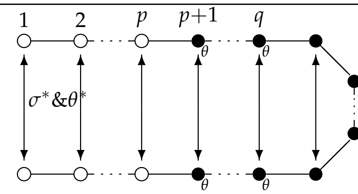
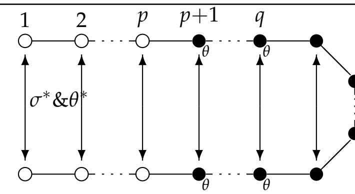
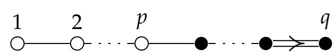
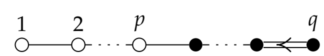
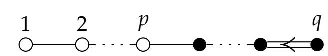
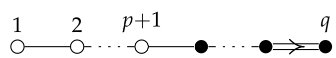
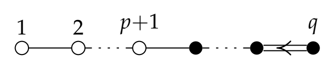
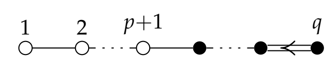
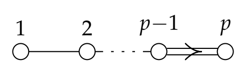
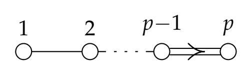
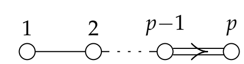
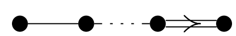
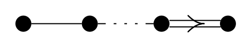
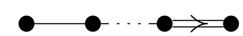
Type (θ,σ)	$A_{4\ell-1}^{2\ell-1,2p}(II,III_a)$ $1\leq 4p\leq 4\ell+2$	$A_{4\ell+1}^{2\ell,2p}(II,III_a)$ $1\leq 4p\leq 4\ell+2$	$A_{2\ell+1}^{\ell,\ell}(II,\epsilon_p)$ $1\leq 2p\leq \ell+1$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

Type (θ,σ)	$A^{\ell,\ell}_{2\ell-1}(III_b,\epsilon_\ell)$	$A^{p,p}_{\ell}(III_a,\epsilon_p)$ $p < \ell+1-2p$	$A^{\ell,p}_{2\ell-1}(III_b,III_a,\epsilon_i)$ $\ell-p+2i \leq 2p$ $0 \leq i \leq p-1$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

Type (θ, σ)	$A^{\ell,p}_{2\ell-1}(III_b, III_a, \epsilon_i)$ $p < \ell - p + 2i \leq \ell$ $0 \leq i \leq p-1$	$A^{\ell,p}_{2\ell-1}(III_b, III_a, \epsilon_i)$ $p \leq \ell + p - 2i \leq \ell$ $0 \leq i \leq p-1$	$A^{\ell,p}_{2\ell-1}(III_b, III_a, \epsilon_i)$ $\ell + p - 2i < p$ $0 \leq i \leq p-1$
(θ, σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

Type (θ,σ)	$A^{p,\ell}_{2\ell-1}(III_a, III_b, \epsilon_i)$ $\ell - p + 2i < p$ $0 \leq i \leq p-1$	$A^{p,\ell}_{2\ell-1}(III_a, III_b, \epsilon_i)$ $p < \ell - p + 2i \leq \ell$ $0 \leq i \leq p-1$	$A^{p,\ell}_{2\ell-1}(III_a, III_b, \epsilon_i)$ $p \leq \ell + p - 2i < \ell$ $0 \leq i \leq p-1$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

Type (θ, σ)	$A^{p,\ell}_{2\ell-1}(III_a, III_b, \epsilon_i)$ $\ell + p - 2i < p$ $0 \leq i \leq p-1$	$A^{q,p}_{\ell}(III_a, III_b, \epsilon_i)$ $q - p + 2i \leq p < q$ $1 \leq p < q \leq \frac{1}{2}(\ell+1)$ $(0 \leq i < p)$	$A^{q,p}_{\ell}(III_a, III_b, \epsilon_i)$ $p < q < q - p + 2i$ $q - p + 2i \leq \frac{1}{2}(\ell+1)$ $1 \leq p < q \leq \frac{1}{2}(\ell+1)$ $(0 \leq i < p)$
(θ, σ)-Diagram			
$\sigma \vert \Phi_\theta$			
$-\sigma \vert \Phi_\theta$			
$\theta \vert \Phi_\sigma$			
$-\theta \vert \Phi_\sigma$			

Type (θ,σ)	$A_\ell^{q,p}(III_a, III_b, \epsilon_i)$ $q \le \ell+1-q+p-2i$ $\ell+1-q+p-2i < \frac{1}{2}(\ell+1)$ $1 \le p < q \le \frac{1}{2}(\ell+1)$ $0 \le i \le p-1$	$A_\ell^{q,p}(III_a, III_a, \epsilon_i)$ $p \le \ell+1-q+p-2i < q$ $1 \le p < q \le \frac{1}{2}(\ell+1)$ $0 \le i \le p$	$A_\ell^{q,p}(III_a, III_a, \epsilon_i)$ $\ell+1-q+p-2i < q$ $1 \le p < q \le \frac{1}{2}(\ell+1)$ $(0 \le i < p)$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

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$1 \le p < q \le \frac{1}{2}(\ell+1)$
$0 \le i \le p$</td><td>$A_\ell^{p,q}(III_a, III_a, \epsilon_i)$
$p < q-p+2i \le \frac{1}{2}(\ell+1)$
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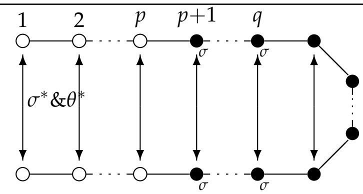
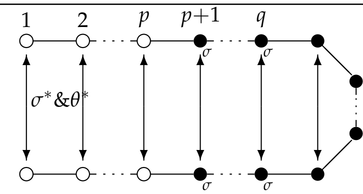
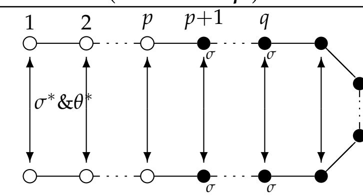
<tr><td>(θ,σ)-Diagram</td><td>
</td><td>
</td><td>
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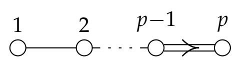
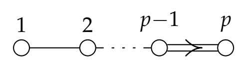
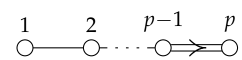
<tr><td>$\sigma\vert\Phi_\theta$</td><td>
</td><td>
</td><td>
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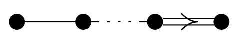
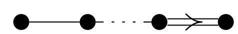
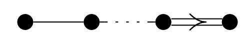
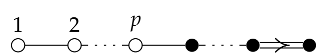
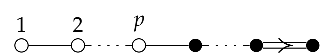
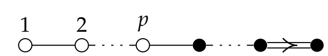
<tr><td>$-\sigma\vert\Phi_\theta$</td><td></td><td></td><td></td></tr>
<tr><td>$\theta\vert\Phi_\sigma$</td><td>
</td><td>
</td><td>
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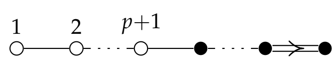
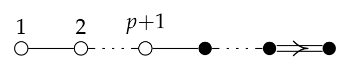
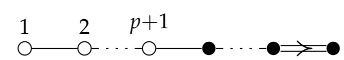
<tr><td>$-\theta\vert\Phi_\sigma$</td><td>
</td><td>
</td><td>
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$0 \leq i \leq p$</td><td>$B_\ell^{p,p}(I_a, \epsilon_i)$
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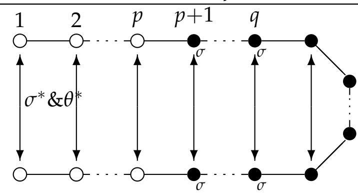
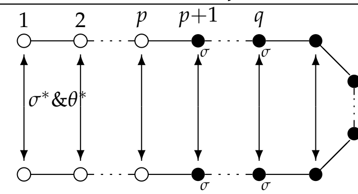
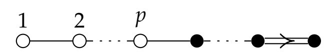
<tr><td>(θ,σ)-Diagram</td><td>
</td><td>
</td><td>
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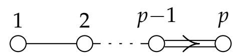
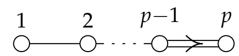
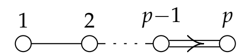
<tr><td>$\sigma|\Phi_\theta$</td><td>
</td><td>
</td><td>
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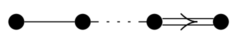
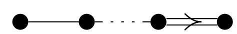
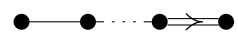
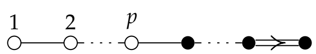
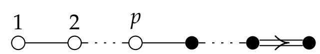
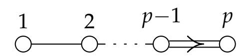
<tr><td>$-\sigma|\Phi_\theta$</td><td></td><td></td><td></td></tr>
<tr><td>$\theta|\Phi_\sigma$</td><td>
</td><td>
</td><td>
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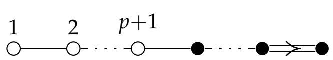
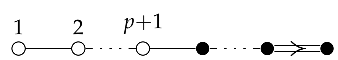
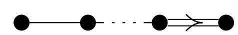
<tr><td>$-\theta|\Phi_\sigma$</td><td>
</td><td>
</td><td></td></tr>
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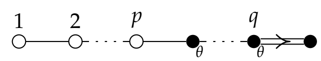
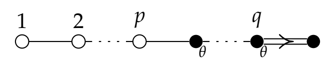
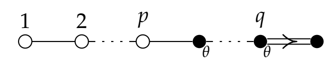
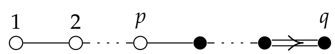
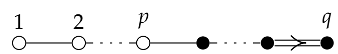
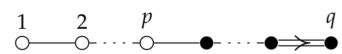
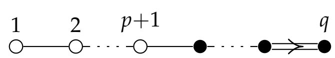
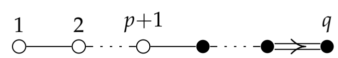
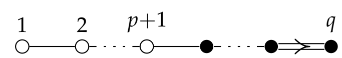
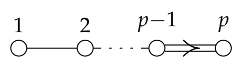
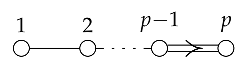
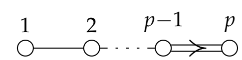
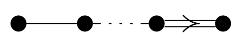
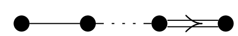
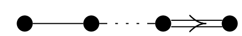
Type (θ, σ)	$B_\ell^{q,p}(I_a, I_a, \epsilon_i)$ $q - p + 2i \leq p < q$ $1 \leq p < q \leq \ell$ $(0 \leq i \leq p)$	$B_\ell^{q,p}(I_a, I_a, \epsilon_i)$ $q < q - p + 2i \leq \ell$ $1 \leq p < q \leq \ell$ $(0 \leq i \leq p)$	$B_\ell^{q,p}(I_a, I_a, \epsilon_i)$ $q \leq 2\ell + 1 - q + p - 2i < \ell$ $1 \leq p < q \leq \ell$ $(0 \leq i \leq p)$
(θ, σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

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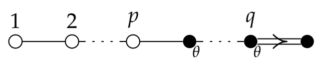
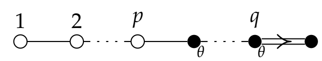
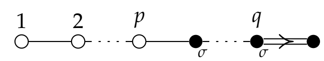
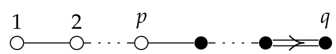
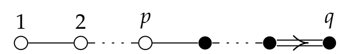
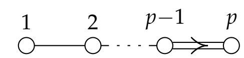
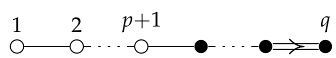
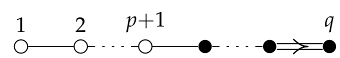
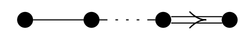
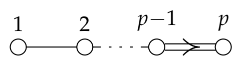
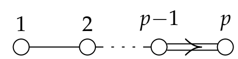
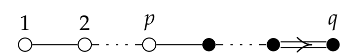
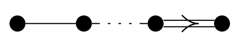
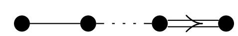
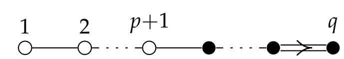
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

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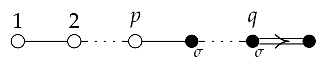
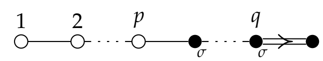
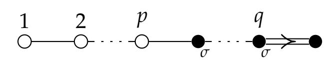
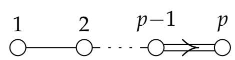
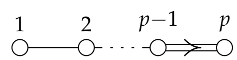
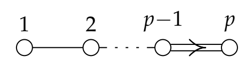
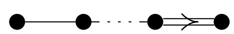
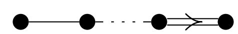
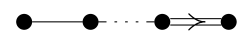
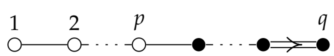
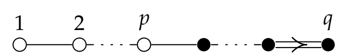
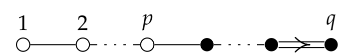
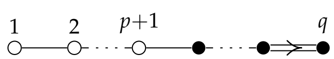
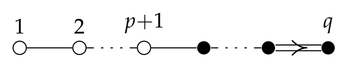
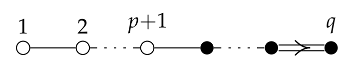
(θ, σ)-Diagram			
$\sigma \vert \Phi_\theta$			
$-\sigma \vert \Phi_\theta$			
$\theta \vert \Phi_\sigma$			
$-\theta \vert \Phi_\sigma$			

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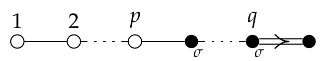
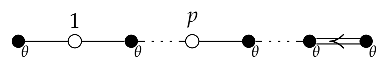
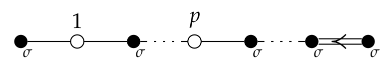
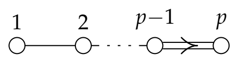
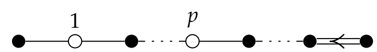
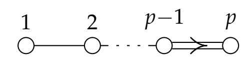
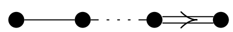
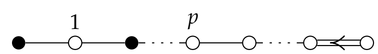
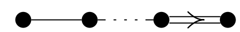
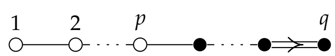
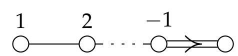
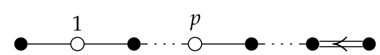
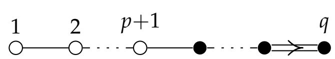
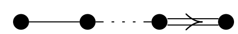
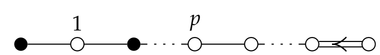
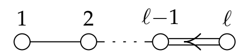
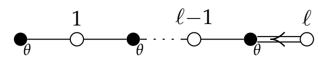
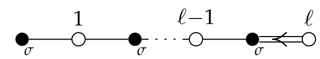
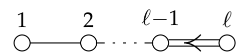
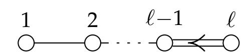
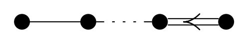
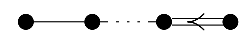
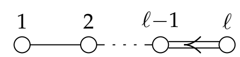
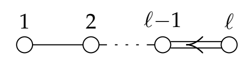
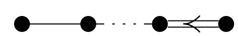
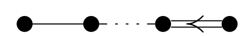
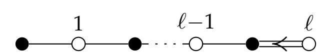
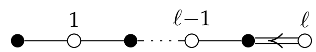
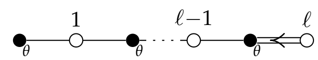
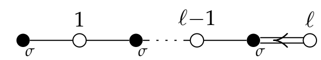
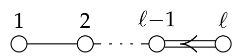
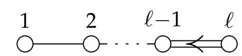
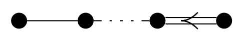
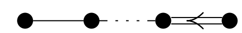
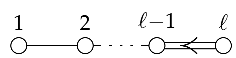
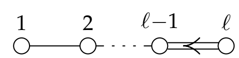
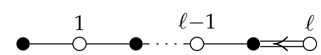
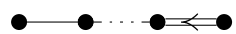
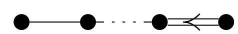
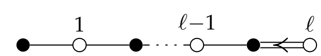
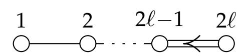
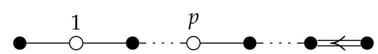
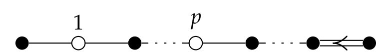
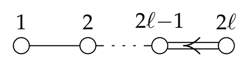
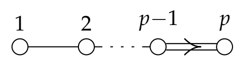
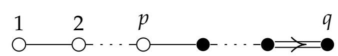
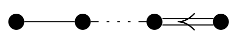
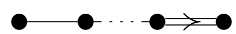
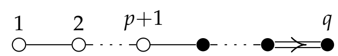
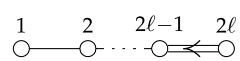
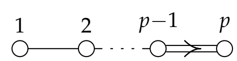
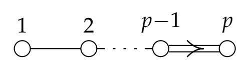
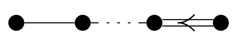
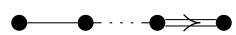
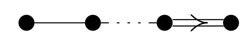
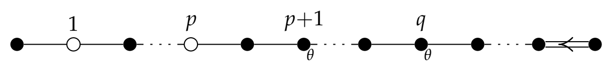
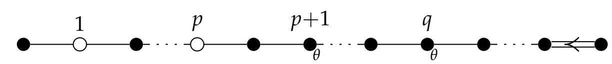
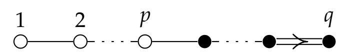
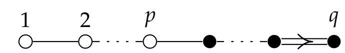
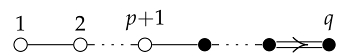
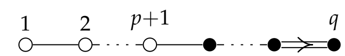
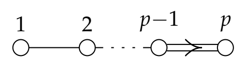
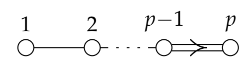
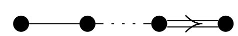
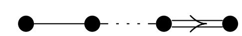
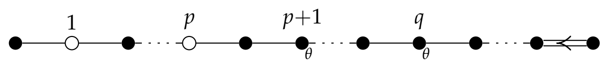
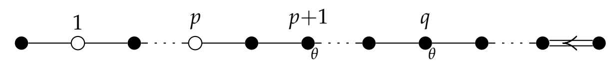
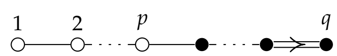
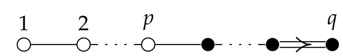
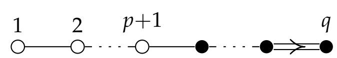
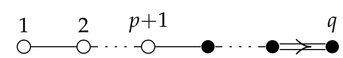
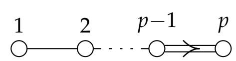
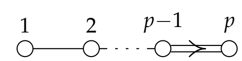
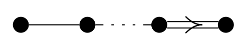
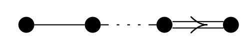
(θ, σ)-Diagram			
$\sigma\|\Phi_{\theta}$			
$-\sigma\|\Phi_{\theta}$			
$\theta\|\Phi_{\sigma}$			
$-\theta\|\Phi_{\sigma}$			

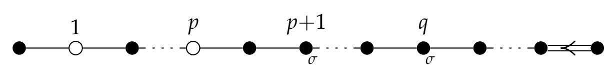
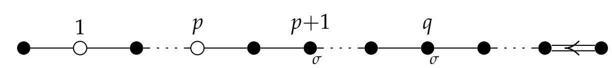
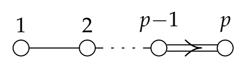
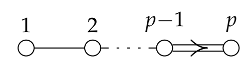
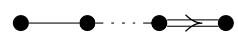
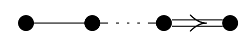
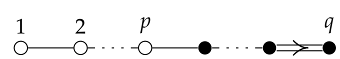
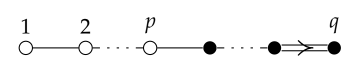
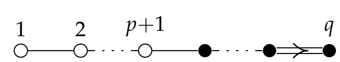
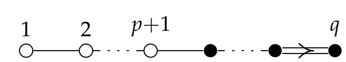
Type (θ, σ)	$C_\ell^{\ell,\ell}(I, \epsilon_p)$ $(2p \leq \ell)$	$C_{2\ell}^{2\ell,\ell}(I, II_b, \epsilon_0)$	$C_{2\ell}^{\ell,2\ell}(II_b, I, \epsilon_0)$
(θ, σ)-Diagram			
$\sigma \vert \Phi_\theta$			
$-\sigma \vert \Phi_\theta$			
$\theta \vert \Phi_\sigma$			
$-\theta \vert \Phi_\sigma$			

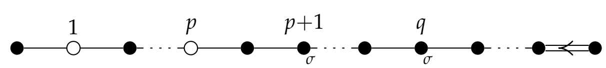
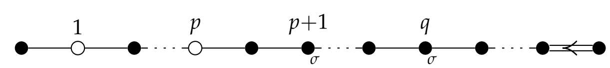
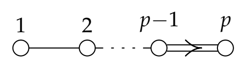
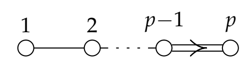
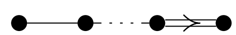
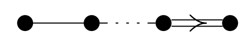
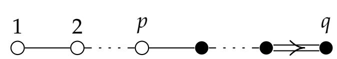
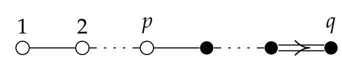
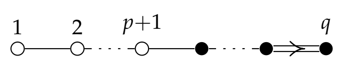
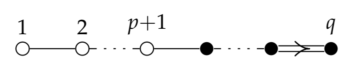
Type (θ,σ)	$C_{2\ell}^{\ell,\ell}(II_b,\epsilon_\ell)$	$C_{2\ell}^{2\ell,\ell}(I,II_b,\epsilon_\ell)$	$C_{2\ell}^{\ell,2\ell}(II_b,I,\epsilon_\ell)$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

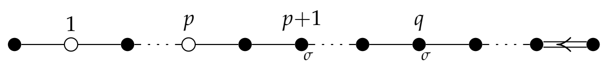
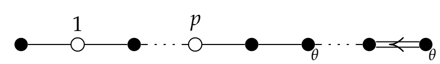
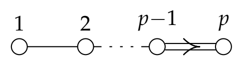
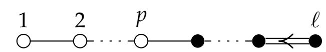
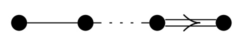
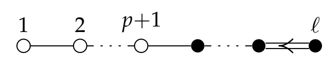
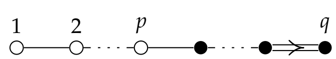
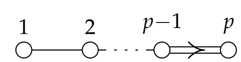
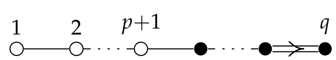
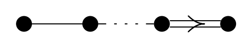
Type (θ, σ)	$C_{2\ell}^{2\ell,2\ell}(I, \epsilon_\ell)$	$C_\ell^{p,p}(II_a, \epsilon_i)$ $p < \ell - 2p + 1$ $1 \leq p \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$	$C_\ell^{q,p}(II_a, II_a, \epsilon_i)$ $q - p + 2i \leq p$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$
(θ, σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

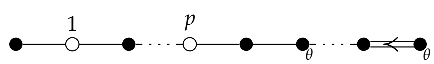
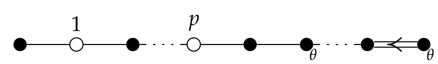
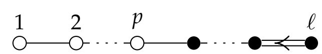
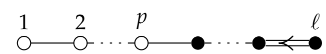
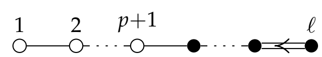
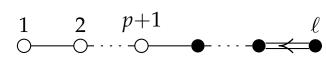
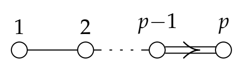
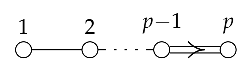
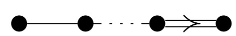
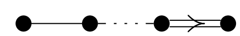
Type (θ,σ)	$C_\ell^{q,p}(II_a, II_a, \epsilon_i)$ $p < q < q-p+2i \le \frac{1}{2}\ell$ $1 \le p < q \le \frac{1}{2}\ell$ $(0 \le i \le p)$	$C_\ell^{q,p}(II_a, II_a, \epsilon_i)$ $q \le \ell - q + p - 2i < \frac{1}{2}\ell$ $1 \le p < q \le \frac{1}{2}\ell$ $(0 \le i \le p)$
(θ,σ)-Diagram		
$\sigma\|\Phi_\theta$		
$-\sigma\|\Phi_\theta$		
$\theta\|\Phi_\sigma$		
$-\theta\|\Phi_\sigma$		

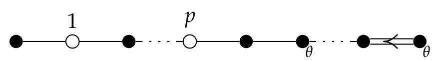
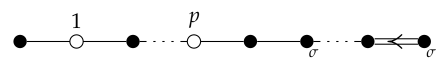
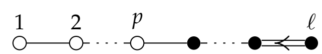
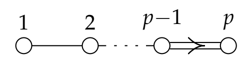
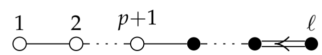
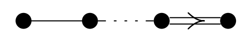
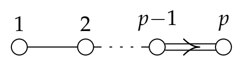
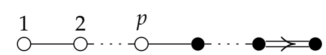
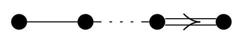
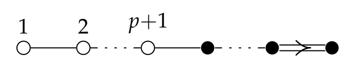
Type (θ,σ)	$C_\ell^{q,p}(II_a,II_a,\epsilon_i)$ $p \leq q+p-2i < q$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$	$C_\ell^{q,p}(II_a,II_a,\epsilon_i)\ell - q + p - 2i < p$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$
(θ,σ)-Diagram		
$\sigma\|\Phi_\theta$		
$-\sigma\|\Phi_\theta$		
$\theta\|\Phi_\sigma$		
$-\theta\|\Phi_\sigma$		

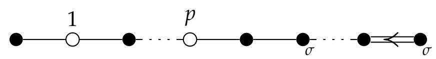
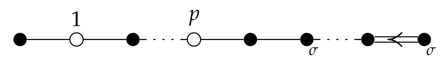
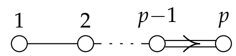
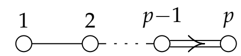
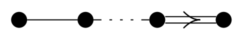
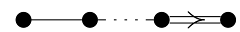
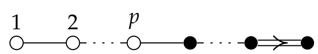
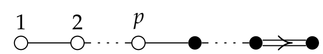
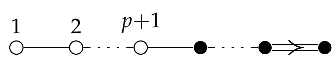
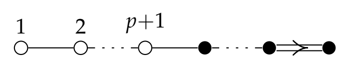
Type (θ, σ)	$C_\ell^{p,q}(II_a, II_a, \epsilon_i)$ $q - p + 2i \leq p$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$	$C_\ell^{p,q}(II_a, II_a, \epsilon_i)$ $p < q - p + 2i < \frac{1}{2}\ell$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$
(θ, σ)-Diagram		
$\sigma\|\Phi_\theta$		
$-\sigma\|\Phi_\theta$		
$\theta\|\Phi_\sigma$		
$-\theta\|\Phi_\sigma$		

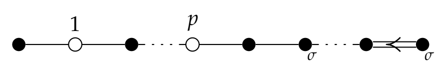
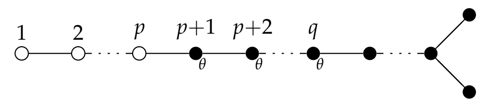
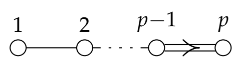
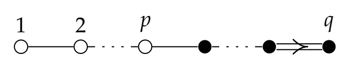
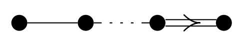
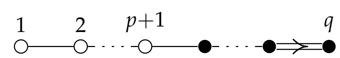
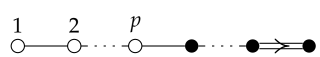
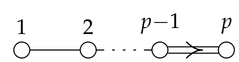
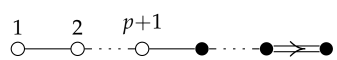
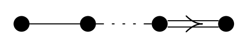
Type (θ, σ)	$C_\ell^{p,q}(II_a, II_a, \epsilon_i)$ $q \leq \ell - q + p - 2i < \frac{1}{2}\ell$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$	$C_\ell^{p,q}(II_a, II_a, \epsilon_i)$ $p \leq \ell - q + p - 2i < q$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$
(θ, σ)-Diagram		
$\sigma\|\Phi_\theta$		
$-\sigma\|\Phi_\theta$		
$\theta\|\Phi_\sigma$		
$-\theta\|\Phi_\sigma$		

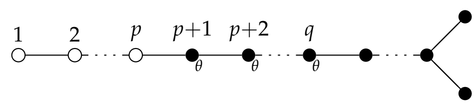
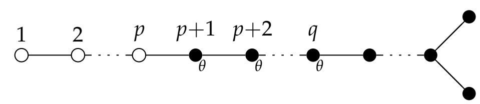
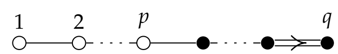
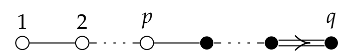
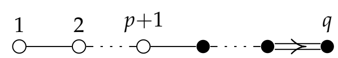
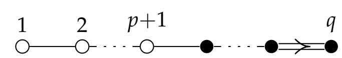
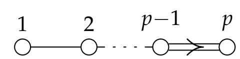
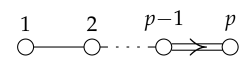
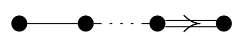
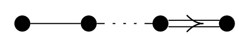
Type (θ, σ)	$C_\ell^{p,q}(II_a, II_a, \epsilon_i)$ $\ell - q + p - 2i < p$ $1 \leq p < q \leq \frac{1}{2}\ell$ $(0 \leq i \leq p)$	$C_{2\ell}^{\ell,p}(II_b, II_a, \epsilon_i)$ $\ell - p + 2i \leq p$ $0 \leq i \leq p-1$ $(1 \leq p \leq \ell)$
(θ, σ)-Diagram		
$\sigma \vert \Phi_\theta$		
$-\sigma \vert \Phi_\theta$		
$\theta \vert \Phi_\sigma$		
$-\theta \vert \Phi_\sigma$		

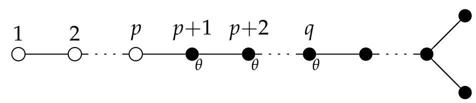
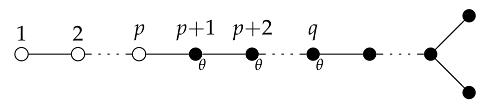
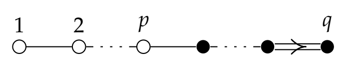
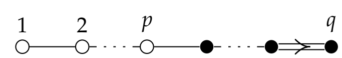
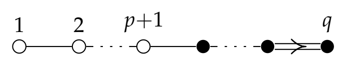
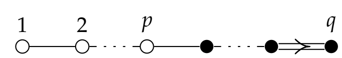
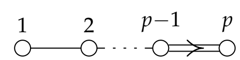
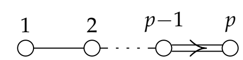
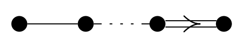
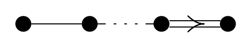
Type (θ, σ)	$C_{2\ell}^{\ell,p}(II_b, II_a, \epsilon_i)$ $p < \ell - p + 2i \leq \ell$ $0 \leq i \leq p-1$ $(1 \leq p \leq \ell)$	$C_{2\ell}^{\ell,p}(II_b, II_a, \epsilon_i)$ $p \leq \ell + p - 2i < \ell$ $0 \leq i \leq p-1$ $(1 \leq p \leq \ell)$
(θ, σ)-Diagram		
$\sigma \vert \Phi_\theta$		
$-\sigma \vert \Phi_\theta$		
$\theta \vert \Phi_\sigma$		
$-\theta \vert \Phi_\sigma$		

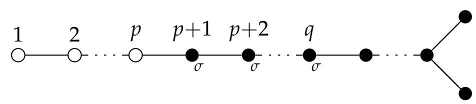
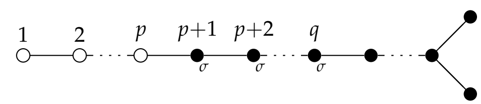
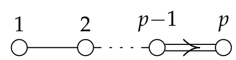
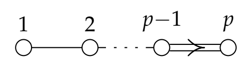
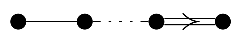
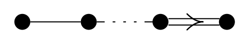
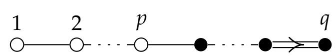
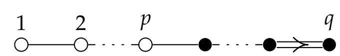
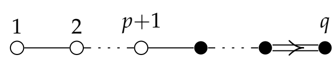
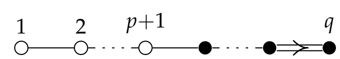
Type (θ, σ)	$C_{2\ell}^{\ell,p}(II_b, II_a, \epsilon_i)$ $\ell + p - 2i < p$ $0 \leq i \leq p-1$ $(1 \leq p \leq \ell)$	$C_{2\ell}^{p,\ell}(II_a, II_b, \epsilon_i)$ $\ell - p + 2i \leq p$ $0 \leq i \leq p-1$ $(1 \leq p \leq \ell)$
(θ, σ)-Diagram		
$\sigma\|\Phi_\theta$		
$-\sigma\|\Phi_\theta$		
$\theta\|\Phi_\sigma$		
$-\theta\|\Phi_\sigma$		

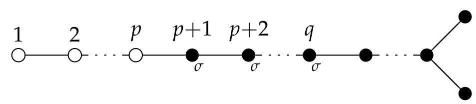
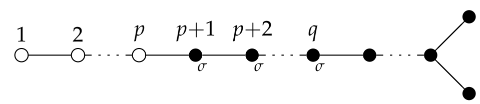
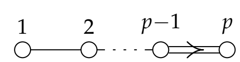
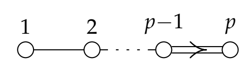
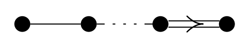
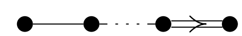
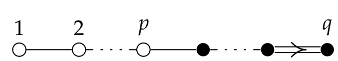
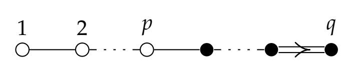
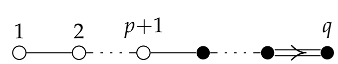
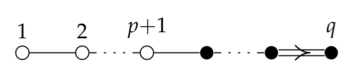
Type (θ,σ)	$C_{2\ell}^{p,\ell}(II_a, II_b, \epsilon_i)$ $p < \ell - p + 2i \leq \ell$ $0 \leq i \leq p-1$ $(1 \leq p \leq \ell)$	$C_{2\ell}^{p,\ell}(II_a, II_b, \epsilon_i)$ $p \leq \ell + p - 2i < \ell$ $0 \leq i \leq p-1$ $(1 \leq p \leq \ell)$
(θ,σ)-Diagram		
$\sigma\|\Phi_\theta$		
$-\sigma\|\Phi_\theta$		
$\theta\|\Phi_\sigma$		
$-\theta\|\Phi_\sigma$		

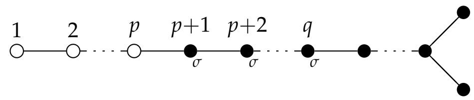
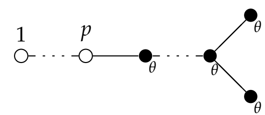
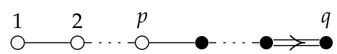
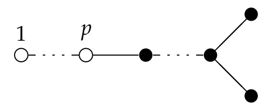
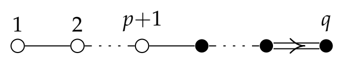
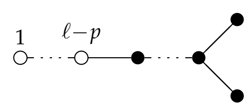
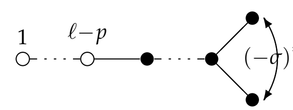
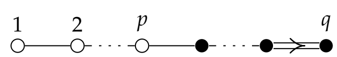
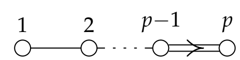
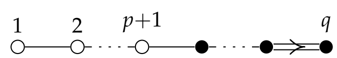
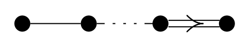
Type (θ,σ)	$C_{2\ell}^{p,\ell}(II_a, II_b, \epsilon_i)$ $\ell+p-2i<p$ $0\le i\le p-1$ $(1\le p\le \ell)$	$D_{\ell}^{q,p}(I_a, I_a, \epsilon_i)$ $q-p+2i\le p$ $1\le p<q\le \ell-1$ $(0\le i\le p-1)$
(θ,σ)-Diagram		
$\sigma\|\Phi_\theta$		
$-\sigma\|\Phi_\theta$		
$\theta\|\Phi_\sigma$		
$-\theta\|\Phi_\sigma$		

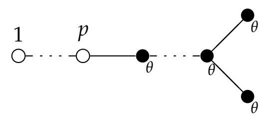
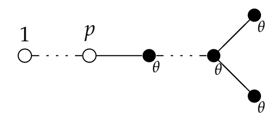
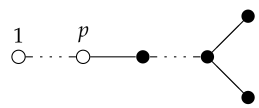
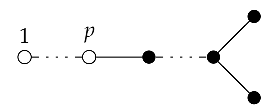
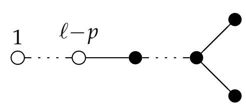
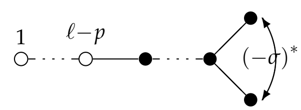
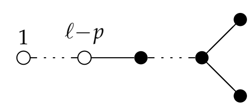
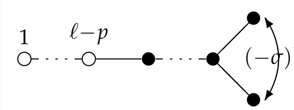
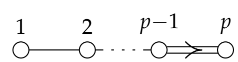
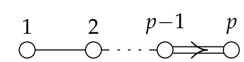
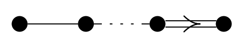
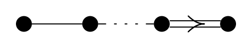
Type (θ,σ)	$D_\ell^{q,p}(I_a, I_a, \epsilon_i)$ $p < q < q-p+2i \leq \ell$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p-1)$	$D_\ell^{q,p}(I_a, I_a, \epsilon_i)$ $q \leq 2\ell - q + p - 2i < \ell$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p-1)$
(θ,σ)-Diagram		
$\sigma \vert \Phi_\theta$		
$-\sigma \vert \Phi_\theta$		
$\theta \vert \Phi_\sigma$		
$-\theta \vert \Phi_\sigma$		

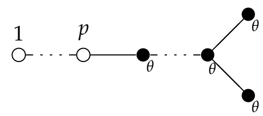
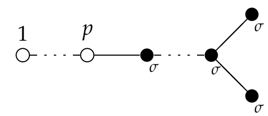
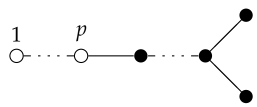
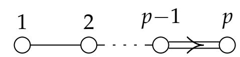
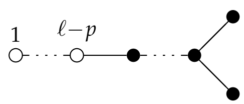
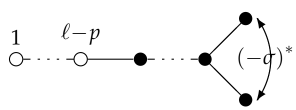
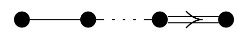
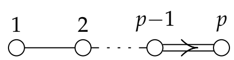
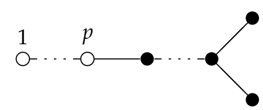
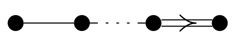
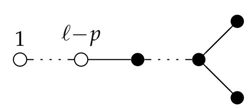
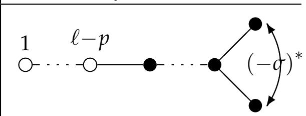
Type (θ, σ)	$D_\ell^{q,p}(I_a, I_a, \epsilon_i)$ $p \leq 2\ell - q + p - 2i < q$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p - 1)$	$D_\ell^{q,p}(I_a, I_a, \epsilon_i)$ $2\ell - q + p - 2i < p$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p - 1)$
(θ, σ)-Diagram		
$\sigma\|\Phi_\theta$		
$-\sigma\|\Phi_\theta$		
$\theta\|\Phi_\sigma$		
$-\theta\|\Phi_\sigma$		

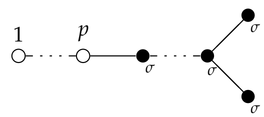
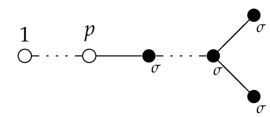
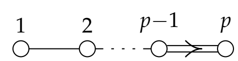
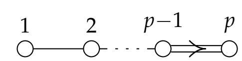
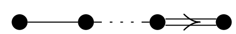
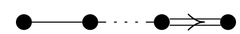
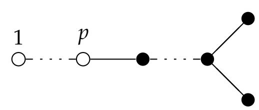
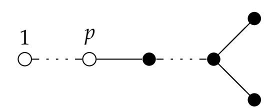
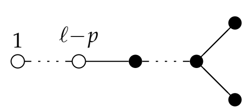
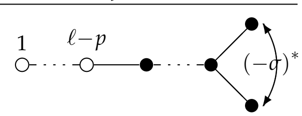
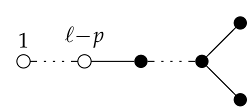
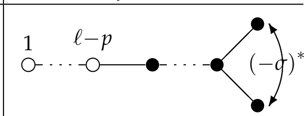
Type (θ,σ)	$D_\ell^{p,q}(I_a,I_a,\epsilon_i)$ $q-p+2i \leq p$ $1 \leq p < q \leq \ell-1$ $(0 \leq i \leq p-1)$	$D_\ell^{p,q}(I_a,I_a,\epsilon_i)$ $p < q-p+2i \leq \ell$ $1 \leq p < q \leq \ell-1$ $(0 \leq i \leq p-1)$
(θ,σ)-Diagram		
$\sigma\|\Phi_\theta$		
$-\sigma\|\Phi_\theta$		
$\theta\|\Phi_\sigma$		
$-\theta\|\Phi_\sigma$		

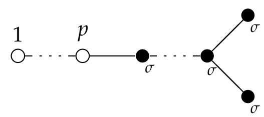
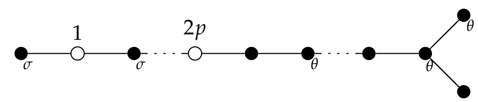
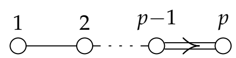
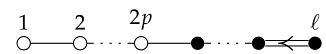
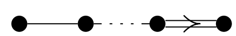
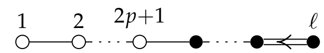
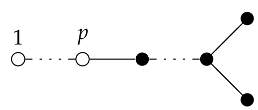
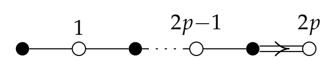
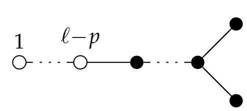
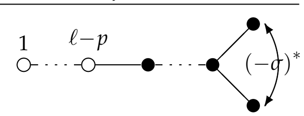
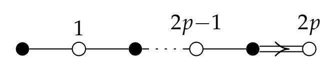
Type (θ, σ)	$D_\ell^{p,q}(I_a, I_a, \epsilon_i)$ $q \leq 2\ell - q + p - 2i < \ell$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p-1)$	$D_\ell^{p,q}(I_a, I_a, \epsilon_i)$ $p \leq 2\ell - q + p - 2i < q$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p-1)$
(θ, σ)-Diagram		
$\sigma \vert \Phi_\theta$		
$-\sigma \vert \Phi_\theta$		
$\theta \vert \Phi_\sigma$		
$-\theta \vert \Phi_\sigma$		

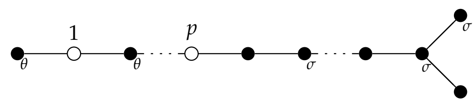
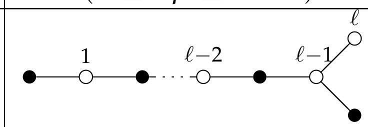
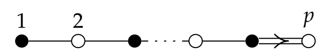
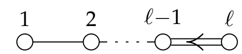
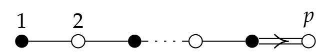
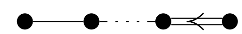
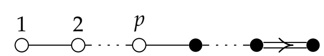
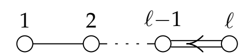
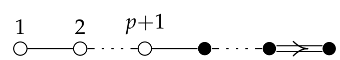
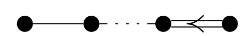
Type (θ,σ)	$D_\ell^{p,q}(I_a, I_a, \epsilon_i)$ $2\ell - q + p - 2i < p$ $1 \leq p < q \leq \ell - 1$ $(0 \leq i \leq p-1)$	$D_\ell^{\ell,p}(I_b, I_a, \epsilon_i)$ $\ell - p + 2i \leq p$ $1 \leq p < \ell$ $(0 \leq i \leq p-1)$
(θ,σ)-Diagram		
$\sigma\vert\Phi_\theta$		
$-\sigma\vert\Phi_\theta$		p even:  p odd: 
$\theta\vert\Phi_\sigma$		
$-\theta\vert\Phi_\sigma$		

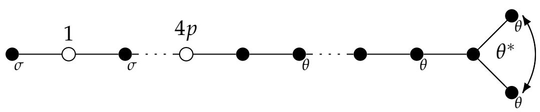
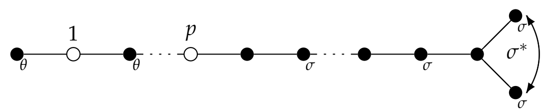
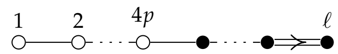
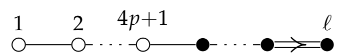
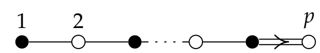
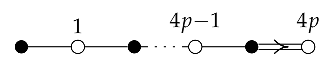
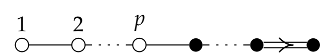
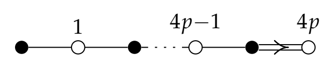
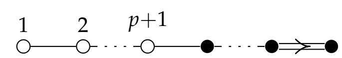
Type (θ, σ)	$D_\ell^{\ell,p}(I_b, I_a, \epsilon_i)$ $p < \ell - p + 2i \leq \ell$ $1 \leq p < \ell$ $(0 \leq i \leq p-1)$	$D_\ell^{\ell,p}(I_b, I_a, \epsilon_i)$ $p \leq \ell + p - 2i < \ell$ $1 \leq p < \ell$ $(0 \leq i \leq p-1)$
(θ, σ)-Diagram		
$\sigma\|\Phi_\theta$		
$-\sigma\|\Phi_\theta$	p even  p odd 	p even  p odd 
$\theta\|\Phi_\sigma$		
$-\theta\|\Phi_\sigma$		

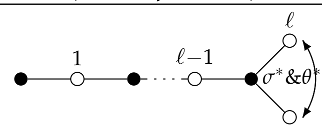
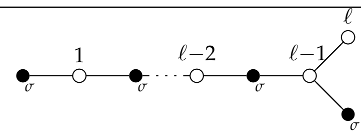
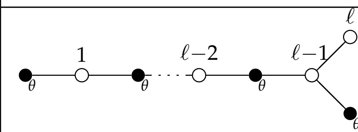
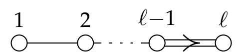
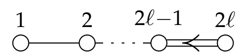
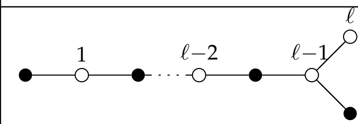
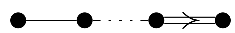
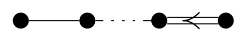
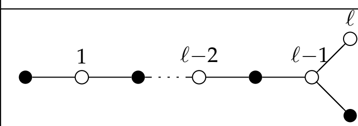
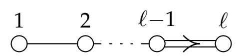
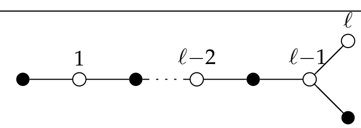
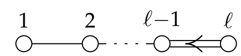
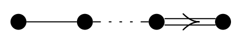
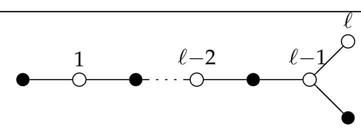
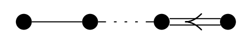
Type (θ,σ)	$D_\ell^{\ell,p}(I_b, I_a, \epsilon_i)$ $\ell + p - 2i < p$ $1 \le p < \ell$ $(0 \le i \le p-1)$	$D_\ell^{p,\ell}(I_a, I_b, \epsilon_i)$ $\ell - p + 2i \le p$ $0 \le p < \ell$ $(0 \le i \le p-1)$
(θ,σ)-Diagram		
$\sigma\|\Phi_\theta$		
$-\sigma\|\Phi_\theta$	p even:  p odd: 	
$\theta\|\Phi_\sigma$		
$-\theta\|\Phi_\sigma$		p even:  p odd: 

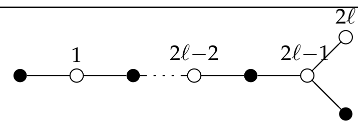
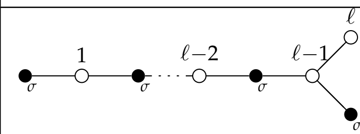
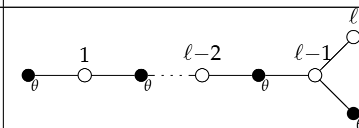
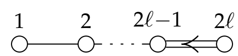
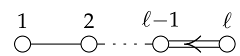
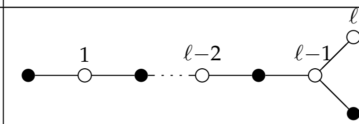
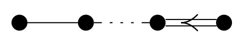
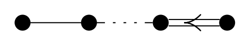
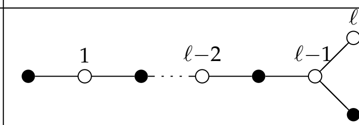
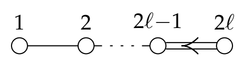
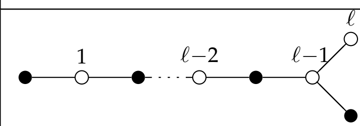
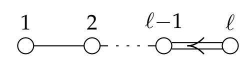
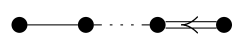
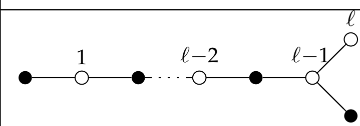
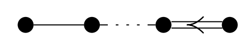
Type (θ, σ)	$D_\ell^{p,\ell}(I_b, I_a, \epsilon_i)$ $p < \ell - p + 2i \leq \ell$ $0 \leq p < \ell$ $(0 \leq i \leq p-1)$	$D_\ell^{p,\ell}(I_a, I_b, \epsilon_i)$ $p \leq \ell + p - 2i < \ell$ $0 \leq p < \ell$ $(0 \leq i \leq p-1)$
(θ, σ)-Diagram		
$\sigma \vert \Phi_\theta$		
$-\sigma \vert \Phi_\theta$		
$\theta \vert \Phi_\sigma$		
$-\theta \vert \Phi_\sigma$	p even:  p odd: 	p even:  p odd: 

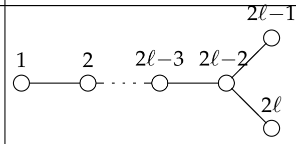
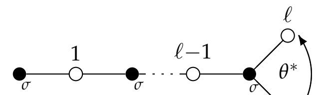
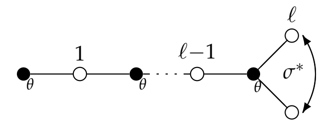
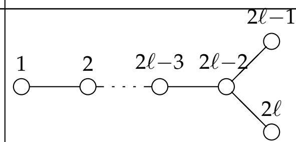
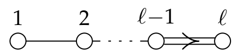
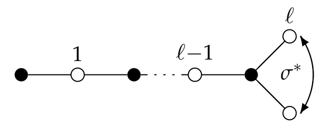
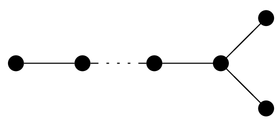
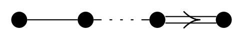
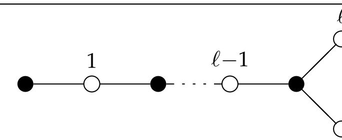
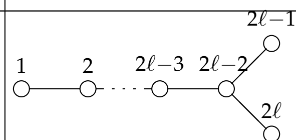
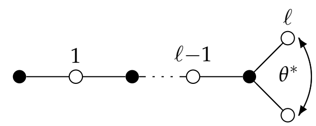
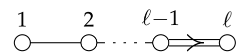
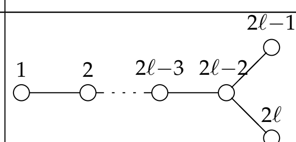
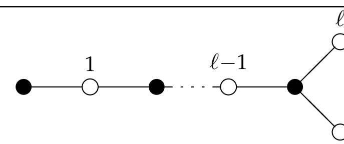
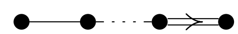
Type (θ,σ)	$D_\ell^{p,\ell}(I_a, I_b, \epsilon_i)$ $\ell + p - 2i < p$ $0 \leq p < \ell$ $(0 \leq i \leq p-1)$	$D_{2\ell}^{\ell,4p}(III_a, I_a)$ $(1 \leq 2p \leq \ell - 1)$
(θ,σ)-Diagram		
$\sigma\|\Phi_\theta$		
$-\sigma\|\Phi_\theta$		
$\theta\|\Phi_\sigma$		
$-\theta\|\Phi_\sigma$	p even:  p odd: 	

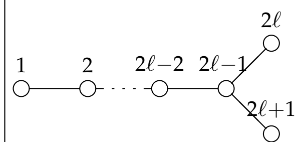
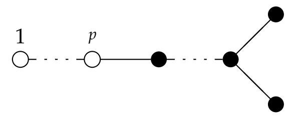
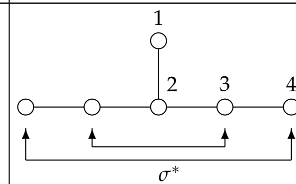
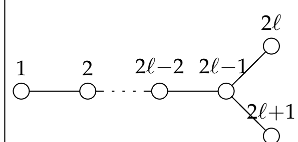
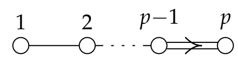
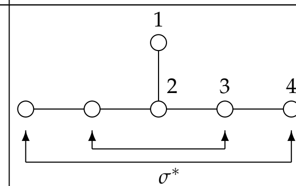
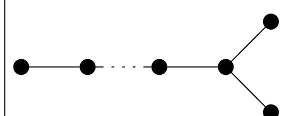
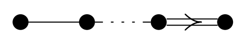
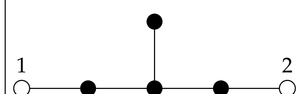
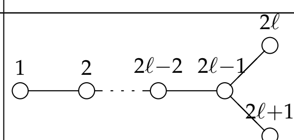
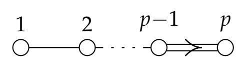
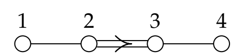
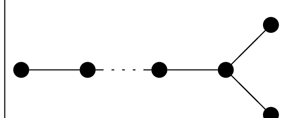
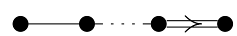
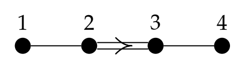
Type (θ, σ)	$D_{2\ell}^{2p,\ell}(I_a, III_a)$ $(1 \leq 2p \leq 2\ell - 1)$	$D_{2\ell}^{\ell,\ell}(III_a, \epsilon_p)$ $(1 \leq 2p \leq \ell - 1)$
(θ, σ)-Diagram		
$\sigma\|\Phi_\theta$		
$-\sigma\|\Phi_\theta$		
$\theta\|\Phi_\sigma$		
$-\theta\|\Phi_\sigma$		

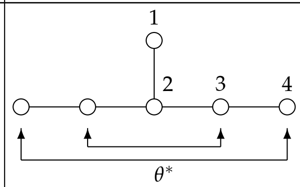
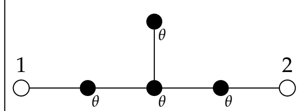
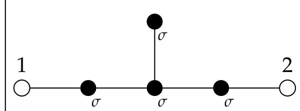
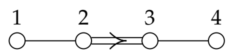
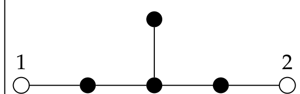
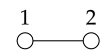
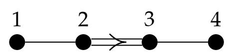
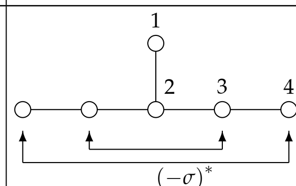
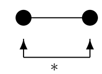
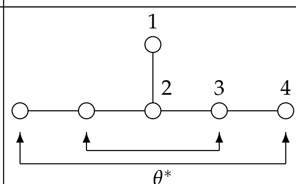
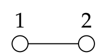
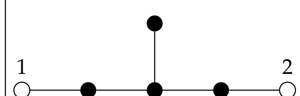
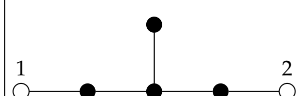
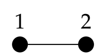
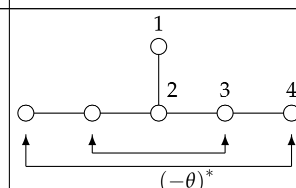
Type (θ, σ)	$D^{\ell,2p}_{2\ell+1}(III_b, I_a)$ $(1 \leq 2p \leq \ell)$	$D^{2p,\ell}_{2\ell+1}(I_a, III_b)$ $(1 \leq 2p \leq 2\ell-1)$
(θ,σ)-Diagram		
$\sigma\|\Phi_\theta$		
$-\sigma\|\Phi_\theta$		
$\theta\|\Phi_\sigma$		
$-\theta\|\Phi_\sigma$		

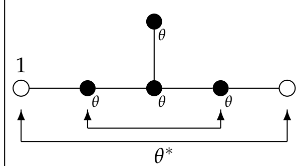
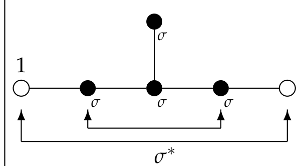
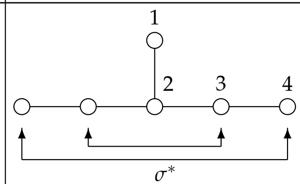
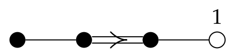
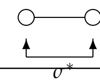
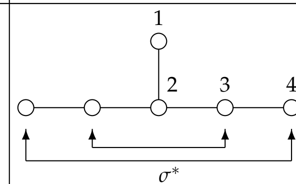
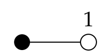
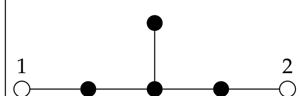
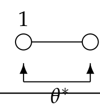
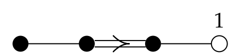
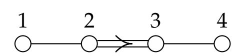
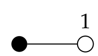
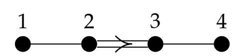
Type (θ, σ)	$D^{\ell,\ell}_{2\ell+1}(III_b, \epsilon_p)$ $(1 \leq 2p \leq 2\ell)$	$D^{2\ell,4\ell}_{4\ell}(III_a, I_b, \epsilon_0)$	$D^{2\ell,\ell}_{2\ell}(I_b, III_a, \epsilon_0)$
(θ, σ)-Diagram			
$\sigma\vert\Phi_\theta$			
$-\sigma\vert\Phi_\theta$			
$\theta\vert\Phi_\sigma$			
$-\theta\vert\Phi_\sigma$			

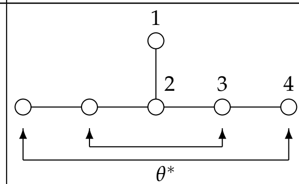
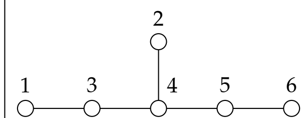
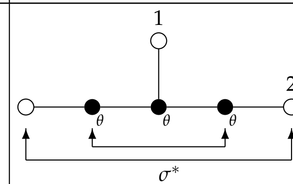
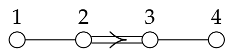
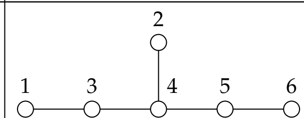
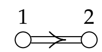
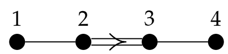
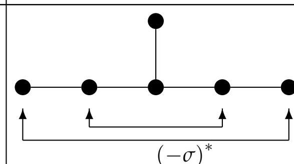
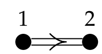
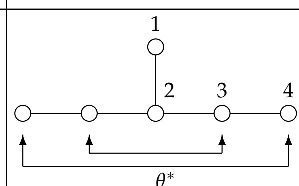
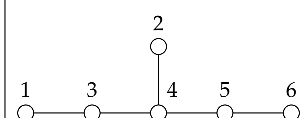
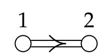
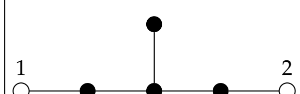
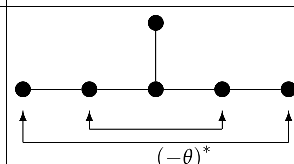
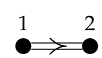
Type (θ,σ)	$D^{2\ell,2\ell}_{4\ell}(III_a,\epsilon_\ell)$	$D^{\ell,2\ell}_{2\ell}(III_a,I_b,\epsilon_\ell)$	$D^{2\ell,\ell}_{2\ell}(I_b,III_a,\epsilon_\ell)$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

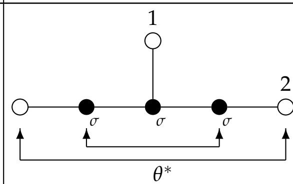
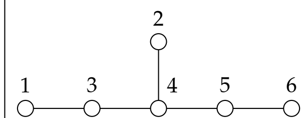
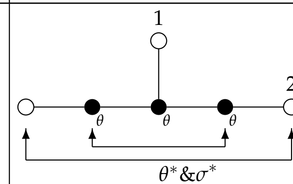
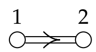
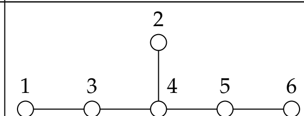
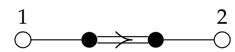
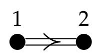
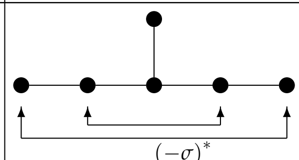
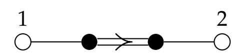
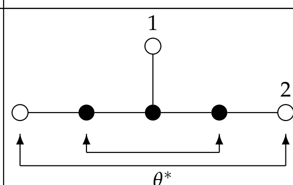
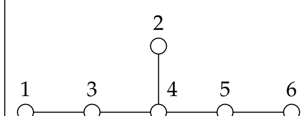
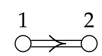
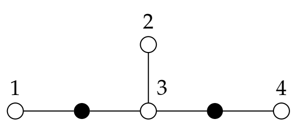
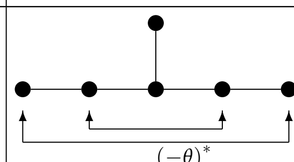
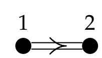
Type (θ,σ)	$D_{2\ell}^{2\ell}(I_b,\epsilon_\ell)$	$D_{2\ell+1}^{\ell,2\ell+1}(III_b,I_b)$	$D_{2\ell+1}^{2\ell+1,\ell}(I_b,III_b)$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

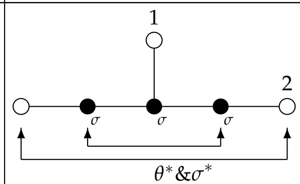
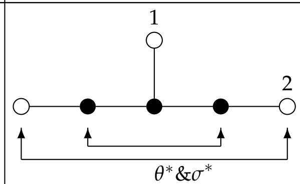
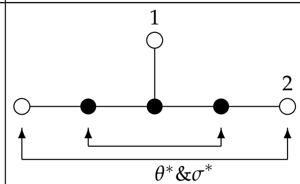
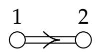
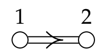
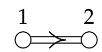
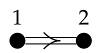
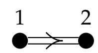
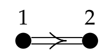
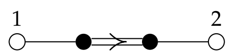
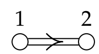
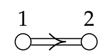
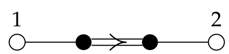
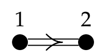
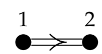
Type (θ,σ)	$D_{2\ell+1}^{2\ell+1,2\ell+1}(I_b,\epsilon_\ell)$	$D_\ell^{p,p}(I_a,\epsilon_p)$ $p<\ell-2p$	$E_6^{6,4}(I,II,\epsilon_0)$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

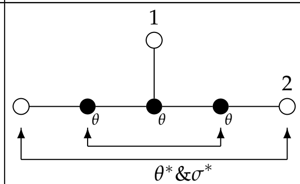
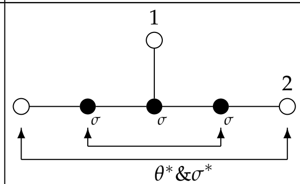
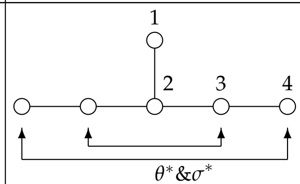
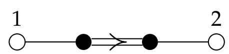
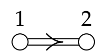
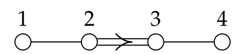
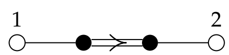
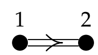
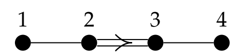
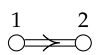
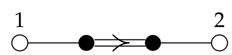
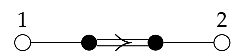
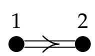
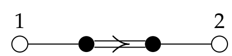
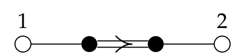
Type (θ, σ)	$E_6^{4,6}(II, I, \epsilon_0)$	$E_6^{6,2}(I, IV)$	$E_6^{2,6}(IV, I)$
(θ, σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

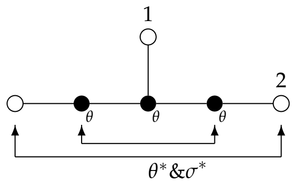
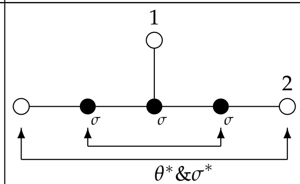
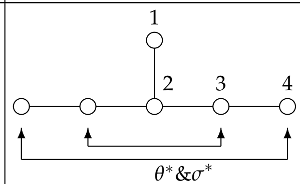
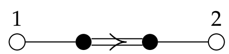
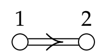
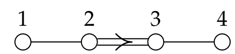
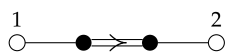
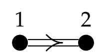
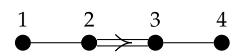
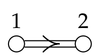
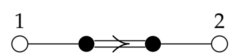
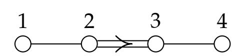
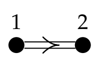
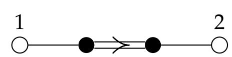
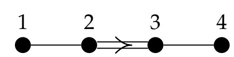
Type (θ,σ)	$E_6^{4,2}(II,IV)$	$E_6^{2,4}(IV,II)$	$E_6^{6,4}(I,II,\epsilon_1)$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

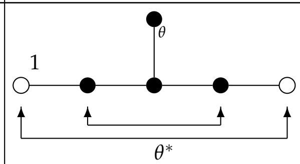
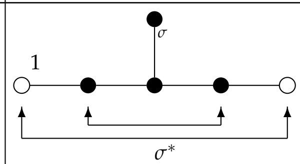
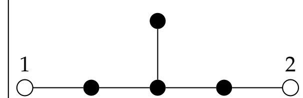
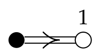
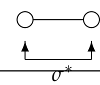
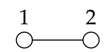
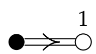
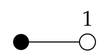
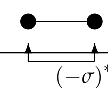
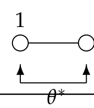
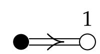
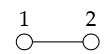
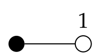
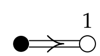
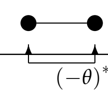
Type (θ,σ)	$E_6^{4,6}(II,I,\epsilon_1)$	$E_6^{6,6}(I,\epsilon_2)$	$E_6^{6,2}(I,III)$
(θ,σ)-Diagram			
$\sigma\vert\Phi_\theta$			
$-\sigma\vert\Phi_\theta$			
$\theta\vert\Phi_\sigma$			
$-\theta\vert\Phi_\sigma$			

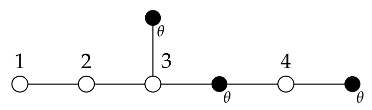
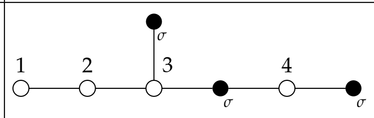
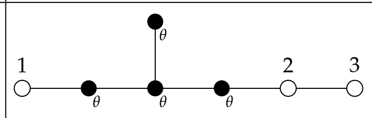
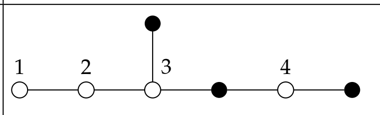
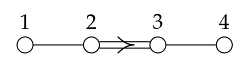
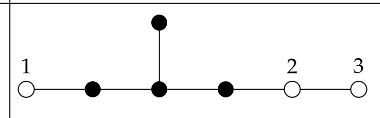
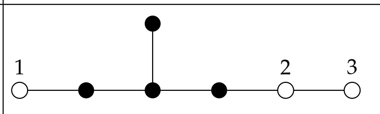
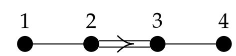
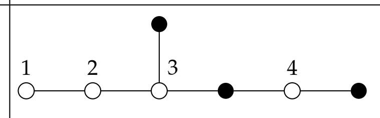
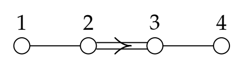
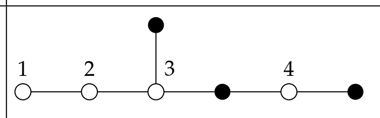
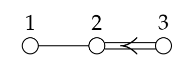
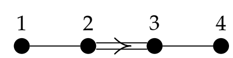
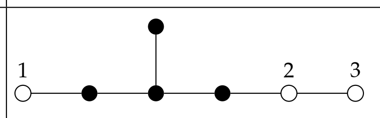
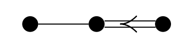
Type (θ,σ)	$E_6^{2,6}(III,I)$	$E_6^{6,6}(I,\epsilon_1)$	$E_6^{4,2}(II,III,\epsilon_0)$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

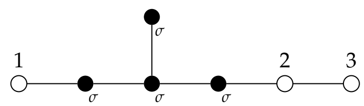
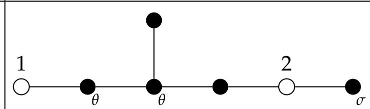
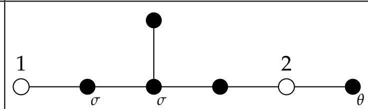
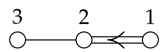
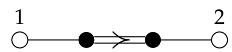
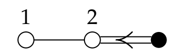
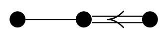
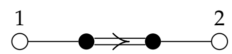
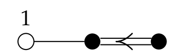
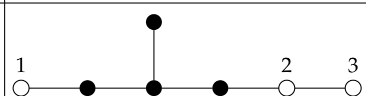
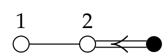
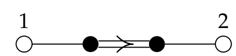
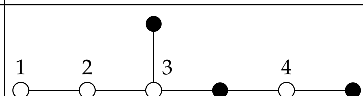
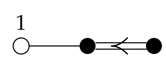
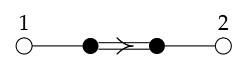
Type (θ,σ)	$E_6^{2,4}(III, II, \epsilon_0)$	$E_6^{2,2}(III, \epsilon_1)$	$E_6^{2,2}(III, \epsilon_2)$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

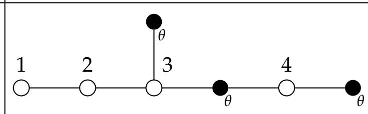
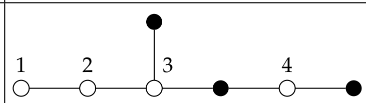
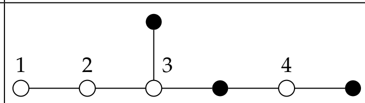
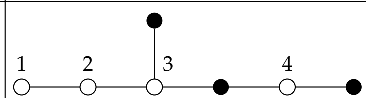
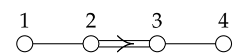
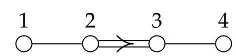
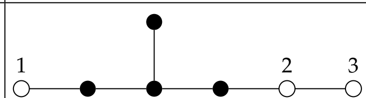
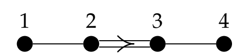
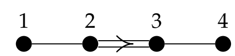
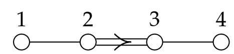
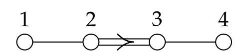
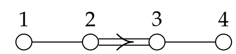
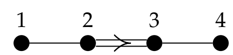
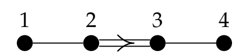
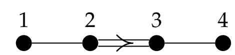
Type (θ, σ)	$E_6^{4,2}(II, III, \epsilon_1)$	$E_6^{2,4}(III, II, \epsilon_1)$	$E_6^{4,4}(II, \epsilon_4)$
(θ, σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

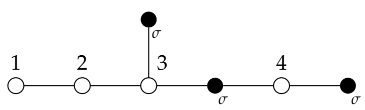
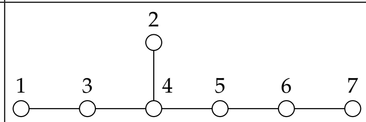
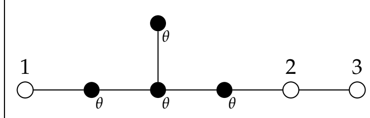
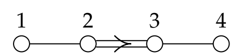
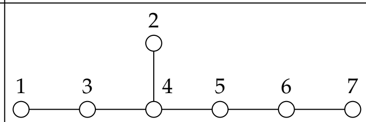
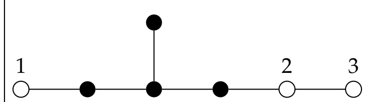
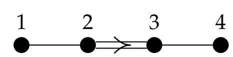
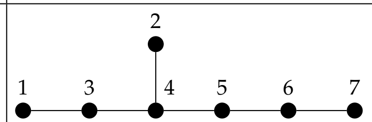
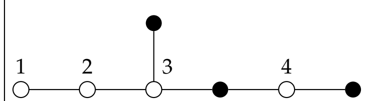
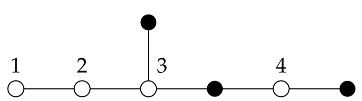
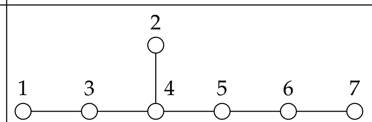
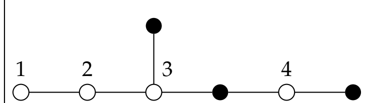
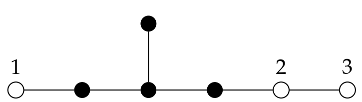
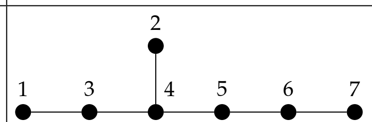
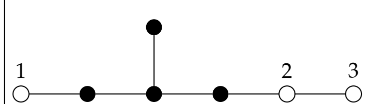
Type (θ,σ)	$E_6^{4,4}(II,\epsilon_1)$	$E_6^{2,4}(III,II,\epsilon_1)$	$E_6^{4,4}(II,\epsilon_4)$
(θ,σ)-Diagram			
$\sigma\vert\Phi_\theta$			
$-\sigma\vert\Phi_\theta$			
$\theta\vert\Phi_\sigma$			
$-\theta\vert\Phi_\sigma$			

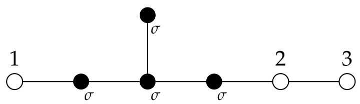
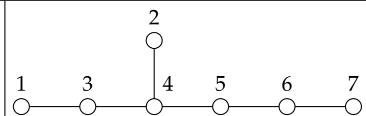
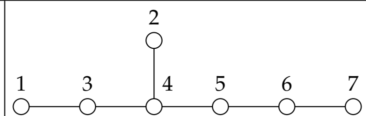
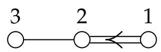
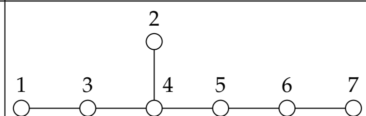
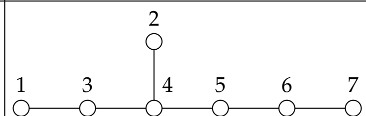
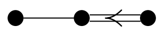
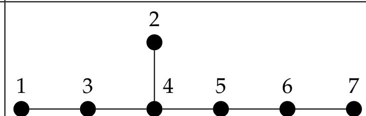
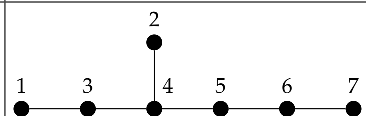
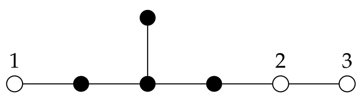
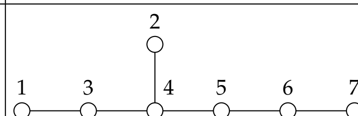
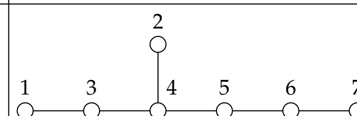
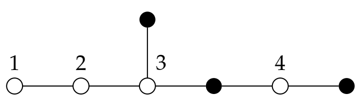
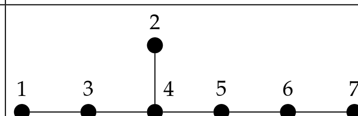
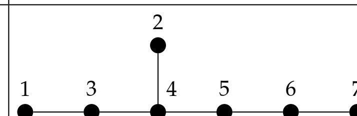
Type (θ, σ)	$E_6^{2,2}(III, IV)$	$E_6^{2,2}(IV, III)$	$E_6^{2,2}(IV, \epsilon_4)$
(θ, σ)-Diagram			
$\sigma\vert\Phi_\theta$			
$-\sigma\vert\Phi_\theta$			
$\theta\vert\Phi_\sigma$			
$-\theta\vert\Phi_\sigma$			

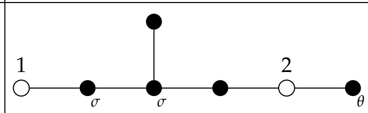
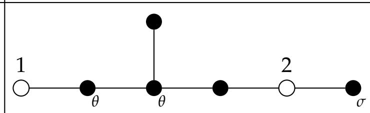
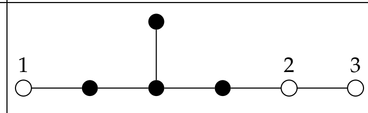
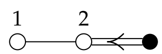
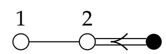
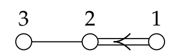
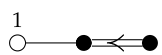
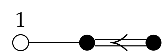
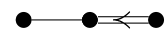
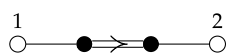
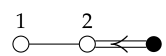
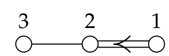
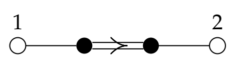
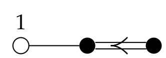
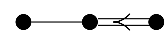
Type (θ,σ)	$E_7^{7,4}(V,VI,\epsilon_0)$	$E_7^{4,7}(VI,V,\epsilon_0)$	$E_7^{7,3}(V,VII)$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

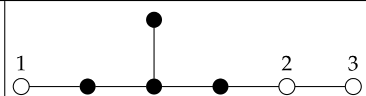
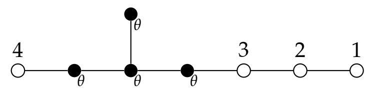
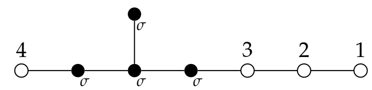
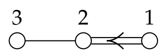
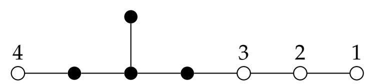
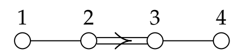
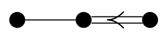
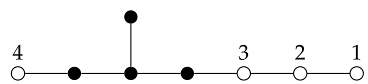
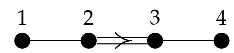
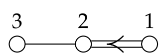
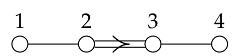
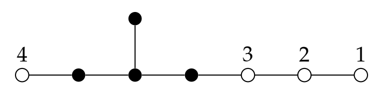
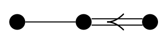
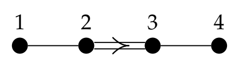
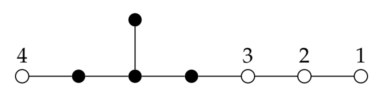
Type (θ,σ)	$E_7^{3,7}(VII,V)$	$E_7^{4,3}(VI,VII,\epsilon_1)$	$E_7^{3,4}(VII,VI,\epsilon_1)$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

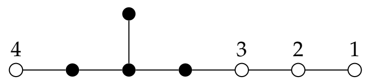
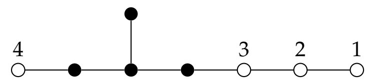
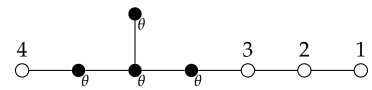
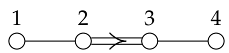
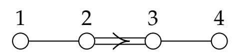
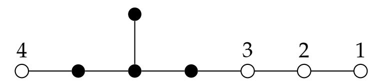
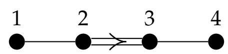
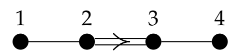
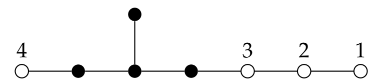
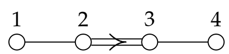
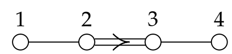
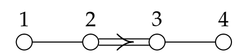
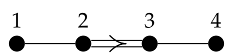
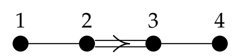
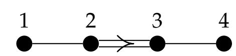
Type (θ, σ)	$E_7^{7,4}(V, VI, \epsilon_1)$	$E_7^{4,4}(VI, \epsilon_1)$	$E_7^{4,4}(VI, \epsilon_4)$
(θ, σ)-Diagram			
$\sigma \vert \Phi_\theta$			
$-\sigma \vert \Phi_\theta$			
$\theta \vert \Phi_\sigma$			
$-\theta \vert \Phi_\sigma$			

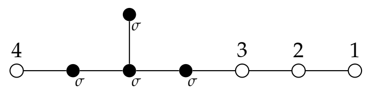
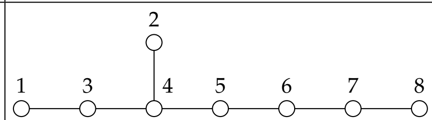
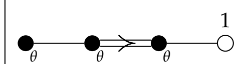
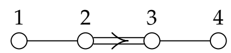
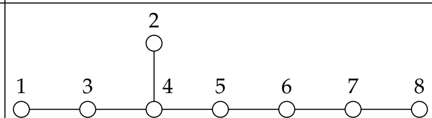
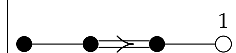
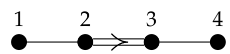
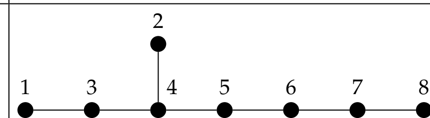
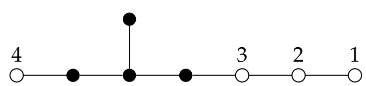
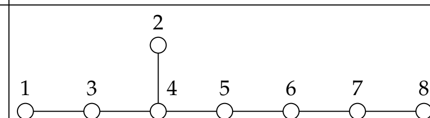
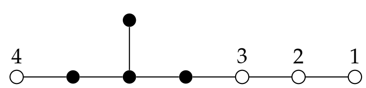
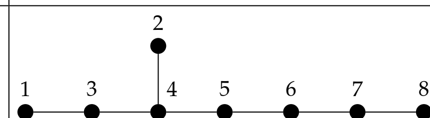
Type (θ,σ)	$E_7^{4,7}(VI, V, \epsilon_1)$	$E_7^{7,7}(V, \epsilon_1)$	$E_7^{7,3}(V, VII, \epsilon_1)$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

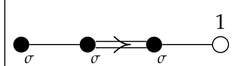
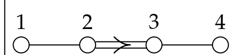
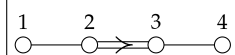
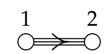
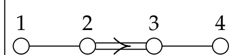
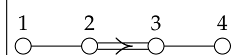
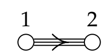
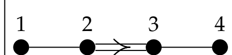
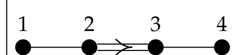
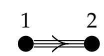
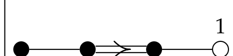
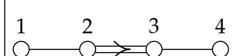
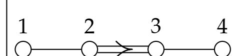
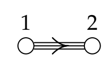
Type (θ, σ)	$E_7^{3,7}(VII, V, \epsilon_1)$	$E_7^{7,7}(V, \epsilon_7)$	$E_7^{7,7}(V, \epsilon_2)$
(θ, σ)-Diagram			
$\sigma \vert \Phi_\theta$			
$-\sigma \vert \Phi_\theta$			
$\theta \vert \Phi_\sigma$			
$-\theta \vert \Phi_\sigma$			

Type (θ, σ)	$E_7^{3,4}(VII, VI)$	$E_7^{4,3}(VI, VII)$	$E_7^{3,3}(VII, \epsilon_3)$
(θ, σ)-Diagram			
$\sigma \vert \Phi_\theta$			
$-\sigma \vert \Phi_\theta$			
$\theta \vert \Phi_\sigma$			
$-\theta \vert \Phi_\sigma$			

Type (θ, σ)	$E_7^{3,3}(VIII, \epsilon_1)$	$E_8^{8,4}(VIII, IX, \epsilon_0)$	$E_8^{4,8}(IX, VIII, \epsilon_0)$
(θ, σ)-Diagram			
$\sigma \vert \Phi_\theta$			
$-\sigma \vert \Phi_\theta$			
$\theta \vert \Phi_\sigma$			
$-\theta \vert \Phi_\sigma$			

Type (θ,σ)	$E_8^{4,4}(IX,\epsilon_4)$	$E_8^{4,4}(IX,\epsilon_1)$	$E_8^{8,4}(VIII,IX,\epsilon_1)$
(θ,σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

Type (θ, σ)	$E_8^{4,8}(IX, VIII, \epsilon_1)$	$E_8^{8,8}(VIII, \epsilon_8)$	$F_4^{4,1}(I, II)$
(θ, σ)-Diagram			
$\sigma\|\Phi_\theta$			
$-\sigma\|\Phi_\theta$			
$\theta\|\Phi_\sigma$			
$-\theta\|\Phi_\sigma$			

Type (θ, σ)	$F_4^{1,4}(II, I)$	$F_4^{4,4}(I, \epsilon_4)$	$F_4^{4,4}(I, \epsilon_1)$	$G_2^{2,2}(I, \epsilon_1)$
(θ, σ)-Diagram				
$\sigma \vert \Phi_\theta$				
$-\sigma \vert \Phi_\theta$				
$\theta \vert \Phi_\sigma$				
$-\theta \vert \Phi_\sigma$