ABSTRACT

BURCH, TIFFANY MICHELLE. Supersolvable Leibniz Algebras. (Under the direction of Dr. Ernest Stitzinger.)

Malcev algebras and Leibniz algebras are generalizations of Lie algebras. Classical theorems in Lie algebra have found extensions to Malcev algebras. It is the purpose of this work to extend some of the Lie and Malcev algebra results, particularly supersolvable results, to Leibniz algebras. In particular, results about lower semi-modular algebras, supersolvable projectors, theta-pairs, two-recognizeable properties, and triangulable properties will be discussed.

Supersolvable Leibniz Algebras

by
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A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Applied Mathematics

Raleigh, North Carolina

2015

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DEDICATION

To my husband, Matthew, and all those who believed in me.

BIOGRAPHY

Tiffany graduated from Coe College in 2011 with a double major in mathematics and chemistry. She attended North Carolina State University where she obtained her masters in mathematics in 2013 before completing her doctorate degree in 2015. While at NC State, Tiffany discovered a passion for teaching mathematics to undergraduate students. It is her career goal to continue teaching math at the collegiate level.

ACKNOWLEDGEMENTS

I would like to thank my advisor, Dr. Ernest Stitzinger, for his support and guidance throughout my graduate career, especially during the research process. He played a crucial role in helping me complete this goal.

I would also like to thank my committee members, Dr. Kailash Misra, Dr. Justin Post, and Dr. Mohan Putcha, for their assistance in completing my coursework as well as with my dissertation.

In addition, I would like to thank my family for their love and support during my graduate career. Matthew, Mom, and Dad, I could not have made it through without your willingness to listen, offer encouragement, and provide assistance when necessary.

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Chapter 1

Introduction

Lie algebras emerged as a result of Lie Groups. Sophus Lie developed the theory of Lie groups in the nineteenth century when working with differential equations [9]. Lie discovered his theory on transformation groups could be applied to the theory of differential equations [2]. As Lie groups were studied in the twentieth century, mathematicians such as Cartan, Weyl, and Borel began to generalize the topic to other areas of mathematics (algebraic topology, Lie algebras, etc.) [24].

Lie algebras have been extensively explored over the past 100 years. Throughout this exploration, a number of known properties, theorems, and results have emerged. A related area of interest involves the behavior of Lie algebra generalizations and their properties. Anatoly Maltsev generalized Lie algebras to Malcev algebras in 1955 [21] and Jean-Louis Loday generalized Lie algebras to Leibniz algebras in 1993 [20].

Lie groups and Lie algebras have become powerful tools in the fields of applied mathematics and physics in studying differential equations, perturbation theory, and dynamical models [9]. Studying the generalizations of Lie algebras allows algebraists to better understand the original concepts in Lie groups and Lie algebras. These generalizations satisfy the defining properties of a Lie algebra but contain at least one structural difference. My research looks at a particular generalization, Leibniz algebras, to see if certain Lie algebra results hold true in the Leibniz algebra case.

Every Lie algebra satisfies the conditions of a Leibniz algebra. However, an algebra can be Leibniz but not Lie. In a Lie algebra, the product, $[x, y]$, is skew-symmetric if $[x, y] = -[y, x]$. This property is not assumed in Leibniz algebras. (In a matrix Lie algebra, the product is defined as $[A, B] = AB - BA$.) This distinction is why the interest in extending Lie algebra results to Leibniz algebras has emerged. While it is already known that Lie's Theorem, Engel's Theorem, and Levi's Theorem for Lie algebras have analogues in Leibniz algebras [15], a number of other Lie algebra results remain to be studied.

In the past five years, North Carolina State University's mathematics department has been extending Lie algebra concepts to Leibniz algebras. The concept of supersolvability is one Lie algebra property of particular interest in Leibniz algebras. Supersolvability can be roughly thought of as the algebra of upper triangular matrices. Supersolvable Lie algebra properties were studied and applied to Malcev algebras by Stitzinger [28]. Since Malcev algebras are a generalization of Lie algebras, the question of whether the same properties would translate to another Lie algebra generalization emerged.

Following the format of Stitzinger [28], I have made progress in finding conditions involving left multiplication which are equivalent to supersolvability. From these results I have been able to extend structure properties of supersolvable Lie algebras to supersolvable Leibniz algebras. These results include supersolvable projectors, theta pairs, and 2-recognizeability.

Chapter 2

Preliminaries

In order to investigate the idea of supersolvability, we need to have a basic understanding of Lie algebra and Leibniz algebra properties. I will present the relevant definitions and results as stated by Demir [15] and Misra [22]. However, I will use the notation of Barnes [7] where the bilinear map

$$[\,,\,] : L \times L \to L$$

representing multiplication is presented without brackets. Thus $[x, y]$ will be presented as xy.

Definition 2.0.1. A *(left) Leibniz algebra* L is an $\mathbb{F}$-vector space equipped with a product satisfying the Leibniz identity

$$a(bc) = (ab)c + b(ac) \qquad \text{for all } a, b, c \in L.$$

Definition 2.0.2. A linear transformation $\delta : L \to L$ is a *derivation* of L if

$$\delta(xy) = \delta(x)y + x\delta(y) \text{ for all } x, y \in L.$$

Thus the left multiplication operator is a derivation. A *(right) Leibniz algebra* can be defined as a vector space equipped with a bilinear multiplication such that the right multiplication operator is a derivation [15].

When referring to a Leibniz algebra, I will be using the definition of a (left) Leibniz algebra as opposed to a (right) Leibniz algebra. Thus I will denote left and right multiplication of L by $x \in L$ as

$$L_x(a) = xa \text{ and } R_x(a) = ax \text{ for all } a \in L.$$

As previously mentioned, every Lie algebra is a Leibniz algebra. However, any Leibniz algebra that contains an element a such that $a^2 \neq 0$ is not a Lie algebra. Leibniz algebras of dimension greater than one which are generated by one element are simple examples of Leibniz algebras that are not Lie. They are called cyclic.

Definition 2.0.3. An algebra $L = \langle a \rangle$ where $a^2 = aa, \ldots, a^n = aa^{n-1}, aa^n = \alpha_1 a + \cdots + \alpha_n a^n$ is called *cyclic*.

Example 2.0.4. Consider the two-dimensional algebra with basis $\{a, a^2\}$ where

$$\begin{array}{c|cc} & a & a^2 \\ \hline a & a^2 & a^2 \\ a^2 & 0 & 0 \end{array}$$

L is cyclic and is a simple example of a Leibniz algebra that is not Lie.

Definition 2.0.5. Leib(L) =span$\{a^2|a \in L\}$

In [6], Leib(L) is shown to be an ideal.

Definition 2.0.6. Let I be a subspace of L.

1. I is a *subalgebra* if $II \subseteq I$.

2. I is an *ideal* of L, $I \triangleleft L$, if $LI \subseteq I$ and $IL \subseteq I$.

The sum and intersection of two ideals of a Leibniz algebra is an ideal, as in Lie algebras. However the result does not hold true for the product of two ideals [15].

Definition 2.0.7. Let $I \triangleleft L$.

$$L/I = \{x + I \,|\, x \in L\}$$

is called the *quotient Leibniz algebra*.

Definition 2.0.8.

1. The *left center of L* is

$$Z^l(L) = \{x \in L \mid xa = 0 \text{ for all } a \in L\}.$$

2. The *center of L* is

$$Z(L) = \{x \in L \mid xa = 0 = ax \text{ for all } a \in L\}.$$

Definition 2.0.9. Let S be a subset of L.

1. The *left centralizer of S* is

$$Z^l_L(S) = \{x \in L \mid xs = 0 \text{ for all } s \in S\}.$$

2. The *centralizer of S* is

$$Z_L(S) = \{x \in L \mid xs = 0 = sx \text{ for all } s \in S\}.$$

If S is an ideal of L, then the left centralizer and the centralizer are also ideals in L.

Definition 2.0.10. The series of ideals

$$L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \dots \text{ where } L^{(1)} = LL, L^{(i+1)} = L^{(i)}L^{(i)}$$

is called the *derived series of L*.

Definition 2.0.11. L is *solvable* if $L^{(m)} = 0$ for some integer $m \geq 0$.

As in Lie algebras, the sum and intersection of two solvable ideals of a Leibniz algebra is solvable and the unique maximal solvable ideal containing all solvable ideals of L is called the *radical of L*, denoted rad(L) [15].

Definition 2.0.12. The series of ideals

$$L \supseteq L^1 \supseteq L^2 \supseteq \dots \text{ where } L^{i+1} = LL^i$$

is called the *lower central series of L*.

Definition 2.0.13. L is *nilpotent* if $L^m = 0$ for some integer m.

As in Lie algebras, the sum and intersection of two nilpotent ideals of a Leibniz algebra is nilpotent and the unique maximal nilpotent ideal of L is called the *nilradical of L*, denoted nil(L) [15]. For our purposes, nil(L) will also be denoted by $N(L)$.

Chapter

3

Supersolvable

3.1 Background

The idea of supersolvability was first explored in groups with the hope that it would provide more insight into groups than just studying solvability. A group G is supersolvable if every homomorphic image $H \neq 1$ of G contains a cyclic normal subgroup different from 1 [3]. A number of properties for supersolvable groups have been presented by Deskins and Venske [31].

The idea was then extended to Lie algebras with a slight change in the definition. A Lie algebra is supersolvable if it has an ascending chain of ideals whose factors are of dimension one [18]. A number of properties for finite dimensional supersolvable Lie algebras have been presented by Barnes [6] and Barnes and Newell [5].

From the study of supersolvablility in groups and Lie algebras, a natural transition is to explore the idea in Lie algebra generalizations. Following the work of Stitzinger [28], I will explore properties of supersolvable Leibniz algebras.

Definition 3.1.1. A Leibniz algebra L is *supersolvable* if there is a chain

$$0 = L_0 \subset L_1 \subset \ldots \subset L_{n-1} \subset L_n = L$$

where L_i is an i-dimensional ideal of L.

Example 3.1.2. Consider an algebra with basis elements $\{a, a^2, a^3, \ldots, a^n\}$ where

$$aa^j = a^{j+1} \text{ for } j = 1, \ldots, n-1 \text{ and } aa^n = 0$$

$$a^i a^j = 0 \text{ for } i = 2, \ldots, n; j = 1, \ldots, n$$

The algebra is clearly Leibniz but not Lie since $a^2 \neq 0$. The algebra is also supersolvable since

$$0 = L_0 \subset L_1 = \{a^n\} \subset L_2 = \{a^{n-1}, a^n\} \subset \ldots \subset L_{n-1} = \{a^2, \ldots, a^n\} \subset L_n = \{a, \ldots, a^n\} = L.$$

3.2 Supersolvable Results

Lemma 3.2.1. *$L_0^*(a)$ is a subalgebra of L for all $a \in L$.*

Proof. Let $a \in L$. Suppose that L_a has the minimum polynomial

$$m(x) = \pi_1(x)^{d_1} \dots \pi_k(x)^{d_k}$$

where $\pi_i(x)$ is linear for $i \leq s$ and nonlinear for $i > s$.

Then

$$\pi_i(x) = (x - c_i)^{d_i} \text{ for } i \leq s.$$

Let $L^*_{c_i}$ for $i \leq s$ and $L^*_{\pi_i}$ for $i > s$ be components in the primary decomposition of L_a acting on L. Let

$$L_0^*(a) = L^*_{c_1} \bigoplus \dots \bigoplus L^*_{c_s} \text{ and } L_1^*(a) = L^*_{\pi_{s+1}} \bigoplus \dots \bigoplus L^*_{\pi_t}$$

and

$$m_1(x) = (x - c_1)\dots(x - c_s).$$

Both $L_0^*(a)$ and $L_1^*(a)$ are invariant under L_a by the general theory of linear transformations. Notice that $m_1(L_a)$ acts nilpotently on $L_0^*(a)$ since

$$m_1(L_a)^{d_i} y = (L_a - c_1 I)^{d_i} \dots (L_a - c_s I)^{d_i} y = 0 \text{ for } y \in L^*_{c_i}$$

due to the terms on the right hand side of the equation commuting. Hence $m_1(L_a)^{max(d_i)} = 0$

on $L_0^*(a)$. Also

$$\gcd(m_1(x), \pi_i(x)^{d_i}) = 1 = m_1(x)u(x) + \pi_i(x)v(x) \text{ for some } u(x), v(x).$$

Let $z \in L^*_{\pi_i}$ Then

$$z = Iz = (m_1(L_a)u(L_a) + \pi_i(L_a)^{d_i} v(L_a))z = m_1(L_a)u(L_a)z.$$

If $m_1(L_a)z = 0$, then $Iz = z = 0$. Hence $z = 0$. Therefore $m_1(L_a)$ is non-singular on each $L^*_{\pi_i}$ for $i > s$ and $m_1(L_a)$ is non-singular on $L_i^*(a)$. Let $x \in L^*_{c_i}$ and $y \in L^*_{c_j}$. Then, using the Leibniz identity for derivations,

$$L_a - ((c_i + c_j)I)^{d_i+d_j}(xy) = \sum_{k=0}^{d_i+d_j} \binom{d_i + d_j}{k} (L_a - c_i I)^k x (L_a - c_j I)^{d_i+d_j-k} y = 0$$

gives that $xy \in L_{c_i+c_j}$. □

It will also be useful to know how the derived algebra, L^2, acts on minimal ideals of L.

Lemma 3.2.2. *Let A be an ideal of L such that dim$(A) = 1$. Then $L^2 \subseteq C_L(A)$.*

Proof. Let $x, y \in L$ and $a \in A$ with $xa = \alpha_x a$. Since A is a minimal ideal of L, then either

$$ax = -\alpha_x a \text{ or } ax = 0 \text{ for all } x \in L.$$

Then

$$(xy)a = x(ya) - y(xa) = x(\alpha_y a) - y(\alpha_x a) = \alpha_x \alpha_y a - \alpha_y \alpha_x a = 0$$

or

$$(xy)a = 0.$$

Therefore $L^2A = 0$. Similarly,

$$a(xy) = (ax)y + x(ay) = (-\alpha_x a)y + x(-\alpha_y a) = \alpha_x \alpha_y a - \alpha_y \alpha_x a = 0$$

or

$$a(xy) = 0.$$

Therefore $AL^2 = 0$. Thus $L^2 \subseteq C_L(A)$ □

Lemma 3.2.3. *L^2 is nilpotent if L is supersolvable.*

Proof. Use induction on the dimension of L. Let A be a minimal ideal of L. Then $L^2 \subseteq C_L(A)$ by Lemma 3.2.2. Now let $A \subseteq L^2$. By induction, $(L/A)^2$ is nilpotent because (L/A) is supersolvable. However, $(L/A)^2 = (L^2 + A)/A$. Thus there exists a k such that $((L^2 + A)/A)^k = 0$. Therefore $(L^2 + A)^k \subseteq A$ for some k. Then

$$(L^2)^{k+1} \subseteq (L^2 + A)^{k+1} = (L^2 + A)(L^2 + A)^k \subseteq (L^2 + A)A = 0.$$

Thus L^2 is nilpotent. □

It is now possible to find a characterization of supersolvable Leibniz algebras in terms of $L_0^*(a) = L$. Since Lie's theorem does not hold in algebraically closed fields of prime characteristic, the following result requires nilpotency of L^2 in order to obtain supersolvability.

Theorem 3.2.4. *L is supersolvable if and only if L^2 is nilpotent and $L_0^*(a) = L$ for all $a \in L$.*

Proof. " $\Longrightarrow$ " Suppose L is supersolvable. Then L^2 is nilpotent by Lemma 3.2.3. Let $v_1, \dots, v_n$ be a basis for L such that $\langle v_1, \dots, v_s \rangle$ is an ideal in L for each s. Then the characteristic polynomial for L_a is the product of linear factors and the result holds.
" $\Longleftarrow$ " Suppose the conditions hold for L. Use induction on the dimension of L. It is sufficient to show that each minimal ideal has dimension one. Let A be a minimal ideal of L. Then $AL^2 + L^2A \trianglelefteq L$ and is contained in both L^2 and A. Since A is minimal, either $AL^2 + L^2A = 0$ or $AL^2 + L^2A = A$. If $AL^2 + L^2A = A$, then

$$A = AL^2 + L^2A = (AL^2 + L^2A)L^2 + L^2(AL^2 + L^2A).$$

Repeating the process shows that the term never becomes 0. This contradicts the fact that L^2 is nilpotent. Hence $AL^2 + L^2A = 0$ and $AL^2 = 0 = L^2A$. Since

$$0 = (xy)a = x(ya) - y(xa),$$

it follows that $L_xL_y = L_yL_x$ on A. Since A is a minimal ideal of L, either $R_x = -L_x$ for all $x \in L$ or $R_x = 0$ for all $x \in L$ when R_x acts on A. Hence all R_x, L_y commute on A. That is

$$R_xL_y = -L_xL_y = -L_yL_x = L_yR_x$$

or

$$R_xL_y = 0 = L_yR_x$$

and

$$R_xR_y = L_xL_y = L_yL_x = R_yR_x$$

on A. Since $L_0^*(a) = L, L_a$ can be triangularized on A as can R_a since $R_a = -L_a$ or $R_a = 0$. Since all L_a, R_a can be triangularized on A and they commute, they can be simultaneously triangularized on A. Hence there is a common eigenvalue z where $A = \langle z \rangle$. Thus $\dim(A) = 1$ and induction on L/A yields L supersolvable. □

It is interesting to note that Theorem 3.2.4 provides a version of Lie's theorem for Leibniz algebras over algebraically closed fields of prime characteristic. In Lie algebras, Lie's theorem does not hold in this case. At characteristic 0, L is solvable if and only if L^2 is nilpotent thus giving the desired result. However, at prime characteristic, L solvable does not imply that L^2 is nilpotent. Thus the result holds for algebraically closed fields of prime characteristic as long as L^2 is nilpotent.

The following theorem follows from a result in both Lie and Malcev algebras as well as in group theory. It is part of a general theory of Schunck classes developed by Barnes in [7]. In particular, the Leibniz algebra result shows that the class of supersolvable Leibniz algebras is a saturated formation. According to Barnes [7], a class $\mathscr{X}$ is *saturated* if $A \triangleleft L$, $B \triangleleft A$, $B \subseteq \Phi(L)$ and $A/B \in \mathscr{X}$, then $A \in \mathscr{X}$.

Definition 3.2.5. The *Frattini ideal of* L, $\Phi(L)$, is the largest ideal that is contained in the intersection of all maximal subalgebras of L. The intersection is called the *Frattini subalgebra*.

Theorem 3.2.6. *Let* $\Phi(L)$ *be the Frattini ideal of* L *and* B *be an ideal contained in* $\Phi(L)$. *If* L/B *is supersolvable, then* L *is supersolvable.*

Proof. Let $x \in L$. Since $L/B = \bar{L}$ is supersolvable, $\bar{L}^2$ is nilpotent by Lemma 3.2.3. Hence $L^2/(L^2 \cap B)$ is nilpotent and L^2 is nilpotent. Since B is an ideal in $\Phi(L)$, $L_1^*(x) \subseteq B$ for all $x \in B$. Since $\bar{L}$ is supersolvable, $L_1^*(x) \subseteq B \subseteq \Phi(L)$. Hence $L_0^*(x) + \Phi(L) = L$ for all $x \in L$. $L_0^*(x)$ is a subalgebra of L by Lemma 3.2.1, which forces $L_0^*(x) = L$. Therefore L is supersolvable by Theorem 3.2.4. □

3.3 Structure Results

The following results are structure results on solvable Leibniz algebras. Similar structure results hold for groups as well as for Lie algebras. In particular, this next result corresponds to a famous result in group theory by B. Huppert stating that a finite group is supersolvable if and only if each of its maximal subgroups has prime index. [26].

Theorem 3.3.1. *Let L be a solvable Leibniz algebra. Then L is supersolvable if and only if all maximal subalgebras of L have codimension one in L.*

Proof. „ $\Longrightarrow$ " Suppose L is supersolvable and let B be a maximal subalgebra of L. Let A be a minimal ideal of L. By induction on the dimension of L, if $A \subseteq B$, then every maximal subalgebra B/A of L/A has codimension one. Thus B has codimension one in L and the result holds. If $A \nsubseteq B$, then $A + B = L$. Since L is solvable, A is abelian and $A \cap B = 0$. Since L is supersolvable and A is a minimal idea of L, $\dim(A) = 1$. Therefore the result holds.

„ $\Longleftarrow$ " Suppose the condition holds. Let A be a minimal ideal of L. If A is contained in the Frattini subalgebra of L, then L/A is supersolvable by induction on the dimension of L and $A \subseteq \Phi(L)$. Thus L is supersolvable by Theorem 3.2.6. If some maximal subalgebra N has $A \nsubseteq N$, then $A \nsubseteq \Phi(L)$ since A is not in the Frattini subalgebra of L. Thus $L = A + N$ and $A \cap N$ is an ideal of L contained in A since A is abelian. Thus $A \cap N = 0$ due to A being a minimal ideal. Thus $\dim(A) = 1$. Then L/A is supersolvable by induction. Therefore L is supersolvable. □

Theorem 3.3.2. *Let L be a solvable Leibniz algebra. Then L is supersolvable if and only if all chains of subalgebras of L have the same length.*

Proof. " $\Longrightarrow$ " If L is supersolvable, then Theorem 3.3.1 gives the desired result that all chains have length equal to the dimension of L.

" $\Longleftarrow$ " Suppose that all chains of subalgebras of L have the same length. In a solvable Leibniz algebra, refining the derived series allows us to add components to the chain giving a chain of length equal to the dimension of L. By the assumption, all maximal subalgebras of L will be composed of chains of subalgebras of equal length. Thus all maximal subalgberas will have codimension one in L. Therefore the result holds by Theorem 3.3.1. □

This next result is similiar to Theorem 3.3.1. However, it follows the variation of Huppert's result proved by O. U. Kramer where a finite solvable group G is supersolvable if and only if, for every maximal subgroup M of G, either $F(G) \subseteq M$ or $M \cap F(G)$ is a maximal subgroup of $F(G)$, where $F(G)$ is the Fitting subgroup of G [30]. In the Leibniz algebra version, the largest nilpotent ideal of L, $N(L)$, will be used in place of the Fitting subgroup of G.

Theorem 3.3.3. *Let L be a solvable Leibniz algebra. Then L is supersolvable if and only if for every maximal subalgebra B of L, either $N(L) \subseteq B$ or $B \cap N(L)$ is a maximal subalgebra of $N(L)$.*

Proof. " $\Longrightarrow$ " Let L be supersolvable and B be a maximal subalgebra of L. Then $\mathrm{codim}_L(B) = 1$ by Theorem 3.3.1. If $N(L) \subseteq B$, the result follows. Thus suppose $N(L) \nsubseteq B$. Then $L = B + N(L)$ implies that $L/B = (B+N(L))/B \cong N(L)/(B \cap N(L))$. Therefore $1 = \mathrm{codim}_L(B) = \mathrm{codim}_{N(L)}(B \cap N(L))$. Hence $B \cap N(L)$ is maximal in $N(L)$ and the result holds.

" $\Longleftarrow$ " Suppose that L satisfies the condition for every maximal subalgebra. Consider the Frattini ideal, $\Phi(L) \lhd L$. Since L is a solvable Leibniz algebra, $N(L/\Phi(L)) = N(L)/\Phi(L) = Asoc(L/\Phi(L))$ where the abelian socle of L, $Asoc(L/\Phi(L))$, is the union of all abelian minimal ideals of $L/\Phi(L)$ [8]. Hence $L/\Phi(L)$ satisfies the condition on the subalgebras. If $\Phi(L) \neq 0$, by induction $L/\Phi(L)$ is supersolvable. Thus L is supersolvable by Theorem 3.2.6.

Suppose that $\Phi(L) = 0$. Then $Asoc(L) = N(L) = Z_L(Soc(L))$ where the socle of L, $Soc(L)$, is the union of all minimal ideals of L and is the direct sum of some of these minimal ideals [8]. Hence $N(L) = H_1 + \ldots + H_R$, where H_i is a minimal ideal that is abelian for each i. Then each H_i is complemented by a maximal subalgebra B_i and $N(L) = N(L) \cap L = N(L) \cap (H_i + B_i) = H_i + (B_i \cap N(L))$. By assumption, $B_i \cap N(L)$ is a maximal subalgebra of $N(L)$. Since $N(L)$ is a nilpotent ideal, $B_i \cap N(L)$ is an ideal of $N(L)$. Thus $1 = \text{codim}_{N(L)}(B_i \cap N(L)) = \dim H_i$. Hence $L^2 \subseteq C_L(H_i)$ by Lemma 3.2.2. Thus $L^2 \subseteq \cap_{i=1}^r C_L(H_i) = C$. Therefore $C \lhd L$ and $C = C_L(N(L))$. Suppose $C \not\subseteq N(L)$. Then $(C + N(L))/N(L)$ contains a minimal ideal $D/N(L)$ where $D^{n+1} \subseteq N(L)^n$ for $n = 1, 2, \ldots$. Hence D is nilpotent, which is a contradiction. Thus $L^2 \subseteq C \subseteq N(L)$ and we see that L^2 is nilpotent.

Suppose B is a maximal subalgebra of L. If $N(L) \subseteq B$, then $L^2 \subseteq B$. Thus B is an ideal in L and B will have codimension one in L. If $N(L) \not\subseteq B$, then $L = B + N(L)$. Thus $\text{codim}_L(B) = \text{codim}_{N(L)}(B \cap N(L)) = 1$, since $B \cap N(L)$ is maximal in $N(L)$ by assumption and $N(L)$ is a nilpotent ideal. Since all maximal subalgebras of L have codimension one in L, L is supersolvable by Theorem 3.3.1. $\square$

Chapter

4

Supersolvable Extensions

4.1 Lower Semi-Modular

In [31], Humphreys and Johnson present results where the lattice of subgroups is lower semi-modular. The following results show that similar properties hold for subalgebras of Leibniz algebras.

Definition 4.1.1. An algebra L is called *lower semi-modular* if for every pair of subalgebras A, B such that A is a maximal subalgebra of $\langle A, B\rangle$, the algebra generations by A and B, $A \cap B$

is a maximal subalgebra of B.

Lemma 4.1.2. *The following are equivalent:*

1. *L is lower semi-modular.*
2. *For every pair H, K of subalgebras of L such that H and K are maximal in $\langle H, K\rangle$, it follows that $H \cap K$ is maximal in both H and K.*

Proof. Suppose L is lower semi-modular and H is maximal in $\langle H, K\rangle$. Then $H\cap K$ is maximal in K. If K is also maximal in $\langle H, K\rangle$, then $K\cap H$ is maximal in H. Thus the result holds.

Assume (2). Let A and B be subalgebras of L with A maximal in $\langle A, B\rangle$. Choose a maximal subalgebra B_1 of $\langle A, B\rangle$ which contains B. Thus $B \subset B_1 \subset \langle A, B\rangle$. Then $\langle A, B\rangle = \langle A, B_1\rangle$. Hence $A\cap B_1$ is maximal in both A and B_1 by assumption. Now choose a maximal subalgebra B_2 of B_1 containing B such that $\langle A\cap B_1, B_2\rangle = B_1$. Thus $B \subset B_2 \subset B_1 \subset \langle A, B\rangle$. Using the assumption, $(A\cap B_1)\cap B_2 = A\cap(B_1\cap B_2) = A\cap B_2$ is maximal in both B_2 and $A\cap B_1$. Continuing the process yields a maximal chain

$$B = B_r \subset \ldots \subset B_2 \subset B_1.$$

Then $A\cap B_r = A\cap B$ is a maximal subalgebra of $B_r = B$. Thus L is lower semi-modular.

□

Theorem 4.1.3. *Let L be a solvable Leibniz algebra. Then L is supersolvable if and only if L is lower semi-modular.*

Proof. " $\Longleftarrow$ " Suppose L is lower semi-modular. By using induction on the dimension of L, it can be shown that all maximal chains of L have the same length. In that case, L will be supersolvable by Theorem 3.3.2. Let $L = L_0 \supset L_1 \supset \ldots \supset L_r = 0$ and $L = N_0 \supset N_1 \supset \ldots \supset N_s = 0$

be two maximal chains. By Lemma 4.1.2, $L_1 \cap N_1$ is maximal in both L_1 and N_1. Let t be the length of a maximal chain of length $t+1$ from L_1 to 0. By induction, L_1 is supersolvable, hence all maximal chains have the same length in L_1. Thus $t+1 = r-1$. Similarly, $t+1 = s-1$ and $r = s$. Hence L is supersolvable.

„ $\Longrightarrow$ " Suppose L is supersolvable and let A be a minimal ideal of L. Then $L^2 \subseteq C_L(A)$ by Lemma 3.2.2 and $C_L(A)$ is an ideal of L of codimension either 0 or 1. By Lemma 4.1.2, it is enough to show that if H and K are maximal subalgebras of $\langle H, K\rangle$, then $H \cap K$ is maximal in H and in K. Since L is supersolvable, any subalgebra of L is also supersolvable. Thus it can be assumed that H and K are maximal in L.

Suppose $A \subseteq H$. If $A \subseteq K$ as well, then by induction, $(H \cap K)/A$ is maximal in H/A and K/A. Thus the desired result holds.

Suppose $A \nsubseteq K$. Then $A + K = L$ and $A \cap K = 0$. Hence $1 = \text{codim}_L(K)$. Also $H = L \cap H = (A + K) \cap H = A + (K \cap H)$. Thus $\text{codim}_H(K \cap H) = 1$ and $K \cap H$ is maximal in H. Since L is supersolvable, $\text{codim}_L(H) = 1$. Therefore $\text{codim}_L(H) + \text{codim}_H(H \cap K) = 2 = \text{codim}_L(K) + \text{codim}_K(H \cap K)$. Therefore $\text{codim}_K(H \cap K) = 1$ and $H \cap K$ is maximal in K.

Now suppose the neither H nor K contains a minimal ideal of L. Since $C_L(A)$ is an ideal of L, $H \cap C_L(A) = 0 = K \cap C_L(A)$. Hence $C_L(A) = A$. But $\dim(A) = 1$ since A is a minimal ideal of L. Thus $\text{codim}_L(C_L(A))$ is either 0 or 1. In the first case, $\dim L = 1$ and the result holds. In the second case, $\dim H = \dim K = 1$ and $H \cap K$ is maximal in both H and K. □

4.2 Supersolvable Projectors

Supersolvable projectors can be discussed for solvable Leibniz algebras following the theory of $\mathscr{F}-$projectors in Lie algebras [4] and Cartan subalgebras of Malcev algebras [27].

Definition 4.2.1. A subalgebra H of L is called a *supersolvable projector* if H is supersolvable and if K is a subalgebra of L such that $H \subseteq K \subseteq L$ and $J \trianglelefteq K$ such that K/J is supersolvable, then $K = H + J$.

Lemma 4.2.2. *Let H be a supersolvable projector of the solvable Leibniz algebra L. Then*

1. *If $H \subseteq U$, then H is a supersolvable projector of U.*
2. *If $N \lhd L$, U/N is a supersolvable projector of L/N and H is a supersolvable projector of U, then H is a supersolvable projector of L.*

Proof. Part (1): Suppose $H \subseteq U$. Choose $K \subseteq U$. Clearly H is a supersolvable projector of U.

Part (2): The result is trivial if dim $L = 0$. Use induction over the dimension of L. Suppose $H \subseteq K \subseteq L$, $K_0 \trianglelefteq K$, and K/K_0 is supersolvable. Then $K + N \supseteq H + N = U$ and U/N is a supersolvable projector of $(K+N)/N$. But $(K+N)/N \cong K/(N \cap K)$ and therefore $(U \cap K)/(N \cap K)$ is a supersolvable projector of $K/(N \cap K)$. Since H is a supersolvable projector of $U \cap K$, if $K \subset L$, then H is a supersolvable projector of K by induction and $H + K_0 = K$. Suppose $K = L$. Since $L/(K_0 + N)$ is supersolvable, $U/N + (K_0 + N)/N = L/N$ and so $U + N + K_0 = L$. Since $N \subseteq U$, $U + K_0 = L$. Therefore $U/(K_0 \cap U) \cong L/K_0$ is supersolvable. This implies $H + (K_0 \cap U) = U$ and $H + K_0 = U + K_0 = L$. □

Theorem 4.2.3. *Each solvable Leibniz algebra contains supersolvable projectors.*

Proof. Suppose L is a minimal counter example and let A be a minimal ideal in L. Then there exists $A \subseteq U$ such that U/A is a supersolvable projector of L/A. Suppose $U \neq L$. Then there exists a supersolvable projector H of U and the result follows from part (2) of Lemma 4.2.2. Therefore let $U = L$. Hence L/A is supersolvable. If $A \subseteq \Phi(L)$, then L is supersolvable by Theorem 3.2.6 and the result holds.

Suppose that $A \nsubseteq \Phi(L)$ and that A is complemented in L by B. Hence $B \cong L/A$ is supersolvable. Let $D \trianglelefteq L$ such that L/D is supersolvable. Then $L/(D \cap A)$ is supersolvable. Since L is not supersolvable, $D \cap A \neq 0$. A is a minimal ideal of L, so $A \subseteq D$. Thus $D \cap A = A$. Hence $D + B = A + B = L$ and B is a supsersolvable projector of L. □

4.3 Theta Pairs

In [25], Salemkar, Chehrazi, and Niri use properties of maximal θ-pairs to characterize solvable and supersolvable Lie algebras. In a similar manner, I will extend the properties to Leibniz algebras by making use of previously stated supersolvable properties.

Definition 4.3.1. Let M be a maximal subalgebra of L. A pair (A, B) of subalgebras of L is said to be a *θ-pair* for M if it satisfies the following conditions:

1. B is an ideal of L contained in A
2. $B \subseteq M$ and $A \nsubseteq M$
3. A/B contains properly no nonzero ideal of L/B.

Definition 4.3.2. If A is an ideal of L, then the pair (A, B) is called an *ideal θ-pair* for M.

Definition 4.3.3. A θ-pair (A, B) for M is said to be *maximal* if M has no θ-pair (C, D) such that $A \subset C$.

The following lemmas and propositions follow the Lie algebra results as presented in [25] and the proofs will therefore be omitted. In the statements, $\theta(M)$ will denote the set of all θ-pairs for a maximal subalgebra M.

Lemma 4.3.4. *Let L be a Leibniz algebra, M a maximal subalgebra of L and $I \trianglelefteq L$ with $I \subseteq M$.*

1. *If (A, B) is a (ideal) θ-pair for M and $I \subseteq B$, then $(A/I, B/I)$ is a (ideal) θ-pair for M/I. Conversely, if $(A/I, B/I)$ is a (ideal) θ-pair for M/I, then (A, B) is a (ideal) θ-pair for M. In particular, (A, B) is a maximal member in $\theta(M)$ if and only if $(A/I, B/I)$ is a maximal member in $\theta(M/I)$.*

2. *If (A,B) is an ideal θ-pair of M, then $\theta(M)$ contains an ideal maximal θ-pair (C,D) such that $(A,B) \subseteq (C,D)$ and $A/B \cong C/D$.*

Definition 4.3.5. For a subalgebra B of L, let B_L, the *core* (with respect to L) of B, be the largest ideal of L contained in B.

Lemma 4.3.6. *Let L be a Leibniz algebra, M a maximal subalgebra of L and (A,B) an ideal maximal θ-pair for M. Then $B = M_L$.*

Lemma 4.3.7. *Let L be a non-simple Leibniz algebra over a field $\mathbb{F}$ of characteristic zero. If L has an abelian maximal subalgebra, then L is solvable.*

Proposition 4.3.8. *Let $\mathscr{X}$ be a class of Leibniz algebras which is closed under taking subalgebras and quotient Leibniz algebras. Let M be a maximal subalgebra of a Leibniz algebra L and I an ideal of L with $I \subseteq M$. If (A,B) is a maximal θ-pair of M such that $A/B \in \mathscr{X}$, then there exists a maximal θ-pair $(A_1/I, B_1/I)$ of M/I such that $(A_1/I)/(B_1/I) \cong A_1/B_1 \in \mathscr{X}$.*

Proposition 4.3.9. *Let M be a maximal subalgebra of a Leibniz algebra L and I an ideal of L with $I \subseteq M$. If $(A,B) \in \theta(M)$ is a maximal member such that $A \cap M = B$, then there exists a maximal member $(A_1/I, B_1/I) \in \theta(M/I)$ such that $(A_1/I) \cap (M/I) = B_1/I$.*

The following proposition was proved in [8].

Proposition 4.3.10. *Let L be a Leibniz algebra with $\Phi(L) = 0$. Then $Asoc(L) = N(L)$.*

Now that the preliminaries are established, it is possible to characterize solvable and supersolvable Leibniz algebras using properties of θ-pairs.

Theorem 4.3.11. *The following properties of a Leibniz algebra L are equivalent:*

1. *L is solvable.*

2. *For each maximal subalgebra M of L and every ideal θ-pair (A,B) for M, A/B is solvable.*

3. *For each maximal subalgebra M of L and every ideal maximal θ-pair (A,B) for M, $C_{L/B}(A/B) \neq 0$.*

4. *Assuming the field $\mathbb{F}$ to be of characteristic zero, for each maximal subalgebra M of L, there exists a maximal θ-pair (A,B) for M such that A/B is abelian.*

Proof. (1) $\implies$ (2)

Since L is solvable, $A, B \subseteq L$ are solvable. Thus A/B is solvable.

(2) $\implies$ (3)

Let M be a maximal subalgebra of L and (A,B) be an ideal θ-pair for M. By part (2) of Lemma 4.3.4, there exists an ideal maximal θ-pair (C,D) for M such that $A/B \cong C/D$. By assumption, C/D is a solvable ideal of L/D. Thus $(C/D)^2$ is an ideal of L/D which is properly contained in C/D since it is solvable. Thus C/D is abelian which implies A/B is abelian. Therefore $C_{L/B}(A/B) \neq 0$.

(3) $\implies$ (1)

Suppose that L is simple. If M is a maximal subalgebra of L, then $(L,0)$ is an ideal maximal θ-pair for M. Thus $Z(L) = C_L(L) \neq 0$. This is a contradiction since L is not abelian. Thus L is non-simple.

Let L be a minimal counter example so that anything smaller than L must be solvable. By Lemma 4.3.4 and induction on the dimension of L, all proper quotient algebras of L are solvable. If L has two minimal ideals N_1 and N_2, then L/N_1 and L/N_2 are solvable by the last statement and $N_1 \cap N_2 = 0$. Therefore L is solvable.

Instead, assume that L has a unique minimal ideal N and note that L/N is solvable. If $N \subseteq \Phi(L)$, then $L/\Phi(L)$ is solvable and hence L is solvable by [6]. If $N \not\subseteq \Phi(L)$, there exists a maximal subalgebra M of L such that $M_L = 0$ and $L = N + M$. But N is a minimal ideal so N is simple or abelian. Clearly, $(N, 0)$ is an ideal maximal θ-pair for M. Then by assumption, $C_{L/0}(N/0) = C_L(N) \neq 0$. Thus N must be contained in $C_L(N)$ and therefore be abelian, thus solvable. Then L/N solvable and N solvable imply that L is solvable.

(1) $\implies$ (4)

Since L is solvable, there exists a series of ideals in L, $L = L_0 \supseteq L_1 \supseteq L_2 \supseteq \ldots \supseteq L_n = 0$, such that L_{i-1}/L_i is an abelian minimal ideal of L/L_i for $i = 1, \ldots, n$. This refinement of the derived series takes ideals inside the chain of ideals. Let M be a maximal subalgebra of L. Then for some $k \geq 1$, $L_k \subseteq M$ but $L_{k-1} \not\subseteq M$. Thus (L_{k-1}, L_k) belongs to $\theta(M)$ by the definition of a θ-pair. If (L_{k-1}, L_k) is a maximal θ-pair for M, then the result holds.

Instead, suppose that $(L_{k-1}, L_k) \subseteq (A, B)$ for some maximal θ-pair $(A, B) \in \theta(M)$. Then $L_{k-1} \not\subseteq B$ since $B = A \cap M$. This results follows from the definition of a θ-pair where $B \subseteq A \cap M$, by definition, and $A \cap M \subseteq B$ since $B \subseteq M$, $B \triangleleft A$, and $A \not\subseteq M$. Since $(L_{k-1} + B)/B$ is an ideal of L/B contained in A/B, the definition of a θ-pair implies that $L_{k-1} + B = A$. Therefore $A \trianglelefteq L$ and $A/B \cong L_{k-1}/(L_{k-1} \cap B)$ is a homomorphic image of the abelian Leibniz algebra L_{k-1}/L_k.

(4) $\implies$ (1)

Let L be a minimal counter example with N a minimal ideal of L. Then the quotient Leibniz algebra L/N satisfies the assumption by Proposition 4.3.8. Thus L/N is solvable and we may assume that N is the unique minimal ideal of L. If $N \subseteq \Phi(L)$, then N is nilpotent and L is solvable. But this is a contradiction since letting L be a minimal counter example implies that L is not solvable.

Instead, suppose that $L = N + M$ for some maximal and core-free subalgebra M of L. By the assumption, there exists a maximal θ-pair $(K,0)$ in $\theta(M)$ such that $K/0 = K$ is abelian. Clearly, $N \nsubseteq K$. Choosing H to be a subalgebra of L such that K is a maximal subalgebra of H yields that H is solvable by Lemma 4.3.7. If $H_L = 0$, then $(H,0) \in \theta(M)$. Thus $K = H$, which is a contradiction. Therefore, H contains N and thus L is solvable. But this is a contradiction since L was assumed to be a minimal counter example. □

Theorem 4.3.12. *A Leibniz algebra L is supersolvable if any one of the following conditions hold:*

1. *For any maximal subalgebra M of L and each maximal θ-pair (A,B) for M, $Z(L/B) \neq 0$.*
2. *For any maximal subalgebra M of L and each maximal θ-pair (A,B) for M, $dim(A/B) = 1$.*
3. *For any maximal subalgebra M of L and each maximal θ-pair (A,B) for M, $\Phi(L/B) \neq 0$.*
4. *L has a supersolvable maximal subalgebra M such that $N(L) \nsubseteq M$ and $\theta(M)$ contains an ideal maximal member (A,B) with $dim(A/B) = 1$.*
5. *L has a supersolvable maximal subalgebra M such that $N(L) \cap M$ is a maximal subalgebra of $N(L)$.*

Proof. Assume (1). If L is simple, then $(L,0)$ is the only θ-pair in L. Thus $Z(L) = 0$, which contradicts our assumption. Thus L is not simple. If L is nilpotent, then L is supersolvable as desired. Therefore, assume that L is a minimal counterexample such that L is neither simple nor nilpotent. If N is a minimal ideal of L, then L/N is supersolvable by induction. If K is another minimal ideal of L, then $L/N \times L/K$ is supersolvable. Since L is isomorphic to a subalgebra of $L/N \times L/K$, L is supersolvable. Hence N is the unique minimal ideal of L.

Let M be a maximal subalgebra of L and suppose that $M_L = 0$. Then $(N, 0)$ is a maximal θ-pair of M. Thus $Z(L/0) \neq 0$ by the hypothesis. Then $L = M + Z(L)$ and by the lower central series, $L^2 = M^2$ is an ideal of L in M. Thus $M^2 = 0$ or M. If $M^2 = 0$ for some M, then $L^2 = M^2 = 0$. Thus L is abelian hence nilpotent hence supersolvable, which is a contradiction. If $M^2 = M$ for all M, then $L^2 = M^2 = M$ so that M is an ideal of L and N is contained in M. Hence N is in $\Phi(L)$. Thus L is supersolvable by Theorem 3.2.6, a contradiction to the minimal counterexample claim.

Assume (2). If L is simple, then $(L, 0)$ is the only θ-pair in L and $\dim(L) \neq 1$, which contradicts our assumption. Thus L is not simple. Assume L is a minimal counterexample such that L is not simple. Let N be a minimal ideal of L such that $(N, 0)$ is an ideal maximal θ-pair for some maximal subalgebra M of L. Then $\dim(N) = 1$ by assumption. Using Lemma 4.3.6 and induction on the dimension of L, L/N is supersolvable. Thus L is supersolvable.

Assume (3). If L is simple, the only θ-pair is $(L, 0)$. Then $\Phi(L/0) = \Phi(L) = 0$, which is a contradiction. Thus L is not simple. Suppose L is a minimal counterexample such that L is not simple. By Lemma 4.3.6 and induction on the dimension of L, L has a unique minimal ideal N and L/N is supersolvable. Suppose that for some maximal subalgebra M of L, $(N, 0)$ is an ideal maximal θ-pair for M. Then $\Phi(L/0) = \Phi(L) \neq 0$ by assumption. Since the unique minimal ideal is contained in every ideal, $N \subseteq \Phi(L)$ and L is supersolvable by Theorem 3.2.6.

Assume (4). Let L be a minimal counterexample and N be a minimal ideal of L. If N is not contained in M, then L/N is a homomorphic image of M. Thus L/N is supersolvable. If $N \subseteq M$, then $B+N$ is a proper ideal of $A+N$. Since $(A+N)/(B+N)$ is a homomorphic image of A/B, it follows that $\dim((A+N)/(B+N)) = 1$ since $\dim(A/B) = 1$ by assumption and $N \subseteq B$. Hence Lemma 4.3.4 establishes the pair $(A/N, B/N)$ is a maximal member in $\theta(M/N)$ such that $\dim((A/N)/(B/N)) = 1$ since (A, B) is a maximal member in $\theta(M)$, by assumption. Therefore, the quotient algebra L/N is supersolvable by part (2) of Theorem 4.3.12.

It remains to show $\dim(N) = 1$. Without loss of generality, assume that N is the unique minimal ideal of L. Since M is a maximal subalgebra with $N(L) \nsubseteq M$, $L = M + N(L)$. Thus M supersolvable yields that $M/(N(L) \cap M)$ is supersolvable and therefore solvable. Since $L/N(L) \cong M/(N(L) \cap M)$, $L/N(L)$ is solvable. This implies that L is solvable and $\text{Asoc}(L) = N$. If $N \subseteq \Phi(L)$, since L/N is supersolvable, then L is supersolvable by Theorem 3.2.6. Therefore assume that $\Phi(L) = 0$. Then $N(L) = Asoc(L) = N$ by [8]. Since $N(L) \nsubseteq M$, N is not contained in M. Thus $M_L = 0$ which implies that $B = 0$ since B is an ideal contained in M. Thus $\dim(A) = \dim(A/B) = 1$. Since $N \subseteq A$, $\dim(N) = 1$.

Assume (5). If L is solvable, the result holds by Theorem 3.3.3. M supersolvable yields that $M/(N(L) \cap M)$ is supersolvable and therefore solvable. Since $M \cap N(L)$ is a maximal subalgebra of $N(L)$ by assumption, $L = M + N(L)$. Thus $L/N(L)$ is a homomorphic image of $M/(N(L) \cap M)$ and therefore solvable. Therefore, L is solvable and the result holds by Theorem 3.3.3. □

Chapter 5

Two-Recognizeable

5.1 Background

In Lie algebras, as well as in groups, it is possible to classify certain classes as 2-recognizeable. In particular, solvable, strongly solvable, and supersolvable are classes deemed 2-recognizeable in these areas. By working with fields of characteristic 0 or algebraically closed fields of characteristic greater than 5, it possible to show the same classes are 2-recognizeable in Leibniz algebras.

Definition 5.1.1. A property of algebras is called *n-recognizeable* if whenever all the n generated subalgebras of algebra L have the property, then L also has the property.

Thus a property is *2-recognizeable* if when all 2 generated subalgebras have the property, the algebra itself has the property. Engel's theorem shows that nilpotentcy is 2-recognizeable in both Lie algebras and Leibniz algebras. Since this work focuses on supersolvable Leibniz algebras, it is natural to explore whether supersolvability is 2-recognizeable in Leibniz algebras. Due to supersolvability's connection to solvability and strong solvability, these two classes can also be explored.

In [11] and [23], solvability, strong solvability, and supersolvability are shown to be 2-recognizeable within Lie algebras taken over a field of characteristic 0 or an algebraically closed field of characteristic greater than 5. Similar group results are shown in [12] and [13]. These results will now be extended from the Lie algebra papers to Leibniz algebras.

5.2 Method One

The following work uses the ideas presented by Moneyhun and Stitzinger in [23]. Consider first the property of solvability for Leibniz algebras.

Theorem 5.2.1. *Solvability is 2-recognizeable for Leibniz algebras over a field of characteristic 0 and over algebraically closed fields of characteristic greater than 5.*

Proof. Let L be a minimal counterexample. Simple Leibniz algebras are Lie algebras and there are no Lie algebra counterexamples, as shown in [23]. Thus, L is not simple. If N is a proper ideal of L, then the hypothesis holds in N and L/N. Hence, both are solvable and the result holds. □

Definition 5.2.2. A Leibniz algebra whose derived algebra is nilpotent is called *strongly solvable.*

In [11] and [23], strong solvability for Lie algebras over the fields mentioned in Theorem 5.2.1 is shown to be 2-recognizeable in the class of solvable Lie algebras. Similar results can be extended to Leibniz algebras using Theorem 5.2.1.

In [23], the authors show that the Lie algebra is strongly solvable if and only if $e_n(x,y)=0$ for all x and y and almost all n, where $e_n(x,y)=L^n_{xy}(x)$. A similar method will be employed here. However, $f_n(x,y)=L^n_{xy}(y)$ will also be used in order to address the lack of symmetry in Leibniz algebras.

Theorem 5.2.3. *Let L be a solvable Leibniz algebra. Then L is strongly solvable if and only if $e_n(x,y)=0$ and $f_n(x,y)=0$ for almost all n and all $x,y\in L$.*

Before proving the theorem, two lemmas will be discussed. These lemmas are used in the proof of the theorem.

Lemma 5.2.4. *Suppose that L is generated by x and y. Suppose that $e_n(x,y)=f_n(x,y)=0$ for almost all n. Then L_{xy} is nilpotent acting on L.*

Proof. Let $u\in L$. Then u is a linear combination of elements of the form $z=z_s(\ldots(z_2z_1)\ldots)$ where each $z_j=x$ or y. Since L_{xy} is a derivation,

$$L^t_{xy}(z)=\sum L^{i_s}_{xy}(z_s)(L^{i_{s-1}}_{xy}(z_{s-1})\ldots(L^{i_2}_{xy}(z_2)L^{i_1}_{xy}(z_1))\ldots)\text{ where } i_s+\ldots+i_1=t.$$

For $t=(n-1)s+1$, at least one of the terms is 0. Since L is finite dimensional, L has a basis made up of terms of the form z, and L_{xy} to a power takes each to 0. Thus L_{xy} is nilpotent on L. □

Definition 5.2.5. A solvable Leibniz algebra is called *primitive* if it has a unique minimal ideal, A, which is its own left centralizer and is complemented.

While all complements of A are conjugate in Lie algebras, [6] shows the complement is unique if L is Leibniz but not Lie. The following lemma illustrates this result for our purposes. It will also be useful to let $L_0(x)$ and $L_1(x)$ for $x \in L$ denote the Fitting 0 and 1 component of L_x acting on L.

Lemma 5.2.6. *Let L be a solvable, primitive, Leibniz algebra, which is not a Lie algebra, such that L is not strongly solvable but all proper subalgebras and quotients of L are strongly solvable. Let A be the unique minimal ideal of L. Then the complement of A in L is unique.*

Proof. Let B be a complement of A in L and $b \in B \cap L^2$. Under the conditions $L_1(b) \subset A$, hence $L_0(b)$ is a supplement of A in L. If $L_0(b) = L$ for all such b, then L^2 is nilpotent, a contradiction. Hence $B = L_0(b)$ for some b. If C is another complement to A in L, then $C = L_0(c)$ for some $c \in C \cap L^2$. Then $c = b + a$ for some $a \in A$ and $b \in B \cap L^2$. Since $a \in A =$Leib(L), $L_b = L_c$. Hence $B = L_0(b) = L_0(c) = C$. □

Proof of Theorem 5.2.3. If L^2 is nilpotent, then the condition clearly holds. For the converse, suppose that L is a minimal counterexample. Let A be a minimal ideal of L. The hypothesis holds in L/A, hence $(L/A)^2$ is nilpotent and each L_{xy} to some power takes L to A. If B is another minimal ideal of L, then $L^2 = (L/A \cap B)^2$ is nilpotent, a contradiction. Hence A is the unique minimal ideal in L. Furthermore, if $A \in \Phi(L)$, then $(L/A)^2$ is nilpotent, which gives that L^2 is nilpotent, a contradiction. Hence, $\Phi(L) = 0$, and A is complemented in L by a subalgebra B as shown in [8]. Thus, L is primitive. Since A is unique, A is its own left centralizer. We may assume that L is not a Lie algebra, for the result is known in that case [23]. Then $A = \text{Leib}(L)$ [8]. Then B is the unique complement of A.

The set $S = \{xy | x, y \in L\} \cup L^{(2)}$ is a Lie set whose span is L^2, where $L^{(2)} = [L^2, L^2]$. We will show that L_s is nilpotent on L for all s in S. Then, using [10], L^2 is nilpotent. Note that L_s is nilpotent for all $s \in L^{(2)}$ since this ideal of L is nilpotent by induction on L^2. Let U be the subalgebra generated by $x, y \in L$. If $U = L$, then L_{xy} is nilpotent on L by Lemma 5.2.4. If $U + A \neq L$, then by induction $U^2 \subset (U + A)^2$ is nilpotent. Then a power of L_{xy} takes L to A, and a further power takes L to 0. Hence, L_{xy} is nilpotent on L.

Suppose that $L = A + U$. Since we have taken care of the case that $U = L$ and $A \cap U$ is an ideal of L, U is a complement of A in L. Let $a \in A$, $a \neq 0$, and take V to be the subalgebra generated by x and $y + a$. Since $x, y = (y + a) - a \in V + A$, it follows that $L = A + U = A + V$. Then $V \cap A = 0$ or A since $V \cap A$ is an ideal of L contained in the minimal ideal A. If $V \cap A = 0$, then using Lemma 5.2.6, $U = V$ since U is the unique complement of A in L. Since y and $y + a$ are in $V = U$, a is also, a contradiction. Hence $V = L$. By Lemma 5.2.4, $L_{x(y+a)}$ acts nilpotently on L. Each s in the Lie set $T = (x(y + a)) \cup L^{(2)}$ has L_s nilpotent on L. Since $xy = x(y + a) - xa$ is in the span of T, it follows that L_{xy} is nilpotent on L. Now every s in the original Lie set S has L_s acting nilpotently on L. Hence, L^2 acts nilpotently on L and L^2 is nilpotent. □

Corollary 5.2.7. *In solvable Leibniz algebras, strong solvability is 2-recognizeable.*

Proof. Let x and y be in L. The subalgebra, H, generated by x and y, has H^2 nilpotent. Thus, L_{xy} acts nilpotently on H. Hence, $e_n(x, y)$ and $f_n(x, y)$ are 0 when $n \geq \dim H$. Therefore, the conditions of the theorem are satisfied, and L^2 is nilpotent. □

It is possible to extend this corollary to larger classes.

Theorem 5.2.8. *For Leibniz algebras over fields of characteristic 0 or algebraically closed fields of characteristic greater than 5, strong solvability is 2-recognizeable.*

Proof. Suppose the base field of the algebra is of characteristic 0 or it is algebraically closed of characteristic greater than 5. Then solvability of all two generated subalgebras yields solvability of the algebra, as seen in Theorem 5.2.1. Thus, if every two generated subalgebra has a nilpotent derived algebra, then all two generated subalgebras are solvable, and hence, L is solvable. Therefore, for the stated fields, if all two generated subalgebras are strongly solvable then the algebra is solvable and Corollary 5.2.7 gives that the algebra is strongly solvable. □

It is now possible to address the supersolvable case.

Corollary 5.2.9. *Supersolvability is 2-recognizeable in the class of solvable Leibniz algebras.*

Proof. Suppose that all two generated subalgebras are supersolvable. Then they are strongly solvable, and thus, so is L. Let $a \in L$. For any b in L, let H be the subalgebra generated by a and b. Then a satisfies the condition in Theorem 3.2.4 in H. This extends to L, yielding $L_0^*(a) = L$. Hence, L is supersolvable. □

Again, the result can easily be extended to larger classes to obtain the following theorem.

Theorem 5.2.10. *For fields of characteristic 0 or algebraically closed fields of characteristic greater than 5, supersolvability is 2-recognizeable.*

While the focus up to this point has been on whether a property is 2-recognizeable, it is worth mentioning that the class of abelian-by-nilpotent Lie algebras is 3-recognizeable. This class is not 2-recognizable, as shown in [23], using the split extension of a Heisenberg Lie algebra, L, spanned by x, y and z with $xy = z$, and a one dimensional space spanned by a semi-simple derivation D of L with $D(x) = x$, $D(y) = y$ and $D(z) = 2z$.

It is possible to extend the result to show that the class of abelian-by-nilpotent Leibniz algebras is 3-recognizable.

Definition 5.2.11. Let $\mathscr{X}$ and $\mathscr{Y}$ be two classes of algebras. An algebra L is said to be $\mathscr{X}$ *-by-*$\mathscr{Y}$ if L has an ideal $I \in \mathscr{X}$ such that $L/I \in \mathscr{Y}$

Thus, for abelian-by-nilpotent, $\mathscr{X}$ represents the class of abelian Leibniz algebras and $\mathscr{Y}$ represents the class of nilpotent Leibniz algebras.

Define $d_k(x, y, z) = (L_z^k(x))(L_z^k(y))$. Let L be a Leibniz algebra such that $d_k(x, y, z) = 0$ for all $x, y, z \in L$ and almost all k. Then $(L_1(z))(L_1(z)) = 0$ and $L_1(z)$ is an abelian ideal in L.

Theorem 5.2.12. *Over an infinite field, the following are equivalent:*

1. *L is abelian-by-nilpotent*

2. *$d_k(x, y, z) = 0$ for all $x, y, z \in L$ and all $k > \dim L$.*

Proof. (1) clearly implies (2).

Assume (2). Let L be a Leibniz algebra such that $d_k(x, y, z) = 0$ for all $x, y, z \in L$ and almost all k.

Let H be a Cartan subalgebra of L. Then H is the Fitting null component, $L_0(z)$, of L_z for some $z \in H$ by Theorem 6.5 in [6], and H is nilpotent. Furthermore, the condition gives that $(L_1(z))(L_1(z)) = 0$ and $L_1(z)$ is an abelian ideal in L. Hence, the result holds. □

Theorem 5.2.13. *The class of abelian-by-nilpotent Leibniz algebras over infinite fields is 3-recognizeable.*

Proof. Suppose that all three generated subalgebras of L are abelian-by-nilpotent, and let H be generated by x, y and z. Condition (2) of Theorem 5.2.12 holds for these x, y and z. Therefore, condition (2) holds in general in L, and L is abelian-by-nilpotent. □

5.3 Method Two

There is an alternative way to prove that the property of being strongly solvable is 2-recognizeable within the class of solvable Leibniz algebras. The alternative makes use of ideas presented by Bowman, Towers, and Varea in [11].

Definition 5.3.1. Let $\mathscr{X}$ be a class of Leibniz algebras. A Leibniz algebra L is *almost* $\mathscr{X}$ if L does not belong to $\mathscr{X}$, but every proper subalgebra of L is in $\mathscr{X}$.

Note that calling L *almost* $\mathscr{X}$ is the same as saying that L is *minimal non-*$\mathscr{X}$.

Proposition 5.3.2. *Let L be a solvable Leibniz algebra such that $\Phi(L) = 0$, and let A be a minimal ideal in L. Then the following hold:*

1. *L has a unique minimal ideal A.*
2. *There exists a maximal subalgebra B of L such that $L = A \oplus B$.*
3. *$C_L^l(A) = A$.*

Proof. Since L is solvable, it must have a minimal ideal, call it A. Suppose that $B \neq A$ is another minimal ideal of L. Then $(L/B)^2$ is nilpotent. Consider the homomorphism

$$\Pi : L \to L/A \times L/B$$

which maps $x \mapsto (x+A, x+B)$. The kernel of Π is $A \cap B$. Then $L/(A \cap B)$ is isomorphic to an algebra contained in $L/A \times L/B$. Since A and B are both minimal ideals, $A \cap B$ is an ideal. Also, since both A and B contain it, $A \cap B = 0$. Thus $L/(A \cap B) = L$, which is not strongly solvable, and hence is a contradiction. Therefore A must be the unique minimal ideal of L.

Since $\Phi(L) = 0$, A is not contained in $\Phi(L)$. Hence, there is a maximal subalgebra B of L such that A is not contained in B. Thus $A + B = L$. Then $A \cap B = 0$ since A is abelian. Therefore $L = A \oplus B$.

Next, consider the map

$$\lambda : L \to Der(A)$$

which maps $x \mapsto \lambda_x$, where $\lambda_x(y) = xy$ for all $x \in L$ and $y \in A$. Notice that λ is a homomorphism. The image of λ is equal to the set of left multiplications by elements of L that act on A, or $Der(A)$. Hence, $Im(\lambda)$ is a Lie algebra. In particular, it is the Lie algebra of derivations of A. Since A is abelian, $Ker(\lambda) = \{x \in L \mid xa = 0 \quad \forall a \in A\}$, which is the left centralizer of A in L, $C_L^l(A)$. Since $C_L^l(A)$ is the kernel of a homomorphism, it is an ideal in L.

Now, $B \cap C_L^l(A)$ is an ideal in L. To see this, consider the following.

Let $x \in B \cap C_L^l(A)$ and let $y \in B$. Consider $a \in A$. Then

$$(yx)a = y(xa) - x(ya) = 0,$$

since $ya \in A$. Thus, $yx \in B \cap C_L^l(A)$. Similarly, $xy \in B \cap C_L^l(A)$.

Next, let $y \in A$. Then

$$(yx)a = 0 = yx,$$

since $y, x \in A$ and A is abelian. Therefore, $yx \in B \cap C_L^l(A)$. Similarly, $xy \in B \cap C_L^l(A)$.

Now, $B \cap C^l_L(A) \subseteq B$. There are no ideals of L contained in B, since A is not contained in B. Thus, $B \cap C^l_L(A) = 0$. Hence, $C^l_L(A) = A$. □

Theorem 5.3.3. *Let L be a solvable Leibniz algebra with $\Phi(L) = 0$ such that L is almost strongly solvable (or minimal non-strongly solvable). Then $\mathbb{F}$ has characteristic $p > 0$ and $L = A \oplus B$ is a semidirect sum, where A is the unique minimal ideal of L, $\dim A \geq 2$, $A^2 = 0$, and B is of one of the following two types:*

- *Type I: $B = M \oplus \langle x \rangle$ is a semidirect sum, where $M \subseteq B$ is a minimal ideal of B such that $M^2 = 0$,*

  or

- *Type II: B is the three-dimensional Heisenberg algebra.*

Proof. Let L be a solvable Leibniz algebra that is Φ-free, i.e. $\Phi(L) = 0$, and let L be almost strongly solvable. This means that L is minimal non-strongly solvable; i.e. L^2 is not nilpotent, but for every proper subalgebra S of L, S^2 is nilpotent.

Clearly, char $\mathbb{F} = p > 0$. This is a result of [17] stating that every solvable Leibniz algebra over a field of characteristic 0 is strongly solvable.

Since $\Phi(L) = 0$, $L = Asoc(L) \oplus B$, where B is a Lie algebra that is isomorphic to $Der(L)$ [8]. Now, $Asoc(L)$ is equal to the sum of all minimal abelian ideals of L. However, by Proposition 5.3.2, L has a unique minimal ideal A. Thus $Asoc(L) = A$. Therefore A is abelian and $L = A \oplus B$.

Let M be any maximal ideal of B that contains B^2. Note that for any maximal ideal M of B, B^2 must be contained in M since L/M abelian and L solvable imply that $L^2 \subseteq M$. Hence, B/M is abelian. Thus, $(B/M)^2 = 0$ and $B^2 \subseteq M$. Note that B^2 is nilpotent since B is a proper subalgebra of L. Thus B is strongly solvable.

Now consider $A+M$. $A+M$ is properly contained in L since $M \neq B$. Also, $A+M$ is an ideal in L, and therefore, $(A+M)^2$ is nilpotent.

Let $N(L)$ denote the nilradical of L. Then $M^2 \subseteq (A+M)^2 \subseteq N(L)$. Since $\Phi(L)=0$, $N(L)= Asoc(L) = C_L(Soc(L))$ by [8]. Now, $C_L(Soc(L)) = C_L(Asoc(L))$ since every minimal ideal is abelian because L is solvable. Hence, $M^2 \subseteq A = C_L(Asoc(L)) = C_L(A)$. Then $M^2 \subseteq A \cap B = 0$. Thus, M is abelian.

Since $B^2 \subseteq M$ and B is a solvable Lie algebra, B/M is abelian. Thus M has codimension 1 in B. Thus, $B = M \oplus \langle x \rangle$ for $x \in B$, $x \notin M$.

Let $C = C_M(x)$. Suppose that $C_M(x) = M$. Then $mx = xm = 0$ for each $m \in M$. Then

$$B^2 = (M + \langle x \rangle)^2 \subseteq M^2 + M\langle x \rangle + \langle x \rangle M + \langle x \rangle \langle x \rangle = 0.$$

If $B^2 = 0$, then

$$L^2 = (A \oplus B)^2 \subseteq A^2 + AB + BA + B^2 \subseteq A$$

since $A^2 = B^2 = 0$. A is abelian, and hence, L^2 is nilpotent, which is a contradiction. Thus, $C \neq M$. Therefore C is properly contained in M.

Let $I \subset M$ be an ideal of B that is properly contained in M. Then $A+I$ is an ideal of L. $AI = A(A+I)$ and $IA = (I+A)A$ are both ideals of L as well, as shown by the following multiplications.

Let $l \in L$, $a \in A$, $i \in I$. Since $L = A \oplus B$, first let $l \in A$.

- $l(ai) = (la)i + a(li) \in AI$.
- $(ai)l = a(il) - i(al) = a(il) \in AI$.

- $l(ia) = (li)a + i(la) = (li)a \in IA$.
- $(ia)l = i(al) - a(il) = a(il) = 0 \in IA$.

Now let $l \in B$.

- $l(ai) = (la)i + a(li) \in AI$.
- $(ai)l = a(il) - i(al) = 0 \in AI$.
- $l(ia) = (li)a + i(la) \in IA$.
- $(ia)l = i(al) - a(il) = i(al) \in IA$.

From Proposition 5.3.2, $C_L(A) = A$. Thus $AI = A = IA$ whenever $I \neq 0$. For AI, and similarly for IA, is certainly contained in A. Suppose that $AI = 0$ for $I \neq 0$. Then $I \subseteq C_L^r(A)$. Thus $C_L(A) \cap I \neq 0$, and $A \cap I \neq 0$. However, $I \subset M \subset B$, and $A \cap B = 0$, which is a contradiction.

Consider such an ideal, $D \neq 0$ of B that is properly contained in M. Then $AD = A = DA$, from above, and $D\langle x\rangle = DB \subset M$ is an ideal of B.

To show that $D\langle x\rangle = DB = BD$, note that

$$DB = D(M \oplus \langle x\rangle) = DM + D\langle x\rangle = D\langle x\rangle$$

since $DM \subset M^2 = 0$. Since D and B are both Lie algebras, $DB = BD$. DB is an ideal of B since D is an ideal of Lie algebra B.

Since $A + D + \langle x\rangle$ is a proper subalgebra of L, $(A + D + \langle x\rangle)^2$ is nilpotent by the assumption. However,

$$(A + D + \langle x\rangle)^2 = A^2 + AD + A\langle x\rangle + DA + D^2 + D\langle x\rangle + \langle x\rangle A + \langle x\rangle D + \langle x\rangle\langle x\rangle = A + D\langle x\rangle.$$

Thus $A+D\langle x\rangle$ is nilpotent.

Consequently, $A(D\langle x\rangle)\neq A$ and $(D\langle x\rangle)A\neq A$. For if either $A(D\langle x\rangle)$ or $(D\langle x\rangle)A$ is equal to A, then $(A+D\langle x\rangle)^2=A$, and hence, $(A+D\langle x\rangle)$ is not nilpotent, which is a contradiction.

Therefore C is the unique largest ideal of B strictly contained in M. Either $C=0$ or $C\neq 0$. The results are trivial if $C=0$.

Thus, suppose that $C\neq 0$. Let $\{c_1,\ldots,c_r\}$ be a basis for C, and let $y\in M$ such that $y\notin C$. Set

$$s_1=y, s_2=xy, s_3=x(xy), s_4=x(x(xy)),$$

and so forth. There is an $n\geq 1$ such that $\{c_1,\ldots,c_r,s_1,\ldots,s_n\}$ is linearly independent, but $\{c_1,\ldots,c_r,s_1,\ldots,s_n,s_{n+1}\}$ is not. Then write

$$s_{n+1}=xs_n=\mu_1c_1+\ldots+\mu_rc_r+\lambda_1s_1+\ldots+\lambda_ns_n.$$

For brevity, denote $c=\sum_{i=1}^{r}\mu_ic_i$.

Now, consider $T=\langle c,s_1,\ldots,s_n\rangle$. T is clearly contained in M and is an ideal of B by construction, for $MT\subseteq M^2=0$ and $xT\subseteq T$ since $xc=0$ and xs_j is a linear combination of other x_k. If $c=0$, then $\langle s_1,\ldots,s_n\rangle$ is properly contained in M. Then T is properly contained in M, which implies that $T\subseteq C$. Hence, $xT=0$. However, $xT=x\langle s_1,\ldots,s_n\rangle\neq 0$ since at least $s_2=xs_1\neq 0$. This is a contradiction. Thus, $c\neq 0$.

Consider $xT=x\langle c,s_1,\ldots,s_n\rangle$. Now $xT\neq 0$, so T is not contained in C. Thus, $T=M$. Note that $\dim T=n+1$, which must be equal to $\dim\{c_1,\ldots,c_r,s_1,\ldots,s_n\}$. Hence, $r=1$, and $\dim C=1$. Thus, we can represent $C=\langle c\rangle$.

Now, we claim that $\Phi(B)=C$. Note that $\Phi(B)$ is a nilpotent ideal of B that is strictly contained in every maximal subalgebra of B by Theorem 6.1 of [29]. In particular, $\Phi(B) \subsetneq M$. Recall that the only ideals of B strictly contained in M are C and 0. Thus, $\Phi(B) \subseteq C$.

Moreover, $C \subseteq Z(B)$ where $Z(B) = \{l \in L \big| l B = 0\}$. Then $0 \neq B^2 \subseteq M$ by definition of M gives that $C \subseteq B^2$. Now, $Z(B) \cap B^2 \subseteq \Phi(B)$. For suppose $Z(B) \cap B^2$ is not contained in $\phi(B)$. Then there is some maximal subalgebra $\mathscr{M}$ of B such that $Z(B) \cap B^2$ is not contained in $\mathscr{M}$. Then $L = (Z(B) \cap B^2) + \mathscr{M}$. Consider two elements of L, l and l_1. Each has the form of $b + m$, where $b \in Z(B) \cap B^2 \subseteq B$ and $m \in \mathscr{M}$.

$$l l_1 = (b + m)(b_1 + m_1) = m m_1 \in \mathscr{M}^{\in}$$

Thus, $L^2 \subseteq \mathscr{M}^{\in}$. Now,

$$Z(B) \cap B^2 \subseteq B^2 \subseteq L^2 \subseteq \mathscr{M}^{\in} \subseteq \mathscr{M},$$

which is a contradiction. Therefore, $Z(B) \cap B^2 \subseteq \phi(B)$.

Thus, $C \subseteq Z(B) \cap B^2 \subseteq \phi(B)$. Therefore $C = \phi(B)$.

Now assume first that $n = 1$. Then $s_1 = y \in M, y \notin C$, but $s_2 = x s_1 = x y$ is a linear combination of c and s_1. So $s_2 = c + \lambda_1 s_1$.

Suppose that $\lambda_1 \neq 0$. Then $\langle c + \lambda_1 s_1 \rangle$ is an ideal of B.

$$x(c + \lambda_1 s_1) = x c + \lambda_1 x s_1 = x c + \lambda_1 s_2,$$

which is equal to

$$x c + \lambda_1 (c + \lambda_1 s_1) = \lambda_1 (c + \lambda_1 s_1).$$

But $C = \langle c \rangle \neq \langle c + \lambda_1 s_1 \rangle \subset M$, which contradicts the fact that C is the unique ideal of B

contained in M.

Thus, λ_1 must be 0. Then $xs_1 = xy = c$, and the following multiplications hold.

$$\begin{array}{lll} xx = 0 & yx = -c & cx = 0 \\ xy = c & yy = 0 & cy \in M^2 = 0 \\ xc = 0 & yc \in M^2 = 0 & cc = 0 \end{array}$$

Hence, B is the three-dimensional Heisenberg algebra.

Assume now that $n \geq 2$.

If $\lambda_1 \neq 0$, then

$$K = \oplus_{i=2}^{n} \langle s_i \rangle \oplus \langle c + \lambda_1 s_1 \rangle + \langle x \rangle$$

is a maximal subalgebra of B. Since $\langle c \rangle = \Phi(B)$ and $\Phi(B)$ is properly contained in K, since K is a maximal subalgebra of B, $\langle c \rangle$ is properly contained in K. However, $\langle c \rangle$ is properly contained in K if and only if $\lambda_1 = 0$, which is a contradiction.

If $\lambda_1 = 0$, then

$$s_{n+1} = xs_n = c + \lambda_2 s_2 + \ldots + \lambda_n s_n.$$

Now let

$$K = \oplus_{i=2}^{n} \langle s_i \rangle \oplus \langle c \rangle \oplus \langle x \rangle.$$

K is a subalgebra of B. Consider

$$BK = (M \oplus \langle x \rangle)K = MK + \langle x \rangle K.$$

The following are true:

- $Ms_i \in M^2 = 0$,

- $Mc \in M^2 = 0$,

  and

- $M = \langle c, s_1, \ldots, s_n \rangle$. $cx = 0$.

- Also, if $m = \oplus_{i=1}^{n} v_i s_i$, then mx is some linear combination of $s_2, \ldots, s_n$, which is contained in K. Hence, $Mx \subset K$.

- Also, $xs_i = s_i + 1 \subseteq K$ for $i = 1, \ldots, n-1$, and $xs_n = s_{n+1} \in K$,

- $xc = 0$ by definition,

  and

- $xx = 0$ since B is a Lie algebra.

Thus, $BK \subseteq K$, and similarly, $KB \subseteq K$. Thus K is an ideal of B. Then K^2 is also an ideal of B. Now, K^2 is properly contained in M since $s_1 \in M$, but $s_1 \notin K^2$. Thus, $K^2 x = 0$ since $K^2 \subset M$. Therefore $K^2 = C$ or $K^2 = 0$ since K^2 is contained in $C_M(x)$ and $\dim C = 1$. However, $K^2 \neq 0$ since if $n = 2$, then $s_3 = xs_2 = c \neq 0$, and hence, $c \in K^2$. Thus, $K^2 = C$.

Now, $A + K$ is an ideal in L. To see this, consider

$$L = A \oplus B = A \oplus M \oplus \langle x \rangle = A \oplus \langle s_i \rangle_{i=1}^{n} \oplus \langle c \rangle \oplus \langle x \rangle.$$

Consider $a + s_j + c + x \in K$. First, let $l \in A$. Then

$$l(a + s_j + c + x) = la + ls_j + lc + lx.$$

Now, $la = 0$ since $l \in A$, and A is self-centralizing. Also, $ls_j, lc, lx \in A$ since A is an ideal in L. Thus,

$$l(a+s_j+c+x) \in A \subseteq A+K.$$

Next, let $l = s_k$ for some $k = 1, \ldots, n$. Then

$$l(a+s_j+c+x) = la+s_j+lc+lx.$$

Now, $la \in A$ since A is an ideal in L, and $ls_j, lc \in M^2 = 0$. Since lx is a linear combination of other s_h for $h = 2, \ldots, n$ and c, $lx \in K$. Thus,

$$l(a+s_j+c+x) \in A+K.$$

Now, let $l = c$. Then $la \in A$ since A is an ideal in L. Also, $ls_j \in M^2 = 0$, $lc = cc = 0$ since B is a Lie algebra, and $lx = cx = 0$ since c kills x. Thus,

$$l(a+s_j+c+x) \in A \subseteq A+K.$$

Finally, let $l = x$. Then $xa \in A$ since A is an ideal in L. Also, $xs_j \in K$ as above, $xc = 0$, and $xx = 0$. Thus,

$$l(a+s_j+c+x) \in A+K.$$

Hence, $L(A+K) \subseteq A+K$. Similarly, $(A+K)L \subseteq A+K$.

AK is also an ideal of L and AK is certainly contained in A. To verify, take $ak \in AK$.

First, let $l \in A$. Then

$$l(ak) = (la)k + a(lk) \in AK$$

since $la = 0$, and $lk \in K$. Also,

$$(ak)l = a(kl) - k(al) \in AK$$

since $kl \in K$ and $al = 0$.

Now, let $l \in B$. Then

$$l(ak) = (la)k + a(lk) \in AK$$

since $la \in A$ and $lk \in K$. Also,

$$(ak)l = a(kl) - k(al) \in AK$$

since $kl \in K$, and $al \in A$.

If L is a Lie algebra, then $KA = AK$, so $(ak)l \in AK$. If L is a Leibniz algebra that is not Lie, then $A =$Leib(L), and left multiplication by any element of A yields 0. Hence, $al = 0$, and $(ak)l = a(kl) \in AK$.

Thus, AK is an ideal in L.

KA is also an ideal in L. Note that KA is contained in A.

First, let $l \in A$. Then

$$l(ka) = (lk)a + k(la) \in KA$$

since $lk \in K$ and $la = 0$. Also,

$$(ka)l = k(al) - a(kl) \in KA$$

since $kl \in K$ and $la = 0$.

Next, let $l \in B$. Then

$$l(ka) = (lk)a + k(la) \in KA$$

since $lk \in K$ and $la \in A$. Also,

$$(ka)l = k(al) - a(kl) \in KA$$

since $al \in A$ and $kl \in K$.

As before, if L is a Lie algebra, then $AK = KA$, and $(ka)l \in KA$. If L is a Leibniz algebra that is not Lie, then $A =$Leib(L), and $al = 0$. Hence, $k(al) - a(kl) = k * 0 - 0 = 0 \in KA$.

Thus, KA is also an ideal in L.

Since A is self centralizing and K is not contained in A, $AK \neq 0$. Similarly, $KA \neq 0$. Thus $AK = KA = A$.

Now, $A + K$ is properly contained in L since $s_1 \notin K$, but $s_1 \in L$. Hence, $(A+K)^2$ must be nilpotent. Notice,

$$(A+K)^2 = A^2 + AK + KA + K^2 = A + K^2 = A + \langle c \rangle.$$

Now, $\langle c \rangle A = A\langle c \rangle = A$ since $A\langle c \rangle$ and similarly, $\langle c \rangle A$, is an ideal in L, $A\langle c \rangle \neq 0$, and $A\langle c \rangle \subseteq A$.

Then

$$(A+\langle c\rangle)^2 = A^2 + A\langle c\rangle + \langle c\rangle + \langle c\rangle\langle c\rangle = A.$$

Thus, $A+\langle c\rangle = (A+K)^2$ is not nilpotent, which is a contradiction.

Hence, the $n \geq 2$ case is not possible. □

Theorem 5.3.4. *Within the class of solvable Leibniz algebras, the property of being strongly solvable is 2-recognizeable.*

Proof. Let L be a solvable Leibniz algebra which is a minimal counterexample to the claim. In other words both of the following statements hold:

1. All 2-generated subalgebras of L are strongly solvable while L is not strongly solvable.

2. All proper subalgebras of L are strongly solvable. This follows from the fact that given a proper subalgebra A of L, every 2-generated subalgebra of A is a 2-generated subalgebra of L and is therefore strongly solvable. Thus A is strongly solvable.

Clearly, char $F = p > 0$ since all Leibniz algebras over fields of characteristic 0 are strongly solvable.

Let $A \neq 0$ be a minimal ideal in L. Then all 2-generated subalgebras of L/A are strongly solvable. Then L/A is strongly solvable since $\dim L/A < \dim L$, which is a minimal counterexample. Then $(L/A)^2$ is nilpotent. Also, $(L/A)^2 = (L^2 + A)/A$ is nilpotent.

Consider $\Phi(L)$, the Frattini ideal of L. If $A \subseteq \phi(L)$, then $L^2 + A$ is nilpotent, and hence, L^2 is nilpotent. Thus $\Phi(L) = 0$. It follows by Theorem 5.3.3 that $L = A \oplus B$ is a semidirect sum, where A is the unique minimal ideal of L, $\dim A \geq 2$, $A^2 = 0$, and B is either Type I or Type II as defined in the theorem.

It will be shown that in either case, L is 2-generated. This will be a contradiction since L is not strongly solvable, but every 2-generated subalgebra of L is strongly solvable. This contradiction will allow us to conclude that no such minimal counterexample exists, and the theorem holds.

1. Case I

   Let $B = M \oplus \langle x \rangle$ be a semidirect sum, where $M \subseteq B$ is a unique minimal idea of B such that $M^2 = 0$. Also, $L = A \oplus B$, with A a unique minimal abelian ideal, Leib(L). The goal is to show L is 2-generated.

   Let $b \in M$, with $b \neq 0$. Allow b to generate the ideal

   $$S = \langle b, xb, x(xb), \ldots \rangle = \langle b_1, xb_1 = b_2, xb_2 = b_3, \ldots, xb_{n-1} = b_n \rangle.$$

   Then $S \subseteq B$. However, the only ideal of B is M. Thus $S = M$.

   If $xb_1 = 0$, then B is abelian. Thus $n > 1$. Since $n > 1$, the smallest ideal that contains $b = b_1$ is M.

   Assume there exists no $a_1 \in A$ such that $b_2 a_1 \neq 0$. Then $b_2 \in Z_L^l(A) = A$. However, $b_2 \in A \cap B = 0$ and $b_2 \neq 0$. Thus there exists such an $a_1 \in A$. Let $T = \langle x, a_1 + b_1 \rangle$. The claim is that $T = L$, and thus L is 2-generated.

   T is composed of the following elements:

   - $x(a_1 + b_1) = a_2 + b_2 \in T$
   - $x(a_2 + b_2) = a_3 + b_3 \in T$

     $\vdots$

- $x(a_{n-1}+b_{n-1})=a_n+b_n\in T$
- $(a_2+b_2)(a_1+b_1)=a_2a_1+a_2b_1+b_2b_1+b_2a_1=b_2a_1\neq 0\in T$

Note since A and M are abelian, $a_2a_1=0=b_2b_1$, and left multiplication by A yields $a_2b_1=0$.

Thus $b_2a_1\in T$ and in A. From above, $a_i+b_i\in T$, and $x\in T$. Multiplying these elements together yields elements of the forms:

- $x(b_2a_1)\in T$
- $(a_i+b_i)(b_2a_1)=b_i(b_2a_1)\in T$.

Repeating this process yields

- $x(x(b_2a_1))\in T$
- $x(b_i(b_2a_1))\in T$
- $(b_j+a_j)(x(b_2a_1))=b_j(x(b_2a_1))\in T$
- $(b_j+a_j)(b_i(b_2a_1))=b_j(b_i(b_2a_1))\in T$.

Continuing in this manner it is clear that for all $t\in T, xt\in T$, and $(a_i+b_i)t\in T$. Thus $A\subseteq T$, and $a\in T$. Since $b_i\in T$ and $x\in T$, we have $B\subseteq T$ as well. Thus $L=A\oplus B=T$. Therefore L is 2-generated as desired.

2. Case II

   Let B be the three-dimensional Heisenberg algebra. B is Lie, and $B=\langle x,y,z\rangle$ with $xy=z=-yx$. Therefore B is 2-generated by x and y. It is also known that $L=A\oplus B$, with $A=$Leib(L). The goal is to show that L is 2-generated.

Assume there does not exist an $a \in A$ such that $xa \neq 0$. Then $xa = 0$ and $x \in C_L^l(A) = A$. This is a contradiction since $x \notin A, x \in B$. Therefore there must exist such an $a \in A$ with $xa \neq 0$. Let $T = \langle x, y + a \rangle$. The claim is that $T = L$, and thus L is 2-generated.

T is composed of the following elements:

- $x(y+a) = xy + xa = z + xa \in T$
- $x(z+xa) = xz + x(xa) = x(xa) \in T$
- $(y+a)(z+xa) = yz + y(xa) + az + a(xa) = y(xa) = y(z+xa) \in T$
- $(z+xa)(z+xa) = zz + z(xa) + xa(z) + xa(xa) = z(xa) = z(z+xa) \in T$

From above, the element $xa \in A$ with $xa \neq 0$, and now $xa \in T$. The smallest ideal that contains xa is the span$\{u_r(u_{r-1}(\ldots(u_1(xa))))\}$, with $u_i \in L$.

If $u_i = a \in A$ for some $1 \leq i \leq r$, then $u_r(\ldots(a(\ldots(u_1(xa))))= 0$ due to the left multiplication of an element in A. Thus $u_i = x, y$, or z. Then

- $x(xa) \in T$
- $y(xa) \in T$
- $z(xa) \in T$,

so the smallest ideal containing xa is in T. A was defined to be minimal, so $A \subseteq T$. Thus $a \in T$ and $x, y \in T$ implies $B \subseteq T$. Therefore, $L = A \oplus B = T$. Therefore L is 2-generated as desired.

□

Now, the aim is to extend Theorem 5.3.4 to nearly all finite-dimensional Leibniz algebras.

Theorem 5.3.5. *Let L be a finite-dimensional Leibniz algebra over an algebraically closed field of characteristic* 0 *or* $p > 5$*. If every 2-generated subalgebra of L is strongly solvable then L is strongly solvable.*

Proof. Let M be a Leibniz algebra such that every 2-generated subalgebra is strongly solvable and such that any Leibniz algebra N, with $\dim N < \dim M$ which has every 2-generated subalgebra strongly solvable, is strongly solvable.

If there exists a proper nonzero ideal I in M, then I and M/I have dimension less than M and are therefore strongly solvable by the assumption. Thus I and M/I are solvable. This implies that M is solvable. Then by Theorem 5.3.4, M is strongly solvable.

If there is not a proper nonzero ideal I in M, then M is a simple Lie algebra. In characteristic 0, all Lie algebras are 2-generated [19]. Consider the case when the underlying field $\mathbb{F}$ has characteristic $p > 5$. Since M is a simple Lie algebra over $\mathbb{F}$ and every subalgebra of M is solvable, it follows that M is either $sl(2)$ or some $W(1,n)$ [32]. But $sl(2)$ and $W(1,n)$ are each 2-generated. Thus M is 2-generated. But then M is strongly solvable by the assumption. □

Chapter

6

Triangulable

6.1 Background

Simultaneous triangulation of the set of linear transformations in a representation of a Leibniz algebra will now be considered. A necessary condition is that the minimum polynomials of the linear transformations are the products of linear factors. If the linear transformations commute, then this condition is also sufficient. Lie's theorem, and its extension to Leibniz algebras, is a generalization of this result, a result that is field dependent. It is possible to find a condition on the action of L on M which is necessary and sufficient for the simultaneous triangulation of the linear transformations in the representation. If, for each $x \in L$, the left multiplication of x on M is nilpotent, then it is shown in [10] that each right multiplication is also nilpotent and L acts nilpotently on M. Then L is said to be *nil* on M.

It can also be useful to have a condition on the action of L on M which guarantees simultaneous triangulation when extending to the algebraic closure, K, of the field of scalars, $\mathbb{F}$. This property will be called *triangulable*, borrowing from Bowman, Towers, and Varea [11]. Thus a Leibniz algebra L over a field $\mathbb{F}$ is triangulable on module M if the induced representation of $K \otimes L$ on $K \otimes M$ admits a basis such that the representing matrices are upper triangular. It is possible to find a condition that is equivalent to being triangulable. This condition is independent of the field. Lie's theorem is a consequence of the result. The condition is on the action of L on M although triangulable refers to action that is over K. As an application, it is possible to consider to what extent an algebra is triangulable if all 2-generated subalgebras have the property. Thus the property of being 2-recognizable will again be explored.

6.2 Results

Let M be a module for the Leibniz algebra L. Let left and right multiplication by $x \in L$ be denoted by T_x and S_x, respectively. If T_x is nilpotent, then S_x is also nilpotent and a generalization of Engle's theorem holds; that is, L acts nilpotently on M (see [10]). Define L to be nil on M if this property holds. If I is an ideal of a Leibniz algebra L, then I is nil on L if and only if I is nilpotent. It will be shown that L is triangulable on itself if and only if L^2 is nilpotent.

Lemma 6.2.1. *Let B be an irreducible submodule of the L-module M such that dim$(B)=1$. Then $L^2 \subseteq C_L(B)=\{a \in L | aB=0=Ba\}$.*

Proof. Let $x, y \in L$, $b \in B$ with $xb = \alpha_x b$ where α_x is a scalar. Then

$$(xy)b = x(yb) - y(xb) = x(\alpha_y b) - y(\alpha_x b) = \alpha_x \alpha_y b - \alpha_y \alpha_x b = 0.$$

Since B is irreducible, it follows that $b(xy) = 0$ ([6]). Hence $L^2 B = BL^2 = 0$ and the result holds. □

Theorem 6.2.2. *Let L be a Leibniz algebra and M be an L-module. There is a basis for M such that the representing matrices are in upper triangular form if and only if the linear transformations in the representation have minimum polynomials with all linear factors and L^2 is nil on M.*

This gives

Corollary 6.2.3. *Let L be a Leibniz algebra over an algebraically closed field and M be an L-module. There is a basis for M such that the representing matrices are in upper triangular form if and only if L^2 is nil on M.*

Proof of Theorem 6.2.2. The condition on the minimum polynomials is equivalent to being able to triangulate all the linear transformations. Suppose also that L^2 is nil on M. Then T_x, and hence S_x, is nilpotent for all $x \in L^2$ (see [10]) and L^2 acts nilpotently on M by Engel's theorem. Let B be an irreducible submodule of M. It is enough to show that dim$(B)=1$ for then the result will follow by induction. Since the submodule $BL^2 + L^2 B$ is contained in B,

$BL^2 + L^2 B$ is 0 or B. If the latter holds, then

$$B = BL^2 + L^2 B = (BL^2 + L^2 B)L^2 + L^2(BL^2 + L^2 B) = \dots,$$

which is never 0, a contradiction. Hence $BL^2 + L^2 B = 0$ and $BL^2 = 0 = L^2 B$. Therefore,

$$0 = (xy)b = x(yb) - y(xb)$$

for all $x, y \in L$, $b \in B$. Hence $x(yb) = y(xb)$ and $T_x T_y = T_y T_x$. Since B is irreducible, either $S_x = 0$ for all $x \in L$ or $S_x = -T_x$ for all $x \in L$ [6]. Then either

$$T_x S_y = -T_x T_y = -T_y T_x = S_y T_x$$

or

$$T_x S_y = 0 = S_y T_x$$

on B and

$$S_x S_y = T_x T_y = T_y T_x = S_y S_x$$

or

$$S_x S_y = 0 = S_y S_x$$

on B. Hence all T_x, S_y commute on B. Thus these transformations are simultaneously triangulable on B. Hence $\dim(B) = 1$ and the result holds in this direction.

Suppose that there is a basis for M such that all the transformations are triangulable on this basis. Then there is a one dimensional submodule B such that $L^2 B = 0 = BL^2$ by

Lemma 6.2.1. L^2 is nil on M/B by induction and hence on M. Since each transformation is triangulable, the minimum polynomials have linear factors. □

Now, these results can be stated when L is acting on itself.

Theorem 6.2.4. *Let L be a Leibniz algebra over an algebraically closed field. Then L is supersolvable if and only if L is strongly solvable.*

Proof. L is supersolvable if and only if L is triangulable on itself. By Corollary 6.2.3, this is equivalent to L^2 being nil on L which in turn is the same as L^2 being nilpotent. The last statement is the definition of strongly solvable. □

The general case looks like the following.

Theorem 6.2.5. *L is triangulable if and only if L^2 is nil on L.*

Proof. Let $L^* = K \otimes L$ where K is the algebraic closure of the base field for L. If L^2 is nil on L, then L^2 is nilpotent as is $(L^2)^* = (L^*)^2$. Hence L^* is supersolvable by the last result and L is triangulable by definition. The converse holds by reversing the steps. □

As mentioned in the previous chapter, a property is n-recognizeable in a class of algebras if whenever all n generated subalgebras have the property, then the algebra has the property. The concept when $n = 2$ has been considered in groups, Lie algebras and Leibniz algebras for classes that are solvable, supersolvable, nilpotent and strongly solvable (see [12], [14], [23]). A version of this has also been considered for triangulable Lie algebras in [11]. It is considered as an application of the result here.

Theorem 6.2.6. *In the class of solvable Leibniz algebras, the property of being triangulable is 2-recognizeable.*

Proof. Let N be a 2-generated subalgebra of L. Since N is triangulable, N^2 is nil on N and, thus, N is strongly solvable. Therefore L is strongly solvable by [14] and L^2 is nil on L. Then, using Theorem 6.2.5, L is triangulable. □

The result can be extended from solvable algebras to all algebras in characteristic 0, again using a result in [14].

Corollary 6.2.7. *For Leibniz algebras over a field of characteristic 0, the property of being triangulable is 2-recognizeable.*

Proof. Let L be a minimal counterexample. Simple Leibniz algebras are Lie algebras. Simple Lie algebras are 2-generated, hence L is triangulable. If I is a proper ideal in L, then all 2-generated subalgebras of I and L/I are strongly solvable by Theorem 6.2.5 and I and L/I are strongly solvable by Theorem 3 of [14] and L is solvable. The result holds by Theorem 6.2.6. □

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